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GROUPS OF COHOMOLOGICAL CODIMENSION ONE

by Alexander J. MARGOLIS (*)

Abstract. — We show that if H is a commensurated subgroup of G such that both H and G are of type VFP and $\mathrm{vcd}(G) = \mathrm{vcd}(H) + 1$, then G is the fundamental group of a graph of groups in which all vertex and edge groups are commensurable to H. We also investigate commensurated subgroups of one-relator groups and duality groups.

Résumé. — Nous montrons que si H est un sous-groupe commensuré de G tel que H et G soient tous deux de type VFP et $\mathrm{vcd}(G) = \mathrm{vcd}(H) + 1$, alors G est le groupe fondamental d'un graphe de groupes dans lequel tous les groupes de sommets et d'arêtes sont commensurables à H. Nous étudions également les sous-groupes commensuré des groupes définis par une seule relation et groupes à dualité.

1. Introduction

In 1968 Stallings proved the following remarkable theorem.

Theorem 1.1 ([42]). — *A finitely generated group of cohomological dimension one is free.*

Swan later proved that all groups of cohomological dimension one are free [44]. This was generalised by Dunwoody to all groups of cohomological dimension one over any commutative ring R [16]. In this article we prove a higher dimensional generalisation of Theorem 1.1.

Let G be a group. Two subgroups H and K are *commensurable* if $H \cap K$ has finite index in both H and K. A subgroup H is *commensurated* in G, denoted $H \lhd\!\!\!\cdot\; G$, if H is commensurable with gHg^{-1} for every $g \in G$. Commensurated subgroups are also known as almost normal subgroups,

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near normal subgroups and inert subgroups. We say that G is of *type VFP* if for some finite index subgroup $G' \leqslant G$, the trivial $\mathbb{Z}G'$-module $\mathbb{Z}$ has a finite projective resolution by finitely generated $\mathbb{Z}G'$ modules. The *virtual cohomological dimension* of G, denoted $\mathrm{vcd}(G)$, is the cohomological dimension of such a finite index subgroup. The main result of this article is the following:

THEOREM 1.2. — *Let H and G be groups of type VFP such that $H \lhd G$ and $\mathrm{vcd}(G) = \mathrm{vcd}(H) + 1$. Then G is the fundamental group of a finite graph of groups in which every vertex and edge group is commensurable to H.*

In Theorem 1.2 we do not assume that H is a codimension one subgroup in the sense of Houghton–Scott or Kropholler–Roller, i.e. we do not assume that $e(G,H) > 1$ or $\widetilde{e}(G,H) > 1$. Theorem 1.2 will be deduced from Theorem 5.8. We also prove Theorem 5.5, a similar result for Gorenstein cohomological dimension, a generalisation of cohomological dimension that is finite for a much wider class of groups (see Proposition 3.5).

The hypothesis that $H \lhd G$ is crucial and cannot be weakened. Indeed, the Euclidean triangle group $G = \langle a,b,c \mid a^2, b^2, c^2, (ab)^3, (ac)^3, (bc)^3 \rangle$ has virtual cohomological dimension two and the infinite cyclic subgroup $H = \langle ab \rangle$ has cohomological dimension one. However, G has Serre's property FA, so the conclusion of Theorem 1.2 does not hold, even though H is normal in some finite index subgroup of G. More generally, there are myriad groups G of cohomological dimension two (for instance, random hyperbolic groups) which contain infinite cyclic subgroups H that do not satisfy the condition $e(G,H) > 1$, ensuring that G does not split over H even though $\mathrm{vcd}(G) = \mathrm{vcd}(H)+1$. This further underlines the importance of the $H \lhd G$ hypothesis in Theorem 1.2.

Similar results have been obtained by Bieri in the case where H is normal, although Theorem 1.2 appears to be stronger than previously known results even in this case [4, 5]. Kropholler proved a special case of this theorem when H is infinite cyclic, and then later when H is a Poincaré duality group that contains no non-abelian free subgroup [27, 29]. Walker also proved a special case of Theorem 1.2 when H is a Poincaré duality group of non-zero Euler characteristic [45]. The methods used in the proof of Theorem 1.2 are more geometric than those used by Kropholler and Walker, giving new proofs of these results.

Groups containing commensurated subgroups that are not normal often have exotic properties and are a frequent source of counterexamples and pathologies in group theory. For instance:

(1) Baumslag and Solitar constructed the first known examples of non-Hopfian groups [2].
(2) Burger and Mozes constructed finitely presented torsion-free simple groups that act cocompactly on the product of two trees [11].
(3) Leary and Minasyan constructed CAT(0) groups that are not biautomatic [31].

These three families of groups all satisfy the hypotheses of Theorem 1.2. In contrast, we note that $SL_n(\mathbb{Q})$ is a commensurated subgroup of $SL_n(\mathbb{R})$, but this pair does not satisfy the remaining hypotheses of Theorem 1.2 regarding finiteness properties and virtual cohomological dimension.

This is the second in a sequence of articles examining geometric and topological properties of groups containing commensurated subgroups. A key ingredient in our approach, used extensively in [33], is that although there is not necessarily a quotient group G/H when $H \lhd G$, there is still a *quotient space* G/H, well-defined up to quasi-isometry, that has many of the properties one might expect from the Cayley graph of the quotient group. In particular, G can be thought of as a coarse fibre bundle over G/H. An earlier article of the author investigated when this coarse fibre bundle structure is preserved by quasi-isometries [33]. In this article, we use the action of G on the quotient space G/H to prove Theorem 1.2.

We briefly outline the proof of Theorem 1.2. We first use the hypothesis that $\mathrm{vcd}(G) = \mathrm{vcd}(H) + 1$ to show that the quotient space G/H is "cohomologically one-dimensional". Then, we formulate this by applying coarse algebraic topological invariants to the quotient space G/H. This approach is fundamentally different from that taken by Bieri and Kropholler. The main tool that is used is Theorem 4.5 (a Künneth theorem for coarse bundles). Cohomological one-dimensionality of G/H implies it has more than one end. We now mirror Stallings proof of Theorem 1.1 and show G splits as an amalgamated free product or HNN extension over a subgroup commensurable to H. It follows from a variant of Dunwoody's accessibility theorem that the process of iteratively splitting G over subgroups commensurable to H eventually terminates [17], from which Theorem 1.2 readily follows.

Applications and other results

The techniques used in the proof of Theorem 1.2 show that if $\mathrm{vcd}(H) = \mathrm{vcd}(G)$, then the quotient space is "cohomologically zero-dimensional" and hence bounded. We thus deduce the following, which is stated as part of Lemma 5.4.

PROPOSITION 1.3. — *Suppose* $H \lhd G$ *are groups of type* VFP*. Then* H *is a finite index subgroup of* G *if and only if* $\mathrm{vcd}(H) = \mathrm{vcd}(G)$.

We can use Theorem 1.2 and Proposition 1.3 to investigate properties of commensurated subgroups in groups of low cohomological dimension. The following statement generalises the fact that a finitely generated normal subgroup of a finite rank free group is either trivial or of finite index.

COROLLARY 1.4. — *If* G *is a finitely generated free group and* $H \lhd G$ *is finitely generated and non-trivial, then* H *is a finite index subgroup of* G.

Corollary 1.4 can also be deduced from Marshall Hall's theorem. Things are more interesting in dimension two, where we obtain the following generalisation of [5, Theorem D].

COROLLARY 1.5. — *Let* G *be a finitely presented group of virtual cohomological dimension two. Suppose* $H \lhd G$ *is infinite and finitely presented. Then either:*

(1) H *is a finite index subgroup of* G*;*
(2) H *is virtually free and* G *splits as a graph of groups in which all vertex and edge groups are commensurable to* H.

The conclusion of Corollary 1.5 is false if we only assume that H is finitely generated but not necessarily finitely presented. Corollary 1.5 can be strengthened when G is a one-relator group:

THEOREM 1.6 (Theorem 6.1). — *Let* G *be a one-relator group and let* $H \lhd G$ *be a finitely presented commensurated subgroup that is infinite and of infinite index. Then* G *is torsion-free and two-generated. Moreover, one of the following holds:*

(1) H *is infinite cyclic and* G *splits as a graph of groups in which all vertex and edge groups are infinite cyclic and commensurable to* H.
(2) G *contains a free normal subgroup* N *that is commensurable to* H *such that* $G/N \cong \mathbb{Z}$ *or* $\mathbb{Z}_2 * \mathbb{Z}_2$.

In particular, G *is either a generalised Baumslag–Solitar group or is virtually a free-by-cyclic group.*

Bieri showed that if a group H is of type VFP and is a normal subgroup of a duality group G, then H and G/H must also be duality groups [4, Theorem A]. We partially generalise this to the setting of commensurated subgroups. The following is a special case of Theorems 7.2 and 7.3.

THEOREM 1.7. — *Suppose G is a virtual (Poincaré) duality group. If $H \lhd G$ is of type VFP, then H is also a virtual (Poincaré) duality group.*

Commensurated subgroups also occur very naturally in the context of groups acting on cell complexes. If G acts cellularly on a connected locally finite CW complex X, then the stabiliser of any cell is a commensurated subgroup. Indeed, suppose that σ is a cell of X with stabiliser H. Then for any $g \in G$, the H-orbit $H \cdot g\sigma$ is finite, so $g^{-1}Hg$ is contained is finitely many left H-cosets. Similarly, H is also contained in finitely many left $g^{-1}Hg$-cosets, so $H \lhd G$.

We remark that if G acts cellularly on a connected locally finite CW complex X, then one can obtain an action of G on a connected locally finite graph by restricting to the 1-skeleton of X. We thus focus on group actions on graphs, equipped with the induced path metric in which edges have length one.

Combining Corollary 1.5 with the preceding observation, we deduce the following rigidity theorem for finitely presented groups of cohomological dimension two:

THEOREM 1.8. — *Let G be a finitely presented group of virtual cohomological dimension two. Suppose G acts cocompactly on a connected locally finite graph X. Suppose also that H, the stabiliser of a vertex of X, is finitely presented. Then exactly one of the following occurs:*

(1) *H is finite and X is quasi-isometric to G;*
(2) *H is of finite index and X is compact;*
(3) *H is an infinite virtually free group and there is a G-quasi-conjugacy $X \to T$, where T is an infinite locally finite tree admitting a cocompact isometric G-action. In particular, X is quasi-isometric to T.*

Recall that if X and Y are metric spaces admitting isometric G-actions, a *G-quasi-conjugacy* is a quasi-isometry $f : X \to Y$ that is coarsely G-equivariant, i.e. there exists a constant $A \geqslant 0$ such that for all $x \in X$ and $g \in G$, $d(g \cdot f(x), f(gx)) \leqslant A$.

Proof. — If H is finite, then $G \curvearrowright X$ is properly discontinuous and so the Milnor–Švarc lemma implies that G is quasi-isometric to X. If H is of finite index, then G-orbits in X are bounded; as $G \curvearrowright X$ is cocompact, this implies that X is compact. We may thus assume that H is infinite and of infinite index. Corollary 1.5 implies H is virtually free and G splits as a finite graph of groups in which every vertex and edge group is commensurable to H. By considering the action of G on the associated Bass–Serre tree T and applying Lemma 2.8, we conclude that X and T are quasi-conjugate. □

Organization of paper

Sections 2 and 3 consist of preliminary material. Firstly, in Section 2 we introduce coarse geometric and topological notions, including the coarse cohomology used in [32]. In Section 3 we discuss various finiteness properties of groups, including Gorenstein cohomological dimension and its relation to coarse cohomology. Then, in Section 4, we introduce the notion of coarse bundles and investigate their cohomological properties. In particular, we prove Theorem 4.5, a Künneth theorem for coarse bundles. We also use Brown's criterion to deduce Proposition 4.4, which demonstrates that quotient spaces are coarsely uniformly acyclic. In Section 5 we bring all these ingredients together to prove Theorems 5.5 and 5.8, from which Theorem 1.2 follows easily. In Sections 6 and 7 we investigate commensurated subgroups of one-relator groups and duality groups respectively.

2. Coarse geometry and topology

We fix a PID R for the remainder of Section 2.

2.1. Large-scale geometry

Given a metric space (X, d), a subset $A \subseteq X$ and $r \geqslant 0$, the *r-metric neighbourhood* of A is defined by

$$N_r(A) \coloneqq \{x \in X \mid d(x, a) \leqslant r \text{ for some } a \in A\}.$$

The *Hausdorff distance* $d_{\text{Haus}}(A, B)$ between $A, B \subseteq X$ is defined to be

$$\inf\{r \geqslant 0 \mid A \subseteq N_r(B) \text{ and } B \subseteq N_r(A)\}.$$

A map $f : X \to Y$ between topological spaces is said to be *proper* if inverse images of compact sets are compact.

Definition 2.1. — *A map* $f : X \to Y$ *is said to be a* coarse embedding *if there exist proper non-decreasing functions* η, $\phi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ *such that*

$$\eta(d_X(x,x')) \leqslant d_Y(f(x), f(x')) \leqslant \phi(d_X(x,x'))$$

We say such an f *is an* (η,ϕ)*-coarse embedding, and we say that* η *and* ϕ *are the* distortion functions. *If there exists an* $r \geqslant 0$ *such that* $N_r(f(X)) = Y$, *then we say that* f *is a* coarse equivalence.

Remark 2.2. — For a proper non-decreasing function $\phi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, there is another proper non-decreasing function $\widetilde{\phi} : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ where $\widetilde{\phi}(R) := \sup(\phi^{-1}([0,R]))$. This is a sort of inverse to ϕ in the following sense: if $\phi(S) \leqslant R$, then $S \leqslant \widetilde{\phi}(R)$, and if $R < \phi(S)$, then $\widetilde{\phi}(R) \leqslant S$.

We say a function $g : Y \to X$ is a *coarse inverse* to $f : X \to Y$ if $\sup_{x\in X} d_X(x, gf(x)) < \infty$ and $\sup_{y \in Y} d_Y(y, fg(y)) < \infty$. It is an easy exercise to verify that every coarse equivalence has a coarse inverse which is also a coarse equivalence. We therefore deduce that being coarsely equivalent defines an equivalence relation on the class of metric spaces. Every discrete countable group admits a proper left-invariant metric, and this metric is unique up to coarse equivalence.

A map $f : X \to Y$ is said to be *coarse Lipschitz* if there exists a constant $A \geqslant 0$ such that $d_Y(f(x), f(x')) \leqslant A d_X(x,x') + A$ for all $x, x' \in X$. A *quasi-isometry* is a coarse Lipschitz with coarse Lipschitz coarse inverse. Quasi-isometries are precisely coarse equivalences in which the distortion functions can be taken to be affine. It is well known that if G is a finitely generated group, then any two word metrics on G with respect to finite generating sets are quasi-isometric. If G and $H \leqslant G$ are finitely generated groups equipped with the word metric with respect to finite generating sets, then the inclusion $H \to G$ is a coarse embedding.

A *net* $N \subset X$ is a set that satisfies the following two properties:

(1) there exists a constant $t > 0$ such that for any distinct $x, y \in N$, $d(x,y) \geqslant t$;
(2) there is a constant $r \geqslant 0$ such that for each $x \in X$, there exists $n \in N$ such that $d(x,n) \leqslant r$.

It is an easy exercise to show that by precomposing with a closest point projection map, a quasi-isometry between nets $N \subset X$ and $M \subset Y$ extends to a quasi-isometry from X to Y.

Given a metric space X and a parameter $r \geqslant 0$, the *Rips complex* $P_r(X)$ is defined to be the simplicial complex with vertex set X, where $\{x_0, \ldots, x_n\}$ spans a simplex if $d(x_i, x_j) \leqslant r$ for all $0 \leqslant i, j \leqslant n$. A metric space X is said to have *bounded geometry* if for all $r \geqslant 0$, there is an $M_r \geqslant 0$ such that $|N_r(x)| \leqslant M_r$ for every $x \in X$. If X is a bounded geometry metric space, then every Rips complex $P_r(X)$ is locally finite. We say that a metric space is *quasi-geodesic* if it is quasi-isometric to a geodesic metric space. We remark that bounded geometry metric spaces containing more than one point will not be geodesic metric spaces, but may be quasi-geodesic.

Example 2.3. — The Cayley graph of a finitely generated group is a geodesic metric space but is not a bounded geometry metric space. However, a finitely generated group equipped with the word metric is a quasi-geodesic, bounded geometry metric space.

Definition 2.4. — *We say that a metric space X is* coarsely uniformly n-acyclic over R *if for every $i \geqslant 0$, $r \geqslant 0$ and $x \in X$, there exist $j = j(i) \geqslant i$ and $s = s(r,i) \geqslant r$ such that the maps*

$$\widetilde{H}_k\big(P_i(N_r(x)); R\big) \longrightarrow \widetilde{H}_k\big(P_j(N_s(x)); R\big),$$

induced by inclusion, are zero for $k \leqslant n$. Here, $\widetilde{H}_*$ denotes the reduced simplicial homology of the Rips complex. We call $j : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ and $s : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ the coarse acyclicity functions. *We say that X is* coarsely uniformly acyclic over R *if it is coarsely uniformly n-acyclic over R for every n.*

The notion of coarse uniformly acyclicity is a geometric generalisation of finiteness properties of groups; see Proposition 3.1. A crucial step in the proof of Theorem 1.2 is Proposition 4.4, which gives sufficient conditions for the quotient space G/H to be coarsely uniformly acyclic.

Remark 2.5. — If X is a bounded geometry metric space admitting a cocompact group action, coarse uniform n-acyclicity is equivalent to the a priori weaker condition that for every $i \geqslant 0$, there exists a $j \geqslant i$ such that the maps

$$\widetilde{H}_k(P_i(X); R) \longrightarrow \widetilde{H}_k\big(P_j(X); R\big)$$

are zero for $k \leqslant n$.

Coarse uniform acyclicity is invariant under coarse equivalences, and hence invariant under quasi-isometries. We recall the following characterisation of coarse uniform 0-acyclicity:

Proposition 2.6 ([32, Proposition 2.18]). — *Let X be a bounded geometry metric space. The following are equivalent:*

(1) *X is coarsely uniformly 0-acyclic over R;*

(2) *X is coarsely equivalent to a connected locally finite graph, equipped with the induced path metric in which edges have length one.*

Suppose Γ is a locally finite graph. If K is a finite subgraph of Γ, let $c(K)$ denote the number of unbounded components of $\Gamma \setminus K$. The number of ends of Γ is defined to be $\sup\{c(K) \mid K \text{ is a finite subgraph}\} \in \mathbb{N} \cup \{\infty\}$.

Definition 2.7. — *Suppose that X is a coarsely uniformly 0-acyclic, bounded geometry metric space. The* number of ends *of X, denoted $e(X)$, is defined to be the number of ends of a locally finite graph that is coarsely equivalent to X.*

We conclude this subsection with a lemma that will be used in the proof of Theorem 1.8.

Lemma 2.8. — *Let X and Y be connected locally finite graphs, each admitting a cocompact G-action. If the stabiliser of a vertex of X is commensurable to the stabiliser of a vertex of Y, then there exists a G-quasi-conjugacy $X \to Y$.*

Proof. — Pick vertices $v \in V(X)$ and $w \in V(Y)$ such that $H_X := \operatorname{stab}(v)$ is commensurable to $H_Y := \operatorname{stab}(w)$. Since $X_0 := G \cdot v$ and $Y_0 := G \cdot w$ are nets in X and Y, it is sufficient to define a G-quasi-conjugacy $X_0 \to Y_0$. We pick R sufficiently large so that $V(X) \subset G \cdot N_R(v)$.

As $\{g \in G \mid d(gv, v) \leqslant 2R + 1\}$ is a finite union of left H_X-cosets, it is contained in a finite union of left H_Y-cosets because H_X and H_Y are commensurable. Thus, there exists a constant $A \geqslant 0$ such that

$$d_Y(gw, kw) \leqslant A \quad \text{whenever} \quad d_X(gv, kv) \leqslant 2R + 1. \tag{2.1}$$

Let $T \subset G$ be a set of left transversals of H_X in G. We define a map $f_0 : X_0 \to Y_0$ by $tv \mapsto tw$ for all $t \in T$. It follows from (2.1) that $d(f_0(gv), gw) \leqslant A$ for all $g \in G$, and so f_0 is coarsely G-equivariant.

Suppose $gv, kv \in X_0$ and $d_X(gv, kv) = n$. Let $gv = v_0, \dots, v_n = kv$ be a minimal edge path from gv to kv. For each i, we pick $g_i \in G$ such that $d_X(g_i v, v_i) \leqslant R$. Without loss of generality, we may assume $g_0 = g$ and $g_n = k$. Thus for every i, $d_X(g_{i-1}v, g_i v) \leqslant 2R+1$, and so (2.1) implies that

$$d_Y(f_0(g_{i-1}v),\ f_0(g_i v)) \leqslant d_Y(g_{i-1}w, g_i w) + 2A \leqslant 3A.$$

Thus $d_Y(f_0(gv), f_0(kv)) \leqslant 3An = 3A d_X(gv, kv)$, and so f_0 is coarse Lipschitz.

Reversing the roles of X and Y and applying the same argument, we construct a coarse Lipschitz, coarsely G-equivariant map $f_1 : Y_0 \to X_0$ that is a coarse inverse to f_0. Thus f_0 is a G-quasi-conjugacy. $\square$

2.2. Proper chain complexes

Definition 2.9. — *A* based free R-module *is a pair* $(\mathbf{M}, \Sigma)$ *consisting of a free R-module* $\mathbf{M}$ *with a distinguished basis* Σ *called the* standard basis. *We define the* support *of* $m = \sum_{\sigma \in \Sigma} n_\sigma \sigma$ *to be* $\operatorname{supp}(m) := \{\sigma \in \Sigma \mid n_\sigma \neq 0\}$. *A* proper map *between based free modules* $(\mathbf{M}, \Sigma)$ *and* $(\mathbf{N}, \Lambda)$ *is a module homomorphism* $f : \mathbf{M} \to \mathbf{N}$ *such that for every* $\lambda \in \Lambda$,

$$\#\{\sigma \in \Sigma \mid \lambda \in \operatorname{supp}(f(\sigma))\} < \infty.$$

A proper chain complex over R *consists of the pair* $(\mathbf{C}_\bullet, \Sigma_\bullet)$, *where* $\mathbf{C}_\bullet$ *is a chain complex of R-modules such that*

(1) Σ_i *is a basis of* $\mathbf{C}_i$ *for every* i;
(2) *every boundary map* $\partial_i : (\mathbf{C}_i, \Sigma_i) \to (\mathbf{C}_{i-1}, \Sigma_{i-1})$ *is a proper map between based free modules.*

To simplify notation, we frequently say that $f : \mathbf{M} \to \mathbf{N}$ is a proper map or that $\mathbf{C}_\bullet$ is a proper chain complex, where the choice of standard bases is implicit. Our motivating example is the cellular chain complex associated to a locally finite CW complex X. Each $\mathbf{C}_i(X)$ is a free module with a standard basis corresponding to the collection of i-cells of X. As X is locally finite, each $(i-1)$-cell is contained in the boundary of finitely many i-cells, and so the boundary maps $\partial_i : \mathbf{C}_i(X) \to \mathbf{C}_{i-1}(X)$ are proper.

Definition 2.10. — *A chain map* $f_\bullet : \mathbf{C}_\bullet \to \mathbf{D}_\bullet$ *between proper chain complexes is said to be* proper *if each map* $f_i : \mathbf{C}_i \to \mathbf{D}_i$ *is a proper map between based free modules. Similarly, a chain homotopy* $h_\bullet : \mathbf{C}_\bullet \to \mathbf{D}_{\bullet+1}$ *is proper if each map* $h_i : \mathbf{C}_i \to \mathbf{D}_{i+1}$ *is proper.*

Recall that a continuous map between topological spaces is said to be proper if the inverse images of compact sets are compact. If X and Y are locally finite CW complexes and $f : X \to Y$ is a proper cellular map, then the induced map $f_\bullet : \mathbf{C}_\bullet(X) \to \mathbf{C}_\bullet(Y)$ is a proper chain map.

If $(\mathbf{M}, \Sigma)$ is a based free R-module, then the *dual module* is $\operatorname{Hom}(\mathbf{M}, R)$. The *support* of $\alpha \in \operatorname{Hom}(\mathbf{M}, R)$ is defined to be $\operatorname{supp}(\alpha) = \{\sigma \in \Sigma | \alpha(\sigma) \neq 0\}$.

Lemma 2.11. — *Suppose that* $(\mathbf{M}, \Sigma)$ *and* $(\mathbf{N}, \Lambda)$ *are based free R-modules and that* $f : \mathbf{M} \to \mathbf{N}$ *is proper. If* $\alpha \in \operatorname{Hom}(\mathbf{N}, R)$ *has finite support, then so does* $f^*(\alpha) := \alpha \circ f \in \operatorname{Hom}(\mathbf{M}, R)$.

Proof. — We observe that

$$\operatorname{supp}(\alpha \circ f) \subseteq \bigcup_{\lambda \in \operatorname{supp}(\alpha)} \{\sigma \in \Sigma \mid \lambda \in \operatorname{supp}(f(\sigma))\}.$$

Since f is proper and α has finite support, $\alpha \circ f$ also has finite support. $\square$

Given a proper chain complex $(\mathbf{C}_\bullet, \Sigma_\bullet)$, we can dualize to obtain the cochain complex $\mathbf{C}^\bullet = \mathrm{Hom}_R(\mathbf{C}_\bullet, R)$ with coboundary maps $\delta^\bullet$. For each $i \in \mathbb{Z}$, let $\mathbf{C}^i_c = \mathrm{cHom}_R(\mathbf{C}_i, R) \subset \mathrm{Hom}_R(\mathbf{C}_i, R)$ denote the set of all cochains with finite support.

Corollary 2.12. — *Let $(\mathbf{C}_\bullet, \Sigma_\bullet)$ be a proper chain complex. Then $\mathbf{C}^\bullet_c$ is a sub-cochain complex of $\mathbf{C}^\bullet$.*

Proof. — If $\alpha \in \mathbf{C}^i_c$, then by Lemma 2.11 and properness of $\partial_{i+1} : \mathbf{C}_{i+1} \to \mathbf{C}_i$, we deduce $\delta^i\alpha \coloneqq \alpha \circ \partial_{i+1}$ has finite support. □

Let $H^k_c(\mathbf{C}_\bullet)$ denote the k^{th} cohomology of this cochain complex. We call this the *cohomology with compact supports* of $\mathbf{C}_\bullet$. When X is a locally finite CW complex, $H^k_c(\mathbf{C}_\bullet(X))$ is (naturally isomorphic to) the standard notion of cohomology with compact supports of X. We deduce the following by applying Lemma 2.11.

Corollary 2.13. — *A proper chain map $f_\bullet : \mathbf{C}_\bullet \to \mathbf{D}_\bullet$ between proper chain complexes induces maps $f^*_c : H^*_c(\mathbf{D}_\bullet) \to H^*_c(\mathbf{C}_\bullet)$ in compactly supported cohomology. Moreover, if two proper chain maps $f_\bullet, g_\bullet : \mathbf{C}_\bullet \to \mathbf{D}_\bullet$ are properly chain homotopic, then $f^*_c = g^*_c$.*

Recall that the tensor product $\mathbf{C}_\bullet \otimes \mathbf{D}_\bullet$ of chain complexes is defined to be $(\mathbf{C}_\bullet \otimes \mathbf{D}_\bullet)_n = \bigoplus_{i+j=n} \mathbf{C}_i \otimes \mathbf{D}_j$ with boundary map

$$\partial(\sigma \otimes \lambda) = \partial\sigma \otimes \lambda + (-1)^i \sigma \otimes \partial\lambda$$

when $\sigma \in \mathbf{C}_i$ and $\lambda \in \mathbf{D}_j$. The tensor product of proper chain complexes is also a proper chain complex. Indeed, suppose that $(\mathbf{C}_\bullet, \Sigma_\bullet)$ and $(\mathbf{D}_\bullet, \Lambda_\bullet)$ are proper chain complexes. We define the standard basis of $(\mathbf{C}_\bullet \otimes \mathbf{D}_\bullet)_n = \bigoplus_{i+j=n} \mathbf{C}_i \otimes \mathbf{D}_j$ to be

$$\{\sigma \otimes \lambda \mid \sigma \in \Sigma_i,\ \lambda \in \Lambda_j,\ i + j = n\},$$

and $\mathbf{C}_\bullet \otimes \mathbf{D}_\bullet$ is a proper chain complex with respect to these bases.

We now restrict to a special class of proper chain complexes called metric complexes, defined by Kapovich and Kleiner in an appendix of [25]. Let X be a bounded geometry metric space. A *free module over X* consists of the tuple $(\mathbf{M}, \Sigma, p)$, where $(\mathbf{M}, \Sigma)$ is a based free R-module and p is a map $\Sigma \to X$. We call X the *control space* and say that $(\mathbf{M}, \Sigma, p)$ has *finite type* if $\sup_{x \in X} |p^{-1}(x)| < \infty$. We define $\mathrm{supp}_X(m) \coloneqq p(\mathrm{supp}(m))$ for every $m \in \mathbf{M}$.

Definition 2.14. — *Let X and X' be bounded geometry metric spaces and let $(\mathbf{M}, \Sigma, p)$ and $(\mathbf{N}, \Lambda, q)$ be free modules over X and X' respectively. We say that a module homomorphism $\widehat{f} : \mathbf{M} \to \mathbf{N}$ has* finite displacement over a function $f : X \to X'$ *if there exists an* $r \geqslant 0$ *such that for every* $\sigma \in \Sigma$,

$$\operatorname{supp}_{X'}\big(\widehat{f}(\sigma)\big) \subseteq N_r(f(p(\sigma))).$$

If the above holds, we say that $\widehat{f}$ has displacement at most r over f.

Maps between free modules over a metric space are often proper.

Proposition 2.15. — *Suppose X and X' are bounded geometry metric spaces. Suppose also that $f : X \to X'$ is a coarse embedding and that $(\mathbf{M}, \Sigma, p)$ and $(\mathbf{N}, \Lambda, q)$ are finite type free modules over X and X' respectively. If $\widehat{f} : \mathbf{M} \to \mathbf{N}$ has finite displacement over f, then $\widehat{f}$ is a proper map between based free modules.*

Proof. — Suppose that $\widehat{f}$ has displacement at most r over f and that f is an (η, ϕ)-coarse embedding. We need to show $A_\lambda := \{\sigma \in \Sigma \mid \lambda \in \operatorname{supp}(\widehat{f}(\sigma))\}$ is finite for every $\lambda \in \Lambda$. There is nothing to show if $A_\lambda = \emptyset$, so we suppose $\sigma_0 \in A_\lambda$. If $\sigma \in A_\lambda$, then $q(\lambda) \in N_r(f(p(\sigma))) \cap N_r(f(p(\sigma_0)))$, so $d_{X'}(f(p(\sigma)),\ f(p(\sigma_0))) \leqslant 2r$. Thus $d_X(p(\sigma), p(\sigma_0)) \leqslant \widetilde{\eta}(2r)$, with $\widetilde{\eta}$ as in Remark 2.2. There are only finitely many such $\sigma \in \Sigma$, since X has bounded geometry and $\mathbf{M}$ has finite type. □

Let X be a bounded geometry metric space. A *metric complex over R* is a tuple $(X, \mathbf{C}_\bullet, \Sigma_\bullet, p_\bullet)$ such that:

1. $\mathbf{C}_\bullet$ is a non-negative chain complex of R-modules, i.e. $\mathbf{C}_i = 0$ for $i < 0$;
2. each $(\mathbf{C}_i, \Sigma_i, p_i)$ is a finite type free R-module over X;
3. each boundary map $\partial_i : \mathbf{C}_i \to \mathbf{C}_{i-1}$ has finite displacement over the identity map on X;
4. the composition $\varepsilon \circ \partial_1$ is zero, where $\varepsilon : C_0 \to R$ is the augmentation map given by $\sigma \mapsto 1_R$ for each $\sigma \in \Sigma_0$;
5. the map $p_0 : \Sigma_0 \to X$ is onto.

We refer the reader to [25, 32] for more details on the theory of metric complexes. When unambiguous, we frequently denote the metric complex $(X, \mathbf{C}_\bullet, \Sigma_\bullet, p_\bullet)$ simply by $(X, \mathbf{C}_\bullet)$, or even by $\mathbf{C}_\bullet$. The metric space X is said to be *control space* of $(X, \mathbf{C}_\bullet)$. We say that $(X, \mathbf{C}_\bullet)$ is *n-dimensional* if $\mathbf{C}_i = 0$ for $i > n$.

A metric complex $(A, \mathbf{C}'_\bullet, \Sigma'_\bullet, p'_\bullet)$ is a *subcomplex* of $(X, \mathbf{C}_\bullet, \Sigma_\bullet, p_\bullet)$ if $A \subseteq X$, $\mathbf{C}'_\bullet$ is a subchain complex of $\mathbf{C}_\bullet$ and for each i, $\Sigma'_i \subseteq \Sigma_i$ and $p'_i = p_i|_{\Sigma'_i}$. Given a subset $A \subseteq X$, we define $(A, \mathbf{C}_\bullet[A])$ to be the largest subcomplex of $(X, \mathbf{C}_\bullet)$ with control space A. The *n-skeleton* of $(A, \mathbf{C}_\bullet[A])$ is the metric complex $(A, \mathbf{C}_\bullet[A]_n)$, where $\mathbf{C}_i[A]_n = \mathbf{C}_i[A]$ when $i \leqslant n$ and $\mathbf{C}_i[A]_n = 0$ when $i > n$.

Definition 2.16. — *We say* $(X, \mathbf{C}_\bullet)$ *is* uniformly n-acyclic *if for all* r, *there exists a* $\mu(r) \geqslant r$ *such that for all* $x \in X$ *and* $k \leqslant n$, *the map*

$$\widetilde{H}_k(\mathbf{C}_\bullet[N_r(x)]) \longrightarrow \widetilde{H}_k(\mathbf{C}_\bullet[N_{\mu(r)}(x)]),$$

induced by inclusion, is zero. We say that $\mu : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ *is the* acyclicity function.

Proposition 2.17 ([25], [32, Proposition 3.20]). — *A bounded geometry metric space* X *is coarsely uniformly* $(n-1)$*-acyclic over* R *if and only if it is the control space of an* n*-dimensional uniformly* $(n-1)$*-acyclic metric complex* $(X, \mathbf{C}_\bullet)$ *over* R. *Moreover, the acyclicity function and the displacement of the boundary maps of* $(X, \mathbf{C}_\bullet)$ *can be bounded as a function of the coarse acyclicity functions of* X.

We recall the notion of *coarse cohomology* used in [32]. We refer the reader to work of Roe and Kapovich–Kleiner for variants of the notion of coarse cohomology [25, 37].

Definition 2.18. — *Fix* $n > 0$. *Suppose* X *is a bounded geometry metric space that is coarsely uniformly* $(n-1)$*-acyclic over* R. *Let* $(X, \mathbf{C}_\bullet)$ *be an* n*-dimensional, uniformly* $(n-1)$*-acyclic metric complex over* R. *For* $k < n$, *we define* $H^k_{\mathrm{coarse}}(X; R) \coloneqq H^k_c(\mathbf{C}_\bullet)$.

When X is coarsely uniformly $(n-1)$-acyclic, it is possible to define $H^n_{\mathrm{coarse}}(X; R)$ using an idea originally due to Geoghegan–Mihalik [23]. This will not be needed here, but we refer the reader to [32] for more details. The preceding definition of coarse cohomology may differ slightly from the coarse cohomology defined in [32] in dimension zero. The former should be thought of as unreduced cohomology, while the latter is reduced (compare Proposition 2.19 below to [32, Remark 3.24]).

In [32, Proposition 3.26], it is shown that coarse equivalences preserve coarse cohomology. In particular, $H^k_{\mathrm{coarse}}(X; R)$ is independent of the choice of metric complex $(X, \mathbf{C}_\bullet)$. Coarse cohomology contains information about the topology at infinity of the space X (see [32, Section 3.5]). In particular, coarse cohomology in dimensions 0 and 1 can be completely characterised:

Proposition 2.19. — *If* X *is a bounded geometry, coarsely uniformly* 0*-acyclic metric space, then*

(1) $H^0_{\mathrm{coarse}}(X; R) \cong \begin{cases} R & \text{if } X \text{ is bounded} \\ 0 & \text{otherwise.} \end{cases}$

(2) *If* X *is unbounded and coarsely uniformly* 1*-acyclic over* R, *then* $H^1_{\mathrm{coarse}}(X; R)$ *is a free* R*-module with* $\operatorname{rank}(H^1_{\mathrm{coarse}}(X; R)) = e(X) - 1$.

The condition that X is coarsely uniformly 1-acyclic over R can be removed, although we do not need this more general formulation of (2) in this article.

Proof. — Let $(X, \mathbf{C}_\bullet, \Sigma_\bullet, p_\bullet)$ be a uniformly 0-acyclic metric complex over R. Suppose $\sigma_0 \in \Sigma$ and $\alpha \in \mathbf{C}_c^0$ is a compactly supported cocycle. For every $\sigma \in \Sigma_0$, we have that $\alpha(\sigma) = \alpha(\sigma_0)$ since $\sigma - \sigma_0$ is a boundary and α is a cocycle. Thus $\alpha \in \mathbf{C}_c^0$ is determined by its value on σ_0. When X is unbounded, since $\operatorname{supp}(\alpha)$ was assumed to be finite, we deduce that $\alpha \equiv 0$. When X is bounded, the map $H_c^0(C_\bullet) \to R$ given by $[\alpha] \mapsto \alpha(\sigma_0)$ is readily seen to be an isomorphism. The proof of (2) is standard and follows from the arguments in [22, Section 13.4] and [32, Section 3.5]. □

We conclude with a universal coefficient theorem for coarse cohomology. Let R and S be PIDs and suppose $\iota : R \to S$ is a ring homomorphism. Then S can be thought of as an R-module via $r \cdot s = \iota(r)s$. We now have the following universal coefficient theorem for coarse cohomology:

Proposition 2.20. — *Let R and S be as above. Suppose X is coarsely uniformly n-acyclic over R. Then X is coarsely uniformly n-acyclic over S and for each $k < n$, we have a split short exact sequence*

$$0 \longrightarrow H^k_{\text{coarse}}(X;R) \otimes_R S \longrightarrow H^k_{\text{coarse}}(X;S) \longrightarrow \operatorname{Tor}_1^R\big(H^{k+1}_{\text{coarse}}(X;R), S\big) \longrightarrow 1.$$

Proof. — Let $(X, \mathbf{C}_\bullet, \Sigma_\bullet, p_\bullet)$ be a uniformly n-acyclic metric complex over R. Then $\mathbf{C}_\bullet \otimes_R S$ is a uniformly n-acyclic metric complex over S. Thus,

$$H^k_{\text{coarse}}(X;S) \cong H^k\big(\operatorname{cHom}_S\big(\mathbf{C}_\bullet \otimes_R S, S\big)\big),$$

where $\operatorname{cHom}_S(\mathbf{C}_\bullet \otimes_R S, S)$ consists of S-module homomorphisms from $\mathbf{C}_\bullet \otimes_R S$ to S with finite supports.

We now define a map

$$\phi : \operatorname{cHom}_R(\mathbf{C}_\bullet, R) \otimes_R S \longrightarrow \operatorname{cHom}_S(\mathbf{C}_\bullet \otimes_R S, S)$$

given by $\phi(\alpha \otimes s)(\rho \otimes s') = \alpha(\rho) \cdot ss'$ for $\alpha \in \operatorname{cHom}_R(\mathbf{C}_\bullet, R)$, $\rho \in \mathbf{C}_\bullet$ and $s, s' \in S$. It is straightforward to verify that ϕ is an isomorphism of cochain complexes. The key point is the standard basis Σ_i of $\mathbf{C}_i$ induces dual bases of $\operatorname{cHom}_R(\mathbf{C}_i, R) \otimes_R S$ and $\operatorname{cHom}_S(\mathbf{C}_i \otimes_R S, S)$. This is not the case if homomorphisms are allowed to have infinite support. We now apply the standard Künneth formula for tensor products of chain complexes (see Proposition 4.7) to obtain the desired short exact sequence. □

3. Cohomological dimension and group cohomology

Let R be a PID. We say that a group G is of *type* $FP_n(R)$ if the trivial RG-module R has a projective resolution $\mathbf{P}_\bullet \to R$ such that $\mathbf{P}_i$ is finitely generated as an RG-module for $i \leqslant n$. We say that a group G is of type $FP_\infty(R)$ if it is of type $FP_n(R)$ for every n. A group is finitely generated if and only if it is of type $FP_1(\mathbb{Z})$. We say that a group is *almost finitely presented* if it is of type $FP_2(\mathbb{Z}_2)$. Every finitely presented group is almost finitely presented. These properties can be characterised geometrically:

Proposition 3.1 (See [15, 25, 32]). — *Suppose that G is a discrete countable group equipped with a left-invariant proper metric. Then:*

(1) *G is of type $FP_n(R)$ if and only if G is coarsely uniformly $(n-1)$-acyclic over R;*
(2) *G is of type $FP_\infty(R)$ if and only if G is coarsely uniformly acyclic over R.*

The *cohomological dimension* of a group G, denoted $\mathrm{cd}_R(G)$, is defined to be the least n such that the trivial RG-module R has a projective resolution of length n, i.e. a projective resolution $\mathbf{P}_\bullet \to R$ with $\mathbf{P}_i = 0$ for $i > n$. A group is of *type* $FP(R)$ if there exists a finite length resolution of R by finitely generated projective RG-modules. A group is said to be of *type* $VFP(R)$ if it has a finite index subgroup of type $FP(R)$. If G is of type $VFP(R)$, then the *virtual cohomological dimension of* G, $\mathrm{vcd}_R(G)$, is the cohomological dimension of a finite index subgroup of type $FP(R)$.

When the ring R is omitted from notation, it will be assumed that $R = \mathbb{Z}$. In other words, $\mathrm{cd}(G)$ denotes $\mathrm{cd}_{\mathbb{Z}}(G)$, type VFP denotes type $VFP(\mathbb{Z})$ etc. If G is of type FP_n, then it is of type $FP_n(R)$ for any R. We frequently make use of the following fact:

Proposition 3.2 ([9]). — *If G is of type $VFP(R)$, then*

$$\mathrm{vcd}_R(G) = \max\{n \mid H^n(G, RG) \neq 0\}.$$

The cohomological dimension of a group over R sometimes depends on the choice of R. For instance, Dicks and Leary have shown that there exists a group G such that $\mathrm{cd}_{\mathbb{F}}(G) < \mathrm{cd}(G) < \infty$ for any field $\mathbb{F}$ [13]. This cannot happen for groups of type $VFP(R)$, as Proposition 3.3 demonstrates.

If R is a PID, let $\mathbb{P}_R$ denote the collection of primes in R. For each $p \in \mathbb{P}_R$, let R_p denote the field R/pR and let R_0 denote the field of fractions of R. The universal coefficient theorem implies that $\mathrm{vcd}_{R_i}(G) \leqslant \mathrm{vcd}_R(G)$ for all $i \in \mathbb{P}_R \cup \{0\}$. Combining this observation with a result of Bieri [4, Lemma 3.6], we deduce:

Proposition 3.3. — *Let R be a PID and suppose G is of type $VFP(R)$. Then $\mathrm{vcd}_{R_i}(G) \leqslant \mathrm{vcd}_R(G)$ for every $i \in \mathbb{P}_R \cup \{0\}$, and $\mathrm{vcd}_{R_i}(G) = \mathrm{vcd}_R(G)$ for some $i \in \mathbb{P}_R \cup \{0\}$.*

Proposition 3.3 plays a crucial role in the proof of Theorem 5.8, allowing us to reduce the general case to the case where R is a field. The following was noted as [32, Proposition 3.28], and is essentially the argument used by Brown [9, Section VIII, Proposition 7.5]. We provide a proof here for the benefit of the reader.

Proposition 3.4. — *If G is a group of type $FP_n(R)$, then $H^k(G, RG)$ and $H^k_{\mathrm{coarse}}(G; R)$ are isomorphic as R-modules for any $k < n$. In particular, $H^k(G, RG)$ is a quasi-isometry invariant amongst groups of type $FP_\infty(R)$.*

Proof. — By [32, Proposition 3.20 (iii)], there exists a uniformly $(n-1)$-acyclic metric complex $(G, \mathbf{C}_\bullet, \mathbf{C}_\bullet, \Sigma_\bullet)$ over R with control space G, such that G acts freely on the chain complex $\mathbf{C}_\bullet$. Thus, $\cdots \to \mathbf{C}_1 \to \mathbf{C}_0 \to R \to 0$ is a projective resolution of the trivial RG-module R. We claim that $\mathrm{Hom}_{RG}(\mathbf{C}_\bullet, RG) \cong \mathrm{cHom}_R(\mathbf{C}_\bullet, R)$ as cochain complexes of R-modules, where cHom is as in the proof of Proposition 2.20. The proposition in question follows from the claim, since $H^k(G, RG)$ is the cohomology of $\mathrm{Hom}_{RG}(\mathbf{C}_\bullet, RG)$, whilst $H^k_{\mathrm{coarse}}(G; R)$ is the cohomology of $\mathrm{cHom}_R(\mathbf{C}_\bullet, R)$.

We define $\Phi : \mathrm{Hom}_{RG}(\mathbf{C}_\bullet, RG) \to \mathrm{cHom}_R(\mathbf{C}_\bullet, R)$ as follows. For each $\psi \in \mathrm{Hom}_{RG}(\mathbf{C}_k, RG)$ and $x \in \mathbf{C}_k$, we can write $\psi(x) = \sum_{g \in G} f^\psi_g(x) g$, where each f^ψ_g is a function $\mathbf{C}_k \to R$ such that for fixed x, $f^\psi_g(x) = 0$ for all but finitely many $g \in G$. Since ψ is G-equivariant, $f^\psi_{g^{-1}h}(x) = f^\psi_h(gx)$ for all $x \in \mathbf{C}_k$ and $g, h \in G$. In particular, for each $x \in \mathbf{C}_k$, $f^\psi_1(gx)$ is zero for all but finitely many $g \in G$. Since Σ_k consists of finitely many G orbits, $f^\psi_1(\sigma) = 0$ for all but finitely many $\sigma \in \Sigma_k$. In other words, f^ψ_1 is an element of $\mathrm{cHom}_R(\mathbf{C}_k, R)$. We define $\Phi : \mathrm{Hom}_{RG}(\mathbf{C}_\bullet, RG) \to \mathrm{cHom}_R(\mathbf{C}_\bullet, R)$ given by $\psi \mapsto f^\psi_1$. This map is surjective, since for every $f \in \mathrm{cHom}_R(\mathbf{C}_k, R)$, we define $\psi(x) = \sum_{g \in G} f(g^{-1}x) g$ such that $\Phi(\psi) = f$. It can be readily checked that Φ defines an isomorphism of cochain complexes as required.

Finally, since $H^k_{\mathrm{coarse}}(G; R)$ is a quasi-isometry invariant, it follows that $H^k(G, RG)$ is also a quasi-isometry when considered as an R-module. □

We now introduce the notion of Gorenstein cohomological dimension of groups, a generalisation of (virtual) cohomological dimension that allows for proper group actions rather than free group actions. The reader is referred to [24] for definitions and properties of Gorenstein projective modules. We say that a group G has finite *Gorenstein cohomological dimension* over R, denoted $\mathrm{Gcd}_R(G)$, if the trivial RG-module R admits a finite length resolution by Gorenstein projective modules.

As the following proposition shows, the class of groups with finite Gorenstein cohomological dimension is much broader than the class of groups with finite virtual cohomological dimension. In particular, such groups are not required to be virtually torsion-free.

Proposition 3.5 ([1, Proposition 3.1] and [20, Proposition 2.1]). — *Suppose a group G acts properly and cellularly on a contractible finite-dimensional CW complex X. Then $\mathrm{Gcd}_R(G) \leqslant \dim(X) < \infty$ for any commutative ring R.*

We now state the key properties of Gorenstein cohomological dimension that will be needed:

Proposition 3.6. — *Suppose that G is a group of type $FP_\infty(R)$ with $\mathrm{Gcd}_R(G) < \infty$, where R is a commutative ring of finite weak global dimension. Then:*

(1) *there exists an n such that $H^n(G, RG) \neq 0$ and*

$$\mathrm{Gcd}_R(G) = \max\{n \mid H^n(G, RG) \neq 0\};$$

(2) *if $H \leqslant G$, then $\mathrm{Gcd}_R(H) \leqslant \mathrm{Gcd}_R(G)$.*

In particular, if $\mathrm{vcd}_R(G) < \infty$, then $\mathrm{Gcd}_R(G) = \mathrm{vcd}_R(G)$.

Proof.

(1). — It follows from [24, Theorem 2.20] that

$$\mathrm{Gcd}_R(G) = \max\big\{i \in \mathbb{N} \,\big|\, H^i(G, P) \neq 0 \text{ for some projective } RG\text{-module } P\big\}.$$

As G is of type $FP_\infty(R)$, the functor $H^i(G, -)$ is continuous. Moreover, since P is a direct summand of some free RG-module, it follows $\mathrm{Gcd}_R(G) = \max\{i \in \mathbb{N} \mid H^i(G, RG) \neq 0\}$ as required.

(2). — This is [20, Proposition 2.4]. □

As we observe in Remark 5.6, these are the only properties of Gorenstein cohomological dimension that we will make use in the proof of Theorem 5.5. In particular, no knowledge of Gorenstein homological algebra will be needed in what follows.

We deduce the following from Propositions 3.4 and 3.6:

Corollary 3.7. — *If G and G' are quasi-isometric groups of type $FP_\infty(R)$ with $\mathrm{Gcd}_R(G), \mathrm{Gcd}_R(G') < \infty$, then $\mathrm{Gcd}_R(G) = \mathrm{Gcd}_R(G')$.*

It is likely that the $FP_\infty(R)$ condition can be removed using the techniques of Sauer [39]. We now state some more properties of Gorenstein cohomological dimension that might be of further interest to geometric group theorists.

Proposition 3.8. — *If G is either a Gromov hyperbolic or a* CAT(0) *group, then* $\mathrm{Gcd}_R(G) < \infty$ *for any commutative ring R.*

Proof. — If G is hyperbolic, then the Rips complex $P_r(G)$ is contractible for r sufficiently large. In the case G is CAT(0), it follows from [8, Remark III.Γ.3.27] that G acts properly and cocompactly on some finite-dimensional simplicial complex. In both cases, Proposition 3.5 ensures that $\mathrm{gcd}_R(G) < \infty$. □

The $\mathcal{Z}$-boundary was first defined by Bestvina and later generalised by Dranishnikov [3, 14]. The $\mathcal{Z}$-boundary of a hyperbolic group is simply its Gromov boundary, and the $\mathcal{Z}$-boundary of a CAT(0) group G is a visual boundary ∂X of a CAT(0) space X admitting a proper cocompact G-action. Gorenstein cohomological dimension has a geometric interpretation for groups admitting a $\mathcal{Z}$-boundary in the sense of [14]. If Z is a topological space, let $\dim(Z)$ be the Lebesgue covering dimension of X and let $\dim_R(Z)$ be the cohomological dimension of Z with coefficients in R.

Proposition 3.9. — *Suppose that G is a group of finite Gorenstein cohomological dimension that admits $\mathcal{Z}$-boundary Z in the sense of* [14]. *Then* $\dim_R(Z) + 1 = \mathrm{Gcd}_R(G)$ *and* $\dim(Z) + 1 = \mathrm{Gcd}(G)$.

Proof. — Suppose $(\overline{X}, Z)$ is a $\mathcal{Z}$-structure on G in the sense of [14]. Since $\overline{X}$ is contractible, the long exact sequence in cohomology gives isomorphisms $H_c^{n+1}(X;R) \cong H^n(Z;R)$ (see [3, Proposition 1.5]). As G is quasi-isometric to X, $H^{n+1}(G,RG) \cong H^{n+1}_{\mathrm{coarse}}(G;R) \cong H^n(Z;R)$ as R-modules. (This argument is used in the proof of [14, Corollary 2], noting $H^{n+1}_{\mathrm{coarse}}(G;R)$ is isomorphic to Roe's coarse cohomology as mentioned in [32, Appendix B].)

It now follows from the main result of [14] and Proposition 3.6 that $\dim_R(Z) + 1 = \mathrm{Gcd}_R(G)$ for any PID R. It was shown in [35] that Z is finite dimensional, so $\dim(Z) = \dim_{\mathbb{Z}}(Z)$. Thus $\dim(Z) + 1 = \mathrm{Gcd}(G)$. □

4. Cohomology of coarse bundles

There have been several adaptations of the theory of fibre bundles and fibrations to the setting of metric spaces, see [21, 25, 34, 46] for more information. The following definition is essentially the same as that used in [46]:

Definition 4.1 ([33]). — *Let X, F and B be bounded geometry, quasi-geodesic metric spaces. We say that $p : X \to B$ is a* coarse bundle *with fibre F if there exist constants $K \geqslant 1, A, E \geqslant 0$ such that the following hold:*

1. $d(p(x), p(x')) \leqslant Kd(x,x') + A$ *for all* $x, x' \in X$.
2. *Let* $D_b := p^{-1}(N_E(b))$ *for all* $b \in B$. *Then there is an* (η, ϕ)*-coarse embedding* $s_b : F \to X$ *such that* $d_{\mathrm{Haus}}(D_b, \mathrm{im}(s_b)) \leqslant A$, *where* η *and* ϕ *can be chosen independently of* b.
3. $d_{\mathrm{Haus}}(D_b, D_{b'}) \leqslant Kd(b,b') + A$ *for all* $b, b' \in B$.

We say that each D_b is a fibre *of X.*

Suppose G is a finitely generated group equipped with the word metric with respect to a finite generating set, and let $H \lhd G$. We define a metric on G/H by $d(gH, kH) := d_{\mathrm{Haus}}(gH, kH)$ for all $gH, kH \in G/H$. The resulting metric space G/H is called the *quotient space*. The following is shown in [33]; see also [12, 26, 46].

Proposition 4.2. — *Let G and H be as above. Then the quotient space G/H is well-defined up to quasi-isometry, and is a bounded geometry, quasi-geodesic metric space. Moreover, the quotient map $p : G \to G/H$ given by $g \mapsto gH$ is a coarse bundle with fibre H.*

In order to apply coarse topological methods to the quotient space, we need to show it is coarsely uniformly acyclic. Fortunately, this can be done by applying Brown's criterion.

An *n-good G-CW complex over R* is a CW complex X admitting a cellular G-action such that:

1. $\widetilde{H}_k(X; R) = 0$ for $k < n$;
2. for $0 \leqslant p \leqslant n$, the stabiliser of any p-cell of X is of type $FP_{n-p}(R)$.

A *filtration* of X is a nested sequence $X_1 \subseteq X_2 \subseteq \ldots$ of G-invariant subcomplexes of X such that $X = \bigcup_{i \in \mathbb{N}} X_i$. We say a filtration (X_i) has *finite n-type* if every X_i has finitely many G-orbits of p-cells for $p \leqslant n$. Moreover, a filtration is said to be *essentially $(n-1)$-acyclic over R* if for every $k < n$ and i, there is a $j \geqslant i$ such that the map $\widetilde{H}_k(X_i; R) \to \widetilde{H}_k(X_j; R)$, induced by inclusion, is trivial. The following is a simplified version of Brown's criterion.

Theorem 4.3 (Brown's Criterion, [10]). — *Suppose X is an n-good G-CW complex over R admitting a finite n-type filtration (X_i). This filtration is essentially $(n-1)$-acyclic over R if and only if G is of type $FP_n(R)$.*

Proposition 4.4. — *Let G be a group of type $FP_n(R)$ containing a commensurated subgroup H of type $FP_n(R)$. Then the quotient space G/H is coarsely uniformly $(n-1)$-acyclic over R.*

Proof. — Consider the infinite-dimensional simplex $P_\infty(G/H)$, whose vertices are all the left cosets of H. The stabiliser of every finite face $\{g_0H, \dots, g_mH\}$ is commensurable to H, hence is of type $FP_n(R)$. As $P_\infty(G/H)$ is contractible, $P_\infty(G/H)$ is an n-good G-CW complex. The filtration $(P_i(G/H))$ is of finite n-type, since G/H has bounded geometry. As G is of type $FP_n(R)$, Theorem 4.3 ensures $(P_i(G/H))$ is essentially $(n-1)$-acyclic over R. Since G/H is proper and admits a cobounded G-action, Remark 2.5 implies that G/H is coarsely uniformly $(n-1)$-acyclic over R. □

The remainder of this section is devoted to a proof of the following.

Theorem 4.5 (A Künneth theorem for coarse bundles). — *Let R be a PID. Suppose $p : X \to B$ is a fibre bundle with fibre F such that F and B are coarsely uniformly acyclic over R. Then X is coarsely uniformly acyclic over R and for every $k \in \mathbb{N}$ there is a split short exact sequence of R-modules*

$$0 \longrightarrow \bigoplus_{i+j=k} H^i_{\mathrm{coarse}}(F;R) \otimes_R H^j_{\mathrm{coarse}}(B;R) \longrightarrow H^k_{\mathrm{coarse}}(X;R)$$
$$\longrightarrow \bigoplus_{i+j=k+1} \mathrm{Tor}_1^R\big(H^i_{\mathrm{coarse}}(F;R), H^j_{\mathrm{coarse}}(B;R)\big) \longrightarrow 1.$$

All the ideas needed to prove Theorem 4.5 are found in [25, Section 11.5]. Indeed, although Theorem 4.5 is not explicitly stated there, a slightly less general version of Theorem 4.5 is implicitly used in [25]. Group theoretic analogues of Theorem 4.5 when $H \lhd G$ are well-known, for instance see [22, Section 17.3].

Let $(B, \mathbf{B}_\bullet, \Sigma_\bullet, p_\bullet)$ be a uniformly acyclic metric complex with control space B. For each $b \in B$, we fix a uniformly acyclic metric complex $(D_b, \mathbf{D}^b_\bullet, \Sigma^b_\bullet, p^b_\bullet)$ with control space D_b. The displacement and acyclicity constants associated to $\mathbf{D}^b_\bullet$ can be chosen independently of b. Moreover, it can also be assumed that the metric complexes $(D_b, \mathbf{D}^b_\bullet)$ have *uniform finite type* independent of b, i.e. $\sup_{b\in B, x\in D_b} |(p^b_i)^{-1}(x)| < \infty$ for every i.

For each $b, b' \in B$, let $f^{b,b'}$ be a closest point projection $D_b \to D_{b'}$, and let $f^{b,b'}_{\#} : \mathbf{D}^b_\bullet \to \mathbf{D}^{b'}_\bullet$ be a chain map with finite displacement over $f^{b,b'}$, where the displacement depends only on $d_B(b, b')$. We may assume that when $b = b'$, $f^{b,b'}$ and $f^{b,b'}_{\#}$ are the identity maps.

Every k-chain in $\tau \in \mathbf{B}_\bullet \otimes \bigoplus_{b\in B} \mathbf{D}^b_\bullet$ can be written uniquely in the form $\tau = \Sigma_r n_r(\sigma_r \otimes \lambda_r)$, where $\sigma_r \in \Sigma_{i_r}$ and $\lambda_r \in \Sigma^{b_r}_{k-i_r}$ for some $b_r \in B$. We define

$$\operatorname{supp}_X(\tau) \coloneqq \bigcup \big\{ p^{b_r}_{k-i_r}(\lambda_r) \;\big|\; n_r \neq 0 \big\} \subseteq X \tag{4.1}$$

and

$$\operatorname{supp}_B(\tau) \coloneqq \bigcup \{ p_{i_r}(\sigma_r) \mid n_r \neq 0 \} \subseteq B. \tag{4.2}$$

Thus, we define $\mathbf{E}_{i,j}$ to be the free module with basis

$$T_{i,j} \coloneqq \big\{ \sigma \otimes \lambda \,\big|\, \sigma \in \Sigma_i, \lambda \in \Sigma^b_j, p_i(\sigma) = b \big\},$$

which we identify with the submodule of $\mathbf{B}_i \otimes \oplus_{b\in B} \mathbf{D}^b_j$ generated by $T_{i,j}$. We set $T_k \coloneqq \bigsqcup_{i+j=k} T_{i,j}$ and $\mathbf{E}_k = \bigoplus_{i+j=k} \mathbf{E}_{i,j}$ and define $q_k : T_k \to X$ by $q_k(\sigma \otimes \lambda) = p^{p_i(\sigma)}_{k-i}(\lambda)$ for every $\sigma \otimes \lambda \in T_{i,k-i}$. Since the metric complexes $(D_b, \mathbf{D}^b_\bullet)$ have uniform finite type, each $(\mathbf{E}_k, T_k, q_k)$ is a finite type free module over X.

It is not necessarily the case that $\mathbf{E}_\bullet$ is a subcomplex of $\mathbf{B}_\bullet \otimes \bigoplus_{b\in B} \mathbf{D}^b_\bullet$. However, we can define boundary maps so that $\mathbf{E}_\bullet$ is a chain complex.

Lemma 4.6. — *For every $k \in \mathbb{Z}$, there exists a boundary map $\partial : \mathbf{E}_k \to \mathbf{E}_{k-1}$ such that*

$$(X, \mathbf{E}_\bullet, T_\bullet, q_\bullet)$$

is a uniformly acyclic metric complex over R. Moreover, $\mathbf{E}_\bullet$ is properly chain homotopic to $\mathbf{B}_\bullet \otimes \mathbf{D}^b_\bullet$ for any $b \in B$.

Before proving Lemma 4.6, we explain how to deduce Theorem 4.5 from it. To do this, we require the ordinary Künneth theorem for chain complexes of R-modules, where R is assumed to be a PID:

Proposition 4.7 ([9, Proposition I.0.8]). — *Let $\mathbf{C}_\bullet$ and $\mathbf{D}_\bullet$ be chain complexes of free R-modules. Then there is a natural short exact sequence*

$$0 \longrightarrow \bigoplus_{i+j=k} H_i(\mathbf{C}_\bullet) \otimes H_j(\mathbf{D}_\bullet) \longrightarrow H_k(\mathbf{C}_\bullet \otimes \mathbf{D}_\bullet) \longrightarrow \bigoplus_{i+j=k-1} \operatorname{Tor}^R_1\big(H_i(\mathbf{C}_\bullet),\ H_j(\mathbf{D}_\bullet)\big) \longrightarrow 1$$

that splits.

Proof of Theorem 4.5. — As $(X, \mathbf{E}_\bullet)$ is a uniformly acyclic metric complex over R, we see that $H^k_{\mathrm{coarse}}(X;R) = H^k_c(\mathbf{E}_\bullet)$ by the definition of coarse cohomology. Since $\mathbf{E}_\bullet$ is properly chain homotopic to $\mathbf{B}_\bullet \otimes \mathbf{D}^b_\bullet$ for some fixed $b \in B$, we see that $H^k_{\mathrm{coarse}}(X;R) \cong H^k_c(\mathbf{B}_\bullet \otimes \mathbf{D}^b_\bullet)$. We conclude by applying Proposition 4.7, which can be done because the cohomology of a cochain complex is simply the homology of the associated chain complex with indices reversed. □

We now prove Lemma 4.6. To define the boundary map on $\mathbf{E}_\bullet$, we define a chain map

$$g^b_\# : \mathbf{B}_\bullet \otimes \mathbf{D}^b_\bullet \longrightarrow \mathbf{E}_\bullet$$

for every $b \in B$. The boundary map on $\mathbf{E}_\bullet$ will then be defined by

$$\partial(\sigma \otimes \lambda) = g^{p_i(\sigma)}_\#(\partial\sigma \otimes \lambda) + (-1)^i \sigma \otimes \partial\lambda$$

for every $\sigma \otimes \lambda \in T_{i,j}$. This might seem rather circular, as the boundary map on $\mathbf{E}_\bullet$ is defined in terms of $g^b_\#$, whilst $g^b_\#$ is defined to be a chain map, which implies boundary maps of $\mathbf{E}_\bullet$ have already been chosen! However, we will inductively define ∂ and $g^b_\#$ on filtrations of $\mathbf{E}_\bullet$ and $\mathbf{B}_\bullet \otimes \mathbf{D}^b_\bullet$, so this makes sense.

Whilst defining each $g^b_\#$, we show that there is a constant $M = M_{i,j} \geqslant 0$ and a function $\phi = \phi_{i,j} : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, both independent of b, such that

$$\operatorname{supp}_X\big(g^b_\#(\sigma \otimes \lambda)\big) \subseteq N_{\phi(d_B(p_i(\sigma),b))}\big(p^b_j(\lambda)\big) \tag{4.3}$$

$$\operatorname{supp}_B\big(g^b_\#(\sigma \otimes \lambda)\big) \subseteq N_M(p_i(\sigma)) \tag{4.4}$$

for all $\sigma \in \Sigma_i$ and $\lambda \in \Sigma^b_j$.

We proceed inductively, defining a nested sequence of chain complexes $\mathbf{E}^0_\bullet \leqslant \mathbf{E}^1_\bullet \leqslant \dots$ with $\mathbf{E}^k_\bullet = \oplus_{i \leqslant k} \mathbf{E}_{i,\bullet - i}$. We show that for each $\sigma \in \Sigma_i$ and $\lambda \in \Sigma^b_j$, we have

$$g^b_\#(\sigma \otimes \lambda) - \sigma \otimes f^{b,p_i(\sigma)}_\#(\lambda) \in \mathbf{E}^{i-1}_\bullet. \tag{4.5}$$

For the base case, we define the boundary map on $\mathbf{E}^0_i = \mathbf{E}_{0,i}$ by $\partial(\sigma \otimes \lambda) = \sigma \otimes \partial\lambda$. For each $b \in B$, we define a chain map

$$g^b_\# : [\mathbf{B}_\bullet]_0 \otimes \mathbf{D}^b_\bullet \longrightarrow \mathbf{E}^0_\bullet$$

given by $g_b(\sigma \otimes \gamma) = \sigma \otimes f^{b,p_0(\sigma)}_\#(\gamma)$. Conditions (4.3), (4.4) and (4.5) are automatically satisfied.

We now assume that the chain map $g^b_\#$ has already been defined on $[\mathbf{B}_\bullet]_{k-1} \otimes \mathbf{D}^b_\bullet$ and that (4.3), (4.4) and (4.5) are satisfied. To extend $g^b_\#$ we make use of the following lemma.

LEMMA 4.8. — *If k is as above, then for every $j \in \mathbb{Z}$ there exists a number $R = R_j$ and a function $\mu_j = \mu : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ such that the following holds. Suppose τ is a j-cycle of the form*

$$\tau = \sum_{\sigma \in \Sigma_k} \sigma \otimes \partial \lambda^\sigma + \sum_{i<k, \rho \in \Sigma_i} \rho \otimes \tau^\rho$$

in either $\mathbf{E}^k_\bullet$ or $[\mathbf{B}_\bullet]_k \otimes \mathbf{D}^b_\bullet$. Let $Y = \operatorname{supp}_X(\tau) \cup \bigcup_{\sigma \in \Sigma_k} p^{p_k(\sigma)}_{j+1-k}(\lambda^\sigma)$ and $Q \coloneqq \operatorname{diam}(Y)$. Then σ is the boundary of a chain of the form

$$\omega = \sum_{\sigma \in \Sigma_k} \sigma \otimes \lambda^\sigma + \sum_{i<k, \rho \in \Sigma_i} \rho \otimes \omega^\rho,$$

where $\operatorname{supp}_X(\omega) \subseteq N_{\mu(Q)}(Y)$ and $\operatorname{supp}_B(\omega) \subseteq N_R(\operatorname{supp}_B(\tau))$.

Proof. — We prove Lemma 4.8 only when $\tau \in \mathbf{E}^k_\bullet$, but a similar argument holds if $\tau \in [\mathbf{B}_\bullet]_k \otimes \mathbf{D}^b_\bullet$. We proceed inductively on k. The base case $k = 0$ is trivial, since then $\partial \sum_{\sigma \in \Sigma_0} \sigma \otimes \lambda^\sigma = \sum_{\sigma \in \Sigma_0} \sigma \otimes \partial \lambda^\sigma$. We assume that Lemma 4.8 holds when $k = j - 1$.

We now suppose $k = j$ and that τ is a cycle of the required form. For each $\sigma \in \Sigma_j$, we write $\partial \sigma = \sum_{\rho \in \Sigma_{j-1}} n^\sigma_\rho \rho$, where $n^\sigma_\rho \in R$. Since $\partial \tau = 0$, we use the definition of the boundary map for $\mathbf{E}^j_\bullet$ and (4.5) to evaluate all the "$\rho \otimes -$ terms" in $\partial \tau$ and deduce that

$$\partial \left(\sum_{\sigma \in \Sigma_j} n^\sigma_\rho f^{p_j(\sigma), p_{j-1}(\rho)}_\#(\lambda^\sigma) + (-1)^{j-1} \tau_\rho \right) = 0.$$

Thus for each $\rho \in \Sigma_{j-1}$, there is some $\omega^\rho \in D^{p_{j-1}(\rho)}_\bullet$ such that

$$\partial \omega^\rho = \sum_{\sigma \in \Sigma_j} (-1)^{j-1} n^\sigma_\rho f^{p_j(\sigma), p_{j-1}(\rho)}_\#(\lambda^\sigma) + \tau_\rho.$$

We set $\nu \coloneqq \sum_{\sigma \in \Sigma_j} \sigma \otimes \lambda^\sigma$. Then

$$\begin{aligned} \partial \nu &= \sum_{\sigma \in \Sigma_j} \left((-1)^j \sigma \otimes \partial \lambda^\sigma + g^{p_j(\sigma)}_\#(\partial \sigma \otimes \lambda^\sigma) \right) \\ &= \sum_{\sigma \in \Sigma_j} \left((-1)^j \sigma \otimes \partial \lambda^\sigma + \sum_{\rho \in \Sigma_{j-1}} n^\sigma_\rho g^{p_j(\sigma)}_\#(\rho \otimes \lambda^\sigma) \right) \end{aligned}$$

for each $\sigma \in \Sigma_j$. We can thus write $\tau + (-1)^{j-1} \partial \nu$ as

$$\sum_{\rho \in \Sigma_{j-1}} \rho \otimes \partial \omega^\rho + \sum_{i<j-1, \alpha \in \Sigma_i} \alpha \otimes \gamma^\alpha$$

for some $\{\gamma_\alpha\}_{i<j-1, \alpha \in \Sigma_i}$. Since $\tau + (-1)^{j-1} \partial \nu$ is a cycle of the required form, we can apply the inductive hypothesis to $\tau + \partial \nu$. □

We use this lemma to extend $g^b_\#$ from $[\mathbf{B}_\bullet]_{k-1} \otimes \mathbf{D}^b_\bullet$ to $[\mathbf{B}_\bullet]_k \otimes \mathbf{D}^b_\bullet$ inductively as follows. We assume that $g^b_\#$ has already been defined on $[\mathbf{B}_\bullet]_k \otimes [\mathbf{D}^b_\bullet]_{j-1}$, and let $\sigma \in \Sigma_k$ and $\lambda \in \Sigma^b_j$. Using (4.5), we see that $g^b_\#(\partial(\sigma\otimes\lambda)) = (-1)^k g^b_\#(\sigma\otimes\partial\lambda)+g^b_\#(\partial\sigma\otimes\lambda)$ is a cycle of the form required by Lemma 4.8. We thus define $g^b_\#(\sigma\otimes\lambda)$ so that $\partial g^b_\#(\sigma\otimes\lambda) = g^b_\#(\partial(\sigma\otimes\lambda))$ and extend linearly. Using Lemma 4.8, we can ensure that $g^b_\#(\sigma\otimes\lambda)$ will satisfy (4.3), (4.4) and (4.5). This then allows us to define a boundary map $\partial : \mathbf{E}_{k+1} \to \mathbf{E}_k$ as above. Since $\mathbf{B}_\bullet$ and $\mathbf{D}^b_\bullet$ each have finite displacement, it follows from (4.3) and the definition of ∂ that the chain complex $\mathbf{E}_\bullet$ has finite displacement. We thus see that $\mathbf{E}_\bullet$ is a metric complex over X.

We now define a chain map $f^b_\# : \mathbf{E}_\bullet \to \mathbf{B}_\bullet \otimes \mathbf{D}^b_\bullet$ for every $b \in B$ so that for every $\sigma\otimes\lambda \in T_{i,j}$ with $w = p_i(\sigma)$, we have

$$f^b_\#(\sigma\otimes\lambda) - \sigma\otimes f^{w,b}_\#(\lambda) \in [\mathbf{B}_\bullet]_{i-1} \otimes \mathbf{D}^b_\bullet. \tag{4.6}$$

This is done using Lemma 4.8, similarly to how we defined $g^b_\#$. We may assume that $f^b_\#$ satisfies analogues of (4.3) and (4.4).

Similarly, we can also define a chain homotopy $h^b_\#$ from $f^b_\# g^b_\#$ to the identity, and a chain homotopy $\overline{h}^b_\#$ from $g^b_\# f^b_\#$ to the identity. This is possible since for all $b, b' \in B$, $f^{b,b'}_\# f^{b',b}_\#$ is chain homotopic to the identity. The chain homotopies $h^b_\#$ and $\overline{h}^b_\#$ also satisfy analogues of (4.3) and (4.4). Since $g^b_\#$, $f^b_\#$, $h^b_\#$ and $\overline{h}^b_\#$ all satisfy analogues of satisfy analogues of (4.3) and (4.4), they are proper chain maps and chain homotopies. In particular $\mathbf{E}_\bullet$ is properly chain homotopic to $\mathbf{B}_\bullet \otimes \mathbf{D}^b_\bullet$ for any $b \in B$.

We now show that $\mathbf{E}_\bullet$ is uniformly acyclic. More specifically, we pick any $n \geqslant 0$ and show that $\mathbf{E}_\bullet$ is uniformly n-acyclic. Let $x \in X$, $r \geqslant 0$ and set $b = p(x)$. There is an $r_1 \geqslant r$ such that

$$f^b_\#(\mathbf{E}_\bullet[N_r(x)]_n) \subseteq \mathbf{B}_\bullet[N_{r_1}(b)] \otimes \mathbf{D}^b_\bullet[N_{r_1}(x)].$$

Using uniform acyclicity of both $\mathbf{B}_\bullet$ and $\mathbf{D}^b_\bullet$ and the naturality of the Künneth formula in Proposition 4.7, there is an $r_2 \geqslant r_1$ such that the map

$$\widetilde{H}_k\big(\mathbf{B}_\bullet[N_{r_1}(b)] \otimes \mathbf{D}^b_\bullet[N_{r_1}(x)]\big) \longrightarrow \widetilde{H}_k\big(\mathbf{B}_\bullet[N_{r_2}(b)] \otimes \mathbf{D}^b_\bullet[N_{r_2}(x)]\big),$$

induced by inclusion, is zero for $k \leqslant n$. Thus there is an $r_3 \geqslant r$, such that $\widetilde{H}_k(\mathbf{E}_\bullet[N_r(x)]) \to \widetilde{H}_k(\mathbf{E}_\bullet[N_{r_3}(x)])$, induced by $g^b_\# f^b_\#$, is zero for $k \leqslant n$. Since $g^b_\# f^b_\#$ is proper chain homotopic to the identity map, there is some $r_4 \geqslant r_3$ such that the map $\widetilde{H}_k(\mathbf{E}_\bullet[N_r(x)]) \to \widetilde{H}_k(\mathbf{E}_\bullet[N_{r_4}(x)])$, induced by inclusion, is zero for $k \leqslant n$. We can use (4.3), (4.4), and the analogous identities for $f^b_\#$ and $h^b_\#$ to show that r_4 can be chosen independently of x, and so $\mathbf{E}_\bullet$ is uniformly n-acyclic. This completes the proof of Lemma 4.6.

5. Groups of cohomological codimension one

We now begin the proof of our main result. We first recall the analogues of Stallings' end theorem and Dunwoody's accessibility theorem for quotient spaces that were used in [33].

If G is finitely generated and $H \lhd G$, it is shown in [12, 33] that $e(G/H) = \widetilde{e}(G, H)$, where $\widetilde{e}(G, H)$ is the Kropholler–Roller number of relative ends of the group pair (G, H) [30]. It thus follows that if G splits over a subgroup commensurable to H, then $e(G/H) > 1$. The converse follows from [18, 41].

Theorem 5.1 ([18, 41]). — *Let G be a finitely generated group containing a commensurated subgroup $H \lhd G$. Then G splits over a subgroup commensurable to H if and only if $e(G/H) > 1$.*

There is also a relative analogue of Dunwoody's accessibility theorem. The following is explicitly stated in [33], although similar applications of Dunwoody accessibility appear in [26, 36].

Theorem 5.2 ([33, Theorem 3.24]). — *Let G be an almost finitely presented containing a finitely generated commensurated subgroup $H \lhd G$. Then G is the fundamental group of a graph of groups such that:*

(1) *every edge group is commensurable to H;*
(2) *every vertex group is finitely generated and does not split over a subgroup commensurable to H.*

We also make use of the following lemma for deducing that vertex groups of graphs of groups are of type $FP_\infty(R)$:

Proposition 5.3 ([6, Proposition 2.13]). — *Suppose that G is a group of type $FP_\infty(R)$ and is the fundamental group of a finite graph of groups $\mathcal{G}$ in which all edge groups are also of type $FP_\infty(R)$. Then all vertex groups of $\mathcal{G}$ are of type $FP_\infty(R)$.*

In order to apply Theorem 4.5 (the coarse Künneth formula) it simplifies matters considerably if we work over a field. This is because for vector spaces, all Tor terms vanish and the tensor product $V \otimes W$ is non-zero if and only if V and W are both non-zero.

Lemma 5.4. — *Let $\mathbb{F}$ be a field. Suppose G and H are groups of type $FP_\infty(\mathbb{F})$ such that $H \lhd G$ and $\operatorname{Gcd}_{\mathbb{F}}(G) \leqslant \operatorname{Gcd}_{\mathbb{F}}(H) + 1$. Then:*

(1) $H^i_{\text{coarse}}(G/H; \mathbb{F}) = 0$ *for* $i > 1$*;*
(2) $H^1_{\text{coarse}}(G/H; \mathbb{F}) \neq 0$ *if and only if* $\operatorname{Gcd}_{\mathbb{F}}(G) = \operatorname{Gcd}_{\mathbb{F}}(H) + 1$*;*
(3) *H has finite index in G if and only if* $\operatorname{Gcd}_{\mathbb{F}}(G) = \operatorname{Gcd}_{\mathbb{F}}(H)$*.*

Proof. — Let B be the quotient space G/H, which is coarsely uniformly acyclic over $\mathbb{F}$ by Proposition 4.4. Since H is of type $FP_\infty(\mathbb{F})$, it is also coarsely uniformly acyclic over $\mathbb{F}$ by Proposition 3.1. Let $n = \mathrm{Gcd}_\mathbb{F}(H)$, so $H^n_{\mathrm{coarse}}(H;\mathbb{F}) \neq 0$ by Propositions 3.4 and 3.6. We now apply Theorem 4.5. As $\mathrm{Gcd}_\mathbb{F}(G) \leqslant n+1$, $H^n_{\mathrm{coarse}}(H;\mathbb{F}) \otimes H^{i-n}_{\mathrm{coarse}}(B;\mathbb{F}) = 0$ for $i > n+1$, and so $H^i_{\mathrm{coarse}}(B;\mathbb{F}) = 0$ for $i > 1$. This shows (1). We now deduce that $H^{n+1}(G,\mathbb{F}G) \cong H^{n+1}_{\mathrm{coarse}}(G;\mathbb{F}) \cong H^n_{\mathrm{coarse}}(H;\mathbb{F}) \otimes H^1_{\mathrm{coarse}}(B;\mathbb{F})$, and (2) readily follows.

We now show (3). If H has finite index in G, then we have an isomorphism $H^k(G,\mathbb{F}G) \cong H^k(H,\mathbb{F}H)$ as $\mathbb{F}$-modules, and so $\mathrm{Gcd}_\mathbb{F}(G) = \mathrm{Gcd}_\mathbb{F}(H)$. Conversely, suppose $\mathrm{Gcd}_\mathbb{F}(G) = \mathrm{Gcd}_\mathbb{F}(H)$. Then by (1) and (2) we see $H^i_{\mathrm{coarse}}(B;\mathbb{F}) = 0$ for $i > 0$, and so $H^n_{\mathrm{coarse}}(G;\mathbb{F}) \cong H^n_{\mathrm{coarse}}(H;\mathbb{F}) \otimes H^0_{\mathrm{coarse}}(B;\mathbb{F})$. Thus, $H^0_{\mathrm{coarse}}(B;\mathbb{F}) \neq 0$ as $\mathrm{Gcd}_\mathbb{F}(G) = n$, and so B is a bounded metric space by Proposition 2.19. It is easy to see that $B = G/H$ is bounded if and only if H has finite index in G. □

We now prove our main theorem in the case the coefficient ring is a field.

Theorem 5.5. — *Let $\mathbb{F}$ be a field. Suppose G and H are groups of type $FP_\infty(\mathbb{F})$ such that $H \lhd G$, G is almost finitely presented and $\mathrm{Gcd}_\mathbb{F}(G) = \mathrm{Gcd}_\mathbb{F}(H) + 1$. Then G is the fundamental group of a graph of groups in which all edge and vertex groups are commensurable to H.*

Proof. — Let $n = \mathrm{Gcd}_\mathbb{F}(H)$. Lemma 5.4 implies $H^1_{\mathrm{coarse}}(G/H;\mathbb{F}) \neq 0$, and so G/H has more than one end by Proposition 2.19. Theorem 5.2 tells us G can be written as the fundamental group of a graph of groups $\mathcal{G}$ in which all edge groups are commensurable to H and no vertex group splits over a subgroup commensurable to H.

Let G_v be a vertex group of G and let H_v be any incident edge group. We need only demonstrate that H_v has finite index in G_v. Since H_v is commensurable to H and $H \lhd G$, we deduce that $H_v \lhd G_v$. By Proposition 5.3, G_v is of type $FP_\infty(\mathbb{F})$. By Proposition 3.6,

$$n = \mathrm{Gcd}_\mathbb{F}(H_v) \leqslant \mathrm{Gcd}_\mathbb{F}(G_v) \leqslant n+1 = \mathrm{Gcd}_\mathbb{F}(H) + 1 = n+1.$$

As G_v does not split over a subgroup commensurable to H_v, Theorem 5.1 and Proposition 2.19 ensure that $H^1_{\mathrm{coarse}}(G_v/H_v;\mathbb{F}) = 0$. By Lemma 5.4, $\mathrm{Gcd}_\mathbb{F}(G_v) = n$ and so H_v is a finite index subgroup of G_v, and hence G_v is commensurable to H. □

Remark 5.6. — Let $d_\mathbb{F}(G) \coloneqq \sup\{n \mid H^n(G,\mathbb{F}G) \neq 0\}$. Theorem 5.5 actually holds whenever G and H are groups of type $FP_\infty(\mathbb{F})$ such that $H \lhd G$, G is almost finitely presented, $d_\mathbb{F}(G) = d_\mathbb{F}(H) + 1 < \infty$ and $d_\mathbb{F}(G') \leqslant d_\mathbb{F}(G)$ whenever $G' \leqslant G$.

Since Gorenstein and virtual cohomological dimension agree whenever the latter is finite, we deduce the following.

Corollary 5.7. — *Let $\mathbb{F}$ be a field. Suppose G and H are groups of type $VFP(\mathbb{F})$ such that $H \lhd G$, G is almost finitely presented and $\mathrm{vcd}_{\mathbb{F}}(G) = \mathrm{vcd}_{\mathbb{F}}(H) + 1$. Then G is the fundamental group of a graph of groups in which all edge and vertex groups are commensurable to H.*

Applying Proposition 3.3 allows us to prove this theorem for virtual cohomological dimension over any PID.

Theorem 5.8. — *Let R be a PID and let G and H be groups of type $VFP(R)$ such that $H \lhd G$, G is almost finitely presented and $\mathrm{vcd}_R(G) = \mathrm{vcd}_R(H) + 1$. Then G is the fundamental group of a graph of groups in which all edge and vertex groups are commensurable to H.*

We note that when $R = \mathbb{Z}$ or $\mathbb{Z}_2$, any group of type $VFP(R)$ is necessarily almost finitely presented.

Proof. — Let $n = \mathrm{vcd}_R(H)$. By Proposition 3.3, there is a field R_i such that $\mathrm{vcd}_{R_i}(H) = n$. It also follows from Proposition 3.3 that $\mathrm{vcd}_{R_i}(G) \leqslant \mathrm{vcd}_R(G)$. As $\mathrm{vcd}_R(G) > \mathrm{vcd}_R(H)$, H cannot be a finite index subgroup of G, so Lemma 5.4 ensures $\mathrm{vcd}_{R_i}(G) = n + 1$. We can now apply Corollary 5.7. □

If H is a duality group of dimension n, then $H^i(H, P) = 0$ whenever $i \neq n$ and P is a projective $\mathbb{Z}H$-module. This follows from the characterisation of duality groups in Proposition 7.1, [9, Proposition 5.2], and the fact that every projective module is the direct summand of a free module. Combining this observation with [29, Proposition 2.6] and Theorem 5.8, we deduce the following:

Theorem 5.9. — *Let G be a finitely generated group of cohomological dimension $n + 1$ and let $H \lhd G$ be a duality group of dimension n. Then G splits as a graph of groups in which every vertex and edge group is commensurable to H.*

6. Commensurated subgroups of one-relator groups

In this section we generalise [5, Corollary 4].

Theorem 6.1. — *Let G be a one-relator group and let $H \lhd G$ be a finitely presented commensurated subgroup that is infinite and of infinite index. Then G is torsion-free and two-generated. Moreover, one of the following holds:*

(1) *H is infinite cyclic and G splits as a graph of groups in which all vertex and edge groups are infinite cyclic and commensurable to H.*
(2) *G contains a free normal subgroup N that is commensurable to H such that $G/N \cong \mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$.*

In particular, G is either a generalised Baumslag–Solitar group or is virtually a free-by-cyclic group.

To prove this, we use the Euler characteristic of a group as defined in [9, Section IX]. This is defined for all groups of type VFP and satisfies the following property: $\chi(G) = \frac{\chi(G')}{[G:G']}$ whenever G is of type VFP and $[G : G'] < \infty$. We can use [9, Proposition IX.7.3, e and e'] to deduce that

$$\chi(A *_C B) = \chi(A) + \chi(B) - \chi(C) \tag{6.1}$$

when $A *_C B$, A, B and C all have type VFP, and

$$\chi(A*_C) = \chi(A) - \chi(C) \tag{6.2}$$

when $A*_C$, A, B and C all have type VFP. We use these observations to calculate the Euler characteristic of certain graphs of groups.

Lemma 6.2. — *Suppose $H \leqslant G$ are groups of type VFP such that G is the fundamental group of a non-trivial reduced graph of groups $\mathcal{G}$ in which every vertex and edge group contains H as a subgroup of finite index. Then:*
(1) *If $\chi(H) = 0$, then $\chi(G) = 0$.*
(2) *If $\chi(H) \neq 0$, then $\frac{\chi(G)}{\chi(H)} \leqslant 0$. Moreover, $\chi(G) = 0$ if and only if the Bass–Serre tree of $\mathcal{G}$ is a line, i.e. either $G = A *_C B$ where $[A : C] = [B : C] = 2$, or $G = A*_\phi$ where $\phi : A \to A$ is an isomorphism.*

Proof. — Since $\chi(G') = \chi(G)[G : G']$ when G' is a finite index subgroup of G, we may assume without loss of generality that G has type FP. If $\chi(H) = 0$, then every edge and vertex group of $\mathcal{G}$ has zero Euler characteristic, so (6.1) and (6.2) ensure $\chi(G) = 0$. We thus assume $\chi(H) \neq 0$.

We proceed by induction on the number of edges of $\mathcal{G}$. For the base case, assume $\mathcal{G}$ has one edge. If $G = A *_C B$, then we deduce from (6.1) that

$$\chi(G) = \frac{\chi(H)}{[C : H]}\left(\frac{1}{[A : C]} + \frac{1}{[B : C]} - 1\right).$$

Since $\mathcal{G}$ is reduced, $[A : C], [B : C] \geqslant 2$ and so the ratio $\frac{\chi(G)}{\chi(H)}$ is non-positive. If $G = A*_C$, then

$$\chi(G) = \frac{\chi(H)}{[C : H]}\left(\frac{1}{[A : C]} - 1\right),$$

and so $\frac{\chi(G)}{\chi(H)}$ is also non-positive. For the inductive step, the same argument shows that if $\frac{\chi(G)}{\chi(H)} \leqslant 0$ and B and C both contain H as a finite index subgroup, then both $\frac{\chi(G*_C)}{\chi(H)}$ and $\frac{\chi(G*_C B)}{\chi(H)}$ are negative as required.

Since $\chi(G)$ is negative when $\mathcal{G}$ has more than one edge, $\chi(G) = 0$ only if $\mathcal{G}$ has one edge. If $G = A *_C B$, then the above formula tells us that $[A : C], [B : C] = 2$ if and only if $\chi(G) = 0$. Similarly, if $G = A*_C$, then $\chi(G) = 0$ if and only if $C = A$. □

We use this to deduce the following:

Proposition 6.3. — *Suppose G and H are groups of type VFP such that* $\mathrm{vcd}(G) = \mathrm{vcd}(H) + 1$, $\chi(G) \leqslant 0$ *and* $\chi(H) \leqslant 0$. *Then* $\chi(G) = 0$ *and one of the following holds:*

(1) $\chi(H) = 0$;
(2) *there is a subgroup $N \lhd G$, commensurable to H, such that $G/N \cong \mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$.*

Proof. — By Theorem 5.5, G splits as a graph of groups in which all vertex and edge groups are commensurable to H. By replacing H with a finite index subgroup, we may assume H is contained in every vertex and edge group; this clearly does not affect the hypothesis that $\chi(H) \leqslant 0$. If $\chi(G) \neq 0$, then it follows from Lemma 6.2 that $\chi(H) \neq 0$ and $\frac{\chi(G)}{\chi(H)} < 0$, which cannot be the case as we assumed $\chi(G) \leqslant 0$ and $\chi(H) \leqslant 0$. Thus $\chi(G) = 0$, and the conclusion follows from Lemma 6.2. □

Proof of Theorem 6.1. — We adapt the proof of [5, Corollary 4]. We first show that G is finitely generated. If not, then we can write $G = G' * F$, where F is free and G' is a finitely generated 1-relator group containing H. If $F \neq 1$, then H cannot be a commensurated subgroup of G, so we deduce that G is finitely generated and so has a presentation of the form $\langle g_1, \dots g_n \mid r^m \rangle$, where $n \geqslant 1$ and $m > 0$. As was noted in [9, Section VIII.11, Example 3], G is of type VFP and has virtual cohomological dimension at most 2. Moreover, $\chi(G) = 1 - n + \frac{1}{m} \leqslant 0$ as in [5].

Since H is an infinite index subgroup of G, it follows from Proposition 3.3 and Lemma 5.4 that $\mathrm{vcd}(H) < \mathrm{vcd}(G)$. As H is infinite, $\mathrm{vcd}(H) > 0$, so $\mathrm{vcd}(H) = 1$ and $\mathrm{vcd}(G) = 2$. By Theorem 5.8, G splits as a graph of groups all of whose edge and vertex groups are commensurable to H. The Euler characteristic of a free group of rank r is $1-r$, so $\chi(H) \leqslant 0$ by Theorem 1.1. By Proposition 6.3, $\chi(G) = 0$ and so $m = 1$ and $n = 1$. Thus G is torsion free and 2-generated. Proposition 6.3 also ensures that either $\chi(H) = 0$, in which case either H is infinite cyclic or H is commensurable to a normal free subgroup N with $G/N \cong \mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$. □

7. Commensurated subgroups of virtual duality groups

We examine commensurated subgroups of virtual duality groups defined in [7]. The following characterisation of duality groups can be taken as a definition for the purposes of this article:

Proposition 7.1 ([4, Proposition 3.1]). — *Let R be a commutative ring. We say a group G is a virtual duality group of dimension n over R if and only if the following hold:*

(1) *G is type $VFP(R)$;*
(2) *$H^i(G, RG) = 0$ for $i \neq n$;*
(3) *$H^n(G, RG)$ is a flat R-module.*

Moreover, G is said to be a Poincaré duality group of dimension n over R if $H^n(G, RG) \cong R$ as R-modules.

In this section we prove the following:

Theorem 7.2. — *Let R be a PID, and let $H \vartriangleleft G$ be groups of type $VFP(R)$.*

(1) *If G is a virtual duality group over R, then so is H.*
(2) *If H is a virtual duality group over R and $\mathrm{vcd}_R(G) = \mathrm{vcd}_R(H) + 1$, then G is a virtual duality group over R.*

Proof.

(1). — We recall that R_0 is the field of fractions of R, and that for each prime $p \in \mathbb{P}_R$, R_p is the field R/pR. It follows from Proposition 2.6 that G is a duality group of dimension n over the field R_i for every $i \in \mathbb{P}_R \cup \{0\}$. We now apply Theorem 4.5 to calculate the coarse cohomology of G in terms of the coarse cohomology of H and G/H. In particular, $H^{m_i}_{\mathrm{coarse}}(H; R_i) \neq 0$ for some unique $m_i \in \mathbb{Z}$. Thus H is an m_i-dimensional virtual duality group over the field R_i. Bieri showed that H is a virtual duality group of dimension m over R if and only if it is a virtual duality group of dimension m over R_i for every $i \in \mathbb{P}_R \cup \{0\}$ [4, Theorem 3.7]. Thus, to show H is a virtual duality group, we need only show that $m_0 = m_i$ for every $i \in \mathbb{P}_R$.

Since H is a duality group of dimension $m := m_0$ over R_0, it follows that $H^i_{\mathrm{coarse}}(H; R_0) = 0$ for $i \neq m$. It follows from Proposition 2.20 that $H^i_{\mathrm{coarse}}(H, R)$ is a torsion R-module for $i \neq m$. In particular, we see that $H^i_{\mathrm{coarse}}(H; R) \otimes_R H^j_{\mathrm{coarse}}(G/H; R)$ is a torsion module for every $i \neq m$. Moreover, for any R-modules A and B, $\mathrm{Tor}_1^R(A, B)$ is always torsion whenever R is an integral domain [38, Theorem 7.15]. As $H^n_{\mathrm{coarse}}(G; R)$ is torsion-free, Theorem 4.5 implies

$$H^n_{\mathrm{coarse}}(G; R) \cong H^m_{\mathrm{coarse}}(H; R) \otimes_R H^{n-m}_{\mathrm{coarse}}(G/H; R).$$

Thus for any $p \in \mathbb{P}_R$, Proposition 2.20 ensures

$$\begin{aligned} H^n_{\mathrm{coarse}}(G, R_p) &\cong H^n_{\mathrm{coarse}}(G, R) \otimes_R R_p \oplus \mathrm{Tor}_1^R\big(H^{n+1}_{\mathrm{coarse}}(G; R), R_p\big) \\ &\cong \Big(H^m_{\mathrm{coarse}}(H; R) \otimes_R H^{n-m}_{\mathrm{coarse}}(G/H; R)\Big) \otimes_R R_p \\ &\cong \Big(H^m_{\mathrm{coarse}}(H; R) \otimes_R R_p\Big) \otimes_{R_p} \Big(H^{n-m}_{\mathrm{coarse}}(G/H; R) \otimes_R R_p\Big). \end{aligned}$$

Hence $H^m_{\mathrm{coarse}}(H; R) \otimes_R R_p \neq 0$, so by Proposition 2.20, $H^m_{\mathrm{coarse}}(H; R_p) \neq 0$ for each $p \in \mathbb{P}_R$. Since H is a duality group of dimension m_p over R_p, we deduce that $m_p = m$ for every $p \in \mathbb{P}_R$ as required.

(2). — This follows from Theorem 5.8 and [6, Section 9.7]. □

Specialising to the case of Poincaré duality groups we have the following.

Theorem 7.3. — *Suppose R is a PID and G is a virtual Poincaré duality group over R. If $H \lhd G$ is of type $VFP(R)$, then H is a virtual Poincaré duality group over R. Moreover, if* $\mathrm{vcd}_R(G) = \mathrm{vcd}_R(H) + 1$, *then H is a commensurable to a subgroup $N \lhd G$ such that $G/N \cong \mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$.*

Proof. — Suppose G is a virtual Poincaré duality group over R of dimension n. As in the proof of Theorem 7.2,

$$R \cong H^n_{\mathrm{coarse}}(G; R) \cong H^m_{\mathrm{coarse}}(H; R) \otimes_R H^{n-m}_{\mathrm{coarse}}(G/H; R),$$

and $H^m_{\mathrm{coarse}}(H; R)$ is a non-zero torsion-free R-module. Pick some non-zero $x \in H^{n-m}_{\mathrm{coarse}}(G/H; R)$. As $H^m_{\mathrm{coarse}}(H; R)$ is torsion-free, hence flat, we have an injection of R-modules

$$H^m_{\mathrm{coarse}}(H; R) \cong H^m_{\mathrm{coarse}}(H; R) \otimes_R Rx \hookrightarrow H^n_{\mathrm{coarse}}(G; R) \cong R.$$

Thus $H^m_{\mathrm{coarse}}(H; R)$ is isomorphic to an ideal of R, hence isomorphic to R.

When $\mathrm{vcd}_R(G) = \mathrm{vcd}_R(H) + 1$, the preceding argument shows that $H^1_{\mathrm{coarse}}(G/H; R) \cong R$, and so $e(G/H) = \widetilde{e}(G, H) = 2$. The result follows from [30, Corollary 3.4]. Alternatively, one can combine Theorem 5.8 and [33, Corollary 3.20] to deduce that G acts on a line with vertex and edge stabilisers commensurable to H. □

We use this to classify finitely generated commensurated subgroups of 3-manifold groups. In the following corollary, a *surface group* is the fundamental group of a closed orientable surface other than the 2-sphere, and a *3-manifold group* is the fundamental group of an irreducible orientable closed 3-manifold.

Corollary 7.4. — *Let G be a 3-manifold group. If $H \lhd G$ is finitely generated, infinite and of infinite index, then H is commensurable to a normal subgroup $N \lhd G$, where N is either infinite cyclic or a surface group.*

Proof. — It follows from the sphere theorem that G is the fundamental group of an aspherical manifold, hence G is a Poincaré duality group of dimension 3. Combining results of Scott and Strebel, H must be of type FP [40, 43] and $m := \mathrm{cd}(H) < \mathrm{cd}(G)$. It follows from Theorem 7.3 and a result of Eckmann–Müller [19] that when $\mathrm{cd}(H) = 2$, $H \lhd G$ and H a surface group.

If $\mathrm{cd}(H) = 1$, then it follows from Theorem 1.1 and Theorem 7.3 that $H = \langle h \rangle \cong \mathbb{Z}$. Let $S = \{s_1, \dots, s_t\}$ be a generating set of G. Since $H \lhd\!\!\!\cdot\, G$, for every $s_i \in S$, there exist integers $n_i, m_i \in \mathbb{Z}$ such that $s_i h^{n_i} s_i^{-1} = h^{m_i}$. By a theorem of Kropholler, $n_i = \pm m_i$ for each i [28]. Thus $\langle h^N \rangle \lhd G$, where $N = \mathrm{lcm}(n_1, n_2, \dots, n_t)$. □

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