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Isabelle Chalendar,
Pavel Gumenyuk & John E. McCarthy

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A NOTE ON COMPOSITION OPERATORS ON MODEL SPACES

**by Isabelle CHALENDAR,
Pavel GUMENYUK & John E. MCCARTHY (*)**

Abstract. — Motivated by the study of composition operators on model spaces launched by Mashreghi and Shabankhah we consider the following problem: for a given inner function $\phi \notin \mathsf{Aut}(\mathbb{D})$, find a non-constant inner function Ψ satisfying the functional equation $\Psi \circ \phi = \tau\Psi$, where τ is a unimodular constant. We prove that this problem has a solution if and only if ϕ is of positive hyperbolic step. More precisely, if this condition holds, we show that there is an infinite Blaschke product B satisfying the equation for $\tau = 1$. If in addition, ϕ is parabolic, we prove that the problem has a solution Ψ for *any* unimodular τ. Finally, we show that if ϕ is of zero hyperbolic step, then no non-constant Bloch function f and no unimodular constant τ satisfy $f \circ \phi = \tau f$.

Résumé. — Motivés par l'étude des opérateurs de composition sur les espaces modèles initiée par Mashreghi et Shabankhah, nous étudions le problème suivant : étant donnée une fonction intérieure ϕ qui n'est pas un automorphisme du disque unité, trouver une fonction intérieure non constante Ψ vérifiant l'équation fonctionnelle $\Psi \circ \phi = \tau\Psi$, où τ est une constante unimodulaire. Nous prouvons que ce problème a une solution si et seulement si ϕ est de pas hyperbolique positif. Plus précisément, si cette condition est satisfaite, nous montrons qu'il existe un produit de Blaschke infini B satisfaisant notre équation avec $\tau = 1$. De plus, si ϕ est parabolique, nous montrons que le problème a une solution Ψ pour tout τ unimodulaire. Enfin nous prouvons que si ϕ est de pas hyperbolique nul alors il n'existe pas de fonction non constante f de Bloch et il n'existe pas de constante unimodulaire τ vérifiant $f \circ \phi = \tau f$.

1. Introduction

A popular and successful subject in operator theory is the study of composition operators on Banach spaces of analytic functions. We refer to the

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monographs [10] by Cowen and McCluer and [22] by Shapiro for a comprehensive presentation. In these two books as well as in the vast literature on this subject, these operators are considered on Banach spaces of analytic functions in the unit disk $\mathbb{D} := \{z\colon |z| < 1\}$ such as the Hardy space $H^2(\mathbb{D})$. As a consequence of the Littlewood subordination principle, for all holomorphic self-maps ϕ of $\mathbb{D}$, the composition operator C_ϕ is linear and bounded on $H^2(\mathbb{D})$.(1)

The seminal Beurling's Theorem describes all the closed invariant subspaces of the forward shift S on $H^2(\mathbb{D})$. They have the form $\Theta H^2(\mathbb{D})$, where Θ is an inner function, that is a bounded analytic function whose radial limits are of modulus one almost everywhere. The so-called "model spaces" denoted by K_Θ are defined to be their orthogonal complement in $H^2(\mathbb{D})$. In other words, for Θ an inner function, $K_\Theta := H^2(\mathbb{D}) \ominus (\Theta H^2(\mathbb{D}))$. The K_Θ's are therefore the closed invariant subspaces of the backward shift S^*. Such model spaces are Hilbert spaces, whose dimension is infinite if and only if Θ is not a finite Blaschke product. Operator Theory on those subspaces has undergone a great development in the past twenty years with the study of the so-called Truncated Toeplitz Operators (TTO) [11, 12, 20].

Much less has been done on composition operators on model spaces. This study was initiated by Mashreghi and Shabankhah in 2013 [16, 17].

The complete characterization of compact composition operators on model spaces was obtained by Lyubarskii and Malinnikova [15, Theorem 1], who proved that the composition operator $C_\phi : K_\Theta \to H^2(\mathbb{D})$ is compact if and only if

$$\lim_{|z|\to 1^-} N_\phi(z)\frac{1-|\Theta(z)|^2}{1-|z|^2} = 0, \tag{1.1}$$

where N_ϕ is the Nevanlinna counting function defined by

$$N_\phi(z) := \begin{cases} \displaystyle\sum_{a\in\mathbb{D},\, \phi(a)=z} (1-|a|) \text{ (counted with multiplicity)} & \text{if } z\in\phi(\mathbb{D}),\\ 0 & \text{if } z\notin\phi(\mathbb{D}).\end{cases}$$

It follows that composition operators induced by inner functions ϕ on an infinite dimensional model space K_Θ are never compact. Indeed, by [21, p. 187], when ϕ is inner, $N_\phi(z) \approx 1-|z|^2$ as $|z|\to 1$. If Θ is not a finite Blaschke product, then by Frostman's theorem, see e.g. [13, Section II.6], for almost every $\alpha\in\mathbb{D}$, the composition $\Psi_\alpha\circ\Theta$, where $\Psi_\alpha(z) := \frac{\alpha - z}{1-\bar{\alpha}z}$, is an infinite Blaschke product. The sequence $(z_n)_{n\in\mathbb{N}}$ formed by the zeros of

(1) In fact, this statement holds for all Hardy spaces $H^p(\mathbb{D})$, $1 \leqslant p \leqslant \infty$; see e.g. [10, p. 123].

$\Psi_\alpha \circ \Theta$ tends to $\partial\mathbb{D}$. In this situation, a simple observation that $\Theta(z_n) = \alpha$ for all $n \in \mathbb{N}$ shows that (1.1) cannot hold.

From now on, suppose that ϕ is not the identity map and that Θ is not a finite Blaschke product. In [17, Theorem 4.1] they proved that if ϕ and Θ are inner functions, then $C_\phi K_\Theta \subset K_\Theta$ in exactly three distinct cases.(2)

Let $\rho_\lambda(z) := \lambda z$.

(I) ϕ is an elliptic automorphism and then either $\Theta(z) = \vartheta(z^n)$ for some integer $n \geqslant 2$, where ϑ is an arbitrary inner function not vanishing at 0, and

$$\phi = \rho_{e^{i2k\pi/n}}, \quad 1 \leqslant k \leqslant n;$$

or

$$\Theta(z) = z\tau_p(z)^m \Psi((\tau_p(z))^n),$$

where $p \in \mathbb{D}$ is the fixed point of ϕ, $\tau_p(z) := \frac{p-z}{1-\bar{p}z}$, $m \in \mathbb{N}_0$, $n \geqslant 2$, Ψ is inner and not a finite Blaschke product, and

$$\phi = \tau_p \circ \rho_{e^{i2k\pi/n}} \circ \tau_p, \quad 1 \leqslant k \leqslant n.$$

(II) ϕ has its Denjoy–Wolff point $\alpha \in \mathbb{T} := \partial\mathbb{D}$ and $\Theta(z) = z\Psi(z)$ where Ψ is an inner function, not a finite Blaschke product, such that $\Psi(\phi(z)) = \tau\Psi(z)$ for some constant τ of modulus one;

(III) ϕ has its Denjoy–Wolff point $\alpha \in \mathbb{T}$ and

$$\Theta(z) = z\Psi(z) \prod_{n \geqslant 0} w\big(\phi^{[n]}(z)\big),$$

where $\phi^{[n]}$ stands for the n^{th} iterate of ϕ, w is an inner function such that the product is convergent, and Ψ is an inner function, not a finite Blaschke product, such that $\Psi(\phi(z)) = \Psi(z)$.

In cases (I) and (II), the smallest model space containing the range of C_ϕ is K_Θ. In case (III) the smallest model space containing the range of C_ϕ is K_μ (see [16, Theorem 2.1]), with $\mu w = \Theta$, which implies that K_μ is strictly included in K_Θ.

The aim of this note is to study the solutions of the equation

$$\Psi \circ \phi = \tau\Psi \tag{1.2}$$

(2) Their paper actually lists 7 cases. We have excluded (i), (ii) and (iv) by our assumptions on ϕ and Θ, their cases (iii) and (v) correspond to (I) above, case (vi) is (II) and case (vii) is (III), with the unimodular factor γ removed as it is absorbed by the inner function Ψ.

whenever ϕ has its Denjoy–Wolff point on the unit circle and Ψ is not a finite Blaschke product. This contribution shows that, surprisingly, the theory of composition operators on model spaces is richer than expected.

Recall that the degree of an inner function Ψ is defined to be infinite when Ψ is not a finite Blaschke product and is $d \in \mathbb{N}$ when Ψ is a finite Blaschke product with d zeroes (taking into account their multiplicity). It is not difficult to check that the degree of the composition of two inner functions is equal to the product of their degrees. Therefore if Ψ is a finite Blaschke product of finite degree $d \in \mathbb{N}$, the existence of an inner function ϕ and $\tau \in \mathbb{T}$ such that (1.2) holds implies that ϕ is an automorphism.

Moreover, as already noticed in [17], it is not difficult to see that when $\phi \neq \mathrm{Id}$ has a fixed point $\alpha \in \mathbb{D}$, then

$$\Psi \circ \phi = \tau \Psi$$

for some non-constant inner function Ψ and $\tau \in \mathbb{T}$ implies that ϕ is an elliptic automorphism. Indeed, comparing the first non-constant terms in the Taylor expansions of $\Psi \circ \phi$ and $\tau\Psi$ at α, we see that $|\phi'(\alpha)| = 1$ and hence, by the Schwarz–Pick lemma, ϕ must be an automorphism of $\mathbb{D}$.

In other words, when ϕ is not an automorphism, the existence of a non-constant inner function Ψ and $\tau \in \mathbb{T}$ such that (1.2) holds implies that Ψ has infinite degree and ϕ has no fixed point in $\mathbb{D}$. We can now formulate the main question at the center of our investigation.

Question 1.1. — *Suppose $\phi : \mathbb{D} \to \mathbb{D}$ is inner, not an automorphism and with no fixed point in $\mathbb{D}$. Does there exist an inner function Ψ satisfying*

$$\Psi \circ \phi = \tau \Psi \tag{1.3}$$

for some unimodular constant τ?

We show that examples can be constructed if and only if ϕ is not of zero hyperbolic step.

The paper is organized as follows. In Section 2 we detail the classification of self-maps of the unit disk, recalling the notion of positive and zero hyperbolic step, as well as properties of Abel's function and Julia's lemma.

In Section 3 we prove our main result, Theorem 3.3, asserting that if ϕ is an inner function with positive hyperbolic step, then there exists an infinite Blaschke product B and $\tau \in \mathbb{T}$ such that $B \circ \phi = \tau B$.

In Section 4 we strenghten Theorem 3.3 to show that τ can always be taken equal to 1. Moreover, if ϕ is a parabolic map with positive hyperbolic step, we show that for every unimodular τ there exists an inner function Ψ satisfying (1.2).

Section 5 is devoted to the case when ϕ is a zero hyperbolic step map. In this case we can answer negatively our main question, and more than that, in Theorem 5.3, we prove that there exists no non-constant Bloch function[(3)] (in particular, no non-constant bounded holomorphic function f in $\mathbb{D}$) and no unimodular number τ satisfying $f \circ \phi = \tau f$.

We conclude the paper with explicit examples of ϕ of degree 2, Ψ a Blaschke product, and τ satisfying (1.3).

2. Background

We let $D(z_0, r)$ denote the open disk in $\mathbb{C}$ centered at z_0 of radius $r > 0$, $\mathbb{D} := D(0, 1)$, and let $\mathbb{H}$ denote the upper half-plane. For $a \in \mathbb{D}$, we let m_a be the Mobius automorphism

$$m_a(z) := -\frac{a}{|a|}\frac{z-a}{1-\bar{a}z} \quad \text{when } a \neq 0, \quad \text{and} \quad m_0 := \mathrm{Id}\,.$$

Denote by $\rho(\cdot,\cdot)$ the pseudo-hyperbolic distance in $\mathbb{D}$, that is

$$\rho(z_1, z_2) := \left|\frac{z_2 - z_1}{1 - \overline{z_2}\, z_1}\right|, \quad z_1, z_2 \in \mathbb{D}.$$

Let $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map. It is said to be *elliptic* if it has a fixed point in $\mathbb{D}$. If not, the Denjoy–Wolff theorem says that there is a unique attracting fixed point on the boundary, called the Denjoy–Wolff point, and the map is called *parabolic* if the angular derivative at the Denjoy–Wolff point equals 1, and otherwise the angular derivative is smaller than 1 and the map is called *hyperbolic*. See [1, 10, 22] for details.

Suppose that ϕ is not an elliptic automorphism. We say ϕ is of *positive hyperbolic step* if for some $z_0 \in \mathbb{D}$, we have

$$\lim_{n\to\infty} \rho\Big(\phi^{[n]}(z_0),\ \phi^{[n+1]}(z_0)\Big) > 0. \tag{2.1}$$

Note that the above limit always exists because by the Schwarz–Pick lemma, the sequence is non-increasing. Moreover, if (2.1) holds for some $z_0 \in \mathbb{D}$, then it holds for every $z_0 \in \mathbb{D}$; see e.g. [1, Corollary 4.6.9 (i)].

If ϕ is not of positive hyperbolic step, we say it is of *zero hyperbolic step*. All elliptic maps are of zero hyperbolic step[(4)], hyperbolic maps are of positive hyperbolic step, and parabolic maps can be either [19].

Our main result is that the answer to Question 1.1 is yes if and only if ϕ is an inner function of positive hyperbolic step.

(3) In fact, this holds also with Bloch functions replaced by normal functions, see Remark 5.5.

(4) We conventionally regard elliptic automorphisms as having zero hyperbolic step.

Remark 2.1. — With the help of conformal mappings, the above classification of holomorphic self-map extends in a natural way to any simply connected domain $D \subset \mathbb{C}$ different from the whole complex plane $\mathbb{C}$. In particular, when dealing with hyperbolic and parabolic self-maps, it is often more convenient to change to a half-plane, with the Denjoy–Wolff point mapped to ∞. Note that in such a case, the self-map is hyperbolic if the angular derivative at ∞ is strictly bigger than 1 and parabolic if it equals 1. The definition of the angular derivative at ∞ can be found in [24, Section 26]. Note also that one can distinguish between parabolic and hyperbolic self-maps using a more intrinsic *divergence rate* instead of the angular derivative at the Denjoy–Wolff point; see e.g. [1, Proposition 4.6.6].

Let ϕ be a non-elliptic self-map of $\mathbb{D}$. Then Cowen [9] proved that there is an Abel function, i.e., a holomorphic function $h : \mathbb{D} \to \mathbb{C}$ that is a solution of the equation

$$h \circ \phi = h + 1 \tag{2.2}$$

and that is univalent on some ϕ-absorbing[(5)] domain $\mathcal{U}$ in $\mathbb{D}$. Moreover, we can arrange that

$$\Omega := \bigcup_{n \in \mathbb{N}} h(\mathbb{D}) - n \tag{2.3}$$

coincides with:

- $\mathbb{H}$ if ϕ is parabolic of positive hyperbolic step,
- $\mathbb{C}$ if it is of zero hyperbolic step, and
- a horizontal strip $\{a < \operatorname{Im}(z) < b\}$ if it is hyperbolic.

The Abel function is then unique up to an additive constant.

For $z_0 \in \mathbb{D}$ we define the *grand orbit of* z_0, denoted $\mathcal{Z}_{z_0}$, by

$$\mathcal{Z}_{z_0} := \Big\{ \zeta \in \mathbb{D} : \exists\, n,\ m \in \mathbb{N}_0 \quad \text{s.t.} \quad \phi^{[n]}(\zeta) = \phi^{[m]}(z_0) \Big\}.$$

For $\omega \in \mathbb{T},\ M > 0$, let

$$\begin{aligned} H(\omega, M) &:= \left\{ z : \frac{|z-\omega|^2}{1-|z|^2} < M \right\} \\ &= D\left(\frac{1}{M+1}\omega,\ \frac{M}{M+1} \right). \end{aligned}$$

Such sets are usually called *horodisks* at ω.

(5) Given a self-map $\phi : X \to X$, a set $Y \subset X$ is called *ϕ-absorbing* if $\phi(Y) \subset Y$ and if for any $x \in X$ there exists $n \in \mathbb{N}$ such that $\phi^{[n]}(x) \in Y$.

Julia's lemma [14] (or e.g. [2, Theorem 5.9]) says that if ϕ admits at ω a n.t.(= non-tangential) limit $\eta \in \mathbb{T}$ and finite angular derivative a, then $\phi(H(\omega, M)) \subseteq H(\eta, aM)$. Moreover, if ϕ is not an automorphism of $\mathbb{D}$, then also $\phi(\partial H(\omega, M) \setminus \{\omega\}) \subset H(\eta, aM)$.

3. Positive Hyperbolic Step

Throughout this section, we let ϕ be a non-elliptic inner function of positive hyperbolic step, with Abel function h as in (2.2). We shall let $\mathcal{Q}$ denote the exceptional set of ϕ, that is

$$\mathcal{Q} := \{a : m_a \circ \phi \text{ has a singular factor}\}.$$

By Frostman's theorem, the set $\mathcal{Q}$ is of logarithmic capacity zero, and in particular it has zero area measure. See [13, Section II.6] for more details.

LEMMA 3.1. — *For each $z_0 \in \mathbb{D}$, the grand orbit $\mathcal{Z}_{z_0}$ satisfies the Blaschke condition (with each element of $\mathcal{Z}_{z_0}$ counted once).*

Proof. — Let h be the Abel map. We have

$$h(\mathcal{Z}_{z_0}) \subseteq \{h(z_0) + n : n \in \mathbb{Z}\}.$$

Since ϕ is of positive hyperbolic step, the function

$$\exp(2\pi i h(z)) - \exp(2\pi i h(z_0))$$

is bounded and non-constant. As the zero set of this function contains $\mathcal{Z}_{z_0}$, we conclude that the grand orbit of z_0 satisfies the Blaschke condition. □

Two sets $\mathcal{Z}_{z_0}$ and $\mathcal{Z}_{z_1}$ intersect if and only if they coincide, and in that case $h(z_1) - h(z_0) \in \mathbb{Z}$.

LEMMA 3.2. — *There exists z_0 with $\mathcal{Z}_{z_0} \cap (\mathcal{Q} \cup \{z \in \mathbb{D} : \phi'(z) = 0\})$ empty.*

Proof. — Let $\mathcal{Q}'$ denote $\mathcal{Q} \cup \{z \in \mathbb{D} : \phi'(z) = 0\}$. Note that $h(\mathcal{Q}')$ has zero area measure. Indeed, otherwise there exists some $r < 1$ so that $h(\mathcal{Q}' \cap D(0,r))$ has positive area. But on $\overline{D(0,r)}$ the function h has bounded derivative, so maps sets of area 0 to sets of area 0.

If $\mathcal{Z}_z \cap \mathcal{Q}'$ is non-empty then

$$h(z) \in \bigcup_{k \in \mathbb{Z}} h(\mathcal{Q}') + k. \tag{3.1}$$

As $\bigcup_{k \in \mathbb{Z}} h(\mathcal{Q}') + k$ has measure 0, it follows that

$$\{h(z) : \mathcal{Z}_z \cap \mathcal{Q}' \neq \emptyset\}$$

has measure zero. Since $h(\mathbb{D})$ has non-zero measure, it follows that there exists z_0 with $\mathcal{Z}_{z_0} \cap \mathcal{Q}'$ empty. $\square$

THEOREM 3.3. — *Let ϕ be a non-elliptic inner function with positive hyperbolic step. Then there exists a Blaschke product B and a unimodular constant τ so that*

$$B \circ \phi = \tau B. \tag{3.2}$$

Proof. — Choose z_0 as in Lemma 3.2. Let B be the Blaschke product whose zero set is $\mathcal{Z}_{z_0}$, with all zeros simple, that is

$$B(z) := \prod_{a \in \mathcal{Z}_{z_0}} m_a(z).$$

This product converges by Lemma 3.1. Since $\phi^{-1}(\mathcal{Z}_{z_0}) = \mathcal{Z}_{z_0}$ and ϕ' does not vanish in $\mathcal{Z}_{z_0}$, the zero-set of the inner function $B \circ \phi$ is also $\mathcal{Z}_{z_0}$, again with all zeros simple. Therefore,

$$B \circ \phi = \tau u B,$$

where τ is unimodular, and u is either 1 or a singular inner function. It remains to show that there is no singular inner factor.

Note that

$$B \circ \phi = \prod_{a \in \mathcal{Z}_{z_0}} m_a \circ \phi.$$

Since $\mathcal{Z}_{z_0}$ is disjoint from $\mathcal{Q}$, each factor on the right-hand side is a Blaschke product times a scalar, and therefore so is their product. Hence $B \circ \phi$ has no singular factor. $\square$

4. Strengthening of Theorem 3.3

We can strengthen Theorem 3.3 in two ways.

THEOREM 4.1. — *Let ϕ be a parabolic inner function with positive hyperbolic step. Then for every unimodular τ there exists an inner function u so that*

$$u \circ \phi = \tau u. \tag{4.1}$$

Moreover, in Theorem 3.3 we can always choose $\tau = 1$.

THEOREM 4.2. — *Let ϕ be a non-elliptic inner function with positive hyperbolic step. Then there exists a Blaschke product B satisfying*

$$B \circ \phi = B.$$

Before proving these theorems, we need the following results. The first one follows from [3, Proposition 3.10]; we include a proof for this special case.

Proposition 4.3. — *Suppose ϕ is a non-elliptic self-map of $\mathbb{D}$ with Abel map h and let $\lambda \in \mathbb{C}\backslash\{0\}$. Then any holomorphic function F satisfying*

$$F \circ \phi = \lambda F \tag{4.2}$$

is of the form $F = G \circ h$ for some holomorphic function G on Ω that satisfies

$$G(w+1) = \lambda G(w). \tag{4.3}$$

Proof. — Note first that $h(a) = h(b)$ if and only if $\phi^{[n]}(a) = \phi^{[n]}(b)$ for some $n \in \mathbb{N}$. Indeed, as

$$h \circ \phi^{[n]} = h + n,$$

if $\phi^{[n]}(a) = \phi^{[n]}(b)$ then $h(a) = h(b)$. Conversely, if $h(a) = h(b)$, let $n \in \mathbb{N}$ be large enough that $\phi^{[n]}(a)$ and $\phi^{[n]}(b)$ lie in $\mathcal{U}$. As

$$h \circ \phi^{[n]}(a) = h \circ \phi^{[n]}(b)$$

and h is univalent on $\mathcal{U}$, it follows that $\phi^{[n]}(a) = \phi^{[n]}(b)$.

Suppose that F satisfies (4.2). Then F is constant on level-sets of h, so there is a well-defined function $G = F \circ h^{-1}$ on $h(\mathbb{D})$. Moreover, G is holomorphic at every point w such that some pre-image of w under h is not a critical point of h. In particular, G is holomorphic in $h(\mathcal{U})$ and satisfies

$$G(w+1) = \lambda G(w) \quad \text{for all} \quad w \in h(\mathcal{U}). \tag{4.4}$$

We extend G holomorphically to all of

$$\Omega = \bigcup_{n\in\mathbb{N}} h(\mathbb{D}) - n = \bigcup_{n\in\mathbb{N}} h(\mathcal{U}) - n$$

by setting[6] $G(w) := \lambda^{-n}\, G(w+n)$ for all $w \in h(\mathcal{U}) - n$ and $n \in \mathbb{N}$. □

Proposition 4.4. — *Suppose ϕ is an inner function with positive hyperbolic step, and with Abel map $h : \mathbb{D} \to \Omega$, where Ω is as in (2.3). At a.e. point of $\mathbb{T}$, the Abel map h has a non-tangential limit that is in $\partial\Omega$.*

Proof. — As $h : \mathbb{D} \to \Omega$, by Fatou's theorem, a.e. in $\mathbb{T}$, it has non-tangential limits that lie in $\overline{\mathbb{H}} \cup \{\infty\}$ if $\Omega = \mathbb{H}$, and in $\overline{\Omega} \cup \{\pm\infty\}$ if Ω is a horizontal strip. Moreover these limits must be finite a.e., which follows from either the F. and M. Riesz theorem [8, Theorem 2.5] or Privalov's theorem [8, Theorem 8.1].

[6] Note that given any $w \in \Omega$, thanks to (4.4), the expression $\lambda^{-n}\, G(w+n)$ has the same value for all $n \in \mathbb{N}$ such that $w \in h(\mathcal{U}) - n$.

By Theorem 3.3, we know that there is a Blaschke product B so that $B\circ\phi = \tau B$. By Proposition 4.3, we have $B = G\circ h$, where G is holomorphic in Ω and satisfies $G(w+n) = \tau^n G(w)$ for any $w \in \Omega$ and any $n \in \mathbb{N}$. Since $G(h(\mathbb{D})) = B(\mathbb{D}) \subset \mathbb{D}$, and since for any $w \in \Omega$ there exists $n \in \mathbb{N}$ such that $w + n \in h(\mathbb{D})$, it follows that $G(\Omega) \subset \mathbb{D}$. Therefore, if at some $\zeta \in \mathbb{T}$, the n.t. limit $h(\zeta)$ of h exists and belongs to Ω, then the n.t. limit of B at ζ is $G(h(\zeta)) \in \mathbb{D}$. Since B is inner, the latter may occur only on a set $E \subset \mathbb{T}$ of linear measure zero. □

Proof of Theorem 4.1. — We are in the case that $\Omega = \mathbb{H}$. By Proposition 4.4, we know that h has real n.t. boundary limits a.e. Define for $0 < \theta \leqslant 2\pi$,

$$u_\theta := \exp(i\theta h).$$

Then each function u_θ is a bounded non-vanishing function on $\mathbb{D}$. Moreover, u_θ has a unimodular boundary value a.e., so is inner. The Abel equation implies

$$u_\theta(\phi(z)) = e^{i\theta} u_\theta(z).$$

This shows that we can solve (4.1) for every unimodular τ. □

Proof of Theorem 4.2. — First we claim there is always an inner function u_0 satisfying

$$u_0(\phi(z)) = u_0(z). \tag{4.5}$$

Indeed, if ϕ is parabolic, this was proved in Theorem 4.1. Assume instead that ϕ is hyperbolic, and define $v := \exp(ih)$. Then v is a bounded non-vanishing function that maps $\mathbb{D}$ into an annulus, satisfies $v(\phi(z)) = v(z)$, and by Proposition 4.4 has non-tangential boundary limits belonging to the boundary of the annulus a.e.

Let f be an Ahlfors map from this annulus to the unit disk, see e.g. [5, Chapter 13]; this is a two-to-one map that maps the boundary of the annulus to the boundary of the disk. Then

$$u_0 := f \circ \exp(ih)$$

is an inner function satisfying (4.5).

Having found an inner function satisfying (4.5), we use a common trick (see for example [6, 7, 23]) to extract a Blaschke product that also satisfies (4.5). For each $a \in \mathbb{D}$, consider the function $\Psi_a := m_a \circ u_0$. By Frostman's theorem [13, Theorem II.6.4], for all a except for an exceptional set of capacity zero, this transformed function will be a Blaschke product times a unimodular scalar, and it is immediate that each Ψ_a is also invariant under composition with ϕ. □

5. Zero hyperbolic step

If ϕ is of zero hyperbolic step, it is either elliptic or parabolic. As explained in Section 1, in the elliptic case, when the Denjoy–Wolff point is in $\mathbb{D}$, there can be no non-constant holomorphic solution f to the equation

$$f \circ \phi = \tau f$$

with unimodular τ unless ϕ is an automorphism.

A holomorphic map $\phi : \mathbb{D} \to \mathbb{D}$ is parabolic if its Denjoy–Wolff point is on the boundary, and it has angular derivative 1 there. Pommerenke [19] showed that there exist parabolic self-maps of zero hyperbolic step as well as parabolic self-maps of positive hyperbolic step; see the definition in Section 2.

For convenience, we will change variables to the right half-plane $\mathbb{P}$ and assume the Denjoy–Wolff point is at ∞. Let us start with some initial point $z_0 = 1$, and define $z_n := x_n + iy_n = \phi^{[n]}(1)$. Pommerenke proved the following [19, Theorem 1].

Theorem 5.1. — *Let ϕ be a parabolic self-map of $\mathbb{P}$. Define*

$$g_n(z) = \frac{\phi^{[n]}(z) - iy_n}{x_n}.$$

Then $\lim_{n\to\infty} g_n(z) =: g(z)$ exists locally uniformly in $\mathbb{P}$, and satisfies $g(\phi(z)) = \psi(g(z))$, where ψ is a Moebius transformation of $\mathbb{P}$ that leaves ∞ fixed. Moreover, if ϕ is of positive hyperbolic step, then ψ is parabolic, and if ϕ is of zero hyperbolic step then $g(z) \equiv 1$.

In [4], Baker and Pommerenke proved the following.

Theorem 5.2. — *Let ϕ be a parabolic self-map of $\mathbb{P}$ of zero hyperbolic step with Denjoy–Wolff point at ∞. Define*

$$h_n(z) = \frac{\phi^{[n]}(z) - z_n}{z_{n+1} - z_n}.$$

Then $h(z) := \lim_{n\to\infty} h_n(z)$ exists locally uniformly in $\mathbb{P}$ and satisfies $h(\phi(z)) = h(z) + 1$.

Using these results, we can show that C_ϕ can never have a Blaschke product as an eigenvector with unimodular eigenvalue if ϕ is of zero hyperbolic step. Indeed, we show slightly more. The Bloch space is the set of holomorphic functions f on the unit disk for which

$$\sup_{z\in\mathbb{D}} \big(1 - |z|^2\big)|f'(z)| < \infty.$$

Note that it follows from the Schwarz–Pick lemma that any bounded holomorphic function in $\mathbb{D}$ is a Bloch function.

THEOREM 5.3. — *Let ϕ be a parabolic self-map of $\mathbb{D}$ with zero hyperbolic step. Then there does not exist a non-constant Bloch function f and a unimodular number τ satisfying*

$$f \circ \phi = \tau f.$$

We will use the following special case of [3, Lemma 3.16]; see also [1, Corollary 4.6.9 (iv)]. An elementary proof is included below.

LEMMA 5.4. — *Let ϕ be a parabolic self-map of $\mathbb{D}$ of zero hyperbolic step. For any pair of points $z_0, w_0 \in \mathbb{D}$, we have*

$$\lim_{n\to\infty} \rho\Big(\phi^{[n]}(z_0),\ \phi^{[n]}(w_0)\Big) = 0.$$

Proof. — Let us change to the right-half plane $\mathbb{P}$ by the conformal map F of $\mathbb{D}$ onto $\mathbb{P}$ that takes the Denjoy–Wolff point of ϕ to ∞ and z_0 to 1. To simplify notation, we will write ϕ, z_0 and w_0 instead of $F \circ \phi \circ F^{-1}$, $F(z_0) = 1$ and $F(w_0)$, respectively. Let $z_n := x_n + iy_n = \phi^{[n]}(z_0) = \phi^{[n]}(1)$, $w_n := \phi^{[n]}(w_0)$. We wish to show that

$$\lim_{n\to\infty} \rho(w_n, z_n) = \lim_{n\to\infty} \left| \frac{w_n - z_n}{w_n + \overline{z_n}} \right| = 0.$$

Write

$$\frac{w_n - z_n}{w_n + \overline{z_n}} = \frac{w_n - z_n}{z_{n+1} - z_n} \frac{z_{n+1} - z_n}{w_n + \overline{z_n}}. \tag{5.1}$$

By Theorem 5.2, the first fraction on the right-hand side of (5.1) tends to $h(w_0)$. We wish to show that the second fraction tends to 0. Let $0 < \varepsilon < 1$. By Theorem 5.1,

$$\frac{\phi^{[n]}(w) - z_n}{x_n} = g_n(w) - 1 \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty. \tag{5.2}$$

Applying (5.2) twice: for $w := w_0$ and for $w := \phi(1)$, we get that there exists N so that for all $n \geqslant N$ we have

$$|w_n - z_n| \leqslant \varepsilon x_n,$$
$$|z_{n+1} - z_n| \leqslant \varepsilon x_n.$$

Therefore,

$$\left| \frac{z_{n+1} - z_n}{w_n + \overline{z_n}} \right| \leqslant \frac{\varepsilon x_n}{(2 - \varepsilon) x_n}.$$

As ε is arbitrary, the limit of (5.1) is 0. ☐

Proof of Theorem 5.3. — Suppose f is a Bloch function satisfying $f \circ \phi = \tau f$, and let

$$M := \sup_{\zeta \in \mathbb{D}} \big(1 - |\zeta|^2\big) |f'(\zeta)|.$$

Then for any points z, w in the disk, we have the inequality

$$|f(z) - f(w)| \leqslant M d_h(z, w),$$

where

$$d_h(z, w) := \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}$$

is the hyperbolic metric. (Simply integrate $|f'(\zeta)| \leqslant M(1 - |\zeta|^2)^{-1}$ along the hyperbolic geodesic segment joining z and w.) As τ is unimodular, we get

$$\left| \frac{f(z) - f(w)}{z - w} \right| = \left| \frac{f(\phi^{[n]}(z)) - f(\phi^{[n]}(w))}{z - w} \right| \leqslant \frac{M}{|z - w|} \, d_h\Big(\phi^{[n]}(z), \ \phi^{[n]}(w)\Big).$$

As $d_h(\phi^{[n]}(z), \phi^{[n]}(w)) \to 0$ by Lemma 5.4, we conclude that f is constant. □

Remark 5.5. — Following essentially the same argument, but with the Euclidean distance replaced by the spherical one, one can show that Theorem 5.3 holds also for the wider class formed by all normal functions(7) in the unit disk.

6. Examples

Example 6.1. — Let

$$b(z) := \frac{z + \alpha}{1 + \alpha z}$$

$$\phi(z) := b(z)^2 = \left(\frac{z + \alpha}{1 + \alpha z} \right)^2,$$

where $\frac{1}{3} < \alpha < 1$. Then there exists a Blaschke product B so that $B \circ \phi = -B$, and hence $B^2 \circ \phi = B^2$.

(7) For the definition and more details on normal functions, we refer readers to [18, Section 9.1].

Proof. — We use the following properties of ϕ:

(1) ϕ has Denjoy–Wolff point at $1 \in \partial\mathbb{D}$.
(2) The angular derivative at the Denjoy–Wolff point is
$$a = 2\,\frac{1-\alpha}{1+\alpha} \; < 1.$$
(3) ϕ is a finite Blaschke product, so its exceptional set is empty.
(4) The only zero of ϕ' is at $-\alpha$.

Let us divide the grand orbit $\mathcal{Z}_0$ into two sets:
$$\mathcal{M}_2 := \Big\{ z \in \mathcal{Z}_0 : \exists \;\; n \geqslant 0 \quad \text{s.t.} \quad \phi^{[n]}(z) = -\alpha \Big\}$$
$$\mathcal{M}_1 := \mathcal{Z}_0 \setminus \mathcal{M}_2.$$

We shall let B be the Blaschke product with zeroes in $\mathcal{Z}_0$, with multiplicity 2 for points in $\mathcal{M}_2$ and multiplicity 1 for points in $\mathcal{M}_1$.

The point $z_0 := 0$ belongs to the boundary of the horodisk $H(1,1)$. Hence by Julia's lemma, see Section 2, the pre-images of z_0 lie outside $H(1, \frac{1}{a})$. Their pre-images lie outside $H(1, a^{-2})$, and so on. For $m \in \mathbb{N}$, let z_m denote $\phi^{[m]}(z_0)$. Consider the pre-images of z_1. There are two: z_0, and the solution to $b(z) = -b(z_0)$. This point is
$$\zeta_0 := -\frac{2\alpha}{1+\alpha^2},$$
which is in the boundary of $H(1, \frac{1-\zeta_0}{1+\zeta_0}) = H(1, \frac{4}{a^2})$.

For each $m \in \mathbb{N}$, consider the pre-images of $\phi(z_m)$. There are two. One is z_{m-1}; call the other ζ_{m-1}. This is the solution to the equation

$$b(\zeta_{m-1}) = -b(z_{m-1}). \tag{6.1}$$

This gives

$$\zeta_{m-1} = \frac{\zeta_0 - z_{m-1}}{1 - \zeta_0 z_{m-1}}. \tag{6.2}$$

Notice that each ζ_{m-1} is negative.

By an argument similar to the proof of Theorem 3.3, we have $B \circ \phi = \tau B$ for some $\tau \in \mathbb{T}$. Indeed, $B \circ \phi$ is a unimodular multiple of a Blaschke product, it has zeroes only in the set $\mathcal{Z}_0$, and their multiplicity is 1 on the set $\mathcal{M}_1$ and 2 on $\mathcal{M}_2$.

Note that $B(0) = 0$. All the zeroes of B are symmetric with respect to the real axis. This is because $\phi^{[n]}(\bar{z}) = \overline{\phi^{[n]}(z)}$, so if $z \in \mathcal{Z}_0$, we have $\phi^{[n]}(z) = z_m$ for some $m, n \in \mathbb{N}_0$, and as z_m is always real, this means that $\phi^{[n]}(\bar{z}) = z_m$ also, so $\bar{z} \in \mathcal{Z}_0$. Therefore B is real on $(-1,1)$. Moreover, as ϕ is real on $(-1,1)$, so is $B \circ \phi$. Therefore τ is real.

There are no zeroes of B on the line segment between 0 and $z_1 = \phi(0) = \alpha^2$. Indeed, all the pre-images of z_0 are outside $H(1,1)$, so in particular do not lie in the set $(0,1)$. Moreover, each ζ_n is negative, so it and all its pre-images also lie outside $H(1,1)$. So the only zeroes of B that lie on the line segment $(0,1)$ are the points z_m, for $m \geqslant 1$. These points increase. Indeed, each z_n is on the boundary of a horodisk H internally tangent at 1, and $\phi(\partial H \setminus \{1\}) \subset H$ by Julia's lemma. Therefore there are no intermediate zeroes between z_0 and z_1.

As each zero of B on $[0,1)$ is of multiplicity 1, the sign of B' will alternate as one moves along the real axis. By the chain rule,

$$B'(z_1)\phi'(0) = \tau B'(0).$$

As $\phi'(0) = 2\alpha(1-\alpha^2) > 0$, and $B'(0)$ and $B'(z_1)$ have opposite signs, we conclude that τ must be negative. □

Example 6.2. — Let $\alpha = \frac{1}{3}$ in the previous example, so

$$\phi(z) = \frac{(z+1/3)^2}{(1+z/3)^2}.$$

Then ϕ has a fixed point at 1, and $\phi'(1) = 1$, so it is parabolic. We show that it is of zero hyperbolic step, and hence according to Theorem 5.3, the answer to Question 1.1 is negative for this inner function ϕ.

Proof. — We calculate, if $z \in (-1,1)$ is real:

$$\begin{aligned} \rho(z,\phi(z)) &= \left| \frac{z - \left(\frac{z+\frac{1}{3}}{1+\frac{1}{3}z}\right)^2}{1 - z\left(\frac{z+\frac{1}{3}}{1+\frac{1}{3}z}\right)^2} \right| \\ &= \frac{(1-z)^2}{9z^2+14z+9}. \end{aligned}$$

If we let $z_0 = 0$, then $z_n \to 1$, and each z_n is real. But we see that as $z_n \to 1$, we have $\rho(z_n, z_{n+1}) \to 0$. □

Remark 6.3. — Another (less direct) way to show that ϕ in the above example is of zero hyperbolic step is based on the fact that all orbits of a parabolic self-map of *positive* hyperbolic step converge to the Denjoy–Wolff point *tangentially* to $\mathbb{T}$; see [19, Remark 1] or [1, Corollary 4.6.10].

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Isabelle CHALENDAR
Université Gustave Eiffel,
LAMA, (UMR 8050),
UPEM, UPEC, CNRS,
F-77454, Marne-la-Vallée (France)
isabelle.chalendar@univ-eiffel.fr

Pavel GUMENYUK
Department of Mathematics,
Politecnico di Milano,
via E. Bonardi 9,
20133 Milan (Italy)
pavel.gumenyuk@polimi.it

John E. MCCARTHY
Department of Mathematics,
Washington University,
One Brookings Drive,
St. Louis, MO 63130 (USA)
mccarthy@wustl.edu