Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

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Article à paraître, mis en ligne le 26 mars 2026, 28 p.

CENTRE MERSENNE

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
Article à paraître
Mis en ligne le 26 mars 2026.

UNIQUE ERGODICITY FOR RANDOM NONINVERTIBLE MAPS ON AN INTERVAL

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ABSTRACT. — In this short note, we investigate noninvertible stochastic dynamical systems on the unit interval $[0,1]$. We provide a handy condition for unique ergodicity for systems that are injective in mean. On the other hand, we give concrete examples where unique ergodicity fails.

RÉSUMÉ. — Dans cette courte note, nous étudions des systèmes dynamiques stochastiques non inversibles de l'intervalle $[0,1]$. Nous proposons une condition maniable pour assurer l'unique ergodicité des systèmes qui sont injectifs en moyenne. D'autre part, nous construisons des exemples concrets où l'unicité ergodique échoue.

1. Introduction

Ergodicity is the key concept in the theory of dynamical systems. It comes from statistical physics but also captures some nice properties of stochastic processes. This note is concerned with the ergodic properties of Markov chains corresponding to random iteration of maps on the interval $[0,1]$.

Let $\mathcal{M}$ be a set of piecewise monotone (with finitely many pieces) continuous functions on $[0,1]$. Let μ be a Borel probability measure on the space $C([0,1])$ (equipped with the topology of uniform convergence) such that $\mu(\mathcal{M})=1$. The pair $(\mathcal{M},\mu)$ will be called a *stochastic dynamical system* (SDS). The dynamical system $(\mathcal{M},\mu)$ induces by recursion a Markov chain on $[0,1]$ given by the formula:

$$X_n^x := \mathbf{f}_n(X_{n-1}^x) \quad \text{for } n \geqslant 1 \text{ and } X_0^x = x.$$

Keywords: unique ergodicity, stationary measures, dynamical systems, random interval maps, entropy.
2020 *Mathematics Subject Classification:* 37E05, 60G10, 37A50, 28D20.
(*) T.S. was supported by the Polish NCN Grant 2022/45/B/ST1/00135. H.O. was partially supported by Early Stage Funding – Vicerector for research, Universität Innsbruck.

Here $(\mathbf{f}_n)_{n\in\mathbb{N}}$ is a sequence of independent $\mathcal{M}$-valued random variables with distribution μ. The transition probability of this Markov chain is given by the formula:

$$\pi(x,A) = \int_{\mathcal{M}} \mathbf{1}_A\big(f(x)\big)\,\mathrm{d}\mu(f) \quad \text{for } x\in[0,1] \text{ and } A\in\mathcal{B}([0,1]).$$

A Borel probability measure ν on $[0,1]$ is called *μ-invariant* (also known as *μ-stationary*) if

$$\nu(A) = \int_{\mathcal{M}} \nu(f^{-1}A)\,\mathrm{d}\mu(f) \quad \text{for every } A\in\mathcal{B}([0,1]).$$

Observe that due to the compactness of $[0,1]$ and the continuity of the functions in $\mathcal{M}$ there always exists at least one μ-invariant probability measure on the closed interval $[0,1]$. Our aim is to formulate sufficient conditions for the existence of a unique μ-invariant probability measure. We will also provide examples where uniqueness fails.

The problem of unique ergodicity for random dynamical systems has been intensively studied recently but mainly in the case when μ is supported on some subgroup of homeomorphisms (see [1, 8, 9, 10, 12, 17, 19]). The paper by A. Homburg et al. [13] is in fact an exception. Namely, the authors prove unique ergodicity for some systems including logistic maps using the criterion for the existence of an invariant measure absolutely continuous with respect to Lebesgue measure derived for unimodal maps with negative Schwarzian derivative (see [20]). Further, under the assumption of some average contractivity, Kloeckner [14] proved unique ergodicity for quite general iterated function systems defined on complete metric spaces. Similar results on this topic were obtained by Czapla in [7].

The paper is aimed at proving unique ergodicity for SDSs that are generated by non-injective maps, but such that a generic point $x\in[0,1]$ has, on average, not more than one preimage. We call this property *μ-injectivity*. For such systems, we can generalize some techniques developed for diffeomorphisms by Avila–Viana [3], Deroin–Kleptsyn–Navas [9], Baxendale [4] and extended to random homeomorphisms on the circle by Malicet [17].

Definition 1.1. — *Let μ be a probability measure on $C([0,1])$ with* $\operatorname{supp}(\mu)=\mathcal{M}$.

(1) *We say that a Borel probability measure ν on $[0,1]$ is* μ-injective *in $x\in[0,1]$ if*

$$\int_{\mathcal{M}} n(f,x)\,\mathrm{d}\mu(f) \leqslant 1,$$

where $n(f,x)$ denotes the cardinality of the preimage $f^{-1}(\{x\})$.

(2) *We say that* $(\mathcal{M},\mu)$ contracts a neighborhood of $x_0 \in [0,1]$ *if there exists* $\varepsilon > 0$ *such that*

$$\mu\big(\big\{f \in \mathcal{M} \,:\, f(x_0) = x_0,\ \big|f(x) - f(x_0)\big| < |x - x_0|,\ \forall\, x \in B_\epsilon(x_0)\big\}\big) > 0,$$

where $B_\epsilon(x_0) := [x_0 - \epsilon, x_0 + \epsilon] \cap [0,1]$.

The two above conditions imply uniqueness as follows.

Theorem 1.2. — *Let* $(\mathcal{M},\mu)$ *be a stochastic dynamical system* μ*-injective in all but countably many* $x \in [0,1]$ *and assume that it contracts a neighborhood of some point* $x_0 \in [0,1]$*. Let* ν *be an atomless, ergodic* μ*-invariant probability measure with* $x_0 \in \operatorname{supp}(\nu)$*. Then any other* μ*-invariant, ergodic Borel probability measure whose support contains* $\operatorname{supp}(\nu)$ *coincides with* ν*.*

While the above theorem does not concern the existence of such measures, we provide conditions for the existence of a μ-invariant probability measure on $(0,1)$ in Corollary 4.1 in Section 4.

The paper is organized as follows. We start by showing in Section 2 that for stochastic dynamical systems defined by piecewise monotone functions, it is possible to extend the notion of Furstenberg entropy $h_\mu(\nu)$ of an invariant probability ν based on Radon–Nikodym derivatives and a pullbackward set function $f^{-1}\nu$. As in the invertible case, if the measure ν is atomless and the entropy is strictly positive, then the SDS is locally contracting (Proposition 2.3) and ergodic measures are explicitly determined by their support (Theorem 2.2).

In Section 3, we prove that for μ-injective systems the entropy is nonnegative and give a handy condition ensuring that it is strictly positive (Theorem 3.6). Together with Theorem 2.2, this proves the above theorem.

In Section 4, we provide explicit examples of noninvertible μ-injective SDSs that admit a unique invariant probability measure (Corollary 4.1).

Finally, in the last part of this note, we present an example of a random system without μ-injectivity that possesses different invariant ergodic measures.

Acknowledgments

The authors are very grateful to the anonymous referee for carefully reading the paper and providing useful and constructive comments, which improved the paper.

2. Generalized entropy for piecewise monotone SDS

Let $\mathcal{M}$ be a closed set of piecewise strictly monotone continuous functions from $[0,1]$ to $[0,1]$ with finitely many pieces, i.e. for every function $f \in \mathcal{M}$ one can find finitely many closed intervals which cover $[0,1]$ such that f is strictly monotone on them. By $\mathcal{I}(f) := \{I_i^f\}_{i=1}^{N(f)} = \big\{[d_{i-1}^f, d_i^f]\big\}_{i=1}^{N(f)}$ we shall denote the collection of closed monotone intervals of f covering $[0,1]$ and such that the monotonicity type of f on $[d_{i-1}^f, d_i^f]$ and $[d_i^f, d_{i+1}^f]$ for $i = 1, \dots, N(f) - 1$ is different. Observe that for every $\mathcal{I}(f)$ there exist homeomorphisms γ_i^f on $[0,1]$ such that $f(x) = \gamma_i^f(x)$ for all $x \in I_i^f$. Let $\mathcal{D}(f)$ denote the set of all points where f changes its monotonicity type, i.e. $\mathcal{D}(f) = \bigcup_{I \in \mathcal{I}(f)} \partial I = \{d_0^f, d_1^f, \dots, d_{N(f)}^f\}$.

2.1. Pushforward and pullbackward measures

Let ν be a Borel probability measure on $[0,1]$, and let $f \in \mathcal{M}$.

We define the *pushforward measure* $f\nu$ as

$$f\nu(A) := \nu\big(f^{-1}(A)\big) \quad \text{for } A \in \mathcal{B}([0,1]),$$

where $f^{-1}(A)$ denotes the preimage of A and $\mathcal{B}([0,1])$ denotes the Borel σ-algebra on $[0,1]$.

Note that if $f \in \mathcal{M}$ the images $f(A) := \big\{f(x) \,:\, x \in A\big\}$, $A \in \mathcal{B}([0,1])$, are Borel measurable and we can define a *pullbackward set function* $f^{-1}\nu$ on Borel sets as

$$f^{-1}\nu(A) := \nu\big(f(A)\big) \quad \text{for } A \in \mathcal{B}([0,1]).$$

In general, $f^{-1}\nu$ is not a measure, since it is not σ-additive (if A_1 and A_2 are two disjoint sets such that $f(A_1) = f(A_2)$ then $f^{-1}\nu(A_1 \cup A_2) = f^{-1}\nu(A_1) = f^{-1}\nu(A_2)$). However, if f coincides with the homeomorphism γ_i^f on an interval $I_i \in \mathcal{I}(f)$, then $f^{-1}\nu$ locally coincides with the measure $\nu_i^f(A) := (\gamma_i^f)^{-1}\nu(A \cap I_i^f)$ (the restriction of $(\gamma_i^f)^{-1}\nu$ to I_i^f) in the sense that

$$f^{-1}\nu(A) = (\gamma_i^f)^{-1}\nu(A) = \nu_i^f(A) \quad \text{for } A \subseteq I_i^f. \tag{2.1}$$

Furthermore, $f^{-1}\nu$ is bounded above by the measure $\bar{\nu}^f$ obtained as the sum of the measures ν_i^f

$$f^{-1}\nu(A) \leqslant \sum_{i=1}^{N(f)} \nu_i^f(A) =: \bar{\nu}^f(A) \quad \text{for all } A \in \mathcal{B}([0,1]). \tag{2.2}$$

2.2. Radon–Nikodym derivative for finite measures on $[0,1]$

Let λ and ν be two probability measures on the interval $[0,1]$. Then, the Radon–Nikodym derivative $\mathrm{d}\lambda/\mathrm{d}\nu$ is defined as

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(x) := \lim_{r\to 0}\frac{\lambda\big(B_r(x)\big)}{\nu\big(B_r(x)\big)} \quad \text{for } \nu\text{-almost every } x\in[0,1],$$

where $B_r(x) := [x-r,x+r]\cap[0,1]$. The limit exists ν-almost everywhere, and the Lebesgue decomposition of λ is given by $\mathrm{d}\lambda = \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}\,\mathrm{d}\nu + \mathrm{d}\lambda_{\mathrm{sing}}$ with $\lambda_{\mathrm{sing}} \perp \nu$. In particular,

$$\int_A \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(x)\,\mathrm{d}\nu(x) \leqslant \lambda(A) \quad \text{for all Borel sets } A\subseteq[0,1],$$

with equality if and only if $\lambda \ll \nu$ (see, for instance, [18, Chapter 2]).

For any $x\in(0,1)$ we denote

$$\mathcal{J}^x := \big\{ I \,:\, I\subset[0,1] \text{ is a closed interval and } x\in\mathrm{int}(I)\big\}. \tag{2.3}$$

Let $|I|$ denote the length of the interval. Let $\ln^+(x) = \max\big\{\ln(x),0\big\}$ for $x>0$.

In the sequel, we will need the following result concerning Radon–Nikodym derivatives.

Proposition 2.1. — *Let λ be a finite Borel measure, and let ν be a Borel probability measure on the interval* $[0,1]$.

(1) *For ν-almost every $x\in[0,1]$ we have*

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(x) = \lim_{\delta\to 0}\sup_{\substack{I\in\mathcal{J}^x\\ |I|<\delta}}\frac{\lambda(I)}{\nu(I)}.$$

(2) *Let $Q^*_\nu(\lambda,x) := \sup_{I\in\mathcal{J}^x}\frac{\lambda(I)}{\nu(I)}$. Then*

$$\int \ln^+ Q^*_\nu(\lambda,x)\,\mathrm{d}\nu(x) \leqslant 2\lambda([0,1]).$$

This result is well-known in the case ν is the Lebesgue measure or for general measures when the supremum is taken over centered intervals (see [18, Chapter 2] or [16, Proposition 5]). The stated result (for general probability measures and non-centered intervals) seems to be a folklore theorem, but since we could not find a precise reference, we provide its proof in the appendix.

2.3. Radon–Nikodym derivative of $f^{-1}\nu$

Since the set function $f^{-1}\nu$ is locally a measure, we can define a *generalized Radon–Nikodym derivative* for ν-almost every x in the interior of $I_i^f \in \mathcal{I}(f)$ by

$$\mathrm{d}_\nu f(x) = \frac{\mathrm{d}f^{-1}\nu}{\mathrm{d}\nu}(x) := \lim_{r\to 0} \frac{\nu\big(f\big(B_r(x)\big)\big)}{\nu\big(B_r(x)\big)} \quad \text{for } \nu\text{-a.e. } x \notin \mathcal{D}(f).$$

This limit is almost everywhere well-defined since "locally" $f^{-1}\nu$ can be viewed as a measure, and

$$\mathrm{d}_\nu f(x) = \frac{\mathrm{d}\nu_i^f}{\mathrm{d}\nu}(x) \quad \text{for } \nu\text{-a.e. } x \in \mathrm{int}(I_i^f) = (d_{i-1}^f, d_i^f).$$

2.4. Entropic criterion for characterizing ergodic measures

Let μ be a Borel probability measure supported on $\mathcal{M}$, and let ν be a Borel probability measure *with no atoms*. Then, for μ-almost every f, the derivative $\mathrm{d}_\nu f(x)$ is well-defined for ν-almost every $x \in [0,1]$ (in fact, $\nu\big(\mathcal{D}(f)\big) = 0$).

If $\ln^+ \mathrm{d}_\nu f(x)$ is $(\nu\times\mu)$-integrable, we can define the *(generalized) Furstenberg entropy* as

$$h_\mu(\nu) := -\int_{\mathcal{M}\times[0,1]} \ln\big(\mathrm{d}_\nu f(x)\big)\,\mathrm{d}\mu(f)\,\mathrm{d}\nu(x) \in (-\infty, +\infty].$$

In the case when μ is supported on invertible functions, for general probability measures μ on $\mathcal{M}$, the entropy can be negative. However, we will see in this section that if one can guarantee that the entropy is strictly positive, then the SDS has nice contraction properties, ensuring that, in some sense, ergodic invariant measures are determined by their support.

Recall that a μ-invariant measure ν is called *ergodic* if for every subset $A \subset [0,1]$ such that ν_A, the restriction of ν to A, is μ-invariant, we have either $\nu(A) = 0$ or $\nu\big([0,1]\setminus A\big) = 0$. For a more detailed survey on ergodic measures for SDS, we refer to the appendix of [6].

Let

$$J(x,f) := \sup\left\{ \frac{\nu\big(f(I)\big)}{\nu(I)} \ :\ I \in \mathcal{J}^x \right\} \in [0, +\infty]$$

with the convention $\frac{0}{0} = 0$. Observe that by definition $\mathrm{d}_\nu f(x) \leqslant J(x,f)$. Thus if $\ln^+ J$ is $(\nu\times\mu)$-integrable then the entropy is well-defined and we have the following:

Theorem 2.2 (Entropic criterion for ergodic measures). — *Let μ be a probability measure on $\mathcal{M}$. Let η and ν be μ-invariant, ergodic Borel probability measures on $[0,1]$. Suppose that ν is atomless and $\operatorname{supp}\nu \subseteq \operatorname{supp}\eta$. If $\ln^+ J \in L^1\big([0,1]\times\mathcal{M}, \nu\times\mu\big)$ and $h_\mu(\nu) > 0$, then we have $\nu = \eta$.*

This theorem is a direct consequence of the following result that ensures that for a given SDS the positive entropy implies that the system contracts small intervals at exponential rate.

Proposition 2.3 (Contractivity). — *Let μ be a Borel probability measure on $\mathcal{M}$, and let ν be a μ-invariant atomless ergodic Borel probability measure on the interval $[0,1]$ such that $\ln^+ J$ is $(\nu\times\mu)$-integrable and $h_\mu(\nu) > 0$. Then for every $h \in \big(0, h_\mu(\nu)\big)$ and $\mu^{\mathbb{N}}$-a.e. $\omega = (f_n)_{n\in\mathbb{N}} \in \mathcal{M}^{\mathbb{N}}$ and ν-a.e. $x \in [0,1]$ there exists a closed interval $I = I(\omega, x)$ with $x \in \operatorname{int}(I)$ such that*

$$\big|f_n \circ \cdots \circ f_1(I)\big| \leqslant \exp(-n\cdot h), \quad \forall\, n \in \mathbb{N}.$$

D. Malicet [17] proved this result when f are homeomorphisms of the circle. With some precaution, his technique can be easily adapted to the case of piecewise monotone functions. For completeness, a proof is given in the appendix.

Proof of Theorem 2.2. — Assume that ν is an ergodic, μ-invariant, atomless Borel probability measure on $[0,1]$ with $\operatorname{supp}(\nu) \subseteq \operatorname{supp}(\eta)$. Assume $\nu \neq \eta$. Then there exists a continuous function ξ on $[0,1]$ such that

$$\int_{[0,1]} \xi \,\mathrm{d}\nu \neq \int_{[0,1]} \xi \,\mathrm{d}\eta.$$

By Birkhoff's ergodic theorem, for $\mu^{\otimes\mathbb{N}}$-a.e. $\omega = (f_1, f_2, \dots) \in \mathcal{M}^{\mathbb{N}}$ we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \xi\big(f_i \circ \cdots \circ f_1(x)\big) = \int_{[0,1]} \xi \,\mathrm{d}\nu \quad \text{for } \nu\text{-a.e. } x \in [0,1]. \tag{2.4}$$

Fix $h \in \big(0, h_\mu(\nu)\big)$. By Proposition 2.3, there exist $x_0 \in \operatorname{supp}\nu$ and $\Xi = \Xi(x) \in \mathcal{B}(\mathcal{M}^{\mathbb{N}})$ with $\mu^{\otimes\mathbb{N}}(\Xi) > 0$ such that, for all $\omega = (f_1, f_2, \dots) \in \Xi$, convergence (2.4) holds and

$$\big|f_n \circ \cdots \circ f_1(I)\big| \leqslant \exp(-n\cdot h) \tag{2.5}$$

for some neighborhood $I = I(x_0, \omega)$ of x_0. Moreover, since $x_0 \in \operatorname{supp}(\nu) \subseteq \operatorname{supp}(\eta)$, we have $\eta(I) > 0$. Thus, there exists $y \in I$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \xi\big(f_i \circ \cdots \circ f_1(y)\big) = \int_{[0,1]} \xi \,\mathrm{d}\eta \neq \int_{[0,1]} \xi \,\mathrm{d}\nu,$$

for $\mu^{\otimes\mathbb{N}}$-a.e. $(f_1, f_2, \dots) \in \Xi$, which is impossible because ξ is continuous and

$$\big|f_n \circ \cdots \circ f_1(x) - f_n \circ \cdots \circ f_1(y)\big| \leqslant \big|f_n \circ \cdots \circ f_1(I)\big| \longrightarrow 0 \quad \text{as } n \to \infty,$$

by condition (2.5). The proof is complete. □

3. μ-injectivity and positive entropy

In the previous section, we saw that positive entropy can be used to prove the stability of SDSs. However, calculating the entropy can be challenging, so we want to provide in this section a more manageable condition.

3.1. Cardinality of pre-images

For $f \in \mathcal{M}$ and $x \in [0,1]$, let us denote

$$n(f,x) \coloneqq \#\big(f^{-1}(x)\big),$$

i.e., the number of preimages of x under the map f, with the convention $\#(\emptyset) \coloneqq 0$. Note that $f \mapsto n(f,x)$ is measurable for every $x \in [0,1]$. Observe also that if $\{I_i^f\}_{i=1}^{N(f)}$ is a collection of covering intervals such that f is monotone on each of the pieces, then we have

$$n(f,x) = \sum_{i=1}^{N(f)} \mathbf{1}_{f(I_i^f)}(x) \quad \text{for all } x \notin \mathcal{D}(f). \tag{3.1}$$

The following lemma shows that if the number of preimages is bounded in mean, then the entropy of the system is well-defined.

Lemma 3.1. — *If ν has no atoms and $n \in L^1\big(\mathcal{M} \times [0,1], \mu \times \nu\big)$, then $\ln^+ J \in L^1\big(\mathcal{M} \times [0,1], \mu \times \nu\big)$ and $h_\mu(\nu)$ is well-defined.*

Proof. — Observe that by (2.2)

$$J(f,x) = \sup_{I \in \mathcal{J}^x} \frac{\nu\big(f(I)\big)}{\nu(I)} \leqslant \sup_{I \in \mathcal{J}^x} \frac{\overline{\nu}^f(I)}{\nu(I)} =: Q_\nu^*(\overline{\nu}^f, x),$$

where $\mathcal{J}^x$ is given by (2.3) and $\overline{\nu}$ is a finite Borel measure defined in (2.2) as sum of finitely many probability measures. Thus by Proposition 2.1

and (3.1),

$$\begin{aligned}\int_{[0,1]} \ln^+ J(f,x)\,\mathrm{d}\nu(x) &\leqslant \int_{[0,1]} \ln^+ Q^*_\nu(\overline{\nu}^f, x)\,\mathrm{d}\nu(x)\\ &\leqslant 2\overline{\nu}^f([0,1])\\ &= 2\sum_{i\in\mathcal{I}(f)} \nu_i^f([0,1])\\ &= 2\sum_{i\in\mathcal{I}(f)} \nu\big(f(I_i^f)\big)\\ &= 2\int_{[0,1]} \sum_{i\in\mathcal{I}(f)} \mathbf{1}_{f(I_i^f)}(x)\,\mathrm{d}\nu(x)\\ &= 2\int_{[0,1]} n(f,x)\,\mathrm{d}\nu(x),\end{aligned}$$

since $\nu\big(\mathcal{D}(f)\big) = 0$. □

3.2. Entropy of μ-injective SDS

Definition 3.2. — *We say that a stochastic dynamical system $(\mathcal{M},\mu)$ is* μ-injective in $x\in[0,1]$ *if*

$$\int_{\mathcal{M}} n(f,x)\,\mathrm{d}\mu(f) \leqslant 1. \tag{3.2}$$

Proposition 3.3. — *Let $(\mathcal{M},\mu)$ be a stochastic dynamical system, and let ν be an atomless Borel probability measure on $[0,1]$. If $(\mathcal{M},\mu)$ is μ-injective for ν-almost all $x\in[0,1]$, then either $h_\mu(\nu) > 0$ (possibly infinite) or $\mathrm{d}_\nu f(x) \equiv 1$ for ν-a.e. $x\in[0,1]$ and μ-a.e. $f\in\mathcal{M}$.*

Proof. — Since $f^{-1}\nu$ coincides with the measure $\nu_i^f := (\gamma_i^f)^{-1}\nu\big|_{I_i^f}$ on each $I_i^f \in I(f)$, we have

$$\begin{aligned}\int_{[0,1]} \mathrm{d}_\nu f(x)\,\mathrm{d}\nu(x) &= \sum_{i\in\mathcal{I}(f)} \int_{I_i^f} \frac{\mathrm{d}f^{-1}\nu}{\mathrm{d}\nu}(x)\,\mathrm{d}\nu(x)\\ &= \sum_{i\in\mathcal{I}(f)} \int_{I_i^f} \frac{\mathrm{d}\nu_i^f}{\mathrm{d}\nu}(x)\,\mathrm{d}\nu(x)\\ &\leqslant \sum_{i\in I} \nu\big(f(I_i^f)\big)\\ &= \int_{[0,1]} n(f,x)\,\mathrm{d}\nu(x),\end{aligned} \tag{3.3}$$

where the last equality follows from (3.1) and the fact that $\nu\big(\mathcal{D}(f)\big) = 0$. Thus, by Jensen's inequality and Fubini's theorem,

$$\begin{aligned} h_\mu(\nu) &= \int_{\mathcal{M}} \int_{[0,1]} -\ln\big(\mathrm{d}_\nu f(x)\big)\,\mathrm{d}\nu(x)\,\mathrm{d}\mu(f) \\ &\geqslant -\ln\left(\int_{\mathcal{M}} \int_{[0,1]} \mathrm{d}_\nu f(x)\,\mathrm{d}\nu(x)\,\mathrm{d}\mu(f)\right) \\ &\geqslant -\ln\left(\int_{[0,1]} \int_{\mathcal{M}} n(f,x)\,\mathrm{d}\mu(f)\,\mathrm{d}\nu(x)\right). \end{aligned}$$

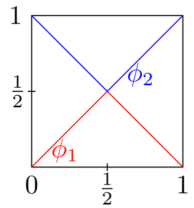
Now let us assume that (3.2) holds for ν-a.e. x. Then, as above by Jensen's inequality, we have $h_\mu(\nu) \geqslant -\ln(1) = 0$ and this inequality is strict unless $\mathrm{d}_\nu f(x) \equiv c$ for ν-a.e. $x \in X$, for μ-almost every $f \in \mathcal{M}$. In the latter case, we obtain again by (3.3) and (3.2) that

$$\begin{aligned} c &= \int_{\mathcal{M}} \int_{[0,1]} \mathrm{d}_\nu f(x)\,\mathrm{d}\nu(x)\,\mathrm{d}\mu(f) \\ &\leqslant \int_{[0,1]} \int_{\mathcal{M}} n(f,x)\,\mathrm{d}\mu(f)\,\mathrm{d}\nu(x) \\ &\leqslant 1. \end{aligned}$$

Now, if $c < 1$ then $h_\mu(\nu) > 0$. In the remaining case $c = 1$. Hence $\mathrm{d}_\nu f(x) = 1$ for ν-a.e. $x \in [0,1]$, for μ-almost every $f \in \mathcal{M}$. □

Proposition 3.3 should be considered a generalization to μ-injective SDSs of the well-known result valid for an SDS driven by invertible maps f. Indeed, then either $h_\mu(\nu) > 0$ or $f\nu = \nu$ for μ-almost every f (see, for instance, [17, Proposition 3.7]). However, in the case of a noninvertible map, the fact that $\mathrm{d}_\nu f \equiv 1$ does not imply that ν is f-invariant. We have, for instance, the following example, which also provides an example of a μ-injective SDS that has several (atomic) ergodic μ-invariant probability measures.

Example 3.4. — Suppose that μ is supported on $\Gamma = \{\phi_1, \phi_2\}$ consisting of a $\frac{1}{2}$-tent-map and an upside-down $\frac{1}{2}$-tent-map, as shown below.



Assume that $\mu(\phi_i) = \frac{1}{2}$ for $i = 1, 2$. This system is μ-injective, since

$$\begin{aligned}\int_{\mathcal{M}} n(f,x)\,\mathrm{d}\mu(f) &= 2\mu(\phi_1)\mathbf{1}_{[0,\frac{1}{2})}(x) + \big(\mu(\phi_1)+\mu(\phi_2)\big)\mathbf{1}_{\{\frac{1}{2}\}}(x) \\ &\quad + 2\mu(\phi_2)\mathbf{1}_{(\frac{1}{2},1]}(x) \\ &= 1.\end{aligned}$$

It is easily checked that any probability measure ν symmetric with respect to 1/2 (that is, invariant under the map $\sigma\colon x \mapsto 1-x$) is μ-invariant. For instance, the Lebesgue measure on $[0,1]$ is μ-invariant, but also the restriction of the Lebesgue measure to the invariant set $Y = [0,1/4] \cup [3/4,1]$.

For all these measures we have $\mathrm{d}_\nu f \equiv 1$. In fact, since $\phi_1(I) = I$ for $I \subseteq [0,1/2]$ and $\phi_1(I) = \sigma(I)$ for $I \subseteq [1/2,1]$ (and reversely for ϕ_2), we have

$$\mathrm{d}_\nu \phi_i(x) = \lim_{r\to 0} \frac{\nu\big(\phi_i\big(B_r(x)\big)\big)}{\nu\big(B_r(x)\big)} = \lim_{r\to 0} \frac{\nu\big(B_r(x)\big)}{\nu\big(B_r(x)\big)} = 1,$$

for $B_r(x) := [x-r, x+r] \cap [0,1]$.

However, a symmetric measure ν is not ϕ_1-invariant, since for $I \subseteq [0,1/2]$ we have

$$\phi_1\nu(I) = \nu\big(\phi_1^{-1}(I)\big) = \nu\big(I \cup \sigma(I)\big) = 2\nu(I),$$

whereas $\phi_1\nu(I) = 0$ for $I \subseteq [1/2,1]$.

As a matter of fact, this SDS has infinitely many ergodic measures, which are atomic. In fact, every measure $\nu_{x_0} = \frac{1}{2}\delta_{x_0} + \frac{1}{2}\delta_{\sigma(x_0)}$ for $x_0 \in [0,1/2]$ is ergodic.

The following natural question arises.

Open question. — *Does there exist an SDS with an* atomless *ergodic μ-invariant measure ν such that $\mathrm{d}_\nu f \equiv 1$, but ν is not f-invariant?*

3.3. Practical condition to ensure $h_\mu(\nu) > 0$

We provide in this section a hands-on condition that ensures that $\mathrm{d}_\nu f \not\equiv 1$.

Definition 3.5. — *We say that a stochastic system $(\mathcal{M},\mu)$* contracts a neighborhood of $x_0 \in [0,1]$ *if there exists $\varepsilon > 0$ such that the set of $f \in \mathcal{M}$ satisfying:*

$$f(x_0) = x_0$$

$$\text{and} \quad \big|f(x) - f(x_0)\big| < |x - x_0| \quad \text{for all } x \in [x_0-\epsilon, x_0+\epsilon] \cap [0,1] \setminus \{x_0\}$$

has positive μ-measure.

If $x_0 = 0$ this means that the stochastic system is *below the diagonal on a neighborhood of* 0, that is there exists $\varepsilon > 0$ such that

$$\mu\big(\{f \in \mathcal{M} \,:\, f(0) = 0 \text{ and } f(x) < x \;\; \forall\, x \in (0, \epsilon]\}\big) > 0.$$

Analogously, if $x_0 = 1$, it means that the system is *above the diagonal on a neighborhood of* 1, that is there exists $\varepsilon > 0$ such that

$$\mu\big(\{f \in \mathcal{M} \,:\, f(1) = 1 \text{ and } f(x) > x \;\; \forall\, x \in [1-\epsilon, 1)\}\big) > 0.$$

If $x_0 \in (0,1)$, it means that it crosses the diagonal with a "slope less than one".

Theorem 3.6 (Theorem 1.2). — *Let $(\mathcal{M}, \mu)$ be a stochastic dynamical system μ-injective in all but countably many $x \in [0,1]$, and assume that it contracts a neighborhood of some point $x_0 \in [0,1]$. Moreover, assume that ν is an atomless μ-invariant probability measure with $x_0 \in \operatorname{supp}(\nu)$. Then $h_\mu(\nu) > 0$.*

In particular if ν is ergodic, any other μ-invariant, ergodic Borel probability measure η whose support contains $\operatorname{supp}(\nu)$ coincides with ν.

The above theorem follows immediately from Proposition 3.3 and the following lemma.

Lemma 3.7. — *Let $(\mathcal{M}, \mu)$ be a stochastic dynamical system μ-injective in all but countably many $x \in [0,1]$, and assume that it contracts a neighborhood of some point $x_0 \in [0,1]$. Moreover, assume that ν is an atomless μ-invariant probability measure with $x_0 \in \operatorname{supp}(\nu)$. Then*

$$\nu \otimes \mu\big(\{(x,f) \in [0,1] \times \mathcal{M} \,:\, \mathrm{d}_\nu f(x) \neq 1\}\big) > 0.$$

Proof. — Let us assume that the system contracts a neighborhood of x_0 and $x_0 \in \operatorname{supp}(\nu)$. Let $J_m := \big[x_0 - \frac{1}{m}, x_0 + \frac{1}{m}\big] \cap [0,1]$ and

$$J_m^+ := \left[x_0, x_0 + \frac{1}{m}\right] \cap [0,1] \qquad \text{and} \qquad J_m^- := \left[x_0 - \frac{1}{m}, x_0\right] \cap [0,1].$$

Since all the maps in $\mathcal{M}$ are piecewise monotone, we can write

$$\{f \,:\, f(x_0) = x_0\} = \bigcup_{m \in \mathbb{N}} \big\{f \,:\, f(x_0) = x_0,\; f\big|_{J_m^+} \text{ and } f\big|_{J_m^-} \text{ are monotone}\big\}.$$

Thus by continuity of measures, there is $m \in \mathbb{N}$ such that $\mu(\Delta_m) > 0$ where Δ_m is the set of $f \in \mathcal{M}$ such that

$$f\big|_{J_m^+} \text{ and } f\big|_{J_m^-} \text{ are monotone}, \;\; f(x_0) = x_0,$$

and

$$\big|f(x) - f(x_0)\big| < |x - x_0| \quad \text{for all } x \in J_m.$$

We claim that $\nu(f^n J_m) \to 0$ for any $f \in \Delta_m$. In fact, we easily check that $f^n J_m = [a_n, b_n]$ with $a_n = \min\{f^n(x_0), f^n(x_0 + \frac{1}{m}), f^n(x_0 - \frac{1}{m})\}$ and $b_n = \max\{f^n(x_0), f^n(x_0 + \frac{1}{m}), f^n(x_0 - \frac{1}{m})\}$, and a_n and b_n converge to x_0. Thus $\nu(f^n J_m) \to \nu(\{x_0\}) = 0$, by atomlessness of ν.

Suppose now that $\mathrm{d}_\nu f(x) = 1$ for ν-a.e. $x \in [0,1]$ and μ-a.e. $f \in \mathcal{M}$. Then for all Borel sets $A \subseteq I_i^f$ we have

$$\nu(A) = \int_A \mathrm{d}_\nu f(x)\, \mathrm{d}\nu(x) = \int_A \frac{\mathrm{d}\nu_i^f}{\mathrm{d}\nu}(x)\, \mathrm{d}\nu(x) \leqslant \nu_i^f(A) = \nu\big(f(A)\big). \tag{3.4}$$

Since for every $f \in \Delta_m$, $f(J_m^+)$ is contained in either J_m^+ or J_m^-, we can iterate (3.4) obtaining

$$\nu(J_m^+) \leqslant \nu\big(f(J_m^+)\big) \leqslant \nu\big(f^n(J_m^+)\big) \leqslant \nu\big(f^n(J_m)\big) \longrightarrow 0.$$

Thus $\nu(J_m^+) = 0$ and similarly $\nu(J_m^-) = 0$. Thus $\nu(J_m) = 0$, which contradicts the hypothesis that $x_0 \in \operatorname{supp} \nu$. □

Proof of Theorem 3.6. — By Proposition 3.3, μ-injectivity implies that either $h_\mu(\nu) > 0$ or $\mu \otimes \nu\big(\{(f,x) : \mathrm{d}_\nu f(x) = 1\}\big) = 1$. The latter case cannot occur due to Lemma 3.7. Thus the entropy $h_\mu(\nu)$ has to be strictly positive, which was to prove.

If ν is also ergodic the claim follows now by Theorem 2.2. □

4. Examples

4.1. A condition for unique ergodicity

In this section, we present a condition for the existence and uniqueness of a μ-invariant probability measure. This condition, derived from our previous results, is not optimal but can be used to construct noninvertible uniquely ergodic SDSs.

Corollary 4.1. — *Let μ be a probability measure with finite support $\Gamma \subseteq \mathcal{M}$ and let Γ^* be the countable semigroup of $\mathcal{M}$ generated by Γ. Suppose that the generated SDS satisfies the following properties:*

- *the set $C := \bigcap_{x\in(0,1)} \overline{\Gamma^* x} \subseteq [0,1]$ is not empty, i.e., there is a point in the closure of all the orbits $\Gamma^* x$, $x \in (0,1)$;*
- *there exists $x_0 \in C \cap (0,1)$ such that the family of measures*

$$\big\{\textstyle\sum_{k=0}^n \pi^k(x_0, \cdot\,)/n\big\}_{n=1}^\infty$$

is tight *in $(0,1)$;*
- *the SDS* contracts a neighborhood of some point $y_0 \in C$;

- *the system is* μ-injective *in all* $x \in (0,1)$*;*
- *every* $x \in (0,1)$ *has* zero or infinitely many preimages, *i.e.*

$$\#\big\{y \,:\, f(y) = x \text{ for some } f \in \Gamma^*\big\} = \infty \text{ or } 0.$$

Then there exists a unique μ*-invariant probability measure on* $(0,1)$*. This probability measure is atomless.*

Proof. — Since the maps f are continuous the transition probability

$$\pi(x,A) = \int_{\mathcal{M}} \mathbf{1}_A\big(f(x)\big)\,\mathrm{d}\mu(f) \quad \text{for } x \in \mathbb{R} \text{ and } A \in \mathcal{B}\big((0,1)\big)$$

is a Feller operator on $[0,1]$. Thus for any $x \in [0,1]$ all limit measures of the family $\big\{\sum_{k=0}^n \pi^k(x,\cdot)/n\big\}_n$ are μ-invariant probability measures. Furthermore, since $\big\{\sum_{k=0}^n \pi^k(x_0,\cdot)/n\big\}_n$ is tight in $(0,1)$ there exists an ergodic probability measure ν_0 with $\nu_0\big((0,1)\big) = 1$, and such that $\operatorname{supp}\nu_0 \subseteq \overline{\Gamma^* x_0}$.

We claim that under the hypothesis any μ-invariant probability ν such that $\nu\big((0,1)\big) = 1$ is atomless. Suppose ν has atoms and let

$$M := \sup_{y\in[0,1]} \nu(y)$$

and $A := \big\{x \in (0,1) \,:\, \nu(x) = M\big\}$. Since ν is a finite measure, A is a finite set. Furthermore μ-injectivity implies that $f^{-1}A \subsetneq A$ for all $f \in \Gamma^*$ and that any $a \in A$ has at least one preimage. In fact, let $a \in A$; then

$$M = \nu(a) = \sum_{f\in\Gamma} \mu(f) \sum_{y\in f^{-1}(a)} \nu(y) \leqslant \sum_{f\in\Gamma} \mu(f) M n(f,a) \leqslant M,$$

where the equality may hold only if $\nu(y) = M$ for all $y \in f^{-1}(a)$ and $\sum_{f\in\Gamma}\mu(f)n(f,a) = 1$. This contradicts the fact that all $x \in (0,1)$ have zero or infinitely many preimages under Γ^*.

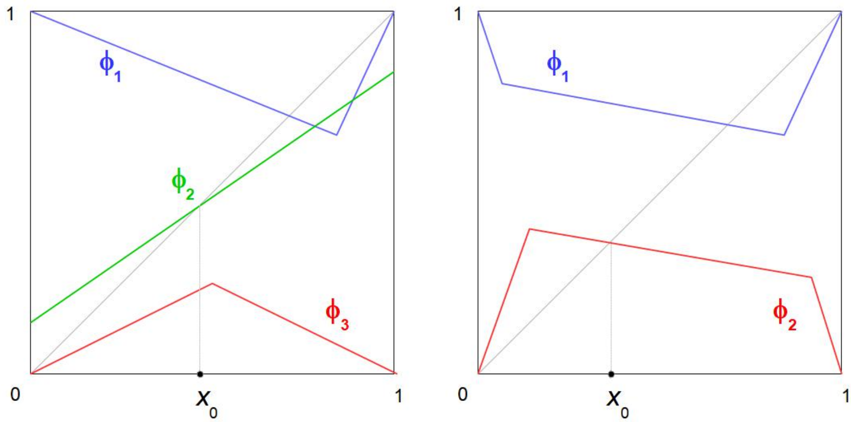
Let ν be an ergodic invariant probability measure. Observe that $\operatorname{supp}\nu$ is a closed Γ^*-invariant set, i.e., $f(\operatorname{supp}\nu) \subseteq \operatorname{supp}\nu$ for all $f \in \Gamma^*$,(1) thus $C \subseteq \operatorname{supp}\nu$. In particular, $x_0 \in \operatorname{supp}\nu$ and $\overline{\Gamma^* x_0} \subseteq \operatorname{supp}\nu$. Hence, $\operatorname{supp}\nu_0 \subseteq \operatorname{supp}\nu$ and since the conditions of Theorem 3.6 are satisfied we obtain $\nu = \nu_0$. □

(1) In fact, since

$$1 = \nu(\operatorname{supp}\nu) = \sum_{g\in\Gamma} \mu(g)\nu\Big(g^{-1}(\operatorname{supp}\nu)\Big),$$

then $\nu\Big(f^{-1}(\operatorname{supp}\nu)\Big) = 1$ for every $f \in \Gamma$, hence $f^{-1}(\operatorname{supp}\nu) \supseteq \operatorname{supp}\nu$, by the fact that $f^{-1}(\operatorname{supp}\nu)$ is closed and $\operatorname{supp}\nu$ is minimal. Thus, $\operatorname{supp}\nu \supseteq f\Big(f^{-1}(\operatorname{supp}\nu)\Big) \supseteq f(\operatorname{supp}\nu)$.

Example 4.2. — Consider the two examples below, where the measure μ is equidistributed on the functions in each graphic.



It can be checked that these two SDSs satisfy the hypothesis of Corollary 4.1, thus admit a unique invariant probability on $(0,1)$.

In both cases $x_0 = y_0$ is in C since $\lim_n \phi_2^n(x) = x_0$ for all $x \in (0,1)$. The family $\left\{\sum_{k=0}^{n} \pi^k(x_0,\cdot)/n\right\}_{n=1}^{\infty}$ is tight since the SDS is strongly repelling at 0 and 1 (see for instance [2, Proposition 9.1]).

4.2. Construction of SDS without unique ergodicity

Let us give here an example where unique ergodicity fails in the setting where μ-injectivity does not hold, but the systems is below the diagonal on a neighborhood of 0 and above the diagonal on a neighborhood of 1.

We start with a general observation and construction.

First observe that every semigroup of continuous functions acts on the set of closed subintervals of $[0,1]$ by the map $I \mapsto g(I)$.

Let $\mathcal{J}$ denote some family of closed subintervals of $[0,1]$. Let μ be a Borel probability measure on $\mathcal{M}$ such that $f\mathcal{J} \subseteq \mathcal{J}$ for μ-almost every $f \in \mathcal{M}$. Assume we have given a family $\left\{\nu_I : I \in \mathcal{J}\right\}$ of Borel probability measures on $[0,1]$, which is equivariant in the sense that

$$\nu_{f(I)} = f\nu_I \quad \bar{\nu} \otimes \mu\text{-a.s.} \tag{4.1}$$

and such that $I \mapsto \nu_I(A)$ is measurable for all $A \in \mathcal{B}([0,1])$.

We start with a simple lemma.

Lemma 4.3. — *Let $(\mathcal{J}, \Xi, \overline{\nu})$ be a probability measure space. Assume that $\overline{\nu}$ is μ-invariant, i.e. it satisfies*

$$\int_{\mathcal{J}\times\mathcal{M}} \xi\big(f(I)\big)\,\mathrm{d}\overline{\nu}(I)\,\mathrm{d}\mu(f) = \int_{\mathcal{J}} \xi(I)\,\mathrm{d}\overline{\nu}(I) \tag{4.2}$$

for any $\xi \in L^\infty(\mathcal{J}, \Xi, \overline{\nu})$.

Then the measure

$$\nu(A) := \int_{\mathcal{J}} \nu_I(A)\,\mathrm{d}\overline{\nu}(I) \quad \textit{for any Borel } A \subset [0,1]$$

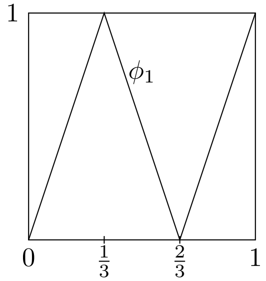
is a μ-invariant measure on $[0,1]$.

Proof. — For $A \in \mathcal{B}([0,1])$ we have

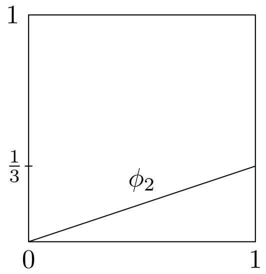
$$\begin{aligned}\int_{\mathcal{M}} \nu\big(f^{-1}(A)\big)\,\mathrm{d}\mu(f) &= \int_{\mathcal{M}}\left(\int_{\mathcal{J}} \nu_I\big(f^{-1}(A)\big)\,\mathrm{d}\overline{\nu}(I)\right)\mathrm{d}\mu(f)\\ &= \int_{\mathcal{M}}\left(\int_{\mathcal{J}} \nu_{f(I)}(A)\,\mathrm{d}\overline{\nu}(I)\right)\mathrm{d}\mu(f)\\ &\overset{(4.2)}{=} \int_{\mathcal{J}} \nu_I(A)\,\mathrm{d}\overline{\nu}(I)\\ &= \nu(A),\end{aligned}$$

which ends the proof. □

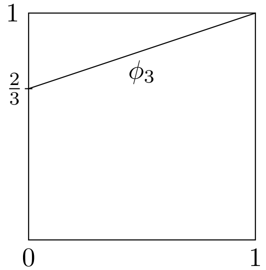
We are now in a position to give a concrete example of an SDS where unique ergodicity fails, i.e., we will construct two different μ-invariant probability measures ν_1, ν_2. On $[0,1]$ consider the following three continuous maps:



$$\phi_1(x) := \begin{cases} 3x & \text{if } x \in \left[0, \frac{1}{3}\right],\\ -3x+2 & \text{if } x \in \left(\frac{1}{3}, \frac{2}{3}\right],\\ 3x-2 & \text{if } x \in \left(\frac{2}{3}, 1\right],\end{cases}$$



$$\phi_2(x) := \tfrac{x}{3},$$



$$\phi_3(x) := \tfrac{x}{3} + \tfrac{2}{3}.$$

Proposition 4.4. — *Let μ be a probability measure on $\mathcal{M}$ such that $\mu(\phi_1) = p$, $\mu(\phi_2) = \frac{1-p}{2} = \mu(\phi_3)$ for some $p \in \left(\frac{1}{2}, 1\right)$. Then there exist at least two different μ-invariant ergodic probability measures ν_1, ν_2 on $(0,1)$.*

Remark 4.5. — Observe that the system is below the diagonal on a neighborhood of 0 and above the diagonal on a neighborhood of 1, but μ-injectivity does not hold, because $\int_{\mathcal{M}} n(f,x)\,\mathrm{d}\mu(f) \geqslant 3p > 1$ for all $x \neq 0,1$.

Proof of Proposition 4.4. — Let

$$K_n := \left\{k \in \mathbb{N} \,:\, k = k_0 + k_1 3 + \cdots + k_{n-1}3^{n-1} \text{ with } k_i = 0 \text{ or } 2\right\}.$$

Consider the collection of all subintervals of $[0,1]$ which are used in the construction of the Cantor-$\frac{1}{3}$-set, i.e.,

$$\mathcal{C} := \left\{\left[\frac{k}{3^n}, \frac{k+1}{3^n}\right] \,:\, k \in K_n,\ \forall\, n \in \mathbb{N} \cup \{0\}\right\}.$$

Observe that $\phi_i(\mathcal{C}) \subseteq \mathcal{C}$ for $i = 1,2,3$, hence $\mathcal{M}$ acts as a semigroup on the discrete set $\mathcal{C}$. Moreover, we can define a μ-invariant probability measure $\overline{\nu}$ on $\left(\mathcal{C}, \mathcal{P}(\mathcal{C})\right)$ — where $\mathcal{P}(\mathcal{C})$ denotes the power set of $\mathcal{C}$ — as follows:

$$\overline{\nu}(I) := c \cdot a^{\mathrm{lev}(I)} \quad \text{for } I \in \mathcal{C},$$

where $\mathrm{lev}(I)$ denotes the level of the interval in the Cantor-construction (i.e. $\mathrm{lev}\left(\left[\frac{k}{3^n}, \frac{k+1}{3^n}\right]\right) := n$) and $a := \frac{1-\sqrt{1-4p(1-p)}}{4p}$ and $c := \left(\sum_{n=0}^{\infty}(2a)^n\right)^{-1}$.

Note that the sum in the latter expression is finite due to the choice of p. Let us verify that $\overline{\nu}$ is μ-invariant: let $I \in \mathcal{C}$, $I \neq [0,1]$, then

$$p\overline{\nu}\big(\phi_1^{-1}(I)\big) + \frac{1-p}{2}\overline{\nu}\big(\phi_2^{-1}(I)\big) + \frac{1-p}{2}\overline{\nu}\big(\phi_3^{-1}(I)\big) \stackrel{?}{=} \overline{\nu}(I)$$
$$\iff \quad 2p\, a^{\mathrm{lev}(I)+1} + \frac{1-p}{2}\, a^{\mathrm{lev}(I)-1} \stackrel{?}{=} a^{\mathrm{lev}(I)}.$$

Note that we have used that I is either a subset of $\left[0,\frac{1}{3}\right]$ or of $\left[\frac{2}{3},1\right]$, hence either $\phi_2^{-1}(I)$ or $\phi_3^{-1}(I)$ is empty. Also, $\phi_1^{-1}(I)$ is a disjoint union of three intervals of one Cantor-level higher, but only two of them will be contained in $\mathcal{C}$. The above condition is thus satisfied by the choice of a. For $I = [0,1]$ the condition of μ-invariance is trivially fulfilled.

Now, using the construction from Lemma 4.3 for the above measure $\overline{\nu}$, we will construct two different μ-invariant ergodic probability measures on $[0,1]$. First, let us consider the uniform probability measures ν_I on I given by

$$\mathrm{d}\nu_I := \frac{1}{|I|}\mathbf{1}_I(x)\,\mathrm{d}x, \quad \text{for } I \in \mathcal{C}.$$

Observe that

$$\nu_{\phi_i(I)} = \phi_i\nu_I, \quad \forall\, i = 1,2,3.$$

Indeed, for f being a continuous map on $[0,1]$, we have

$$\begin{aligned}\nu_{\phi_1(I)}(f) &= \int_{[0,1]} f\,\mathrm{d}\nu_{\phi_1(I)} \\ &= \frac{1}{3|I|}\int_{\phi_1(I)} f(x)\,\mathrm{d}x \\ &= \frac{1}{3|I|}\int_I f\big(\phi_1(y)\big)\,3\,\mathrm{d}y \\ &= \nu_I\big(\phi_1^{-1}(f)\big)\end{aligned}$$

and similarly for ϕ_2, ϕ_3. Therefore, we can apply Lemma 4.3 and obtain that the measure

$$\nu_1 := \sum_{I\in\mathcal{C}} \overline{\nu}(I)\nu_I$$

is μ-invariant. Moreover, ν_1 is absolutely continuous with respect to the Lebesgue measure, $\nu_1 \ll \mathrm{Leb}$, and $\mathrm{supp}(\nu_1) = [0,1]$.

In order to construct a second (different) μ-invariant probability measure ν_2 on $(0,1)$, let us first consider the Cantor measure η on $[0,1]$, i.e. the unique probability measure such that (see for instance [15])

$$\eta = \frac{1}{2}\phi_2\eta + \frac{1}{2}\phi_3\eta. \tag{4.3}$$

We set

$$\widetilde{\nu}_I := \frac{\eta(\,\cdot\,\cap I)}{\eta(I)} \quad \text{for } I \in \mathcal{C}.$$

We shall prove that for $i = 1, 2, 3$

$$\widetilde{\nu}_{\phi_i(I)}(A) := \frac{\eta\big(A \cap \phi_i(I)\big)}{\eta\big(\phi_i(I)\big)} = \frac{\eta\big(\phi_i^{-1}(A) \cap I\big)}{\eta(I)} =: \phi_i \widetilde{\nu}_I(A) \tag{4.4}$$

for all $I \in \mathcal{C}$ and $A \subseteq [0,1]$. Note that

$$\phi_2^{-1}([2/3,1]) = \emptyset \qquad \text{and} \qquad \phi_3^{-1}([0,1/3]) = \emptyset. \tag{4.5}$$

For $i = 2$ the relation (4.4) follows the fact that

$$\begin{aligned}
\eta\big(A \cap \phi_2(I)\big) &\overset{(4.3)}{=} \frac{1}{2}\eta\Big(\phi_2^{-1}\big(A \cap \phi_2(I)\big)\Big) + \frac{1}{2}\eta\Big(\phi_3^{-1}\big(A \cap \phi_2(I)\big)\Big) \\
&\overset{(4.5)}{=} \frac{1}{2}\eta\Big(\phi_2^{-1}\big(A \cap \phi_2(I)\big)\Big) \\
&= \frac{1}{2}\eta\Big(\phi_2^{-1}(A) \cap \phi_2^{-1}\big(\phi_2(I)\big)\Big) \\
&= \frac{1}{2}\eta\big(\phi_2^{-1}(A) \cap I\big).
\end{aligned}$$

and that, for $A = [0,1]$, we obtain :

$$\eta\big(\phi_2(I)\big) = \frac{1}{2}\eta(I) \quad \text{for } I \in \mathcal{C}.$$

For $i = 3$, a similar argument holds.

We now prove (4.4) for $i = 1$. First observe that for $j = 2, 3$, since $\phi_1 \circ \phi_j = \mathrm{id}$, we have

$$\phi_j^{-1}\big(\phi_1^{-1}(A) \cap I\big) = \phi_j^{-1}\big(\phi_1^{-1}(A)\big) \cap \phi_j^{-1}(I) = A \cap \phi_j^{-1}(I).$$

Thus

$$\begin{aligned}
\eta\big(\phi_1^{-1}(A) \cap I\big) &= \frac{1}{2}\eta\Big(\phi_2^{-1}\big(\phi_1^{-1}(A) \cap I\big)\Big) + \frac{1}{2}\eta\Big(\phi_3^{-1}\big(\phi_1^{-1}(A) \cap I\big)\Big) \\
&= \frac{1}{2}\eta\big(A \cap \phi_2^{-1}(I)\big) + \frac{1}{2}\eta\big(A \cap \phi_3^{-1}(I)\big).
\end{aligned}$$

If $I \subseteq [0,1/3]$, then (4.4) follows from the fact that $\eta\big(\phi_1(I)\big) = 2\eta(I)$ and the relation

$$\eta\big(\phi_1^{-1}(A) \cap I\big) = \frac{1}{2}\eta\big(A \cap \phi_1(I)\big),$$

which holds since $\phi_2^{-1}(I)=\phi_1(I)$ and $\phi_3^{-1}(I)=\emptyset$. Similarly if $I\subseteq[2/3,1]$. For $I=[0,1]$, in turn, we obtain

$$\begin{aligned}\eta\big(\phi_1^{-1}(A)\cap[0,1]\big)&=\frac{1}{2}\eta\big(A\cap\phi_2^{-1}[0,1]\big)+\frac{1}{2}\eta\big(A\cap\phi_3^{-1}[0,1]\big)\\&=\eta\big(A\cap[0,1]\big)\\&=\eta\big(A\cap\phi_1([0,1])\big),\end{aligned}$$

which also leads to (4.4), since $\eta([0,1])=\eta\big(\phi_1([0,1])\big)=1$.

Thus, by Lemma 4.3,

$$\nu_2:=\sum_{I\in\mathcal{C}}\overline{\nu}(I)\widetilde{\nu}_I$$

is a μ-invariant probability measure, which by construction is absolutely continuous with respect to the Cantor measure η and $\operatorname{supp}(\nu_2)=\partial\mathcal{C}$ is the Cantor-$\frac{1}{3}$-set. Moreover, ν_2 is atomless, thus we can view it as a probability measure on $(0,1)$.

The μ-invariant probability measures ν_1, ν_2 might not be ergodic, but we obtain the existence of two different μ-invariant ergodic Borel probability measures by Rohklin's ergodic decomposition theorem. □

Appendix A. Proof of Proposition 2.1

Proposition (Proposition 2.1). — *Let λ be a finite Borel measure, and let ν be a Borel probability measure on the interval $[0,1]$.*

(1) *For ν-almost every $x\in[0,1]$ we have*

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(x)=\lim_{\delta\to 0}\sup_{\substack{I\in\mathcal{J}^x\\|I|<\delta}}\frac{\lambda(I)}{\nu(I)}.$$

(2) *Let $Q_\nu^*(\lambda,x):=\sup_{I\in\mathcal{J}^x}\frac{\lambda(I)}{\nu(I)}$. Then*

$$\int\ln^+Q_\nu^*(\lambda,x)\,\mathrm{d}\nu(x)\leqslant 2\lambda([0,1]).$$

The first part of the proposition is a consequence of the following version of the non-centered Vitali covering theorem for a general finite measure on the interval.

Lemma A.1. — *Let ν be a finite regular measure on $[0,1]$, and let $A\subseteq[0,1]$ be a Borel set. Let $\mathcal{J}$ be a family of closed intervals such that*

$$\inf\big\{|I|\ :\ I\in\mathcal{J}\text{ and }x\in\operatorname{int}(I)\big\}=0\qquad\forall\,x\in A.$$

Then there exists a countable family $\{I_i\}\subset\mathcal{J}$ of disjoint intervals such that $\nu\big(A\setminus\bigcup_i I_i\big)=0$.

Proof.

Step 1. — Let K be a compact set, and let $U \supset K$ be open in $[0,1]$. We claim that there exists a finite family of disjoint intervals $\{I_i\} \subset \mathcal{J}$ such that $I_i \subset U$ and $\nu\big(K \setminus \bigcup_i I_i\big) \leqslant \frac{1}{2}\nu(K)$.

In fact, since the intervals of $\mathcal{J}$ included in U form a covering of the compact set K, we can find a finite covering family $\{\widehat{I}_j\}_{j=1}^N$. Passing to a subfamily if necessary, we can assume that every interval in this family intersects at most two other intervals by Besicovitch's covering theorem. Additionally, we may assume that

$$\widehat{I}_i \cap \widehat{I}_j = \emptyset \quad \text{if } j \neq i \pm 1.$$

Consider the family $I'_i = \widehat{I}_{2i}$ for $i = 1, \dots, \lfloor N/2 \rfloor$ and $I''_i = \widehat{I}_{2i+1}$ for $i = 1, \dots, \lfloor N/2 \rfloor$. Then $\mathcal{I}'$ and $\mathcal{I}''$ are disjoint families and since their union covers K, either $\nu\big(K \setminus \bigcup_i I'_i\big) < \frac{1}{2}\nu(K)$ or $\nu\big(K \setminus \bigcup_i I''_i\big) < \frac{1}{2}\nu(K)$.

Step 2. — For any $A \subset U \subseteq [0,1]$ we claim that there exists a finite family of disjoint intervals $\mathcal{I} = \{I_i\} \subset \mathcal{J}$ such that $I_i \subset U$ and $\nu\big(A \setminus \bigcup_i I_i\big) < \frac{3}{4}\nu(A)$.

In fact, there exists a compact set $K \subset A$ such that $\nu(A \setminus K) < \frac{1}{4}\nu(A)$. Applying Step 1, we have a finite disjoint family of intervals such that

$$\nu\big(A \setminus \textstyle\bigcup_i I_i)\big) \leqslant \nu\big(K \setminus \bigcup_i I_i)\big) + \nu(A \setminus K) \leqslant \dfrac{1}{2}\nu(K) + \dfrac{1}{4}\nu(A) \leqslant \dfrac{3}{4}\nu(A).$$

Step 3. — We prove by induction that for any $n \in \mathbb{N}$ there exists a finite family of disjoint intervals $\mathcal{I}^n := \{I^n_i\} \subset \mathcal{J}$ such that $I^n_i \subset U$ and

$$\nu\big(A \setminus \textstyle\bigcup_i I^n_i\big) < \left(\dfrac{3}{4}\right)^n \nu(A). \tag{A.1}$$

Step 2 gives a proof for $n = 1$. Suppose that (A.1) holds for n. Take $A_n = A \setminus \bigcup_i I^n_i$ and $U_n = [0,1] \setminus \bigcup_i I^n_i$. Then, by Step 2, there exists $\widehat{\mathcal{J}}^n = \big\{J^n_i\big\} \subset \mathcal{J}$ such that $I^n_i \subset U_n$ (therefore the intervals from $\mathcal{J}^n$ are disjoint from the intervals $\mathcal{I}^n$) and $\nu\big(A_n \setminus \bigcup_i J^n_i\big) < \frac{3}{4}\nu(A_n)$. Setting $\mathcal{I}^{n+1} = \mathcal{J}^n \cup \mathcal{I}^n$ we easily check (A.1) for $n + 1$.

Finally, set $\mathcal{I} := \bigcup_n \mathcal{I}^n$. We conclude the proof by taking the limit in (A.1) for $n \to \infty$. □

Proof of Proposition 2.1.

(1). — We refer to Federer's book [11]. Indeed, Lemma A.1 proves that $V = \big\{(x, I) \;:\; I \text{ a closed interval and } x \in \operatorname{int}(I)\big\}$ is a ν-Vitali relation according to [11, Definition 2.8.16]. Let

$$D(\lambda, \nu, x) := \lim_{\delta \to 0} \sup_{I \in \mathcal{I}^x_\delta} \frac{\lambda(I)}{\nu(I)}.$$

By [11, Theorem 2.9.7], $D(\lambda,\nu,x)\,\mathrm{d}\nu(x)$ is an absolutely continuous part in the Lebesgue decomposition of λ, thus $D(\lambda,\nu,x)=\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(x)$.

(2). — Consider the set:

$$A_a \coloneqq \{x\in[0,1]\ :\ Q^*(\lambda,\nu,x)>a\}.$$

For every $x\in A_a$ choose a closed interval I_x such that $x\in\operatorname{int}(I_x)$ and $\lambda(I_x)>a\nu(I_x)$. It is possible to extract from the family $\{I_x\}_{x\in A_a}$ a finite covering $\{I_i\}_i$ of A_a such that every interval may overlap with at most one other interval at each point, that is

$$\mathbf{1}_{A_a}\leqslant\sum_i \mathbf{1}_{I_i}\leqslant 2,$$

by Besicovitch's covering theorem in dimension 1. Then

$$\nu(A_a)\leqslant\sum_i\nu(I_i)\leqslant\sum_i\lambda(I_i)/a\leqslant\int_{[0,1]}\sum_i\mathbf{1}_{I_i}/a\,\mathrm{d}\lambda\leqslant 2\lambda([0,1])/a.$$

Now by Fubini's theorem, we easily see that

$$\begin{aligned}\int_{[0,1]}\ln^+ Q^*(\lambda,\nu,x)\,\mathrm{d}\nu(x) &= \int_{[0,1]}\int_1^\infty \mathbf{1}_{[Q^*(\lambda,\nu,x)>a]}\frac{\mathrm{d}a}{a}\,\mathrm{d}\nu(x)\\ &= \int_1^\infty \nu(A_a)\frac{\mathrm{d}a}{a}\\ &\leqslant 2\lambda([0,1])\int_1^\infty\frac{\mathrm{d}a}{a^2}\\ &= 2\lambda([0,1]).\end{aligned}$$

This completes the proof. □

Appendix B. Proof of exponential contraction

Proposition (Proposition 2.3). — *Let μ be a Borel probability measure on $\mathcal{M}$, and let ν be a μ-invariant atomless ergodic Borel probability measure on the interval $[0,1]$ such that $\ln^+ J$ is $(\nu\times\mu)$-integrable and $h_\mu(\nu)>0$. Then for every $h\in\big(0,h_\mu(\nu)\big)$ and $\mu^{\mathbb{N}}$-a.e. $\omega=(f_n)_{n\in\mathbb{N}}\in\mathcal{M}^{\mathbb{N}}$ and ν-a.e. $x\in[0,1]$ there exists a closed interval $I=I(\omega,x)$ with $x\in\operatorname{int}(I)$ such that*

$$\big|f_n\circ\cdots\circ f_1(I)\big|\leqslant\exp(-n\cdot h),\quad\forall\, n\in\mathbb{N}.$$

Define

$$J_\epsilon(x,f)\coloneqq\sup\left\{\frac{\nu\big(f(I)\big)}{\nu(I)}\ \middle|\ \begin{array}{l}I\subset[0,1]\text{ closed interval,}\\ x\in\operatorname{int}(I)\text{ and }\nu(I)\leqslant\epsilon\end{array}\right\}.$$

Lemma B.1. — *Suppose that ν is an atomless probability measure and $g \in \mathcal{M}$. Then we have*

$$\lim_{\varepsilon\to 0} J_\epsilon(x,f) = \mathrm{d}_\nu f(x) \quad \mathrm{d}\nu(x)\text{-a.s.} \tag{B.1}$$

Furthermore, if $\ln^+ J$ *is* $(\nu\times\mu)$*-integrable, then*

$$h^\epsilon_\mu(\nu) \coloneqq \int_{[0,1]}\int_{\mathcal{M}} \ln\big(J_\epsilon(x,f)\big)\,\mathrm{d}\nu(x)\,\mathrm{d}\mu(f)$$

is well-defined, $h^\epsilon_\mu(\nu)\in(-\infty,+\infty]$, *and*

$$\lim_{\varepsilon\to 0} h^\epsilon_\mu(\nu) = h_\mu(\nu). \tag{B.2}$$

Proof. — Denote $J_0 \coloneqq \lim_{\varepsilon\to 0} J_\epsilon = \inf_{\varepsilon>0} J_\epsilon$. First observe that since ν has no atoms $\varepsilon(r,x)\coloneqq \nu\big(B_r(x)\big)\to 0$ as $r\to 0$, where $B_r(x)\coloneqq[x-r,x+r]\cap[0,1]$. Thus

$$\mathrm{d}_\nu f(x) = \lim_{r\to 0}\frac{\nu\big(f\big(B_r(x)\big)\big)}{\nu\big(B_r(x)\big)} \leqslant \lim_{r\to 0} J_{\varepsilon(r,x)}(x,f) = J_0(x,f).$$

To prove the reverse inequality observe that

$$Q^*(x) \coloneqq \sup\left\{\frac{|I|}{\nu(I)}\ :\ I \text{ closed interval of } [0,1] \text{ s.t. } x\in\operatorname{int}(I)\right\}$$

coincides with $Q^*_\nu(\lambda,x)$ when λ is the Lebesgue measure on $[0,1]$ and thus it is ν-a.s. finite, by Proposition 2.1(2). Let I^x_n be a sequence of intervals such that $J_0(x)=\lim_n \nu\big(f(I^x_n)\big)/\nu(I^x_n)$ and such that $\nu(I^x_n)\to 0$. Then $\delta(x,n)\coloneqq|I^x_n|\leqslant Q^*(x)\nu(I^x_n)\to 0$. Finally, since locally $f^{-1}\nu$ coincides with the measure $\nu^f_i \coloneqq (\gamma_i^{-1}\nu)\big|_{I^f_i}$ we can apply Proposition 2.1(1) to $\lambda=\nu^f_i$ and get

$$\begin{aligned}
J_0(x,f) &= \lim_{n\to\infty}\frac{\nu\big(f(I^x_n)\big)}{\nu(I^x_n)}\\
&= \lim_{n\to\infty}\frac{\nu_i(I^x_n)}{\nu(I^x_n)}\\
&\leqslant \lim_{n\to\infty}\sup_{I\in\mathcal{I}_{\delta(x,n)}}\frac{\nu_i(I^x_n)}{\nu(I^x_n)}\\
&= \frac{\mathrm{d}\nu_i}{\mathrm{d}\nu}(x)\\
&= \mathrm{d}_\nu f(x).
\end{aligned}$$

This proves (B.1).

To prove condition (B.2) observe that $\ln^+ J_\epsilon(x,f)$ converges $\nu\otimes\mu^{\otimes\mathbb{N}}$-a.s. to $\ln^+\big(\mathrm{d}_\nu f(x)\big)$, and since $\ln^+ J_\epsilon(x,f)\leqslant \ln^+ J(x,f)$ and $\ln^+ J(x,f)\in L^1\big([0,1]\times\mathcal{M}^{\mathbb{N}},\nu\otimes\mu^{\otimes\mathbb{N}}\big)$ we have

$$\lim_{\varepsilon\to 0}\int_{[0,1]}\int_{\mathcal{M}}\ln^+\big(J_\epsilon(x,f)\big)\,\mathrm{d}\nu(x)\,\mathrm{d}\mu(f)=\int_{[0,1]}\int_{\mathcal{M}}\ln^+\big(\mathrm{d}_\nu f(x)\big)\,\mathrm{d}\nu(x)\,\mathrm{d}\mu(f).$$

On the other hand, since the sequence $\ln^-\big(J_\epsilon(x,g)\big)$ is decreasing as $\varepsilon\to 0$, by the Lebesgue theorem we also have

$$\lim_{\varepsilon\to 0}\int_{[0,1]}\int_{\mathcal{M}}\ln^-\big(J_\epsilon(x,f)\big)\,\mathrm{d}\nu(x)\,\mathrm{d}\mu(f)=\int_{[0,1]}\int_{\mathcal{M}}\ln^-\big(\mathrm{d}_\nu f(x)\big)\,\mathrm{d}\nu(x)\,\mathrm{d}\mu(f),$$

and consequently (B.2) holds. □

Proof of Proposition 2.3. — Consider the transformation $T\colon X\times\mathcal{M}^{\mathbb{N}}\to X\times\mathcal{M}^{\mathbb{N}}$ given by the formula

$$T(x,\omega):=\big(f_1(x),\sigma(\omega)\big),$$

where σ denotes the left shift on $\mathcal{M}^{\mathbb{N}}$, i.e. $\sigma(f_1,f_2,\dots)=(f_2,f_3,\dots)$. The transformation T is $(\nu\otimes\mu^{\mathbb{N}})$-measure preserving and ergodic for ν ergodic and μ-invariant (see [5]). For any $\varepsilon>0$, let $\widetilde{J}_\varepsilon(x,\omega)=J_\varepsilon(x,f_1)$ and thus $\widetilde{J}_\varepsilon\big(T^k(x,\omega)\big)=J_\varepsilon(X_k^x,f_{k+1})$.

Fix $h\in\big(0,h_\mu(\nu)\big)$ and take $\widetilde{h}\in\big(h,h_\mu(\nu)\big)$. Choose $\varepsilon>0$ such that $\widetilde{h}<h_\mu^\varepsilon(\nu)$. By Birkhoff's ergodic theorem,

$$\frac{\sum_{k=0}^{n-1}\ln\big(\widetilde{J}_\varepsilon\big(T^k(x,\omega)\big)\big)}{n}\xrightarrow[n\to\infty]{}\int_{[0,1]\times\mathcal{M}}\ln\big(J_\epsilon(y,f)\big)\,\mathrm{d}(\nu\otimes\mu)(y,f) = -h_\mu^\varepsilon(\nu)$$

for $(\nu\otimes\mu^{\otimes\mathbb{N}})$-a.e. $(x,\omega)\in[0,1]\times\mathcal{M}^{\mathbb{N}}$. The above is still true even if $\ln\circ J_\epsilon\notin L^1\big([0,1]\times\mathcal{M}^{\mathbb{N}},\nu\otimes\mu^{\otimes\mathbb{N}}\big)$, (thus $h_\mu^\varepsilon(\nu)=+\infty$) because we can consider the function $\max\big\{\ln\circ J_\epsilon,-M\big\}$ for arbitrary large $M>0$, which is in $L^1\big([0,1]\times\mathcal{M}^{\mathbb{N}},\nu\otimes\mu^{\otimes\mathbb{N}}\big)$.

Then there exists $N\in\mathbb{N}$ such that

$$\widetilde{h}\leqslant-\frac{1}{n}\log\left(\prod_{k=0}^{n-1}\widetilde{J}_\epsilon\big(T^k(x,\omega)\big)\right)\quad\text{for all } n\geqslant N.$$

This is equivalent to

$$\prod_{k=0}^{n-1}J_\epsilon\big(X_k^x,f_{k+1}\big)\big)=\prod_{k=0}^{n-1}\widetilde{J}_\epsilon\big(T^k(x,\omega)\big)\leqslant e^{-n\tilde{h}}\quad\text{for all } n\geqslant N.$$

Thus for any (x,ω) such that the Birkhoff limit holds there exists a constant $C = C(x,\omega,\widetilde{h}) > 0$ such that

$$\prod_{k=0}^{n-1} J_\epsilon\big(X_k^x, f_{k+1})\big) \leqslant C(x,\omega,\widetilde{h})e^{-n\tilde{h}} \quad \text{for all } n \in \mathbb{N}. \tag{B.3}$$

Let

$$\Omega_1 \coloneqq \big\{(x,\omega) \,:\, \text{(B.3) holds and } x \notin \textstyle\bigcup_{n\in\mathbb{N}} \mathcal{D}(f_n \dots f_1)\big\}.$$

Since ν is atomless and $\bigcup_{n\in\mathbb{N}} \mathcal{D}(f_n \circ \cdots \circ f_1)$ is countable, Ω_1 has full measure.

Take $(x,\omega) \in \Omega_1$. By induction we show that: for any closed interval I_0 containing x in its interior such that

$$\nu(I_0) < \delta(x,\omega,\widetilde{h}) \coloneqq \frac{\varepsilon}{1 + C(x,\omega,\widetilde{h})}$$

we have

$$\nu\big(f_n \circ \cdots \circ f_1(I_0)\big) \leqslant Ce^{-n\cdot\tilde{h}}\nu(I_0) \quad \text{for all } n \in \mathbb{N}. \tag{B.4}$$

For $n = 0$ the above is trivial.

Assume it is true for $k = 0,\dots,n-1$. Let $I_k \coloneqq f_k \circ \cdots \circ f_1(I_0)$. Hence, by induction hypothesis

$$\nu(I_k) = \nu\big(f_k \circ \cdots \circ f_1(I_0)\big) \leqslant Ce^{-k\cdot\tilde{h}}\nu(I_0) \leqslant Ce^{-k\cdot\tilde{h}}\delta \leqslant \varepsilon$$

since $e^{-k\cdot\tilde{h}} \leqslant 1$ using $\widetilde{h} > 0$.

Observe that $X_k^x = f_k \circ \cdots \circ f_1(x) \in I_k$ since $x \in I_0$. Furthermore, the fact that the functions f_i are piecewise monotone and $x \notin D(f_{k+1} \circ \cdots \circ f_1)$, ensures also that $X_k^x \in \operatorname{int}(I_k)$. Therefore,

$$\begin{aligned}
\frac{\nu\big(f_{k+1} \circ \cdots \circ f_1(I_0)\big)}{\nu\big(f_k \circ \cdots \circ f_1(I_0)\big)} &= \frac{\nu\big(f_{k+1}(I_k)\big)}{\nu(I_k)} \\
&\leqslant \sup\left\{ \frac{\nu\big(f_{k+1}(I)\big)}{\nu(I)} \;\middle|\; \begin{array}{l} I \text{ an interval} \\ \text{with } X_k^x \in \operatorname{int}(I), \\ \nu(I) \leqslant \varepsilon \end{array} \right\} \\
&= J_\epsilon(X_k^x, f_{k+1}) \\
&= \widetilde{J}_\varepsilon\big(T^k(x,\omega)\big).
\end{aligned}$$

Now, applying the above inequalities, we obtain

$$\begin{aligned}\nu\big(f_n\circ\cdots\circ f_1(I_0)\big) &= \nu(I_0)\prod_{k=0}^{n-1}\frac{\nu\big(f_{k+1}\circ\cdots\circ f_1(I_0)\big)}{\nu\big(f_k\circ\cdots\circ f_1(I_0)\big)}\\ &\leqslant \nu(I_0)\prod_{k=0}^{n-1}J_\epsilon\big(T^k(x,\omega)\big)\\ &\leqslant C\nu(I_0)e^{-n\tilde h},\end{aligned}\tag{B.5}$$

where the last inequality follows by (B.3) and the proof of (B.4) is complete.

Let now

$$Q^*(x) \coloneqq \sup\left\{\frac{|I|}{\nu(I)}\ :\ I \text{ a closed interval such that } x\in \operatorname{int}(I)\right\}.$$

By Proposition 2.1(2) $\ln^+ Q^* \in L^1\big([0,1],\nu\big)$. Therefore, by the Birkhoff ergodic theorem, we obtain that $\frac{1}{n}\sum_{k=0}^{n-1}\ln^+ Q^*(X_k^x)$ converges to

$$\int_{[0,1]}\ln^+ Q^*\,\mathrm{d}\nu$$

for $(\nu\otimes\mu^{\otimes\mathbb{N}})$-almost every (x,ω). Hence $\ln^+ Q^*(X_n^x)/n$ tends to 0 for $(\nu\otimes\mu^{\otimes\mathbb{N}})$-almost every (x,ω). On the other hand, as observed above, $X_n^x\in\operatorname{int}(I_n)$, if $x\notin\mathcal{D}(f_n\circ\cdots\circ f_1)$, thus for ν-a.e. x (since ν is atomless). Hence for every $\kappa>0$, there exists a constant $\widetilde C=\widetilde C(x,\omega,\kappa)>0$ such that

$$\frac{\big|f_n\circ\cdots\circ f_1(I_0)\big|}{\nu\big(f_n\circ\cdots\circ f_1(I_0)\big)} = \frac{|I_n|}{\nu(I_n)} \leqslant Q^*(X_n^x)\leqslant \widetilde C e^{n\kappa}\quad\text{for all } n\tag{B.6}$$

for $(\nu\otimes\mu^{\otimes\mathbb{N}})$-almost every (x,ω). Consequently, taking $\kappa=\widetilde h-h$, it follows from (B.5) that for every $n\in\mathbb{N}$ we have

$$\big|f_{n+1}\circ f_n\circ\cdots\circ f_1(I_0)\big|\leqslant C\widetilde C\nu(I_0)e^{-nh}\tag{B.7}$$

for any closed interval I_0 containing x in its interior such that $\nu(I_0)<\frac{\varepsilon}{1+C(x,\omega,\tilde h)}$. Thus, if we decrease the interval I_0, we have

$$\big|f_{n+1}\circ f_n\circ\cdots\circ f_1(I_0)\big|\leqslant e^{-nh}\quad\text{for all } n\in\mathbb{N}\tag{B.8}$$

for any closed interval I_0 containing x in its interior such that $\nu(I_0)<\min\left\{\frac{1}{C(x,\omega,\tilde h)\tilde C(x,\omega,\kappa)}\frac{\varepsilon}{1+C(x,\omega,\tilde h)}\right\}$. This completes the proof. □

BIBLIOGRAPHY

[1] L. Alsedà & M. Misiurewicz, "Random interval homeomorphisms", *Publ. Mat., Barc.* **58** (2014), p. 15-36.

[2] G. Alsmeyer, S. Brofferio & D. Buraczewski, "Asymptotically linear iterated function systems on the real line", *Ann. Appl. Probab.* **33** (2023), no. 1, p. 161-199.

[3] A. Avila & M. Viana, "Extremal Lyapunov exponents: an invariance principle and applications", *Invent. Math.* **181** (2010), no. 1, p. 115-189.

[4] P. H. Baxendale, "Lyapunov exponents and relative entropy for a stochastic flow of diffeomorphisms", *Probab. Theory Relat. Fields* **81** (1989), no. 4, p. 521-554.

[5] Y. Benoist & J.-F. Quint, *Random walks on reductive groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 62, Springer, 2016.

[6] S. Brofferio, D. Buraczewski & T. Szarek, "On uniqueness of invariant measures for random walks on $\mathrm{HOMEO}^+(\mathbb{R})$", *Ergodic Theory Dyn. Syst.* **42** (2022), no. 7, p. 2207-2238.

[7] D. Czapla, "On the existence and uniqueness of stationary distributions for some piecewise deterministic Markov processes with state-dependent jump intensity", *Results Math.* **79** (2024), no. 5, article no. 177 (32 pages).

[8] K. Czudek & T. Szarek, "Ergodicity and central limit theorem for random interval homeomorphisms", *Isr. J. Math.* **239** (2020), no. 1, p. 75-98.

[9] B. Deroin, V. Kleptsyn & A. Navas, "Sur la dynamique unidimensionnelle en régularité intermédiaire", *Acta Math.* **199** (2007), no. 2, p. 199-262.

[10] B. Deroin, V. Kleptsyn, A. Navas & K. Parwani, "Symmetric random walks on $\mathrm{Homeo}^+(\mathbb{R})$", *Ann. Probab.* **41** (2013), no. 3B, p. 2066-2089.

[11] H. Federer, *Geometric measure theory*, Classics in Mathematics, vol. 153, Springer, 1996.

[12] M. Gharaei & A. J. Homburg, "Random interval diffeomorphisms", *Discrete Contin. Dyn. Syst., Ser. S* **10** (2017), no. 2, p. 241-272.

[13] A. J. Homburg, C. Kalle, M. Ruziboev, E. Verbitskiy & B. Zeegers, "Critical intermittency in random interval maps", *Commun. Math. Phys.* **394** (2022), no. 1, p. 1-37.

[14] B. R. Kloeckner, "Optimal transportation and stationary measures for iterated function systems", *Math. Proc. Camb. Philos. Soc.* **173** (2022), no. 1, p. 163-187.

[15] A. Lasota & M. C. Mackey, *Chaos, fractals, and noise*, 2nd ed., Applied Mathematical Sciences, vol. 97, Springer, 1994.

[16] F. Ledrappier, "Positivity of the exponent for stationary sequences of matrices", in *Lyapunov exponents (Bremen, 1984)*, Lecture Notes in Mathematics, vol. 1186, Springer, 1986, p. 56-73.

[17] D. Malicet, "Random walks on $\mathrm{Homeo}(S^1)$", *Commun. Math. Phys.* **356** (2017), no. 3, p. 1083-1116.

[18] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, 1995.

[19] A. Navas, *Groups of circle diffeomorphisms*, Chicago Lectures in Mathematics, University of Chicago Press, 2011.

[20] T. Nowicki & S. van Strien, "Invariant measures exist under a summability condition for unimodal maps", *Invent. Math.* **105** (1991), no. 1, p. 123-136.

Manuscrit reçu le 21 avril 2024,
révisé le 26 juillet 2024,
accepté le 29 novembre 2024.

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