Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Eva A. Gallardo-Gutiérrez,
Jonathan R. Partington & William T. Ross

Invariant subspaces of the Cesàro operator

Article à paraître, mis en ligne le 26 mars 2026, 37 p.

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
Article à paraître
Mis en ligne le 26 mars 2026.

INVARIANT SUBSPACES OF THE CESÀRO OPERATOR

by Eva A. GALLARDO-GUTIÉRREZ, Jonathan R. PARTINGTON & William T. ROSS (*)

ABSTRACT. — This paper explores various classes of invariant subspaces of the classical Cesàro operator C on the Hardy space H^2. We provide a characterization of the finite co-dimensional C-invariant subspaces, based on earlier work of the first two authors, and determine exactly which model spaces are C-invariant subspaces; using this, we describe the C-invariant subspaces contained in model spaces, which we show are all cyclic. Along the way, we re-examine an associated Hilbert space of analytic functions on the unit disk developed by Kriete and Trutt. We also make a connection between the adjoint of the Cesàro operator and certain composition operators on H^2 which have universal translates in the sense of Rota.

RÉSUMÉ. — Cet article explore différentes classes de sous-espaces invariants de l'opérateur classique de Cesàro C sur l'espace de Hardy H^2. Nous fournissons une caractérisation des sous-espaces C-invariants de codimension finie, basée sur des travaux antérieurs des deux premiers auteurs, et nous déterminons exactement quels espaces modèles sont des sous-espaces C-invariants ; en utilisant cela, nous décrivons les sous-espaces C-invariants contenus dans les espaces modèles, dont nous montrons qu'ils sont tous cycliques. En passant, nous réexaminons un espace de Hilbert associé de fonctions analytiques sur le disque unité développé par Kriete et Trutt. Nous faisons également un lien entre l'adjoint de l'opérateur de Cesàro et certains opérateurs de composition sur H^2 qui ont des translations universelles au sens de Rota.

1. Introduction

This paper examines the invariant subspaces of the *Cesàro operator*

$$(Cf)(z) := \frac{1}{z}\int_0^z \frac{f(\xi)}{1-\xi}\mathrm{d}\xi$$

Keywords: Cesàro operator, Hardy space, invariant subspace.
2020 *Mathematics Subject Classification:* 47A15, 47A55, 47B15.
(*) First author is partially supported by Plan Nacional I+D grant no. PID2022-137294NB-I00 (Spain), the Spanish Ministry of Science and Innovation, through the "Severo Ochoa Programme for Centres of Excellence in R&D" (CEX2019-000904-S) and from the Spanish National Research Council, through the "Ayuda extraordinaria a Centros de Excelencia Severo Ochoa" (20205CEX001).

on the *Hardy space* H^2 [21] of the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Recall that H^2 is the Hilbert space of analytic functions f on $\mathbb{D}$ with

$$\|f\|_{H^2} := \left(\sup_{0<r<1} \int_{\mathbb{T}} |f(r\xi)|^2 \mathrm{d}m(\xi) \right)^{\frac{1}{2}} < \infty.$$

In the above, m denotes Lebesgue measure on the unit circle $\mathbb{T} := \partial\mathbb{D}$, normalized so that $m(\mathbb{T}) = 1$. Note also that $\|f\|_{H^2}^2 = \sum_{n \geqslant 0} |a_n|^2$, where $\{a_n\}_{n \geqslant 0}$ is the sequence of Taylor coefficients of f.

Hardy's inequality [28] (see also [29]) says that C is a bounded operator on H^2. Moreover, a calculation shows that the matrix representation of C with respect to the orthonormal basis $\{z^n\}_{n \geqslant 0}$ for H^2, i.e., the infinite matrix whose entries are $\langle Cz^n, z^m \rangle_{H^2}$, $m, n \geqslant 0$, is the well known *Cesàro matrix*

$$C := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \cdots \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{1.1}$$

In this matrix setting, one thinks of C as acting on column vectors $\mathbf{a} = (a_0, a_1, a_2, \ldots)^T$ in ℓ^2 via matrix multiplication

$$C\mathbf{a} = \left(a_0, \frac{a_0 + a_1}{2}, \frac{a_0 + a_1 + a_2}{3}, \ldots \right)^T.$$

As was known for quite some time, the Cesàro matrix is the basis for an important summability method for Fourier series [39]. A more detailed analysis from a paper of Brown, Halmos, and Shields establishes the following basic operator theory facts about C and its adjoint C^* (which we record here for later use).

Proposition 1.1 (Brown–Halmos–Shields [7]). — *For the Cesàro operator C on H^2,*

(a) $\|I - C\| = 1$;
(b) $\|C\| = 2$;
(c) $\sigma(C) = \{z : |z - 1| \leqslant 1\}$;
(d) $\sigma_p(C) = \varnothing$;
(e) $\sigma_p(C^*) = \{z : |z - 1| < 1\}$.

Here, $\sigma(C)$ denotes the spectrum of C and $\sigma_p(C)$ the point spectrum (eigenvalues of C).

The Cesàro operator is hyponormal, namely, $C^*C - CC^* \geqslant 0$ [7, Theorem 3] (see also [17]). One of the gems in the study of the Cesàro operator, and to be explored further in this paper, is a theorem of Kriete and Trutt [30] which extends the above hyponormality result and says that $I - C$ is unitarily equivalent to the operator $M_z f = zf$ (multiplication by the independent variable z) on $\mathcal{H}^2(\mu)$, where μ is a certain positive finite Borel measure on $\overline{\mathbb{D}}$, the closure of $\mathbb{D}$, and $\mathcal{H}^2(\mu)$ denotes the closure of the analytic polynomials $\mathbb{C}[z]$ in $L^2(\mu)$. This shows that the Cesàro operator is subnormal (a normal operator restricted to one of its invariant subspaces).

A follow up paper of Kriete and Trutt [31] began a discussion of some of the complexities of the invariant subspaces of the Cesàro operator via a discussion of the M_z-invariant subspaces of $\mathcal{H}^2(\mu)$. We find this space quite interesting and this paper will continue this line of inquiry. A recent paper [24] of the first two authors of this current paper rekindled the discussion of the complexities of the Cesàro invariant subspaces from the vantage point of semigroups of composition operators and of translation operators $f(x) \mapsto f(x - t)$ on a certain weighted L^2 space of the real line $\mathbb{R}$.

The purpose of this paper is to continue this invariant subspace discussion. One particularly interesting class of invariant subspaces of the Cesàro operator will be the model spaces $(u_\alpha H^2)^\perp$, where $\alpha > 0$ and u_α is the atomic inner function

$$u_\alpha(z) = \exp\left(\alpha \frac{z+1}{z-1}\right).$$

Using semigroup ideas from [24], along with a Beurling–Lax theorem developed in that same paper, we will show in Theorem 6.1 that indeed the model space $(u_\alpha H^2)^\perp$, where $\alpha > 0$, is an invariant subspace of the Cesàro operator. Moreover, in Theorem 6.2 we will show that $(u_\alpha H^2)^\perp$, $\alpha > 0$, are the *only* model spaces which are Cesàro invariant. One can also arrive at these two results using a discussion in [2] involving the Volterra operator on H^2. In Proposition 8.4 we prove that the vector

$$\frac{1 - u_\alpha(z)}{1 + z}$$

belongs to $(u_\alpha H^2)^\perp$ and is a cyclic vector for the Cesàro operator when restricted to $(u_\alpha H^2)^\perp$. In Proposition 7.1 we prove that these model spaces $(u_\alpha H^2)^\perp$ correspond to a curious class of M_z-invariant subspaces of the associated Kriete–Trutt space $\mathcal{H}^2(\mu)$ that are different from the "standard"

Beurling-type ones consisting of the closure of uH^2 in $\mathcal{H}^2(\mu)$ (as explored in [31]), where u is an inner function.

An integral calculation will verify that the adjoint C^* of the Cesàro operator on H^2 is given by the integral formula

$$(C^* f)(z) = \frac{1}{1-z} \int_z^1 f(\xi) \mathrm{d}\xi. \tag{1.2}$$

A semigroup discussion in [24] characterized the finite dimensional C^*-invariant subspaces (equivalently the finite co-dimensional C-invariant subspaces) in the setting of translation invariant subspaces of a certain weighted L^2 space on $\mathbb{R}$. In Theorem 3.1 we use this discussion to recast this characterization in the H^2 setting and show that every finite dimensional C^*-invariant subspace is the span of finite unions of the functions

$$(1-z)^{\mu}(\log(1-z))^{j}, \quad 0 \leqslant j \leqslant k,$$

where $\Re\mu > -\frac{1}{2}$ and $k \in \mathbb{N}_0$ is fixed. We will prove this in Section 3, using some semigroup techniques from [24] (see Theorem 4.2 below for a different approach).

In Section 9 we establish a connection between the Cesàro operator and the concept of a universal operator in the sense of Rota. In Theorem 9.1 we show that although C^* is not universal, there is a bounded analytic function F on the disk $\{z : |z-1| < 1\}$ such that $F(C^*)$ is universal. We prove this by using an interesting result from [10] which says that certain linear translates $C_\varphi - \lambda I$ of a class of composition operators C_φ on H^2 are universal in the sense of Rota.

Through the results in this paper, we hope to make the case that the Kriete–Trutt space $\mathcal{H}^2(\mu)$ is an important Hilbert space of analytic functions on $\mathbb{D}$ that is very much worthy of further study. It has a rich variety of "nonstandard" M_z-invariant subspaces and the overall complexity of these invariant subspaces is yet unknown.

2. Semigroups of operators

For each $t \geqslant 0$ let

$$\varphi_t(z) := e^{-t}z + 1 - e^{-t}, \quad z \in \mathbb{D}.$$

These are analytic self maps of the open unit disk $\mathbb{D}$ and $\varphi_t(\mathbb{D})$ is an internally tangent disk at $\xi = 1$ (see Figure 2.1). This collection of maps $\{\varphi_t\}_{t \geqslant 0}$ defines a *holomorphic flow* in that

(i) $\varphi_0(z) = z$ for all $z \in \mathbb{D}$;

(ii) $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for all $s, t \geqslant 0$ and all $z \in \mathbb{D}$; and
(iii) for any fixed $s \geqslant 0$ and any $z \in \mathbb{D}$, $\lim_{t \to s} \varphi_t(z) = \varphi_s(z)$.

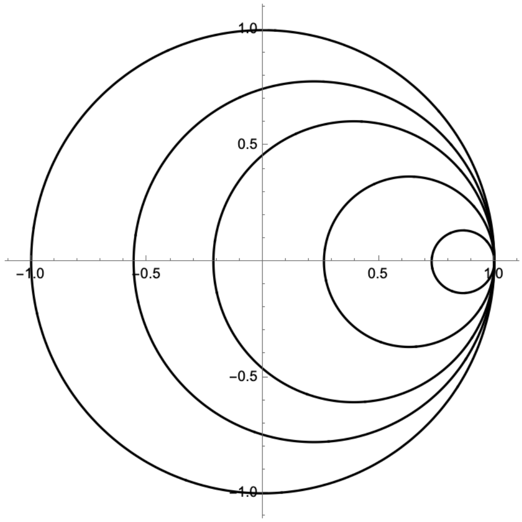

Figure 2.1. The circles $\varphi_t(\mathbb{T})$ for the maps $\varphi_t(z) = e^{-t}z + 1 - e^{-t}$. The circles get smaller as $t \to \infty$.

The Littlewood subordination theorem [35, p. 13] implies that the composition operator

$$C_{\varphi_t} : H^2 \longrightarrow H^2, \quad C_{\varphi_t} f = f \circ \varphi_t,$$

defines a bounded linear operator which satisfies $\|C_{\varphi_t}\| = e^{t/2}$ [15, Theorem 9.4] and $\sigma(C_{\varphi_t}) = \{z : |z| \leqslant e^{t/2}\}$ [15, Theorem 7.26]. In fact, one can quickly verify that the collection $\{C_{\varphi_t}\}_{t \geqslant 0}$ defines a strongly continuous (or C_0) semigroup of operators on H^2 in that

(i) $C_{\varphi_0} = I$;
(ii) $C_{\varphi_t} C_{\varphi_s} = C_{\varphi_{s+t}}$ for all $s, t \geqslant 0$; and
(iii) $\lim_{t \to 0^+} C_{\varphi_t} f = f$ for all $f \in H^2$.

Using an analysis of the generator and co-generator of the C_0 semigroup $\{C_{\varphi_t}\}_{t \geqslant 0}$, the authors in [24, Corollary 2.3] connect the above composition operators with the adjoint C^* of the Cesàro operator from (1.2) by means of the following formula:

$$(C^* f)(z) = -\int_0^\infty e^{-t} (C_{\varphi_t} f)(z) \mathrm{d}t.$$

The formula above has the following important consequence which is key to some of our results concerning the invariant subspaces of the Cesàro operator.

THEOREM 2.1. — *A closed subspace $\mathcal{M}$ of H^2 satisfies $C\mathcal{M} \subseteq \mathcal{M}$ if and only if $C_{\varphi_t}\mathcal{M}^{\perp} \subseteq \mathcal{M}^{\perp}$ for all $t \geqslant 0$.*

We pause to mention the papers [14, 37, 38], which also use semigroups to obtain information about the Cesàro operator.

In a way, Theorem 2.1 is a Beurling–Lax theorem for the Cesàro operator, reminiscent of the Beurling–Lax theorem for the invariant subspaces for the semigroup of shifts $f(x) \mapsto f(x-t)$, $t \in \mathbb{R}$, on $L^2(\mathbb{R})$ [33, p. 204].

There is another associated C_0 semigroup of shift operators from [24] which will play an important role in our discussion. Let $\mathbb{C}^+$ denote the right half plane

$$\mathbb{C}^+ := \{z \in \mathbb{C} : \Re z > 0\}$$

and let $H^2(\mathbb{C}^+)$ denote the *Hardy space of* $\mathbb{C}^+$ [27, 33]. These are the analytic functions F on $\mathbb{C}^+$ for which

$$\|F\|_{H^2(\mathbb{C}^+)} := \left(\sup_{0<x<\infty} \int_{-\infty}^{\infty} |F(x+iy)|^2 \mathrm{d}y\right)^{\frac{1}{2}} < \infty.$$

Remark 2.2. — As to not be overly pedantic, we will use the notation H^2 (and not $H^2(\mathbb{D})$) to denote the Hardy space of the disk $\mathbb{D}$.

By a version of the Paley–Wiener theorem [33, p. 203], every $F \in H^2(\mathbb{C}^+)$ can be realized as

$$F(s) = (\mathcal{L}f)(s) := \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(x) e^{-sx} \mathrm{d}x, \quad s \in \mathbb{C}^+, \tag{2.1}$$

(the normalized Laplace transform) where $f \in L^2(\mathbb{R}_+)$ (and $\mathbb{R}_+ = (0, \infty)$). Furthermore,

$$\|F\|^2_{H^2(\mathbb{C}^+)} = \int_0^{\infty} |f(x)|^2 \mathrm{d}x.$$

Conversely, $\mathcal{L}f \in H^2(\mathbb{C}^+)$ whenever $f \in L^2(\mathbb{R}_+)$. In other words, the normalized Laplace transform $\mathcal{L}$ is a unitary operator from $L^2(\mathbb{R}_+)$ onto $H^2(\mathbb{C}^+)$.

The function

$$\gamma(z) = \frac{1+z}{1-z}, \quad z \in \mathbb{D}, \tag{2.2}$$

is a conformal map from $\mathbb{D}$ onto $\mathbb{C}^+$ which induces the unitary operator $\mathcal{U} : H^2 \to H^2(\mathbb{C}^+)$ given by

$$\mathcal{U}g(s) = \frac{1}{\sqrt{\pi}(1+s)} g\left(\frac{s-1}{s+1}\right), \quad g \in H^2(\mathbb{D}) \tag{2.3}$$

with

$$\mathcal{U}^{-1}G(z) = \frac{2\sqrt{\pi}}{1-z} G\left(\frac{1+z}{1-z}\right), \quad G \in H^2(\mathbb{C}^+). \tag{2.4}$$

Moreover, as discussed in [11, Lemma 2.1], if ψ is an analytic self map of $\mathbb{C}^+$ such that $C_\psi F = F \circ \psi$ defines a bounded composition operator on $H^2(\mathbb{C}^+)$ (and this happens if and only if $\psi(\infty) = \infty$ and

$$\lim_{z\to\infty} \frac{z}{\psi(z)}$$

exists and is finite) [23, 32]), then with

$$\Phi(z) := \gamma^{-1} \circ \psi \circ \gamma,$$

which defines an analytic self map of $\mathbb{D}$, the linear transformation

$$(\mathcal{L}_\Phi f)(z) := \frac{1-\Phi(z)}{1-z} f(\Phi(z))$$

defines a weighted composition operator on H^2 and

$$\mathcal{U}^{-1} C_\psi \mathcal{U} = \mathcal{L}_\Phi.$$

Applying this discussion to the family of functions defined on $\mathbb{C}^+$ by

$$\psi_t(s) = e^t s + (e^t - 1), \quad t \geqslant 0,$$

which are analytic self maps of $\mathbb{C}^+$ (see Figure 2.2) that induce bounded composition operators C_{ψ_t} on $H^2(\mathbb{C}^+)$, a discussion in [10] shows that

$$\mathcal{U} C_{\varphi_t} \mathcal{U}^{-1} = e^t C_{\psi_t}, \quad \forall\, t \geqslant 0.$$

Now consider the semigroup $\{V_t\}_{t\geqslant 0}$ of operators on $L^2(\mathbb{R}_+)$ given by

$$(V_t g)(x) := e^{-t} e^{-(1-e^{-t})x} g(e^{-t}x), \quad t \geqslant 0.$$

Using the unitary operator

$$T : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}_+), \quad (Th)(x) := \frac{1}{\sqrt{x}} h(\log x), \quad x > 0, \tag{2.5}$$

(for which $T^{-1}g(y) = e^{y/2} g(e^y)$), one sees that

$$\big(T^{-1} V_t T h\big)(y) = e^{-t/2} e^{-(1-e^{-t})e^y} h(y-t), \quad y \in \mathbb{R}.$$

Finally, with the weight function

$$w(y) := e^{-2(e^y - 1)} \tag{2.6}$$

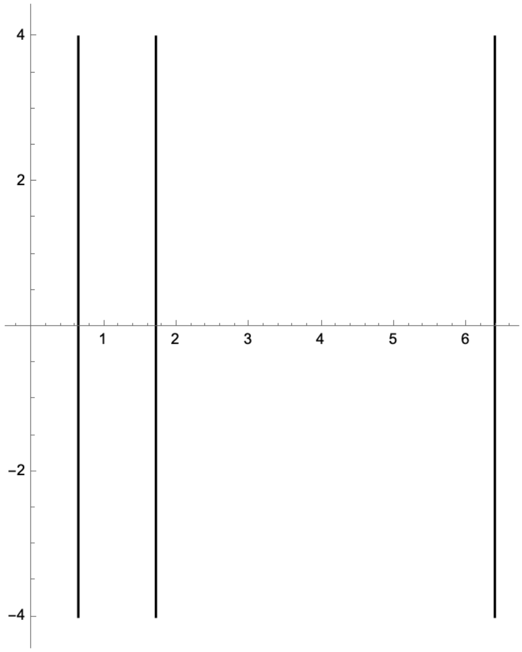

Figure 2.2. The lines $\psi_t(\{\Re s = 0\})$ for the maps $\psi_t(s) = e^t s + (e^t - 1)$. These lines drift off to infinity as $t \to \infty$.

(see Figure 2.3), one can show that the operator

$$W : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}, w(y)\mathrm{d}y), \quad (Wh)(y) := \frac{h(y)}{\sqrt{w(y)}}, \tag{2.7}$$

is unitary and a calculation from [24, Proposition 2.4] shows that

$$\big(W\sigma_t W^{-1} f\big)(y) = f(y-t), \quad f \in L^2(\mathbb{R}, w(y)\mathrm{d}y), \quad t > 0.$$

This gives us the following result from [24, Proposition 2.4].

Proposition 2.3. — *The semigroup $\{\sigma_t\}_{t\geqslant 0}$ on $L^2(\mathbb{R})$ given by*

$$(\sigma_t h)(y) := e^{-(1-e^{-t})e^y} h(y-t), \quad y \in \mathbb{R},$$

is unitarily equivalent, via

$$\mathfrak{F} := WT^{-1}\mathcal{L}^{-1}\mathcal{U} : H^2 \longrightarrow L^2(\mathbb{R}, w(y)dy),$$

to the semigroup

$$(S_t f)(y) := f(y-t)$$

acting on $L^2(\mathbb{R}, w(y)dy)$.

Putting this all together, yields the following result from [24, Theorem 2.5].

Theorem 2.4. — *A closed subspace $\mathcal{M}$ of H^2 is invariant for the Cesàro operator if and only if the closed subspace $\mathfrak{F}\mathcal{M}^{\perp}$ of $L^2(\mathbb{R}, w(y)\mathrm{d}y)$ satisfies $S_t(\mathfrak{F}\mathcal{M}^{\perp}) \subseteq \mathfrak{F}\mathcal{M}^{\perp}$ for all $t \geqslant 0$.*

For the unweighted $L^2(\mathbb{R})$ space, the closed subspaces $\mathcal{F}$ of $L^2(\mathbb{R})$ for which $S_t\mathcal{F} \subseteq \mathcal{F}$ for all $t \geqslant 0$ are described by the Beurling–Lax theorem (see [33, p. 204]). For a wide class of weights v on $\mathbb{R}$, the same result is true for the $\{S_t\}_{t\geqslant 0}$-invariant subspaces of $L^2(\mathbb{R}, v(y)\mathrm{d}y)$. However, for the weight w from (2.6), there is a different lattice of $\{S_t\}_{t\geqslant 0}$- invariant subspaces. Some restrictions on these subspaces were known by Domar [19, Eq. 8] and the paper [24] gives some specific examples. The complexity of the $\{S_t\}_{t\geqslant 0}$-invariant subspaces stems from the fact that the weight w is uniformly bounded above and below on every interval $(-\infty, a)$ but decreases rapidly on (a, ∞). Thus, the space $L^2((-\infty, a), w(y)\mathrm{d}y)$ can be seen as a renormed version of $L^2(-\infty, a)$ whereas $L^2((a, \infty), w(y)\mathrm{d}y)$ is a much larger space than $L^2(a, \infty)$.

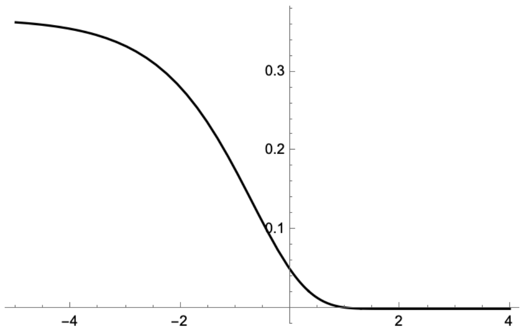

Figure 2.3. The weight function $w(y) = e^{-2(e^y-1)}$ from (2.6).

3. Finite co-dimensional invariant subspaces – first proof

An integral computation, using the fact that the adjoint C^* of the Cesàro operator C on H^2 given by

$$(C^* f)(z) = \frac{1}{1-z}\int_z^1 f(\xi)\mathrm{d}\xi, \tag{3.1}$$

shows that the linear span of the functions

$$(1-z)^{\mu}(\log(1-z))^{j}, \quad 0 \leqslant j \leqslant k,$$

where $\Re\mu > -\frac{1}{2}$ and $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ is fixed, is a finite dimensional invariant subspace for C^*. For example, the one dimensional space

$$\operatorname{span}(1-z)^{\mu}$$

is an eigenspace for C^*, with eigenvalue $(\mu+1)^{-1}$, while the spaces

$$\operatorname{span}\{(1-z)^{\mu},\ (1-z)^{\nu}\} \quad \text{and} \quad \operatorname{span}\{(1-z)^{\mu},\ (1-z)^{\mu}\log(1-z)\}$$

are two dimensional invariant subspaces for C^*. Theorem 3.1 below describes all of the finite dimensional C^*-invariant subspaces. There are several ways of proving this result, and we shall use the semigroup techniques developed in Section 2. An alternate path to this result runs through some techniques of Aleman [1] (see Theorem 4.2 below).

Theorem 3.1. — *The finite dimensional C^*-invariant subspaces of H^2 are spans of finite unions of sets of the form*

$$\left\{(1-z)^{\mu}(\log(1-z))^{j},\ 0 \leqslant j \leqslant k\right\},$$

where $\Re\mu > -\frac{1}{2}$ and $k \in \mathbb{N}_0$.

Proof. — Our starting point is the result in [24, Theorem 2.9] which says that the finite dimensional $\{S_t\}_{t\geqslant 0}$-invariant subspaces of $L^2(\mathbb{R}, w(y)\mathrm{d}y)$ (recall the translations S_t from Proposition 2.3) are all spanned by finite unions of sets of the form

$$\left\{e^{\lambda y}, ye^{\lambda y}, \dots y^{n}e^{\lambda y}\right\}$$

for some $n \in \mathbb{N}_0$ and $\Re\lambda > 0$. We now translate this into information about invariant subspaces of the Cesàro operator via our semigroup discussion from Section 2.

Under the unitary operator $W^{-1} : L^2(\mathbb{R}, w(y)\mathrm{d}y) \to L^2(\mathbb{R})$ from (2.7), the function $y^k e^{\lambda y}$ is sent to

$$y^k e^{\lambda y} e^{-(e^{y}-1)}.$$

Next, under the unitary operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R}_+)$ from (2.5), this last function is sent to

$$(\log x)^k x^{\lambda-1/2} e^{-(x-1)}.$$

We may multiply by a meaningless constant and instead consider the function

$$(\log x)^k x^{\lambda-1/2} e^{-x}.$$

The next step is to use the normalized Laplace transform $\mathcal{L} : L^2(\mathbb{R}_+) \to H^2(\mathbb{C}_+)$ from (2.1). Observe that

$$\int_0^\infty e^{-sx} x^{\lambda-1/2} e^{-x}\, \mathrm{d}x = (s+1)^{-\lambda-1/2}\Gamma\left(\lambda+\frac{1}{2}\right)$$

(where Γ is the standard gamma function) and, differentiating with respect to λ, we deduce that

$$\int_0^\infty e^{-sx}(\log x)^k x^{\lambda-1/2} e^{-x}\, \mathrm{d}x$$

is equal to

$$(s+1)^{-\lambda-1/2}\Big\{a_0 + a_1\log(s+1) + \ldots + a_k(\log(s+1))^k\Big\}, \tag{3.2}$$

for constants $a_0, \ldots, a_k$ (depending on λ but not s).

Finally, with the unitary operator $\mathcal{U}^{-1} : H^2(\mathbb{C}^+) \to H^2$ from (2.4), the functions from (3.2) are mapped to functions of the form

$$(1-z)^{\lambda-1/2}\Big\{b_0 + b_1\log(1-z) + \ldots + b_k(\log(1-z))^k\Big\}.$$

Note that $\Re(\lambda - \frac{1}{2}) > -\frac{1}{2}$ and we pick up the (only) eigenvectors $(1-z)^\mu$ with $\Re\mu > -\frac{1}{2}$.

Next, two-dimensional invariant subspaces are spanned either by $(1-z)^{\mu_1}$ and $(1-z)^{\mu_2}$ with $\mu_1 \neq \mu_2$, or by $(1-z)^\mu$ and $(1-z)^\mu(\log(1-z))$. Since $1-z \mapsto 1-e^{-t}z-1+e^{-t} = e^{-t}(1-z)$ under the composition semigroup, it is easy to see that this is an invariant subspace. And so on, for higher dimensions. □

Corollary 3.2. — *The finite co-dimensional closed invariant subspaces of the Cesàro operator on H^2 are orthogonal complements of spans of finite unions of sets of the form*

$$\big\{(1-z)^\mu(\log(1-z))^j,\ 0 \leqslant j \leqslant k\big\},$$

where $\Re\mu > -\frac{1}{2}$ and $k \in \mathbb{N}_0$.

4. The Kriete–Trutt transform

A result of Kriete and Trutt [30], used to prove that the Cesàro operator on H^2 is subnormal, involves the following transform. Observe that

$$\Re\left(\frac{w}{1-w}\right) > -\frac{1}{2}, \quad \forall \quad w \in \mathbb{D}$$

and so the family of functions

$$q_w(z) := (1-z)^{\frac{w}{1-w}}, \quad w \in \mathbb{D}, \tag{4.1}$$

belong to H^2. Since

$$q_{\frac{n}{n+1}}(z) = (1-z)^n, \quad \forall\, n \geqslant 0,$$

and the linear span of $(1-z)^n, n \geqslant 0$, is dense in H^2, we see that

$$\overline{\operatorname{span}}\{q_w : w \in \mathbb{D}\} = H^2. \tag{4.2}$$

The integral formula from (3.1) verifies the eigenvalue identity

$$C^* q_w = (1-w) q_w, \quad \forall\, w \in \mathbb{D}.$$

For $f \in H^2$, define the function Kf by

$$(Kf)(w) := \langle f, q_{\overline{w}} \rangle_{H^2}, \quad w \in \mathbb{D}. \tag{4.3}$$

Note that Kf is analytic on $\mathbb{D}$ and (4.2) enables us to create a Hilbert space $\mathcal{H}$ of analytic functions on $\mathbb{D}$, we will call the *Kriete–Trutt space*, as the range of K, i.e.,

$$\mathcal{H} := \{Kf : f \in H^2\}.$$

The Hilbert space structure on $\mathcal{H}$ comes from the range norm

$$\|Kf\|_{\mathcal{H}} := \|f\|_{H^2}.$$

With this norm, the *Kriete–Trutt transform* K defined in (4.3) is a unitary map from H^2 onto $\mathcal{H}$.

For $f \in H^2$ and $z \in \mathbb{D}$, use the eigenvalue identity from (3.1) to see that

$$\begin{aligned}
(KCf)(z) &= \langle Cf, q_{\overline{z}} \rangle_{H^2}\\
&= \langle f, C^* q_{\overline{z}} \rangle_{H^2}\\
&= \langle f, (1-\overline{z}) q_{\overline{z}} \rangle_{H^2}\\
&= (1-z)\langle f, q_{\overline{z}} \rangle_{H^2}\\
&= (1-z)(Kf)(z).
\end{aligned}$$

This shows that the multiplication operator $M_z f = zf$ on $\mathcal{H}$ is well defined, bounded, and $\|M_z\| = 1$ (since $\|I - C\| = 1$ from Proposition 1.1). We summarize the discussion above with the following.

PROPOSITION 4.1 (Kriete–Trutt [30]). — *The operator $K : H^2 \to \mathcal{H}$ is unitary and $K(I-C)K^* = M_z$, multiplication by z, on $\mathcal{H}$.*

A further analysis of Kriete and Trutt from [30] shows that $\mathcal{H}$ contains the analytic polynomials $\mathbb{C}[z]$ as a dense set and, more importantly (and quite difficult to prove), there is a positive finite Borel measure μ on $\overline{\mathbb{D}}$ such that

$$\int_{\mathbb{D}} |p|^2 \mathrm{d}\mu = \|p\|_{\mathcal{H}}^2, \quad \forall\, p \in \mathbb{C}[z]. \tag{4.4}$$

This measure μ is supported on the sequence of circles

$$\gamma_n = \left\{ z : \left| z - \frac{n}{n+1} \right| = \frac{1}{n+1} \right\} \quad \text{for } n \geqslant 0;$$

(see Figure 4.1).

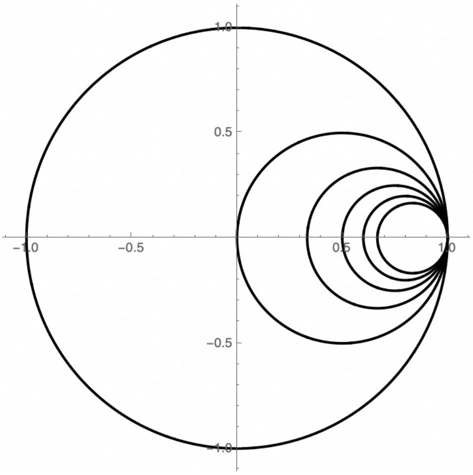

Figure 4.1. The circles (from left to right) $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$.

Furthermore, $\mu(\gamma_n) = 2^{-n-1}$ and $\mu|_{\gamma_n}$ is mutually absolutely continuous with respect to arc length measure on γ_n. This says that $\mathcal{H} = \mathcal{H}^2(\mu)$ (the closure of $\mathbb{C}[z]$ in $L^2(\mu)$) and thus, C is unitarily equivalent to $I - M_z$ on $\mathcal{H}^2(\mu)$. This last operator is subnormal (with M_{1-z} on $L^2(\mu)$ being a normal extension) and hence C is subnormal.

A special case of a theorem of Aleman [1] is the following theorem, which can be proved by an analogue of an argument from [4] (see also [13, Chapter 10]).

Theorem 4.2. — *A closed subspace $\mathcal{M}$ of $\mathcal{H}^2(\mu)$ is a finite co-dimensional M_z-invariant subspace if and only if there is a polynomial p all of whose zeros lie in $\mathbb{D}$ such that $\mathcal{M} = p\mathcal{H}^2(\mu)$.*

The following two propositions are relevant to such considerations (and for some examples in Section 5), so we give short self-contained proofs.

PROPOSITION 4.3. — *The space $\mathcal{H}^2(\mu)$ has the so-called* division property*: if $\lambda \in \mathbb{D}$ and $f \in \mathcal{H}^2(\mu)$ with $f(\lambda) = 0$, then $f(z)/(z-\lambda) \in \mathcal{H}^2(\mu)$.*

Proof. — The measure μ from $\mathcal{H}^2(\mu)$ is a finite positive Borel measure on $\overline{\mathbb{D}}$ with no point masses in $\mathbb{D}$ (recall the discussion at the end of Section 4). Moreover, using the fact that $\mathcal{H}^2(\mu)$ is a reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$ [30, Lemma 3], it follows that for each $\lambda \in \mathbb{D}$, there is a positive constant C_λ such that

$$|f(\lambda)|^2 \leqslant C_\lambda \int |f|^2 \mathrm{d}\mu, \quad \forall\, f \in \mathcal{H}^2(\mu). \tag{4.5}$$

Fix $\lambda \in \mathbb{D}$ and, for $f \in \mathcal{H}^2(\mu)$, consider the difference quotient

$$(Q_\lambda f)(z) := \frac{f(z) - f(\lambda)}{z - \lambda}, \quad z \in \mathbb{D}. \tag{4.6}$$

Note that $Q_\lambda f$ is analytic on $\mathbb{D}$ (since f is analytic on $\mathbb{D}$).

Let us first show that $Q_\lambda f \in L^2(\mu)$. Fix $r > 0$ small enough so that $\overline{D(\lambda, r)} \subseteq \mathbb{D}$ (where $D(\lambda, r) = \{z : |z - \lambda| < r\}$). Then

$$\int |Q_\lambda f|^2 \mathrm{d}\mu = \int_{\overline{D(\lambda,r)}} |Q_\lambda f|^2 \mathrm{d}\mu + \int_{\overline{\mathbb{D}} \setminus \overline{D(\lambda,r)}} |Q_\lambda f|^2 \mathrm{d}\mu.$$

The first integral above is bounded above by

$$\mu(\overline{D(\lambda, r)}) \cdot \sup_{z \in D(\lambda,r)} \left| \frac{f(z) - f(\lambda)}{z - \lambda} \right|^2$$

which is finite. The second integral is bounded above by

$$\frac{1}{r^2} \int_{\overline{\mathbb{D}}} |f - f(\lambda)|^2 \mathrm{d}\mu$$

which is also finite.

Next we show that $Q_\lambda f \in \mathcal{H}^2(\mu)$. Let $\{p_n\}_{n \geqslant 1}$ be a sequence polynomials which approximate $f \in \mathcal{H}^2(\mu)$ in $L^2(\mu)$ norm. From (4.5) one can see that $p_n \to f$ pointwise in $\mathbb{D}$. By [31, Lemma 2], there is a positive constant C_w such that

$$\|(z - \lambda) q\|_{\mathcal{H}^2(\mu)} \geqslant C_w \|q\|_{\mathcal{H}^2(\mu)}, \quad \forall\, q \in \mathbb{C}[z]. \tag{4.7}$$

Apply this inequality to the sequence of polynomials

$$q_n(z) = \frac{p_n(z) - p_n(\lambda)}{z - \lambda},$$

to see that the sequence $\{q_n\}_{n \geqslant 1}$ is norm bounded. Thus, a subsequence of it converges weakly to g and since weak convergence in $\mathcal{H}^2(\mu)$ implies pointwise convergence, we see that $g = Q_\lambda f \in \mathcal{H}^2(\mu)$. Moreover, by the closed graph theorem, the map $f \mapsto Q_\lambda f$ defines a bounded linear operator on $\mathcal{H}^2(\mu)$. □

See a discussion from [34] for more on the division property for general Banach spaces of analytic functions.

Finally, we make the following observation.

Proposition 4.4. — *The vector space $(z-\xi)\mathcal{H}^2(\mu)$ is dense in $\mathcal{H}^2(\mu)$ whenever $\xi \in \mathbb{T}$.*

Proof. — By [31, Theorem 4] the inclusion map $i : H^2 \to \mathcal{H}^2(\mu)$ is continuous. Thus, there is a positive constant C such that

$$\|p(z)(z-\xi) - 1\|_{H^2} \geqslant C\|p(z)(z-\xi) - 1\|_{\mathcal{H}^2(\mu)}, \quad \forall\, p \in \mathbb{C}[z]. \tag{4.8}$$

Since $(z-\xi)$ is an outer function, Beurling's theorem [21, Theorem 7.4] shows that the left hand side of the above can be made as small as desired by choosing an appropriate $p \in \mathbb{C}[z]$.

The previous paragraph shows that $1 \in \mathcal{S} := \overline{(z-\xi)\mathcal{H}^{\in}(\mu)}$ and the M_z-invariance of $\mathcal{S}$ will show that $\mathcal{S} = \mathcal{H}^2(\mu)$. □

5. Other interesting invariant subspaces

Although the finite co-dimensional M_z-invariant subspaces of the Kriete–Trutt space $\mathcal{H}^2(\mu)$ are thoroughly understood (Theorem 4.2), a general M_z-invariant subspace of $\mathcal{H}^2(\mu)$ can be quite complicated. Recall the family of H^2 functions q_w, $w \in \mathbb{D}$, from (4.1).

Proposition 5.1. — *If $A \subseteq \mathbb{D}$ is a zero sequence for $\mathcal{H}^2(\mu)$, meaning there is an $f \in \mathcal{H}^2(\mu) \setminus \{0\}$ which vanishes on A, then*

$$\overline{\operatorname{span}}\{q_{\overline{a}} : a \in A\}$$

is a nontrivial invariant subspace of the adjoint of the Cesàro operator.

Proof. — Notice that

$$\begin{aligned}(\overline{\operatorname{span}}\{q_{\overline{a}} : a \in A\})^{\perp} &= \big\{f \in H^2 : \langle f, q_{\overline{a}}\rangle = 0 \ \forall\, a \in A\big\}\\ &= \big\{f \in H^2 : (Kf)(a) = 0 \ \forall\, a \in A\big\}\\ &= \big\{g \in \mathcal{H}^2(\mu) : g|_A \equiv 0\big\} \neq 0.\end{aligned}$$

This shows that $\overline{\operatorname{span}}\{q_{\overline{a}} : a \in A\} \neq H^2$. From the fact that

$$C^* q_a = (1-\overline{a})q_{\overline{a}},$$

it follows that $\overline{\operatorname{span}}\{q_{\overline{a}} : a \in A\}$ is C^*-invariant. □

Corollary 5.2. — *If $A \subseteq \mathbb{D}$ is a zero sequence for $\mathcal{H}^2(\mu)$, then*

$$(\overline{\operatorname{span}}\{q_{\overline{a}} : a \in A\})^{\perp}$$

is a nontrivial invariant subspace of the Cesàro operator.

The zero sequences of $\mathcal{H}^2(\mu)$ can be quite complicated (see Section 10) and they do not have a complete characterization.

One can use the discussion in the previous section (Proposition 4.3 in particular) to create interesting "chains" of invariant subspaces for the Cesàro operator. Since the functions $(1-z)^{\alpha}$, $\Re\alpha > -\frac{1}{2}$, are eigenfunctions for C^*, we may construct invariant subspaces by joining together eigenspaces. The following result [31, Corollary 7] will be of use here.

Corollary 5.3. — *Let $\{\lambda_n\}_{n\geqslant 1}$ be a sequence of positive numbers satisfying $\lambda_{n+1} - \lambda_n \geqslant \delta > 0$ for all $n \in \mathbb{N}$ and consider the quantity*

$$b_r := \sum_{\lambda_n < r} \frac{1}{\lambda_n} - a \log r$$

for $r > 0$, where $a > 0$ is independent of r. If b_r is unbounded above as $r \to \infty$ for some $a \geqslant \frac{1}{2}$ then

$$\operatorname{span}\big\{(1-z)^{\lambda_n} : n \geqslant 1\big\}$$

is a dense subset of H^2. However, if b_r is bounded above as $r \to \infty$ for some $a < \frac{1}{2}$ then

$$\operatorname{span}\big\{(1-z)^{\lambda_n} : n \geqslant 1\big\}$$

is not a dense subset of H^2.

For example, for each $k \in \mathbb{N}$, the set

$$\Lambda_k := \{k + 3n : n \geqslant 1\}$$

yields a proper closed C^*-invariant subspace V_k of H^2 defined by

$$V_k := \overline{\operatorname{span}}\big\{(1-z)^{\lambda_n} : n \geqslant 1\big\}.$$

(take $\frac{1}{3} < a < \frac{1}{2}$ in the above corollary). However, the join (the closure of $V_k + V_\ell$) of any two such subspaces V_k and V_ℓ with $k \not\equiv \ell \bmod 3$ will be all of H^2 (take $\frac{1}{2} < a < \frac{2}{3}$).

We can actually say a bit more about these C^*-invariant subspaces V_k. Translating this into the Kriete–Trutt space $\mathcal{H}^2(\mu)$ determined by $f \mapsto Kf$, upon which C is unitarily equivalent to M_{1-z} (recall Proposition 4.1), the images (under K) of the orthogonal complements of the subspaces V_k above (which are clearly invariant under M_{1-z}) are subspaces of $\mathcal{H}^2(\mu)$ consisting of functions that vanish at $w = j/(j+1)$, $j \geqslant 1$.

Now the division property for $\mathcal{H}^2(\mu)$ (recall that the difference quotient $Q_\lambda f$ belongs to $\mathcal{H}^2(\mu)$ whenever $f \in \mathcal{H}^2(\mu)$ and $\lambda \in \mathbb{D}$, from (4.6)) can be used to show that, for example,

$$V_1 \supsetneq V_4 \supsetneq V_7 \supsetneq \dots$$

(or equivalently, that $(1-z) \notin V_4$, etc.). Indeed, by the division property, it is possible to construct functions in $\mathcal{H}^2(\mu)$ that vanish at $j/(j+1)$ for $j = 4, 7, 10, \dots$ but not for $j = 1$.

6. The role of model spaces

This section contains a very curious class of invariant subspaces for the Cesàro operator coming from the well studied theory of model spaces. For $\alpha > 0$ let u_α be the atomic inner function

$$u_\alpha(z) := \exp\left(\alpha \frac{z+1}{z-1}\right), \quad z \in \mathbb{D}. \tag{6.1}$$

Notice that u_α is a bounded analytic function on $\mathbb{D}$ and $|u_\alpha(e^{i\theta})| = 1$ for all $0 < \theta < 2\pi$. Thus, $u_\alpha H^2$ is a closed subspace of H^2. From here one can consider the *model space*

$$(u_\alpha H^2)^\perp := \{f \in H^2 : f \perp u_\alpha H^2\}.$$

We refer the reader to [25] for the basics of model spaces.

Let us denote $H^\infty(\mathbb{D})$ to be the set of bounded analytic functions on $\mathbb{D}$ and for $g \in H^\infty(\mathbb{D})$, define

$$\|g\|_\infty := \sup_{z\in\mathbb{D}} |g(z)|.$$

As noted by Professor Aleman, the following result can also be derived from a similar result on the Volterra operator, which can be found in [2]. We give a direct proof using semi-groups since they will be used to obtain further results below.

Theorem 6.1. — *For each $\alpha > 0$, the model space $(u_\alpha H^2)^\perp$ is a closed invariant subspace of the Cesàro operator.*

Proof. — By Theorem 2.1 it suffices to show that

$$C_{\varphi_t}(u_\alpha H^2) \subseteq u_\alpha H^2, \quad \forall\, t > 0.$$

This comes from a special case of a result of Cowen and Wahl [16, Lemma 5]. For the sake of completeness, we outline the proof here. Observe that

$$\Re\left(\frac{\varphi_t(z)+1}{\varphi_t(z)-1} - \frac{z+1}{z-1}\right) \leqslant 0, \quad \forall\, z \in \mathbb{D}. \tag{6.2}$$

This follows from the simple fact that $\varphi_t(\mathbb{T})$ is a circle inside $\overline{\mathbb{D}}$ that is internally tangent to $\mathbb{T}$ at $\{1\}$ and that the linear fractional transformation

$$z \longmapsto \frac{z+1}{z-1}$$

maps $\mathbb{D}$ onto the left half plane $-\mathbb{C}^+$ (One can also see (6.2) via Julia's Lemma). For any $g \in H^2$ we have

$$\begin{aligned}
(6.3)\quad & C_{\varphi_t}(u_\alpha(z)g(z)) \\
&= \exp\left(\alpha \frac{\varphi_t(z)+1}{\varphi_t(z)-1}\right) g(\varphi_t(z)) \\
&= \left(\exp\left(\alpha \frac{z+1}{z-1}\right) \exp\left(\alpha\left(\frac{\varphi_t(z)+1}{\varphi_t(z)-1} - \frac{z+1}{z-1}\right)\right)\right) g(\varphi_t(z)) \\
&= u_\alpha(z) \exp\left(\alpha\left(\frac{\varphi_t(z)+1}{\varphi_t(z)-1} - \frac{z+1}{z-1}\right)\right) g(\varphi_t(z)).
\end{aligned}$$

From (6.2) it follows that the function

$$\exp\left(\alpha\left(\frac{\varphi_t(z)+1}{\varphi_t(z)-1} - \frac{z+1}{z-1}\right)\right)$$

belongs to $H^\infty(\mathbb{D})$. Since $g(\varphi_t) \in H^2$, (6.3) shows that

$$C_{\varphi_t}(u_\alpha g) \in u_\alpha H^2, \quad \forall\, t \geqslant 0,$$

which completes the proof. □

As it turns out (see Theorem 6.2 below and a discussion of the Volterra operator in [2]), the family of model spaces

$$\left\{(u_\alpha H^2)^\perp : \alpha > 0\right\}$$

are the *only* model spaces that are invariant under the Cesàro operator. To prove this, we need a few reminders about inner functions.

Recall [21, Chapter 2] that an analytic function u on $\mathbb{D}$ is *inner* if $u \in H^\infty(\mathbb{D})$ and the radial boundary function

$$u(\xi) := \lim_{r \to 1^-} u(r\xi),$$

which exists for almost every $\xi \in \mathbb{T}$ by Fatou's theorem [21, Theorem 1.3], satisfies $|u(\xi)| = 1$ for almost every $\xi \in \mathbb{T}$. Any inner function can be factored (uniquely up to a unimodular constant factor) as $u = Bs_\mu$, where B is the Blaschke product

$$B(z) := z^N \prod_{n=1}^{\infty} \frac{\overline{z_n}}{|z_n|} \frac{z_n - z}{1 - \overline{z_n} z}, \quad z \in \mathbb{D},$$

with zeros at $z = 0$ and at $z_n \in \mathbb{D} \setminus \{0\}$ satisfying the Blaschke condition $\sum_{n\geqslant 1}(1 - |z_n|) < \infty$, and s_μ is the singular inner function

$$s_\mu(z) := \exp\left(-\int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \mathrm{d}\mu(\xi)\right), \quad z \in \mathbb{D},$$

with associated singular measure (with respect to Lebesgue measure m) on $\mathbb{T}$. Note that $u_\alpha = s_{\alpha\delta_1}$, where $\alpha > 0$. Both B and s_μ are inner. An important tool needed below is *boundary spectrum*

$$\Sigma(u) := \left\{\xi \in \mathbb{T} : \liminf_{z\in\mathbb{D}, z\to\xi} |u(z)| = 0\right\} \tag{6.4}$$

of u. It is known (see [25, Theorem 7.18 and Proposition 7.19]) that $\Sigma(u)$ is a closed subset of $\mathbb{T}$, u has an analytic continuation across $\mathbb{T} \setminus \Sigma(u)$, and $\Sigma(u)$ consists of the accumulation points of the zeros of B and the support of the singular measure μ.

There is a corresponding notion of inner function for $H^2(\mathbb{C}^+)$. A bounded analytic function Θ on $\mathbb{C}^+$ is inner if $|\Theta(iy)| = 1$ for almost every $y \in \mathbb{R}$. Note that Θ is inner for $\mathbb{C}^+$ if and only if $u = \Theta \circ \gamma$ is inner for $\mathbb{D}$, where γ is the conformal map from $\mathbb{D}$ to $\mathbb{C}^+$ from (2.2). Though there is an analogous factorization theorem for inner functions Θ on $\mathbb{C}^+$ as there was for inner functions u on $\mathbb{D}$ (see [27, Chapter 2] or [33, Chapter 6]), we will only need the fact that the function

$$\Theta(s) = e^{-as}, \quad s \in \mathbb{C}^+, \tag{6.5}$$

where $a > 0$, is inner on $\mathbb{C}^+$.

As we did for the atomic inner functions u_α, we can define the *model space* $(uH^2)^\perp$ for any inner function u. As noted earlier, the following result can be derived from a similar result on the Volterra operator on H^2 given in [2].

Theorem 6.2. — *If u is a nonconstant inner function and $(uH^2)^\perp$ is an invariant subspace for the Cesàro operator, $u = \xi u_\alpha$ for some $\alpha > 0$ and some constant $\xi \in \mathbb{T}$.*

Proof. — We use Theorem 2.1. For the closed subspace uH^2 to be invariant under each composition operator C_{φ_t}, $t \geqslant 0$, it is necessary for u to divide $u \circ \varphi_t$ for each t (meaning that $(u \circ \varphi_t)/u \in H^\infty(\mathbb{D})$). Thus, if u has a zero $z_0 \in \mathbb{D}$ then $u(\varphi_t(z_0)) = 0$ for every $t \geqslant 0$. As $t \to 0^+$, these zeros accumulate at $z = 0$ which is impossible by the isolated zeros theorem for analytic functions.

By the classical factorization theorem for inner functions mentioned earlier (and the argument in the previous paragraph), we may now suppose that u is a singular inner function s_μ and consider what happens if a point $\xi_0 \in \mathbb{T} \setminus \{1\}$ lies in its boundary spectrum $\Sigma(u)$ from (6.4). Let us examine what happens to $C_{\varphi_t}(uH^2)$ when $t = \log 2$. Assuming that $u(\varphi_{\log 2}) \in uH^2$, we see that $u(z)$ divides the bounded analytic function $u((1+z)/2)$, say $u((1+z)/2) = u(z)g(z)$ where $g \in H^\infty(\mathbb{D})$ (removing inner factors does not change the H^∞ norm).

Now observe that

$$\liminf_{z \in \mathbb{D}, z \to \xi_0} |u((1+z)/2)| = |u((1+\xi_0)/2)| \neq 0 \tag{6.6}$$

(since u is a singular inner function and thus has no zeros in $\mathbb{D}$). However, by the definition of boundary spectrum from (6.4),

$$\liminf_{z \in \mathbb{D}, z \to \xi_0} |u(z)| = 0. \tag{6.7}$$

Since $u((1+z)/2) = u(z)g(z)$, the facts from (6.6) and (6.7) contradict the fact that g must be bounded near ξ_0. □

From Theorem 6.1, the Cesàro operator has the model spaces $(u_\alpha H^2)^\perp$ as invariant subspaces for $\alpha > 0$ (and no other model spaces are invariant under C). Thus, its adjoint C^* has the invariant subspaces $u_\alpha H^2$ and no other Beurling subspaces uH^2 are invariant.

With the notation of [24], we can see that these Beurling spaces $u_\alpha H^2$ are equivalent to the standard $\{S_t\}_{t \geqslant 0}$-invariant subspaces

$$L^2((a,\infty), w(y)\mathrm{d}y) \quad \text{with} \quad a = \log \alpha.$$

For under

$$TW^{-1} : L^2(\mathbb{R}, w(y)\mathrm{d}y) \longrightarrow L^2(0,\infty)$$

(recall the operators T and W from Section 2) such a subspace is sent to $L^2(e^a,\infty)$. Now, under the normalized Laplace transform $\mathcal{L}$, it maps to the space $\Theta_a H^2(\mathbb{C}^+)$, where Θ_a is the inner function $\Theta_a(s) = \exp(-e^a s)$ from (6.5). Finally, under the unitary operator $\mathcal{U}^{-1} : H^2(\mathbb{C}^+) \to H^2$ from (2.4) we arrive at $u_\alpha H^2$ with $\alpha = e^a$.

The model spaces $(u_\alpha H^2)^\perp$, $\alpha > 0$, also have the following interesting property.

Theorem 6.3. — *If $\mathcal{M}$ is a closed invariant subspace for the Cesàro operator such that $(u_\alpha H^2)^\perp \subseteq \mathcal{M}$ for some $\alpha > 0$, then $\mathcal{M} = (u_\beta H^2)^\perp$ for some $\beta \geqslant \alpha$.*

Proof. — As we have just seen, the transformation $\mathfrak{F} = WT^{-1}\mathcal{L}^{-1}\mathcal{U}$ maps the subspace $u_\alpha H^2$ to $L^2((a,\infty), w(y)\mathrm{d}y)$, with $a = \log \alpha$. This has the remarkable property that the only translation-invariant subspaces contained in it are standard, that is, of the form $L^2((b,\infty), w(y)\mathrm{d}y)$, for $b \geqslant a$

(this follows from Domar's work and appears in the discussion preceding Theorem 2.9 in [24]). On applying $\mathfrak{F}^{-1}$ the result follows. ☐

We now examine the C-invariant subspaces which are *contained* in a fixed model space $(u_\alpha H^2)^\perp$.

THEOREM 6.4. — *Fix $\alpha > 0$. Up to isomorphism, the restriction of $I-C$ to the model space $(u_\alpha H^2)^\perp$ is a shift of multiplicity one and its invariant subspaces are parameterized by Beurling spaces of the form $\Theta H^2(\mathbb{C}^+)$ where $\Theta \in H^\infty(\mathbb{C}^+)$ is inner.*

Proof. — To examine the C-invariant subspaces contained in a model space $(u_\alpha H^2)^\perp$, let us return to the translation model on $L^2(\mathbb{R}, w(y)\mathrm{d}y)$ from Section 2 and note that the adjoint of the weighted translation $e^{t/2}S_t$ is given by

$$(T_t g)(y) = e^{t/2} g(y+t)\frac{w(y+t)}{w(y)}, \quad y \in \mathbb{R}.$$

We seek the common invariant subspaces for $\{T_t\}_{t\geqslant 0}$ that are contained in the space $L^2((-\infty, a), w(y)\,\mathrm{d}y)$.

Note that the weight w is uniformly bounded above and below on the interval $(-\infty, a)$ (see Figure 2.3), and thus the operator

$$V : L^2\big((-\infty, a), w(y)\,\mathrm{d}y\big) \longrightarrow L^2(-\infty, a), \quad (Vg)(y) := \frac{g(y)}{w(y)}$$

is an isomorphism (but not an isometry).

The question therefore reduces to parameterizing the common invariant subspaces of the semigroup $\{VT_tV^{-1}\}_{t\geqslant 0}$ on $L^2(-\infty, a)$, and we see that

$$\big(VT_tV^{-1}g\big)(y) = e^{t/2} g(y+t),$$

is a backward translation. Now we can use the Beurling–Lax theorem to find the lattice of common invariant subspaces, but there is an alternative method that gives us more information about the action of C on $(u_\alpha H^2)^\perp$.

Namely, we use a reversal operator

$$R : L^2(-\infty, a) \longrightarrow L^2(0, \infty), \quad (Rf)(y) = f(a-y)$$

and then, applying the normalized Laplace transform $\mathcal{L}$, we have the isomorphism

$$\mathcal{L}R : L^2(-\infty, a) \longrightarrow H^2(\mathbb{C}^+).$$

Under this, the weighted left translation $e^{t/2}S_t$ becomes

$$(\mathcal{L}R)VT_tV^{-1}(\mathcal{L}R)^{-1}G(s) = e^{t/2}e^{-st}G(s), \quad s \in \mathbb{C}^+.$$

Let A be the infinitesimal generator of the composition semigroup $\{C_{\varphi_t}\}_{t\geqslant 0}$, namely

$$(Af)(z) = (1-z)f'(z), \quad z \in \mathbb{D},$$

(see the pioneering work by Berkson and Porta [6], for instance). Then we may represent the resolvent $(I-A^*)^{-1}$ of the adjoint semigroup – which is the restriction of the Cesàro operator – equivalently as multiplication by the function

$$F(s) = \int_0^\infty e^{-t}e^{t/2}e^{-st}\,\mathrm{d}t = \frac{1}{s+\frac{1}{2}}.$$

It is more productive now to use $I-C$ in which case we have the operator of multiplication by the function

$$\frac{s-\frac{1}{2}}{s+\frac{1}{2}}.$$

That is, up to isomorphism, the restriction of $I-C$ is a shift of multiplicity one and its invariant subspaces are parameterized by Beurling spaces of the form $\Theta H^2(\mathbb{C}^+)$ where $\Theta \in H^\infty(\mathbb{C}^+)$ is inner. □

From Proposition 1.1 we know that

$$\sigma(I-C) = \overline{\mathbb{D}}.$$

The construction above yields the following.

Corollary 6.5. — *For $\alpha > 0$, the spectrum of $I-C$ restricted to the model space $(u_\alpha H^2)^\perp$, or any of its invariant subspaces, is $\overline{\mathbb{D}}$.*

Proof. — From the proof of Theorem 6.4, the operator $I-C$, when restricted to a closed C-invariant subspace of a model space $(u_\alpha H^2)^\perp$ will be the same as the spectrum of the operator of multiplication by

$$b(s) = \frac{s-\frac{1}{2}}{s+\frac{1}{2}}$$

on $\Theta H^2(\mathbb{C}^+)$ for an appropriate inner function Θ on $\mathbb{C}^+$. This is independent of Θ as we can solve

$$(b-\lambda)\Theta f = \Theta g$$

for $f, g \in H^2(\mathbb{C}^+)$ precisely when we can solve $(b-\lambda)f = g$. The spectrum of the analytic Toeplitz operator M_b on $H^2(\mathbb{C}^+)$ is easily seen to be $\overline{b(\mathbb{C}^+)} = \overline{\mathbb{D}}$ (this is a special case of what is sometimes known as Wintner's theorem [26, p. 365]). □

We see also that

$$\dim(\mathcal{M}/(I-C)\mathcal{M}) = 1$$

for all C-invariant subspaces $\mathcal{M} \subseteq (u_\alpha H^2)^\perp$, since

$$\dim\Big(\big(u_\alpha H^2\big)^\perp/(I-C)\big(u_\alpha H^2\big)^\perp\Big) = \dim\big(\Theta H^2(\mathbb{C}^+)/b\Theta H^2(\mathbb{C}^+)\big),$$

and this last quantity is equal to one. Indeed, it can be shown, more generally, that every invariant subspace for $I-C$ has index 1, using the main theorem of Aleman, Richter and Sundberg [3]. We discuss this further in Section 10.

Remark 6.6. — Suppose that $\mathcal{M}$ is a closed C-invariant subspace of H^2. For fixed $\alpha > 0$ consider

$$\overline{\mathcal{M} + (u_\alpha H^2)^\perp}.$$

Notice how this is a C-invariant subspace that contains $(u_\alpha H^2)^\perp$. Thus, by Theorem 6.3, either

$$\overline{\mathcal{M} + (u_\alpha H^2)^\perp} = \big(u_\beta H^2\big)^\perp$$

for some $\beta \geqslant \alpha$ or

$$\overline{\mathcal{M} + (u_\alpha H^2)^\perp} = H^2.$$

In the first case we have $\mathcal{M} \subseteq (u_\beta H^2)^\perp$ and, by Theorem 6.4, we can, in a sense, describe $\mathcal{M}$. Notice how the second possibility above says that $\mathcal{M}^\perp \cap u_\alpha H^2 = \{0\}$. An example of when this occurs, one can consider a finite co-dimensional invariant subspace $\mathcal{M}$ consisting of eigenfunctions of C. Then $\mathcal{M}^\perp \cap u_\alpha H^2 = \{0\}$ and so $\overline{\mathcal{M} + (u_\alpha H^2)^\perp} = H^2$.

7. Model spaces and the Kriete–Trutt space

Results from the previous section bring up further interesting things one can say about the lattice of invariant subspaces for M_z on the Kriete–Trutt space $\mathcal{H}^2(\mu)$. The Kriete–Trutt transform

$$K : H^2 \longrightarrow \mathcal{H}^2(\mu), \quad (Kf)(z) = \langle f, q_{\overline{w}} \rangle_{H^2},$$

where

$$q_w(z) = (1-z)^{\frac{w}{1-w}}$$

from (4.3) is a unitary operator which maps C-invariant subspaces of H^2 to M_z-invariant subspaces of $\mathcal{H}^2(\mu)$. What type of M_z-invariant subspace is $K((u_\alpha H^2)^\perp)$?

To begin to unpack this, we require a few further details about $\mathcal{H}^2(\mu)$ from [31]. The first [31, Theorem 4] is that the inclusion map

$$i : H^2 \longrightarrow \mathcal{H}^2(\mu), \quad i(f) = f,$$

is bounded. The second [31, Lemma 4] is that if u is an inner function whose boundary spectrum $\Sigma(u)$ from (6.4) does not contain the point 1, then the multiplication operator $f \mapsto uf$ is bounded below on $\mathcal{H}^2(\mu)$ and so $u\mathcal{H}^2(\mu)$ is a closed subspace of $\mathcal{H}^2(\mu)$. Third [31, Corollary 1], for inner functions A and B,

$$\overline{A\mathcal{H}^2(\mu)} = \overline{B\mathcal{H}^2(\mu)} \quad \text{if and only if} \quad u_s A = c u_t B$$

for some $|c| = 1$ and some $s, t \geqslant 0$, where u_t is the standard atomic inner function from (6.1). Thus, an inner function B is cyclic for M_z on $\mathcal{H}^2(\mu)$ if and only if $B = cu_t$. They also note that

$$K \frac{1}{1-(1-a)z} = e^t u_t, \quad t = -(\log a)/2, \quad 0 < a < 1.$$

This shows that

$$\frac{1}{1-(1-a)z}$$

is a *cyclic vector* for C, meaning

$$\overline{\operatorname{span}}\left\{ C^n \frac{1}{1-(1-a)z} : n \geqslant 0 \right\} = H^2.$$

Fourth [31, Theorem 5], for a closed M_z-invariant subspace $\mathcal{M}$ of $\mathcal{H}^2(\mu)$, we have that $\mathcal{M} = \overline{u\mathcal{H}^2(\mu)}$ for some inner function u if and only if $\mathcal{M} \cap i(H^2)$ is dense in $\mathcal{M}$.

One can describe $K((u_\alpha H^2)^\perp)$ as a "liminf" space as follows. A theorem of Tumarkin [20, Theorem 4.3.1] says that for a given $\alpha > 0$, there is a sequence $\{B_n\}_{n \geqslant 1}$ of finite Blaschke products with simple zeros such that

$$(u_\alpha H^2)^\perp = \underline{\lim} (B_n H^2)^\perp,$$

meaning that given any $f \in (u_\alpha H^2)^\perp$ there are $f_n \in (B_n H^2)^\perp$ such that $f_n \to f$ in H^2 norm. Note that

$$(B_n H^2)^\perp = \operatorname{span}\left\{ \frac{1}{1-\overline{\lambda} z} : \lambda \in B_n^{-1}(\{0\}) \right\}$$

is a convenient space of rational functions [25, Proposition 5.6].

If $\lambda \in \mathbb{D}$ and $k_\lambda(z) = (1-\overline{\lambda} z)^{-1}$, then [31, p. 203]

$$(K k_\lambda)(z) = (1-\lambda)^{\frac{z}{1-z}}.$$

When $0 < \lambda < 1$,

$$(Kk_\lambda)(z) = c_\lambda \exp\left(-a_\lambda \frac{1+z}{1-z}\right) \tag{7.1}$$

for some positive constants c_λ and a_λ. As mentioned earlier, k_λ is cyclic for C and so Kk_λ is cyclic for M_z on $\mathcal{H}^2(\mu)$.

The discussion above says that

$$K\big((u_\alpha H^2)^\perp\big) = \underline{\lim}\Big\{\operatorname{span}\big\{(1-\lambda)^{\frac{z}{1-z}} : \lambda \in B_n^{-1}(\{0\})\big\}\Big\},$$

meaning that given any $g \in K((u_\alpha H^2)^\perp)$, there are

$$g_n \in \operatorname{span}\big\{(1-\lambda)^{\frac{z}{1-z}} : \lambda \in B_n^{-1}(\{0\})\big\}$$

with $g_n \to g$ in $\mathcal{H}^2(\mu)$. But what does $K((u_\alpha H^2)^\perp)$ really contain? Is

$$K\big((u_\alpha H^2)^\perp\big) = \overline{u\mathcal{H}^2(\mu)}$$

for some inner function u? The answer is no.

Proposition 7.1. — *For any $\alpha > 0$, the M_z-invariant subspace $K((u_\alpha H^2)^\perp)$ of $\mathcal{H}^2(\mu)$ is not equal to $\overline{u\mathcal{H}^2(\mu)}$ for any inner function u.*

Proof. — First note that $\overline{uH^2} = \overline{u\mathcal{H}^2(\mu)}$, since H^2 is a dense subspace of $\mathcal{H}^2(\mu)$ and multiplication by u is a bounded operator on both H^2 and $\mathcal{H}^2(\mu)$.

Certainly

$$K\big((u_\alpha H^2)^\perp\big) \neq \overline{u_t H^2}$$

for any $t > 0$ since, by (7.1), u_t corresponds to a Cauchy kernel via K and these are cyclic vectors for C.

Is $K((u_\alpha H^2)^\perp) = \overline{uH^2}$ where the inner function u has a Blaschke factor? No, since otherwise, $u(\lambda_0) = 0$ for some $\lambda_0 \in \mathbb{D}$ and thus

$$0 = (Kf)(\lambda_0) = \Big\langle f(w), (1-w)^{\frac{\lambda_0}{1-\lambda_0}} \Big\rangle_{H^2}, \quad \forall\, f \in \big(u_\alpha H^2\big)^\perp.$$

This would mean that

$$(1-w)^{\frac{\lambda_0}{1-\lambda_0}} \in u_\alpha H^2.$$

But this cannot be since $(1-w)^{\lambda_0/(1-\lambda_0)}$ is outer (Beurling's theorem).

Thus, if $K((u_\alpha H^2)^\perp) = \overline{uH^2} = \overline{u\mathcal{H}^2(\mu)}$, then the inner function u must be a singular inner function s_ν. Now, unless the measure ν consists of a single atom, u has nonconstant inner factors J_1 and J_2 with associated singular measures with disjoint supports. In this case, it follows that the C-invariant subspaces containing $\overline{uH^2}$ are not totally ordered by inclusion, and so, by Theorem 6.2, we cannot have $K((u_\alpha H^2)^\perp) = \overline{uH^2}$.

Finally, suppose that $K((u_\alpha H^2)^\perp) = \overline{uH^2} = u\mathcal{H}^2(\mu)$ where u is a singular inner function corresponding to a point mass at $p \in \mathbb{T} \setminus \{1\}$. From our discussion above, note that $u\mathcal{H}^2(\mu)$ is closed in $\mathcal{H}^2(\mu)$. Then there is a function $g \in (u_\alpha H^2)^\perp$ such that $Kg = u$. Now, and this is interesting on its own, if $f \in H^2$, then

$$\left\langle S^* f, (1-w)^{\frac{\bar z}{1-\bar z}} \right\rangle_{H^2} = \left\langle f, (w-1+1)(1-w)^{\frac{\bar z}{1-\bar z}} \right\rangle_{H^2}.$$

So that $K(S^*f - f)(z)$ is $-Kf(\varphi(z))$ where

$$\frac{\varphi(z)}{1-\varphi(z)} = \frac{z}{1-z} + 1;$$

that is, $\varphi(z) = 1/(2-z)$. But now $S^*g - g$ belongs to $(u_\alpha H^2)^\perp$ and so $K(S^*g - g) \in u\mathcal{H}^2(\mu)$. However, $-Kg(\varphi(z))$ is analytic in an open neighborhood of p, which is a contradiction unless $S^*g - g = 0$, which is impossible. □

Remark 7.2. — The M_z-invariant subspaces $\mathcal{M}_\alpha := K((u_\alpha H^2)^\perp)$ are not the "standard" Beurling-type M_z-invariant subspaces $\overline{u\mathcal{H}^2(\mu)}$. Moreover, via Theorem 6.3, they also inherit the interesting property that if $\mathcal{M}$ is an M_z-invariant subspace that contains $\mathcal{M}_\alpha$ for some α, then $\mathcal{M} = \mathcal{M}_\beta$ for some $\beta \geqslant \alpha$.

8. Cyclic subspaces

Recall that $I - C$ is unitarily equivalent to multiplication by the function z on the Kriete–Trutt space $\mathcal{H}^2(\mu)$ and is bounded below (recall (4.7)). Thus, we may discuss the *index*

$$\dim(\mathcal{M}/z\mathcal{M})$$

of an invariant subspace $\mathcal{M}$. See [34] for some general facts about the index of an invariant subspace of M_z on Banach spaces of analytic functions. We start with some preliminary observations about C itself, which is not bounded below.

PROPOSITION 8.1. — *Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}^2(\mu)$ that is invariant under M_{1-z}, and hence M_z. Then $(1-z)\mathcal{M}$ is dense in $\mathcal{M}$.*

Proof. — Let

$$p_n(z) = 1 - \big(z + z^2 + \cdots + z^n\big)/n.$$

Since $p_n(1) = 0$, we have that $p_n(z) = (1-z)q_n(z)$ for some $q_n \in \mathbb{C}[z]$.

By Parseval's theorem, observe that $\|p_n h - h\|_{H^2} \to 0$ for every monomial $h(z) = z^k$ (hence, by (4.8), in $\mathcal{H}^2(\mu)$ norm as well). Use this, along with the facts that $|p_n(z)| \leqslant 2$ for all $z \in \mathbb{D}$ and the density of $\mathbb{C}[z]$ in $\mathcal{H}^2(\mu)$, to see that

$$(1-z)q_n h = p_n h \longrightarrow h$$

in $\mathcal{H}^2(\mu)$ norm for all $h \in \mathcal{H}^2(\mu)$. Now, if $h \in \mathcal{M}$ then

$$(1-z)q_n(z)h \in (1-z)\mathcal{M}$$

and the result follows. □

Corollary 8.2. — *Let $\mathcal{M} \subseteq H^2$ be a closed invariant subspace for the Cesàro operator C. Then $C\mathcal{M}$ is dense in $\mathcal{M}$.*

Recall that a vector $\mathbf{x}$ in a Hilbert space $\mathcal{H}$ is a *cyclic vector* for a bounded operator T on $\mathcal{H}$ if

$$\overline{\operatorname{span}}\{T^n \mathbf{x} : n \geqslant 0\} = \mathcal{H}.$$

From Theorem 6.4 we know that the operator $I - C$, when restricted to $(u_\alpha H^2)^\perp$, is similar to the operator M_b of multiplication by $b(s) = (s - 1/2)/(s + 1/2)$ on $H^2(\mathbb{C}^+)$. From this we can determine some cyclic vectors for C when restricted to the model spaces $(u_\alpha H^2)^\perp$, $\alpha > 0$.

Proposition 8.3. — *For every $\lambda \in \mathbb{C}^+$ the function*

$$\frac{1}{s+\lambda}$$

is a cyclic vector for M_b on $H^2(\mathbb{C}^+)$. Moreover, for every inner function $\Theta \in H^\infty(\mathbb{C}^+)$ the function

$$\frac{\Theta(s)}{s+\lambda}$$

is cyclic for the restriction of M_b to $\Theta H^2(\mathbb{C}^+)$.

Proof. — There is an orthonormal basis of $H^2(\mathbb{C}^+)$ given by

$$\frac{1}{\sqrt{2\pi}} \frac{\left(s - \frac{1}{2}\right)^n}{\left(s + \frac{1}{2}\right)^{n+1}}, \quad n \geqslant 0.$$

This can be seen most easily by transforming to $\mathbb{D}$ using the conformal mapping

$$s \longmapsto \frac{s - \frac{1}{2}}{s + \frac{1}{2}}.$$

With respect to this basis, M_b is a unilateral shift (so we see again that its spectrum is $\overline{\mathbb{D}}$). This tells us that $1/(s + 1/2)$ is cyclic, as a nonzero multiple of the first basis vector. Now a vector remains cyclic for M_b when multiplied by an invertible function in $H^\infty(\mathbb{C}^+)$, and in particular,

$$\frac{s+\frac{1}{2}}{s+\lambda}.$$

(We have many other choices but these ones lead to simpler expressions.) The result for $\Theta H^2(\mathbb{C}^+)$ follows easily. □

We now track the vectors from Proposition 8.3 back to the original H^2 setting. For $\alpha > 0$ write $a = \log\alpha$. There is a surjective isomorphism L_a between the spaces $L^2((-\infty, a), w(y)\,\mathrm{d}y)$ and $H^2(\mathbb{C}^+)$ given by

$$(L_a f)(s) = \int_{-\infty}^{a} f(y)e^{-s(a-y)}\,\mathrm{d}y.$$

Taking $f(y) = e^{\lambda y}$, we conclude from Proposition 8.3 that this is a cyclic vector for the operator $L_a^{-1}M_bL_a$ on $L^2(-\infty, a)$ corresponding to $I - C$ (and hence for the operator corresponding to C) as in Theorem 6.4. Note that, since we are restricting w to $(-\infty, a)$, on which it is bounded above and below (Figure 2.3), we can drop all references to w from now on.

We now recall the operator $(Th)(x) = x^{-1/2}h(\log x)$ from (2.5) to transfer this to $L^2(0, \alpha)$ and obtain the cyclic vectors

$$f_\lambda(x) = x^{-\frac{1}{2}}x^\lambda.$$

Let us calculate the Laplace transform of $f_\lambda \in L^2(0, \alpha)$, i.e.,

$$\int_0^\alpha x^{-1/2}x^\lambda e^{-sx}\,\mathrm{d}x$$

to obtain a cyclic vector for the unitarily equivalent form of C on the model space $(e^{-\alpha s}H^2(\mathbb{C}^+))^\perp$. The answer is too complicated to be usable for most values of λ but if we take $\lambda = \frac{1}{2}$ we obtain

$$\frac{1-e^{-\alpha s}}{s}.$$

Finally, using $\mathcal{U}^{-1}$ defined in (2.4), we obtain a cyclic vector for C on $(u_\alpha H^2)^\perp$, namely

$$g_\alpha(z) = \frac{1-u_\alpha(z)}{1+z}.$$

The singularity at -1 for g_α is removable. Also note that g_α belongs to $(u_\alpha H^2)^\perp$ since if $h = (z+1)k \in (z+1)H^2$ then

$$\langle g_\alpha, u_\alpha h\rangle_{H^2} = \langle 1-u_\alpha, u_\alpha zk\rangle_{H^2} = 0,$$

and this is true for a dense set of h hence for all $h \in H^2$. We summarize this discussion with the following.

Proposition 8.4. — *For $\alpha > 0$, the vector*

$$\frac{1 - u_\alpha(z)}{1 + z}$$

belongs to $(u_\alpha H^2)^\perp$ and is a cyclic vector for the restriction of C to $(u_\alpha H^2)^\perp$.

Even though $C|_{\mathcal{M}}$, where $\mathcal{M}$ is a closed C-invariant subspace contained in $(u_\alpha H^2)^\perp$, is cyclic, it seems that finding cyclic vectors cannot be done explicitly, except perhaps for some very simple inner functions Θ in Proposition 8.3.

For $w \in \mathbb{D}$ and

$$\lambda = \frac{\overline{w}}{1 - \overline{w}} \in \left\{ z \in \mathbb{C} : \operatorname{Re} z > -\frac{1}{2} \right\},$$

we want an expression for

$$\left\langle g_\alpha, (1 - z)^\lambda \right\rangle_{H^2},$$

the cyclic vector for M_z when restricted to $K((u_\alpha H^2)^\perp)$. Since $\mathcal{U}$ from (2.3) is a unitary operator, and thus preserves inner products, we may do the calculation in $H^2(\mathbb{C}^+)$. Thus, to within irrelevant constants, we obtain

$$\left\langle \frac{1 - e^{-\alpha s}}{s}, \frac{2^\lambda}{(s+1)^{\lambda+1}} \right\rangle_{H^2(\mathbb{C}^+)}.$$

The first term in the inner product is the Laplace transform of the characteristic function of $(0, \alpha)$, as we have seen already, while the second is the Laplace transform of the function

$$\frac{2^\lambda e^{-t} t^\lambda}{\Gamma(\lambda + 1)},$$

where Γ is the standard gamma function.

Thus, since the normalized Laplace transform is unitary, we obtain the Kriete–Trutt cyclic vectors given by the function

$$U_\alpha(w) := \frac{2^{w/(1-w)}}{\Gamma(1/(1-w))} \int_0^\alpha e^{-t} t^{w/(1-w)} \, \mathrm{d}t, \quad w \in \mathbb{D}.$$

This is summarized with the following.

Proposition 8.5. — *For $\alpha > 0$, the function U_α is a cyclic vector for M_z when restricted to $K((u_\alpha H^2)^\perp)$.*

The function U_α does not have a usable formula. However, we can say a few things.

Proposition 8.6. — *The function U_α has no zeros on $\mathbb{D}$.*

Proof. — This will follow from the general fact that if $f \in (u_\alpha H^2)^\perp$ and

$$\overline{\operatorname{span}}\{C^n f : n \geqslant 0\} = \left(u_\alpha H^2\right)^\perp,$$

then Kf, which will be a generator for the M_z-invariant subspace $K((u_\alpha H^2)^\perp)$, has no zeros in $\mathbb{D}$.

Indeed, suppose $(Kf)(\lambda) = 0$ for some $\lambda \in \mathbb{D}$. Since $KCf(z) = (1-z)Kf(z)$ (Proposition 4.1), every function in

$$\overline{\operatorname{span}}\{(1-z)^n Kf(z) : n \geqslant 0\}$$

would have a zero at λ (recall (4.5)). The previous identity would say that every function in

$$\overline{\operatorname{span}}\{KC^n f : n \geqslant 0\}$$

vanishes at λ. However, f is a cyclic vector for $C|_{(u_\alpha H^2)^\perp}$ and so every function from $K((u_\alpha H^2)^\perp)$ would vanish at λ. It follows that

$$\left\langle f, (1-w)^{\frac{\bar\lambda}{1-\bar\lambda}} \right\rangle_{H^2} = 0, \quad \forall\, f \in \left(u_\alpha H^2\right)^\perp$$

which, in turn, implies that

$$(1-z)^{\frac{\bar\lambda}{1-\bar\lambda}} \in u_\alpha H^2.$$

This last statement is impossible since $(1-z)^{\frac{\bar\lambda}{1-\bar\lambda}}$ is an outer function and this can not belong to any Beurling subspace $u_\alpha H^2$. □

Proposition 8.7. — *The function U_α does not belong to H^2.*

Proof. — From Proposition 7.1, we know that $K((u_\alpha H^2)^\perp)$ is never equal to $\overline{u\mathcal{H}^2(\mu)}$ for any inner function u. If $U_\alpha \in H^2$, then, since $\{pU_\alpha : p \in \mathbb{C}[z]\}$ is dense in $K((u_\alpha H^2)^\perp)$ (recall Proposition 8.5), then [31, Theorem 5] (mentioned above) would say that $K((u_\alpha H^2)^\perp) = \overline{u\mathcal{H}^2(\mu)}$, which we know is not the case. □

9. A connection to universal operators

A bounded operator U on a Hilbert space $\mathcal{H}$ is *universal* for $\mathcal{H}$ if given any bounded operator T on $\mathcal{H}$, there exists a constant $a \neq 0$ and a closed invariant subspace $\mathcal{M}$ for U such that $U|_{\mathcal{M}}$ is similar to aT. Universal operators thus have an extraordinarily rich class of invariant subspaces. Though universal operators might not seem to exist at all, a theorem of Caradus [9] (see also [12, Chapter 8]) shows they exist in abundance. Indeed, if $\mathcal{H}$ is an infinite dimensional separable Hilbert space and U is a bounded operator on $\mathcal{H}$ such that $\ker(U)$ is infinite dimensional and U is surjective, then U is universal.

For example, if u is an inner function that is not a finite Blaschke product, then the co-analytic Toeplitz operator $T_{\overline{u}}$ on H^2 satisfies Caradus' criterion and is thus universal [26, Proposition 16.7.1]. Although neither C nor C^* is universal (because they are injective), we do have the following.

Theorem 9.1. — *There exists a bounded analytic function F on the disk $D(1,1)$ such that $F(C^*)$ is universal.*

Note that C^* has an H^∞ functional calculus (see the remarks in Section 10) and so the operator $F(C^*)$ makes sense. Our proof of Theorem 9.1 uses a universality result for composition operators from [10], an analysis from [18], and a commutant result from [36].

The following theorem is from Carmo and Noor [10, Theorem 4.3].

Theorem 9.2. — *Let φ be a hyperbolic non-automorphism of $\mathbb{D}$ with a fixed point $\zeta \in \mathbb{T}$ and the other outside the closed unit disk $\overline{\mathbb{D}}$, possibly at ∞. If $a := \varphi'(\zeta) \in (0,1)$, then for each λ with $0 < |\lambda| < a^{-1/2}$ the operator $C_\varphi - \lambda I$ is universal on the Hardy space H^2.*

In our case, each of the symbols in the semigroup $\varphi_t(z) = e^{-t}z + 1 - e^{-t}, t > 0$, from Section 2 induces a composition operator with a universal translate. Here $a_t = \varphi_t'(1) = e^{-t}$ and the fixed points are $1 \in \mathbb{T}$ and at ∞. Clearly, the lattice of invariant subspaces of $C_{\varphi_t} - \lambda I$ and C_{φ_t} are the same.

Proof of Theorem 9.1. — To help with the typesetting, fix $t > 0$ and let $\alpha = e^{-t}$. The composition operator induced by φ_t becomes $C_{\alpha z + (1-\alpha)}$ and, with respect to the usual orthonormal basis $\{z^n\}_{n \geqslant 0}$ for H^2, has the matrix representation

$$\begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \cdots \\ 0 & (1-\alpha) & 2\alpha(1-\alpha) & 3\alpha^2(1-\alpha) & \cdots \\ 0 & 0 & (1-\alpha)^2 & 3\alpha(1-\alpha)^2 & \cdots \\ 0 & 0 & 0 & (1-\alpha)^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

See a paper of Deddens [18, p. 862] for the details of this. Note that C^* has the matrix representation

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{5} & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

One can check, as Deddens did, that the two operators above commute. By Theorem 9.2 there exists a constant β_α for which

$$C_{\alpha z+(1-\alpha)} - \beta_\alpha I$$

is universal. Moreover, this operator also commutes with C^* and has the matrix representation

$$\begin{bmatrix} 1-\beta_\alpha & \alpha & \alpha^2 & \alpha^3 & \cdots \\ 0 & (1-\alpha)-\beta_\alpha & 2\alpha(1-\alpha) & 3\alpha^2(1-\alpha) & \cdots \\ 0 & 0 & (1-\alpha)^2-\beta_\alpha & 3\alpha(1-\alpha)^2 & \cdots \\ 0 & 0 & 0 & (1-\alpha)^3-\beta_\alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

By a result of Shields and Wallen [36] (which describes the commutant of C – and thus C^*), there is a bounded analytic function F on $D(1,1)$ such that

$$F(C^*) = C_{\alpha z+(1-\alpha)} - \beta_\alpha I.$$

This says that $F(C^*)$ is universal. □

By a trick from [18, p. 863], we can actually compute F. Note that

$$F(C^*) = \begin{bmatrix} F(1) & * & * & * & * & \cdots \\ 0 & F(\frac{1}{2}) & * & * & * & \cdots \\ 0 & 0 & F(\frac{1}{3}) & * & * & \cdots \\ 0 & 0 & 0 & F(\frac{1}{4}) & * & \cdots \\ 0 & 0 & 0 & 0 & F(\frac{1}{5}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where the $*$ entries in the matrix above are not important. Comparing the diagonal entries of the matrix representations of $C_{\alpha z+(1-\alpha)} - \beta_\alpha I$ and

$F(C^*)$, we see that

$$F\left(\frac{1}{n}\right) = (1-\alpha)^{n-1} - \beta_\alpha, \quad \forall\, n \geqslant 1,$$

and thus, since $\{\frac{1}{n}\}_{n\geqslant 1}$ is not a Blaschke sequence for the disk $D(1,1)$, we see that

$$F(z) = (1-\alpha)^{1/z-1} - \beta_\alpha, \quad z \in D(1,1).$$

Remark 9.3. — The lattice of invariant subspaces for C^* is strictly contained in the corresponding lattice for $F(C^*)$. For the function F is not injective on $\sigma_p(C^*) = D(1,1)$ (Proposition 1.1), and so there exist distinct eigenvalues λ, μ for C^*, with corresponding eigenvectors $g, h \in H^2$, such that $F(\lambda) = F(\mu)$. Now the one-dimensional space $\mathcal{E}$ spanned by $g+h$ is not invariant under C^*, but $F(C^*)(g+h) = F(\lambda)g + F(\mu)h$ and so $\mathcal{E}$ is an invariant subspace for $F(C^*)$.

10. Some final remarks

So how complicated is the invariant subspace structure for the Cesàro operator? Since the invariant subspaces for the Cesàro operator are in one-to-one (and order preserving) correspondence with the invariant subspaces for the multiplication operator M_z on the Kriete–Trutt space $\mathcal{H}^2(\mu)$, the complexity of the invariant subspaces of the Cesàro operator is reflected in the complexity of the M_z-invariant subspaces of $\mathcal{H}^2(\mu)$.

So what kind of space is $\mathcal{H}^2(\mu)$? In some ways it is more like the Hardy space H^2 than the Bergman space A^2 of analytic functions on $\mathbb{D}$ which are square integrable with respect to area measure [22], since, by the main theorem of [3], $\dim(\mathcal{M}/z\mathcal{M}) = 1$ for any nonzero M_z-invariant subspace $\mathcal{M}$ of $H^2(\mu)$. On the other hand, there are M_z-invariant subspaces which are Beurling-like, and of the form $\overline{u\mathcal{H}^2(\mu)}$, and those which are not.

Some further parallels between the Hardy shift, the Bergman shift, and the operator $T = I - C$ can be observed as follows:

- $\|T\| = \|T^*\| = 1$ and $\sigma(T) = \sigma(T^*) = \overline{\mathbb{D}}$ (Proposition 1.1);

- Using the matrix representations of C and C^* from (1.1), a calculation shows that
$$TT^* = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{2}{3} & 0 & \cdots \\ 0 & 0 & 0 & \frac{3}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$
and so $\|T^*\mathbf{x}\| < \|\mathbf{x}\|$ for all $\mathbf{x} \in \ell^2$. Hence T^* is completely non-unitary.
- By [5, Proposition 4.6], the properties above imply that T (and hence T^*) has an isometric H^∞ functional calculus. That is, the operators lie in the class $\mathbb{A}$.
- As shown in [8], every operator of class $\mathbb{A}$ is reflexive, which means that every operator fixing all T's invariant subspaces can be approximated in the weak operator topology by polynomials in T. This is a property that guarantees a rich subspace lattice.

There are other curious facts from [31] which only add to the mystery of whether or not $\mathcal{H}^2(\mu)$ is closer to a Hardy space or a Bergman space. If $f \in \mathcal{H}^2(\mu) \setminus \{0\}$ vanishes on a sequence $\{z_n\}_{n\geqslant 1}$ in $\mathbb{D}$ which does not accumulate at the point 1, then $\{z_n\}_{n\geqslant 1}$ must satisfy the Blaschke condition $\sum_{n\geqslant 1}(1-|z_n|) < \infty$ (which seems to point towards $\mathcal{H}^2(\mu)$ being closer to H^2). Moreover, if J is a closed arc of $\mathbb{T}$ which does not contain 1 and $f \in \mathcal{H}^2(\mu)\setminus\{0\}$, then $\int_J \log|f|\mathrm{d}m > -\infty$ [21, Theorem 2.2] (again pointing towards $\mathcal{H}^2(\mu)$ being closer to H^2). However, there are $f \in \mathcal{H}^2(\mu) \setminus \{0\}$ which vanish on sequences $\{z_n\}_{n\geqslant 1}$ which do not satisfy the Blaschke condition (pointing towards $\mathcal{H}^2(\mu)$ being closer to A^2).

Furthermore [30, Lemma 2], the functions
$$g_0(z) = 1;$$
$$g_n(z) = \frac{1}{(1-z)^n} z\left(z - \frac{1}{2}\right)\cdots\left(z - \frac{n-1}{n}\right), \quad n \geqslant 1,$$
form an orthonormal basis for $\mathcal{H}^2(\mu)$. In particular,
$$\frac{1}{(1-z)^n} \in \mathcal{H}^2(\mu), \quad \forall\, n \geqslant 1$$
which shows that $\mathcal{H}^2(\mu)$ is a space much bigger than H^2 (and even bigger than A^2). It is often the case that "large" Hilbert spaces of analytic functions on $\mathbb{D}$ have a rich class of M_z-invariant subspaces.

BIBLIOGRAPHY

[1] A. Aleman, "Finite codimensional invariant subspaces in Hilbert spaces of analytic functions", *J. Funct. Anal.* **119** (1994), no. 1, p. 1-18.

[2] A. Aleman & B. Korenblum, "Volterra invariant subspaces of H^p", *Bull. Sci. Math.* **132** (2008), no. 6, p. 510-528.

[3] A. Aleman, S. Richter & C. Sundberg, "Nontangential limits in $\mathcal{P}^t(\mu)$-spaces and the index of invariant subspaces", *Ann. Math. (2)* **169** (2009), no. 2, p. 449-490.

[4] S. Axler & P. Bourdon, "Finite-codimensional invariant subspaces of Bergman spaces", *Trans. Am. Math. Soc.* **306** (1988), no. 2, p. 805-817.

[5] H. Bercovici, C. Foiaş & C. Pearcy, *Dual algebras with applications to invariant subspaces and dilation theory*, CBMS Regional Conference Series in Mathematics, vol. 56, American Mathematical Society, 1985, xi+108 pages.

[6] E. Berkson & H. Porta, "Semigroups of analytic functions and composition operators", *Mich. Math. J.* **25** (1978), no. 1, p. 101-115.

[7] A. Brown, P. R. Halmos & A. L. Shields, "Cesàro operators", *Acta Sci. Math.* **26** (1965), p. 125-137.

[8] S. Brown & B. Chevreau, "Toute contraction à calcul fonctionnel isométrique est réflexive", *Comptes Rendus. Mathématique* **307** (1988), no. 5, p. 185-188.

[9] S. R. Caradus, "Universal operators and invariant subspaces", *Proc. Am. Math. Soc.* **23** (1969), p. 526-527.

[10] J. R. Carmo & S. Waleed Noor, "Universal composition operators", *J. Oper. Theory* **87** (2022), no. 1, p. 137-156.

[11] I. Chalendar & J. R. Partington, "On the structure of invariant subspaces for isometric composition operators on $H^2(\mathbb{D})$ and $H^2(\mathbb{C}_+)$", *Arch. Math.* **81** (2003), no. 2, p. 193-207.

[12] ———, *Modern approaches to the invariant-subspace problem*, Cambridge Tracts in Mathematics, vol. 188, Cambridge University Press, 2011, xii+285 pages.

[13] R. Cheng, J. Mashreghi & W. T. Ross, *Function theory and ℓ^p spaces*, University Lecture Series, vol. 75, American Mathematical Society, 2020, xvi+219 pages.

[14] C. C. Cowen, "Subnormality of the Cesàro operator and a semigroup of composition operators", *Indiana Univ. Math. J.* **33** (1984), no. 2, p. 305-318.

[15] C. C. Cowen & B. D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, 1995, xii+388 pages.

[16] C. C. Cowen & R. G. Wahl, "Shift-invariant subspaces invariant for composition operators on the Hardy–Hilbert space", *Proc. Am. Math. Soc.* **142** (2014), no. 12, p. 4143-4154.

[17] H. j. Crawford Rhaly, "Posinormal operators", *J. Math. Soc. Japan* **46** (1994), no. 4, p. 587-605.

[18] J. A. Deddens, "Analytic Toeplitz and composition operators", *Can. J. Math.* **24** (1972), p. 859-865.

[19] Y. Domar, "Translation invariant subspaces of weighted l^p and L^p spaces", *Math. Scand.* **49** (1981), no. 1, p. 133-144.

[20] R. G. Douglas, H. S. Shapiro & A. L. Shields, "Cyclic vectors and invariant subspaces for the backward shift operator", *Ann. Inst. Fourier* **20** (1970), no. 1, p. 37-76.

[21] P. L. Duren, *Theory of H^p Spaces*, Pure and Applied Mathematics, vol. 38, Academic Press Inc., 1970, xii+258 pages.

[22] P. L. Duren & A. Schuster, *Bergman spaces*, Mathematical Surveys and Monographs, vol. 100, American Mathematical Society, 2004, x+318 pages.

[23] S. Elliott & M. T. Jury, "Composition operators on Hardy spaces of a half-plane", *Bull. Lond. Math. Soc.* **44** (2012), no. 3, p. 489-495.

[24] E. A. Gallardo-Gutiérrez & J. R. Partington, "Insights on the Cesàro operator: Shift semigroups and invariant subspaces", *J. Anal. Math.* **152** (2024), no. 2, p. 595-614.

[25] S. R. Garcia, J. Mashreghi & W. T. Ross, *Introduction to model spaces and their operators*, Cambridge Studies in Advanced Mathematics, vol. 148, Cambridge University Press, 2016, xv+322 pages.

[26] ———, *Operator theory by example*, Oxford Graduate Texts in Mathematics, vol. 30, Oxford University Press, 2023, xxiii+494 pages.

[27] J. B. Garnett, *Bounded analytic functions*, revised 1st ed., Graduate Texts in Mathematics, vol. 236, Springer, 2007, xiv+459 pages.

[28] G. H. Hardy, "Note on a theorem of Hilbert", *Math. Z.* **6** (1920), no. 3-4, p. 314-317.

[29] G. H. Hardy, J. E. Littlewood & G. Pólya, *Inequalities*, reprint of the 1952 ed., Cambridge Mathematical Library, Cambridge University Press, 1988, xii+324 pages.

[30] T. L. Kriete, III & D. Trutt, "The Cesàro operator in l^2 is subnormal", *Am. J. Math.* **93** (1971), p. 215-225.

[31] ———, "On the Cesàro operator", *Indiana Univ. Math. J.* **24** (1974), p. 197-214.

[32] V. Matache, "Weighted composition operators on H^2 and applications", *Complex Anal. Oper. Theory* **2** (2008), no. 1, p. 169-197.

[33] N. K. Nikolski, *Hardy spaces*, translated from the french ed., Cambridge Studies in Advanced Mathematics, vol. 179, Cambridge University Press, 2019, xviii+277 pages.

[34] S. Richter, "Invariant subspaces in Banach spaces of analytic functions", *Trans. Am. Math. Soc.* **304** (1987), no. 2, p. 585-616.

[35] J. H. Shapiro, *Composition operators and classical function theory*, Universitext: Tracts in Mathematics, Springer, 1993, xvi+223 pages.

[36] A. L. Shields & L. J. Wallen, "The commutants of certain Hilbert space operators", *Indiana Univ. Math. J.* **20** (1971), p. 777-788.

[37] A. G. Siskakis, "Composition semigroups and the Cesàro operator on H^p", *J. Lond. Math. Soc. (2)* **36** (1987), no. 1, p. 153-164.

[38] ———, "The Cesàro operator is bounded on H^1", *Proc. Am. Math. Soc.* **110** (1990), no. 2, p. 461-462.

[39] A. Zygmund, *Trigonometric series. Vol. I, II*, 3rd ed., Cambridge Mathematical Library, Cambridge University Press, 2002, With a foreword by Robert A. Fefferman.

Manuscrit reçu le 9 décembre 2023,
révisé le 15 juillet 2024,
accepté le 27 janvier 2025.

Eva A. GALLARDO-GUTIÉRREZ
Departamento de Análisis
Matemático y Matemática Aplicada
Facultad de Matemáticas
Universidad Complutense de Madrid
Plaza de Ciencias 3
28040 Madrid (Spain)
and
Instituto de Ciencias
Matemáticas ICMAT
(CSIC-UAM-UC3M-UCM)
Madrid (Spain)
eva.gallardo@mat.ucm.es

Jonathan R. PARTINGTON
School of Mathematics
University of Leeds
Leeds LS2 9JT (United Kingdom)
J.R.Partington@leeds.ac.uk

William T. ROSS
Department of Mathematics
and Statistics,
University of Richmond
Richmond, VA 23173 (USA)
wross@richmond.edu