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ON ALMOST-FUCHSIAN SUBMANIFOLDS OF HADAMARD SPACES AND THE ASYMPTOTIC PLATEAU PROBLEM

by Samuel BRONSTEIN

Abstract. — We consider minimal submanifolds of negatively curved spaces with small curvature. We show that in a Hadamard space with negatively pinched curvature $-C \leqslant K \leqslant -1$, complete minimal submanifolds with second fundamental form less than 1 everywhere bound a class of spheres at infinity for which the asymptotic Plateau problem is uniquely solvable.

Résumé. — Nous étudions les sous-variétés minimales de petite seconde forme fondamentale dans un espace de courbure de négative. Nous montrons que dans un espace de Hadamard de courbure négativement pincée entre $-C$ et -1, les sous-variétés minimales complètes de seconde forme fondamentales uniformément bornée par 1 bordent asymptotiquement une famille de sphères pour lesquelles le problème de Plateau asymptotique admet une unique solution.

1. Introduction

In 1983, Uhlenbeck [30] introduced the notion of *almost-fuchsian representation*. A representation of a surface group in $\mathrm{PSL}(2,\mathbb{C})$ is almost-fuchsian if it is discrete, faithful and there is an equivariant minimal disc in $\mathbb{H}^3$ whose principal values lie in a compact set of $(-1,1)$. Since then, almost-fuchsian representations have been a very fruitful field of study, see for instance [5, 9, 10, 16, 17, 21, 27, 28] We consider here a generalization of almost-fuchsian submanifolds in negatively curved spaces: Let $\mathbb{X}$ be a complete simply connected riemannian manifold whose sectional curvature is less or equal than -1. Let $k \geqslant 2$ and Y be a k-dimensional manifold.

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Definition 1.1. — *An immersion* $f : Y \to \mathbb{X}$ *is* almost-fuchsian *if it is minimal, proper, and it satisfies*

$$\sup_{y \in Y} |\mathbb{I}_f| < 1. \tag{1.1}$$

Remark that Y can always be endowed with the induced metric which will be complete by the properness assumption. By extension, a submanifold $Y \subset \mathbb{X}$ will be said to be almost-fuchsian if the inclusion map is almost-fuchsian.

In this paper, we prove the following embedding theorem:

Theorem 1.2. — *Let* $f : Y \to \mathbb{X}$ *be an almost-fuchsian immersion. Then* Y *is homeomorphic to a ball. Denote by* NY *the normal bundle to* Y*. The exponential map* $\exp : NY \to \mathbb{X}$ *is a diffeomorphism. Also,* f *extends at infinity to an embedding from a sphere* $S \to \partial_\infty \mathbb{X}$.

We also consider the following asymptotic Plateau problem: given a sphere $S \subset \partial_\infty \mathbb{X}$, can we count the complete minimal submanifolds of $\mathbb{X}$ asymptotically bounding S? Jiang [17] and Huang–Lowe–Seppi [15] considered this problem for spheres in $\partial_\infty \mathbb{H}^n$ bounding respectively an almost-fuchsian disc or an almost-fuchsian hypersurface. They proved that in that case, the unique minimal submanifold of the given dimension bounding S is the almost-fuchsian one, which is unique.

Here we prove that this statement holds in a broader generality, provided that $\mathbb{X}$ has negatively pinched curvature between $-C$ and -1.

Theorem 1.3. — *Let* $\mathbb{X}$ *be an* n*-dimensional negatively pinched Hadamard space with sectional curvature less or equal than* -1*. Let* $Y \subset \mathbb{X}$ *be a* k*-dimensional almost-fuchsian submanifold bounding a* $(k-1)$*-dimensional sphere* $S \subset \partial_\infty \mathbb{X}$*. Then* Y *is the unique* k*-dimensional complete minimal submanifold of* $\mathbb{X}$ *asymptotically bounding* S.

We don't know whether this statement holds when the sectional curvature of $\mathbb{X}$ has no lower bound, because we lack of convex sets to work with. Note that Huang–Lowe–Seppi [15] also cover the case of weakly-almost-fuchsian discs in $\mathbb{H}^3$, that is when the supremum of the principal values is allowed to be one.

This paper is divided in three parts. The first part is about a very standard fact, that minimal submanifolds of a space $\mathbb{X}$ with negatively pinched sectional curvature remain in the convex hull of their asymptotic boundary. Surprisingly, we couldn't prove it without a lower bound on the curvature of $\mathbb{X}$. The second part is devoted to the proof of the embedding theorem, with some explicit estimates on the geometry of an almost-fuchsian submanifold. Finally, the third part is devoted to the proof of the uniqueness to the asymptotic Plateau problem for spheres bounding an almost-fuchsian submanifold.

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2. The convex hull barrier

This section is devoted to the proof of the following theorem:

THEOREM 2.1. — *Let $\mathbb{X}$ be a Hadamard space with negatively pinched curvature. Let Y be a minimal submanifold of $\mathbb{X}$, bounding a subset $F \subset \partial_\infty \mathbb{X}$. Denote C the convex hull of F. Then $Y \subset C$.*

While this theorem is well known when $\mathbb{X} = \mathbb{H}^n$, we didn't find a general proof of this fact in negative curvature. A very elegant proof in $\mathbb{H}^n$ uses totally geodesic hypersurfaces as barriers, as the convex hull is the intersection of all half spaces containing C [17]. While this proof can be word for word done in a symmetric space of a semisimple Lie group of rank 1, here we need to be more careful, as totally geodesic hypersurfaces might not exist.

In all this section $\mathbb{X}$ will denote a negatively pinched Hadamard space, its boundary $\partial_\infty \mathbb{X}$ will be its ideal boundary which equals its Gromov boundary in that case [3, 8, 14]. The main idea is to use the following property:

PROPOSITION 2.2. — *Let F be a closed set of $\partial_\infty \mathbb{X}$, and denote by C its convex hull. Let $p \in X - C$. Then there is a closed domain $D \subset \mathbb{X} \cup \partial_\infty \mathbb{X}$ disjoint from $C \cup F$ such that $p \in D$, and D is foliated by smooth strictly convex hypersurfaces.*

Proof. — We use a theorem of Bowditch [4]. As $\mathbb{X}$ has negatively pinched curvature, there is a metric on $\mathbb{X} \cup \partial_\infty \mathbb{X}$ compatible with the topologies such that the map $F \mapsto CH(F)$ is continuous on the closed sets of $\mathbb{X} \cup \partial_\infty \mathbb{X}$ equipped with the Hausdorff distance. Therefore, there is a neighborhood V of F such that $\widetilde{C}$ the convex hull of V does not contain p. The complement of

$\widetilde{C}$ is then foliated by the equidistant to $\widetilde{C}$, which are strictly convex : indeed, in a Cartan–Hadamard space, the distance function to a closed convex set is convex (cf. [3, 1.6]), here strongly convex as the curvature is nonzero, and the neighborhoods to $\widetilde{C}$ defined as sublevels to $d(\,\cdot\,,\widetilde{C})$ are strongly convex sets, hence their boundaries are embedded convex hypersurfaces.

Now a priori the convex $\widetilde{C}$ has no smooth boundary, and so our foliation is only $C^{1,1}$: Indeed the boundary of a convex set is $C^{1,1}$-regular if and only if it is the union of balls with unifomly lower bounded radius (see Kiselman [19] for the statement in $\mathbb{R}^n$, Lemma 2.6. of Ghomi–Spruck [11] for the precise statement in Cartan–Hadamard spaces), which is the case here for the uniform neighborhoods of closed convex sets as a consequence of the property of projection on closed convex sets. However, Parkkonen–Paulin [24, Proposition 6] proved that a ε-neighborhood of $\widetilde{C}$ contains a smooth convex set, so up to replacing $\widetilde{C}$ by this smooth convex set we assume the foliation of the complement to be by smooth convex hypersurfaces. Denote D the closure of the complement of $\widetilde{C}$. □

Proof of Theorem 2.1. — Let Y be a minimal submanifold and let C denote the convex hull of $\partial_\infty Y$. Let $p \in \mathbb{X} - C$ and D denote the domain provided by Proposition 2.2.

Suppose that Y intersects D. Because D does not bound $\partial_\infty Y$, the intersection $Y \cap D$ is compact. Then there must be some point q at which Y is tangent to one of the strictly convex hypersurfaces foliating D. This contradicts the minimality of Y. Hence $Y \cap D$ is empty. Repeating this argument for any $p \in \mathbb{X} - C$ proves that $Y \subset C$. □

Remark 2.3. — The lower bound assumption on the curvature of $\mathbb{X}$ is crucial here, otherwise there might not be enough convex sets to pursue our proof. See for instance Ancona [1], there are spaces with curvature less than -1 such that the convex hull of any nontrivial open set at infinity is the full space $\mathbb{X}$.

3. Almost-fuchsian submanifolds

This part is about the notion of almost-fuchsian submanifold. First introduced for surface group representations in $\mathrm{PSL}(2,\mathbb{C})$ by Uhlenbeck [30], almost-fuchsian discs and hypersurfaces have a rich literature [5, 9, 10, 15, 17, 20, 27, 28].

Throughout this section $\mathbb{X}$ will be a Hadamard space with sectional curvature less or equal than -1. Note that we do not require its curvature to admit a lower bound. We consider an immersion $f : Y \to \mathbb{X}$, where Y is connected. We assume f to be complete, i.e. the induced metric on Y is complete.

We will prove the following theorem.

THEOREM 3.1. — *Assume $f : Y \to \mathbb{X}$ is minimal and satisfies:*

$$\sup_{y \in Y} |\mathbb{I}_f| < 1. \tag{3.1}$$

Then Y is homeomorphic to a ball. Denote by NY the normal bundle to Y. The exponential map $\exp : NY \to \mathbb{X}$ is a diffeomorphism. Also, f extends at infinity to an embedding from a sphere $S \to \partial_\infty \mathbb{X}$, with S the boundary at infinity of Y endowed with the pullback metric by f.

The proof is divided in several steps. First, we show that the induced metric by exp is nondegenerate on NY. This will imply that exp is a covering map to $\mathbb{X}$, and so it is a diffeomorphism. Finally, we will see that f is a quasi-isometric embedding from Y to $\mathbb{X}$, so it extends at infinity to an embedding $S \to \partial_\infty \mathbb{X}$.

First note that because of the condition on f, the induced metric has negatively pinched curvature on Y, and it makes sense to talk about its boundary $\partial_\infty Y$, which is constructed in the works of Gromov [14], Eberlein–O'Neill [8] and Ballmann–Gromov–Schroeder [3].

LEMMA 3.2. — *Let $f : Y \to \mathbb{X}$ be an almost-fuchsian immersion. Let $\varepsilon > 0$ such that*

$$\sup_{y \in Y} |\mathbb{I}_f| \leqslant 1 - \varepsilon \,. \tag{3.2}$$

Endow Y with the induced metric by f. Then its sectional curvature is less than $-\varepsilon(2-\varepsilon)$. Moreover, if the sectional curvature of $\mathbb{X}$ is lower bounded by $-C$, then the sectional curvature of $f^* g_{\mathbb{X}}$ is bigger than $-C - 2(1-\varepsilon)^2$.

Remark 3.3. — When Y is a surface, the minimality condition ensures that we have more precise bounds: The sectional curvature of an almost-fuchsian surface is then less than -1 and bigger than $-C - (1-\varepsilon)^2$.

Proof. — This is a consequence of the Gauss condition: For u, v an orthonormal basis of a tangent plane to Y, By the Gauss' equation, the curvature of $f^* g_{\mathbb{X}}$ is then

$$\begin{aligned} R_{f^* g_{\mathbb{X}}}(u,v,v,u) \\ = R_{g_{\mathbb{X}}}(u,v,v,u) + g_{\mathbb{X}}(\mathbb{I}_f(u,u), \mathbb{I}_f(v,v)) - g_{\mathbb{X}}(\mathbb{I}_f(u,v), \mathbb{I}_f(u,v)). \end{aligned} \tag{3.3}$$

As $R_{g_{\mathbb{X}}}(u,v,v,u) \leqslant -1$ and $|\mathbb{I}_f(u,u)| \leqslant 1-\varepsilon$, we directly get

$$R_{f^*g_{\mathbb{X}}}(u,v,v,u) \leqslant -1 + (1-\varepsilon)^2 = -\varepsilon(2-\varepsilon)\,. \tag{3.4}$$

as desired. Moreover, if we have a lower bound $R_{g_{\mathbb{X}}}(u,v,v,u) \geqslant -C$, we directly get that

$$R_{f^*g_{\mathbb{X}}}(u,v,v,u) \geqslant -C - 2(1-\varepsilon)^2\,, \tag{3.5}$$

as claimed. □

The next point requires some notation: we want to study the behavior of the induced metric by the exponential map from the normal bundle to an almost-fuchsian immersion to $\mathbb{X}$. In order to do so, we consider the following notations:

Notation 3.4. — Let $f : Y \to \mathbb{X}$ be an almost-fuchsian immersion. We denote by NY the normal bundle to Y, and by UY the unit normal bundle. For $t > 0$, we consider $\Phi_t : UY \to \mathbb{X}$ the map defined by

$$\Phi_t(x,v) = \exp_{f(p)}(tv)\,.$$

We denote by $P_t : T_{f(p)}X \to T_{\exp_{(f(p)}(tv)}\mathbb{X}$ the parallel transport along the geodesic $\gamma(s) = \Phi_s(x,v)$. Then we denote by B_t the endomorphism

$$B_t = P_t^{-1}\, d\Phi_t$$

from $T_{p,v}(UY) \approx T_pY \oplus (N_pY \cap v^{\perp})$ into itself.

The endomorphisms B_t carry the data of the induced metric of exp from NY to $\mathbb{X}$. Note that by design, the vector field ∂_t is preserved by parallel transport, so that $\dot{B}_t = P_t^{-1}\nabla_{\partial_t} d\Phi_t$. They satisfy the following inequations:

Proposition 3.5. — *Let $f : Y \to \mathbb{X}$ be an almost-fuchsian immersion. With the notation 3.4, consider $(x,v) \in UY$. Fix $w \in T_v(UY)$ and let φ be the function:*

$$\varphi(t) = |B_t w|^2\,. \tag{3.6}$$

If $\mathbb{X}$ is the hyperbolic space, φ satisfies the following control:

$$(\sqrt{\varphi})'' = \sqrt{\varphi}. \tag{3.7}$$

If $\mathbb{X}$ has sectional curvature strictly less than -1, then

$$(\sqrt{\varphi})'' > \sqrt{\varphi}. \tag{3.8}$$

Also, in both cases

$$(\ln\varphi)'' + \frac{(\ln\varphi)'^2}{2} \geqslant 2 \tag{3.9}$$

with the same equality and inequality conditions.

Proof. — With the introduced notations, fix $\gamma : I \to UY$ a smooth path such that $\gamma(0) = v$ and $\gamma'(0) = w$. The family $c_s(t)$ is a smooth variation of geodesics, hence $J(t) = \partial_s c_s(t)|_{s=0}$ is a Jacobi field along c_0. Evaluate the characteristic equation of Jacobi fields against J:

$$g_{\mathbb{X}}(\ddot{J}, J) + R_{\mathbb{X}}(J, \partial_t c, \partial_t c, J) = 0. \tag{3.10}$$

In the second term, we recognize the sectional curvature of the tangent plane spanned by J and $\partial_t c$. Choose a local orthonormal chart of $T_x\mathbb{X}$ such that $B_0 = I_k \oplus 0_{n-k-1}$ and $\dot{B}_0 = B(v) \oplus I_{n-k-1}$, where $B(v)$ is the shape operator of f at v. Then Equation (3.10) rewrites as

$${}^{\perp}w\ddot{B}_t B_t w + R_{\mathbb{X}}(J, \partial_t c, \partial_t c, J) = 0 \tag{3.11}$$

and as the sectional curvature is less than -1, we replace the second term with the control

$$R_{\mathbb{X}}(J, \partial_t c, \partial_t c, J) \leqslant -{}^{\perp}w^{\perp}B_t B_t w \tag{3.12}$$

to get the following

$${}^{\perp}w^{\perp}(\ddot{B}_t - B_t)B_t w \geqslant 0. \tag{3.13}$$

Now remark that

$$\begin{cases} \varphi'(t) = 2^{\perp}w^{\perp}B_t\dot{B}_t w \\ \varphi''(t) = 2^{\perp}w^{\perp}B_t\ddot{B}_t w + 2^{\perp}w^{\perp}\dot{B}_t\dot{B}_t w. \end{cases} \tag{3.14}$$

So the characteristic equation rewrites as

$$\varphi''(t) \geqslant 2\varphi(t) + \frac{\varphi'(t)^2}{2\varphi(t)} \tag{3.15}$$

with equality when $\mathbb{X}$ is the hyperbolic space, and strict inequality if $\mathbb{X}$ has sectional curvature strictly less than -1. As a direct consequence,

$$(\sqrt{\varphi})'' \geqslant \sqrt{\varphi} \tag{3.16}$$

with the same equality and inequality conditions. Finally, introduce $g = \ln(\varphi)$. As long as it is well defined, we deduce from (3.15) that

$$g'' + \frac{(g')^2}{2} \geqslant 2. \tag{3.17}$$

□

Remark 3.6. — When $\mathbb{X}$ has varying curvature less or equal than -1, up to rescaling the metric we can assume its sectional curvature to be strictly less than -1.

Proof of Theorem 3.1. — With the notations introduced, Fix $w \in T_{x,v}(UY)$ and consider the function φ defined in Proposition 3.5. Under the splitting $T_{x,v}(UY) \approx T_xY \oplus (v^\perp \cap N_xY)$, decompose $w = w_1 \oplus w_2$. We first assume $w_1 \neq 0$. The initial conditions are, by construction of φ,

$$\begin{cases} \varphi(0) = \alpha^2 = |w_1|^2 > 0 \\ \varphi'(0) = 2\alpha\beta = 2^\perp w_1 B(v) w_1 < 2\alpha. \end{cases} \tag{3.18}$$

From Proposition 3.5, we deduce that

$$\begin{cases} \varphi(t) \geqslant (\alpha\cosh(t) + \beta\sinh(t))^2 \\ \varphi'(t) \geqslant 2(\alpha\cosh(t) + \beta\sinh(t))(\alpha\sinh(t) + \beta\cosh(t)) \\ g'(t) = \frac{\varphi'(t)}{\varphi(t)} \geqslant 2\frac{\alpha\sinh(t)+\beta\cosh(t)}{\alpha\cosh(t)+\beta\sinh(t)} \end{cases} \tag{3.19}$$

with equality when $\mathbb{X}$ is hyperbolic, and strict inequality when the curvature of $\mathbb{X}$ is strictly less than -1.

A first consequence is that φ never vanishes. Hence (B_t) has trivial kernel, and the metric induced by exp on NY is nondegenerate. So exp is a local diffeomorphim from NY to $\mathbb{X}$.

The control on φ also shows that the induced metric by exp is complete: Indeed, denote by g_t the induced metric by Φ_t, the control on φ shows that:

$$g_t(w,w) \geqslant g_0(w_1,w_1)(\cosh(t) - (1-\varepsilon)\sinh(t))^2 \geqslant g_0(w_1,w_1)(1-(1-\varepsilon)^2)^2.$$

Now let $w_n = w_n^1 \oplus w_n^2$ be a Cauchy sequence in NY for the induced metric. the previous control implies that w_n^1 is Cauchy for the induced metric $f^*g_{\mathbb{X}}$. As f is assumed complete, w_n^1 converges. So we can assume that w_n^1 is in a small ball on which the induced metric on the fibers of NY are uniformly comparable to $|\cdot|$. Then the characteristic equation on φ implies the control

$$g_t(w_n,w_n) \geqslant \sinh(t)^2 |w_n^2|^2\,,$$

so that (w_n^2) is Cauchy and converges. Hence the induced metric on NY is Cauchy-complete, and by Hopf–Rinow it is complete. Now we apply a classical Lemma of riemannian geometry (cf. [22, Lemma 11.6]) to obtain that exp is indeed a covering map. But $\mathbb{X}$ is homeomorphic to a ball and Y is connected, so exp is a global diffeomorphism and Y is embedded, homeomorphic to a ball too.

It remains to prove that f is a quasi-isometric embedding. In order to do so we introduce a metric on NY for which the inverse map of the exponential map will be Lipschitz, and such that Y sits in NY totally geodesically with the induced metric by f:

Consider $(x, tv) \in NY$ and decompose its tangent space into $T_{(x,tv)}NY = T_t(x, \mathbb{R}v) \oplus T_xY \oplus T_v\mathbb{S}^{n-k-1}$.

Introduce the metric h on NY which in that decomposition is:

$$h_{(x,tv)} = 1 \oplus \cosh^2(t) f^* g_{\mathbb{X},x} \oplus \sinh^2(t) g_{\mathbb{S}^{n-k-1}}. \tag{3.20}$$

By construction, Y sits inside NY totally geodesically with the metric induced by f, that we denote g_Y.

The equation on φ (3.19) ensures that:

$$\exp^* g_{\mathbb{X}} \geqslant 1 \oplus (\cosh(t) I_k + B(v)\sinh(t))^2 g_Y \oplus \sinh^2(t) g_{\mathbb{S}^{n-k-1}}. \tag{3.21}$$

Denote $\delta = 1 - \sup |\mathbb{I}_f| > 0$. Then

$$\exp^* g_{\mathbb{X}} \geqslant \delta^2 (1 \oplus \cosh^2(t) g_Y \oplus \sinh^2(t) g_{\mathbb{S}^{n-k-1}}) = \delta^2 h. \tag{3.22}$$

Hence the inverse map of the exponential map is $\frac{1}{\delta}$-Lipschitz.

As a consequence, f is a quasi-isometric embedding, as it satisfies, for $x_1, x_2 \in Y$:

$$d_{\mathbb{X}}(f(x_1), f(x_2)) \leqslant d_Y(x_1, x_2) \leqslant \frac{1}{\delta} d_{\mathbb{X}}(f(x_1), f(x_2)). \tag{3.23}$$

The first inequality comes from the fact that Y sits totally geodesically in (NY, h), and the other is from the comparison between h and $\exp^* g_{\mathbb{X}}$. Combined with the fact that Y has negative sectional curvature proven in Lemma 3.2, it ensures (see for instance [12, Chapter 7 Proposition 14]) that f extends at infinity to an embedding from a sphere $\mathbb{S}^{k-1} \to \partial_\infty \mathbb{X}$. □

As a corollary of the computations we made, we have some explicit estimates of the geometry of NY, endowed with the induced metric by exp.

Corollary 3.7. — *Let $f : Y \to \mathbb{X}$ be a complete almost-fuchsian immersion, with $\mathbb{X}$ a Hadamard space. Denote by N_tY the distance t hypersurface to Y. At a point $\exp_x(tv) \in N_tY$, consider $\lambda_1 \leqslant \cdots \leqslant \lambda_k$ the eigenvalues of $B(v)$ the shape operator of Y evaluated at (x, v).*

Then the second fundamental form $\mathbb{I}_t$ of N_tY has eigenvalues $\lambda_1^t \leqslant \cdots \leqslant \lambda_{n-1}^t$ which satisfy

$$\forall i \leqslant k, \quad \sum_{j=1}^{i} \lambda_j^t \geqslant \sum_{j=1}^{i} \frac{\lambda_j + \tanh(t)}{1 + \lambda_j \tanh(t)} \tag{3.24}$$

$$\forall i > k, \quad \sum_{j=1}^{i} \lambda_j^t \geqslant \sum_{j=1}^{k} \frac{\lambda_j + \tanh(t)}{1 + \lambda_j \tanh(t)} + (i-k)\frac{1}{\tanh(t)}. \tag{3.25}$$

In particular, N_tY is k-convex in the sense of Sha [29].

Proof. — This is a consequence of the control on $\frac{f'}{f}$ gotten in (3.19). The trace of $\mathbb{I}_t$ on a k-plane P is the sum of $\frac{1}{2}\frac{f'(t)}{f(t)}$ for w spanning a basis of the plane P.

The almost-fuchsian condition ensures that the sum of the λ_i is 0, and that they live in a compact set of $(-1,1)$, which ensures the k-convexity of $\mathbb{I}_t$. □

4. The asymptotic Plateau problem

The Plateau problem was introduced in 1847, originally about a mathematical proof of the existence of soap films bounding a wire frame. It was solved by Douglas [7] and Rado [26]. Here we are interested in a noncompact analogous problem: the asymptotic Plateau problem. Fix $\mathbb{X}$ a Hadamard space of sectional curvature less than -1, the asymptotic Plateau problem can be formulated as

Problem 4.1 (Asymptotic Plateau Problem). — *Let S be a $(k-1)$-sphere embedded in $\partial_\infty \mathbb{X}$. Is there a minimal k-dimensional submanifold M of $\mathbb{X}$ asymptotically bounding S? Is M unique?*

The asymptotic Plateau problem has been studied by Anderson [2]. The problem has since had a rich history, see the survey of Cozkunuzer [6].

Recently, Huang–Lowe–Seppi [15] considered the asymptotic Plateau problem for a class of almost-fuchsian submanifolds of $\mathbb{H}^n$. If a sphere in $\partial_\infty \mathbb{H}^n$ bounds an almost-fuchsian hypersurface, then it is the unique minimal hypersurface bounding it. Jiang [17] also solved the asymptotic Plateau problem for Jordan curves bounding an almost-fuchsian disc in $\partial_\infty \mathbb{H}^n$. In this section, we generalize these results to any sphere bounding an almost-fuchsian submanifold in a space $\mathbb{X}$, under the assumption that $\mathbb{X}$ has negatively pinched curvature.

Theorem 4.2 (Asymptotic Plateau Problem for almost-fuchsian spheres). — *Let $\mathbb{X}$ be a negatively pinched Hadamard space with sectional curvature less or equal than -1. Let $Y \subset \mathbb{X}$ be an almost-fuchsian submanifold of dimension k, bounding at infinity a $(k-1)$-dimensional sphere $S \subset \partial_\infty \mathbb{X}$. Then Y is the unique complete k-dimensional minimal submanifold of $\mathbb{X}$ bounding S.*

In all the remainder of the article, $\mathbb{X}$ is assumed to have pinched sectional curvature between $-C$ and -1. $Y \subset \mathbb{X}$ is a complete almost-fuchsian k-dimensional submanifold, and S is its asymptotic boundary in $\partial_\infty \mathbb{X}$.

The first step in the proof is that any other minimal submanifolds bounding S remain at bounded distance from Y:

PROPOSITION 4.3. — *Let Z be a minimal submanifold of $\mathbb{X}$ bounding S. Then $Z \subset \mathcal{N}_r Y$, where $\mathcal{N}_r Y$ is the r-uniform neighborhood of Y, and* $r = \tanh^{-1}(1 - \sup|\mathbb{I}_f|)$.

Proof. — Thanks to Theorem 2.1, we know that Z is included in the convex hull of S. But the explicit bounds shown in Corollary 3.7 show that for $r \geqslant \tanh^{-1}(1 - \sup|\mathbb{I}_f|)$, the uniform neighborhoods $\mathcal{N}_r(Y)$ are convex, and contain Y so bound S too. As a consequence, $\mathcal{N}_r(Y)$ contains the convex hull of Y which contains Z. □

Remark 4.4. — The same Corollary 3.7 shows that all uniform neighborhoods are k-convex, which directly implies that the function $z \in Z \mapsto d(z, Y)$ cannot have a local maximum. When the induced metric of Z has lower bounded sectional curvature, we could apply the Omori–Yau [23, 31] maximum principle to conclude that $Z = Y$. Here we don't prove that Z has lower bounded sectional curvature, but we prove that the maximum principle is still applicable. This property is called *stochastical completeness*, cf Pigola–Rigoli–Setti [25].

We want to apply a maximum principle to the distance function to Y restricted on a minimal submanifold Z, so we first prove it satisfies a strong subharmonicity condition.

PROPOSITION 4.5. — *Let $Z \subset \mathbb{X}$ be a minimal k-dimensional submanifold. Consider the function $u : z \in Z \mapsto d(z, Y)^2$. Then there is $C > 0$ such that*

$$\Delta u \geqslant Cu. \tag{4.1}$$

Proof. — Consider d the distance function to Y, defined on the whole space X. Because Y is almost-fuchsian, for any point of $\mathbb{X} - Y$, the distance is attained at a unique point of $\mathbb{X}$, so the function d is smooth on $X - Y$.

Furthermore, at a point x where $d(x) = t > 0$, we have the formula

$$\nabla^2 d = 0 \oplus \mathbb{I}_{N_t(Y)} \tag{4.2}$$

the 0 is simply because d is linear in the direction of the minimizing geodesic to Y. Using the formula

$$\nabla^2(d^2) = 2\nabla d \cdot \nabla d + 2d\nabla^2 d. \tag{4.3}$$

We deduce that

$$\nabla^2(u) \geqslant 2 \oplus 2d\mathbb{I}_{N_t(Y)}. \tag{4.4}$$

Now, because Z is a minimal submanifold of X, the laplacian of the restriction of u equals the trace of the restriction of its Hessian to the tangent space of Z

$$\Delta(u|_Z) = \mathrm{Tr}((\nabla^2 u)|_{TZ}). \tag{4.5}$$

Hence the bounds on the eigenvalues of $\mathbb{I}_{N_t(Y)}$ computed in Corollary 3.7 and the control

$$\frac{\tanh(t) + \lambda_i}{1 + \lambda_i \tanh(t)} \leqslant 1 \tag{4.6}$$

allow us to get the control:

$$\Delta(u|_Z) \geqslant 2d \inf_{\lambda_1 + \cdots + \lambda_k = 0, |\lambda_i| \leqslant 1-\varepsilon} \sum_{i=1,\dots,k} \frac{\tanh(d) + \lambda_i}{1 + \tanh(d)\lambda_i} = 2d\Phi(d)\,, \tag{4.7}$$

with $\Phi(d)$ defined to be that infimum. Remark that $\Phi(d)$ is continuous, positive, vanishes only when $d = 0$, and satisfies

$$\Phi'(0) \geqslant k(1 - (1 - \sup|\mathbb{I}_f|)^2) > 0. \tag{4.8}$$

As d is bounded by a constant $r = \tanh^{-1}(1 - \sup|\mathbb{I}_f|)$, there is a constant $C > 0$ such that on $[0, r]$, $\Phi(d) \geqslant Cd$. We deduce that the restriction of u to Z satisfies

$$\Delta u \geqslant 2Cu, \tag{4.9}$$

as claimed. □

Now we want to prove that we can apply the maximum principle to u. In order to do so, we will use the Khas'minskii test [18], as stated in [25, Theorem 3.1 and Proposition 3.2].

Theorem 4.6. — *Let M, g be a Riemannian manifold, and assume that M supports a C^2 function γ, which tends to infinity at infinity, and satisfies*

$$\Delta\gamma \leqslant \lambda\gamma \text{ off a compact set} \tag{4.10}$$

for some $\lambda > 0$. Then M, g is stochastically complete, or equivalently for every $\lambda > 0$ the only non-negative bounded smooth solution u of $\Delta u \geqslant \lambda u$ on M is zero.

Note that the equivalence between stochastical completeness and the applicability of the maximum principle is due to Grigor'Yan [13].

The following proposition will ensure that we can apply our maximum principle on Z:

Proposition 4.7. — *Let $Z \subset \mathbb{X}$ be a proper minimal submanifold of $\mathbb{X}$ complete contractible space with sectional curvature pinched between $-b$ and $-a$, $b \geqslant a \geqslant 0$. Fix $p \in \mathbb{X}$ and denote f the distance function to the point p. Then out of a compact set K, there is a constant $C > 0$ such that the restriction of f to Z satisfies*

$$\Delta(f|_Z) \leqslant C. \tag{4.11}$$

Proof. — As $\mathbb{X}$ is negatively pinched, there are $b \geqslant a \geqslant 0$ such that $-b \leqslant K_{\mathbb{X}} \leqslant -a$. By the Hessian comparison Theorem, the Hessian of f is bounded between the hessian of the distance functions in the spaceforms of sectional curvature $-b$ and $-a$. Explicitely, this means

$$a \coth(af) g_{\mathbb{X}} \leqslant \nabla^2 f \leqslant b \coth(bf) g_{\mathbb{X}}. \tag{4.12}$$

If a is zero, one has to replace $a \coth(af)$ by $\frac{1}{f}$, but it doesn't change anything to the proof. In particular, out of a compact set K containing p, the hessian of f is bounded by a constant C. Now because Z is proper, the intersection $K \cap Z$ is a compact subset of Z. Also, because Z is minimal, the laplacian of the restriction of f satisfies

$$\Delta(f|_Z) = \mathrm{Tr}(\nabla^2 f|_{TZ}) \leqslant C \dim Z\,. \tag{4.13}$$

as desired. □

Applying Theorem 4.6, we deduce that the maximum principle is applicable on a proper submanifold of such a space $\mathbb{X}$.

Corollary 4.8. — *Let $\mathbb{X}$ be a complete contractible space whose sectional curvature is bounded between $-b$ and 0 for some $b > 0$. Let $Z \subset \mathbb{X}$ a proper minimal submanifold. Then for any $u \in C^2(Z)$ such that $\sup u < \infty$, for every $c < \sup u$,*

$$\inf_{z \in Z : u(z) > c} \Delta u \leqslant 0. \tag{4.14}$$

Proof. — Thanks to Proposition 4.7, the function $f|_Z$ tends to ∞ at ∞, and out of a compact set K, it satisfies

$$\Delta f \leqslant C \leqslant C' f \tag{4.15}$$

for some constant $C' > 0$. In particular, we can apply Theorem 4.6 to deduce that Z is stochastically complete, which is equivalent to the statement of our corollary by [25, Theorem 3.1]. □

We now have everything needed to prove our main theorem.

Proof of Theorem 4.2. — Introduce $u : Z \to \mathbb{R}$, such that $u(z) = d(z,Y)^2$. Thanks to Proposition 4.3, u is bounded. But thanks to Proposition 4.5, there is $C > 0$ such that

$$\Delta u \geqslant Cu. \tag{4.16}$$

Now because of Corollary 4.8, we can apply the maximum principle on u to get that $u = 0$. Hence $Z \subset Y$, and by completeness $Z = Y$, as desired. □

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