Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

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Article à paraître, mis en ligne le 26 mars 2026, 60 p.

CENTRE MERSENNE
Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
Article à paraître
Mis en ligne le 26 mars 2026.

MALLE'S CONJECTURE FOR NONIC HEISENBERG EXTENSIONS

by Étienne FOUVRY & Peter KOYMANS (*)

ABSTRACT. — We prove Malle's conjecture for nonic Heisenberg extensions over $\mathbb{Q}$. Our main algebraic result shows that the number of nonic Heisenberg extensions over $\mathbb{Q}$ with discriminant bounded by X is given by a character sum. We then extract the main term from this sum by exploiting oscillation of characters.

RÉSUMÉ. — On démontre la conjecture de Malle pour les extensions de $\mathbb{Q}$ de degré 9 et de type Heisenberg. Le principal résultat algébrique montre que le nombre de telles extensions de discriminants bornés par X est donné par une somme de caractères. On extrait le terme principal en exploitant les oscillations de ces caractères.

1. Introduction

A fundamental problem in arithmetic statistics is to count algebraic extensions over $\mathbb{Q}$ with bounded discriminant. This subject has its roots in a famous theorem due to Hermite that there are only finitely many number fields with bounded discriminant.

Let $K/\mathbb{Q}$ be an extension of degree n and write L for the normal closure of K. Then $\mathrm{Gal}(L/\mathbb{Q})$ acts on the n embeddings $K \hookrightarrow \overline{\mathbb{Q}}$, which gives a homomorphism from $\mathrm{Gal}(L/\mathbb{Q})$ to S_n. By abuse of notation we define $\mathrm{Gal}(K/\mathbb{Q}) \subseteq S_n$ to be the image of this homomorphism. We then define for every transitive group $G \subseteq S_n$ the counting function

$$N(G, X) \coloneqq |\{K/\mathbb{Q} : \mathrm{Gal}(K/\mathbb{Q}) \cong G, \Delta_{K/\mathbb{Q}} \leqslant X\}|,$$

where $\Delta_{K/\mathbb{Q}}$ is the absolute discriminant and the fields K are taken inside a fixed algebraic closure of $\mathbb{Q}$ (up to isomorphism). Here we stress that the

Keywords: Malle's conjecture, Heisenberg group.
2020 *Mathematics Subject Classification:* 11N37, 11R20, 11R34.
(*) Peter Koymans wishes to thank the Max Planck Institute for Mathematics in Bonn for its financial support, great work conditions and an inspiring atmosphere.

isomorphism is not just an isomorphism of finite groups but as permutation groups; this is equivalent to G and $\mathrm{Gal}(K/\mathbb{Q})$ being conjugate subgroups of S_n. This counting function is the subject of Malle's conjecture [24, 25], who conjectured an asymptotic of the form

$$N(G,X) \sim c(G)X^{a(G)}(\log X)^{b(G)-1}, \tag{1.1}$$

where $c(G)$ is an unspecified constant and where $a(G)$ and $b(G)$ can be computed as follows. Let $G \subseteq S_n$, so that G has a natural action on the set $\{1,\dots,n\}$. Then put for $\sigma \in G$

$$\mathrm{ind}(\sigma) := n - |\{\text{orbits of } \sigma\}|,$$

where the orbits are with respect to the action on $\{1,\dots,n\}$. We define

$$a(G)^{-1} := \min_{\sigma \in G\setminus\{\mathrm{id}\}} \mathrm{ind}(\sigma).$$

To define $b(G)$, we consider the following action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on G. Let $c : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\mathbb{Z}}^*$ be the cyclotomic character. For $g \in G$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we define

$$g^\sigma := g^{c(\sigma)}.$$

It is easy to see that this induces an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $C(G)$, the conjugacy classes of G. We remark that $\mathrm{ind}(\sigma)$ is constant as σ varies through a conjugacy class C, which allows us to define $\mathrm{ind}(C)$ in the obvious way. Furthermore, the index of g is the same as the index of g^σ. Then we define

$$b(G) := |\{C \in C(G) : \mathrm{ind}(C) = a(G)^{-1}\}/\sim|,$$

where two conjugacy classes are equivalent if they are in the same orbit under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $C(G)$. As stated the exponent $b(G)$ in Malle's conjecture is not always correct, see the work of Klüners [18] for a counterexample. Türkelli [32] proposed a modified version of Malle's conjecture, with a different $b(G)$, to take into account the counterexample found by Klüners.

Equation (1.1) is known in a limited number of cases. The authors are aware of the following results

- the work of Mäki [23] for abelian G, see also Wright [34] for the generalization to arbitrary number fields,
- Davenport–Heilbronn [10] for S_3,
- Klüners [19] for generalized quaternion groups,
- Klüners [20] for many wreath products $C_2 \wr H$,
- Bhargava [5, 6] for S_4 and S_5,

- Bhargava–Wood [7] and independently Belabas–Fouvry [4] for $S_3 \subseteq S_6$,
- Wang [33] for direct products $G \times A$ with $G \in \{S_3, S_4, S_5\}$ and A abelian with some conditions on $|A|$,
- Masri, Thorne, Tsai and Wang [26] for all direct products $G \times A$ with $G \in \{S_3, S_4, S_5\}$ and A abelian.

Alberts [1, 2] made progress for many solvable groups. Finally, Equation (1.1) is also known for quartic D_4-extensions, see the work [8] that we reproduce now.

THEOREM 1.1 (Cohen–Diaz y Diaz–Olivier). — *The number of degree 4 extensions L of $\mathbb{Q}$, up to isomorphism, such that the normal closure has Galois group isomorphic to D_4, with absolute discriminant at most X is asymptotic to $c(D_4)X$, where*

$$c(D_4) = \frac{3}{\pi^2} \sum_D \frac{2^{-i(D)}}{D^2} \frac{L(1,D)}{L(2,D)}.$$

Here the sum is over fundamental discriminants different from 1, and $i(D) = 0$ if $D > 0$ and $i(D) = 1$ if $D < 0$.

The error term in Theorem 1.1 is of exceptional quality, namely of size $O_\epsilon(X^{\frac{3}{4}+\epsilon})$. In this paper we are interested in nonic Heisenberg extensions, which bear some similarities with quartic D_4-extensions. Let Heis_3 be the Heisenberg group with 27 elements, i.e. the multiplicative group of upper triangular matrices with coefficients in $\mathbb{F}_3$ and ones on the diagonal. Denote by $N(\mathrm{Heis}_3, X)$ the number of degree 9 extensions L of $\mathbb{Q}$, up to isomorphism, such that the normal closure has Galois group isomorphic to Heis_3 and such that the absolute discriminant is bounded by X. Our main result is the following.

THEOREM 1.2. — *There exists a constant $c(\mathrm{Heis}_3) > 0$ such that*

$$N(\mathrm{Heis}_3, X) \sim c(\mathrm{Heis}_3)\, X^{1/4}.$$

We give a completely explicit formula for $c(\mathrm{Heis}_3)$, which we postpone until Equation (3.16). In Remark 3.8, we will compare the constants $c(\mathrm{Heis}_3)$ and $c(D_4)$. Actually, our proof leads to the asymptotic formula

$$N(\mathrm{Heis}_3, X) = c(\mathrm{Heis}_3)\, X^{1/4} + O_A(X^{1/4}(\log X)^{-A})$$

for all $A > 0$.

Our main theorem implies Malle's conjecture for nonic Heisenberg extensions (note that, up to conjugation, there is precisely one transitive subgroup of S_9 isomorphic to Heis_3). One of the challenges is to find an explicit expression for the constant $c(\mathrm{Heis}_3)$. Indeed, it is substantially easier to show that there exists a constant $c(\mathrm{Heis}_3)$. This phenomenon can already be observed in the work of [9], where the strong form of Malle's conjecture is proved, with an explicit constant $c(G, K)$, for cyclic degree ℓ extensions over an arbitrary base field K.

Despite the superficial similarities between Theorem 1.1 and Theorem 1.2, the proof techniques employed in Theorem 1.1 break down completely for nonic Heisenberg extensions. The key principle used in the proof of Theorem 1.1 is the following: take a quadratic extension $K/\mathbb{Q}$ and take a quadratic extension L/K. Then typically L is a quartic D_4-extension of $\mathbb{Q}$. The problem then reduces to uniformly counting quadratic extensions.

However, a degree ℓ cyclic extension L/K of a degree ℓ cyclic extension $K/\mathbb{Q}$ is almost never a degree ℓ^2 Heisenberg extension. For this reason we must take an entirely different approach, where we estimate a certain character sum that counts the number of Heisenberg extensions. Our approach is in spirit of the work of Heath-Brown [16] and Fouvry–Klüners [11, 12, 13, 14], although the technical details are somewhat different than these works.

We believe that Theorem 1.2 can be extended in various directions. As a first generalization one can consider the Heisenberg group Heis_ℓ of order ℓ^3, where $\ell \geqslant 3$ is a prime. Our algebraic results are in fact stated in this more general setting. However our analytic results currently use that $\mathbb{Z}[\zeta_3]$ is a principal ideal domain. It is possible to extend our analytic results to any odd prime ℓ for which $\mathbb{Z}[\zeta_\ell]$ is a principal ideal domain (so $\ell \in \{3, 5, 7, 11, 13, 17, 19\}$), and perhaps it is possible to extend them to all odd primes ℓ.

Another direction to consider is to count Heisenberg extensions in the regular representation. The resulting counting function has some similarities to the ones considered in Fouvry–Luca–Pappalardi–Shparlinski [15] and Klys [21]. We are optimistic that our techniques also apply here. A final direction that we shall discuss in this introduction is to count extensions by conductor instead of discriminant. This was done in [3] for quartic D_4-extensions. Perhaps it is possible to extend our results to this setting as well.

Acknowledgements

We thank Carlo Pagano for several inspiring conversations that led to the proof of Theorem 2.8. We would also like to express our admiration for the thorough reading of the manuscript by the anonymous referee and the many excellent suggestions.

2. The Heisenberg group

In this section we develop the algebraic theory for the Heisenberg group Heis_ℓ of order ℓ^3 with $\ell \geqslant 3$ a prime. We start by fixing an algebraic closure $\overline{\mathbb{Q}}$ once and for all. We also fix algebraic closures $\overline{\mathbb{Q}_p}$ for all prime numbers p. All our number fields and local fields are implicitly taken inside these fixed algebraic closures. All our cohomology groups have to be interpreted as profinite group cohomology.

2.1. The different ideal

For a local or global field K, we write $\mathcal{O}_K$ for its ring of integers. If L/K is an extension of local or global fields, we write $\mathfrak{d}_{L/K}$ for the different ideal and $\Delta_{L/K}$ for the relative discriminant. Recall that $\mathfrak{d}_{L/K}$ is an ideal of L, while $\Delta_{L/K}$ is an ideal of K. Denote by f_α the minimal polynomial of an element α and denote by $e_{\mathfrak{q}/\mathfrak{p}}$ the ramification index of the prime $\mathfrak{q}$ of L lying above a prime $\mathfrak{p}$ of K. We now record the following well-known properties of the different ideal.

Lemma 2.1. — *Let L/K be an extension of local or global fields. Let $\mathfrak{q}$ be a prime of L and let $\mathfrak{p}$ be the prime of K below $\mathfrak{q}$. The different ideal satisfies the following properties*

(i) *we have* $\mathrm{N}_{L/K}(\mathfrak{d}_{L/K}) = \Delta_{L/K}$;
(ii) *we have* $\mathfrak{d}_{M/L}\mathfrak{d}_{L/K} = \mathfrak{d}_{M/K}$;
(iii) *we have* $\mathfrak{q} \mid \mathfrak{d}_{L/K}$ *if and only if* $\mathfrak{q}$ *is ramified in* L/K. *Furthermore, in case that* $\mathfrak{q}$ *is not wildly ramified, we have that* $\mathfrak{q}^{e_{\mathfrak{q}/\mathfrak{p}}-1}$ *exactly divides* $\mathfrak{d}_{L/K}$;
(iv) *we have*

$$v_{\mathfrak{q}}(\mathfrak{d}_{L/K}) = v_{\mathfrak{q}}(\mathfrak{d}_{L_{\mathfrak{q}}/K_{\mathfrak{p}}});$$

(v) $\mathfrak{d}_{L/K}$ *is generated by the elements* $f'_\alpha(\alpha)$ *as* α *ranges over all elements of* $\mathcal{O}_L$ *such that* $L = K(\alpha)$. *Now suppose additionally that* α *is an element of* L *such that* $\mathcal{O}_L = \mathcal{O}_K[\alpha]$. *Then* $\mathfrak{d}_{L/K} = (f'_\alpha(\alpha))$.

Our next result is known in the case $k = \mathbb{Q}$, but we were unable to find a reference for general k.

Lemma 2.2. — *Let ℓ be a prime number. Suppose that K/k is an extension of local or global fields such that $\mathrm{Gal}(K/k) \cong \mathbb{F}_\ell^2$. Write $k_1, \dots, k_{\ell+1}$ for the intermediate fields. Then we have*

$$\Delta_{K/k} = \prod_{i=1}^{\ell+1} \Delta_{k_i/k}.$$

Proof. — Let for now K/k be any finite Galois extension of local or global fields. The conductor–discriminant formula states that

$$\mathfrak{d}_{K/k} = \prod_{\chi \in \mathrm{Irr}(G)} \mathfrak{f}(\chi)^{\chi(1)}, \tag{2.1}$$

where $\mathrm{Irr}(G)$ denotes the set of irreducible characters of $G = \mathrm{Gal}(K/k)$ and $\mathfrak{f}(\chi)$ denotes the Artin conductor of χ, see [30, Chapter VI] for the definition of the Artin conductor.

If K/k is bicyclic, then there are ℓ^2 irreducible characters. Except for the trivial character, there are $\ell - 1$ non-trivial characters coming from each $\mathrm{Gal}(k_i/k)$ for $i = 1, \dots, \ell + 1$. Choose one non-trivial character χ_i for $\mathrm{Gal}(k_i/k)$, so that all non-trivial characters for $\mathrm{Gal}(k_i/k)$ are χ_i^j for $j = 1, \dots, \ell - 1$. It is also proven in [30, Chapter VI, Proposition 6] that the Artin conductor of χ_i^j is the same as the Artin conductor of χ_i^j restricted to $\mathrm{Gal}(k_i/k)$. We conclude that

$$\mathfrak{d}_{K/k} = \prod_{i=1}^{\ell+1} \prod_{j=1}^{\ell-1} \mathfrak{f}(\chi_i^j) = \prod_{i=1}^{\ell+1} \mathfrak{d}_{k_i/k}$$

by two applications of Equation (2.1). The lemma follows once we take norms. □

2.2. General theory

Let ℓ be an odd prime. The Heisenberg group Heis_ℓ is the multiplicative group of upper triangular matrices with coefficients in $\mathbb{F}_\ell$ (and ones on the diagonal). Heis_ℓ is a non-commutative group of size ℓ^3 with center $Z(\mathrm{Heis}_\ell)$ of size ℓ. Furthermore, every non-trivial element has order ℓ.

We will now classify the subgroups of Heis_ℓ. Write z for a non-trivial element of the center $Z(\mathrm{Heis}_\ell)$. There are two types of subgroups of order ℓ:

- the central (and therefore normal) subgroup $\langle z \rangle$ generated by z,
- the subgroup $\langle g \rangle$ generated by any $g \notin \langle z \rangle$. This subgroup is not normal and its normalizer is equal to $\langle g, z \rangle$.

If g_1 and g_2 are conjugate, then they have the same image in the abelianization of Heis_ℓ and thus must differ by a central element. In particular, we find that the conjugacy class of an element $g \notin \langle z \rangle$ is precisely equal to $\{g, gz, \dots, gz^{\ell-1}\}$. Moreover, there are precisely $\ell + 1$ subgroups of order ℓ^2. These are all normal and of the shape $\langle g, z \rangle$ for some $g \notin \langle z \rangle$. Finally, two subgroups $\langle g_1 \rangle$ and $\langle g_2 \rangle$ with $g_1, g_2 \notin \langle z \rangle$ are conjugate if and only if $\langle g_1, g_2\rangle$ is a proper subgroup of Heis_ℓ, in which case we must have $\langle g_1, g_2\rangle = \langle g_1, z\rangle = \langle g_2, z\rangle$ or $\langle g_1\rangle = \langle g_2\rangle$.

The quotient $\mathrm{Heis}_\ell / Z(\mathrm{Heis}_\ell)$ is bicyclic so that Heis_ℓ is a central $\mathbb{F}_\ell$-extension of $\mathbb{F}_\ell^2$. Recall that the central extensions of $\mathbb{F}_\ell^2$ by $\mathbb{F}_\ell$ are parametrized by the group $H^2(\mathbb{F}_\ell^2, \mathbb{F}_\ell)$, where we view $\mathbb{F}_\ell$ as a trivial $\mathbb{F}_\ell^2$-module. Write χ_1 and χ_2 for the two natural projection maps from $\mathbb{F}_\ell^2$ to $\mathbb{F}_\ell$. Then it is shown in [22, Section 4.1] that the classes $\theta \in H^2(\mathbb{F}_\ell^2, \mathbb{F}_\ell)$, such that the extension group E in the corresponding exact sequence

$$0 \longrightarrow \mathbb{F}_\ell \longrightarrow E \longrightarrow \mathbb{F}_\ell^2 \longrightarrow 0$$

satisfies $E \cong \mathrm{Heis}_\ell$, are precisely the non-trivial multiples of the 2-cocycle $(\sigma, \tau) \mapsto \chi_1(\sigma)\chi_2(\tau)$. We will from now on denote this last 2-cocycle by θ_{χ_1,χ_2}.

The inflation–restriction exact sequence will play an important role throughout this section. Let G be a profinite group, N a normal open subgroup and A a discrete G-module. Then the quotient G/N naturally acts on the fixed points A^N. We have a long exact sequence

$$(2.2)\quad 0 \longrightarrow H^1(G/N, A^N) \xrightarrow{\text{inf}} H^1(G, A) \xrightarrow{\text{res}} H^1(N, A)^{G/N} \xrightarrow{\text{tr}} H^2(G/N, A^N) \xrightarrow{\text{inf}} H^2(G, A).$$

Here the map tr is known as the *transgression* map, while the other maps are the usual *inflation* and *restriction* maps. We remark that G/N naturally acts on $H^1(N, A)$ by sending a cocycle $f : N \to A$ to $(g * f)(n) = g * f(g^{-1}ng)$.

Over a field F of characteristic 0, the Heisenberg group is realized as follows. Take two linearly independent characters $\chi, \chi' : G_F \to \mathbb{F}_\ell$ and let K be the bicyclic extension given by χ and χ'. We apply Equation (2.2) with $A = \mathbb{F}_\ell$, $G = G_F$ and $N = G_K$. Here, and for the remainder of this paper, we view $\mathbb{F}_\ell$ as a discrete Galois module with trivial action. In this case we get an isomorphism

$$(2.3)\quad \frac{\mathrm{Hom}(G_K, \mathbb{F}_\ell)^{\mathrm{Gal}(K/F)}}{\mathrm{res}(\mathrm{Hom}(G_F, \mathbb{F}_\ell))} \cong \ker(H^2(\mathrm{Gal}(K/F), \mathbb{F}_\ell) \xrightarrow{\text{inf}} H^2(G_F, \mathbb{F}_\ell)).$$

We are now ready to define the space of characters of interest to us.

Definition 2.3. — *For an extension K/F with $\mathrm{Gal}(K/F) \cong \mathbb{F}_\ell^2$, we define $\mathrm{Heis}(K/F)$ to be the subspace of $\rho \in \mathrm{Hom}(G_K, \mathbb{F}_\ell)^{\mathrm{Gal}(K/F)}$ that maps to the 1-dimensional subspace of $H^2(\mathbb{F}_\ell^2, \mathbb{F}_\ell)$ generated by the 2-cocycle θ_{χ_1,χ_2} under the transgression map. If $\rho \in \mathrm{Heis}(K/F)$ and $\chi : G_F \to \mathbb{F}_\ell$, we call $\rho + \chi \in \mathrm{Heis}(K/F)$ the twist of ρ by χ.*

Remark 2.4. — The transgression map naturally lands in $H^2(\mathrm{Gal}(K/F), \mathbb{F}_\ell)$, not in $H^2(\mathbb{F}_\ell^2, \mathbb{F}_\ell)$. Hence we are implicitly choosing an isomorphism $\mathrm{Gal}(K/F) \cong \mathbb{F}_\ell^2$ in the above definition, which allows us to identify

$$H^2(\mathrm{Gal}(K/F), \mathbb{F}_\ell) \cong H^2(\mathbb{F}_\ell^2, \mathbb{F}_\ell).$$

Take any character $\chi : \mathbb{F}_\ell^2 \to \mathbb{F}_\ell$. Observe that the 2-cocycle $\theta_{\chi,\chi} \in H^2(\mathbb{F}_\ell^2, \mathbb{F}_\ell)$ is trivialized by the 1-cochain that sends σ to $\chi(\sigma)^2/2$. Using this, we directly verify that the choice of isomorphism does not change the set $\mathrm{Heis}(K/F)$.

If K is a Galois extension of F and $\rho : G_K \to \mathbb{F}_\ell$ is a character, we write $K(\rho)$ for the field extension of K corresponding to ρ.

THEOREM 2.5. — *Let F be a field of characteristic 0 and let K be a Galois extension with Galois group $\mathbb{F}_\ell^2$. Let χ_1, χ_2 be the natural projection maps $\mathbb{F}_\ell^2 \to \mathbb{F}_\ell$.*

(i) Let $\rho : G_K \to \mathbb{F}_\ell$ be a non-trivial homomorphism. Then $K(\rho)$ is a Galois extension of F with Galois group isomorphic to Heis_ℓ and $K(\rho)^{Z(\mathrm{Gal}(K(\rho)/F))} = K$ if and only if $\rho \in \mathrm{Heis}(K/F)$.

(ii) We have

$$\frac{\mathrm{Heis}(K/F)}{\mathrm{res}(\mathrm{Hom}(G_F, \mathbb{F}_\ell))} = \begin{cases} \mathbb{F}_\ell & \text{if } \inf(\theta_{\chi_1,\chi_2}) = 0 \text{ in } H^2(G_F, \mathbb{F}_\ell), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, there is a Heisenberg extension containing K if and only if $\inf(\theta_{\chi_1,\chi_2}) = 0$. If there is such a Heisenberg extension, the set of all such Heisenberg extensions may be obtained by twisting with a character $G_F \to \mathbb{F}_\ell$.

Proof. — The key step is to explicitly describe the transgression map. The space $\mathrm{Hom}(G_K, \mathbb{F}_\ell)^{\mathrm{Gal}(K/F)}$ consists of those characters $\rho \in \mathrm{Hom}(G_K, \mathbb{F}_\ell)$ satisfying the following two properties. Firstly, $K(\rho)/F$ is a Galois extension. Secondly, there is an exact sequence

$$1 \longrightarrow \mathrm{Gal}(K(\rho)/K) \longrightarrow \mathrm{Gal}(K(\rho)/F) \longrightarrow \mathrm{Gal}(K/F) \longrightarrow 1 \tag{2.4}$$

with $\mathrm{Gal}(K(\rho)/K)$ central in $\mathrm{Gal}(K(\rho)/F)$. As explained in [22, Section 4], the isomorphism in Equation (2.3) is then explicitly given as follows. Viewing ρ as a map from $\mathrm{Gal}(K(\rho)/K)$ to $\mathbb{F}_\ell$ in Equation (2.4), we naturally get a class in the second cohomology group $H^2(\mathrm{Gal}(K/F), \mathbb{F}_\ell)$.

Part (i) and part (ii) now follow upon unwinding the definition of $\mathrm{Heis}(K/F)$ and using the isomorphism (2.3). □

Our final lemma gives a convenient way to decide if two degree ℓ^2 Heisenberg extensions of a field F of characteristic 0 are isomorphic.

LEMMA 2.6. — *Let ℓ be an odd prime number and let F be a field of characteristic* 0*. Let L and L' be two degree ℓ^2 extensions of F such that the Galois groups of the normal closures $N(L)$ and $N(L')$ are isomorphic to the Heisenberg group* Heis_ℓ*. Then L and L' are isomorphic if and only if $N(L)$ is isomorphic to $N(L')$ and L and L' contain the same degree ℓ subfield.*

Proof. — Certainly, if L and L' are isomorphic, then $N(L)$ and $N(L')$ are isomorphic. Furthermore, since the degree ℓ subfield is Galois over F, they must be the same.

Reversely, suppose that L and L' are as in the lemma. By Galois theory, L and L' correspond to non-normal subgroups H and H' of the Heisenberg group Heis_ℓ of order ℓ. Then, since L and L' contain the same degree ℓ subfield, it follows that either $H = H'$ or that H and H' together generate a subgroup of order ℓ^2. From the structure of the Heisenberg group, we see that H and H' are then conjugate in Heis_ℓ. This implies that L and L' are isomorphic. □

2.3. Heisenberg extensions of $\mathbb{Q}_\ell$

Let us first analyze the situation locally at ℓ. Since every non-trivial element of Heis_ℓ has order ℓ, its ramification theory is relatively simple. We further profit from the fact that for $\ell \neq 2$, there are only two linearly independent characters $G_{\mathbb{Q}_\ell} \to \mathbb{F}_\ell$, which fails for $\ell = 2$.

Given two characters $\rho_1, \rho_2 \in H^1(G_K, \mathbb{F}_\ell)$, we define $\theta'_{\rho_1,\rho_2} \in H^2(G_K, \mathbb{F}_\ell)$ to be the inflation of θ_{χ_1,χ_2} via $(\rho_1, \rho_2) : G_K \to \mathbb{F}_\ell^2$, where χ_1 and χ_2 are the natural projection maps $\mathbb{F}_\ell^2 \to \mathbb{F}_\ell$.

LEMMA 2.7. — *Let K be any field of characteristic* 0 *containing a primitive ℓ-th root of unity ζ_ℓ. For $\alpha \in K^*$, we write χ_α for a character corresponding to $K(\sqrt[\ell]{\alpha})$. Let χ_α, χ_β be linearly independent. Then $\theta'_{\chi_\alpha,\chi_\beta}$ is trivial in $H^2(G_K, \mathbb{F}_\ell)$ if and only if there exists $\omega \in K(\sqrt[\ell]{\alpha})$ such that $\mathrm{N}_{K(\sqrt[\ell]{\alpha})/K}(\omega) = \beta$. Such a Heisenberg extension can be obtained by adjoining the ℓ-th root of the element*

$$\prod_{i=0}^{\ell-2} \sigma^i(\omega^{\ell-i-1})$$

to $K(\chi_\alpha, \chi_\beta)$, where σ is a generator of $\mathrm{Gal}(K(\sqrt[\ell]{\alpha})/K)$*.*

Proof. — This is [27, Theorem 3.1]. □

THEOREM 2.8. — *There exists precisely one Galois extension* $M/\mathbb{Q}_\ell$ *with the property that* $\mathrm{Gal}(M/\mathbb{Q}_\ell)$ *is isomorphic to* Heis_ℓ. *Its discriminant ideal equals*

$$(\ell)^{\ell(\ell+1)(2\ell-2)}.$$

Proof. — Since $\mathbb{Q}_\ell^*/\mathbb{Q}_\ell^{*\ell}$ is a 2-dimensional vector space, it follows from local class field theory that there are two linearly independent characters $G_{\mathbb{Q}_\ell} \to \mathbb{F}_\ell$. In particular it follows that there is precisely one extension K of $\mathbb{Q}_\ell$ with $\mathrm{Gal}(K/\mathbb{Q}_\ell) \cong \mathbb{F}_\ell^2$. Therefore it follows from part (ii) of Theorem 2.5 that there exists at most one Heisenberg extension M of $\mathbb{Q}_\ell$.

Let $\chi_{\mathrm{un}} : G_{\mathbb{Q}_\ell} \to \mathbb{F}_\ell$ be a non-trivial unramified character and let $\chi_{\mathrm{ram}} : G_{\mathbb{Q}_\ell} \to \mathbb{F}_\ell$ be a ramified character. By Theorem 2.5, the existence of M is equivalent to the vanishing of $\theta'_{\chi_{\mathrm{un}},\chi_{\mathrm{ram}}}$ in $H^2(G_{\mathbb{Q}_\ell}, \mathbb{F}_\ell)$. There are natural maps

$$H^2(G_{\mathbb{Q}_\ell}, \mathbb{F}_\ell) \xrightarrow{\mathrm{res}} H^2(G_{\mathbb{Q}_\ell(\zeta_\ell)}, \mathbb{F}_\ell) \xrightarrow{\mathrm{cores}} H^2(G_{\mathbb{Q}_\ell}, \mathbb{F}_\ell).$$

The composition $\mathrm{cores} \circ \mathrm{res}$ is multiplication by $[\mathbb{Q}_\ell(\zeta_\ell) : \mathbb{Q}_\ell] = \ell - 1$. Hence the map res is injective. Over $\mathbb{Q}_\ell(\zeta_\ell)$ we see that χ_{ram} is in the span of χ_{un} and the character χ_{ζ_ℓ} corresponding to the extension $\mathbb{Q}_\ell(\zeta_{\ell^2})/\mathbb{Q}_\ell(\zeta_\ell)$. By local class field theory we know that the norm map $\mathcal{O}^*_{\mathbb{Q}_\ell(\zeta_\ell)(\chi_{\mathrm{un}})} \to \mathcal{O}^*_{\mathbb{Q}_\ell(\zeta_\ell)}$ is surjective. Since ζ_ℓ is a unit, it follows from Lemma 2.7 that $\theta'_{\chi_{\mathrm{un}},\chi_{\mathrm{ram}}}$ is trivial in $H^2(G_{\mathbb{Q}_\ell}, \mathbb{F}_\ell)$ as desired.

We now compute the discriminant of M. Define $L := \mathbb{Q}_\ell(\zeta_\ell)(\chi_{\mathrm{un}})$. Take ω to be an element of $\mathcal{O}_L^*$ such that $\mathrm{N}_{L/\mathbb{Q}_\ell(\zeta_\ell)}(\omega) = \zeta_\ell$. Observe that $\zeta_\ell - 1$ is a uniformizer of $\mathbb{Q}_\ell(\zeta_\ell)$ and therefore also of L. Now we expand

$$\omega = a_0 + a_1(\zeta_\ell - 1) + a_2(\zeta_\ell - 1)^2 + \cdots,$$

where the digits a_i are the Teichmüller lifts of $\mathbb{F}_{\ell^\ell}$ in L. Since the a_i are Teichmüller lifts, the equation

$$\mathrm{N}_{L/\mathbb{Q}_\ell(\zeta_\ell)}(\omega) = \zeta_\ell = 1 + (\zeta_\ell - 1)$$

implies that

$$a_0\sigma(a_0) \cdot \cdots \cdot \sigma^{\ell-1}(a_0) = 1$$

with σ a generator of $\mathrm{Gal}(L/\mathbb{Q}_\ell(\zeta_\ell))$. Now define $\omega_1 := \omega/a_0$, which still satisfies $\mathrm{N}_{L/\mathbb{Q}_\ell(\zeta_\ell)}(\omega_1) = \zeta_\ell$. Expand ω_1 as

$$\omega_1 = 1 + b_1(\zeta_\ell - 1) + b_2(\zeta_\ell - 1)^2 + \cdots.$$

From $\mathrm{N}_{L/\mathbb{Q}_\ell(\zeta_\ell)}(\omega_1) = \zeta_\ell = 1 + (\zeta_\ell - 1)$ we deduce that

$$\sum_{i=0}^{\ell-1} \sigma^i(b_1) \equiv 1 \bmod \zeta_\ell - 1. \tag{2.5}$$

Consider the element

$$\prod_{i=0}^{\ell-2} \sigma^i(\omega_1^{\ell-i-1}) = 1 + \left(\sum_{i=0}^{\ell-2} (\ell - i - 1)\sigma^i(b_1)\right)(\zeta_\ell - 1) + \cdots.$$

We claim that

$$x := \sum_{i=0}^{\ell-2} (\ell - i - 1)\sigma^i(b_1)$$

does not reduce to an element in $\mathbb{F}_\ell$ modulo the maximal ideal of $\mathcal{O}_L$. Indeed, we have the equality

$$\sigma(x) - x \equiv \sum_{i=0}^{\ell-1} \sigma^i(b_1) \equiv 1 \bmod \zeta_\ell - 1$$

according to Equation (2.5). Hence $\sigma(x) \not\equiv x \bmod \zeta_\ell - 1$, so

$$x \bmod \zeta_\ell - 1 \notin \mathbb{F}_\ell.$$

Having established the claim, we write

$$\omega_2 := \prod_{i=0}^{\ell-2} \sigma^i(\omega_1^{\ell-i-1}), \quad \omega_2 = 1 + c_1(\zeta_\ell - 1) + c_2(\zeta_\ell - 1)^2 + \cdots,$$

where $\mathrm{red}_L(c_1) \notin \mathbb{F}_\ell$. From Lemma 2.7, we see that $L(\zeta_{\ell^2}, \sqrt[\ell]{\omega_2})$ is a Galois extension of $\mathbb{Q}_\ell(\zeta_\ell)$ with Galois group isomorphic to Heis_ℓ and whose subfield fixed by the center of its Galois group is the $\mathbb{F}_\ell^2$-extension $\mathbb{Q}_\ell(\zeta_\ell)(\chi_{\mathrm{un}}, \chi_{\mathrm{ram}})/\mathbb{Q}_\ell(\zeta_\ell)$. But so is $M\mathbb{Q}_\ell(\zeta_\ell)$. Then, by Theorem 2.5, it follows that

$$L(\zeta_{\ell^2}, \sqrt[\ell]{t\omega_2}) = M\mathbb{Q}_\ell(\zeta_\ell)$$

for some twist $t \in \mathbb{Q}_\ell(\zeta_\ell)^*$.

Suppose that $t = (\zeta_\ell - 1)^s \cdot u$, where $u \in \mathcal{O}^*_{\mathbb{Q}_\ell(\zeta_\ell)}$ and $s \in \mathbb{Z}$. We claim that $\ell \mid s$. Assume for the sake of contradiction that $\ell \nmid s$. Denote by ρ the automorphism of L that sends ζ_ℓ to ζ_ℓ^2 but fixes the field corresponding to χ_{un}. We claim that $L(\zeta_{\ell^2})$, $L(\sqrt[\ell]{t\omega_2})$ and $L(\sqrt[\ell]{\rho(t\omega_2)})$ are three independent extensions in this case, which is impossible as $M\mathbb{Q}_\ell(\zeta_\ell)/L$ is bicyclic. Indeed, suppose that

$$\zeta_\ell^{x_1}(t\omega_2)^{x_2}(\rho(t\omega_2))^{x_3} \in L^{*\ell}.$$

Inspecting valuations, we certainly find that $x_2 + x_3 \equiv 0 \bmod \ell$. Then modulo ℓ-th powers, the above becomes

$$\omega_2^{x_2} \rho(\omega_2)^{x_3} u' \in L^{*\ell}$$

with $u' \in \mathcal{O}^*_{\mathbb{Q}_\ell(\zeta_\ell)}$, which we expand as

$$u' \cdot (1 + c_1(x_2 + x_3(\zeta_\ell + 1))(\zeta_\ell - 1) + \cdots).$$

Since $\mathrm{red}_L(c_1) \notin \mathbb{F}_\ell$, it follows that

$$\mathrm{red}_L(c_1(x_2 + x_3(\zeta_\ell + 1))) \notin \mathbb{F}_\ell \quad \text{or} \quad \mathrm{red}_L(x_2 + x_3(\zeta_\ell + 1)) = 0.$$

We first dispose with the second case. But $\mathrm{red}_L(x_2+x_3(\zeta_\ell+1)) = 0$ implies that $x_2 + 2x_3 \equiv 0 \bmod \ell$ and hence $x_2 \equiv x_3 \equiv 0 \bmod \ell$. In this case we conclude that

$$x_1 \equiv x_2 \equiv x_3 \equiv 0 \bmod \ell$$

as desired. From now on we suppose that

$$\mathrm{red}_L(c_1(x_2 + x_3(\zeta_\ell + 1))) \notin \mathbb{F}_\ell.$$

In this case $\omega_2^{x_2} \rho(\omega_2)^{x_3} u'$ is of the shape

$$d_0 + d_1(\zeta_\ell - 1) + \cdots \quad \text{with } \mathrm{red}_L(d_0) \in \mathbb{F}_\ell \setminus \{0\} \text{ and } \mathrm{red}_L(d_1) \notin \mathbb{F}_\ell, \tag{2.6}$$

where the digits d_i are the Teichmüller lifts. We claim that such elements are never ℓ-th powers in L. Suppose that α is such an element and consider the polynomial

$$f(x) = x^\ell - \alpha.$$

Then $f(x+d_0)$ is irreducible by Eisenstein's criterion. This finishes the proof of both claims, and we conclude that $\ell \mid s$. Furthermore, $\omega_3 := \frac{t\omega_2}{(\zeta_\ell - 1)^s}$ has an expansion of the shape displayed in Equation (2.6).

Finally, we compute the discriminant of the extension $L(\sqrt[\ell]{\omega_3})/L$. We just showed that

$$f(x + d_0) = (x + d_0)^\ell - \omega_3 = -\omega_3 + \sum_{i=0}^{\ell} \binom{\ell}{i} x^i d_0^{\ell - i}$$

is Eisenstein, i.e. $f(x+d_0)$ satisfies Eisenstein's criterion. Write r for a root of the polynomial $f(x + d_0)$. Since $f(x + d_0)$ is Eisenstein, it follows that

$$\mathcal{O}_{L(\sqrt[\ell]{\omega_3})} = \mathcal{O}_L[r],$$

so we are in the position to apply Lemma 2.1(v). We conclude that

$$\mathfrak{d}_{L(\sqrt[\ell]{\omega_3})/L} = \left(\sum_{i=1}^{\ell} \binom{\ell}{i} i r^{i-1} d_0^{\ell - i} \right) = (\ell).$$

By construction we have that $L(\zeta_{\ell^2}, \sqrt[\ell]{\omega_3}) = M\mathbb{Q}_\ell(\zeta_\ell)$. Then there exists some degree ℓ cyclic extension M' of $\mathbb{Q}_\ell(\chi_{\mathrm{un}})$ such that the Galois closure of M' over $\mathbb{Q}_\ell$ is M and furthermore $M' \subseteq L(\sqrt[\ell]{\omega_3})$. This implies that

$$\Delta^{\ell-1}_{M'/\mathbb{Q}_\ell(\chi_{\mathrm{un}})} \mathrm{N}_{M'/\mathbb{Q}_\ell(\chi_{\mathrm{un}})}(\Delta_{L(\sqrt[\ell]{\omega_3})/M'}) = \Delta^{\ell}_{L/\mathbb{Q}_\ell(\chi_{\mathrm{un}})} \mathrm{N}_{L/\mathbb{Q}_\ell(\chi_{\mathrm{un}})}(\Delta_{L(\sqrt[\ell]{\omega_3})/L})$$

The extensions $L(\sqrt[\ell]{\omega_3})/M'$ and $L/\mathbb{Q}_\ell(\chi_{\mathrm{un}})$ are tamely ramified and of degree $\ell - 1$. We conclude that

$$\Delta^{\ell-1}_{M'/\mathbb{Q}_\ell(\chi_{\mathrm{un}})} \cdot (\ell)^{\ell-2} = (\ell)^{(\ell-2)\ell} \cdot (\ell)^{(\ell-1)\ell}$$

and hence

$$\Delta_{M'/\mathbb{Q}_\ell(\chi_{\mathrm{un}})} = (\ell)^{2\ell-2}. \tag{2.7}$$

Lemma 2.2 yields

$$\Delta_{M/\mathbb{Q}_\ell(\chi_{\mathrm{un}})} = \prod_{i=1}^{\ell+1} \Delta_{M_i/\mathbb{Q}_\ell(\chi_{\mathrm{un}})},$$

where the M_i are the subfields $\mathbb{Q}_\ell(\chi_{\mathrm{un}}) \subsetneq M_i \subsetneq M$ of the bicyclic extension $M/\mathbb{Q}_\ell(\chi_{\mathrm{un}})$. One of the M_i is the field $\mathbb{Q}_\ell(\chi_{\mathrm{un}}, \chi_{\mathrm{ram}})$, while the other M_i are all isomorphic to M' by Lemma 2.6. We deduce that

$$\Delta_{M/\mathbb{Q}_\ell(\chi_{\mathrm{un}})} = (\ell)^{\ell(2\ell-2)} \cdot \Delta_{\mathbb{Q}_\ell(\chi_{\mathrm{un}},\chi_{\mathrm{ram}})/\mathbb{Q}_\ell(\chi_{\mathrm{un}})} = (\ell)^{(\ell+1)(2\ell-2)}$$

as desired. □

2.4. Minimal Heisenberg extensions

In this subsection we will study Heisenberg extensions from a global perspective. We start by defining minimal Heisenberg extensions, which is analogous to the definition of minimal dihedral extensions given by Stevenhagen [31].

Definition 2.9. — *Let $\chi, \chi' : G_{\mathbb{Q}} \to \mathbb{F}_\ell$ be two linearly independent characters. Let M be a Heisenberg extension of $\mathbb{Q}$ containing $\mathbb{Q}(\chi, \chi')$. We say that M is minimal if the following two conditions are satisfied*

- *M is unramified at every place v that is unramified in $\mathbb{Q}(\chi, \chi')$;*
- *$M/\mathbb{Q}(\chi, \chi')$ is unramified at all primes above ℓ in case ℓ has residue field degree 1 in $\mathbb{Q}(\chi, \chi')$.*

Suppose that the residue field degree of ℓ in $\mathbb{Q}(\chi, \chi')$ is 1 and further assume that ℓ ramifies in $\mathbb{Q}(\chi, \chi')$. As we shall see, the second condition is then automatically satisfied for all Heisenberg extensions M containing $\mathbb{Q}(\chi, \chi')$. From this it follows that any Heisenberg extension M that satisfies the first condition also satisfies the second condition.

Lemma 2.10. — *Let $\chi, \chi' : G_{\mathbb{Q}} \to \mathbb{F}_\ell$ be two linearly independent characters. Then $\theta'_{\chi,\chi'}$ is trivial in $H^2(G_{\mathbb{Q}}, \mathbb{F}_\ell)$ if and only if all ramified primes not equal to ℓ have residue field degree 1 in $\mathbb{Q}(\chi, \chi')$.*

Proof. — We first prove the backward implication. There are natural maps

$$H^2(G_{\mathbb{Q}}, \mathbb{F}_\ell) \xrightarrow{\text{res}} H^2(G_{\mathbb{Q}(\zeta_\ell)}, \mathbb{F}_\ell) \xrightarrow{\text{cores}} H^2(G_{\mathbb{Q}}, \mathbb{F}_\ell).$$

The composition cores $\circ$ res is multiplication by $[\mathbb{Q}(\zeta_\ell) : \mathbb{Q}] = \ell - 1$. It follows that the map res is injective. From class field theory, we get another injective map

$$H^2(G_{\mathbb{Q}(\zeta_\ell)}, \mathbb{F}_\ell) \longrightarrow \bigoplus_w H^2(G_{\mathbb{Q}(\zeta_\ell)_w}, \mathbb{F}_\ell),$$

where w runs over the places of $\mathbb{Q}(\zeta_\ell)$. Hence it suffices to check that $\theta'_{\chi,\chi'}$ is trivial in $H^2(G_{\mathbb{Q}(\zeta_\ell)_w}, \mathbb{F}_\ell)$ for each place w.

Denote by v the place of $\mathbb{Q}$ below w. If v is the infinite place, then the restriction of $\theta'_{\chi,\chi'}$ is certainly trivial in $H^2(G_{\mathbb{Q}(\zeta_\ell)_w}, \mathbb{F}_\ell)$, since $\mathbb{Q}(\zeta_\ell)_w$ is isomorphic to $\mathbb{C}$. If v is unramified in $\mathbb{Q}(\chi, \chi')$, then the restriction of $\theta'_{\chi,\chi'}$ to $G_{\mathbb{Q}(\zeta_\ell)_w}$ equals $\theta'_{a\chi_{\text{un}}, a'\chi_{\text{un}}}$ for some $a, a' \in \mathbb{F}_\ell$, where $\chi_{\text{un}} : G_{\mathbb{Q}(\zeta_\ell)_w} \to \mathbb{F}_\ell$ is a non-trivial unramified character (which is unique up to non-zero scalars). Therefore the desired triviality follows from Remark 2.4.

Now suppose that $v \neq \ell$ ramifies in $\mathbb{Q}(\chi, \chi')$. By assumption v has residue field degree 1 in $\mathbb{Q}(\chi, \chi')$. Therefore, if χ and χ' are both ramified at v, then χ' must be a non-trivial multiple of χ locally at v. Then the 2-cocycle $\theta'_{\chi,\chi'}$ is trivial in $H^2(G_{\mathbb{Q}(\zeta_\ell)_w}, \mathbb{F}_\ell)$ by Remark 2.4, since it is in the span of $\theta'_{\chi,\chi}$ locally at v. If instead χ is ramified at v, while χ' is not, then the assumptions imply that χ' is the trivial character. Therefore the map $(\sigma, \tau) \mapsto \chi(\sigma)\chi'(\tau)$ becomes the zero map locally at v. In particular, the 2-cocycle $\theta'_{\chi,\chi'}$ is trivial in $H^2(G_{\mathbb{Q}(\zeta_\ell)_w}, \mathbb{F}_\ell)$.

It remains to deal with the case $v = \ell$. But the analysis in Theorem 2.8 shows that $\theta'_{\chi,\chi'}$ is always locally trivial at ℓ.

For the forward implication, suppose that $\theta'_{\chi,\chi'}$ is trivial in $H^2(G_{\mathbb{Q}}, \mathbb{F}_\ell)$. Take a place v of $\mathbb{Q}$, not equal to ℓ, that ramifies in $\mathbb{Q}(\chi, \chi')$. By Theorem 2.5 there exists a Heisenberg extension M containing $\mathbb{Q}(\chi, \chi')$. Since v is tamely ramified and since every non-trivial element of Heis_ℓ has order ℓ, any inertia subgroup of v in M must be cyclic of order ℓ. Using that every non-trivial element has order ℓ once more, we conclude that any decomposition group of v in M has size ℓ or ℓ^2. Since every subgroup of size ℓ^2 of Heis_ℓ intersects non-trivially with the center, the decomposition group of v in $\mathbb{Q}(\chi, \chi')$ must have size ℓ, which gives the lemma. □

Let $\chi, \chi' : G_{\mathbb{Q}} \to \mathbb{F}_\ell$ be two linearly independent characters. We define

$$\mu(\chi,\chi') = \begin{cases} \ell^0 & \text{if } \mathbb{Q}(\chi,\chi') \text{ is unramified at } \ell \\ \ell^{(\ell-1)(2\ell-2)} & \text{if } \ell \text{ splits in } \mathbb{Q}(\chi) \text{ and ramifies in } \mathbb{Q}(\chi,\chi') \\ \ell^{\ell(2\ell-2)} & \text{if } \ell \text{ is inert in } \mathbb{Q}(\chi) \text{ and ramifies in } \mathbb{Q}(\chi,\chi') \\ \ell^{\ell(2\ell-2)} & \text{if } \ell \text{ ramifies in } \mathbb{Q}(\chi) \text{ and } \mathfrak{l} \text{ splits in } \mathbb{Q}(\chi,\chi')/\mathbb{Q}(\chi) \\ \ell^{(\ell+1)(2\ell-2)} & \text{if } \ell \text{ ramifies in } \mathbb{Q}(\chi) \text{ and } \mathfrak{l} \text{ is inert in } \mathbb{Q}(\chi,\chi')/\mathbb{Q}(\chi), \end{cases}$$

where $\mathfrak{l}$ is by definition the unique prime of $\mathbb{Q}(\chi)$ above ℓ. Denote by $\widetilde{\Delta}(\chi)$ the product of the ramifying primes in $\mathbb{Q}(\chi)$ that are coprime to ℓ and let $\mathrm{free}(d,a)$ be the largest squarefree integer dividing d and coprime with a.

Theorem 2.11. — *Let $\chi, \chi' : G_{\mathbb{Q}} \to \mathbb{F}_\ell$ be two linearly independent characters. Suppose that $\theta'_{\chi,\chi'}$ is trivial in $H^2(G_{\mathbb{Q}}, \mathbb{F}_\ell)$. Then there exists a minimal Heisenberg extension $M/\mathbb{Q}$ containing $\mathbb{Q}(\chi,\chi')$.*

Now suppose that $\mathbb{Q}(\chi) \subsetneq L \subsetneq M$ and suppose that the Galois closure of L is M. Then

$$\Delta_{L/\mathbb{Q}} = \widetilde{\Delta}(\chi)^{\ell(\ell-1)} \mathrm{free}(\widetilde{\Delta}(\chi'), \widetilde{\Delta}(\chi))^{(\ell-1)^2} \mu(\chi,\chi').$$

Proof. — By Theorem 2.5 it follows that there exists a Heisenberg extension M of $\mathbb{Q}$ containing $\mathbb{Q}(\chi,\chi')$. It is then a general fact about central extensions that there exists a Heisenberg extension M containing $\mathbb{Q}(\chi,\chi')$ that is unramified at every place v that is unramified in $\mathbb{Q}(\chi,\chi')$, see [22, Proposition 4.8]. We claim that such an extension M is minimal.

Since M satisfies the first bullet point of Definition 2.9 by construction, it suffices to verify the second bullet point. We will analyze the situation depending on the splitting behavior of ℓ. If ℓ is unramified in $K := \mathbb{Q}(\chi,\chi')$, then ℓ is also unramified in M by construction.

It remains to treat the case where ℓ ramifies in K and has residue field degree 1. Let w be a place of K above ℓ. We are going to show that M is unramified at w. Assume that it does ramify at w. Write $I \subseteq D \subseteq \mathrm{Heis}_\ell$ for the inertia and decomposition group in $\mathrm{Gal}(M/\mathbb{Q})$ corresponding to the unique prime above w. Denoting by Z the center of Heis_ℓ, our assumptions imply that $Z \subsetneq I = D$. Then we have either $I = D = \mathrm{Heis}_\ell$, so that we in particular get a totally ramified $\mathbb{F}_\ell^2$-extension of $\mathbb{Q}_\ell$, or we have $Z \subsetneq I = D \subsetneq \mathrm{Heis}_\ell$, in which case $I = D \cong \mathbb{F}_\ell^2$ so we get once more a totally ramified $\mathbb{F}_\ell^2$-extension of $\mathbb{Q}_\ell$. But such an extension does not exist, which gives the desired contradiction in both cases.

We will now further analyze the ramification properties of M. Take a place $v \neq \ell$ that ramifies in K. Let w be a place of K above v. We claim that w is unramified in M. If not, we see that any inertia subgroup I_v of v must be of size ℓ^2. But v is tamely ramified and therefore I_v is a cyclic group. This is plainly impossible, since every non-trivial element of the Heisenberg group has order ℓ.

We are now ready to compute the discriminant of L. Take a place $v \neq \ell$ that ramifies in $\mathbb{Q}(\chi)$ and recall the formula

$$\Delta_{L/\mathbb{Q}} = \mathrm{N}_{\mathbb{Q}(\chi)/\mathbb{Q}}(\Delta_{L/\mathbb{Q}(\chi)})\Delta_{\mathbb{Q}(\chi)/\mathbb{Q}}^{\ell}. \tag{2.8}$$

From the above we see that the v-adic valuation of $\mathrm{N}_{\mathbb{Q}(\chi)/\mathbb{Q}}(\Delta_{L/\mathbb{Q}(\chi)})$ is 0. Furthermore, since v is tamely ramified, we have that the v-adic valuation of $\Delta_{\mathbb{Q}(\chi)/\mathbb{Q}}^{\ell}$ is $\ell(\ell-1)$. Next we compute the contribution from those $v \neq \ell$ that are unramified in $\mathbb{Q}(\chi)$ but ramify in K. In this case the formula (2.8) simplifies to

$$\Delta_{L/\mathbb{Q}} = \mathrm{N}_{\mathbb{Q}(\chi)/\mathbb{Q}}(\Delta_{L/\mathbb{Q}(\chi)}). \tag{2.9}$$

Furthermore, we know by Lemma 2.10 that v splits completely in $\mathbb{Q}(\chi)$. Suppose that $w_1, \dots, w_\ell$ are the places above v. We claim that precisely $\ell - 1$ of them ramify in L. To this end, we consider the bicyclic extension $M/\mathbb{Q}(\chi)$. By the structure of Heis_ℓ, the intermediate fields in this extension are K and $L_1, \dots, L_\ell$, where the L_i are all conjugate and $L_1 = L$ say. Since the L_i are all conjugate, we know that

$$|\{1 \leqslant j \leqslant \ell : w_j \text{ splits in } L_i/\mathbb{Q}(\chi)\}|$$

does not depend on i. Furthermore, all the w_j ramify in K, so that every given w_j ramifies in precisely $\ell - 1$ of the L_i. Then a double counting argument establishes the claim. Using Equation (2.9) and the claim, we deduce that the v-adic valuation of $\Delta_{L/\mathbb{Q}}$ equals $(\ell-1)^2$.

It remains to deal with the case $v = \ell$. We distinguish four cases

(i) Suppose that ℓ ramifies in $\mathbb{Q}(\chi)$ and has residue field degree 1 in K. In this case any prime above ℓ is unramified in L. By class field theory we see that

$$v_\ell(\Delta_{\mathbb{Q}(\chi)/\mathbb{Q}}) = 2\ell - 2.$$

Therefore Equation (2.8) yields

$$v_\ell(\Delta_{L/\mathbb{Q}}) = \ell(2\ell - 2).$$

(ii) Suppose that ℓ splits in $\mathbb{Q}(\chi)$ but ramifies in K. Then

$$v_\ell(\Delta_{L/\mathbb{Q}}) = \mathrm{N}_{\mathbb{Q}(\chi)/\mathbb{Q}}(\Delta_{L/\mathbb{Q}(\chi)}).$$

Note that

$$v_\ell(\Delta_{M/\mathbb{Q}}) = \ell^2(2\ell - 2)$$

and hence $w(\Delta_{M/\mathbb{Q}(\chi)}) = \ell(2\ell - 2)$ for any place w of $\mathbb{Q}(\chi)$ above v. Suppose that w ramifies in L. Consider the bicyclic extension $M/\mathbb{Q}(\chi)$. There are $\ell + 1$ intermediate fields $K, L_1, \ldots, L_\ell$, where the L_i are all isomorphic by Lemma 2.6. Furthermore, w ramifies in K and precisely $\ell - 1$ of the L_i. Therefore Lemma 2.2 implies that

$$(\ell - 1) \cdot w(\Delta_{L/\mathbb{Q}(\chi)}) + w(\Delta_{K/\mathbb{Q}(\chi)}) = w(\Delta_{M/\mathbb{Q}(\chi)}) = \ell(2\ell - 2).$$

We conclude that

$$w(\Delta_{L/\mathbb{Q}}) = 2\ell - 2, \quad v_\ell(\Delta_{L/\mathbb{Q}}) = (\ell - 1)(2\ell - 2).$$

(iii) Suppose that ℓ ramifies in $\mathbb{Q}(\chi)$ and has residue field degree ℓ in K. Denote by w the unique place of $\mathbb{Q}(\chi)$ above ℓ. Arguing as above we get

$$\ell \cdot w(\Delta_{L/\mathbb{Q}(\chi)}) = w(\Delta_{M/\mathbb{Q}(\chi)}) = \ell(2\ell - 2),$$

where the last equality follows from Theorem 2.8. Hence we have

$$v_\ell(\Delta_{L/\mathbb{Q}}) = (2\ell - 2) + \ell(2\ell - 2) = (\ell + 1)(2\ell - 2).$$

(iv) Suppose that ℓ is inert in $\mathbb{Q}(\chi)$ but ramifies in K. Inspecting the proof of Theorem 2.8, see Equation (2.7), we conclude that

$$v_\ell(\Delta_{L/\mathbb{Q}}) = \ell(2\ell - 2).$$

This completes the proof. □

2.5. Counting Heisenberg extensions by discriminant

Let $\chi, \chi' : G_\mathbb{Q} \to \mathbb{F}_\ell$ be two linearly independent characters. Define for an integer $d > 0$

$$\mu(\chi, \chi', d) = \begin{cases} \ell^{\ell(2\ell-2)} & \text{if } \mathbb{Q}(\chi, \chi') \text{ is unramified at } \ell \text{ and } \ell \mid d, \\ \mu(\chi, \chi') & \text{otherwise.} \end{cases}$$

We also put

$$\begin{aligned} D(d,\chi,\chi',\ell) &\coloneqq \widetilde{\Delta}(\chi)^{\ell(\ell-1)}\mathrm{free}(\widetilde{\Delta}(\chi'),\widetilde{\Delta}(\chi))^{(\ell-1)^2}\mu(\chi,\chi',d)\\ S_1(X,\ell) &\coloneqq \{d\in\mathbb{Z}_{>0} : d\leqslant X,\ d \text{ squarefree},\ p\mid d\Rightarrow p\equiv 0,1 \bmod \ell\}\\ S_2(X,\chi,\chi',\ell) &\coloneqq \{d\in S_1(X,\ell) : \gcd(d,\widetilde{\Delta}(\chi)\widetilde{\Delta}(\chi'))=1\}\\ S_3(X,\chi,\chi',\ell) &\coloneqq \sum_{\substack{d\in S_2(X,\chi,\chi',\ell)\\ \mathrm{free}(d,\ell)^{\ell(\ell-1)}\leqslant \frac{X}{D(d,\chi,\chi',\ell)}}} (\ell-1)^{\omega_\ell^*(d)}, \end{aligned}$$

where ω_ℓ^* is the number of prime divisors (counted without multiplicity) not equal to ℓ. Recall that $N(\mathrm{Heis}_\ell,X)$ denotes the number of degree ℓ^2 extensions L of $\mathbb{Q}$, up to isomorphism, with $\mathrm{Gal}(N(L)/\mathbb{Q})\cong\mathrm{Heis}_\ell$ and absolute discriminant bounded by X.

THEOREM 2.12. — *Let ℓ be an odd prime number. Then*

$$N(\mathrm{Heis}_\ell,X) = (\ell-1)^{-2}\sum_{\substack{\chi,\chi':G_\mathbb{Q}\to\mathbb{F}_\ell\\ \chi,\chi' \text{ lin. indep.}}} \mathbb{1}_{\theta'_{\chi,\chi'}\text{ trivial}} \cdot \ell^{\omega(\tilde{\Delta}(\chi)\tilde{\Delta}(\chi'))-3}\cdot S_3(X,\chi,\chi',\ell). \tag{2.10}$$

Proof. — We recall that $\theta_{\chi,\chi'}$ and $\theta_{\chi,\chi'+a\chi}$ give the same class in $H^2(\mathbb{F}_\ell^2,\mathbb{F}_\ell)$ for all $a\in\mathbb{F}_\ell$ by Remark 2.4. Since we are counting our fields up to isomorphism, our goal will be to apply Lemma 2.6.

First, we fix χ and compute the contribution from those degree ℓ^2 Heisenberg extensions L containing $\mathbb{Q}(\chi)$. Since χ and $a\chi$ both have fixed field $\mathbb{Q}(\chi)$ for any $a\in\mathbb{F}_\ell^*$, we are overcounting by a factor $\ell-1$. Next, let us further restrict to those L such that the normal closure of L contains $\mathbb{Q}(\chi,\chi')$ with χ' linearly independent from χ. This certainly implies that $\theta'_{\chi,\chi'}$ is trivial.

Hence further fix a χ' linearly independent from χ with $\theta'_{\chi,\chi'}$ trivial. Note that there are in fact $\ell(\ell-1)$ choices of χ' that all give the same bicyclic extension $\mathbb{Q}(\chi,\chi')$, namely $a\chi'+b\chi$ with $a\in\mathbb{F}_\ell^*$ and $b\in\mathbb{F}_\ell$. Hence we are overcounting by another factor $\ell(\ell-1)$.

Now we compute the contribution from the fields L containing $\mathbb{Q}(\chi)$ such that the normal closure of L contains $\mathbb{Q}(\chi,\chi')$. Fix a minimal extension M containing $\mathbb{Q}(\chi,\chi')$. Then any field L' satisfying $\mathbb{Q}(\chi)\subsetneq L'\subsetneq M$ has discriminant

$$\widetilde{\Delta}(\chi)^{\ell(\ell-1)}\mathrm{free}(\widetilde{\Delta}(\chi'),\widetilde{\Delta}(\chi))^{(\ell-1)^2}\mu(\chi,\chi')$$

by Theorem 2.11. Let $\rho \in \mathrm{Heis}(\mathbb{Q}(\chi,\chi')/\mathbb{Q})$ be a character with fixed field M. Twisting ρ by characters $\chi'' : G_{\mathbb{Q}} \to \mathbb{F}_\ell$, we get all degree ℓ^3 Heisenberg extensions containing $\mathbb{Q}(\chi,\chi')$. However, we get every extension ℓ^2 times, since the characters χ and χ' are trivial when restricted to $G_{\mathbb{Q}(\chi,\chi')}$.

Suppose that we twist ρ by a character $\chi'' : G_{\mathbb{Q}} \to \mathbb{F}_\ell$ that is ramified precisely at the primes dividing d. From class field theory we immediately get that $d \in S_1(\infty,\ell)$. Furthermore for such an integer d, there are precisely $(\ell-1)^{\omega(d)}$ characters that are ramified at exactly those primes dividing d. We claim that the discriminant of any field L' such that $\mathbb{Q}(\chi) \subsetneq L' \subsetneq \mathbb{Q}(\chi,\chi')(\rho+\chi'')$ equals

$$\widetilde{\Delta}(\chi)^{\ell(\ell-1)}\mathrm{free}(\widetilde{\Delta}(\chi'),\widetilde{\Delta}(\chi))^{(\ell-1)^2}\mathrm{free}(d,\ell\widetilde{\Delta}(\chi)\widetilde{\Delta}(\chi'))^{\ell(\ell-1)}\mu(\chi,\chi',d).$$

The factor $\mathrm{free}(d,\ell\widetilde{\Delta}(\chi)\widetilde{\Delta}(\chi'))^{\ell(\ell-1)}$ is easily computed. Let us now focus on the factor $\mu(\chi,\chi',d)$. If there is precisely one place above ℓ in $\mathbb{Q}(\chi,\chi')$, twisting does not change the discriminant locally at ℓ by Theorem 2.8. Indeed, the two twists have the same normal closure (since there is only one Heisenberg field locally at ℓ) and share the same cyclic subfield, so we can apply Lemma 2.6. Similarly, if ℓ ramifies in $\mathbb{Q}(\chi)$, twisting does not change the discriminant locally at ℓ. If ℓ splits in $\mathbb{Q}(\chi)$ and ramifies in $\mathbb{Q}(\chi,\chi')$, then

$$L \otimes \mathbb{Q}_\ell \cong \mathbb{Q}_\ell(\chi') \oplus \cdots \oplus \mathbb{Q}_\ell(\chi') \oplus \mathbb{Q}_\ell^\ell$$

or

$$L \otimes \mathbb{Q}_\ell \cong \mathbb{Q}_\ell(\chi_{\mathrm{un}}) \oplus \mathbb{Q}_\ell(\chi_{\mathrm{un}}+\chi') \oplus \cdots \oplus \mathbb{Q}_\ell(\chi_{\mathrm{un}}+(\ell-1)\chi'),$$

where χ_{un} is an unramified degree ℓ character of $G_{\mathbb{Q}_\ell}$. Since χ' is a ramified character, we see once more that twisting does not change the discriminant locally at ℓ. A similar analysis works if ℓ is unramified in $\mathbb{Q}(\chi,\chi')$.

Having established the claim, we are now ready to complete the proof. There are

$$\ell^{\omega(\tilde{\Delta}(\chi)\tilde{\Delta}(\chi'))}$$

characters only ramified at the places dividing $\widetilde{\Delta}(\chi)\widetilde{\Delta}(\chi')$. Twisting with such characters clearly does not change the discriminant. Furthermore, they give

$$\ell^{\omega(\tilde{\Delta}(\chi)\tilde{\Delta}(\chi'))-2}$$

different fields, because the characters χ and χ' are trivial characters of $G_{\mathbb{Q}(\chi,\chi')}$. This gives the theorem. □

3. Analytic prerequisites

3.1. The general question

From now on we shall mostly focus on the case $\ell = 3$. The aim of this section is to transform Equation (2.10) into the character sum $\mathrm{Heis}(X, 3)$ (see Proposition 3.6 below). The definition of $\mathrm{Heis}(X, 3)$ is given in Definition 3.5 below. Since this character sum is rather delicate, we take some time to present its definition.

By convention we reserve the letters p and ℓ for usual rational primes. The letter r will also designate a prime particularly in Definition 3.1 and in the formulas deduced from it. When $\ell \geqslant 3$ is a prime, we introduce the following sets of integers

$$\mathbb{P}_\ell := \{p : p \equiv 0,\, 1 \bmod \ell\},$$

$$\mathbb{P}_\ell^* := \{p : p \equiv 1 \bmod \ell\},$$

$$\mathbb{N}_\ell := \{n : n \geqslant 1,\, n \text{ squarefree},\, p \mid n \Rightarrow p \in \mathbb{P}_\ell\},$$

and

$$\mathbb{N}_\ell^* := \{n : n \geqslant 1,\, n \text{ squarefree},\, p \mid n \Rightarrow p \in \mathbb{P}_\ell^*\}.$$

For $d \geqslant 1$, we denote by $\omega_\ell^*(d)$ the number of distinct prime divisors of d belonging to $\mathbb{P}_\ell^*$ and, as usual, $\omega(d)$ is the total number of distinct prime divisors of d.

3.2. Standard primes, standard decomposition and characters

Let

$$j = \frac{-1 + i\sqrt{3}}{2},$$

be a cubic root of unity. For $z \in \mathbb{Z}[j]$, let $\mathrm{N}(z) = z \cdot \bar{z}$ be the norm of z. Every $p \in \mathbb{P}_3^*$ can be uniquely written as

$$p = \pi\,\overline{\pi} \tag{3.1}$$

where

$$\begin{cases} \pi \text{ and } \overline{\pi} \text{ belong to } \mathbb{Z}[j], \\ \pi \text{ is } \textit{primary} \text{ (which means } \pi \equiv 2 \bmod 3), \\ \operatorname{Im} \pi > 0. \end{cases}$$

This decomposition is named the *standard decomposition of p,* and π is a *standard prime.* For $p \in \mathbb{P}_3^*$, there are two Dirichlet characters modulo p with order 3. One of these is

$$\chi_p(n) := \left(\frac{n}{\pi}\right)_3, \tag{3.2}$$

which is defined without ambiguity as soon as π is given by the standard decomposition (3.1). Recall that the cubic character $\left(\frac{\alpha}{\pi}\right)_3$ is defined, for $\alpha \in \mathbb{Z}[j]$ not divisible by π, by the formula

$$\left(\frac{\alpha}{\pi}\right)_3 := j^m,$$

where $0 \leqslant m \leqslant 2$ is the unique integer such that $\alpha^{\frac{p-1}{3}} \equiv j^m \bmod \pi$ (see [17, Chapter 9 Section 3], for instance).

Modulo 9, there are also two Dirichlet characters with order 3. One of these is the character χ_3 defined by its value

$$\chi_3(2) = j,$$

which also defines χ_3 without ambiguity. In conclusion, for every $p \in \mathbb{P}_3$ we have fixed a Dirichlet character χ_p of order 3.

Let $f : \mathbb{P}_3 \to \mathbb{F}_3$ be a function. By definition, the *support of f* is the set

$$\text{supp } f := \{p \in \mathbb{P}_3 : f(p) \neq 0\},$$

and $\text{supp}_3\, f$ is the support of the restriction of f to $\mathbb{P}_3^*$. We introduce the sets of functions

$$V := \{f : \mathbb{P}_3 \to \mathbb{F}_3, \text{supp } f \text{ is finite}\},$$

and

$$V^* := \{f : \mathbb{P}_3 \to \mathbb{F}_3, \text{supp } f \text{ is finite and } f(3) = 0\}. \tag{3.3}$$

The sets V and V^* naturally have a structure of $\mathbb{F}_3$–vector space with infinite dimension.

Given an f in V, we define the Dirichlet character $\chi(f)$ over $\mathbb{Z}$ by the formula

$$\chi(f) := \prod_{p \in \mathbb{P}_3} \chi_p^{f(p)}. \tag{3.4}$$

This has a meaning since this is a finite product and since all χ_p have order 3. To evaluate $\chi(f)$ at some number $m \in \mathbb{Z}$, we naturally have

$$\chi(f)(m) = \prod_{p \in \mathbb{P}_3} \left[\, \chi_p(m) \,\right]^{f(p)} \tag{3.5}$$

with the convention that $z^0 = 1$ for any $z \in \mathbb{C}$. In particular, we have

$$\chi(f)(p) = \begin{cases} 0 & \text{if } p \in \text{supp } f, \\ 1,\ j \text{ or } j^2 & \text{if } p \notin \text{supp } f. \end{cases} \tag{3.6}$$

To any $f \in V$ we associate an integer $\Delta(f) \in \mathbb{N}_3^*$ defined by

$$\Delta(f) := \prod_{p \in \text{ supp}_3 f} p.$$

If $f(3) = 0$, then $\Delta(f)$ is the conductor of the Dirichlet character $\chi(f)$. On the other hand, if $f(3) \neq 0$, the conductor of $\chi(f)$ is equal to $9 \cdot \Delta(f)$. For $\Delta \in \mathbb{N}_3^*$, we will meet the following sets of functions, with cardinalities $3 \cdot 2^{\omega(\Delta)}$ and $2^{\omega(\Delta)}$

$$V(\Delta) := \{f \in V : \Delta(f) = \Delta\} \text{ and } V^*(\Delta) := \{f \in V^* : \Delta(f) = \Delta\}. \tag{3.7}$$

Finally, we introduce the function $\mathbb{1}(f, f')$ which can be interpreted as a characteristic function since it takes only values 0 and 1 (see Lemma 3.2 below).

Definition 3.1. — *For linearly independent f and f' in V, let $\mathbb{1}(f, f')$ be the number defined by*

$$\mathbb{1}(f, f') := 3^{-|\text{supp}_3 f \cup \text{supp}_3 f'|} \prod_{r \mid \Delta(f)\Delta(f')} \left(\sum_{\substack{(z,z') \in \mathbb{F}_3^2 \\ f(r)z + f'(r)z' = 0}} \big(\chi(zf + z'f')\big)(r) \right).$$

It follows from Lemma 2.10 that

$$\mathbb{1}(f, f') = \mathbb{1}_{\theta'_{\chi(f),\chi(f')} \text{ trivial}}. \tag{3.8}$$

In particular the following lemma is now obvious.

Lemma 3.2. — *For every linearly independent f and f' in V, one has the property*

$$\mathbb{1}(f, f') \in \{0, 1\}.$$

3.3. The μ–functions

To each pair $(f, f') \in V^2$ we associate an integer denoted by $\mu(f, f')$. This integer is a power of 3 but it is not a symmetric function of f and f'.

Definition 3.3. — *For every f and f' in V, we define*[(1)]

$$\mu(f,f') = \begin{cases} 1 & \text{if } f(3)=f'(3)=0,\\ 3^8 & \text{if } f(3)=0,\ f'(3)\neq 0, \text{ and } \chi(f)(3)=1,\\ 3^{12} & \text{if } f(3)=0,\ f'(3)\neq 0, \text{ and } \chi(f)(3)\in\{j,j^2\},\\ 3^{12} & \text{if } f(3)\neq 0,\ f'(3)=0, \text{ and } \chi(f')(3)=1,\\ 3^{16} & \text{if } f(3)\neq 0,\ f'(3)=0, \text{ and } \chi(f')(3)\in\{j,j^2\},\\ 3^{12} & \text{if } f(3)\neq 0,\ f'(3)\neq 0, \text{ and } \big(\chi(f'(3)\cdot f+2f(3)\cdot f')\big)(3)=1,\\ 3^{16} & \text{if } f(3)\neq 0,\ f'(3)\neq 0, \text{ and } \big(\chi(f'(3)\cdot f+2f(3)\cdot f')\big)(3)\in\{j,j^2\}.\end{cases}$$

We give another definition

Definition 3.4. — *Let $f,f'\in V$ and let $d\in\mathbb{N}_3$. We denote by $\mu(f,f',d)$ the positive integer defined by*

$$\mu(f,f',d) := \begin{cases} 3^{12} & \text{if } 3\mid d,\ f(3)=f'(3)=0,\\ \mu(f,f') & \text{otherwise.}\end{cases}$$

3.4. The crucial sum

For positive integers d and a, recall that $\mathrm{free}(d,a)$ is the largest squarefree integer dividing d and coprime with a. In other words, we have

$$\mathrm{free}(d,a) = \prod_{\substack{p\mid d\\ p\nmid a}} p,$$

which simplifies to $\mathrm{free}(d,a)=d/(d,a)$, when d is squarefree.

For $f,f'\in V$ and $d\in\mathbb{N}_3$, we introduce the integer

$$D(d,f,f') := \Delta(f)^6\ \mathrm{free}\big(\Delta(f'),\Delta(f)\big)^4\ \mu(f,f',d), \tag{3.9}$$

the set

$$\mathcal{S}(f,f') := \big\{d\in\mathbb{N}_3 : \big(d,\Delta(f)\Delta(f')\big)=1\big\},$$

and the associated summatory function

$$S(X,f,f') := \sum_{d} 2^{\omega_3^*(d)}, \tag{3.10}$$

(1) Actually, as the referee noticed, the fourth formula in this definition can be absorbed by the sixth one and the same remark applies to the fifth and the seventh formulas. This gathering process also works between several cases in the list (3.20),...,(3.33). For reasons of clarity of exposition, we have preferred not to benefit from this possible shortening.

where the sum is over

$$d \in \mathcal{S}(f, f') \text{ with } \mathrm{free}(d, 3) \leqslant \left(X \Big/ D(d, f, f')\right)^{1/6}.$$

Gathering the above notations, we define the crucial sum $\mathrm{Heis}(X, 3)$ announced in Section 3.1.

Definition 3.5. — *For $X \geqslant 2$ and the prime $\ell = 3$, the associated Heisenberg sum* $\mathrm{Heis}(X, 3)$ *is*

$$\mathrm{Heis}(X, 3) := 2^{-2}3^{-3} \sum_{\substack{f, f' \in V \\ f, f' \text{ lin. indep.}}}\sum 3^{|\mathrm{supp}_3 f \cup \mathrm{supp}_3 f'|} \cdot \mathbb{1}(f, f') \cdot S(X, f, f').$$

It is an exercise to verify that Definition 3.5 does not dependent on the way we have chosen the characters χ_p of order 3 for each $p \in \mathbb{P}$. Combining Theorem 2.12 (with $\ell = 3$) and Equation (3.8), we obtain

Proposition 3.6. — *We have for every $X \geqslant 2$ the equality*

$$N(\mathrm{Heis}_3, X) = \mathrm{Heis}(X, 3).$$

To state our main result we introduce the following notations

- $\mathbb{1}_{\{3\}}$ is the characteristic function of the set $\{3\}$,
- ψ_3 is the multiplicative function defined on squarefree positive integers, satisfying

$$\psi_3(p) = p/(p + 2) \tag{3.11}$$

  (see the general definition given in (4.48)),
- λ is the multiplicative function defined on squarefree positive integers, satisfying

$$\lambda(p) = \left(1 + 2/\left(p^{1/2}(p + 2)\right)\right)^{-1},$$

- α_3 is the infinite product

$$\alpha_3 := \frac{3}{4} \prod_p \left\{ \left(1 + \frac{1}{p} + \frac{\left(\frac{p}{3}\right)}{p}\right) \cdot \left(1 - \frac{1}{p}\right) \right\} \tag{3.12}$$

  (see the general definition given in (4.47)),

- H_0 is the constant defined by[2]

$$H_0 := \sum_{\substack{\Delta\in\mathbb{N}_3^* \\ \Delta>1}} \lambda(\Delta)\,\psi_3(\Delta) \cdot \frac{3^{\omega(\Delta)}}{\Delta^{3/2}} \sum_{f\in V^*(\Delta)} \left\{ \prod_{p\in\mathbb{P}_3^*} \left(1 + 2\frac{\chi(f)(p)+\chi(2f)(p)}{p+2} + \frac{2}{p^{1/2}(p+2)}\right)\right\}, \tag{3.13}$$

- H_1 is the constant defined by

$$H_1 := \sum_{\substack{\Delta\in\mathbb{N}_3^* \\ \Delta>1}} \lambda(\Delta)\,\psi_3(\Delta) \cdot \frac{3^{\omega(\Delta)}}{\Delta^{3/2}} \sum_{\substack{f\in V^*(\Delta) \\ \chi(f)(3)=1}} \left\{ \prod_{p\in\mathbb{P}_3^*} \left(1 + 2\frac{\chi(f)(p)+\chi(2f)(p)}{p+2} + \frac{2}{p^{1/2}(p+2)}\right)\right\}, \tag{3.14}$$

- H_2 is the constant defined by

$$H_2 := \sum_{\substack{\Delta\in\mathbb{N}_3^* \\ \Delta\geqslant 1}} \lambda(\Delta)\,\psi_3(\Delta) \cdot \frac{3^{\omega(\Delta)}}{\Delta^{3/2}} \sum_{f\in V^*(\Delta)} \sum_{\eta=1,2} \left\{ \prod_{p\in\mathbb{P}_3^*} \left(1 + 2\frac{\chi(f+\eta\mathbb{1}_{\{3\}})(p)+\chi(2f+2\eta\mathbb{1}_{\{3\}})(p)}{p+2} + \frac{2}{p^{1/2}(p+2)}\right)\right\}. \tag{3.15}$$

We now have all the tools to define the constant

$$c(\mathrm{Heis}_3) := 2^{-2}\left(\frac{32}{3^6}\cdot H_0 + \frac{8}{3^6}\cdot H_1 + \frac{10}{3^7}\cdot H_2\right)\alpha_3. \tag{3.16}$$

We will prove the following theorem, which combined with Proposition 3.6 gives Theorem 1.2.

Theorem 3.7. — *Uniformly for $X \geqslant 2$, we have the equality*

$$\mathrm{Heis}(X,3) = c(\mathrm{Heis}_3)\cdot X^{1/4}\big(1 + O\big((\log X)^{-1}\big)\big).$$

By utilizing the full strength of the Siegel–Walfisz Theorem one can improve the above error term to $O_A\big((\log X)^{-A}\big)$ where $A > 0$ is arbitrary.

[2] In H_0 we are summing over all primitive Dirichlet characters with order 3 and with squarefree conductor $\Delta > 1$ coprime to 3, while in the sum H_2 we are summing over all primitive Dirichlet characters with order 3 and with conductor 9Δ, where $\Delta \geqslant 1$ is squarefree and coprime to 3.

Remark 3.8. — In Section 4.7, we will prove that the Euler product appearing in the definition of H_0 is essentially the product of the square of the modulus of cubic L–functions at the point 1, see Equation (4.57). This leads to the observation that the constant $c(\mathrm{Heis}_3)$ has obvious similarities with the constant $c(D_4)$, the value of which is given in Theorem 1.1. These two constants are defined as series of values of Dirichlet L–functions at the point 1. In the case of $c(D_4)$ the associated characters have order 2, in the case of $c(\mathrm{Heis}_3)$ this order is 3.

3.5. The archetypical sum

We first consider the subsum $\mathrm{Heis}^*(X)$ defined by[(3)]

$$\mathrm{Heis}^*(X) := 2^{-2}3^{-3} \sum_{\substack{f,f' \in V^* \\ f,\, f' \text{ lin. indep.}}}\sum 3^{|\mathrm{supp}_3 f \,\cup\, \mathrm{supp}_3\, f'|} \cdot \mathbb{1}(f,f') \cdot S^*(X,f,f'),$$

where

- V^* is defined in (3.3),
- $S^*(X,f,f')$ is the subsum of $S(X,f,f')$, where we exclude all the d divisible by 3 (see (3.10)).

Note that the subsum $\mathrm{Heis}^*(X)$ contains exactly those terms from $\mathrm{Heis}(X,3)$ with $\mu(f,f',d) = 1$. Algebraically, this subsum corresponds to nonic Heisenberg extensions unramified at 3. This is a convenient first sum to consider, since it avoids the many case distinctions in the definition of the function $\mu(f,f')$. We have the equality

$$\mathrm{Heis}^*(X) = 2^{-2}3^{-3} \sum_{\substack{f,f' \in V^* \\ f,\, f' \text{ lin. indep.}}}\sum 3^{|\mathrm{supp}\, f \cup \mathrm{supp}\, f'|} \cdot \mathbb{1}(f,f') \cdot \Bigl(\sum_d 2^{\omega(d)}\Bigr), \tag{3.17}$$

where d satisfies the following conditions

$$\begin{cases} d \in \mathbb{N}_3^*, \\ \bigl(d, \Delta(f)\Delta(f')\bigr) = 1, \\ 1 \leqslant d \leqslant X^{1/6}\Delta(f)^{-1}\,\Delta(f')^{-2/3}\,\bigl(\Delta(f),\Delta(f')\bigr)^{2/3}. \end{cases} \tag{3.18}$$

(3) From now on, many notations will be shortened by omitting the dependency on the prime $\ell = 3$.

Let

$$C_{\mathrm{Heis}^*} := 2^{-2}3^{-3}\alpha_3 \sum_{\substack{\Delta\in\mathbb{N}_3^*\\ \Delta>1}} \psi_3(\Delta) \tag{3.19}$$

$$\cdot \frac{3^{\omega(\Delta)}}{\Delta^{3/2}} \sum_{f\in V^*(\Delta)} \left\{ \prod_{p\in\mathbb{P}_3^*} \left(1+2\frac{\chi(f)(p)+\chi(2f)(p)}{p+2}\right)\right\}$$

$$\times \left\{ \prod_{\substack{p\in\mathbb{P}_3^*\\ p\nmid\Delta}} \left(1+\frac{2}{p^{1/2}\,(p+2(1+\chi(f)(p)+\chi(2f)(p)))}\right)\right\}$$

where α_3 and ψ_3 are defined in (3.12) and in (3.11). Thanks to (3.6) and easy transformations, C_{Heis^*} can also be written as

$$C_{\mathrm{Heis}^*} := 2^{-2}3^{-3}\alpha_3 H_0,$$

with H_0 defined in (3.13). We will prove the following

Proposition 3.9. — *Uniformly for $X \geqslant 2$ one has the equality*

$$\mathrm{Heis}^*(X) = C_{\mathrm{Heis}^*}\cdot X^{1/4} + O\big(\,X^{1/4}(\log X)^{-1}\,\big).$$

We will prove in Proposition 4.16 that C_{Heis^*} is positive, which implies that the above formula is an asymptotic one.

3.6. The other sums

The subsum $\mathrm{Heis}^*(X)$ will be a model to treat the other subsums constituting $\mathrm{Heis}(X,3)$. According to the definition of the μ–functions, it is natural to consider the following fourteen subsums of $\mathrm{Heis}(X,3)$, denoted by $\mathrm{Heis}^{(3.20)}(X)$, $\mathrm{Heis}^{(3.21)}(X)$, $\mathrm{Heis}^{(3.22)},\dots,\mathrm{Heis}^{(3.33)}(X)$ where the exponent of Heis corresponds to the additional restrictions imposed to the variables of summation d in $S(X,f,f')$ and to the pair (f,f') in the first double summation in the Definition 3.5:

(3.20) $3\nmid d$, $f(3)=f'(3)=0$,

(3.21) $3\nmid d$, $f(3)=0$, $f'(3)\neq 0$, $\chi(f)(3)=1$,

(3.22) $3\nmid d$, $f(3)=0$, $f'(3)\neq 0$, $\chi(f)(3)\in\{j,j^2\}$,

(3.23) $3\nmid d$, $f(3)\neq 0$, $f'(3)=0$, $\chi(f')(3)=1$,

(3.24) $3\nmid d$, $f(3)\neq 0$, $f'(3)=0$, $\chi(f')(3)\in\{j,j^2\}$,

$$(3.25)\quad 3\nmid d,\ f(3)\neq 0,\ f'(3)\neq 0,\ \big(\chi(f'(3)\cdot f+2f(3)\cdot f')\big)(3)=1,$$
$$(3.26)\quad 3\nmid d,\ f(3)\neq 0,\ f'(3)\neq 0,\ \big(\chi(f'(3)\cdot f+2f(3)\cdot f')\big)(3)\in\{j,j^2\},$$
$$(3.27)\quad 3\mid d,\ f(3)=f'(3)=0,$$
$$(3.28)\quad 3\mid d,\ f(3)=0,\ f'(3)\neq 0,\ \chi(f)(3)=1,$$
$$(3.29)\quad 3\mid d,\ f(3)=0,\ f'(3)\neq 0,\ \chi(f)(3)\in\{j,j^2\},$$
$$(3.30)\quad 3\mid d,\ f(3)\neq 0,\ f'(3)=0,\ \chi(f')(3)=1,$$
$$(3.31)\quad 3\mid d,\ f(3)\neq 0,\ f'(3)=0,\ \chi(f')(3)\in\{j,j^2\},$$
$$(3.32)\quad 3\mid d,\ f(3)\neq 0,\ f'(3)\neq 0,\ \big(\chi(f'(3)\cdot f+2f(3)\cdot f')\big)(3)=1,$$
$$(3.33)\quad 3\mid d,\ f(3)\neq 0,\ f'(3)\neq 0,\ \big(\chi(f'(3)\cdot f+2f(3)\cdot f')\big)(3)\in\{j,j^2\}.$$

In each of these cases, the factor $\mu(d,f,f')$ is constant. We have the obvious equalities

$$\mathrm{Heis}^*(X)=\mathrm{Heis}^{(3.20)}(X),$$

and

$$(3.34)\quad \mathrm{Heis}(X,3)=\mathrm{Heis}^{(3.20)}(X)+\mathrm{Heis}^{(3.21)}(X)+\cdots+\mathrm{Heis}^{(3.33)}(X).$$

By following the proof of Proposition 3.9 and by indicating the alterations between the different cases, we will prove in Section 5.

Proposition 3.10. — *Let $(i,j)=$ (3.20), (3.21), (3.22), …, or (3.33). Then there exists a constant $C^{(i,j)}>0$ such that*

$$\mathrm{Heis}^{(i,j)}(X)=2^{-2}3^{-3}\alpha_3\,C^{(i,j)}X^{1/4}\big(1+O\big((\log X)^{-1}\big)\big).$$

Furthermore, we have the equalities

$$\begin{aligned}
C^{(3.20)}&=H_0, & C^{(3.27)}&=3^{-3}\cdot H_0,\\
C^{(3.21)}&=2\cdot 3^{-2}\cdot H_1, & C^{(3.28)}&=2\cdot 3^{-2}\cdot H_1,\\
C^{(3.22)}&=2\cdot 3^{-3}\cdot(H_0-H_1), & C^{(3.29)}&=2\cdot 3^{-3}\cdot(H_0-H_1),\\
C^{(3.23)}&=3^{-4}\cdot H_2, & C^{(3.30)}&=3^{-4}\cdot H_2,\\
C^{(3.24)}&=2\cdot 3^{-5}\cdot H_2, & C^{(3.31)}&=2\cdot 3^{-5}\cdot H_2,\\
C^{(3.25)}&=2\cdot 3^{-4}\cdot H_2, & C^{(3.32)}&=2\cdot 3^{-4}\cdot H_2,\\
C^{(3.26)}&=4\cdot 3^{-5}\cdot H_2, & C^{(3.33)}&=4\cdot 3^{-5}\cdot H_2.
\end{aligned}$$

Gathering the decomposition given by (3.34) and the explicit values given by Proposition 3.10, we complete the proof of Theorem 3.7 through the equality

$$c(\mathrm{Heis}_3)=2^{-2}3^{-3}\,\alpha_3\big(C^{(3.20)}+\cdots+C^{(3.33)}\big),$$

which gives the explicit value announced in (3.16). The inequality $c(\mathrm{Heis}_3) > 0$ is a consequence of the inequalities $H_0 > 0$ (see Proposition 4.16 below) and of the trivial inequality

$$\mathrm{Heis}(X,3) \geqslant \mathrm{Heis}^*(X),$$

since every subsum $\mathrm{Heis}^{(3.21)}(X), \ldots, \mathrm{Heis}^{(3.33)}(X)$ is non-negative.

4. Study of the archetypical sum

In this section we will prove Proposition 3.9 concerning the sum $\mathrm{Heis}^*(X)$ as it appears in (3.17) with the conditions of summation (3.18).

4.1. Trivial bounds and restrictions

The number of positive divisors of the integer $n \geqslant 1$ is denoted by $\tau(n)$ and for $X \geqslant 1$, we write

$$\mathcal{L} := \log 2X.$$

In the course of the statements or proofs, the reader will find constants $A_0, A_1, \ldots$ (particularly as exponents of $\mathcal{L}$) for which it is possible to give explicit values, but we will refrain from doing so.

4.1.1. Classical lemmas from analytic number theory

We will use the following bounds.

Lemma 4.1. — *Let $b > 0$ be given. Then uniformly for $X \geqslant 1$ one has*

$$\sum_{n \leqslant X} b^{\omega(n)} = O(X\mathcal{L}^{b-1}) \quad \text{and} \quad \sum_{\substack{n \leqslant X \\ n \in \mathbb{N}_3^*}} b^{\omega(n)} = O(X\mathcal{L}^{b/2-1}).$$

The following lemma shows that in the sums we will meet, the contribution of the integers with a huge number of prime factors is small.

Lemma 4.2. — *Let b and $b' > 0$ be given. Then there exists $B_0 = B_0(b,b')$ such that uniformly for $X \geqslant 1$ one has*

$$\sum_{\substack{n \leqslant X \\ \omega(n) > B_0 \log\log X}} b^{\omega(n)} = O\big(X\mathcal{L}^{-b'}\big).$$

Proof. — Let $\mathcal{E}_{B_0}(X)$ be the set of integers $n \leqslant X$ such that $\omega(n) > B_0 \log\log X$. We trivially have

$$|\mathcal{E}_{B_0}(X)| \cdot 2^{B_0 \log\log X} \leqslant \sum_{n \leqslant X} \tau(n) \sim X\mathcal{L},$$

which gives the bound $|\mathcal{E}_{B_0}(X)| \ll X\,\mathcal{L}^{1-B_0 \log 2}$. Now, by the Cauchy–Schwarz inequality and by the first bound given by Lemma 4.1, we have the inequalities

$$\sum_{\substack{n \leqslant X \\ \omega(n) > B_0 \log\log X}} b^{\omega(n)} \ll |\mathcal{E}_{B_0}(X)|^{1/2} \left(\sum_{n \leqslant X} b^{2\omega(n)} \right)^{1/2} \ll X\mathcal{L}^{b^2/2-(B_0 \log 2)/2},$$

which is $\ll X\mathcal{L}^{-b'}$ with the choice $B_0 = (b^2 + 2b')/\log 2$. □

4.1.2. A trivial bound for $\mathrm{Heis}^*(X)$

We first consider the sum (see (3.17))

$$S^*(X, f, f') = \sum_{d} 2^{\omega(d)},$$

where the integer d satisfies the conditions (3.18). The last condition of (3.18) implies the inequality

$$\Delta(f)\,\Delta(f')^{2/3}\,(\Delta(f), \Delta(f'))^{-2/3} \leqslant X^{1/6}, \tag{4.1}$$

which also implies

$$\Delta(f) \leqslant X^{1/6} \text{ and } \Delta(f') \leqslant X^{1/4}. \tag{4.2}$$

A direct application of the second part of Lemma 4.1 leads to the bound

$$S^*(X, f, f') \ll X^{1/6} \Delta(f)^{-1}\,\Delta(f')^{-2/3} \big(\Delta(f), \Delta(f')\big)^{2/3}. \tag{4.3}$$

Later, in this paper, we will give a more precise formula for this quantity (see Proposition 4.15 below).

We insert the bound (4.3) into (3.17). However, given $\Delta \in \mathbb{N}_3^*$, there are $2^{\omega(\Delta)} = 2^{|\mathrm{supp}\, f|}$ functions $f \in V^*$ such that $\Delta(f) = \Delta$. These remarks and Lemma 3.2 lead to the bound

$$\mathrm{Heis}^*(X) \ll X^{1/6} \sum_{\Delta,}\sum_{\Delta'} 3^{\omega(\Delta\,\Delta')} \cdot 2^{\omega(\Delta)} \cdot 2^{\omega(\Delta')} \Delta^{-1} \Delta'^{-2/3} \,(\Delta, \Delta')^{2/3}, \tag{4.4}$$

where Δ and Δ' belong to $\mathbb{N}_3^*$ and satisfy (4.1).

To study this sum, we put $\Delta_0 = (\Delta, \Delta')$, $\Delta = \Delta_0\Delta_1$ and $\Delta' = \Delta_0\Delta_1'$ to write the inequality

$$\text{Heis}^*(X) \ll X^{1/6} \sum_{\Delta_0} 12^{\omega(\Delta_0)} \Delta_0^{-1} \sum_{\Delta_1} 6^{\omega(\Delta_1)} \Delta_1^{-1} \sum_{\Delta_1'} 6^{\omega(\Delta_1')} {\Delta_1'}^{-2/3}. \tag{4.5}$$

By a repeated application of Lemma 4.1, by partial summations and by the crude inequalities (4.2), we arrive at the inequality

$$\text{Heis}^*(X) \ll X^{1/4}\mathcal{L}^5. \tag{4.6}$$

This trivial bound just misses the expected order of magnitude of $\text{Heis}^*(X)$ announced in Proposition 3.9 by a power of $\mathcal{L}$.

4.1.3. Restriction on the size of $\Delta(f)$

Let $K > 1$ be given. We denote by $\text{Heis}^*(X; \Delta > K)$ the subsum of $\text{Heis}^*(X)$ corresponding to the following restrictions of summations over f and f' (compare with the conditions in (3.17))

$$\begin{cases} f,\ f' \in V^*, \\ f,\ f' \text{ linearly independent}, \\ \Delta(f) > K. \end{cases} \tag{4.7}$$

We will prove the following

Proposition 4.3. — *There exists $A_0 > 0$ such that, uniformly for $X \geqslant 2$, one has the upper bound*

$$\text{Heis}^*(X; \Delta > \mathcal{L}^{A_0}) \ll X^{1/4}\mathcal{L}^{-1}.$$

Proof. — By a computation similar to (4.5), one has the inequality

$$\text{Heis}^*(X; \Delta > K) \ll X^{1/6} \sum_{\Delta_0} 12^{\omega(\Delta_0)} \Delta_0^{-1} \sum_{\Delta_1} 6^{\omega(\Delta_1)} \Delta_1^{-1} \sum_{\Delta_1'} 6^{\omega(\Delta_1')} {\Delta_1'}^{-2/3},$$

where the sum is over the triples of positive integers $(\Delta_0, \Delta_1, \Delta_1')$ such that

$$\begin{cases} \Delta_0\,\Delta_1 > K, \\ \Delta_0\,\Delta_1\,{\Delta_1'}^{2/3} \leqslant X^{1/6}, \end{cases}$$

(see (4.1) for the last condition). Summing first over Δ_1' we get, for some constant $A_1 > 0$, the bound

$$\begin{aligned} \text{Heis}^*(X; \Delta > K) &\ll X^{1/4}\mathcal{L}^{A_1} \sum_{\Delta_0} 12^{\omega(\Delta_0)} \Delta_0^{-3/2} \sum_{\Delta_1} 6^{\omega(\Delta_1)} \Delta_1^{-3/2}, \\ &\ll X^{1/4}\mathcal{L}^{A_1} \sum_{\Delta > K} 18^{\omega(\Delta)}\, \Delta^{-3/2}, \end{aligned}$$

since $\Delta_0\Delta_1 = \Delta$. If we choose $K = \mathcal{L}^{A_0}$ for a sufficiently large value of A_0, Lemma 4.1 and partial summation show that the above expression is $\ll X^{1/4}\mathcal{L}^{-1}$. □

4.1.4. Restriction on the size of $\Delta(f')$

In this paragraph, we show that we can restrict ourselves to large values of $\Delta(f')$ which means $\Delta(f') > X^{1/4}\mathcal{L}^{-A_2}$.

To be more precise, let A_0 be as in Proposition 4.3. For $K' > 1$ let

$$\mathrm{Heis}^*(X; \Delta \leqslant \mathcal{L}^{A_0}, \Delta' < K')$$

be the subsum of $\mathrm{Heis}^*(X)$ corresponding to the restriction of summations (compare with (3.17) and with (4.7))

$$\begin{cases} f,\ f' \in V^*, \\ f,\ f' \text{ linearly independent}, \\ \Delta(f) \leqslant \mathcal{L}^{A_0}, \\ \Delta(f') < K'. \end{cases} \tag{4.8}$$

We will prove

Proposition 4.4. — *Let A_0 be as in Proposition 4.3. There exists $A_2 > 0$ such that, uniformly for $X \geqslant 2$, one has the upper bound*

$$\mathrm{Heis}^*\big(X; \Delta \leqslant \mathcal{L}^{A_0}, \Delta' < X^{1/4}\mathcal{L}^{-A_2}\big) \ll X^{1/4}\mathcal{L}^{-1}.$$

Proof. — The proof mimics the proof of the crude bound (4.6). It suffices to replace the conditions (4.2) by the two present hypotheses: $\Delta(f) \leqslant \mathcal{L}^{A_0}$ and $\Delta(f') < X^{1/4}\mathcal{L}^{-A_2}$ and to choose A_2 sufficiently large to replace the exponent 5 by -1 on the right–hand side of (4.6). □

4.1.5. Restriction on the number of prime factors of $\Delta(f')$

Thanks to Propositions 4.3 and 4.4, it remains to study the contribution of the pairs $(f, f') \in V^* \times V^*$, linearly independent, with $\Delta(f)$ small (which means $\leqslant \mathcal{L}^{A_0}$) and with $\Delta(f')$ of size almost maximal (which means between $X^{1/4}\mathcal{L}^{-A_2}$ and $X^{1/4}$). We continue our preparation of the pairs (f, f') by controlling the number of prime factors of $\Delta(f')$. Let A_0 and A_2 be as in Propositions 4.3 and 4.4. Let $A_3 > 0$ to be fixed later. Let

$$\mathrm{Heis}^*\big(X; 1 < \Delta \leqslant \mathcal{L}^{A_0}, \Delta' \geqslant X^{1/4}\mathcal{L}^{-A_2}, \omega(\Delta') \geqslant A_3 \log\log X\,\big)$$

be the subsum of $\mathrm{Heis}^*(X)$ corresponding to the restriction of summations (compare with (3.17) and (4.8))

$$\begin{cases} f,\ f' \in V^*, \\ 1 < \Delta(f) \leqslant \mathcal{L}^{A_0}, \\ \Delta(f') \geqslant X^{1/4}\mathcal{L}^{-A_2}, \\ \omega\big(\Delta(f')\big) \geqslant A_3 \log\log X. \end{cases} \tag{4.9}$$

Remark 4.5. — The second and third condition of (4.9) imply that f and f' have distinct supports for sufficiently large X. So these functions are linearly independent, as soon as $\Delta(f) > 1$.

We will prove

Proposition 4.6. — *Let A_0 and A_2 be as in Propositions 4.3 and 4.4. Then there exists A_3 such that, uniformly for $X \geqslant 2$, one has the upper bound*

$$\mathrm{Heis}^*\big(X; 1 < \Delta \leqslant \mathcal{L}^{A_0}, \Delta' > X^{1/4}\mathcal{L}^{-A_2}, \omega(\Delta') \geqslant A_3 \log\log X\big) \ll X^{1/4}\mathcal{L}^{-1}.$$

Proof. — We go back to the inequality (4.4) to perform a trivial summation over $\Delta \leqslant \mathcal{L}^{A_0}$. Hence, for some A_4, we have the inequality

$$\begin{aligned} &\mathrm{Heis}^*\big(X; 1 < \Delta \leqslant \mathcal{L}^{A_0}, \Delta' > X^{1/4}\,\mathcal{L}^{-A_2}, \omega(\Delta') \geqslant A_3 \log\log X\big) \\ &\qquad \ll X^{1/6}\mathcal{L}^{A_4} \sum_{\substack{\Delta' < X^{1/4} \\ \omega(\Delta') \geqslant (A_3/2)\log\log X^{1/4}}} 6^{\omega(\Delta')}\Delta'^{-2/3} \ll X^{1/4}\mathcal{L}^{-1}, \end{aligned}$$

by Lemma 4.2, by a partial summation and by choosing A_3 sufficiently large. □

We have finished with the technical preparation of $\Delta(f)$ and $\Delta(f')$. So it is natural to define the subsum $\mathrm{Heis}^\dagger(X)$ of $\mathrm{Heis}^*(X)$, defined in (3.17), by imposing the following additional restrictions of summation on f and f'

$$\begin{cases} f,\ f' \in V^*, \\ 1 < \Delta(f)) \leqslant \mathcal{L}^{A_0}, \\ \Delta(f') \geqslant X^{1/4}\,\mathcal{L}^{-A_2}, \\ \omega(\Delta(f')) \leqslant A_3 \log\log X, \\ \Delta(f)\,\Delta(f')^{2/3}\,(\Delta(f), \Delta(f'))^{-2/3} \leqslant X^{1/6}, \end{cases} \tag{4.10}$$

where A_0, A_2 and A_3 are defined in Propositions 4.3, 4.4 and 4.6. Gathering Propositions 4.3, 4.4 and 4.6, we see that the proof of Proposition 3.9 is reduced to the proof of the formula

$$\mathrm{Heis}^\dagger(X) = C_{\mathrm{Heis}^*}\, X^{1/4} + O\big(X^{1/4}\mathcal{L}^{-1}\big), \tag{4.11}$$

where C_{Heis^*} is defined in (3.19) and where the O–constant is uniform for $X \geqslant 1$.

4.2. Inverting summations in $\mathrm{Heis}^\dagger(X)$

We now benefit from the control of the sizes of the variables appearing in $\mathrm{Heis}^\dagger(X)$ which is a subsum of $\mathrm{Heis}^*(X)$. By the last line of (3.18) and by the second and third lines of (4.10) we see that d satisfies the inequalities

$$1 \leqslant d \leqslant X^{1/6}\Delta(f)^{-1}\left(X^{1/4}\mathcal{L}^{-A_2}\right)^{-2/3}\Delta(f)^{2/3} \leqslant \mathcal{L}^{2A_2/3} = \mathcal{L}^{A_4},$$

by definition. This means that the variable d is almost constant and it is wise to perform the summation over this variable at the very end of the proof. We decompose $\mathrm{Heis}^\dagger(X)$ as

$$\mathrm{Heis}^\dagger(X) = \sum_{\substack{d\in\mathbb{N}_3^*\\ d\leqslant \mathcal{L}^{A_4}}} 2^{\omega(d)}\, U(X,d) \tag{4.12}$$

with

$$U(X,d) = 2^{-2}3^{-3}\sum_{f,}\ \sum_{f'} 3^{|\mathrm{supp}\, f\,\cup\, \mathrm{supp}\ f'|}\cdot \mathbb{1}(f,f'), \tag{4.13}$$

where the pair of functions (f,f') satisfies (4.10), the inequality

$$\Delta(f)\,\Delta(f')^{2/3}\left(\Delta(f),\Delta(f')\right)^{-2/3} \leqslant X^{1/6}d^{-1}, \tag{4.14}$$

(which both come from (3.18)), and the coprimality condition

$$\big(d,\Delta(f)\Delta(f')\big) = 1.$$

4.3. Factorisation of the function $\mathbb{1}(f,f')$

To facilitate the study of the function $\mathbb{1}(f,f')$, we put

$$\mathcal{E} := \mathrm{supp}\, f \quad\text{and}\quad \mathcal{E}' := \mathrm{supp}\, f'.$$

In a unique way, we decompose $\mathcal{E}$ and $\mathcal{E}'$ as a disjoint union

$$\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1 \quad \text{and} \quad \mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1', \tag{4.15}$$

where, furthermore $\mathcal{E}_1$ and $\mathcal{E}_1'$ are disjoint. This decomposition incites to write the functions f and f' as

$$f = f_0 \oplus f_1 \quad \text{and} \quad f' = f_0' \oplus f_1', \tag{4.16}$$

where $\operatorname{supp} f_0 = \operatorname{supp} f_0' = \mathcal{E}_0$, $\operatorname{supp} f_1 = \mathcal{E}_1$ and $\operatorname{supp} f_1' = \mathcal{E}_1'$. We define

$$\Delta_0 := \Delta(f_0) = \Delta(f_0') = \prod_{p \in \mathcal{E}_0} p, \tag{4.17}$$

and we define Δ_1 and Δ_1' analogously. The integers Δ_0, Δ_1 and Δ_1' belong to $\mathbb{N}_3^*$ and are coprime in pairs. The numbers $\Delta = \Delta_0\Delta_1$ and $\Delta' = \Delta_0\Delta_1'$ also belong to $\mathbb{N}_3^*$. We now start rewriting $\mathbb{1}(f, f')$ in terms of characters.

Lemma 4.7. — *Let $f, f' \in V^*$. We adopt the notations* (4.15), (4.16) *and* (4.17). *We then have the equalities*

$$\begin{aligned} &\sum_{\substack{(z,z') \in \mathbb{F}_3^2 \\ f(r)z + f'(r)z' = 0}} \big(\chi(zf + z'f')\big)(r) \\ &= 1 + \begin{cases} \chi(f_0' + f_1')(r) + \chi(2(f_0' + f_1'))(r) & \text{if } r \in \mathcal{E}_1, \\ \chi(f_0 + f_1)(r) + \chi(2(f_0 + f_1))(r) & \text{if } r \in \mathcal{E}_1', \\ \chi\big(f_0'(r)(f_0 + f_1) + 2f_0(r)(f_0' + f_1')\big)(r) & \\ \qquad + \chi\big(2f_0'(r)(f_0 + f_1) + f_0(r)(f_0' + f_1')\big)(r) & \text{if } r \in \mathcal{E}_0. \end{cases} \end{aligned} \tag{4.18}$$

Proof. — Solve the equation $f(r)z + f'(r)z' = 0$ in each of the three cases. □

Remark 4.8. — Recall that the value of the left–hand side of (4.18) is 0 or 3.

4.4. Decomposition of $U(X,d)$

We incorporate the decompositions (4.15), (4.16) and (4.17) in (4.13). Combining Lemma 4.7 with the notation introduced in (3.7) and with Definition 3.1 we arrive at the equality

$$U(X,d) = 2^{-2}3^{-3} \sum_{\Delta_0,}\sum_{\Delta_1,}\sum_{\Delta_1'} \quad \sum\sum\sum\sum_{\substack{f_0,\, f_0'\in V^*(\Delta_0)\\ f_1\in V^*(\Delta_1)\\ f_1'\in V^*(\Delta_1')}} \prod_{r\mid\Delta_0}\Big\{1+\chi\big(f_0'(r)(f_0+f_1)+2f_0(r)(f_0'+f_1')\big)(r) + \chi\big(2f_0'(r)(f_0+f_1)+f_0(r)(f_0'+f_1')\big)(r)\Big\} \times \prod_{r\mid\Delta_1}\{1+\chi(f_0'+f_1')(r)+\chi(2(f_0'+f_1'))(r)\} \prod_{r\mid\Delta_1'}\{1+\chi(f_0+f_1)(r)+\chi(2(f_0+f_1))(r)\}, \tag{4.19}$$

where the conditions of summation (4.10) and (4.14) become

$$\begin{cases} \Delta_0,\, \Delta_1,\, \Delta_1' \in \mathbb{N}_3^*,\\ (\Delta_0,\, \Delta_1) = (\Delta_0, \Delta_1') = (\Delta_1, \Delta_1') = (d, \Delta_0\Delta_1\Delta_1') = 1,\\ 1 < \Delta_0\Delta_1 \leqslant \mathcal{L}^{A_0},\\ \Delta_0\Delta_1' \geqslant X^{1/4}\mathcal{L}^{-A_2},\\ \omega\big(\Delta_0\Delta_1'\big) \leqslant A_3 \log\log X,\\ \Delta_0\, \Delta_1\, {\Delta_1'}^{2/3} \leqslant X^{1/6}/d. \end{cases} \tag{4.20}$$

In a condensed way, we write (4.19) as

$$U(X,d) = 2^{-2}3^{-3} \sum_{\boldsymbol{\Delta}} \quad \sum_{\boldsymbol{f}} \prod_{r\mid\Delta_0}\{\cdots\} \prod_{r\mid\Delta_1}\{\cdots\} \prod_{r\mid\Delta_1'}\{\cdots\},$$

and we decompose $U(X,d)$ as

$$U(X,d) = \mathrm{MT}(X,d) + \mathrm{ET}(X,d), \tag{4.21}$$

where

$$\mathrm{MT}(X,d) \coloneqq 2^{-2}3^{-3} \sum_{\boldsymbol{\Delta}} \quad \sum_{\boldsymbol{f}} \prod_{r\mid\Delta_1'}\{\cdots\}, \tag{4.22}$$

and

$$\text{(4.23)}\quad \mathrm{ET}(X,d) := 2^{-2}3^{-3}\sum_{\boldsymbol{\Delta}}\ \sum_{\boldsymbol{f}}\left(-1+\prod_{r\mid\Delta_0}\{\cdots\}\prod_{r\mid\Delta_1}\{\cdots\}\right)\prod_{r\mid\Delta_1'}\{\cdots\}.$$

To describe the scenery of these sums we insist on the fact that Δ_0 and Δ_1 are very small variables. In contrast, Δ_1' is a large variable, and since Δ_1' has few prime divisors (see the fifth line of (4.20)), its largest prime divisor, that we will denote by $p_\infty := p_\infty(\Delta_1')$, is also large. When summing over p_∞, we will obtain cancellation between cubic characters as a consequence of a theorem of Siegel–Walfisz type (see Lemma 4.12). We will obtain Proposition 4.13 below, which shows that $\mathrm{ET}(X,d)$ is an error term. In the other direction, the term $\mathrm{MT}(X,d)$, roughly speaking, appears to be the product of $X^{1/4}$ by a convergent series for which we will search for a concise value, which will lead to the value of C_{Heis^*} given in (3.19).

4.5. Study of $\mathrm{ET}(X,d)$

We factorize Δ_1' as

$$\text{(4.24)}\qquad \begin{cases}\Delta_1' = \Delta_1'' p_\infty,\\ p\mid\Delta_1'' \Rightarrow p < p_\infty.\end{cases}$$

Correspondingly, there are two possible decompositions of the function f_1'

$$\text{(4.25)}\qquad f_1' := f_1''\oplus \mathbb{1}_{p_\infty} \text{ or } f_1' = f_1''\oplus 2\cdot\mathbb{1}_{p_\infty},$$

where $\Delta(f_1'') = \Delta_1''$, and $\mathbb{1}_{p_\infty}$ is the characteristic function of the set $\{p_\infty\}$. We also have

$$\chi(f_1') = \chi(f_1'')\,\chi_{p_\infty} \text{ or } \chi(f_1') = \chi(f_1'')\,\chi_{p_\infty}^2,$$

according to the cases listed in (4.25). We return to (4.23) to highlight the summation over p_∞:

$$\text{(4.26)}\quad \mathrm{ET}(X,d) \ll \sum\sum\sum_{\Delta_0,\Delta_1,\Delta_1''} 3^{\omega(\Delta_1'')} \sum\sum\sum\sum_{\substack{f_0,\, f_0'\in V^*(\Delta_0)\\ f_1\in V^*(\Delta_1)\\ f_1''\in V^*(\Delta_1'')}}$$

$$\left|\sum_{p_\infty}\left(-1+\prod_{r\mid\Delta_0}\{\cdots\}\prod_{r\mid\Delta_1}\{\cdots\}\right)\big(1+\chi(f_0+f_1)(p_\infty)+\chi(2(f_0+f_1))(p_\infty)\big)\right|$$

$$+\ \text{similar term},$$

where, in the second line of (4.26), we have chosen, for f_1', the first decomposition written in (4.25). The *similar term* corresponds to the second decomposition in (4.25). In (4.26), the conditions of summation are deduced from (4.20) by applying the decomposition (4.24).

We develop the product on the second line of (4.26) to bring out

$$3(3^{\omega(\Delta_0\Delta_1)} - 1) \ (= O(\mathcal{L}^{A_5})) \tag{4.27}$$

products of cubic characters. This means that the sum over p_∞ appearing in (4.26) is the sum of $O(\mathcal{L}^{A_5})$ sums of the form

$$\begin{aligned} A(\boldsymbol{\eta},\, \boldsymbol{\zeta}, \boldsymbol{\epsilon}, f_0, f_0', f_1, f_1'') &= \sum_{p_\infty} \prod_{r\mid\Delta_0} \Big\{ \big[\chi\big(f_0'(r)(f_0+f_1) + 2f_0(r)(f_0'+f_1')\big)(r)\big]^{\eta_{1r}} \\ &\qquad \times \big[\chi\big(2f_0'(r)(f_0+f_1) + f_0(r)(f_0'+f_1')\big)(r)\big]^{\eta_{2r}} \Big\} \\ &\qquad \prod_{r\mid\Delta_1} \big\{ \big[\chi(f_0'+f_1')(r)\big]^{\zeta_{1r}} \cdot \big[\chi(2(f_0'+f_1'))(r)\big]^{\zeta_{2r}} \big\} \\ &\qquad \cdot \big[\chi(f_0+f_1)(p_\infty)\big]^{\epsilon_1} \cdot \big[\chi(2(f_0+f_1))(p_\infty)\big]^{\epsilon_2} \end{aligned}$$

where the exponents are non-negative integers and satisfy the inequalities

$$\begin{cases} 0 \leqslant \eta_{1r} + \eta_{2r} \leqslant 1 & \text{for each } r \mid \Delta_0, \\ 0 \leqslant \zeta_{1r} + \zeta_{2r} \leqslant 1 & \text{for each } r \mid \Delta_1, \\ 0 \leqslant \epsilon_1 + \epsilon_2 \leqslant 1, \\ \sum_{r\mid\Delta_0}(\eta_{1r}+\eta_{2r}) + \sum_{r\mid\Delta_1}(\zeta_{1r}+\zeta_r) \geqslant 1. \end{cases} \tag{4.28}$$

We return to the definition of $\chi(f)$ given in (3.4) and recall the equalities $f_1'(p_\infty) = 1$ and $f_0(p_\infty) = f_0'(p_\infty) = f_1(p_\infty) = 0$. Keeping only the factors depending on p_∞, we get an equality

$$|A(\boldsymbol{\eta},\, \boldsymbol{\zeta}, \boldsymbol{\epsilon}, f_0, f_0', f_1, f_1'')| = |\widetilde{A}(\boldsymbol{\eta},\, \boldsymbol{\zeta}, \boldsymbol{\epsilon}, f_0, f_1)| \tag{4.29}$$

between moduli, where

$$\begin{aligned} \widetilde{A}(\boldsymbol{\eta},\, \boldsymbol{\zeta}, \boldsymbol{\epsilon}, f_0, f_1) &= \sum_{p_\infty} \big[\chi(f_0+f_1)(p_\infty)\big]^{\epsilon_1+2\epsilon_2} \\ &\qquad \times \prod_{r\mid\Delta_0} \big[\chi_{p_\infty}(r)\big]^{f_0(r)(2\,\eta_{1r}+\eta_{2r})} \prod_{r\mid\Delta_1} \big[\chi_{p_\infty}(r)\big]^{\zeta_{1r}+2\,\zeta_{2r}} \qquad (4.30) \\ &= \sum_{p_\infty} \widetilde{M}(p_\infty), \qquad (4.31) \end{aligned}$$

by definition. Of course p_∞ satisfies the conditions of summation deduced from (4.20) by applying the factorization (4.24). The exponent $f_0(r)$ appearing in (4.30) can take the value 1 or 2 mod 3. If its value is 2, we have the equality

$$f_0(r)(2\,\eta_{1r}+\eta_{2r}) = 2\,\eta_{2r}+\eta_{1r}.$$

So in that case, we can invert the roles of η_{1r} and η_{2r} without affecting the conditions (4.28). So we can always suppose that $f_0(r) = 1$ in the definition of $\widetilde{M}(p_\infty)$. We also replace f_0+f_1 by f (see (4.16)) and $\Delta_0\Delta_1$ by Δ (see (4.17)). So $\widetilde{M}(p_\infty)$ equals

$$\widetilde{M}(p_\infty) = \big[\,\chi(f)(p_\infty)\,\big]^{\epsilon_1+2\epsilon_2}\prod_{r\mid\Delta}\big[\,\chi_{p_\infty}(r)\,\big]^{e_{1r}+2e_{2r}}, \tag{4.32}$$

where $p_\infty \in \mathbb{P}_3^*$ does not divide Δ and where the non-negative exponents ϵ_i and e_{ir} satisfy

$$\begin{cases} 0 \leqslant e_{1r}+e_{2r} \leqslant 1, & \text{for all } r \mid \Delta,\\ 0 \leqslant \epsilon_1+\epsilon_2 \leqslant 1 \\ \sum_{r\mid\Delta}(e_{1r}+e_{2r}) \geqslant 1. \end{cases} \tag{4.33}$$

To obtain the desired cancellation when summing over p_∞, we will show that $\widetilde{M}$ is a character of $\mathbb{Z}[j]$. As a first step we use the following

Lemma 4.9. — *For every distinct primes p and r in $\mathbb{P}_3^*$, decomposed in the standard way: $p = \pi\cdot\overline{\pi}$ and $r = \rho\cdot\bar{\rho}$, we have the equality*

$$\chi_p(r) = \overline{\chi_r(p)}\left(\frac{\pi}{\rho}\right)_3^2.$$

Proof. — Combine the multiplicative properties of the cubic character, the cubic reciprocity law $(\pi/\rho)_3 = (\rho/\pi)_3$ (see [17, Theorem 1 p. 114], for instance) and the conjugation property $\overline{(\pi/\rho)_3} = (\overline{\pi}/\bar{\rho})_3$. ☐

We use Lemma 4.9 to write the equality

$$\chi_{p_\infty}(r) = \overline{\chi_r(p_\infty)}\left(\frac{\pi_\infty}{\rho}\right)_3^2,$$

where we decomposed in a standard way $p_\infty = \pi_\infty\cdot\overline{\pi_\infty}$ and $r = \rho\cdot\bar{\rho}$.

Let $f\in V^*$, let $\boldsymbol{\epsilon}$ be a pair (ϵ_1,ϵ_2) of positive integers and let $\boldsymbol{e} = (e_{1r},e_{2r})_{r\in\operatorname{supp} f}$ be a $2\cdot|\operatorname{supp} f|$-tuple of positive integers. Let $r\in\mathbb{P}_3^*$ decomposed in the standard way $r=\rho\cdot\bar{\rho}$. For $z\in\mathbb{Z}[j]$, we define

$$\begin{aligned} M(z) &= M(z,f,\boldsymbol{\epsilon},\boldsymbol{e}) \\ &:= \big[\,\chi(f)(z\bar{z})\,\big]^{\epsilon_1}\ \big[\,\chi(2f)(z\bar{z})\,\big]^{\epsilon_2}\prod_{r\in\operatorname{supp} f}\left[\chi_r(z\bar{z})\left(\frac{z}{\rho}\right)_3\right]^{2e_{1r}+e_{2r}}. \end{aligned}$$

Lemma 4.10. — *Let $f \in V^*$ with a non-empty support. Let $p = \pi \cdot \overline{\pi}$ be the standard decomposition of a prime p belonging to $\mathbb{P}_3^*$ but not to* $\operatorname{supp} f$. *We then have the equality*

$$\widetilde{M}(p) = M(\pi). \tag{4.34}$$

Suppose furthermore that the following conditions are satisfied:

$$\begin{cases} 0 \leqslant \epsilon_1 + \epsilon_2 \leqslant 1, \\ 0 \leqslant e_{1,r} + e_{2,r} \leqslant 1, \quad \text{for each } r \in \operatorname{supp} f, \\ \sum_{r \in \operatorname{supp} f} (e_{1r} + e_{2r}) \geqslant 1. \end{cases}$$

Then the function M is a non-trivial multiplicative character over $\mathbb{Z}[j]$, with period dividing $\prod_{r \in \operatorname{supp} f} r$.

Proof. — The equality (4.34) is a consequence of the construction of the function M and Lemma 4.9.

For the second part, it is clear that the function M is a multiplicative character over $\mathbb{Z}[j]$, and it is also clear that its period divides $\prod_{r \in \operatorname{supp} f} r = \Delta(f)$. It remains to show that it is a non-trivial character.

Suppose that $M(z)$ is the trivial character. Note that $M(z)$ is a product

$$\Bigg[\prod_{r \in \operatorname{supp} f} \chi_r^{f(r)}(z\overline{z})\Bigg]^{\epsilon_1} \Bigg[\prod_{r \in \operatorname{supp} f} \chi_r^{2f(r)}(z\overline{z})\Bigg]^{\epsilon_2} \prod_{r \in \operatorname{supp} f} \left[\chi_r(z\overline{z}) \left(\frac{z}{\rho}\right)_3\right]^{2e_{1r}+e_{2r}},$$

where all the factors have coprime period. Hence $M(z)$ trivial implies that

$$\chi_r^{\epsilon_1 f(r)}(z\overline{z}) \chi_r^{2\epsilon_2 f(r)}(z\overline{z}) \left[\chi_r(z\overline{z}) \left(\frac{z}{\rho}\right)_3\right]^{2e_{1r}+e_{2r}}$$

is the trivial character for any r in the support of f. Now recall the inequalities

$$0 \leqslant \epsilon_1 + \epsilon_2 \leqslant 1, \quad 0 \leqslant e_{1r} + e_{2r} \leqslant 1.$$

But $\chi_r(z\overline{z})$ and $(\frac{z}{\rho})_3$ are linearly independent characters. This forces

$$\epsilon_1 = \epsilon_2 = e_{1r} = e_{2r} = 0,$$

contrary to our third assumption. □

4.5.1. A Siegel–Walfisz type Theorem for standard primes

The famous Siegel–Walfisz Theorem for rational primes gives equidistribution of primes $p \leqslant X$ in arithmetic progressions $a + kq$ (with $(a, q) = 1$) uniformly for the modulus $q \leqslant \mathcal{L}^A$ for any arbitrary given A. Such a phenomenom of equidistribution also holds for prime ideals in number fields since the associated L–functions have properties similar to those of Dirichlet L–functions. On that subject, among other references, an interesting general one is [28, Main Theorem p.35], which was used in [13, Lemma 32 and Proposition 7] in the context of *privileged primes* of the ring $\mathbb{Z}[i]$, the ring of Gaussian integers. The methods presented in [13] are easily translated in the context of $\mathbb{Z}[j]$ which is the theatre of our paper. We introduce the following notations:

Let a and w be two elements of $\mathbb{Z}[j]$ such that w is coprime with $3a$. For $x \geqslant 2$, let

$$\pi_{\mathbb{Z}[j]}(x; w, a) \\ := \Big|\{\pi \in \mathbb{Z}[j] : \pi \text{ is a standard prime, } \mathrm{N}(\pi) \leqslant x,\ \pi \equiv a \bmod w\}\Big|,$$

and let $\phi(w)$ be the number of invertible classes in $\mathbb{Z}[j]/(w\mathbb{Z}[j])$. We then have

Proposition 4.11. — *For every $A > 0$, there exists $c(A) > 0$ such that, uniformly for*

$$x \geqslant 2,\ a,\ w \in \mathbb{Z}[j],\ (w, 3a) = 1,\ \mathrm{N}(w) \leqslant (\log x)^A,$$

one has the equality

$$\pi_{\mathbb{Z}[j]}(x; w, a) = \frac{1}{\phi(w)} \pi_{\mathbb{Z}[j]}(x; 1, 0) + O\Big(x \exp(-c(A)\sqrt{\log x}\,)\Big).$$

This proposition gives the desired cancellation in sums over multiplicative characters χ on $\mathbb{Z}[j]$.

Lemma 4.12. — *For every $A > 0$, there exists $c(A) > 0$, such that, uniformly for $x \geqslant 2, w \in \mathbb{Z}[j]$, coprime with 3 and satisfying $1 < \mathrm{N}(w) \leqslant (\log x)^A$, χ a non-trivial character modulo w, one has the inequality*

$$\sum_{\substack{\pi \text{ standard prime} \\ \mathrm{N}(\pi) \leqslant x}} \chi(\pi) = O\left(x \exp\left(-c(A)\sqrt{\log x}\right)\right).$$

In particular, for any $A > 0$, there exists $C(A)$ such that, for any non-trivial character χ over $\mathbb{Z}[j]$, with period w, for every $x \geqslant 2$, one has the inequality

$$\left| \sum_{\substack{\pi \text{ standard prime} \\ \mathrm{N}(\pi) \leqslant x}} \chi(\pi) \right| \leqslant C(A)\, x\, \mathrm{N}^{1/4}(w)\, (\log x)^{-A}.$$

Proof. — We write the sum in question as

$$\sum_{\substack{a \bmod w \\ (a,w)=1}} \chi(a)\, \pi_{\mathbb{Z}[j]}(x; w, a)$$

and then apply Proposition 4.11. Recalling that $\sum_{(a,w)=1} \chi(a) = 0$ for a non-trivial character χ modulo w finishes the proof. □

4.5.2. Bounding $|\widetilde{A}(\boldsymbol{\eta}, \boldsymbol{\zeta}, \boldsymbol{\epsilon}, f_0, f_1)|$

Returning to the definitions (4.31) and (4.32) and applying Lemmas 4.10 and 4.12, we deduce that, for any $A > 0$, for any $f \in V^*$, for any $2 < U < Z$, for any $\boldsymbol{\epsilon}$ and $\boldsymbol{e}$ satisfying (4.33), we have

$$\begin{aligned} \sum_{U < p_\infty < Z} \widetilde{M}(p_\infty) &= \sum_{\substack{\pi \text{ standard prime} \\ U < \mathrm{N}(\pi) < Z}} M(\pi, f, \boldsymbol{\epsilon}, \boldsymbol{e}) \\ &= O_A\big(\Delta(f)^{1/2} Z (\log U)^{-A}\big). \end{aligned} \tag{4.35}$$

The constant implicit in the O–symbol depends on A only. By the third and fourth lines of (4.20) we know that Δ_1' is large, since it satisfies the inequality

$$\Delta_1' \geqslant X^{1/4} \mathcal{L}^{-A_0 - A_2}.$$

Furthermore, p_∞ is the largest prime divisor of Δ_1' (see (4.24)) and Δ_1' has few prime factors (see the fifth line of (4.20)) so we deduce the lower bound

$$p_\infty \geqslant \big(X^{1/4} \mathcal{L}^{-A_0 - A_2}\big)^{1/A_3 \log\log X} \gg \exp\left(\frac{\mathcal{L}}{A_6 \log\log X}\right),$$

for some positive A_6. So we apply (4.35) by choosing U satisfying $\log U \gg \mathcal{L}^{1/2}$ and $1 < \Delta(f) \leqslant \mathcal{L}^{A_0}$ (see the third line of (4.20)). The value of Z is given by the last line of (4.20)

$$Z = X^{1/4} \big/ \big(d^{3/2} \Delta_0^{3/2} \Delta_1^{3/2} \Delta_1''\big).$$

Inserting these values in (4.35), we deduce, by (4.31), that

$$|\widetilde{A}(\boldsymbol{\eta}, \boldsymbol{\zeta}, \boldsymbol{\epsilon}, f_0, f_1)| \ll_A X^{1/4} \big/ \big(d^{3/2}\, \Delta_0\, \Delta_1\, \Delta_1''\, \mathcal{L}^A\big)$$

for any $A > 0$. Combining with (4.29), with (4.27) and with (4.26), we obtain the bound

$$\begin{aligned} \mathrm{ET}(X, d) \ll X^{1/4} \mathcal{L}^{A_5} \sum_{\Delta_0,} \sum_{\Delta_1,} \sum_{\Delta_1''} & 4^{\omega(\Delta_0)} \cdot 2^{\omega(\Delta_1)} \\ & \cdot 6^{\omega(\Delta_1'')} \big(d^{3/2}\, \Delta_0\, \Delta_1\, \Delta_1''\, \mathcal{L}^A\big)^{-1}, \end{aligned}$$

where A is arbitrary. It remains to perform a crude summation over Δ_0, Δ_1, Δ_1'' ($< \Delta_1'$) satisfying (4.20) and over $d \leqslant \mathcal{L}^{A_4}$. By choosing A sufficiently large we complete the proof of the following proposition

PROPOSITION 4.13. — *Uniformly for $X \geqslant 2$ one has*

$$\sum_{\substack{d \in \mathbb{N}_3^* \\ d \leqslant \mathcal{L}^{A_4}}} 2^{\omega(d)} \,\mathrm{ET}(X, d) = O(X^{1/4}\mathcal{L}^{-1}).$$

Remark 4.14. — The orders of magnitude of the variables of summation $\Delta_0\Delta_1$ and Δ_1' are completly different (see (4.20)). So Lemma 4.12 is the unique tool to exploit oscillation of characters. This situation is quite different from [12] or from [13], for instance, where the case of variables with comparable sizes also has to be treated. This is accomplished by appealing to bounds of *double oscillation type* (see [12, Lemmas 14 and 15], [13, Section 6] for instance).

4.6. Study of $\mathrm{MT}(X, d)$

We now turn our attention to the term $\mathrm{MT}(X, d)$, defined in (4.22). In order to prove that it behaves like a main term, we shall give the asymptotic formula

$$\sum_{d \leqslant \mathcal{L}^{A_4}} 2^{\omega(d)} \mathrm{MT}(X, d) = C_{\mathrm{Heis}^*} X^{1/4} + O\big(X^{1/4}\mathcal{L}^{-1}\big),$$

(see Section 4.6.4). By the definition (4.22) we have

$$\mathrm{MT}(X, d) = 2^{-2}3^{-3} \sum_{\Delta_0,} \sum_{\Delta_1,} \sum_{\Delta_1'} \quad \sum\sum\sum\sum_{\substack{f_0,\, f_0' \in V^*(\Delta_0) \\ f_1 \in V^*(\Delta_1) \\ f_1' \in V^*(\Delta_1')}} \prod_{r \mid \Delta_1'} \big\{1 + \chi(f_0 + f_1)(r) + \chi(2(f_0 + f_1))(r)\big\}. \tag{4.36}$$

The factor $\prod_{r\mid\Delta_1'}\{\cdots\}$ is independent of the choice of $f_0' \in V^*(\Delta_0)$ and of $f_1' \in V^*(\Delta_1')$. So we can replace the summations over f_0' and f_1' by the factor $2^{\omega(\Delta_0)} \cdot 2^{\omega(\Delta_1')}$. Furthermore the functions f_0 and f_1 only appear through their sum $f := f_0 + f_1$. We rewrite $\Delta = \Delta_0\Delta_1$. With this notation we have $f \in V^*(\Delta)$ and $\Delta(f) = \Delta$. Instead of summing over Δ_0, Δ_1, f_0 and f_1 we sum over Δ and f $f \in V^*(\Delta)$ replacing the factor $2^{\omega(\Delta_0)}$ by

$$\sum_{\Delta_0 \mid \Delta} 2^{\omega(\Delta_0)} = 3^{\omega(\Delta)}.$$

Gathering these remarks, (4.36) becomes

$$\begin{aligned}\mathrm{MT}(X,d) = 2^{-2}3^{-3}\sum_{\Delta,}\sum_{\Delta_1'} 3^{\omega(\Delta)}\cdot 2^{\omega(\Delta_1')}\\ \times \sum_{f\in V^*(\Delta)}\prod_{r\mid\Delta_1'}\bigl\{1+\chi(f)(r)+\chi(2f)(r)\bigr\} + O(X^{1/4}\mathcal{L}^{-2}).\end{aligned}\tag{4.37}$$

The conditions of summation in (4.37) are inferred from (4.20):

$$\begin{cases} d\Delta\,\Delta_1' \in \mathbb{N}_3^*,\\ 1<\Delta\leqslant \mathcal{L}^{A_0},\\ \Delta\,{\Delta_1'}^{2/3}\leqslant X^{1/6}/d.\end{cases}\tag{4.38}$$

The error term in (4.37) comes from forgetting the fourth and the fifth lines of (4.20). We control the induced error as it was done in the proofs of Propositions 4.4 and 4.6. The first condition of (4.38) implies that d, Δ and Δ_1' are coprime in pairs.

4.6.1. Expanding the product over primes r

By the multiplicativity of characters, the product appearing in (4.37) equals

$$\prod_{r\mid\Delta_1'}\bigl\{1+\chi(f)(r)+\chi(2f)(r)\bigr\} = \sum_{d_0}\sum_{d_1}\sum_{d_2=\Delta_1'}\chi(f)(d_1)\,\chi(2f)(d_2).\tag{4.39}$$

We insert this expression in (4.37) and we invert summations to obtain

$$\begin{aligned}\mathrm{MT}(X,d) = 2^{-2}3^{-3}\sum_{\Delta}3^{\omega(\Delta)}\sum_{f\in V^*(\Delta)}\sum_{d_1,}\sum_{d_2}\Bigl(2^{\omega(d_1)}\chi(f)(d_1)\Bigr)\\ \times\Bigl(2^{\omega(d_2)}\chi(2f)(d_2)\Bigr)\cdot\left(\sum_{d_0}2^{\omega(d_0)}\right) + O(X^{1/4}\mathcal{L}^{-2}),\end{aligned}\tag{4.40}$$

where the conditions of summation are deduced from (4.38)

$$\begin{cases} (dd_0d_1d_2)\,\Delta \in \mathbb{N}_3^*,\\ 1<\Delta\leqslant\mathcal{L}^{A_0},\\ \Delta\,(d_0\,d_1\,d_2)^{2/3}\leqslant X^{1/6}/d.\end{cases}\tag{4.41}$$

4.6.2. Controlling the sizes of d_1 and d_2

The last line of (4.41) implies that the product $d_1 d_2$ can be as large as $X^{1/4}$. In that case (4.41) shows that the variable d_0 has no room for variation and Proposition 4.15 below is useless in that situation (see formula (4.52)). To circumvent this particular difficulty we invert summations as in the hyperbola method, to exploit the presence of the oscillating coefficients $\chi(f)(d_1)$ and $\chi(2f)(d_2)$. These non-trivial Dirichlet characters, with moduli $\ll \mathcal{L}^{A_0}$, allow us to restrict the summation to

$$d_1,\, d_2 < D_0,$$

where D_0 is a small power of X:

$$D_0 := X^{1/100}.$$

Indeed the contribution of the (Δ, d_0, d_1, d_2) to the right–hand side of (4.40) satisfying $\max(d_1, d_2) > D_0$ is negligible. To see this, consider for instance the case when $d_1 > D_0$. The corresponding contribution, denoted by $\Xi(D_0, d)$, is bounded by

(4.42)
$$\Xi(D_0, d) \ll \sum_{\Delta} 3^{\omega(\Delta)} \sum_{f \in V^*(\Delta)} \sum_{d_0} 2^{\omega(d_0)} \sum_{d_2} 2^{\omega(d_2)} \left| \sum_{d_1} 2^{\omega(d_1)} \chi(f)(d_1) \right|$$

where $D_0 < d_1 \leqslant D_1 := X^{1/4} \Delta^{-3/2} d^{-3/2} d_0^{-1}\, d_2^{-1}$.

The Siegel–Walfisz Theorem allows us to save any power of $\mathcal{L}$ over the trivial bound in the sum over d_1. More precisely, for any $A > 0$, one has the bound

$$\sum_{D_0 < d_1 < D_1} 2^{\omega(d_1)} \chi(f)(d_1) \ll D_1 (\log D_0)^{-A} \ll D_1\, \mathcal{L}^{-A}. \tag{4.43}$$

Inserting this bound in (4.42), summing over Δ, d_0 and d_2, and choosing A sufficiently large, we obtain the bound

$$\Xi(D_0, d) \ll X^{1/4} \mathcal{L}^{-2}. \tag{4.44}$$

We give some details about the proof of (4.43). The process is similar to what was explained in Section 4.5.2. First of all, one can restrict to d_1 with a reasonable number of prime factors, which means $\omega(d_1) \leqslant B_0 \log\log X$ for some B_0 with acceptable error by Lemma 4.2. The remaining d_1 are then factorized as $d_1 = p_\infty \delta_1$, where p_∞ is the greatest prime factor of d_1. The prime p_∞ is a large variable to which we can apply a Siegel–Walfisz Theorem related to the Dirichlet L–functions $L(s, \chi(f))$ and $L(s, \chi(f)\, (\frac{\cdot}{3}))$. The second line of (4.41) ensures that the conductor of these L–functions is larger than 1 but less than $3\mathcal{L}^{A_0}$, which is the adequate situation to apply the Siegel–Walfisz Theorem. We omit the details.

In conclusion, by (4.44), we proved that (4.40) remains true, with the conditions of summations (4.41) replaced by

$$\begin{cases} (dd_0d_1d_2)\,\Delta \in \mathbb{N}_3^*, \\ 1 < \Delta \leqslant \mathcal{L}^{A_0}, \\ d_1,\, d_2 \leqslant D_0, \\ \Delta\,(d_0\,d_1\,d_2)^{2/3} \leqslant X^{1/6}/d. \end{cases} \tag{4.45}$$

4.6.3. Summing a multiplicative function on $\mathbb{N}_3^*$

To continue our study of the main term $\mathrm{MT}(X,d)$, as presented in (4.40), we have to give a precise asymptotic expansion for $\sum_{d_0} 2^{\omega(d_0)}$. Actually we will study the following more general problem which is obviously linked with the possible extension of Theorem 1.2 to any odd prime ℓ: let $\ell \geqslant 3$ be prime, $d \geqslant 1$ an integer and $x \geqslant 1$ be a real number. We consider the sum

$$\mathrm{K}(x;\ell,d) := \sum_{\substack{n\leqslant x,\, n\in\mathbb{N}_\ell^* \\ (n,d)=1}} (\ell-1)^{\omega(n)}.$$

Without loss of generality, we assume that

$$d \in \mathbb{N}_\ell^*. \tag{4.46}$$

For the statement of our result, we denote by $\chi_0, \chi_1, \dots, \chi_{\ell-2}$, the $\ell-1$ Dirichlet characters modulo ℓ, χ_0 being the principal character. There is no risk of confusion with the notation introduced by (3.2). Let α_ℓ be the infinite product

$$\alpha_\ell := \frac{\ell}{\ell+1}\prod_p\left\{\left(1+\frac{1}{p}+\frac{\chi_1(p)}{p}+\cdots+\frac{\chi_{\ell-2}(p)}{p}\right)\cdot\left(1-\frac{1}{p}\right)\right\}, \tag{4.47}$$

and let $\psi_\ell(d)$ be the multiplicative function

$$\psi_\ell(d) := \prod_{p\mid d}\left(1+\frac{\ell-1}{p}\right)^{-1}. \tag{4.48}$$

We will prove the following

Proposition 4.15. — *Let $\ell \geqslant 3$ be a fixed prime. There exists $\nu = \nu_\ell > 0$ such that, uniformly for $d \geqslant 1$ satisfying (4.46) and $x \geqslant 2$, one has the equality*

$$\mathrm{K}(x;\ell,d) = \alpha_\ell\,\psi_\ell(d)\,x + O\Big(\tau(d)^{\ell-1}x^{1-\nu}\Big).$$

Proof. — Consider the Dirichlet series

$$F(s)=F_{\ell,d}(s):=\sum_{\substack{n\in\mathbb{N}_\ell^*\\(n,d)=1}}\frac{(\ell-1)^{\omega(n)}}{n^s}=\sum_n\frac{a_n}{n^s},$$

by definition. This series is absolutely convergent in the half-plane $\{s:\sigma>1\}$. In this region, $F(s)$ has an expression as an Euler product

$$F(s)=\prod_{\substack{p\in\mathbb{P}_\ell^*\\p\nmid d}}\left(1+\frac{\ell-1}{p^s}\right).$$

For a prime $p\neq\ell$ we detect the condition $p\equiv 1 \bmod \ell$, by the sum

$$\frac{1}{\ell-1}\big(\chi_0(p)+\cdots+\chi_{\ell-2}(p)\big).$$

Thus $F(s)$ has the following expression

$$\begin{aligned}F(s)&=\prod_{p\nmid\ell d}\left(1+\frac{\chi_0(p)}{p^s}+\cdots+\frac{\chi_{\ell-2}(p)}{p^s}\right)\\&=\left(1+\frac{1}{\ell^s}\right)^{-1}\prod_{p\mid d}\left(1+\frac{\ell-1}{p^s}\right)^{-1}\\&\qquad\prod_p\left(1+\frac{1}{p^s}+\frac{\chi_1(p)}{p^s}+\cdots+\frac{\chi_{\ell-2}(p)}{p^s}\right).\end{aligned}\tag{4.49}$$

Recall the following Euler products for $\sigma>1$:

$$\zeta(s)^{-1}=\prod_p\left(1-\frac{1}{p^s}\right),$$

and

$$L(s,\chi_j)^{-1}=\prod_p\left(1-\frac{\chi_j(p)}{p^s}\right)\ (1\leqslant j\leqslant \ell-2).$$

Inserting these products into (4.49), we have the equality

$$F(s)=\zeta(s)\left[\left(1+\frac{1}{\ell^s}\right)^{-1}\cdot\prod_{p\mid d}\left(1+\frac{\ell-1}{p^s}\right)^{-1}\cdot L(s,\chi_1)\cdots L(s,\chi_{\ell-2})\right]G(s),\tag{4.50}$$

where $G(s)$ is defined by the Euler product

$$\begin{aligned}G(s):=\prod_p\Bigg\{&\left(1+\frac{1}{p^s}+\frac{\chi_1(p)}{p^s}+\cdots+\frac{\chi_{\ell-2}(p)}{p^s}\right)\cdot\left(1-\frac{1}{p^s}\right)\\&\cdot\left(1-\frac{\chi_1(p)}{p^s}\right)\cdots\left(1-\frac{\chi_{\ell-2}(p)}{p^s}\right)\Bigg\}.\end{aligned}$$

The Euler product $G(s)$ is absolutely convergent in the half–plane $\{s : \sigma > 1/2\}$. Returning to (4.50) we proved that the Dirichlet series $F(s)$ has a meromorphic continuation of the form

$$F(s) = \zeta(s)H(s),$$

where $H(s) = H_{\ell,d}(s)$ is holomorphic on the half plane

$$\Omega := \bigl\{s : \sigma > \log(\ell - 1)/\log(\ell + 1)\bigr\},$$

since every factor $(1 + (\ell - 1)/p^s)$ (for $p \mid d$) is different from zero in this region. On this half–plane, $F(s)$ has a unique pole at $s = 1$. This pole is induced by the singularity of ζ at $s = 1$. Hence this pole of F is simple with residue

$$\begin{aligned}\operatorname{Res}(F; s = 1) = H_{\ell,d}(1) = \psi_\ell(d) \cdot \frac{\ell}{\ell+1} \cdot \bigl[L(1,\chi_1)\cdots L(1,\chi_{\ell-2})\bigr] \\ \times \prod_p \Bigl\{\Bigl(1 + \frac{1}{p} + \frac{\chi_1(p)}{p} + \cdots + \frac{\chi_{\ell-2}(p)}{p}\Bigr) \\ \cdot \Bigl(1 - \frac{1}{p}\Bigr)\Bigl(1 - \frac{\chi_1(p)}{p}\Bigr)\cdots\Bigl(1 - \frac{\chi_{\ell-2}(p)}{p}\Bigr)\Bigr\},\end{aligned}$$

which equals

$$\operatorname{Res}(F; s = 1) = \alpha_\ell\, \psi_\ell(d).$$

The number α_ℓ is not zero as a consequence of the fact that $L(1, \chi_j) \neq 0$. We apply an effective version of Perron's formula (see for instance [29, Corollary 5.3, p. 140]) to obtain the equality

$$\begin{aligned}\mathrm{K}(x; \ell, d) = \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} F(s)\frac{x^s}{s}\,\mathrm{d}s \\ + O\Biggl(\sum_{\substack{x/2<n<2x\\ n\neq x}} |a_n| \min\Bigl(1, \frac{x}{T|x-n|}\Bigr)\Biggr) \\ + O\Bigl(\frac{4^\kappa + x^\kappa}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^\kappa}\Bigr) + O(x^\varepsilon).\end{aligned} \tag{4.51}$$

If we choose $\kappa = 1 + 2\varepsilon$, and $T = x^\vartheta$ $(\vartheta > 0)$, we have the equality

$$\mathrm{K}(x; \ell, d) = \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} F(s)\frac{x^s}{s}\,\mathrm{d}s + O(x^{1-\vartheta+\varepsilon})$$

by the inequality $|a_n| \ll n^\varepsilon$ and by separating the cases $|x - n| < x/T$ and $|x - n| \geqslant x/T$ in the first sum on the right–hand side of (4.51).

We transform the path of integration into a vertical segment $\sigma_0 + it$ with $\sigma_0 < 1$ and $|t| \leqslant T$ belonging to Ω and two horizontal segments belonging to the lines with equations $t = T$ and $t = -T$. On these segments, the function $G(s)$, defined in (4.50), is uniformly bounded and we also have

$$\left(1 + \frac{1}{\ell^s}\right)^{-1} \cdot \prod_{p \mid d} \left(1 + \frac{(\ell - 1)}{p^s}\right)^{-1} = O\big(\tau(d)^{\ell - 1}\big).$$

By classical bounds for the functions $L(s, \chi_j)$ on these segments, by an optimal choice of ϑ and σ_0, we complete the proof of Proposition 4.15. □

We apply Proposition 4.15 with the values

$$n \longleftarrow d_0, \quad \ell \longleftarrow 3, \quad d \longleftarrow dd_1d_2\Delta, \quad x \longleftarrow X^{1/4}d^{-3/2}d_1^{-1}d_2^{-1}\Delta^{-3/2}$$

to obtain the equality

$$\sum_{d_0} 2^{\omega(d_0)} = \alpha_3\, \psi_3\big(dd_1d_2\Delta\big)\, \frac{X^{1/4}}{d^{3/2}d_1d_2\Delta^{3/2}} + O\left(\tau^2(dd_1d_2\Delta)\left(\frac{X^{1/4}}{d^{3/2}d_1d_2\Delta^{3/2}}\right)^{1-\nu}\right). \tag{4.52}$$

Denote by $\mathcal{E}r(X, d, d_1, d_2, \Delta)$ the error term in the above formula. Since we have the inequalities $d \leqslant \mathcal{L}^{A_4}$ (see (4.12)), d_1, $d_2 \leqslant D_0$ and $\Delta \leqslant \mathcal{L}^{A_0}$ (see (4.45)), we see that the total contribution to $\mathrm{Heis}^\dagger(X)$ will be negligible, since we have (see (4.12), (4.21) and (4.40))

$$\sum_{d} \sum_{d_1} \sum_{d_2} \sum_{\Delta} 2^{\omega(d)} 2^{\omega(d_1)} 2^{\omega(d_2)} 6^{\omega(\Delta)} \mathcal{E}r(X, d, d_1, d_2, \Delta) = O\big(X^{1/4 - \delta}\big), \tag{4.53}$$

for some positive δ. This contribution is compatible with the error term that we claim in (4.11).

4.6.4. The final step

We insert the equality (4.52) in (4.40). By (4.12), (4.21), (4.42), (4.44), (4.53) and Proposition 4.13, we see that, in order to prove (4.11), it is sufficient to prove that the sum $\mathrm{Heis}^{\ddagger}(X)$ defined by

$$\mathrm{Heis}^{\ddagger}(X) := 2^{-2}3^{-3}\alpha_3 \sum_{d} \psi_3(d)\cdot \frac{2^{\omega(d)}}{d^{3/2}} \sum_{\Delta} \psi_3(\Delta)\cdot \frac{3^{\omega(\Delta)}}{\Delta^{3/2}} \sum_{f\in V^*(\Delta)} \left\{ \sum_{d_1} \psi_3(d_1) 2^{\omega(d_1)} \frac{\chi(f)(d_1)}{d_1} \left(\sum_{d_2} \psi_3(d_2) 2^{\omega(d_2)} \frac{\chi(2f)(d_2)}{d_2} \right) \right\}, \tag{4.54}$$

where the conditions of summation are successively

$$\begin{cases} d \leqslant \mathcal{L}^{A_4} \text{ and } d \in \mathbb{N}_3^*, \\ 1 < \Delta \leqslant \mathcal{L}^{A_0} \text{ and } d\Delta \in \mathbb{N}_3^*, \\ d_1 \leqslant D_0 \text{ and } (dd_1)\,\Delta \in \mathbb{N}_3^*, \\ d_2 \leqslant D_0 \text{ and } (dd_1d_2)\,\Delta \in \mathbb{N}_3^*, \end{cases} \tag{4.55}$$

satisfies the equality

$$\mathrm{Heis}^{\ddagger}(X) = C_{\mathrm{Heis}^*} + O(\mathcal{L}^{-1}). \tag{4.56}$$

Once again by the Siegel–Walfisz Theorem, we can drop the conditions $d_1, d_2 \leqslant D_0$ in (4.55) with an error in $O(\mathcal{L}^{-1})$ so that complete series over d_1 and d_2 appear. We write the double series

$$\left\{ \sum_{d_1} \left(\sum_{d_2} \right) \right\} = \sum_{d_1} \sum_{d_2} \psi_3(d_1d_2) 2^{\omega(d_1)} 2^{\omega(d_2)} \chi(f)(d_1)\chi(2f)(d_2)/d_1d_2,$$

where we keep as conditions of summation over d_1 and d_2 the last two lines of (4.55), with $D_0 = +\infty$. To deal with this double sum over multiplicative functions, we remark that for a prime $p \nmid d$, there are exactly three possibilities : $p \mid d_1$, $p \mid d_2$, $p \nmid d_1d_2$. The corresponding p–factor is respectively equal to

$$2\psi_3(p)\chi(f)(p)/p,\ 2\psi_3(p)\chi(2f)(p)/p \text{ and } 0.$$

By the equality $\psi_3(p) = p/(p+2)$ we finally obtain the equality

$$\left\{ \sum_{d_1} \left(\sum_{d_2} \right) \right\} = \prod_{p\mid d} \left(1 + 2\frac{\chi(f)(p) + \chi(2f)(p)}{p+2} \right)^{-1} \prod_{p\in\mathbb{P}_3^*} \left(1 + 2\frac{\chi(f)(p) + \chi(2f)(p)}{p+2} \right).$$

We insert this value in (4.54), and invert the summations. We extend the summation to all $d \in \mathbb{N}_3^*$ and all $\Delta \in \mathbb{N}_3^*$ with $\Delta > 1$ and $(d, \Delta) = 1$. With an acceptable error in $O(\mathcal{L}^{-1})$, we have the equality

$$\begin{aligned}\mathrm{Heis}^{\ddagger}(X) = 2^{-2}3^{-3}\alpha_3 \sum_{\substack{\Delta\in\mathbb{N}_3^* \\ \Delta>1}} \psi_3(\Delta) \\ \cdot \frac{3^{\omega(\Delta)}}{\Delta^{3/2}} \sum_{f\in V^*(\Delta)} \left\{ \prod_{p\in\mathbb{P}_3^*} \left(1+2\frac{\chi(f)(p)+\chi(2f)(p)}{p+2}\right)\right\} \\ \times \left\{ \prod_{\substack{p\in\mathbb{P}_3^* \\ p\nmid\Delta}} \left(1+\frac{2}{p^{1/2}\bigl(p+2(1+\chi(f)(p)+\chi(2f)(p))\bigr)}\right)\right\} + O(\mathcal{L}^{-1}).\end{aligned}$$

We recognize the constant C_{Heis^*} defined in (3.19). So we proved (4.56) and the proof of Proposition 3.9 is now complete.

4.7. Comments on the constant C_{Heis^*}

We will prove the following

Proposition 4.16. — *The constant C_{Heis^*} is a real positive number.*

Proof. — It follows from definition (3.19) that C_{Heis^*} is a real non-negative number, since it is a sum of non-negative real numbers. To prove that $C_{\mathrm{Heis}^*} > 0$, it is sufficient to prove that for at least one $\Delta \in \mathbb{N}_3^*$, $\Delta > 1$ and one $f \in V^*(\Delta)$, we have

$$\begin{aligned}\left\{ \prod_{p\in\mathbb{P}_3^*} \left(1+2\frac{\chi(f)(p)+\chi(2f)(p)}{p+2}\right)\right\} \\ \times \left\{ \prod_{\substack{p\in\mathbb{P}_3^* \\ p\nmid\Delta}} \left(1+\frac{2}{p^{1/2}\bigl(p+2(1+\chi(f)(p)+\chi(2f)(p))\bigr)}\right)\right\} > 0.\end{aligned}$$

By the inequality $1+\chi(f)(p)+\chi(2f)(p) \geqslant 0$, the second product is an absolutely convergent product, the limit of which is positive. We will prove the following lemma which implies Proposition 4.16

Lemma 4.17. — *We have for every $\Delta \in \mathbb{N}_3^*$ with $\Delta > 1$ and for every $f \in V^*(\Delta)$*

$$\prod_{p\in\mathbb{P}_3^*} \left(1+2\frac{\chi(f)(p)+\chi(2f)(p)}{p+2}\right) > 0.$$

To prove this lemma, we will approximate this infinite product, that we denote by $\mathcal{P}(f)$, by a product of the values at the point $s = 1$ of four Dirichlet L–series attached to characters of orders 3 or 6. Each factor of $\mathcal{P}(f)$ is a positive real number. If $p \neq 3$, we detect the congruence $p \equiv 1 \bmod 3$ by the sum $(1 + (p/3))/2$. We have

$$\begin{aligned}\mathcal{P}(f) &= \prod_{p \neq 3} \left(1 + \left(1 + \left(\frac{p}{3}\right)\right) \cdot \frac{\chi(f)(p) + \chi(2f)(p)}{p+2}\right) \\ &= \prod_{p \neq 3} \left(1 + \frac{\chi(f)(p)}{p}\right) \overline{\left(1 + \frac{\chi(f)(p)}{p}\right)} \\ &\quad \times \left(1 + \frac{(p/3)\chi(p)}{p}\right) \overline{\left(1 + \frac{(p/3)\chi(f)(p)}{p}\right)} \left(1 + \frac{\xi(p)}{p^2}\right),\end{aligned} \tag{4.57}$$

where $\xi(p)$ is some unspecified real number satisfying $1 + \xi(p)/p^2 > 0$ and $\xi(p) = O(1)$. We introduce the factor corresponding to the prime $p = 3$ and we continue the transformations of $\mathcal{P}(f)$ to arrive at the equality

$$\mathcal{P}(f) = \left|L\big(1, \chi(f)\big)\right|^2 \cdot \left|L\big(1, (\,\cdot\,/3)\chi(f)\big)\right|^2 \prod_{p \geqslant 2} \left(1 + \frac{\xi'(p)}{p^2}\right),$$

where $\xi'(p)$ is another unspecified real number satisfying $1 + \xi'(p)/p^2 > 0$ and $\xi'(p) = O(1)$. The inequalities $\left|L\big(1, \chi(f)\big)\right|^2 > 0$, $\left|L\big(1, (\,\cdot\,/3)\chi(f)\,\big)\right|^2 > 0$ and $\prod_{p \geqslant 2}\big(1 + \frac{\xi'(p)}{p^2}\big) > 0$ imply $\mathcal{P}(f) > 0$. This gives Lemma 4.17 and also Proposition 4.16. □

5. Study of the other sums

We now study the thirteen sums $\mathrm{Heis}^{(i,j)}(X)$ for $(i,j) \neq (3.20)$ by comparison with $\mathrm{Heis}^{(3.20)}(X) = \mathrm{Heis}^*(X)$, the asymptotic value of which is given in Proposition 3.9.

5.1. Easy observations between pairs of $\mathrm{Heis}^{(i,j)}(X)$

By inspecting the list of conditions (3.20), …, (3.33), we see that we pass from (3.20) to (3.27), from (3.21) to (3.28), …, from (3.26) to (3.33), by replacing the condition $3 \nmid d$ by $3 \mid d$. By studying Definition 3.4 and definition (3.9), we easily get

LEMMA 5.1. — *Let d be an element of $\mathbb{N}_3^*$ and let $f, f' \in V$. Then we have the equality*

$$D(3d, f, f') = \begin{cases} 3^{12} \cdot D(d, f, f') & \text{if } f(3) = f'(3) = 0, \\ D(d, f, f') & \text{otherwise.} \end{cases}$$

We now follow the influence of the conditions $3 \nmid d$ and $3 \mid d$ in the value of the sum $S(X, f, f')$ defined in (3.10) (recall that $\Delta(f)\Delta(f')$ is coprime with 3 and that $\mathrm{free}(3d, 3) = d$ for $d \in \mathbb{N}_3^*$). This gives the following

PROPOSITION 5.2. — *We have the equalities*

$$\mathrm{Heis}^{(3.20)}(3^{-12}X) = \mathrm{Heis}^{(3.27)}(X),$$

and

$$\mathrm{Heis}^{(3.21)}(X) = \mathrm{Heis}^{(3.28)}(X),\ \mathrm{Heis}^{(3.22)}(X) = \mathrm{Heis}^{(3.29)}(X),$$
$$\mathrm{Heis}^{(3.23)}(X) = \mathrm{Heis}^{(3.30)}(X),\ \mathrm{Heis}^{(3.24)}(X) = \mathrm{Heis}^{(3.31)}(X),$$
$$\mathrm{Heis}^{(3.25)}(X) = \mathrm{Heis}^{(3.32)}(X),\ \mathrm{Heis}^{(3.26)}(X) = \mathrm{Heis}^{(3.33)}(X).$$

The first part of this proposition, combined with Proposition 3.9, shows that

$$C^{(3.27)} = 3^{-3} H_0.$$

Moreover the second part of Proposition 5.2 reduces the proof of Proposition 3.10 to the study of six sums: $\mathrm{Heis}^{(3.21)}(X)$, $\mathrm{Heis}^{(3.22)}(X)$, $\mathrm{Heis}^{(3.23)}(X)$, $\mathrm{Heis}^{(3.24)}(X)$, $\mathrm{Heis}^{(3.25)}(X)$ and $\mathrm{Heis}^{(3.26)}(X)$.

5.2. Preparation of the functions f and f'

In the six remaining sums, we remark that the prime 3 belongs to $\mathrm{supp}\, f \cup \mathrm{supp}\, f'$. We generalize the decomposition (4.16) as follows

$$\begin{cases} f = \eta\, \mathbb{1}_{\{3\}} \oplus f_0 \oplus f_1, \\ f' = \eta'\, \mathbb{1}_{\{3\}} \oplus f'_0 \oplus f'_1, \end{cases} \tag{5.1}$$

- where $\eta, \eta' \in \{0, 1, 2\}$,
- where $\mathbb{1}_{\{3\}}$ is defined in Section 3.4,
- where the functions f_0, f'_0, f_1 and f'_1 do not contain 3 in their support,
- where we have $\mathrm{supp}\, f_0 = \mathrm{supp}\, f'_0$ $(:= \mathcal{E}_0)$,
- where the three sets $\mathcal{E}_1$ $(:= \mathrm{supp}\, f_1)$, $\mathcal{E}'_1$ $(:= \mathrm{supp}\, f'_1)$ and $\mathcal{E}_0$ are disjoint.

This decomposition is unique and the definitions of Δ_0, Δ_1 and Δ_1' (see (4.17)) remain valid. Observe that $\Delta_0\Delta_1\Delta_1'$ is never divisible by 3. We now state a generalization of Lemma 4.7, which can be proven in the same way as Lemma 4.7.

Lemma 5.3. — *Let $f, f' \in V$ decomposed as in* (5.1). *We then have the equalities*

$$\sum_{\substack{(z,z')\in\mathbb{F}_3^2 \\ f(r)z+f'(r)z'=0}} \big(\chi(zf+z'f')\big)(r) \\ = 1 + \begin{cases} \chi(f')(r)+\chi(2f')(r) & \text{if } r\in\mathcal{E}_1, \\ \chi(f)(r)+\chi(2f)(r) & \text{if } r\in\mathcal{E}_1', \\ \chi\big(f_0'(r)f+2f_0(r)f'\big)(r)+\chi\big(2f_0'(r)f+f_0(r)f'\big)(r) & \text{if } r\in\mathcal{E}_0. \end{cases}$$

As a consequence of this lemma, we deduce that the triple product appearing at the end of (4.19) now has the shape

$$\Pi(f,f') := \prod_{r\mid\Delta_0}\big\{1+\chi\big(f_0'(r)f+2f_0(r)f'\big)(r)+\chi\big(2f_0'(r)f+f_0(r)f'\big)(r)\big\} \\ \times \prod_{r\mid\Delta_1}\big\{1+\chi(f')(r)+\chi(2f')(r)\big\}\prod_{r\mid\Delta_1'}\big\{1+\chi(f)(r)+\chi(2f)(r)\big\}. \tag{5.2}$$

As in Section 4.4, we write this product in a schematic way as

$$\Pi(f,f') = \prod_{r\mid\Delta_0}\{\cdots\}\prod_{r\mid\Delta_1}\{\cdots\}\prod_{r\mid\Delta_1'}\{\cdots\}.$$

In the six sums, that we will study below, the main term will correspond to the contribution of the subproduct $\Pi^{\mathrm{mt}}(f,f')$ of $\Pi(f,f')$ defined by

$$\Pi^{\mathrm{mt}}(f,f') := \prod_{r\mid\Delta_1'}\{\cdots\}, \tag{5.3}$$

while the complementary product $\Pi^{\mathrm{et}}(f,f')$, defined by

$$\Pi^{\mathrm{et}}(f,f') := \Big(-1+\prod_{r\mid\Delta_0}\{\cdots\}\prod_{r\mid\Delta_1}\{\cdots\}\Big)\prod_{r\mid\Delta_1'}\{\cdots\},$$

is absorbed in the error term after summation over d, Δ_0, Δ_1, Δ_1', f, f'.

5.3. Study of $\mathrm{Heis}^{(3.21)}(X)$

In this case we have $\mu(f,f',d)=3^8$ which incites to compare $\mathrm{Heis}^{(3.21)}(X)$ with $\mathrm{Heis}^*(X/3^8)$. By (3.21), we need to impose three conditions on the functions f and f' that we decompose as in (5.1). The first condition is $f(3)=\eta=0$ and is equivalent to $f\in V^*$. The second condition $f'(3)\neq 0$ (i.e. $\eta'=1$ or 2) does not affect the treatment of the error terms $\Pi^{\mathrm{et}}(f,f')$. More precisely, we separate the cases $\eta'=1$ and $\eta'=2$. Then we follow the technique used in Section 4.5, which benefits, after some preparation, from the oscillation of a non principal Dirichlet character (with modulus less than some fixed power of $\mathcal{L}$). Then we obtain an analogue of Proposition 4.13.

To deal with the contribution of the main term $\Pi^{\mathrm{mt}}(f,f')$ defined in (5.3), we use the decomposition (5.1) of f'. This means that in (4.36), we have to introduce an extra summation over $\eta'\in\{1,2\}$. Gathering these remarks, taking care of the third condition $\chi(f)(3)=1$ in (3.21) and appealing to the definition (3.14) of H_1, we conclude that

Proposition 5.4. — *Uniformly for $X\geqslant 2$, one has the equality*

$$\mathrm{Heis}^{(3.21)}(X)=2^{-1}\cdot 3^{-5}\alpha_3\,H_1\,X^{1/4}+O(X^{1/4}\mathcal{L}^{-1}).$$

5.4. Study of $\mathrm{Heis}^{(3.22)}(X)$

We now have $\mu(f,f',d)=3^{12}$, which incites to compare $\mathrm{Heis}^{(3.22)}(X)$ with $\mathrm{Heis}^*(X/3^{12})$. Furthermore, as in Section 5.3 we have $\eta=0$ and $\eta'\in\{1,2\}$. Following the proof of Proposition 5.4, we get

$$\mathrm{Heis}^{(3.22)}(X)=2^{-1}\cdot 3^{-6}\alpha_3\,H_1'\,X^{1/4}+O(X^{1/4}\mathcal{L}^{-1})$$

with

$$H_1':=\sum_{\substack{\Delta\in\mathbb{N}_3^*\\ \Delta>1}}\lambda(\Delta)\,\psi_3(\Delta)$$

$$\cdot\frac{3^{\omega(\Delta)}}{\Delta^{3/2}}\sum_{\substack{f\in V^*(\Delta)\\ \chi(f)(3)=j,\,j^2}}\left\{\prod_{p\in\mathbb{P}_3^*}\left(1+2\frac{\chi(f)(p)+\chi(2f)(p)}{p+2}+\frac{2}{p^{1/2}(p+2)}\right)\right\}.$$

Applying (3.6) and returning to the definitions of H_0 and H_1 (see (3.13) and (3.14)), we trivially have the equality

$$H_1+H_1'=H_0.$$

So we proved the following

Proposition 5.5. — *Uniformly for $X \geqslant 2$, one has the equality*

$$\mathrm{Heis}^{(3.22)}(X) = 2^{-1} \cdot 3^{-6} \alpha_3 \, (H_0 - H_1) \, X^{1/4} + O(X^{1/4} \mathcal{L}^{-1}).$$

5.5. Study of $\mathrm{Heis}^{(3.23)}(X)$

In this case we have

$$\mu(f, f', d) = 3^{12}. \tag{5.4}$$

By the conditions (3.23), we know that in the decomposition (5.1), we have $\eta \in \{1, 2\}$ and $\eta' = 0$. Furthermore the functions f and f' are linearly independent if and only if $\Delta(f) \geqslant 1$ and $\Delta(f') > 1$. Since $\chi(f')(3) \neq 0$ (see (3.6)) we detect the condition $\chi(f')(3) = 1$ by the sum

$$\frac{1}{3}\Big(1 + \chi(f')(3) + \chi(2f')(3)\Big),$$

and this factor is easily integrated in the second product on the right–hand side of (5.2) by replacing the product over $r \mid \Delta_1$ by $r \mid 3\Delta_1$. This extra factor causes no new difficulty in the treatment of the error term: one follows the method explained in Section 5.3.

The treatment of the main term requires more care. Up to some error in $O(X^{1/4}\mathcal{L}^{-1})$ the main term has the shape (compare with (4.36))

$$2^{-2}3^{-4} \sum_{\substack{d \in \mathbb{N}_3^* \\ d \leqslant \mathcal{L}^{A_4}}} 2^{\omega(d)} \sum_{(\eta,\eta') \in \{(1,0),(2,0)\}} \sum_{\Delta_0,} \sum_{\Delta_1,} \sum_{\Delta_1'} \sum_{\substack{f_0, \, f_0' \in V^*(\Delta_0) \\ f_1 \in V^*(\Delta_1) \\ f_1' \in V^*(\Delta_1')}} \sum \sum \sum \Pi^{\mathrm{mt}}(f, f'),$$

where

- we use the notations of (5.1),
- the conditions of summations are given by (4.20), but with X replaced by $X/3^{12}$ (consequence of (5.4)).

When we expand the product over $r \mid \Delta_1'$ appearing in the definition (5.3) we have the following analogue of (4.39)

$$\begin{aligned}\Pi^{\mathrm{mt}}(f, f') &= \prod_{r \mid \Delta_1'} \{\cdots\} \\ &= \sum\sum\sum_{d_0 d_1 d_2 = \Delta_1'} \chi(f_0 + f_1 + \eta \mathbb{1}_{\{3\}})(d_1) \chi(2(f_0 + f_1 + \eta \mathbb{1}_{\{3\}}))(d_2)\end{aligned}$$

(we recall that $\eta \in \{1, 2\}$). We now write $f = f_0 + f_1$ to mimic the notations used in Section 4.6 and we follow the method given in that section. By the definition (3.15), we finally arrive at

Proposition 5.6. — *Uniformly for $X \geqslant 2$, one has the equality*

$$\mathrm{Heis}^{(3.23)}(X) = 2^{-2}3^{-7}\alpha_3\, H_2\, X^{1/4} + O(X^{1/4}\mathcal{L}^{-1}).$$

5.6. Study of $\mathrm{Heis}^{(3.24)}(X)$

We now have

$$\mu(f, f', d) = 3^{16}, \tag{5.5}$$

$\eta \in \{1, 2\}$ and $\eta' = 0$. By (3.6), the event $\chi(f')(3) \in \{j, j^2\}$ is complementary to the event $\chi(f')(3) = 1$ treated in Section 5.5. We detect the condition $\chi(f')(3) = j$ and the condition $\chi(f') = j^2$, by the respective indicators

$$\frac{1}{3}\Big(1+j^2\,\chi(f')(3)+j\,\chi(f')(3)\Big) \text{ and } \frac{1}{3}\Big(1+j\,\chi(f')(3)+j^2\,\chi(f')(3)\Big), \tag{5.6}$$

which can also be incorporated in the right–hand side of (5.2) by replacing the product over $r \mid \Delta_1$ by $r \mid 3\Delta_1$. We now follow the proof of Proposition 5.6. By taking into account the value of $\mu(f, f', d)$ given in (5.5) and the two cases listed in (5.6), we complete the proof of

Proposition 5.7. — *Uniformly for $X \geqslant 2$, one has the equality*

$$\mathrm{Heis}^{(3.24)}(X) = 2^{-1}\cdot 3^{-8}\alpha_3\, H_2\, X^{1/4} + O(X^{1/4}\mathcal{L}^{-1}).$$

5.7. Study of $\mathrm{Heis}^{(3.25)}(X)$

We now have

$$\mu(f, f', d) = 3^{12} \tag{5.7}$$

and $\eta, \eta' \in \{1, 2\}$. This condition implies that

$$\chi(f'(3)\cdot f + 2f(3)\cdot f')(3) \neq 0$$

by (3.5). We detect the equality $\chi(f'(3)\cdot f + 2f(3)\cdot f')(3) = 1$ by the sum

$$\frac{1}{3}\Big(1 + \chi(f'(3)\cdot f + 2f(3)\cdot f')(3) + \chi(2f'(3)\cdot f + f(3)\cdot f')(3)\Big),$$

which is easily inserted in the first product on the right–hand side of (5.2) by changing the product $\prod_{r\mid\Delta_0}$ to $\prod_{r\mid 3\Delta_0}$. The treatment of the error term is the same as for the archetype sum. For the main term we take into account the four values $(\eta, \eta') \in \{1, 2\}^2$ and the value of μ given in (5.7). Following the method leading to Proposition 5.7 we arrive at

Proposition 5.8. — *Uniformly for $X \geqslant 2$, one has the equality*

$$\mathrm{Heis}^{(3.25)}(X) = 2^{-1}\cdot 3^{-7}\alpha_3\, H_2\, X^{1/4} + O(X^{1/4}\mathcal{L}^{-1}).$$

5.8. Study of $\mathrm{Heis}^{(3.26)}(X)$

In our final case $\mu(f,f',d)$ satisfies (5.5). The proof mimics what was done for $\mathrm{Heis}^{(3.25)}(X)$ since we also have $\eta,\eta'\in\{1,2\}$. To detect the last condition of (3.26) we use the sums

$$\frac{1}{3}\Big(1+j^2\cdot\chi(f'(3)\cdot f+2f(3)\cdot f')(3)+j\cdot\chi(2f'(3)\cdot f+f(3)\cdot f')(3)\Big)$$

and

$$\frac{1}{3}\Big(1+j\cdot\chi(f'(3)\cdot f+2f(3)\cdot f')(3)+j^2\cdot\chi(2f'(3)\cdot f+f(3)\cdot f')(3)\Big)$$

that we insert in the first product on the right–hand side of (5.2) by changing the product $\prod_{r\mid\Delta_0}$ to $\prod_{r\mid 3\Delta_0}$. Finally, we conclude that

Proposition 5.9. — *Uniformly for $X\geqslant 2$, one has the equality*

$$\mathrm{Heis}^{(3.26)}(X)=3^{-8}\alpha_3\,H_2\,X^{1/4}+O(X^{1/4}\mathcal{L}^{-1}).$$

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Manuscrit reçu le 20 octobre 2021,
révisé le 26 mai 2023,
accepté le 29 novembre 2024.

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