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NEW CHARACTERIZATIONS FOR FOCK SPACES

by Guanlong BAO, Pan MA & Kehe ZHU (*)

Abstract. — We show that the maximal Fock space F_α^∞ on $\mathbb{C}^n$ is a Lipschitz space, that is, there exists a distance d_α on $\mathbb{C}^n$ such that an entire function f on $\mathbb{C}^n$ belongs to F_α^∞ if and only if

$$|f(z) - f(w)| \leqslant C d_\alpha(z, w)$$

for some constant C and all $z, w \in \mathbb{C}^n$. This can be considered the Fock space version of the following classical result in complex analysis: a holomorphic function f on the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ belongs to the Bloch space if and only if there exists a positive constant C such that $|f(z) - f(w)| \leqslant C\beta(z, w)$ for all $z, w \in \mathbb{B}^n$, where $\beta(z, w)$ is the distance on $\mathbb{B}^n$ in the Bergman metric. We also present a new approach to Hardy–Littlewood type characterizations for F_α^p.

Résumé. — Nous montrons que l'espace de Fock maximal F_α^∞ sur $\mathbb{C}^n$ est un espace de Lipschitz, c'est-à-dire qu'il existe une distance d_α sur $\mathbb{C}^n$ telle qu'une fonction entière f sur $\mathbb{C}^n$ appartient à F_α^∞ si et seulement si

$$|f(z) - f(w)| \leqslant C d_\alpha(z, w)$$

pour une constante C et pour tous $z, w \in \mathbb{C}^n$. Cela peut être considéré comme la version de l'espace de Fock du résultat classique suivant en analyse complexe : une fonction holomorphe f sur la boule unité $\mathbb{B}^n$ dans $\mathbb{C}^n$ appartient à l'espace de Bloch si et seulement s'il existe une constante positive C telle que $|f(z)-f(w)| \leqslant C\beta(z, w)$ pour tous $z, w \in \mathbb{B}^n$, où $\beta(z, w)$ est la distance sur $\mathbb{B}^n$ dans la métrique de Bergman. Nous présentons également une nouvelle approche des caractérisations de type Hardy–Littlewood pour F_α^p.

Keywords: Fock spaces, Gaussian measure, induced distance, Lipschitz space, Hardy–Littlewood type theorem.
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Introduction

For $\alpha > 0$ and $0 < p \leqslant \infty$ we use L^p_α or $L^p_\alpha(\mathbb{C}^n)$ to denote the space of all Lebesgue measurable functions f on the complex Euclidean space $\mathbb{C}^n$ such that the function $f(z)e^{-\alpha|z|^2/2}$ belongs to $L^p(\mathbb{C}^n, \mathrm{d}v)$, where dv is ordinary volume measure on $\mathbb{C}^n$. For $f \in L^p_\alpha$ we write

$$\|f\|^p_{p,\alpha} = \left(\frac{p\alpha}{2\pi}\right)^n \int_{\mathbb{C}^n} \left|f(z)e^{-\alpha|z|^2/2}\right|^p \mathrm{d}v(z)$$

when $0 < p < \infty$ and

$$\|f\|_{\infty,\alpha} = \operatorname{ess\,sup}\left\{|f(z)|e^{-\alpha|z|^2/2} \,\middle|\, z \in \mathbb{C}^n\right\}$$

when $p = \infty$.

Let $H(\mathbb{C}^n)$ denote the space of all entire functions on $\mathbb{C}^n$. The spaces

$$F^p_\alpha = L^p_\alpha \cap H(\mathbb{C}^n), \quad 0 < p \leqslant \infty, \quad \alpha > 0,$$

are usually called Fock spaces. Each F^p_α is closed in the Lebesgue space L^p_α. In particular, F^p_α is a Banach space when $1 \leqslant p \leqslant \infty$.

It is clear that $L^2_\alpha = L^2(\mathbb{C}^n, \mathrm{d}\lambda_\alpha)$, where

$$\mathrm{d}\lambda_\alpha(z) = \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|z|^2} \mathrm{d}v(z)$$

is the Gaussian measure. The orthogonal projection $P_\alpha : L^2(\mathbb{C}^n, \mathrm{d}\lambda_\alpha) \to F^2_\alpha$ is an integral operator, namely,

$$P_\alpha f(z) = \int_{\mathbb{C}^n} e^{\alpha z\overline{w}} f(w)\, \mathrm{d}\lambda_\alpha(w), \tag{0.1}$$

where $z\overline{w} = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$. It is well known that $F^p_\alpha = P_\alpha L^p_\alpha$ for all $1 \leqslant p \leqslant \infty$.

The function $K_w(z) = K(z,w) = e^{\alpha z\overline{w}}$ is the reproducing kernel of the Hilbert space F^2_α. We will need to use the normalized reproducing kernels $k_w = K_w/\|K_w\|_{2,\alpha}$, which are unit vectors in F^2_α. It is clear that

$$k_w(z) = \frac{K(z,w)}{\sqrt{K(w,w)}} = e^{\alpha z\overline{w} - \left(\alpha|w|^2/2\right)}.$$

See [10] for an introduction to Fock spaces.

The main result of the paper is Theorem A below.

Theorem A. — *Suppose $\alpha > 0$, $f \in H(\mathbb{C}^n)$, and d is the distance function on $\mathbb{C}^n$ defined by*

$$d(z,w) = \int_{\mathbb{C}^n} \left|e^{\alpha z\overline{u}} - e^{\alpha w\overline{u}}\right| \mathrm{d}\lambda_{\alpha/2}(u).$$

Then the following conditions are equivalent:

(a) $f \in F^\infty_\alpha$.

(b) *There exists a positive constant* C *such that*

$$|f(z) - f(w)| \leqslant Cd(z,w), \quad z,w \in \mathbb{C}^n.$$

(c) *The function* $Rf(z)/(1+|z|^2)$ *belongs to* L^∞_α, *where*

$$Rf(z) = z_1\partial_1 f(z) + \cdots + z_n\partial_n f(z)$$

is the radial derivative of f *with* $\partial_k f = \partial f/\partial z_k$ *for* $1 \leqslant k \leqslant n$.

This result has a well-known analogue in the more classical theory of Bergman spaces. Recall that the Bergman space A^p, $0 < p < \infty$, of the open unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ is the space of all holomorphic functions in $L^p(\mathbb{B}^n, \mathrm{d}v)$. If $P : L^2(\mathbb{B}^n, \mathrm{d}v) \to A^2$ is the Bergman projection, then it is well known that $A^p = PL^p(\mathbb{B}^n, \mathrm{d}v)$ for $1 < p < \infty$. When $p = \infty$, the space $\mathcal{B} = PL^\infty(\mathbb{B}^n)$ is called the Bloch space of $\mathbb{B}^n$, which can be shown to consist of all holomorphic f on $\mathbb{B}^n$ such that

$$\sup_{z\in\mathbb{B}^n} \left(1-|z|^2\right)|Rf(z)| < \infty.$$

The Bergman space analogue of Theorem A is the following: a holomorphic function f on $\mathbb{B}^n$ belongs to $\mathcal{B}$ if and only if $|f(z) - f(w)| \leqslant C\beta(z,w)$ for some positive constant C and all $z, w \in \mathbb{B}^n$, where $\beta(z,w)$ is the distance function on $\mathbb{B}^n$ in the Bergman metric. See [9].

The Bergman spaces A^p and the Bloch space $\mathcal{B}$ on $\mathbb{B}^n$ can also be described in terms of higher order derivatives. More precisely, if f is holomorphic on $\mathbb{B}^n$ and N is a positive integer, then $f \in A^p$ if and only if the functions $(1-|z|)^{|m|}\partial^m f(z)$ belong to $L^p(\mathbb{B}^n, \mathrm{d}v)$ for all $|m| = N$. Here $m = (m_1, \ldots, m_n)$ is an n-tuple of non-negative integers and

$$|m| = m_1 + \cdots + m_n, \qquad \partial^m f = \frac{\partial^{|m|} f}{\partial z_1^{m_1}, \ldots, \partial z_n^{m_n}}.$$

Similarly, $f \in \mathcal{B}$ if and only if the functions $(1-|z|)^{|m|}\partial^m f(z)$ belong to $L^\infty(\mathbb{B}^n, \mathrm{d}v)$ for all $|m| = N$. Such results are usually called Hardy–Littlewood theorems, especially in the one-dimensional case of the unit disc. See [9] again.

It turns out that these Hardy–Littlewood type theorems also hold for Fock spaces. The following theorem can be found in [1, 2, 3, 5, 6].

Theorem B. — *Suppose* $\alpha > 0$, $0 < p \leqslant \infty$, N *is positive integer, and* $f \in H(\mathbb{C}^n)$. *Then* $f \in F^p_\alpha$ *if and only if the functions* $\partial^m f(z)/(1+|z|)^{|m|}$, $|m| = N$, *all belong to* L^p_α.

We will present a new approach to Theorem B above when $n = 1$. Our proof is based on the main theorem in [4] and is much different and simpler than the existing proofs in the literature. We will conclude the paper with two open problems.

1. Some distance functions on $\mathbb{C}^n$

Fix any two positive parameters α and β and define a function

$$d_{\alpha,\beta} : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow [0, \infty)$$

by

$$d_{\alpha,\beta}(z, w) = \int_{\mathbb{C}^n} \left| e^{\beta z\bar{u}} - e^{\beta w\bar{u}} \right| \mathrm{d}\lambda_\alpha(u).$$

By the "rotation invariance" of the Gaussian measure, we clearly have

$$d_{\alpha,\beta}(z, w) = d_{\alpha,\beta}(Uz, Uw)$$

for all $z, w \in \mathbb{C}^n$ and all unitary transformations $U : \mathbb{C}^n \to \mathbb{C}^n$. The following lemma is obvious.

Lemma 1.1. — *Each $d_{\alpha,\beta}$ is a distance on $\mathbb{C}^n$, that is, it satisfies the following three axioms:*

(a) *$d_{\alpha,\beta}(z, w) \geqslant 0$ for all $z, w \in \mathbb{C}^n$, and $d_{\alpha,\beta}(z, w) = 0$ iff $z = w$.*
(b) *$d_{\alpha,\beta}(z, w) = d_{\alpha,\beta}(w, z)$ for all $z, w \in \mathbb{C}^n$.*
(c) *$d_{\alpha,\beta}(z, w) \leqslant d_{\alpha,\beta}(z, u) + d_{\alpha,\beta}(u, w)$ for all $z, w, u \in \mathbb{C}^n$.*

The computation of the precise distance $d_{\alpha,\beta}(z, w)$ between z and w is often difficult. But we have the following estimate when one of the two points is the origin. Here $F(z) \sim G(z)$ means that there exist positive constants c and C (independent of z but dependent on other parameters) such that $cF(z) \leqslant G(z) \leqslant CF(z)$ for all $z \in \mathbb{C}^n$.

Proposition 1.2. — *For any fixed positive α and β we have*

$$d_{\alpha,\beta}(z, 0) \sim \sqrt{e^{\beta^2|z|^2/(2\alpha)} - 1}$$

for $z \in \mathbb{C}^n$. In particular,

$$d_{\alpha/2,\alpha}(z, 0) \sim \sqrt{e^{\alpha|z|^2} - 1}, \quad z \in \mathbb{C}^n.$$

Proof. — It follows from the reproducing property in F^2_α that

$$\int_{\mathbb{C}^n} \left| e^{\beta z\bar{u}} \right| \mathrm{d}\lambda_\alpha(u) = \int_{\mathbb{C}^n} \left| e^{\alpha(\beta z/2\alpha)\bar{u}} \right|^2 \mathrm{d}\lambda_\alpha(u) = e^{\alpha|\beta z/2\alpha|^2} = e^{\beta^2|z|^2/(4\alpha)}.$$

Thus we have the following estimates:

$$d_{\alpha,\beta}(z,0) \leqslant \int_{\mathbb{C}^n} \left|e^{\beta z\overline{u}}\right| \,\mathrm{d}\lambda_\alpha(u) + 1 = e^{\beta^2|z|^2/(4\alpha)} + 1, \quad z \in \mathbb{C}^n,$$

and

$$d_{\alpha,\beta}(z,0) \geqslant \int_{\mathbb{C}^n} \left|e^{\beta z\overline{u}}\right| \,\mathrm{d}\lambda_\alpha(u) - 1 = e^{\beta^2|z|^2/(4\alpha)} - 1, \quad z \in \mathbb{C}^n.$$

Consequently,

$$\lim_{|z|\to\infty} \frac{d_{\alpha,\beta}(z,0)}{e^{\beta^2|z|^2/(4\alpha)}} = 1,$$

and for any $r > 0$ there exists a constant $C > 0$ such that

$$C^{-1} e^{\beta^2|z|^2/(4\alpha)} \leqslant d_{\alpha,\beta}(z,0) \leqslant C e^{\beta^2|z|^2/(4\alpha)}, \quad |z| \geqslant r.$$

This clearly implies that there is another positive constant C such that

$$C^{-1}\sqrt{e^{\beta^2|z|^2/(2\alpha)} - 1} \leqslant d_{\alpha,\beta}(z,0) \leqslant C\sqrt{e^{\beta^2|z|^2/(2\alpha)} - 1}, \quad |z| \geqslant r.$$

On the other hand, it is clear from the unitary invariance of the Gaussian measure (or the distance $d_{\alpha,\beta}$) that we can use the special points $z = (z_1, 0, \ldots, 0)$ below to obtain

$$\lim_{|z|\to 0} \frac{d_{\alpha,\beta}(z,0)}{|z|} = \lim_{|z|\to 0} \int_{\mathbb{C}^n} \frac{\left|e^{\beta z\overline{u}} - 1\right|}{|z|} \,\mathrm{d}\lambda_\alpha(u) = \beta \int_{\mathbb{C}^n} |u_1| \,\mathrm{d}\lambda_\alpha(u). \tag{1.1}$$

It follows that

$$d_{\alpha,\beta}(z,0) \sim \sqrt{e^{\beta^2|z|^2/(2\alpha)} - 1}, \quad |z| \longrightarrow 0.$$

The proof of Proposition 1.2 is complete when we combine this with the estimate at the end of the previous paragraph. □

To simplify notation, we will write

$$d_\alpha(z,w) = d_{\alpha/2,\alpha}(z,w)$$

from this point on. This distance function arises naturally in the study of the spaces F^p_α. For example, we have the following result.

Lemma 1.3. — *For any $\alpha > 0$ we have*

$$d_\alpha(z,w) \sim \sup\left\{|f(z) - f(w)| : f \in F^\infty_\alpha,\ \|f\|_{\infty,\alpha} \leqslant 1\right\}$$

for $z, w \in \mathbb{C}^n$.

Proof. — It is well known that the integral operator P_α defined in (0.1) maps $L^\infty_\alpha(\mathbb{C}^n)$ boundedly onto F^∞_α. It follows easily that for $f \in F^\infty_\alpha$ we have

$$\|f\|_{\infty,\alpha} \sim \inf\left\{\|g\|_{\infty,\alpha} : f = P_\alpha g,\ g \in L^\infty_\alpha(\mathbb{C}^n)\right\}.$$

Let $d'_\alpha(z,w)$ denote the supremum in the Lemma 1.3, which is also a distance on $\mathbb{C}^n$ (see [8] for many other examples of distance functions induced by spaces of analytic functions). Then

$$\begin{aligned} d'_\alpha(z,w) &\sim \sup_{\|g\|_{\infty,\alpha}\leqslant 1} |P_\alpha g(z)-P_\alpha g(w)| \\ &= \sup_{\|g\|_{\infty,\alpha}\leqslant 1} \left|\int_{\mathbb{C}^n} \left(e^{\alpha z\bar u}-e^{\alpha w\bar u}\right) g(u)\,\mathrm{d}\lambda_\alpha(u)\right| \\ &= \int_{\mathbb{C}^n} \left|e^{\alpha z\bar u}-e^{\alpha w\bar u}\right| e^{\alpha|z|^2/2}\,\mathrm{d}\lambda_\alpha(u) \\ &= 2^n \int_{\mathbb{C}^n} \left|e^{\alpha z\bar u}-e^{\alpha w\bar u}\right|\,\mathrm{d}\lambda_{\alpha/2}(u). \end{aligned}$$

This proves the desired estimates. □

It is natural to wonder if the limit in (1.1) can be computed at points away from the origin. When $n=1$, it is clear that

$$\lim_{w\to z}\frac{d_\alpha(z,w)}{|z-w|} = \lim_{w\to z}\int_{\mathbb{C}} \frac{\left|e^{\alpha z\bar u}-e^{\alpha w\bar u}\right|}{|z-w|}\,\mathrm{d}\lambda_{\alpha/2}(u) = \alpha\int_{\mathbb{C}} |u|\left|e^{\alpha z\bar u}\right|\,\mathrm{d}\lambda_{\alpha/2}(u).$$

However, the limit above does NOT exist when $n>1$ and $z\neq 0$. In fact, if we choose $w=z+(t,0,\dots,0)$, then

$$\lim_{t\to 0}\int_{\mathbb{C}^n} \frac{\left|1-e^{\alpha(w-z)\bar u}\right|}{|w-z|}\left|e^{\alpha z\bar u}\right|\,\mathrm{d}\lambda_{\alpha/2}(u) = \alpha\int_{\mathbb{C}^n}\left|u_1 e^{\alpha z\bar u}\right|\,\mathrm{d}\lambda_{\alpha/2}(u).$$

Other similar "partial derivatives" will yield the following sub-limits:

$$\alpha\int_{\mathbb{C}^n}\left|u_k e^{\alpha z\bar u}\right|\,\mathrm{d}\lambda_{\alpha/2}(u),\quad 1\leqslant k\leqslant n,$$

which clearly depend on k. More specifically, it follows from polar coordinates and the one-dimensional case that

$$\int_{\mathbb{C}^n}\left|u_k e^{\alpha z\bar u}\right|\,\mathrm{d}\lambda_{\alpha/2}(u)\sim (1+|z_k|)e^{\alpha|z|^2/2}.$$

On the other hand, we can write

$$\frac{d_\alpha(z,w)}{|z-w|} = \int_{\mathbb{C}^n}\frac{\left|1-e^{\alpha(w-z)\bar u}\right|}{|(w-z)\bar u|}\,\frac{|(w-z)\bar u|}{|z-w|}\,\left|e^{\alpha z\bar u}\right|\,\mathrm{d}\lambda_{\alpha/2}(u).$$

It follows from the Cauchy–Schwarz inequality for vectors in $\mathbb{C}^n$ that

$$\limsup_{w\to z}\frac{d_\alpha(z,w)}{|z-w|}\leqslant \alpha\int_{\mathbb{C}^n}|u|\left|e^{\alpha z\bar u}\right|\,\mathrm{d}\lambda_{\alpha/2}(u). \tag{1.2}$$

Thus we want to determine the growth rate of the integral

$$E(z)=\int_{\mathbb{C}^n}|u|\;\left|e^{\alpha z\bar u}\right|\,\mathrm{d}\lambda_{\alpha/2}(u)$$

as $|z| \to \infty$, which will be used several times later on.

Lemma 1.4. — *We have*

$$\frac{\alpha}{2}\int_{\mathbb{C}^n} |u|^2 \left|e^{\alpha z\overline{u}}\right| \,\mathrm{d}\lambda_{\alpha/2}(u) = \left(n + \frac{\alpha}{2}|z|^2\right) e^{\alpha|z|^2/2}$$

for all $z \in \mathbb{C}^n$.

Proof. — It follows from the reproducing property in $F^2_{\alpha/2}$ that

$$\int_{\mathbb{C}^n} \left|e^{\frac{\alpha}{2}z\overline{u}}\right|^2 \,\mathrm{d}\lambda_{\alpha/2}(u) = e^{\frac{\alpha}{2}|z|^2}$$

for all $z \in \mathbb{C}^n$. Apply $\partial^2/\partial z_k \partial \overline{z}_k$ to both sides of the above identity. The result is

$$\left(\frac{\alpha}{2}\right)^2 \int_{\mathbb{C}^n} |u_k|^2 \left|e^{\frac{\alpha}{2}z\overline{u}}\right|^2 \,\mathrm{d}\lambda_{\alpha/2}(u) = \frac{\alpha}{2}\left(1 + \frac{\alpha}{2}|z_k|^2\right) e^{\frac{\alpha}{2}|z|^2}.$$

Summing over k, we obtain

$$\frac{\alpha}{2}\int_{\mathbb{C}^n} |u|^2 \left|e^{\alpha z\overline{u}}\right| \,\mathrm{d}\lambda_{\alpha/2}(u) = \left(n + \frac{\alpha}{2}|z|^2\right) e^{\frac{\alpha}{2}|z|^2},$$

completing the proof of Lemma 1.4. □

Lemma 1.5. — *For any $\alpha > 0$ we have $E(z) \sim (1+|z|)e^{\alpha|z|^2/2}$ on $\mathbb{C}^n$.*

Proof. — We write $z = rw$, where $r = |z|$ (so $|w| = 1$). Then

$$\int_{\mathbb{C}^n} e^{\alpha r w\overline{u}/2} e^{\alpha r\overline{w}u/2} \,\mathrm{d}\lambda_{\alpha/2}(u) = \int_{\mathbb{C}^n} \left|e^{\alpha z\overline{u}/2}\right|^2 \,\mathrm{d}\lambda_{\alpha/2}(u) = e^{\alpha|z|^2/2} = e^{\alpha r^2/2}.$$

Take the derivative with respect to r on both sides. We obtain

$$\frac{\alpha}{2}\int_{\mathbb{C}^n} (\overline{w}u + w\overline{u}) \left|e^{\alpha r\overline{u}}\right| \,\mathrm{d}\lambda_{\alpha/2}(u) = \alpha r e^{\alpha r^2/2},$$

or

$$\int_{\mathbb{C}^n} \mathrm{Re}(w\overline{u}) \left|e^{\alpha z\overline{u}}\right| \,\mathrm{d}\lambda_{\alpha/2}(u) = |z| e^{\alpha|z|^2/2}.$$

Therefore,

$$|z|e^{\alpha|z|^2/2} \leqslant \int_{\mathbb{C}^n} |u| \left|e^{\alpha z\overline{u}}\right| \,\mathrm{d}\lambda_{\alpha/2}(u).$$

It is clear that the integral on the right-hand side above is a strictly positive continuous function of z for $|z| \leqslant 1$, so there is a positive constant c such that

$$c\,(1+|z|)e^{\alpha|z|^2/2} \leqslant \int_{\mathbb{C}^n} |u| \left|e^{\alpha z\overline{u}}\right| \,\mathrm{d}\lambda_{\alpha/2}(u)$$

for all $z \in \mathbb{C}^n$.

On the other hand, it follows from Hölder's inequality and Lemma 1.4 that

$$\begin{aligned}\left[\int_{\mathbb{C}^n} |u| \left|e^{\alpha z\bar{u}}\right| \,\mathrm{d}\lambda_{\alpha/2}(u)\right]^2 &\leqslant \int_{\mathbb{C}^n} |u|^2 \left|e^{\alpha z\bar{u}}\right| \,\mathrm{d}\lambda_{\alpha/2}(u) \int_{\mathbb{C}^n} \left|e^{\alpha z\bar{u}}\right| \,\mathrm{d}\lambda_{\alpha/2}(u)\\ &= \left(\frac{2n}{\alpha} + |z|^2\right) e^{\frac{\alpha}{2}|z|^2} e^{\frac{\alpha}{2}|z|^2}\\ &= \left(\frac{2n}{\alpha} + |z|^2\right) e^{\alpha|z|^2}.\end{aligned}$$

Thus there exists a positive constant $C = C_\alpha$ such that

$$\int_{\mathbb{C}^n} |u| \left|e^{\alpha z\bar{u}}\right| \,\mathrm{d}\lambda_{\alpha/2}(u) \leqslant C(1+|z|)\, e^{\frac{\alpha}{2}|z|^2}$$

for all $z \in \mathbb{C}$. Combining this with the estimate in the previous paragraph, we complete the proof of Lemma 1.5. □

Finally in this section we note that for any $\alpha > 0$, $\beta > 0$, and $p > 1$, the function

$$d(z,w) = \left[\int_{\mathbb{C}^n} \left|e^{\beta z\bar{u}} - e^{\beta w\bar{u}}\right|^p \,\mathrm{d}\lambda_\alpha(u)\right]^{\frac{1}{p}}$$

is also a distance on $\mathbb{C}^n$. For $0 < p < 1$ the following is a distance on $\mathbb{C}^n$:

$$d(z,w) = \int_{\mathbb{C}^n} \left|e^{\beta z\bar{u}} - e^{\beta w\bar{u}}\right|^p \,\mathrm{d}\lambda_\alpha(u).$$

It is not clear what these more general distances might be good for. It would be nice to find some applications for them.

2. Characterizations of F_α^∞

In this section we prove two characterizations for the space F_α^∞. It is well known that F_α^∞ is maximal in some sense among Banach spaces of entire functions under the action of the Heisenberg group; see [8]. Our first characterization of F_α^∞ below shows that it is a Lipschitz space.

Theorem 2.1. — *Let $\alpha > 0$ and $f \in H(\mathbb{C}^n)$. Then $f \in F_\alpha^\infty$ if and only if there exists a positive constant C such that*

$$|f(z) - f(w)| \leqslant C d_\alpha(z,w)$$

for all $z, w \in \mathbb{C}^n$.

Proof. — The "only if" direction is a direct consequence of Lemma 1.3. Alternatively, for any $f \in F^\infty_\alpha$ we have

$$\begin{aligned}|f(z) - f(w)| &= \left|\int_{\mathbb{C}^n} \left(e^{\alpha z\bar{u}} - e^{\alpha w\bar{u}}\right) f(u)\,\mathrm{d}\lambda_\alpha(u)\right| \\ &\leqslant \left(\frac{\alpha}{\pi}\right)^n \int_{\mathbb{C}^n} \left|e^{\alpha z\bar{u}} - e^{\alpha w\bar{u}}\right| \left|f(u)e^{-\alpha|u|^2/2}\right| e^{-\alpha|u|^2/2}\,\mathrm{d}v(u) \\ &\leqslant 2^n \|f\|_{\infty,\alpha} \int_{\mathbb{C}^n} \left|e^{\alpha z\bar{u}} - e^{\alpha w\bar{u}}\right| \mathrm{d}\lambda_{\alpha/2}(u) \\ &= 2^n \|f\|_{\infty,\alpha} d_\alpha(z,w)\end{aligned}$$

for all $f \in F^\infty_\alpha$ and $z, w \in \mathbb{C}^n$.

On the other hand, if f satisfies the Lipschitz condition, then in particular, $|f(z) - f(0)| \leqslant C d_\alpha(z, 0)$ for all $z \in \mathbb{C}^n$. By Proposition 1.2, there is another positive constant C such that $|f(z) - f(0)| \leqslant C\sqrt{e^{\alpha|z|^2} - 1}$ for all $z \in \mathbb{C}^n$, which clearly implies that the function $f(z)e^{-\alpha|z|^2/2}$ is bounded on $\mathbb{C}^n$. This completes the proof of Theorem 2.1. □

Again, to put the result above in proper perspective, we should think of it as the Fock space version of the following well-known result for holomorphic functions f on the unit ball $\mathbb{B}^n$: f belongs to the Bloch space $\mathcal{B}$ if and only if there exists a constant C such that $|f(z) - f(w)| \leqslant C\beta(z, w)$ for all $z, w \in \mathbb{B}^n$, where $\beta(z, w)$ is the distance between z and w in the Bergman metric. See [9].

For a function $f \in H(\mathbb{C}^n)$ we will write $\nabla f = (\partial_1 f, \ldots, \partial_n f)$ for the holomorphic gradient of f. The equivalence of conditions (a), (b), and (c) below is known, and we include a simple proof here. But condition (d) appears to be new, interesting, and non-trivial.

THEOREM 2.2. — *Suppose $\alpha > 0$ and f is an entire function on $\mathbb{C}^n$. Then the following conditions are equivalent:*

(a) $f \in F^\infty_\alpha$.

(b) *There exists a positive constant C such that*

$$|\partial_k f(z)| \leqslant C(1 + |z|)e^{\alpha|z|^2/2}$$

for all $z \in \mathbb{C}^n$ and $1 \leqslant k \leqslant n$.

(c) *There exists a positive constant C such that*

$$|\nabla f(z)| \leqslant C(1 + |z|)e^{\alpha|z|^2/2}$$

for all $z \in \mathbb{C}^n$.

(d) *There exists a positive constant C such that*

$$|Rf(z)| \leqslant C\left(1+|z|^2\right) e^{\alpha|z|^2/2}$$

for all $z \in \mathbb{C}^n$.

Proof. — It is obvious that conditions (b) and (c) are equivalent.

If $f \in F^\infty_\alpha$, then by Theorem 2.1, there exists a positive constant C such that

$$|f(z)-f(w)| \leqslant C\int_{\mathbb{C}^n} \left|e^{\alpha z\bar u}-e^{\alpha w\bar u}\right| \,\mathrm{d}\lambda_{\alpha/2}(u)$$

for all z and w in $\mathbb{C}$. Let $w=z+(h,0,\dots,0)$, divide both sides of the inequality above by $|h|$, and then let $h\to 0$. The result is

$$|\partial_1 f(z)| \leqslant C\alpha\int_{\mathbb{C}^n} \left|u_1 e^{\alpha z\bar u}\right| \,\mathrm{d}\lambda_{\alpha/2}(u).$$

It is clear that we also have

$$|\partial_k f(z)| \leqslant C\alpha\int_{\mathbb{C}^n} \left|u_k e^{\alpha z\bar u}\right| \,\mathrm{d}\lambda_{\alpha/2}(u)$$

for all $1\leqslant k\leqslant n$. By Lemma 1.5, there exists another constant C such that

$$|\partial_k f(z)| \leqslant C(1+|z|)e^{\alpha|z|^2/2}$$

for all $1\leqslant k\leqslant n$ and $z\in\mathbb{C}$. Thus condition (a) implies (b).

By the Cauchy–Schwarz inequality, we have $|Rf(z)|\leqslant |z||\nabla f(z)|$, which yields

$$\frac{|Rf(z)|}{1+|z|^2} \leqslant \frac{|z||\nabla f(z)|}{1+|z|^2} \leqslant \frac{2|\nabla f(z)|}{1+|z|} \tag{2.1}$$

for all $z\in\mathbb{C}^n$. This shows that condition (c) implies (d).

Finally, we assume that condition (d) holds. For any $z\in\mathbb{C}^n$ we can write

$$\begin{aligned} f(z)-f(0) &= \sum_{k=1}^n z_k\int_0^1 \partial_k f(tz)\,\mathrm{d}t = \int_0^1 Rf(tz)\,\frac{\mathrm{d}t}{t} \\ &= \int_0^{1/2} Rf(tz)\,\frac{\mathrm{d}t}{t} + \int_{1/2}^1 Rf(tz)\,\frac{\mathrm{d}t}{t}. \end{aligned} \tag{2.2}$$

Let $I_1(z)$ and $I_2(z)$ denote the two integrals above, respectively. Then

$$\begin{aligned}|I_2(z)| &\leqslant \int_{1/2}^1 |Rf(tz)|\,\frac{\mathrm{d}t}{t} \leqslant 2\int_{1/2}^1 |Rf(tz)|\,\mathrm{d}t \\ &\leqslant 2C\int_0^1 \left(1+t^2|z|^2\right) e^{\alpha t^2|z|^2/2}\,\mathrm{d}t \\ &\leqslant 2C\left[e^{\alpha|z|^2/2} + \int_0^{|z|} s e^{\alpha s^2/2}\,\mathrm{d}s\right] \\ &= 2C\left[e^{\alpha|z|^2/2} + \frac{1}{\alpha}\left(e^{\alpha|z|^2/2}-1\right)\right] \\ &\sim e^{\alpha|z|^2/2}, \quad z\in\mathbb{C}^n.\end{aligned}$$

To estimate $I_1(z)$, note that the assumption

$$|Rf(z)| \leqslant C\left(1+|z|^2\right) e^{\alpha|z|^2/2}$$

is clearly equivalent to

$$|Rf(z)| \leqslant C|z|(1+|z|)e^{\alpha|z|^2/2}$$

(with a possibly different constant). Thus

$$\begin{aligned}|I_1(z)| &\leqslant \int_0^{1/2} |Rf(tz)|\,\frac{\mathrm{d}t}{t} \\ &\leqslant C\int_0^{1/2} |z|(1+t|z|)e^{\alpha t^2|z|^2/2}\,\mathrm{d}t \\ &\leqslant \frac{C}{2}|z|\left(1+\frac{|z|}{2}\right) e^{\alpha|z|^2/8} \\ &\leqslant C' e^{\alpha|z|^2/2}, \quad z\in\mathbb{C}^n\end{aligned}$$

for another positive constant C'. Combining the estimates for $I_1(z)$ and $I_2(z)$, we find another positive constant C such that

$$|f(z)-f(0)| \leqslant Ce^{\alpha|z|^2/2}$$

for all $z\in\mathbb{C}^n$. This shows $f\in F_\alpha^\infty$ and completes the proof of Theorem 2.2. □

We warn any inexperienced reader that the factor $1-|z|^2$ on the unit ball $\mathbb{B}^n$ can be replaced by $1-|z|$, while $1+|z|^2$ is critically different from $1+|z|$ in Theorem 2.2 above!

3. Hardy–Littlewood type theorems for Fock spaces

It is known that Theorem 2.2 can be extended to all Fock spaces F^p_α in terms of higher order derivatives.

Theorem 3.1. — *Suppose* $0 < p \leqslant \infty$, $\alpha > 0$, $f \in H(\mathbb{C}^n)$, *and* N *is a positive integer. Then* $f \in F^p_\alpha$ *if and only if the functions* $\partial^m f(z)/(1+|z|)^m$, *where* $|m| = N$, *all belong to* L^p_α.

Proof. — See [1, 2, 3, 5, 6]. □

In this section, we provide a new approach to Theorem 3.1 in the one-dimensional case. This approach is based on the main theorem in [4] and is much easier than the arguments used in other papers in the literature. We begin with the case of first order derivatives.

Theorem 3.2. — *Suppose* f *is an entire function on* $\mathbb{C}$, $\alpha > 0$, *and* $0 < p \leqslant \infty$. *Then* $f \in F^p_\alpha$ *if and only if the function* $f'(z)/(1+|z|)$ *belongs to* L^p_α.

Proof. — We will prove the result with the help of [4], where it was proved that $f' \in F^p_\alpha$ if and only if the function $zf(z)$ is in F^p_α. What we want to prove here is that $f \in F^p_\alpha$ if and only if the function $[f'(z)-f'(0)]/z$ is in F^p_α.

Without loss of generality, we may assume that $f(0) = f'(0) = 0$. In the equivalence

$$zf(z) \in F^p_\alpha \Longleftrightarrow f'(z) \in F^p_\alpha,$$

if we replace $f(z)$ by $f(z)/z$, then

$$f(z) \in F^p_\alpha \Longleftrightarrow \frac{f'(z)}{z} - \frac{f(z)}{z^2} \in F^p_\alpha.$$

Since $f(z) \in F^p_\alpha$ clearly implies that $f(z)/z^2 \in F^p_\alpha$, it follows that

$$f(z) \in F^p_\alpha \Longrightarrow f'(z)/z \in F^p_\alpha \Longleftrightarrow f'(z)/(1+|z|) \in L^p_\alpha.$$

On the other hand, if $f'(z)/z \in F^p_\alpha \subset F^\infty_\alpha$, then by what we have proved about F^∞_α, we must have $f \in F^\infty_\alpha$, which implies that $f(z)/z^2 \in F^p_\alpha$ when $p > 1/2$. Thus $f'(z)/z \in F^p_\alpha$ implies $f'(z)/z - f(z)/z^2 \in F^p_\alpha$, which yields $f \in F^p_\alpha$ for $p > 1/2$.

For $0 < p \leqslant 1/2$ (actually, the argument below works for $p > 1/2$ as well), we write

$$\frac{f'(z)}{z} = \left[\frac{f'(z)}{z} - \frac{f(z)}{z^2}\right] + \frac{f(z)}{z^2}.$$

By the triangle inequality for the distance

$$d(f_1, f_2) = \int_{\mathbb{C}} |f_1(z) - f_2(z)|^p \, \mathrm{d}\lambda_\alpha(z)$$

in F^p_α (technically, we should first work with $f_r(z) = f(rz)$, $0 < r < 1$, and then use an approximation argument in order to make sure that all the integrals below all converge) and one of the main results in [4], there exists a positive constant c such that

$$\begin{aligned}\int_{\mathbb{C}} \left|\frac{f'(z)}{z}\right|^p \mathrm{d}\lambda_\alpha(z) &\geqslant \int_{\mathbb{C}} \left|\frac{f'(z)}{z} - \frac{f(z)}{z^2}\right|^p \mathrm{d}\lambda_\alpha(z) - \int_{\mathbb{C}} \left|\frac{f(z)}{z^2}\right|^p \mathrm{d}\lambda_\alpha(z)\\ &\geqslant c \int_{\mathbb{C}} |f(z)|^p \, \mathrm{d}\lambda_\alpha(z) - \int_{\mathbb{C}} \left|\frac{f(z)}{z^2}\right|^p \mathrm{d}\lambda_\alpha(z).\end{aligned}$$

Choose a positive radius R such that $c - 1/R^{2p} > 0$. Then

$$\begin{aligned}\int_{\mathbb{C}} \left|\frac{f'(z)}{z}\right|^p \mathrm{d}\lambda_\alpha(z) &\geqslant \left(c - \frac{1}{R^{2p}}\right) \int_{|z|>R} |f(z)|^p \, \mathrm{d}\lambda_\alpha(z)\\ &\quad + c \int_{|z|\leqslant R} |f(z)|^p \, \mathrm{d}\lambda_\alpha(z) - \int_{|z|\leqslant R} \left|\frac{f(z)}{z^2}\right|^p \mathrm{d}\lambda_\alpha(z)\end{aligned}$$

whenever $f(0) = f'(0) = 0$. This shows that

$$\int_{|z|>R} |f(z)|^p \, \mathrm{d}\lambda_\alpha(z) < \infty$$

and hence

$$\int_{\mathbb{C}} |f(z)|^p \, \mathrm{d}\lambda_\alpha(z) < \infty$$

if the function $f'(z)/z$ belongs to F^p_α. □

Corollary 3.3. — *Suppose $\alpha > 0$, $0 < p \leqslant \infty$, and $f \in H(\mathbb{C})$ with $f(0) = f'(0) = 0$. Then for any constant c (including $c = 0$) we have $f \in F^p_\alpha$ if and only if the function*

$$\frac{f'(z)}{z} + c\,\frac{f(z)}{z^2}$$

belongs to F^p_α.

Proof. — This follows from Theorem 3.2 and its proof. □

Theorem 3.4. — *Suppose f is an entire function on $\mathbb{C}$, N is a positive integer, and $0 < p \leqslant \infty$. Then $f \in F^p_\alpha$ if and only if the functions $f^{(N)}(z)/(1 + |z|)^N$ belongs to L^p_α.*

Proof. — We prove this by induction on N. The case $N=1$ has already been proved.

Suppose the result holds for some positive integer N. We proceed to show that $f \in F^p_\alpha$ if and only if the function $f^{(N+1)}(z)/(1+|z|^{N+1})$ belongs to L^p_α. Without loss of generality. we may assume that

$$f(0) = f'(0) = \cdots = f^{(2N+2)}(0) = 0.$$

In this case, we just need to show that $f \in F^p_\alpha$ if and only if the function $f^{(N+1)}(z)/z^{N+1}$ belongs to L^p_α.

By the induction hypothesis, we have that $f \in F^p_\alpha$ if and only if the function $f^{(N)}(z)/(1+|z|)^N$ belongs to L^p_α, which is the same as the function $g(z) = f^{(N)}(z)/z^N$ belonging to F^p_α. By Corollary 3.3, this is equivalent to the function

$$\frac{g'(z)}{z} + N\,\frac{g(z)}{z^2} = \frac{f^{(N+1)}(z)}{z^{N+1}} - \frac{Nf^{(N)}(z)}{z^{N+2}} + \frac{Nf^{(N)}(z)}{z^{N+2}} = \frac{f^{(N+1)}(z)}{z^{N+1}}$$

belonging to F^p_α. So the desired result is true for $N+1$, and the proof of Theorem 3.4 is complete. □

Serious obstacles arise when we try the arguments above in higher dimensions, although several steps still work. In particular, the "only if part" of Theorem 3.2 in the higher dimensional case follows easily from the one-dimensional case. In fact, by Theorem 3.2 and the closed-graph theorem, there exists a positive constant C (independent of f) such that

$$\int_{\mathbb{C}} \left| \frac{f'(z)e^{-\alpha|z|^2/2}}{1+|z|} \right|^p \mathrm{d}A(z) \leqslant C \int_{\mathbb{C}} \left| f(z)e^{-\alpha|z|^2/2} \right|^p \mathrm{d}A(z)$$

for all $f \in H(\mathbb{C})$, where $\mathrm{d}A$ is ordinary area measure on $\mathbb{C}$. Now if $f \in H(\mathbb{C}^n)$ and $1 \leqslant k \leqslant n$, then

$$\int_{\mathbb{C}} \left| \frac{\partial_k f(z_1, \ldots, z_k, \ldots, z_n)e^{-\alpha|z_k|^2/2}}{1+|z_k|} \right|^p \mathrm{d}A(z_k) \\ \leqslant C \int_{\mathbb{C}} \left| f(z_1, \ldots, z_k, \ldots, z_n)e^{-\alpha|z_k|^2/2} \right|^p \mathrm{d}A(z_k),$$

where C is independent of the $n-1$ variables $\{z_1, \ldots, z_n\} \setminus \{z_k\}$. Since $1/(1+|z|) \leqslant 1/(1+|z_k|)$ for $1 \leqslant k \leqslant n$, $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$, and the Gaussian measure on $\mathbb{C}^n$ is a product measure, we easily deduce that

$$\int_{\mathbb{C}^n} \left| \frac{\partial_k f(z)e^{-\alpha|z|^2/2}}{1+|z|} \right|^p \mathrm{d}v(z) \leqslant C \int_{\mathbb{C}^n} \left| f(z)e^{-\alpha|z|^2/2} \right|^p \mathrm{d}v(z).$$

It then follows that there exists another positive constant C such that

$$\int_{\mathbb{C}^n}\left[\frac{|\nabla f(z)|}{1+|z|}\,e^{-\alpha|z|^2}\right]^p \mathrm{d}v(z) \leqslant C\int_{\mathbb{C}^n}\left|f(z)e^{-\alpha|z|^2/2}\right|^p \mathrm{d}v(z)$$

for all $f\in H(\mathbb{C}^n)$. Since $|Rf(z)|\leqslant |z||\nabla f(z)|$, we can also find a positive constant C such that

$$\int_{\mathbb{C}^n}\left|\frac{Rf(z)}{1+|z|^2}\,e^{-\alpha|z|^2/2}\right|^p \mathrm{d}v(z) \leqslant C\int_{\mathbb{C}^n}\left|f(z)e^{-\alpha|z|^2/2}\right|^p \mathrm{d}v(z)$$

for all $f\in F^p_\alpha$.

When $p=1$, the other direction of the inequalities above can also be proved using elementary arguments. In fact, it follows from (2.2) and Fubini's theorem that

$$\begin{aligned}
\int_{\mathbb{C}^n}|f(z)-f(0)|e^{-\beta|z|^2}\,\mathrm{d}v(z) &\leqslant \int_{\mathbb{C}^n} e^{-\beta|z|^2}\,\mathrm{d}v(z)\int_0^1\frac{|Rf(tz)|}{t}\,\mathrm{d}t\\
&=\int_0^1\frac{\mathrm{d}t}{t}\int_{\mathbb{C}^n}|Rf(tz)|e^{-\beta|z|^2}\,\mathrm{d}v(z)\\
&=\int_0^1\frac{\mathrm{d}t}{t^{2n+1}}\int_{\mathbb{C}^n}|Rf(z)|e^{-\beta|z|^2/t^2}\,\mathrm{d}v(z)\\
&=\int_{\mathbb{C}^n}|Rf(z)|\,\mathrm{d}v(z)\int_0^1\frac{e^{-\beta|z|^2/t^2}}{t^{2n+1}}\,\mathrm{d}t\\
&=\frac{1}{2\beta^n}\int_{\mathbb{C}^n}\frac{|Rf(z)|}{|z|^{2n}}\,\mathrm{d}v(z)\int_{\beta|z|^2}^\infty s^{n-1}e^{-s}\,\mathrm{d}s.
\end{aligned}$$

An argument using mathematical induction shows that the incomplete gamma function

$$\Gamma(n,x)=\int_x^\infty s^{n-1}e^{-s}\,\mathrm{d}s,\quad x\in(0,\infty),$$

has the property that $\Gamma(n,x)\sim x^{n-1}e^{-x}$ as $x\to\infty$, where n is any positive integer. It follows that there exists a positive constant C such that

$$\int_{\mathbb{C}^n}|f(z)-f(0)|e^{-\beta|z|^2}\,\mathrm{d}v(z)\leqslant C\int_{\mathbb{C}^n}\frac{|Rf(z)|}{1+|z|^2}e^{-\beta|z|^2}\,\mathrm{d}v(z).$$

This together with (2.1) shows that we also have

$$\int_{\mathbb{C}^n}|f(z)-f(0)|e^{-\beta|z|^2}\,\mathrm{d}v(z)\leqslant C\int_{\mathbb{C}^n}\frac{|\nabla f(z)|}{1+|z|}e^{-\beta|z|^2}\,\mathrm{d}v(z),$$

where the positive constant C only depends on n and β.

4. Further remarks

It follows from the analysis in previous sections that we have the following results about Fock spaces in terms of the radial derivative.

Corollary 4.1. — *Suppose $f \in H(\mathbb{C}^n)$ and $\alpha > 0$.*

(a) *If $0 < p \leqslant \infty$ and $f \in F^p_\alpha$, then the function $Rf(z)/(1+|z|^2)$ belongs to L^p_α.*
(b) *If $p = 1$ or $p = \infty$, and if the function $Rf(z)/(1+|z|^2)$ belongs to L^p_α, then $f \in F^p_\alpha$.*

It is therefore very natural for us to make the following conjecture.

Conjecture 4.2. — *Suppose $0 < p \leqslant \infty$, $\alpha > 0$, and $f \in H(\mathbb{C}^n)$. Then $f \in F^p_\alpha$ if and only if the function $Rf(z)/(1+|z|^2)$ belongs to L^p_α. More generally, if N is any positive integer, then $f \in F^p_\alpha$ if and only if the function $R^N f(z)/(1+|z|^2)^N$ belongs to L^p_α.*

The "only if" parts above follows from Theorem 3.1 and the expression of $R^N f$ in terms of partial derivatives. For example, if $N = 2$, we have

$$\begin{aligned} R^2 f &= R(z_1\partial_1 f + \cdots + z_n \partial_n f) \\ &= \sum_{k=1}^n R(z_k \partial_k f) = \sum_{k=1}^n \sum_{j=1}^n z_j \partial_j (z_k \partial_k f) \\ &= \sum_{k=1}^n \left[z_k \partial_k f + \sum_{j=1}^n z_j z_k \frac{\partial^2 f}{\partial z_j \partial z_k} \right] \\ &= Rf + \sum_{j,k=1}^n z_j z_k \frac{\partial^2 f}{\partial z_j \partial z_k}. \end{aligned}$$

Similar formulas can be obtained for $R^N f$ when N is any positive integer.

It is clear from the previous sections that, for each $\alpha > 0$, the distance function $d_\alpha(z,w)$ plays a signficant role in the study of the Fock spaces F^p_α. However, we have very limited information about these distance functions. Proposition 1.2 gives a good estimate for $d_\alpha(0,z)$. A natural question is whether or not we can use the estimate in Proposition 1.2 together with Weyl unitary operators (see [10]) to obtain optimal estimates for the distance function $d_\alpha(z,w)$. Our attempts so far have been unsuccessful.

Recall that Bergman spaces A^p can be characterized by Lipschitz type conditions

$$|f(z) - f(w)| \leqslant \beta(z,w)\left[g(z) + g(w)\right],$$

where $g \in L^p(\mathbb{B}^n, dv)$ and $\beta(z,w)$ is the Bergman distance between z and w. See [7]. It is natural to ask whether or not something similar is also true for Fock spaces. We make the following conjecture here.

CONJECTURE 4.3. — *Suppose $\alpha > 0$, $0 < p \leqslant \infty$, and f is an entire function on $\mathbb{C}^n$. Then $f \in F^p_\alpha$ if and only if there exists a non-negative continuous function $g \in L^p(\mathbb{C}^n, dv)$ such that*

$$|f(z) - f(w)| \leqslant d_\alpha(z,w)\,[g(z) + g(w)] \tag{4.1}$$

for all $z, w \in \mathbb{C}^n$.

If f satisfies the Lipschitz type condition in (4.1), then

$$\frac{|f(z) - f(w)|}{|z-w|} \leqslant \frac{d_\alpha(z,w)}{|z-w|}\,[g(z) + g(w)]$$

for all $z \neq w$ in $\mathbb{C}^n$. Fix z, let $w \to z$, and use (1.2) and Lemma 1.5. We obtain a positive constant C such that

$$|\partial_k f(z)| \leqslant C(1+|z|)e^{\alpha|z|^2/2} g(z), \quad 1 \leqslant k \leqslant n, z \in \mathbb{C}^n.$$

It follows that the functions $\partial_k f(z)/(1+|z|)$, $1 \leqslant k \leqslant n$, all belong to L^p_α. By Theorem 3.1, we have $f \in F^p_\alpha$.

To prove the other direction, it seems that we need more detailed information and more properties of the distance function $d_\alpha(z,w)$, which are not available at this point. We intend to pursue these issues in a future paper.

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