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NORMALITY OF MINIMAL LOG CANONICAL CENTERS OF THREEFOLDS IN MIXED AND POSITIVE CHARACTERISTIC

by Emelie ARVIDSSON & Quentin POSVA (*)

ABSTRACT. — We prove the normality of minimal log canonical centers on threefold pairs whose residue fields are perfect of residue characteristics $p \neq 2, 3$ and 5. We also show that the union of all log canonical centers on threefold pairs with standard coefficients are seminormal provided that the residue characteristics are large enough. In contrast, we provide an example of a non-seminormal log canonical center on a threefold in characteristic 3, and give sufficient conditions to construct similar examples.

RÉSUMÉ. — On prouve la normalité des centres log canoniques minimaux sur les paires de dimension trois dont les caractéristiques résiduelles sont différentes de 2, 3 et 5. On montre également que l'union de tous les centres log canoniques sur les paires de dimension trois à coefficients standards est semi-normale dès que les caractéristiques résiduelles sont assez grandes. Par contraste, on décrit un exemple de centre log canonique non-semi-normal sur une variété tri-dimensionnelle en caractéristique 3, et on donne des conditions suffisantes pour construire de tels exemples.

1. Introduction

Log canonical centers play a central role in the birational geometry of varieties. The worst singularities of a log canonical variety are concentrated at its log canonical centers, and the understanding of those is oftentimes a key step to the understanding of the whole variety. In particular, the cohomological and topological properties of log canonical centers are worth investigating.

General properties of log canonical centers are well-understood in characteristic 0: see for example [25, Section 4.3]. It is known that in characteristic 0 log canonical centers are always seminormal and that the minimal

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ones are actually normal. These properties can be established in two steps. First, one studies the log canonical centers of dlt pairs, which are particularly simple [25, Section 4.2]. Then one relates the log canonical centers on lc pairs to the ones on dlt pairs, using dlt modifications and some vanishing theorems.

The situation in positive characteristic and mixed characteristic is more complicated and not as well-understood. In particular, some crucial vanishing theorems are simply false. Without restriction on the characteristics, even log canonical centers on dlt pairs need not be well-behaved [7, 11]. While the surface theory is quite similar in equicharacteristics 0 and $p > 0$ and in mixed characteristic [25, Sections 2.2–3.3], already for threefolds significant differences appear.

Let us give a summary of the current understanding of log canonical centers in positive and mixed characteristics.

Plt centers in arbitrary dimensions. — For every prime $p > 0$ we have examples of non-normal plt centers in characteristic p and dimension $2p + 2$ [7].

Plt centers on threefolds. — Normality in characteristic > 5 is shown in [18, Theorem 3.11, Proposition 4.1]. Without restriction on the characteristic, plt centers on threefolds are known to be normal up to universal homeomorphism [17, Theorem 1.2] (see also [16, Section 3.2]). This is the best we can hope for in general, since in [11] an example of non-normal plt center on a threefold in characteristic 2 is given.

Minimal log canonical centers on threefolds. — Normality up to universal homeomorphism in mixed characteristic different from 2, 3 and 5 follows from [15]. For minimal log canonical centers on semi-log canonical threefolds in characteristic > 5 , this is also proved in [29, Theorem 7]. Minimal log-canonical centers of dlt-pairs are normal in equicharacteristic > 5 , as well as in mixed characteristic of residue characteristic > 5 [9, Theorem 3.1]

One could hope that in a fixed dimension, log canonical centers satisfy (semi-)normality properties after bounding from below the residue characteristics. This hope falls short: recently Kollár [26] constructed in every positive characteristic examples of non-weakly normal log canonical centers on fourfold pairs. Still, the question remains open for unions of log canonical centers on threefolds. The main result of this note is the normality of minimal ones in mixed and positive characteristics different from 2, 3 and 5:

THEOREM 1.1 (Theorem 3.2). — *Let (X, Δ) be a log canonical threefold whose closed points have perfect residue fields of characteristic different*

from 2, 3 and 5. Then the minimal log canonical centers of (X, Δ) are normal.

If a minimal lc center is a surface, then the pair is plt around that surface and normality follows from the aforementioned works. So the only case left to study is when the minimal lc center is a curve, say $C \subset X$. We follow the strategy used in characteristic 0. Indeed, when the residue characteristics are greater than five, threefold dlt singularities have all the desired properties. We would like to descend these properties from dlt modifications to log canonical pairs. In order to do so we will need some vanishing theorem. The sufficient vanishing theorem is established in [3]. More precisely, given a threefold log canonical singularity (X, Δ) with minimal log canonical center a curve C , by [3, Proposition 14] there exists a dlt model $g: Y \rightarrow X$ with exceptional divisor E contracting to C , such that E is weakly normal and $R^1g_*\mathcal{O}(-E) = 0$. Combining this vanishing with a factorization of $E \rightarrow C$ through the normalization of C , whose existence is guaranteed by Proposition 2.10, we obtain that C is normal.

Building on [3, Proposition 15] and on the recent work of Tatsuro Kawakami on Kawamata–Viehweg vanishing on log Calabi–Yau surface pairs [21], we are also able to prove some semi-normality property, provided that the characteristic is large enough:

THEOREM 1.2 (Theorem 3.5). — *There exists an integer p_0 such that if Δ has standard coefficients and (X, Δ) is a log canonical threefold whose closed points have perfect residue fields of characteristic $p > p_0$, then the union of all the log canonical centers of (X, Δ) is semi-normal.*

Whether an arbitrary union of log canonical centers is semi-normal is not a formal consequence of this theorem, even in characteristic 0 (see for example the proof given in [24, Sections 4.3–4.4]).

Also, for threefold pairs whose reduced boundary is \mathbb{Q} -Cartier with index not divisible by any of the residue characteristics, it is possible to show that the boundary is demi-normal as long as the residue characteristics at closed points are different from 2, 3 and 5. This can be obtained by combining the main theorem of [5] and cyclic covers (see also [3, Corollary 23]). We include it here in Proposition 3.3 for completeness.

Let us now discuss counter-examples to (semi)-normality properties. The only counter-example currently known to normality of plt centers for threefolds, obtained in [11], stems from a failure of Kawamata–Viehweg vanishing on some del Pezzo surface. After taking a cone over that surface, the authors obtain a plt pair whose boundary is not S_2 at the vertex.

Kawamata–Viehweg vanishing holds on del Pezzo surfaces in characteristic > 5 by [4], but we have counter-examples in characteristic 3 [8] and in characteristic 5 by [4]. Thus one can wonder if it is possible to produce non-normal plt centers on some cones over these surfaces, following the method of [11]. Unfortunately, a sole instance of Kawamata–Viehweg vanishing failure is not sufficient to produce such an example. We were unable to find the right configuration of ample divisor and boundary for the del Pezzo surface in characteristic 5 described in [4]: actually, our calculations led us to believe that no example of non-normal plt center can be produced by taking an appropriate cone over that surface.

For the characteristic 3 del Pezzo surface constructed in [8], we find a configuration leading to a non-normal cone boundary. However the cone pair we obtain has a minimal lc center at the vertex, so we do not get a plt pair. Nonetheless we can show that the cone divisor is not seminormal at the vertex, contrarily to what would happen in characteristic 0 (see [25, 4.20(3)]).

THEOREM 1.3 (Corollary 4.9). — *There exists a threefold log canonical pair (X, E) with reduced boundary over an algebraically closed field of characteristic 3, such that E is not S_2 nor seminormal.*

In the example we construct, the reduced boundary E contains every log canonical center of the pair (X, E) : this shows that Theorem 1.2 can fail in characteristic 3.

In Section 4, we give two sets of sufficient conditions on a Picard rank one variety X (of any dimension) for the existence of a non-normal cone boundary over X . We believe these conditions to be of independent interest. Indeed, the calculations of [11] are quite involved, rely on a precise local analysis of the relative cone, and require the vanishing of many cohomology groups. In particular, it is not immediate to adapt their method to a potential new example. By contrast, our conditions are short and only concern the global geometry of X : in one case a single failure of Kawamata–Viehweg vanishing for a nef divisor, in the other a failure of Kawamata–Viehweg for an anti-ample divisor and the existence of a boundary with small numerical class. Our counterexample in characteristic 3 is obtained by using one of these conditions.

Let us also mention that examples of pathological behaviour of threefold singularities in positive characteristic are often constructed by considering cones constructed from \mathbb{Q} -Cartier polarizations (see [4, 8, 11]). The birational singularities of cones with respect to Cartier divisors are well-understood (see for example [25, Section 3.1]). However, to the best of the

authors' knowledge the case of \mathbb{Q} -Cartier polarizations in arbitrary characteristic has not received a similar treatment in the literature, although this construction is well-known to the experts. For the convenience of the reader we gather in Section 2.5.1 some foundational results about singularities of cones, and more precise statements about three-dimensional cones are given in Section 2.5.2.

Lastly, as far as threefolds in positive characteristic are concerned, our results leave us with the following interrogations:

QUESTION 1.4. — *Let (X, Δ) be a threefold pair over a perfect field of characteristic $p > 0$.*

- (1) *If $p = 5$, are plt centers normal? What about minimal lc centers?*
- (2) *If $p \geq 5$, are unions of lc centers seminormal? or even weakly normal? What about union of slc centers?*
- (3) *Are Cartier boundaries demi-normal in characteristics ≤ 5 ? I.e., does Proposition 3.3 fail in characteristic $p \leq 5$?*

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2. Preliminaries

2.1. Conventions and notations

We will work with schemes that are not necessarily defined over a field, but we keep most of the usual terminology.

Throughout this article we work over an affine base B which is excellent, of pure finite dimension, and admits a dualizing complex. Unless stated otherwise, all the schemes appearing in this note are essentially of finite type over B .

By a *variety* we shall mean a separated reduced equidimensional scheme of finite type over B . Note that a variety in our sense might be reducible, even disconnected. A *curve* (resp. a *surface*, resp. a *threefold*) is a variety of absolute dimension one (resp. two, resp. three). In particular the dimension

shall always mean the absolute dimension and not the relative dimension over B .

Let X be a variety and \mathcal{F} a coherent \mathcal{O}_X -module. We say that \mathcal{F} is S_i if $\text{depth}_{\mathcal{O}_{X,x}} \mathcal{F}_x \geq \min\{i, \dim \mathcal{F}_x\}$ for every $x \in X$.

If X is a reduced Noetherian scheme, its *normalization* is defined to be its relative normalization along the structural morphism $\bigsqcup_{\eta} \text{Spec}(k(\eta)) \rightarrow X$ where η runs through the generic points of X . Recall that X is normal if and only if it is regular in codimension one and \mathcal{O}_X is S_2 .

If X is a (demi-)normal variety, we use the language of \mathbb{Q} -divisors, \mathbb{Q} -Cartier divisors and reflexive sheaves in the usual way. Recall that $[\otimes]$ denotes the reflexified tensor product.

A scheme X essentially of finite type over B has a dualizing complex ω_X^\bullet and a dualizing sheaf $\omega_X := h^{-i}(\omega_X^\bullet)$ where $i := \max\{j \mid h^{-j}(\omega_X^\bullet) \neq 0\}$. See [10, Section 2.1] for more details. If X is a variety and ω_X is invertible in codimension one, it defines a canonical divisor K_X .

A *pair* is the data (X, Δ) where X is a demi-normal variety, Δ is a \mathbb{Q} -divisor with coefficients in $[0, 1]$ such that no component of Δ is contained in $\text{Sing}(X)$, and satisfying that $K_X + \Delta$ is \mathbb{Q} -Cartier. We say that Δ has *standard coefficients* if its coefficients belong to $\{1 - \frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$. If $f: Y \rightarrow X$ is a proper birational morphism from a normal variety, then there is a \mathbb{Q} -divisor Δ_Y on Y such that

$$K_Y + \Delta_Y = f^*(K_X + \Delta)$$

and Δ_Y is uniquely defined if we assume $f_*K_Y = K_X$, which we always do. Running through all such $f: Y \rightarrow X$, we define the singularity of the pair (X, Δ) (e.g. *lc*, *klt*) the usual way as in [27]. See [10, Section 2.5] for details.

Let (X, Δ) be a pair. We say that a reduced closed subscheme $Z \subset X$ is an *lc center* of (X, Δ) if there exists a proper birational morphism $f: Y \rightarrow X$ and an f -exceptional prime divisor $E \subset Y$ such that $\text{coeff}_E \Delta_Y = 1$ and $f(E) = Z$.

We refer to [10, Section 2.5] for the various notions of positivity (ampleness, nefness, etc.) for \mathbb{Q} -Cartier \mathbb{Q} -divisors on proper B -schemes.

2.2. Universal homeomorphisms

Recall that a morphism $f: X \rightarrow Y$ is a *universal homeomorphism* if for every $T \rightarrow Y$, the base-change $f_T: X_T \rightarrow T$ is an homeomorphism. Extensions of rings inducing universal homeomorphisms on spectra have the following algebraic property:

PROPOSITION 2.1 ([6, 0CNE]). — *Let $A \subset B$ be an extension of rings. Then the following are equivalent:*

- (1) *the induced morphism $\text{Spec } B \rightarrow \text{Spec } A$ is a universal homeomorphism;*
- (2) *every finite subset of B is contained in a sub-extension $A[b_1, \dots, b_n] \subset B$ such that for every $i = 1, \dots, n$ we have either:*
 - $b_i^2, b_i^3 \in A[b_1, \dots, b_{i-1}]$, or
 - *there exists a prime $p > 0$ such that $b_i^p, pb_i \in A[b_1, \dots, b_{i-1}]$.*

Next we recall some sufficient conditions on a locally Noetherian scheme X for its normalization morphism to be a universal homeomorphism.

DEFINITION 2.2 ([6, 0BPZ]). — *A local ring \mathcal{O} is called geometrically unibranch if \mathcal{O}_{red} is a domain, its integral closure \mathcal{O}' in $\text{Frac}(\mathcal{O}_{\text{red}})$ is local, and the extension of residue fields $k(\mathcal{O}) \subset k(\mathcal{O}')$ is purely inseparable.*

A scheme X is called geometrically unibranch if every local ring $\mathcal{O}_{X,x}$ is geometrically unibranch.

LEMMA 2.3 ([6, 0BQ4]). — *Let X be a scheme, and $x \in X$ a point. Assume that: if $(x' \in X') \rightarrow (x \in X)$ is any étale neighbourhood, then X' is irreducible at x' . Then $\mathcal{O}_{X,x}$ is geometrically unibranch.*

LEMMA 2.4. — *Let X be an excellent geometrically unibranch scheme. Then the normalization morphism $X^\nu \rightarrow X$ is a universal homeomorphism.*

The excellent hypothesis implies that of local Noetherianity, and in this context the normalization is defined: see [6, 035E].

Proof. — By [6, 0C1S] the normalization $\nu: X^\nu \rightarrow X$ is universally bijective. Since ν is surjective [6, 035Q], and moreover universally closed by the excellence hypothesis [6, 035S], we obtain that ν is a universal homeomorphism. □

2.3. Weak normality and semi-normality

We now recall the definition of semi-normal and weakly normal schemes. These notions originally appeared in [1, 2, 33]. We follow the more recent treatment given in [23, I.7].

DEFINITION 2.5 ([23, I.7.2.1]). — *Let X be an excellent reduced scheme with normalization $X^\nu \rightarrow X$. A scheme X' with a morphism $\varphi: X' \rightarrow X$ is called the weak normalization (resp. the semi-normalization) of X if the following properties hold:*

- (1) *the normalization $X^\nu \rightarrow X$ factors through φ ,*
- (2) *φ is finite surjective, and an isomorphism over the generic points of X ,*
- (3) *φ is an homeomorphism, and for any $x \in X$ with unique preimage $x' \in X'$, the field extension $k(x) \subset k(x')$ is purely inseparable (resp. an isomorphism),*
- (4) *if $\varphi': Y \rightarrow X$ also satisfies the above properties, then there is a factorization*

$$\varphi: X' \longrightarrow Y \xrightarrow{\varphi'} X.$$

We say that X is weakly normal (resp. semi-normal) if $\varphi: X' \rightarrow X$ is the identity morphism.

Let us also recall the notion of demi-normality: it will appear in Section 3.2.

DEFINITION 2.6. — *A reduced Noetherian scheme X is demi-normal if it is S_2 and for every codimension one point $\eta \in X$, the local ring $\mathcal{O}_{X,\eta}$ is either regular or a nodal singularity.*

See [25, 1.41] for the definition of nodal singularity, and [30, Section 2.3.1] for the relations between demi-normality, semi-normality and weak normality.

LEMMA 2.7 ([23, I.7.2.3]). — *Let X be a reduced excellent scheme. Then X has a weak normalization and a semi-normalization. They coincide if X is a \mathbb{Q} -scheme.*

An example of weakly normal schemes that will be relevant in this note is given by the following lemma.

LEMMA 2.8. — *Let $(X, \Delta + E)$ be an lc pair, where the residue fields of X have characteristics $\neq 2$, and where E is a (possibly reducible) divisor with coefficients 1. Assume that E is S_2 . Then E is weakly normal.*

Proof. — By the classification of lc singularities on surfaces [25, Theorem 2.31], we see that E has at worse nodal singularities in codimension 1. Since E is also S_2 , it is demi-normal. Now demi-normal schemes in char $\neq 2$ are weakly normal [30, Corollary 2.3.8]. \square

Let us next give an algebraic description of weak normality. We stick to reduced excellent rings for simplicity.

PROPOSITION 2.9 ([34, Section 1]). — *Let A be a reduced excellent ring with normalization \bar{A} .*

- (1) *A is semi-normal if and only if: for every $x \in \bar{A}$, the fact that $x^2, x^3 \in A$ implies that $x \in A$.*
- (2) *A is weakly normal if and only if: A is semi-normal, and for every $x \in \bar{A}$ the fact that $x^p, px \in A$ for some prime $p > 0$ implies that $x \in A$.*

Notice the similarity between the algebraic conditions for semi-normality and weak normality given above, and the algebraic description of universal homeomorphisms given in Proposition 2.1. We take advantage of this similarity in order to obtain the following factorization result:

PROPOSITION 2.10. — *Let $f: Y \rightarrow X$ be a morphism of excellent reduced schemes. Assume that:*

- (1) *each irreducible component of Y dominates a component of X , and*
- (2) *Y is weakly normal.*

Then f factors through the weak normalization $w: X^{\text{wn}} \rightarrow X$.

Proof. — To give a morphism $g: Y \rightarrow X^{\text{wn}}$ means giving a continuous map $|g|: |Y| \rightarrow |X^{\text{wn}}|$ of topological spaces, together with a map of sheaves $g^\#: \mathcal{O}_{X^{\text{wn}}} \rightarrow g_*\mathcal{O}_Y$. By definition $X^{\text{wn}} \rightarrow X$ is an homeomorphism, so we take $|g| = |f|$, and to conclude we need to show that the map $\mathcal{O}_X \hookrightarrow f_*\mathcal{O}_Y$ admits a factorization $\mathcal{O}_X \rightarrow w_*\mathcal{O}_{X^{\text{wn}}} \rightarrow f_*\mathcal{O}_Y$.

Let $Y^\nu \rightarrow Y$ be the normalization of Y and $\nu_X: X^\nu \rightarrow X$ the normalization of X . By [6, 035Q(4)] there is a commutative diagram

$$\begin{array}{ccc}
 Y^\nu & \longrightarrow & Y \\
 \downarrow f^\nu & & \downarrow f \\
 X^\nu & \longrightarrow & X^{\text{wn}} \xrightarrow{w} X
 \end{array}$$

which induces the commutative diagram of \mathcal{O}_X -algebras

$$\begin{array}{ccccc}
 (\nu_X)_* f^\nu_* \mathcal{O}_{Y^\nu} & \longleftarrow & & \longrightarrow & f_* \mathcal{O}_Y \\
 \uparrow & \swarrow \pi & & \searrow \pi|_{\mathcal{O}_X} & \uparrow \\
 (\nu_X)_* \mathcal{O}_{X^\nu} & \longleftarrow & w_* \mathcal{O}_{X^{\text{wn}}} & \longleftarrow & \mathcal{O}_X
 \end{array}$$

We claim that the arrow $\pi: w_*\mathcal{O}_{X^{\text{wn}}} \rightarrow (\nu_X)_*f_*^\nu\mathcal{O}_{Y^\nu}$ factors through $f_*\mathcal{O}_Y$. Indeed, take $s \in w_*\mathcal{O}_{X^{\text{wn}}}(U)$ for some affine open set U of X . By Proposition 2.1 we can find $t_1, \dots, t_n \in \mathcal{O}_X(U)$ such that $s \in \mathcal{O}_X(U)[t_1, \dots, t_n]$ where for every $i = 1, \dots, n$ we have either

- (1) $t_i^2, t_i^3 \in \mathcal{O}_X(U)[t_1, \dots, t_{i-1}]$, or
- (2) $t_i^p, pt_i \in \mathcal{O}_X(U)[t_1, \dots, t_{i-1}]$ for some prime $p > 0$.

Let $V = f^{-1}(U)$ and V^ν be its preimage in Y^ν . Then we can say that $\pi(s) \in \mathcal{O}_{Y^\nu}(V^\nu)$ belongs to $\mathcal{O}_Y(V)[\pi(t_1), \dots, \pi(t_n)]$ where for every $i = 1, \dots, n$ we have either

- (1) $\pi(t_i)^2, \pi(t_i)^3 \in \mathcal{O}_Y(V)[\pi(t_1), \dots, \pi(t_{i-1})]$, or
- (2) $\pi(t_i)^p, p\pi(t_i) \in \mathcal{O}_Y(V)[\pi(t_1), \dots, \pi(t_{i-1})]$ for some prime $p > 0$.

Since Y is weakly normal, it follows from Proposition 2.9 (applied on an affine cover of V) that $\pi(t_1) \in \mathcal{O}_Y(V)$. In turn, it follows that $\pi(t_2) \in \mathcal{O}_Y(V)$, and by induction we arrive at the conclusion that all the $\pi(t_i)$'s belong to $\mathcal{O}_Y(V)$. Therefore $\pi(s)$ belongs to $\mathcal{O}_Y(V) = f_*\mathcal{O}_Y(U)$. As s was arbitrary, our claim is proved, and we are provided with the morphism $Y \rightarrow X^{\text{wn}}$ that we were looking for. □

COROLLARY 2.11. — *In the situation of Proposition 2.10, if the normalization $X^\nu \rightarrow X$ is a universal homeomorphism, then $Y \rightarrow X$ factorizes through X^ν .*

Proof. — Indeed, if $X^\nu \rightarrow X$ is a universal homeomorphism then X^ν is equal to the weak normalization of X . □

Remark 2.12. — One can see Proposition 2.10 as a universal factorization property of the weak normalization, similar to that of semi-normalization (see [23, I.7.2.3.3]). However, one cannot drop the assumption that every component of Y dominates a component of X . The following counter-example is given in [23, I.7.2.2, I.7.2.3]. Let k be a field of characteristic $p > 0$ and let $R_n = k[x^{p^n}, yx^i : 0 \leq i \leq p^n - 1]$. The weak normalization of R_n is $k[x, y]$. If we quotient R_n by the ideal $(yx^i : 0 \leq i \leq p^n - 1)$, we obtain the (weakly) normal ring $k[x^{p^n}]$. However there is no morphism $\alpha: \text{Spec } k[x^{p^n}] \rightarrow \text{Spec } k[x, y]$ such that the diagram

$$\begin{array}{ccc}
 & & \text{Spec } k[x^{p^n}] \\
 & \swarrow \alpha & \downarrow \\
 \text{Spec } k[x, y] & \longrightarrow & \text{Spec } R_n
 \end{array}$$

is commutative. (Indeed there is no valid choice for the image $\alpha^*(x) \in k[x^{p^n}]$.) So the non-dominant morphism $Y = \text{Spec } k[x^{p^n}] \rightarrow \text{Spec } R_n = X$ does not satisfy the conclusions of Proposition 2.10.

2.4. Birational vanishing theorems for threefolds

In this section we collect the vanishing theorems for a dlt modification of a log canonical threefold that we will need in the proof of Theorem 1.1 and Theorem 1.2.

PROPOSITION 2.13 ([3, Proposition 14]). — *Let (X, Δ) be a quasi-projective log canonical threefold without zero-dimensional log canonical centers and with closed points of perfect residue field of characteristics $p > 5$. There exists a proper birational morphism $f: (X^{dlt}, \Delta_{X^{dlt}} + E) \rightarrow (X, \Delta)$ such that $E = \text{Exc}(f)$ is the reduced exceptional divisor of f , the pair $(X^{dlt}, \Delta_{X^{dlt}} + E)$ is dlt, $f^*(K_X + \Delta) = K_X + \Delta_{X^{dlt}} + E$ and $R^1 f_* \mathcal{O}_X(-E) = 0$.*

The above proposition will allow us to say something about *minimal* log canonical centers. On the contrary, the proposition below (which takes the main result of [21] as an essential input) will be applied to the study of the union of all log canonical centers for large residue characteristics.

PROPOSITION 2.14 ([3, Proposition 15]). — *There exists an integer p_0 with the following property. Assume that $\Delta \geq 0$ is a boundary with standard coefficients. Let (X, Δ) be a quasi-projective log canonical threefold with closed points of perfect residue fields, each of characteristic $p > p_0$. Then there exists a proper birational morphism $f: (X^{dlt}, \Delta_{X^{dlt}} + E) \rightarrow (X, \Delta)$ such that $E = \text{Exc}(f)$ is the reduced exceptional divisor of f , the pair $(X^{dlt}, \Delta_{X^{dlt}} + E)$ is dlt, $f^*(K_X + \Delta) = K_X + \Delta_{X^{dlt}} + E$ and $R^1 f_* \mathcal{O}_X(-E) = 0$.*

2.5. Some generalities on cones

For the convenience of the reader, we recall some basic facts about cones. The results of this section will be used in Section 4. Cone constructions for Cartier polarizations are reviewed in [25, Section 3.1], but we shall also use \mathbb{Q} -Cartier polarizations. In addition to what is presented below, see also [8, Section 2.3] and [25, Section 9.3].

Let k be an arbitrary field and let X be a normal connected excellent separated k -scheme of positive dimension. Let A be a \mathbb{Q} -Cartier \mathbb{Z} -divisor on X . We define the *Seifert bundle* over X with respect to A to be

$$\mathrm{BC}_a(X, A) = \mathrm{Spec}_X \bigoplus_{d \geq 0} \mathcal{O}(dA) \xrightarrow{\rho} X$$

where the \mathcal{O}_X -algebra structure on $\bigoplus_d \mathcal{O}(dA)$ is given by the reflexified tensor product. The morphism ρ has a section $X \cong X^- \subset \mathrm{BC}_a(X, A)$, defined by the ideal $\bigoplus_{d > 0} \mathcal{O}(dA)$ of $\mathcal{O}_{\mathrm{BC}_a(X, A)}$.

In case X is proper and A is ample, we also define the *affine cone* of X with respect to A to be

$$\mathrm{C}_a(X, A) = \mathrm{Spec}_k \bigoplus_{d \geq 0} H^0(X, \mathcal{O}(dA)).$$

It fits into a diagram

$$(2.1) \quad \begin{array}{ccc} \mathrm{BC}_a(X, A) & \xrightarrow{f} & \mathrm{C}_a(X, A) \\ \downarrow \rho & & \\ X & & \end{array}$$

The affine cone $\mathrm{C}_a(X, A)$ has a closed point v , called its *vertex*, defined by the irrelevant ideal $\bigoplus_{d > 0} H^0(X, \mathcal{O}(dA))$. The following claim is standard.

LEMMA 2.15. — *With the notations as above:*

- (1) ρ is of finite type of relative dimension one, and over a big open subset of X is it an \mathbb{A}^1 -bundle;
- (2) $\mathrm{BC}_a(X, A)$ is normal and X^- is \mathbb{Q} -Cartier;
- (3) $\mathrm{BC}_a(X, A) \setminus X^- \cong \mathrm{Spec}_X \bigoplus_{d \in \mathbb{Z}} \mathcal{O}(dA)$;
- (4) under the canonical isomorphism $X^- \cong X$ we have

$$\mathcal{O}_{\mathrm{BC}_a(X, A)}(X^-)|_{X^-} \cong \mathcal{O}_X(-A).$$

- (5) if $U \subset X$ is open, then we have a cartesian square

$$\begin{array}{ccc} \mathrm{BC}_a(U, A_U) & \hookrightarrow & \mathrm{BC}_a(X, A) \\ \downarrow \rho_U & & \downarrow \rho \\ U & \hookrightarrow & X \end{array}$$

with the obvious morphisms.

If moreover X is proper and A is ample:

- (6) $\mathrm{C}_a(X, A)$ is affine, connected, normal and of finite type over k ;

(7) *the morphism f induces an isomorphism*

$$\mathrm{BC}_a(X, A) \setminus X^- \cong \mathrm{C}_a(X, a) \setminus \{v\}.$$

Let us record the following results about divisors on the schemes we have defined:

LEMMA 2.16. — *There is an isomorphism $\rho^{[-1]}: \mathrm{Cl}(X) \cong \mathrm{Cl}(\mathrm{BC}_a(X, A))$.*

Proof. — Let $\mathcal{U} \subset X$ be the big dense open subset where A is Cartier. Then $\mathcal{V} = \rho^{-1}\mathcal{U}$ is big and dense in $\mathrm{BC}_a(X, A)$ and $\rho|_{\mathcal{V}}: \mathcal{V} \rightarrow \mathcal{U}$ is an \mathbb{A}^1 -bundle. Therefore by [19, II.6.5-6] we obtain isomorphisms of class groups

$$\rho^{[-1]}: \mathrm{Cl}(X) = \mathrm{Cl}(\mathcal{U}) \xrightarrow{\sim} \mathrm{Cl}(\mathcal{V}) = \mathrm{Cl}(\mathrm{BC}_a(X, A))$$

given by

$$\rho^{[-1]}([D]) = \left[\overline{\rho|_{\mathcal{V}}^{-1}(D_{\mathcal{U}})} \right]$$

where the bar denotes the closure in $\mathrm{BC}_a(X, A)$ and the brackets denote the images in the class groups. Notice that the inverse of $\rho^{[-1]}$ can be described as follows: for any \mathbb{Z} -divisor on $\mathrm{BC}_a(X, A)$, restrict it to the preimage of $\mathrm{Reg}(X)$ (which belongs to the regular locus of $\mathrm{BC}_a(X, A)$), pullback the associated line bundle to $\mathrm{Reg}(X)$ (alternatively to $\mathrm{Reg}(X^-)$), take the associated \mathbb{Z} -divisor on $\mathrm{Reg}(X)$ (alternatively, on $\mathrm{Reg}(X^-)$) and finally take the class of its closure in the class group of X (alternatively, on X^- and then use $\mathrm{Cl}(X^-) \cong \mathrm{Cl}(X)$). \square

LEMMA 2.17. — *Suppose that X is \mathbb{Q} -factorial. Then $\mathrm{BC}_a(X, A)$ is also \mathbb{Q} -factorial.*

Proof. — Take a \mathbb{Z} -divisor F on $\mathrm{BC}_a(X, A)$: by Lemma 2.16, there is a \mathbb{Z} -divisor G on X such that $\rho^{[-1]}([G]) = [F]$. As X is \mathbb{Q} -factorial, there exists an integer $n > 0$ such that nG is Cartier. Choose an open covering $\{X_i\}$ of X trivializing $\mathcal{O}_X(nG)$. The construction of the isomorphism $\rho^{[-1]}$ above is compatible with restriction to any X_i , as the intersection $\mathcal{U} \cap X_i$ is still big in X_i : so for each i , the cartesian square of Lemma 2.15 induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Cl}(X) & \xrightarrow[\sim]{\rho^{[-1]}} & \mathrm{Cl}(\mathrm{BC}_a(X, A)) \\ \downarrow \mathrm{res}_i & & \downarrow \mathrm{res}_i \\ \mathrm{Cl}(X_i) & \xrightarrow[\sim]{\rho_i^{[-1]}} & \mathrm{Cl}(\mathrm{BC}_a(X_i, A_{X_i})). \end{array}$$

By choice of X_i , we have $0 = [nG_{X_i}] = \mathrm{res}_i([nG])$ in $\mathrm{Cl}(X_i)$. By commutativity it follows that $0 = \mathrm{res}_i(\rho^{[-1]}[nG]) = \mathrm{res}_i([nF])$. This implies that

$\mathcal{O}(nF)$ is trivial on the open subset $\mathrm{BC}_a(X_i, A_{X_i})$. Hence F is n -Cartier. As F is arbitrary we deduce that $\mathrm{BC}_a(X, A)$ is \mathbb{Q} -factorial. \square

LEMMA 2.18. — *Assume that X is proper and that A is ample. Then:*⁽¹⁾

- (1) $\mathrm{Cl}(\mathrm{C}_a(X, A)) = \mathrm{Cl}(X)/\langle A \rangle$, and
- (2) $\mathrm{Pic}(\mathrm{C}_a(X, A)) = 0$.

Proof. — Since the vertex $v \in \mathrm{C}_a(X, A)$ has codimension ≥ 2 , we have

$$\mathrm{Cl}(\mathrm{C}_a(X, A)) = \mathrm{Cl}(\mathrm{C}_a(X, A) \setminus \{v\}) \xrightarrow{\sim} \mathrm{Cl}(\mathrm{BC}_a(X, A) \setminus X^-)$$

where the isomorphism is induced by $\mathrm{BC}_a(X, A) \setminus X^- \cong \mathrm{C}_a(X, a) \setminus \{v\}$. Now by [19, II.6.5] we have an exact sequence

$$\mathbb{Z} \xrightarrow{1 \rightarrow [X^-]} \mathrm{Cl}(\mathrm{BC}_a(X, A)) \longrightarrow \mathrm{Cl}(\mathrm{BC}_a(X, A) \setminus X^-) \longrightarrow 0.$$

By Lemma 2.16 there is an isomorphism $\rho^{[-1]}: \mathrm{Cl}(X) \cong \mathrm{Cl}(\mathrm{BC}_a(X, A))$: by the description of the inverse Ψ given in the proof, as $\mathcal{O}(X^-)|_{X^-} \cong \mathcal{O}_X(-A)$, we see that $[X^-]$ is sent to $-[A]$. The first point follows.

Now let \mathcal{L} be a line bundle on $\mathrm{C}_a(X, A)$. Its pullback $f^*\mathcal{L}$ to $\mathrm{BC}_a(X, A)$ restricts to the trivial line bundle on X^- . Consider the map

$$\alpha = \left(\mathrm{Pic}(\mathrm{BC}_a(X, A)) \longleftarrow \mathrm{Cl}(\mathrm{BC}_a(X, A)) \xrightarrow{\Psi} \mathrm{Cl}(X) \right)$$

where the embedding is given by $\mathrm{Pic} \cong \mathrm{CaCl}$ [19, II.6.15]. By the description of Ψ given in the proof of Lemma 2.16, we see that $\alpha(f^*\mathcal{L}) = \alpha(\mathcal{O}_{\mathrm{BC}_a(X, A)})$. As α is injective, we deduce that $f^*\mathcal{L}$ is trivial. As $\mathrm{BC}_a(X, A) \setminus X^-$ is isomorphic to $\mathrm{C}_a(X, A)^* = \mathrm{C}_a(X, A) \setminus \{v\}$, we obtain an isomorphism

$$\mathcal{O}_{\mathrm{C}_a(X, A)^*} \cong \mathcal{L}|_{\mathrm{C}_a(X, A)^*}.$$

As $\mathcal{O}_{\mathrm{C}_a(X, A)}$ and \mathcal{L} are reflexive, and as $\mathrm{C}_a(X, A)^*$ is big in $\mathrm{C}_a(X, A)$, by [6, 0E9I] this isomorphism extends to $\mathcal{O}_{\mathrm{C}_a(X, L)} \cong \mathcal{L}$. This proves the point (2). \square

The construction of Seifert bundles over non-proper schemes was only needed for the proof of Lemma 2.17. So for the rest of the section, we assume that X is proper and that A is ample.

Let $E \subset X$ be a prime divisor (not necessarily \mathbb{Q} -Cartier). We define a prime divisor $E_{\mathrm{C}_a(X, A)} \subset \mathrm{C}_a(X, A)$ by $E_{\mathrm{C}_a(X, A)} = f_*\rho^{-1}E$. We also define

(1) This is stated in [8, Proposition 2.4]. However there are minor issues with the proof: in the first paragraph, the claimed isomorphism $\mathrm{Pic}(V) \cong \mathrm{Pic}(X)$ usually does not hold, as there could be non-invertible divisorial sheaves on X that become invertible on V . Similarly, the implicitly used $\mathrm{Pic}(\mathrm{BC}_a(X, A)) \cong \mathrm{Pic}(\rho^{-1}V)$ need not be true. This affects the proof of (2). Therefore we give a different argument.

a prime divisor $E_{BC_a(X,A)} \subset BC_a(X,A)$ by $E_{BC_a(X,A)} = \text{red}(\rho^{-1}E)$. Then

$$\mathcal{I}_{E_{BC_a(X,A)}} = \bigoplus_{d \geq 0} \mathcal{O}_X(dA - E) \hookrightarrow \bigoplus_{d \geq 0} \mathcal{O}_X(dA) = \mathcal{O}_{BC_a(X,A)}$$

is the (graded) ideal defining $E_{BC_a(X,A)}$ inside $BC_a(X,A)$.

Consider for all $d \in \mathbb{Z}$ the exact sequence of \mathcal{O}_X -modules

$$(2.2) \quad 0 \longrightarrow \mathcal{O}_X(dA - E) \longrightarrow \mathcal{O}_X(dA) \longrightarrow \mathcal{E}_d \longrightarrow 0,$$

where \mathcal{E}_d is defined by the exactness of the sequence. Then \mathcal{E}_d is a rank one S_1 -sheaf supported on E [25, Lemma 2.60], and hence a torsion free \mathcal{O}_E -module [31, Proposition 2.6].

LEMMA 2.19. — *The direct sum $\bigoplus_{d \in \mathbb{Z}} \mathcal{E}_d$ has a structure of \mathbb{Z} -graded \mathcal{O}_E -module.*

Proof. — This direct sum is the quotient of the module $\mathcal{A} = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_X(dA)$ by the graded sub-module $\mathcal{I} = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_X(dA - E)$. Since A is a \mathbb{Z} -divisor, the module \mathcal{A} obtains a graded \mathcal{O}_X -algebra structure via the pairing

$$\mathcal{O}(dA) \otimes \mathcal{O}(eA) \longrightarrow \mathcal{O}(dA)[\otimes] \mathcal{O}(eA) = \mathcal{O}((d+e)A).$$

So we only have to show that \mathcal{I} is a graded ideal of \mathcal{A} . We observe that the product structure takes $\mathcal{O}(dA - E) \otimes \mathcal{O}(eA)$ to $\mathcal{O}((d+e)A - E)$, so \mathcal{I} is indeed an ideal. \square

Similarly,

$$I_{E_{C_a(X,A)}} = \bigoplus_{d \geq 0} H^0(X, \mathcal{O}(dA - E))$$

is the (graded) ideal defining $E_{C_a(X,A)}$ inside $C_a(X,A)$ and the cokernel $\mathcal{O}_{E_{C_a(X,A)}}$ is by definition the k -algebra $\bigoplus_{d \geq 0} \text{im } \psi_d$, where

$$\psi_d: H^0(X, \mathcal{O}(dA)) \longrightarrow H^0(E, \mathcal{E}_d)$$

is the map on cohomology induced by (2.2).

PROPOSITION 2.20. — *$E_{C_a(X,A)}$ is S_2 at the vertex if and only if the map $H^1(X, \mathcal{O}(dA - E)) \rightarrow H^1(X, \mathcal{O}(dA))$ induced by (2.2) is injective for every $d \geq 0$.*

Proof. — Let U be the complement of the vertex $v \in C_a(X,A)$. Applying [19, III.exerc.2.3] to $\mathcal{O}_{E_{C_a(X,L)}} = \mathcal{O}$, as $H^1(E_{C_a(X,L)}, \mathcal{O}) = 0$ we find the exact sequence

$$H^0(E_{C_a(X,L)}, \mathcal{O}) \longrightarrow H^0(E_{C_a(X,L)} \cap U, \mathcal{O}) \longrightarrow H^1_v(E_{C_a(X,L)}, \mathcal{O}) \longrightarrow 0.$$

So $E_{C_a(X,L)}$ is S_2 if and only if $H^0(E_{C_a(X,L)}, \mathcal{O}) \rightarrow H^0(E_{C_a(X,L)} \cap U, \mathcal{O})$ is surjective. Now $H^0(E_{C_a(X,L)}, \mathcal{O}) = \bigoplus_{d \geq 0} \text{im } \psi_d$ as we have seen above. The isomorphism $U \cong \text{Spec}_X \bigoplus_{d \in \mathbb{Z}} \mathcal{O}(dA)$ induces an isomorphism $E_{C_a(X,A)} \cap U \cong \text{Spec}_X \bigoplus_{d \in \mathbb{Z}} \mathcal{E}_d$. So the restriction π of ρ to $E_{C_a(X,A)} \cap U$ is affine, and we obtain

$$\begin{aligned} H^0(E_{C_a(X,L)} \cap U, \mathcal{O}) &= H^0(X, \pi_* \mathcal{O}_{E_{C_a(X,A)} \cap U}) \\ &= H^0\left(X, \bigoplus_{d \in \mathbb{Z}} \mathcal{E}_d\right) = \bigoplus_{d \in \mathbb{Z}} H^0(E, \mathcal{E}_d), \end{aligned}$$

where the last equality comes from that the sheaves \mathcal{E}_d are supported on E .

Therefore, the morphism $H^0(E_{C_a(X,L)}, \mathcal{O}) \rightarrow H^0(E_{C_a(X,L)} \cap U, \mathcal{O})$ is the direct sum

$$\bigoplus_{d \in \mathbb{Z}} [\text{im } \psi_d \hookrightarrow H^0(E, \mathcal{E}_d)].$$

In other words, $E_{C_a(X,A)}$ is S_2 at the vertex if and only if ψ_d is surjective for every $d \in \mathbb{Z}$. Since we have the exact sequences

$$H^0(X, \mathcal{O}(dA)) \xrightarrow{\psi_d} H^0(E, \mathcal{E}_d) \longrightarrow H^1(X, \mathcal{O}(dA - E)) \xrightarrow{\alpha_d} H^1(X, \mathcal{O}(dA))$$

induced by (2.2), we see that ψ_d is surjective if and only if α_d is injective.

Finally, we claim that the group $H^0(E, \mathcal{E}_d)$ is trivial for $d < 0$, which implies that α_d is injective for every $d < 0$. This will conclude the proof.

Fix $d < 0$. For $n > 0$ divisible enough, the sheaf \mathcal{E}_{nd} is the restriction of an invertible sheaf whose dual is very ample. Thus \mathcal{E}_{nd}^{-1} is ample, and so $H^0(E, \mathcal{E}_{nd}) = 0$. Now consider the n -fold reflexified product

$$\mathcal{E}_d \longrightarrow \mathcal{E}_{nd}, \quad s \longmapsto s^{[\otimes]n}$$

induced by the structure of graded algebra of $\bigoplus_{d \in \mathbb{Z}} \mathcal{E}_d$ (see Lemma 2.19). Since generically along E the sheaf $\mathcal{O}_X(A)$ is invertible, this n -fold product is generically an isomorphism. Since the \mathcal{E}_i 's are torsion-free, this implies that the induced product on global sections

$$H^0(E, \mathcal{E}_d) \longrightarrow H^0(E, \mathcal{E}_{nd}), \quad s \longmapsto s^{[\otimes]n}$$

is injective. But $H^0(E, \mathcal{E}_{nd})$ is trivial, so $H^0(E, \mathcal{E}_d)$ is also trivial. \square

LEMMA 2.21. — ψ_d is surjective for $d \gg 0$.

Proof. — Say that m is the Cartier index of A . Then $\mathcal{E}_{d+m} = \mathcal{E}_d \otimes \mathcal{O}_E((mA)|_E)$. For every $i = 0, \dots, m-1$, by Serre vanishing for the ample Cartier divisor mA there exists $n_i > 0$ such that $H^1(T, \mathcal{O}((nm+i)A - E)) = 0$ for every $n \geq n_i$. If $N = \max\{n_i\}$ this implies that

$$\psi_{nm+i}: H^0(T, \mathcal{O}((nm+i)A)) \longrightarrow H^0(E, \mathcal{E}_{nm+i})$$

is surjective for all $n \geq N$ and $i = 0, \dots, m - 1$. In other words, $\text{im } \psi_d = H^0(E, \mathcal{E}_d)$ for $d \geq Nm$. \square

COROLLARY 2.22. — *The morphism $\text{Spec}_k \bigoplus_{d \geq 0} H^0(E, \mathcal{E}_d) \rightarrow E_{C_a(X,A)}$ is a finite birational homeomorphism inducing equalities of residue fields.*

Proof. — Consider the graded rings

$$R = \bigoplus_{d \geq 0} \text{im } \psi_d \quad \text{and} \quad R' = \bigoplus_{d \geq 0} H^0(E, \mathcal{E}_d).$$

Denote by $v' \in \text{Spec } R'$ the closed point given by the irrelevant ideal. We have an inclusion $R \hookrightarrow R'$ preserving the degrees. By Lemma 2.21 there exists $N \gg 0$ such that $R_d = R'_d$ for $d \geq N$. This shows that $\text{Spec } R' \rightarrow \text{Spec } R = E_{C_a(X,A)}$ is finite.

Take $f \in R_d = R'_d$ for $d \geq N$. Then for any $s \in R'_e$ we have $fs \in R_{e+d} = R'_{e+d}$, so $s = fs/f \in R[f^{-1}]$. This shows that $R[f^{-1}] = R'[f^{-1}]$. Since $\{\text{Spec } R[f^{-1}] \mid f \in R_d, d \geq N\}$ is an open cover of $\text{Spec } R \setminus \{v\}$, and similarly $\{\text{Spec } R'[f^{-1}] \mid f \in R_d, d \geq N\}$ is an open cover of $\text{Spec } R' \setminus \{v'\}$, we obtain that $\text{Spec } R' \setminus \{v'\} \rightarrow \text{Spec } R \setminus \{v\}$ is an isomorphism.

To conclude we only have to show that $k(v') = k(v)$. We have

$$\text{Spec } R' \times_{\text{Spec } R} k(v) = \text{Spec}_k \left(H^0(E, \mathcal{O}_E) \oplus \bigoplus_{d=1}^{N-1} H^0(E, \mathcal{E}_d) / \text{im } \psi_d \right).$$

Thus

$$\text{Spec } k(v') = \text{red}(\text{Spec } R' \times_{\text{Spec } R} k(v)) = \text{Spec}_k H^0(E, \mathcal{O}_E) = \text{Spec } k(v),$$

which concludes the proof. \square

2.5.1. Birational singularities of Seifert bundles

Consider anew a proper pair (X, E) such that $K_X + E \sim_{\mathbb{Q}} rA$ where A is an ample \mathbb{Q} -Cartier \mathbb{Z} -divisor and $r \in \mathbb{Q}$. The assumption $K_X + E \sim_{\mathbb{Q}} rA$ ensures that $K_{C_a(X,A)} + E_{C_a(X,A)}$ is \mathbb{Q} -Cartier [8, Proposition 2.4(3)]. In this subsection we consider the birational singularities of the affine cone around its vertex. We have (see [8, (2.4.2)]) that

$$f^*(K_{C_a(X,A)} + E_{C_a(X,A)}) = K_{\text{BC}_a(X,A)} + E_{\text{BC}_a(X,A)} + (1+r)X^-,$$

where $f: \text{BC}_a(X,A) \rightarrow C_a(X,A)$ is as in (2.1), so it is equivalent to consider the birational singularities of the relative cone along its section.

DEFINITION 2.23. — We will say that inversion of adjunction holds for the pair $(\mathrm{BC}_a(X, A), E_{\mathrm{BC}_a(X, A)} + X^-)$ along X^- , if the following implications hold:

- If $(X^-, \mathrm{Diff}_{X^-}(E_{\mathrm{BC}_a(X, A)}))$ is klt then $(\mathrm{BC}_a(X, A), E_{\mathrm{BC}_a(X, A)} + X^-)$ is plt around X^- , and
- If $(X^-, \mathrm{Diff}_{X^-}(E_{\mathrm{BC}_a(X, A)}))$ is lc then $(\mathrm{BC}_a(X, A), E_{\mathrm{BC}_a(X, A)} + X^-)$ is lc around X^- .

Since $X^- \cong X$ we see that X^- is normal. Moreover, note that under this isomorphism we have

$$(2.3) \quad K_{X^-} + \mathrm{Diff}_{X^-}(E_{\mathrm{BC}_a(X, A)}) = K_X + E.$$

Indeed, this equality holds over the Cartier locus of A where the Seifert bundle is an \mathbb{A}^1 -bundle, and since the preimage of that Cartier locus in $\mathrm{BC}_a(X, A)$ contains every codimension one point of X^- .

The following proposition is standard.

PROPOSITION 2.24. — Assume that (X, E) is a pair such that $K_X + E \sim_{\mathbb{Q}} rA$ where A is an ample \mathbb{Q} -Cartier \mathbb{Z} -divisor and $r \in \mathbb{Q}$. Assume inversion of adjunction for the pair $(\mathrm{BC}_a(X, A), E_{\mathrm{BC}_a(X, A)} + X^-)$ along X^- . Then:

- (1) If $r < 0$ and (X, E) is klt, then $(C_a(X, A), E_{C_a(X, A)})$ is klt at the vertex.
- (2) If $r \leq 0$ and (X, E) is lc, then $(C_a(X, A), E_{C_a(X, A)})$ is lc at the vertex.

Proof. — Let $f: \mathrm{BC}_a(X, A) \rightarrow C_a(X, A)$ be the natural morphism. As mentioned above, we have

$$(2.4) \quad f^*(K_{C_a(X, A)} + E_{C_a(X, A)}) = K_{\mathrm{BC}_a(X, A)} + E_{\mathrm{BC}_a(X, A)} + (1+r)X^-.$$

It is therefore sufficient to prove that $(\mathrm{BC}_a(X, A), E_{\mathrm{BC}_a(X, A)} + (1+r)X^-)$ is klt, respectively lc.

Assume that (X, E) is klt and $r < 0$. Then by (2.3), since inversion of adjunction holds by hypothesis, we obtain that $(\mathrm{BC}_a(X, A), E_{\mathrm{BC}_a(X, A)} + X^-)$ is plt in a neighbourhood of X^- . Since $r < 0$ we get that the pair $(\mathrm{BC}_a(X, A), E_{\mathrm{BC}_a(X, A)} + (1+r)X^-)$ is klt on the same neighbourhood.

If we only assume that (X, E) is lc and $r \leq 0$, then the same argument shows that the pair $(\mathrm{BC}_a(X, A), E_{\mathrm{BC}_a(X, A)} + (1+r)X^-)$ is lc in a neighbourhood of X^- . \square

Remark 2.25. — If the index of A is not divisible by the characteristic, then the conclusions of Proposition 2.24 hold *without assuming inversion of adjunction*. Indeed, this can be deduced from the Cartier case [25, Lemma 3.1] by means of [25, Corollary 2.43(1)].

2.5.2. Birational singularities of cones over surfaces.

For cones over surfaces, inversion of adjunction in the form of Definition 2.23 is available:

LEMMA 2.26. — *Let (X, E) be a normal \mathbb{Q} -factorial surface pair over a perfect field. Let A be an ample \mathbb{Q} -Cartier \mathbb{Z} -divisor on X such that $K_X + E \sim_{\mathbb{Q}} rA$ with $r \in \mathbb{Q}$. Then inversion of adjunction holds for the threefold pair $(\text{BC}_a(X, A), E_{\text{BC}_a(X, A)} + X^-)$ along X^- .*

Proof. — Thanks to Lemma 2.17, our assumptions imply that $\text{BC}_a(X, A)$ is \mathbb{Q} -factorial and that the divisor $K_{\text{BC}_a(X, A)} + E_{\text{BC}_a(X, A)}$ is \mathbb{Q} -Cartier ([8, Proposition 2.4(3)]). Therefore, thanks to [17, Corollary 1.4], the pair $(\text{BC}_a(X, A), E_{\text{BC}_a(X, A)} + X^-)$ admits a dlt-modification. Now, the proof of [18, Theorem 6.2] carries over in all characteristic and without assumption on the boundary. \square

Remark 2.27. — If (X, E) is dlt then X is \mathbb{Q} -factorial. Indeed, as X is a potentially klt surface it is \mathbb{Q} -factorial [32, Corollary 4.11].

COROLLARY 2.28. — *Assume that (X, E) is a \mathbb{Q} -factorial surface pair such that $K_X + E \sim_{\mathbb{Q}} rA$ where A is an ample \mathbb{Q} -Cartier \mathbb{Z} -divisor and $r \in \mathbb{Q}$. Then;*

- (1) *If $r < 0$ and (X, E) is klt, then $(C_a(X, A), E_{C_a(X, A)})$ is klt everywhere (not only at the vertex).*
- (2) *If $r \leq 0$ and (X, E) is lc, then $(C_a(X, A), E_{C_a(X, A)})$ is lc everywhere (not only at the vertex).*

Proof. — In both cases the pair is has the claimed singularity type at the vertex by Proposition 2.24, since by Lemma 2.26 inversion of adjunction holds for threefold pairs. So we look away from the vertex.

First assume that $r < 0$ and that (X, E) is klt. To show that the pair $(C_a(X, A), E_{C_a(X, A)})$ is klt everywhere, by the crepant equation (2.4) it suffices to show that the pair $(\text{BC}_a(X, A), E_{\text{BC}_a(X, A)})$ is klt everywhere.

For the rest of the proof, let us write

$$\mathcal{V} = \mathrm{BC}_a(X, A) \setminus X^-, \quad \rho|_{\mathcal{V}} = \rho^0: \mathcal{V} \longrightarrow X$$

where $\rho: \mathrm{BC}_a(X, A) \rightarrow X$ is the structural morphism.

We have seen that $(\mathrm{BC}_a(X, A), E_{\mathrm{BC}_a(X, A)})$ is klt along the section X^- . Since log resolution for threefolds is available in positive and mixed characteristics [12, 13], we can apply the arguments of [27, Corollary 2.31 and Corollary 2.32] to deduce that the non-klt locus of $(\mathrm{BC}_a(X, A), E_{\mathrm{BC}_a(X, A)})$ is closed. Therefore we obtain that: if $U_{\mathrm{klt}} \subset \mathrm{BC}_a(X, A)$ is the klt locus of the pair $(\mathrm{BC}_a(X, A), E_{\mathrm{BC}_a(X, A)})$, then U_{klt} is open and $X^- \subset U_{\mathrm{klt}}$. Therefore, $U_{\mathrm{klt}}^0 = U_{\mathrm{klt}} \cap \mathcal{V}$ intersects non-trivially every fiber of the morphism ρ^0 .

By Lemma 2.15 we have

$$\mathcal{V} \cong \mathrm{Spec}_X \bigoplus_{d \in \mathbb{Z}} \mathcal{O}(dA),$$

and the \mathbb{Z} -grading induces a \mathbb{G}_m -action on \mathcal{V} such that ρ^0 is invariant. If D is any prime divisor supported on $\mathrm{Supp}(E)$, then the ideal of $D_{\mathrm{BC}_a(X, A)} \cap \mathcal{V}$ is the graded ideal $\bigoplus_{d \in \mathbb{Z}} \mathcal{O}_X(dA - D)$. In particular the \mathbb{G}_m -action preserves $D_{\mathrm{BC}_a(X, A)} \cap \mathcal{V}$. Therefore \mathbb{G}_m acts on the log pair $(\mathcal{V}, E_{\mathrm{BC}_a(X, A)} \cap \mathcal{V})$. That action preserves the open subset $U_{\mathrm{klt}}^0 \subseteq \mathcal{V}$, because isomorphisms of log pairs preserve klt singularities.

Since ρ^0 is \mathbb{G}_m -invariant, the \mathbb{G}_m -action preserves the fibers of ρ^0 . Moreover the restricted action on any fiber of ρ^0 is transitive. Since U_{klt}^0 intersects every fiber of ρ^0 non-trivially, we deduce that $U_{\mathrm{klt}}^0 = \mathcal{V}$. Therefore $U_{\mathrm{klt}} = \mathrm{BC}_a(X, A)$, and we have proved that $(\mathrm{BC}_a(X, A), E_{\mathrm{BC}_a(X, A)})$ is klt everywhere.

In the case where $r \leq 0$ and (X, E) is lc, the proof is similar. \square

3. Normality in characteristic $p > 5$

3.1. Normality of minimal lc centers

The following result follows essentially from [15].

LEMMA 3.1. — *Let (X, Δ) be a semi-log canonical threefold whose closed points have perfect residue fields of characteristic different from 2, 3 and 5. Then the minimal lc centers of (X, Δ) are normal up to universal homeomorphism.*

Proof. — By Lemma 2.4 it is sufficient to show that the minimal lc centers are geometrically unibranch. The case where (X, Δ) is lc follows from [15, Corollary 1.7] and stability of lc singularities under étale base-changes.

In general, let $Z \subset X$ is a minimal lc center, let $z \in Z$ and consider an étale neighborhood $(z' \in Z') \rightarrow (z \in Z)$. We have to show that Z' is irreducible at z' . After shrinking it if necessary, we may assume that there is an étale neighborhood $(z' \in X') \rightarrow (z \in X)$ such that $Z' = Z \times_X X'$ [6, 02GI, 02GU]. As normalization commutes with étale base-changes [6, 07TD], if $(\bar{X}, \bar{D} + \bar{\Delta})$ is the normalization of X then $Y = \bar{X} \times_X X' \rightarrow X'$ is the normalization morphism. We also obtain crepant divisors, defined by the following diagram:

$$\begin{array}{ccc} (Y, D_Y + \Delta_Y) & \longrightarrow & (\bar{X}, \bar{D} + \bar{\Delta}) \\ \downarrow \mu & & \downarrow \\ (X', \Delta') & \xrightarrow{f} & (X, \Delta) \end{array}$$

Since μ is crepant, the lc centers of (X', Δ') are exactly the images of the lc centers of $(Y, D_Y + \Delta_Y)$. By construction Z' is an lc center of (X', Δ') . Since intersections of lc centers on Y are union of lc centers by [15, Corollary 1.7], we see that if Z' is reducible then it is not a minimal lc center. But f is crepant, so it means that Z is not a minimal lc center of (X, Δ) , which is a contradiction. Thus Z' is irreducible, as desired. \square

THEOREM 3.2. — *Let (X, Δ) be a log canonical threefold whose closed points have perfect residue fields of characteristic different from 2, 3 and 5. If $C \subset X$ is a minimal log canonical center then C is normal.*

Proof. — The strategy of proof is similar to [14, Theorem 3.6]. Firstly, if C is a surface, then (X, Δ) is plt around C and by [9, Theorem 19] the surface C is normal. If C is a point there is nothing to prove. So for the rest of the proof we may assume that C is a curve. We may also localize at any closed point of C , so that we can assume that $(x \in X, \Delta)$ is local and that C is the unique minimal log canonical center of (X, Δ) .

Let $g: (Z, \Delta_Z + E) \rightarrow (X, \Delta_X)$ be the dlt model described in Proposition 2.13. We can assume that $g(E_i) = C$ for each component E_i of E . Since $(Z, \Delta_Z + E)$ is dlt [9, Theorem 19] implies that E is S_2 . Therefore E is weakly normal by Lemma 2.8. Consider the short exact sequence associated to E inside Z :

$$0 \longrightarrow \mathcal{O}_Z(-E) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_E \longrightarrow 0.$$

By Proposition 2.13 it holds that $R^1g_*\mathcal{O}_Z(-E) = 0$ and hence, pushing forward along g , on X there is a short exact sequence on

$$0 \longrightarrow g_*\mathcal{O}_Z(-E) \longrightarrow g_*\mathcal{O}_Z \longrightarrow g_*\mathcal{O}_E \longrightarrow 0.$$

We have $g_*\mathcal{O}_Z = \mathcal{O}_X$, so $g_*\mathcal{O}_Z(-E)$ is an ideal of \mathcal{O}_X co-supported on $g(E) = C$. The cokernel of $g_*\mathcal{O}_Z(-E) \hookrightarrow \mathcal{O}_X$ is isomorphic to the \mathcal{O}_X -algebra $g_*\mathcal{O}_E$. Since \mathcal{O}_E is a reduced algebra, so is $g_*\mathcal{O}_E$, and thus we see that $g_*\mathcal{O}_Z(-E)$ is a radical ideal. It follows that $g_*\mathcal{O}_Z(-E)$ is the ideal sheaf of C .

Now let $\nu: C^\nu \rightarrow C$ denote the normalization. By Lemma 3.1 this normalization morphism is a universal homeomorphism. Since E is weakly normal, Corollary 2.11 implies that there exists a factorization $\mathcal{O}_C \hookrightarrow \nu_*\mathcal{O}_{C^\nu} \hookrightarrow g_*\mathcal{O}_E$. Therefore we have the commutative diagram

$$\begin{array}{ccc} & \mathcal{O}_C & \\ & \nearrow & \searrow \\ \mathcal{O}_X & & \nu_*\mathcal{O}_{C^\nu} \\ & \searrow & \downarrow \\ & g_*\mathcal{O}_E & \end{array}$$

Since $\mathcal{O}_X \twoheadrightarrow g_*\mathcal{O}_E$ is surjective, we see that $\mathcal{O}_C \twoheadrightarrow \nu_*\mathcal{O}_{C^\nu}$ is also a surjection. Therefore $C \cong C^\nu$, as claimed. \square

3.2. Demi-normality of $\mathbb{Z}_{(p)}$ -Cartier boundaries

In the Cartier case, the following result appeared in [3, 5]. We can build on it using cyclic covers to obtain a slight improvement.

PROPOSITION 3.3. — *Let $(X, S + \Delta)$ be an semi-log canonical threefold pair, where S is an effective divisor. Assume that the residue fields at closed points are perfect of characteristics different from 2, 3 and 5. If S is \mathbb{Q} -Cartier with index not divisible by any of the residue characteristics, then S is demi-normal.*

Proof. — First let us assume that S is Cartier. Let $\nu: (\bar{X}, \bar{S} + \bar{\Delta} + \bar{D}) \rightarrow (X, S + \Delta)$ be the normalization morphism, where $\bar{D} \subset \bar{X}$ is the conductor divisor. By the classification of lc surface singularities we know that \bar{S} is nodal in codimension one, and by [5, Theorem 1.5] we know it is S_2 (see also [3, Corollary 23]). Thus \bar{S} is demi-normal, and by adjunction the boundary $(\bar{\Delta} + \bar{D})|_{\bar{S}}$ gives to \bar{S} a structure of slc pair. We claim this

implies that S is also demi-normal. This is a local question, so we may localize on X and assume that $X = \text{Spec } A$ and $S = V(s)$ for some $s \in A$. The normalization \bar{X} is also affine, say $\bar{X} = \text{Spec } \bar{A}$, and $\bar{S} = V(s) \subset \bar{X}$. The section s defines a morphism to \mathbb{A}^1 on both X and \bar{X} , fitting into the commutative diagram

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\nu} & X \\ & \searrow \bar{\varphi} & \swarrow \varphi \\ & \mathbb{A}^1 & \end{array}$$

Notice that $\varphi^{-1}(0) = S$ and $\bar{\varphi}^{-1}(0) = \bar{S}$. Localizing at $0 \in \mathbb{A}^1$, we may assume that the target of φ and $\bar{\varphi}$ is the spectrum of a DVR. Then we are in situation to apply [30, Theorem 4 and Lemma 2.2.1], which concludes the Cartier case.

Next assume that S is \mathbb{Q} -Cartier, with Cartier index $m > 0$ not divisible by the residue characteristics. We have that S is nodal in codimension one by Lemma 3.4 below. To show that S is S_2 is a local property, so we may localize at any closed point. In particular we may assume that $\mathcal{O}(-mS) \cong \mathcal{O}_X$. Let $\pi: X' \rightarrow X$ be the cyclic cover induced by this isomorphism [27, Definition 2.52, Lemma 2.53 and Definition 5.19]. The scheme X' is endowed with a μ_m -action and the morphism μ_m is a geometric quotient. By assumption on m , the morphism π is étale in codimension one. So X' is demi-normal. If $S' = \text{red}(\pi^{-1}S)$ and $\Delta' = \text{red}(\pi^{-1}\Delta)$ then $\pi^*(K_X + S + \Delta) = K_{X'} + S' + \Delta'$ and S' is Cartier. So by [25, Corollary 2.43] the pair $(X', S' + \Delta')$ is slc and by the first part of the proof $\mathcal{O}_{S'}$ is S_2 . Now observe that

$$\pi_*\mathcal{I}_{S'} = \bigoplus_{i=0}^{m-1} \mathcal{O}(iS)[\otimes]\mathcal{O}(-S).$$

The μ_m -coaction on $\mathcal{O}_{X'} = \bigoplus_{i=0}^{m-1} \mathcal{O}(iS)$ makes $\pi_*\mathcal{I}_{S'}$ into an μ_m -equivariant sheaf, with invariant part $\mathcal{O}(-S) = \mathcal{I}_S$. Since μ_m is diagonalizable, taking invariants is an exact functor (see for example [20, I.2.5, I.2.11] for basics about diagonalizable group schemes). Thus the exact sequence

$$0 \longrightarrow \mathcal{I}_{S'} \longrightarrow \mathcal{O}_{X'} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0$$

yields the exact sequence

$$0 \longrightarrow \mathcal{I}_S \longrightarrow \mathcal{O}_X \longrightarrow (\mathcal{O}_{S'})^{\mu_m} \longrightarrow 0.$$

From this we see that $(\mathcal{O}_{S'})^{\mu^m} = \mathcal{O}_S$. Since $(\mathcal{O}_{S'})^{\mu^m}$ is a direct summand of $\mathcal{O}_{S'}$ [20, I.2.11], we deduce that \mathcal{O}_S is S_2 . This concludes the proof of Proposition 3.3. \square

LEMMA 3.4. — *Let $(X, S + \Delta)$ be a two-dimensional excellent semi-log canonical pair, where S is a reduced divisor, and whose residue fields at closed points are perfect of characteristic $\neq 2$. Then S has at worst nodal singularities.*

Proof. — Let $\nu: (\bar{X}, \bar{D} + \bar{S} + \bar{\Delta}, \tau) \rightarrow (X, S + \Delta)$ be the normalization. Then \bar{D} is a nodal curve and \bar{S} does not pass through the nodes p_1, \dots, p_n of \bar{D} [25, Theorem 2.31]. This implies that S does not contain the images $\nu(p_i)$. So we may restrict on $X \setminus \{\pi(p_1), \dots, \pi(p_n)\}$ and assume that the curve \bar{D} is normal. In particular τ is an involution on \bar{D} . Thus the morphism $\bar{X} \rightarrow X$ is the quotient by the finite equivalence relation $R(\tau)$ where the equivalence classes have cardinality at most 2. Moreover \bar{D} does not pass through the nodes of \bar{S} , by [25, Theorem 2.31] again. Therefore $\bar{S} \rightarrow S$ is obtained by gluing pairs of smooth points of \bar{S} . It follows that S has at worst nodal singularities. \square

3.3. Semi-normality of unions of lc centers in large characteristics

THEOREM 3.5. — *Let p_0 be the positive integer occurring in Proposition 2.14. Let (X, Δ) be a log canonical threefold with standard coefficients whose closed points have perfect residue fields of characteristic $p > p_0$. Then the union of all log canonical centers of (X, Δ) is semi-normal.*

Proof. — The proof follows similar lines as Theorem 3.2, however using instead the vanishing theorem of Proposition 2.14. Let C denote the union of all log canonical centers of (X, Δ) . Let $g: (Z, \Delta_Z + E) \rightarrow (X, \Delta)$ be the dlt model described in Proposition 2.14. Since $(Z, \Delta_Z + E)$ is dlt [9, Theorem 19] implies that E is S_2 . Therefore E is weakly normal by Lemma 2.8. Consider the short exact sequence associated to E inside Z :

$$0 \longrightarrow \mathcal{O}_Z(-E) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_E \longrightarrow 0.$$

By Proposition 2.14 it holds that $R^1 g_* \mathcal{O}_Z(-E) = 0$ and hence, pushing forward along g , on X there is a short exact sequence

$$0 \longrightarrow g_* \mathcal{O}_Z(-E) \longrightarrow g_* \mathcal{O}_Z \longrightarrow g_* \mathcal{O}_E \longrightarrow 0.$$

We have $g_*\mathcal{O}_Z = \mathcal{O}_X$ and, as in the proof of Theorem 3.2, we see that $g_*\mathcal{O}_Z(-E)$ is the ideal sheaf of C . Let $\varphi: C^{\text{sn}} \rightarrow C$ denote the semi-normalization.

Since E is semi-normal, the universal property of the semi-normalization [23, I, Proposition 7.2.3(3)] implies that there exists a factorization $\mathcal{O}_C \hookrightarrow \varphi_*\mathcal{O}_{C^{\text{sn}}} \hookrightarrow g_*\mathcal{O}_E$. Therefore we have the commutative diagram

$$\begin{array}{ccc}
 & \mathcal{O}_C & \\
 & \nearrow & \searrow \\
 \mathcal{O}_X & & \varphi_*\mathcal{O}_{C^{\text{sn}}} \\
 & \xrightarrow{\quad} & \downarrow \\
 & & g_*\mathcal{O}_E.
 \end{array}$$

Since $\mathcal{O}_X \twoheadrightarrow g_*\mathcal{O}_E$ is surjective, we see that $\mathcal{O}_C \hookrightarrow \varphi_*\mathcal{O}_{C^{\text{sn}}}$ is also a surjection. Therefore $C \cong C^{\text{sn}}$, as claimed. \square

4. Sufficient conditions for a non-(semi-)normal boundary

To conclude this article, we describe two sets of conditions that are sufficient to obtain examples of cone pairs with non-(semi-)normal reduced boundary. In both cases it boils down to a single cohomology statement on the base of the cone. Our criteria are independent of the dimension.

Let us fix the notations. The assumption on the Picard rank is not strictly speaking necessary, but it makes our conditions easier to state.

Notation 4.1. — We let X be a projective normal variety of Picard rank one over an algebraically closed field and E a prime effective \mathbb{Q} -Cartier \mathbb{Z} -divisor on X . We assume that (X, E) is lc and that $K_X + E$ is numerically trivial or anti-ample.

If A is an ample \mathbb{Q} -Cartier \mathbb{Z} -divisor on X , then we have $K_X + E \sim_{\mathbb{Q}} -rA$ for some positive rational number $r \in \mathbb{Q}_{\geq 0}$. So $(C_a(X, A), E_{C_a(X, A)})$ is a pair [8, Proposition 2.4]. If we additionally assume inversion of adjunction in the form of Definition 2.23 then it follows that this pair is actually lc by Proposition 2.24. Below we give two sets of sufficient conditions for $E_{C_a(X, A)}$ to be non- S_2 and non-semi-normal at the vertex.

Remark 4.2. — Neither of the two conditions discussed here applies to the instance of Kodaira non-vanishing obtained in [4, Theorem 1.6]. However, there are other surfaces in characteristic 5 that does not lift to $W(k)$,

and that could serve as candidates [28]. Similarly, further pathologies in characteristic 2 and 3 beyond what is discussed here could possibly be found by studying the classification in [22].

4.1. Failure of Kawamata–Viehweg vanishing for a nef divisor

PROPOSITION 4.3. — *Situation as in Notation 4.1. Assume that there exists a nef \mathbb{Q} -Cartier \mathbb{Z} -divisor L such that $H^1(X, \mathcal{O}(L)) \neq 0$. Then there exists $n \geq 1$ such that the cone boundary $E_{C_a(X, L+nE)}$ is not S_2 nor seminormal.*

Proof. — Take n minimal such that

$$H^1(X, \mathcal{O}(L + nE)) = 0 \quad \text{but} \quad H^1(X, \mathcal{O}(L + (n-1)E)) \neq 0.$$

By assumption on L we have $n \geq 1$. Then $A = L + nE$ is ample, and it follows from Proposition 2.20 that $E_{C_a(X, A)}$ is not S_2 at the vertex. To prove that it is non-seminormal, consider the exact sequence of cohomology

$$\begin{aligned} H^0(X, \mathcal{O}_T(dA)) \xrightarrow{\psi_d} H^0(E, \mathcal{E}_d) &\longrightarrow H^1(X, \mathcal{O}_T(dA - E)) \\ &\xrightarrow{\alpha_d} H^1(X, \mathcal{O}_T(dA)) \end{aligned}$$

obtained from the restriction sequences (2.2) in our setting. Let $E' = \text{Spec}_k \bigoplus_{d \geq 0} H^0(E, \mathcal{E}_d)$ with vertex v' . Then by Corollary 2.22 we have a finite homeomorphism $\pi: E' \rightarrow E_{C_a(X, A)}$ which is an isomorphism outside the vertex and induces an isomorphism $k(v) \cong k(v')$. Now α_1 is not injective by assumption, so ψ_1 is not surjective: this implies that the inclusion $\bigoplus_{d \geq 0} \text{im } \psi_d \subset \bigoplus_{d \geq 0} H^0(E, \mathcal{E}_d)$ is not an equality. This implies in turn that $\pi: E' \rightarrow E_{C_a(X, A)}$ is not an isomorphism, and therefore $E_{C_a(X, A)}$ is not seminormal. \square

Let us describe an example where such non-vanishing occurs: it is the surface constructed in [8], whose notation we follow. Let k be an algebraically closed field of characteristic 3. Then there exists a normal connected projective surface T over k such that:

- (1) T is a projective \mathbb{Q} -factorial Picard rank one del Pezzo surface;
- (2) there exists a proper curve E_1 such that $-K_T \equiv E_1$;
- (3) there is an ample \mathbb{Q} -Cartier \mathbb{Z} -divisor A such that $h^1(T, \mathcal{O}_T(-A)) \neq 0$ and $A \equiv -K_T$.

The pair (T, E_1) is constructed in [8, Section 3.1]. The ample divisor A is defined in [8, Section 3.2] and the non-vanishing is shown in [8, Theorem 3.6]. The fact that $A \equiv -K_T$ follows from [8, Section 3.2].

LEMMA 4.4. — *The pair (T, E_1) is a plt log Calabi–Yau surface pair.*

Proof. — Let $\Psi: S \rightarrow T$ denote the minimal resolution of T . An easy computation shows that, in the notations of [8, Section 3.2],

$$\psi^*(E_1) = \psi_*^{-1}E_1 + \frac{1}{3}C + \frac{2}{3}H_1 + \frac{1}{3}G_1 + \frac{1}{3}F_1.$$

Since $\psi^*(K_T) = K_S + \sum_{i=1}^3 \frac{1}{3}F_i + \frac{1}{3}C$ [8, Proof of Proposition 3.2], it follows that (T, E_1) is plt. Since $K_T + E_1$ is numerically trivial, it follows that $K_T + E_1 \sim_{\mathbb{Q}} 0$. \square

Now let us define the divisors $L = K_T + A$ and $B = L + E_1$.

LEMMA 4.5. — *L is numerically trivial and $h^1(T, \mathcal{O}_T(L)) \neq 0$.*

Proof. — Numerical triviality holds by construction, and the non-vanishing holds by Serre duality. \square

PROPOSITION 4.6. — *B is ample. Moreover $h^1(T, \mathcal{O}_T(B)) = 0$ and $h^1(T, \mathcal{O}_T(B - E_1)) \neq 0$.*

Proof. — Since L is numerically trivial and E_1 is ample, we see that B is ample. We have $h^1(T, \mathcal{O}_T(B - E_1)) = h^1(T, \mathcal{O}_T(L)) \neq 0$ by Lemma 4.5. If E_2 and E_3 are the two other curves on T involved in the definition of A , then $B = K_T + E_2 + E_3$ [8, Section 3.2]. By Serre duality $h^1(T, \mathcal{O}_T(B)) = h^1(T, \mathcal{O}_T(-E_2 - E_3))$, and the latter h^1 is 0 by [11, Proposition 3.3]. \square

COROLLARY 4.7. — *$E_{1, C_a(T, B)}$ is not S_2 nor semi-normal.*

Proof. — Combine Proposition 4.6 and Proposition 4.3. \square

Next, we study the lc centers of the cone pair $(C_a(T, B), E_{1, C_a(T, B)})$.

PROPOSITION 4.8. — *The pair $(C_a(T, B), E_{1, C_a(T, B)})$ is lc, and $E_{1, C_a(T, B)}$ is a non-minimal lc center.*

Proof. — The pair $(C_a(T, B), E_{1, C_a(T, B)})$ is lc by Corollary 2.28, and obviously $E_{1, C_a(T, B)}$ is an lc center. However it is not minimal. Consider the partial resolution $f: BC_a(T, B) \rightarrow C_a(T, B)$ with exceptional divisor T^- . As $K_X + E \sim_{\mathbb{Q}} 0$, by [8, (2.4.2)] we have

$$K_{BC_a(T, B)} + f_*^{-1}E_{1, C_a(T, B)} + T^- = f^*(K_{C_a(T, B)} + E_{1, C_a(T, B)}).$$

Therefore T^- is an lc place of $(C_a(T, B), E_{1, C_a(T, B)})$ and the minimal lc center is the vertex of the cone. \square

COROLLARY 4.9. — *There exists a log canonical threefold singularity $(y \in Y, \Delta_Y)$ over a perfect field of characteristic 3 such that the union of all log canonical centers of $(y \in Y, \Delta_Y)$ is an irreducible divisor which is not semi-normal.*

Proof. — We let $Y = C_a(T, B)$ and $\Delta_Y = E_{1, C_a(T, B)}$, in the notation of this subsection. We let $y \in Y$ be the cone point. We have already showed that (Y, Δ_Y) is log canonical around y and that Δ_Y is a log canonical center of this pair. We will show that any other log canonical center of (Y, Δ_Y) is contained in Δ_Y . Suppose in order to arrive at a contradiction that $C \subset Y$ is a log canonical center of (Y, Δ_Y) and that C is not contained in Δ_Y . Since we work in a neighbourhood of y we may assume that $y \in C$.

As already mentioned in the proof of Proposition 2.24, the morphism

$$f: (\mathrm{BC}_a(T, B), E_{\mathrm{BC}_a(T, B)} + (1+r)T^-) \longrightarrow (C_a(T, B), E_{C_a(T, B)})$$

is crepant, where $r \in \mathbb{Q}$ is such that $K_T + E \sim_{\mathbb{Q}} rB$. By Lemma 4.5 we have $r = 0$. Thus $(\mathrm{BC}_a(T, B), E_{\mathrm{BC}_a(T, B)} + T^-)$ has a log canonical center C' such that $f(C') = C$. Since $C \neq \{y\}$ we have $C' \not\subseteq T^-$, and since $C \not\subseteq E_{C_a(T, B)}$ we have $C' \not\subseteq E_{\mathrm{BC}_a(T, B)}$.

Since T is \mathbb{Q} -factorial, the Seifert bundle $\mathrm{BC}_a(T, B)$ is \mathbb{Q} -factorial by Lemma 2.17: in particular T^- is \mathbb{Q} -Cartier. So if E is an lc place over $(\mathrm{BC}_a(T, B), E_{\mathrm{BC}_a(T, B)} + T^-)$ with center C' , we have the equality of discrepancies

$$-1 = a(E; \mathrm{BC}_a(T, B), E_{\mathrm{BC}_a(T, B)} + T^-) = a(E; \mathrm{BC}_a(T, B), T^-)$$

by [27, Lemma 2.27]. This shows that C' is also a log canonical center of $(\mathrm{BC}_a(T, B), T^-)$.

Now we apply inversion of adjunction for $(\mathrm{BC}_a(T, B), T^-)$ along T^- . Notice that $K_T \sim_{\mathbb{Q}} -A$ is anti-ample, so $K_T \sim_{\mathbb{Q}} rB$ for some $r \in \mathbb{Q}_-$. So the desired inversion of adjunction holds by Proposition 2.26. As T is klt we obtain by Proposition 2.24 that $(\mathrm{BC}_a(T, B), T^-)$ is plt in a neighbourhood of T^- . This fact contradicts what we obtained above, namely that $C' \neq T^-$ is also a log canonical center of $(\mathrm{BC}_a(T, B), T^-)$. We have reached a contradiction, and therefore the proof is complete. \square

4.2. Failure of Kawamata–Viehweg vanishing for an anti-ample divisor

The second set of sufficient conditions grew out of our efforts to find an alternative to Proposition 4.3.

PROPOSITION 4.10. — *Situation as in Notation 4.1. Assume that there exists an ample \mathbb{Q} -Cartier \mathbb{Z} -divisor A with the following properties:*

- (1) $H^1(X, \mathcal{O}(-A)) \neq 0$, and
- (2) $A - (\mathrm{ind}(E) - 1)E$ is ample, where $\mathrm{ind}(E)$ is the Cartier index of E .

Then there exist $d > 0$ and $0 \leq i \leq \text{ind}(E) - 1$ such that $B = dA - iE$ is ample and the cone boundary $E_{C_a(X,B)}$ is not S_2 nor seminormal.

The second condition means that E is numerically small compared to A . For the del Pezzo surface studied in [4] this criterion cannot be applied, since the ample A belongs to the smallest effective numerical class.

Proof. — Consider the threefold cone pair $(C_a(X, A), E_{C_a(X,A)})$. By the first condition the cone $C_a(X, A)$ is not S_3 at the vertex point (see [8, Proposition 2.6]). By Lemma 2.18 we see that $m = \text{ind}(E_{C_a(X,A)})$ divides $\text{ind}(E)$. If $E_{C_a(X,A)}$ is Cartier we obtain that $E_{C_a(X,A)}$ cannot be S_2 . By Proposition 2.20 and Corollary 2.22 we deduce that $E_{C_a(X,A)}$ is not seminormal either. Thus we may assume that $m > 1$ for the rest of the proof.

Denote by D the divisor $E_{C_a(X,A)}$ localized at the vertex v . We have an isomorphism $\mathcal{O}(mD) \cong \mathcal{O}_{C_a(X,A),v} = \mathcal{O}$. This isomorphism gives a structure of \mathcal{O} -algebra to $\bigoplus_{i=0}^{m-1} \mathcal{O}(-iD)$. Let

$$\pi: X' = \text{Spec}_{\mathcal{O}} \bigoplus_{i=0}^{m-1} \mathcal{O}(-iD) \longrightarrow \text{Spec } \mathcal{O}$$

be the induced finite morphism (see [27, Definition 2.52]). Then X' is an integral $S_2 + G_1$ semi-local scheme. Denote by D' the divisorial pullback of D : it holds that $\pi^{[*]} \mathcal{O}(D) = \mathcal{O}_{X'}(D')$. By [27, Lemma 2.53] we see that $\mathcal{O}_{X'}(D')$ is Cartier (actually principal).

We claim that the reduced subscheme $D' \subset X'$ cannot be S_2 . Suppose for the sake of contradiction that it is. Then $\text{depth}(\mathcal{O}_{X',\mathfrak{m}'}) = 3$ for every closed point $\mathfrak{m}' \in X'$ by [27, Proposition 5.3], hence X' would be S_3 . By [27, Proposition 5.4] it would follow that $\bigoplus_{i=0}^{m-1} \mathcal{O}(-iD)$ is an S_3 \mathcal{O} -module. But \mathcal{O} is a direct summand, so \mathcal{O} would be S_3 . In other words $C_a(X, A)$ is S_3 at its vertex, which we know not to be the case.

Thus $\mathcal{O}_{D'}$ is not S_2 , and neither is its pushforward $\pi_* \mathcal{O}_{D'}$ as \mathcal{O} -module. Let us identify $\pi_* \mathcal{O}_{D'}$. On X we have the exact sequences

$$0 \longrightarrow \mathcal{O}(-(i+1)D) \longrightarrow \mathcal{O}(-iD) \longrightarrow (\mathcal{O}(-iD) \otimes \mathcal{O}_{E_{C_a(X,A)}}) / \text{tors} \longrightarrow 0$$

and for ease of notation we denote the cokernel by \mathcal{B}_i . Then \mathcal{B}_i is a torsion-free rank one $\mathcal{O}_{E_{C_a(X,A),v}}$ -module. Since

$$\pi_* \mathcal{O}_{X'}(-D') = \bigoplus_{i=0}^{m-1} \mathcal{O}(-iD)[\otimes] \mathcal{O}_{E_{C_a(X,A)}}$$

we obtain that $\pi_* \mathcal{O}_{D'} = \bigoplus_{i=0}^{m-1} \mathcal{B}_i$. We know that $\pi_* \mathcal{O}_{D'}$ is not S_2 , thus some \mathcal{B}_i is not S_2 as \mathcal{O} -module. We fix this index i for the rest of the proof.

Next we want a description of the non-zero local cohomology group $H_v^1(C_a(X, A), \mathcal{B}_i)$. To simplify notations, from now on and unless otherwise specified, the cohomology is taken on $C_a(X, A)$. We also let $U = C_a(X, A) \setminus \{v\}$. Since $H_v^0(\mathcal{B}_i) = 0$ and $H^1(\mathcal{B}_i) = 0$ we have an exact sequence

$$(4.1) \quad 0 \longrightarrow H^0(\mathcal{B}_i) \longrightarrow H^0(U \cap E_{C_a(X, A)}, \mathcal{B}_i) \longrightarrow H_v^1(\mathcal{B}_i) \longrightarrow 0.$$

Let us identify the first two non-zero groups in this sequence:

- On the affine $C_a(X, A)$ we have the exact sequence

$$0 \longrightarrow H^0(\mathcal{O}(-(i+1)E_{C_a(X, A)})) \longrightarrow H^0(\mathcal{O}(-iE_{C_a(X, A)})) \longrightarrow H^0(\mathcal{B}_i) \longrightarrow 0.$$

As for any $j \in \mathbb{Z}$ we have

$$\begin{aligned} H^0(\mathcal{O}(jE_{C_a(X, A)})) &= H^0(\mathrm{BC}_a(X, A) \setminus X^-, \mathcal{O}(jE_{\mathrm{BC}_a(X, A)})) \\ &= \bigoplus_{d \in \mathbb{Z}} H^0(X, \mathcal{O}(dA + jE)) \end{aligned}$$

we see that

$$H^0(\mathcal{B}_i) = \bigoplus_{d \in \mathbb{Z}} \mathrm{coker}[H^0(\mathcal{O}_X(dA - (i+1)E)) \longrightarrow H^0(\mathcal{O}_X(dA - iE))].$$

- The isomorphism $U \cong \mathrm{BC}_a(X, A) \setminus X^-$ induces an isomorphism

$$\mathcal{B}_i|_{E_{C_a(X, A)} \cap U} \cong \bigoplus_{d \in \mathbb{Z}} \mathrm{coker}[\mathcal{O}_X(dA - (i+1)E) \longrightarrow \mathcal{O}_X(dA - iE)].$$

Thus if for any $d, j \in \mathbb{Z}$ we let the sheaf $\mathcal{E}_{d, j}$ on E be defined by

$$\mathcal{E}_{d, j} = \mathrm{coker}[\mathcal{O}_X(dA - (j+1)E) \longrightarrow \mathcal{O}_X(dA - jE)],$$

we obtain

$$H^0(U \cap E_{C_a(X, A)}, \mathcal{B}_i) = \bigoplus_{d \in \mathbb{Z}} H^0(E, \mathcal{E}_{d, -i}).$$

Now consider the exact sequences of cohomology groups

$$\begin{array}{ccc} H^0(X, \mathcal{O}(dA - iE)) & \xrightarrow{\psi_{d, -i}} & H^0(E, \mathcal{E}_{d, -i}) \longrightarrow H^1(X, \mathcal{O}(dA - (i+1)E)) \\ & & \downarrow \alpha_{d, -i} \\ & & H^1(X, \mathcal{O}(dA - iE)). \end{array}$$

It follows from the exact sequence (4.1) and from the explicit descriptions of the first two terms of the said exact sequence, that

$$\begin{aligned} H_v^1(\mathcal{B}_i) \neq 0 &\iff \exists d \in \mathbb{Z} : \psi_{d, -i} \text{ not surjective} \\ &\iff \exists d \in \mathbb{Z} : \alpha_{d, -i} \text{ not injective.} \end{aligned}$$

Now A and E are ample, so $H^0(E, \mathcal{E}_{d,-i}) = 0$ for $d \leq 0$. (The case $d = 0 = i$ follows simply from that $H^0(X, \mathcal{O}_X) \cong H^0(E, \mathcal{O}_E)$ are equal to the algebraically closed base-field). Thus

$$H_v^1(\mathcal{B}_i) \neq 0 \iff \exists d_i > 0 : \alpha_{d_i, -i} \text{ not injective.}$$

Since we chose \mathcal{B}_i with the property that $H_v^1(\mathcal{B}_i) \neq 0$, we would like to take as new ample divisor $B = d_i A - iE$. In general B need not be ample. But if the second hypothesis of Proposition 4.10 is satisfied, as $0 \leq i \leq \text{ind}(E_{C_a(X,A)}) - 1 < \text{ind}(E)$, we obtain that B is ample. In this case we conclude by Proposition 2.20 and Corollary 2.22. \square

BIBLIOGRAPHY

- [1] A. ANDREOTTI & E. BOMPIERI, “Sugli omeomorfismi delle varietà algebriche”, *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (3)* **23** (1969), p. 431-450.
- [2] A. ANDREOTTI & F. NORGUET, “La convexité holomorphe dans l’espace analytique des cycles d’une variété algébrique”, *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (3)* **21** (1967), p. 31-82.
- [3] E. ARVIDSSON, “On a vanishing theorem for birational morphisms of threefolds in positive characteristics”, 2023, <https://arxiv.org/abs/2302.04420>.
- [4] E. ARVIDSSON, F. BERNASCONI & J. LACINI, “On the Kawamata–Viehweg vanishing theorem for log del Pezzo surfaces in positive characteristic”, *Compos. Math.* **158** (2022), no. 4, p. 750-763.
- [5] E. ARVIDSSON, F. BERNASCONI & Z. PATAKFALVI, “On the properness of the moduli space of stable surfaces over $\mathbb{Z}[1/30]$ ”, *Moduli* **1** (2024), article no. e3 (37 pages).
- [6] AUTHORS, “The Stacks Project”.
- [7] F. BERNASCONI, “Non-normal purely log terminal centres in characteristic $p \geq 3$ ”, *Eur. J. Math.* **5** (2019), no. 4, p. 1242-1251.
- [8] ———, “Kawamata–Viehweg vanishing fails for log del Pezzo surfaces in characteristic 3”, *J. Pure Appl. Algebra* **225** (2021), article no. 106727 (16 pages).
- [9] F. BERNASCONI & J. KOLLÁR, “Vanishing theorems for three-folds in characteristic $p > 5$ ”, *Int. Math. Res. Not.* **2023** (2023), no. 4, p. 2846-2866.
- [10] B. BHATT, L. MA, Z. PATAKFALVI, K. SCHWEDE, K. TUCKER, J. WALDRON & J. WITASZEK, “Globally $+$ -regular varieties and the minimal model program for threefolds in mixed characteristic”, *Publ. Math., Inst. Hautes Étud. Sci.* **138** (2023), p. 69-227.
- [11] P. CASCINI & H. TANAKA, “Purely log terminal threefolds with non-normal centres in characteristic two”, *Am. J. Math.* **141** (2019), no. 4, p. 941-979.
- [12] V. COSSART & O. PILTANT, “Resolution of singularities of threefolds in positive characteristic. I. Reduction to local uniformization on Artin-Schreier and purely inseparable coverings”, *J. Algebra* **320** (2008), no. 3, p. 1051-1082.
- [13] ———, “Resolution of singularities of arithmetical threefolds”, *J. Algebra* **529** (2019), p. 268-535.
- [14] O. DAS & C. D. HACON, “On the adjunction formula for 3-folds in characteristic $p > 5$ ”, *Math. Z.* **284** (2016), no. 1-2, p. 255-269.
- [15] S. FILIPAZZI & J. WALDRON, “Connectedness Principle for 3-Folds in Characteristic $p > 5$ ”, *Mich. Math. J.* **74** (2024), no. 4, p. 1-27.

- [16] Y. GONGYO, Y. NAKAMURA & H. TANAKA, “Rational points on log Fano threefolds over a finite field”, *J. Eur. Math. Soc.* **21** (2019), no. 12, p. 3759-3795.
- [17] C. D. HACON & J. WITASZEK, “On the relative Minimal Model Program for threefolds in low characteristics”, *Peking Math. J.* **5** (2022), no. 2, p. 365-382.
- [18] C. D. HACON & C. XU, “On the three dimensional minimal model program in positive characteristic”, *J. Am. Math. Soc.* **28** (2015), no. 3, p. 711-744.
- [19] R. HARTSHORNE, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer, 1977, xvi+496 pages.
- [20] J. C. JANTZEN, *Representations of algebraic groups*, 2nd ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, 2003, xiv+576 pages.
- [21] T. KAWAKAMI, “On the Kawamata–Viewheg vanishing theorem for log Calabi–Yau surfaces in large characteristic”, *Ann. Inst. Fourier* **75** (2025), no. 6, p. 2657-2675.
- [22] T. KAWAKAMI & M. NAGAOKA, “Pathologies and liftability of Du Val del Pezzo surfaces in positive characteristic”, *Math. Z.* **301** (2022), no. 3, p. 2975-3017.
- [23] J. KOLLÁR, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 32, Springer, 1996, viii+320 pages.
- [24] ———, “Singularities of pairs”, in *Algebraic geometry. Proceedings of the Summer Research Institute Santa Cruz 1995* (J. Kollár et al., eds.), Proceedings of Symposia in Pure Mathematics, vol. 62, Part 1, American Mathematical Society, 1997, p. 221-287.
- [25] ———, *Singularities of the Minimal Model Program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, 2013.
- [26] ———, “Families of stable 3-folds in positive characteristic”, *Épjournal de Géom. Algébr.*, *EPIGA* **7** (2023), article no. 6 (11 pages).
- [27] J. KOLLÁR & S. MORI, *Birational geometry of algebraic varieties*, translated from the 1998 japanese original ed., Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, 1998, with the collaboration of C. H. Clemens and A. Corti, viii+254 pages.
- [28] M. NAGAOKA, “Non-liftable log del Pezzo surfaces of rank one in characteristic five”, *Nagoya Math. J.* **259** (2025), p. 399-422.
- [29] Q. POSVA, “Gluing theory for slc surfaces and threefolds in positive characteristic”, *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)* **25** (2024), no. 2, p. 811-886.
- [30] ———, “Gluing for stable families of surfaces in mixed characteristic”, *Algebr. Geom.* **12** (2025), no. 2, p. 145-172.
- [31] K. SCHWEDE, “Generalized Divisors and Reflexive Sheaves”, 2008.
- [32] H. TANAKA, “Minimal model program for excellent surfaces”, *Ann. Inst. Fourier* **68** (2018), no. 1, p. 345-376.
- [33] C. TRAVERSO, “Seminormality and Picard group”, *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (3)* **24** (1970), p. 585-595.
- [34] H. YANAGIHARA, “Some results on weakly normal ring extensions”, *J. Math. Soc. Japan* **35** (1983), no. 4, p. 649-661.

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