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THE FOURIER TRANSFORM FOR TRIPLES OF QUADRATIC SPACES

by Jayce R. GETZ & Chun-Hsien HSU (*)

ABSTRACT. — Let V_1, V_2, V_3 be a triple of even dimensional vector spaces over a number field F equipped with nondegenerate quadratic forms $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$, respectively. Let $Y \subset \prod_{i=1}^3 V_i$ be the closed subscheme consisting of (v_1, v_2, v_3) such that $\mathcal{Q}_1(v_1) = \mathcal{Q}_2(v_2) = \mathcal{Q}_3(v_3)$. One has a Poisson summation formula for this scheme under suitable assumptions on the functions involved, but the relevant Fourier transform was previously only defined as a correspondence. In the current paper we employ a novel global-to-local argument to prove that this Fourier transform is well-defined on the Schwartz space of $Y(\mathbb{A}_F)$. To execute the global-to-local argument, we introduce boundary terms and thereby extend the Poisson summation formula to a broader class of test functions. This is the first time a summation formula with boundary terms has been proven for a spherical variety that is not a Braverman–Kazhdan space.

RÉSUMÉ. — Soit V_1, V_2, V_3 un triplet d'espaces vectoriels de dimension paire sur un corps de nombres F équipés de formes quadratiques non dégénérées $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$, respectivement. Soit $Y \subset \prod_{i=1}^3 V_i$ le sous-schéma fermé constitué de (v_1, v_2, v_3) tels que $\mathcal{Q}_1(v_1) = \mathcal{Q}_2(v_2) = \mathcal{Q}_3(v_3)$. On dispose d'une formule sommatoire de Poisson pour ce schéma sous des hypothèses appropriées concernant les fonctions impliquées, mais la transformation de Fourier pertinente n'était auparavant définie que comme une correspondance. Dans le présent article, nous employons un argument global-local novateur pour prouver que cette transformation de Fourier est bien définie sur l'espace de Schwartz de $Y(\mathbb{A}_F)$. Pour exécuter cet argument global-local, nous introduisons des termes de frontière et étendons ainsi la formule sommatoire de Poisson à une classe plus large de fonctions de test. C'est la première fois qu'une formule avec des termes de frontière est prouvée pour une variété sphérique qui n'est pas un espace de Braverman–Kazhdan.

Keywords: Fourier transform, Poisson summation conjecture, spherical varieties.

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1. Introduction

Let d_1, d_2, d_3 be three positive even integers, let $V_i := \mathbb{G}_a^{d_i}$, $V := \bigoplus_{i=1}^3 V_i$, and let F be a number field. For each i , let \mathcal{Q}_i be a nondegenerate quadratic form on $V_i(F)$. Let $Y \subset V$ be the closed subscheme whose points in an F -algebra R are given by

$$Y(R) := \{(v_1, v_2, v_3) \in V(R) : \mathcal{Q}_1(v_1) = \mathcal{Q}_2(v_2) = \mathcal{Q}_3(v_3)\}.$$

In [14] the first author and Liu proved a Poisson summation formula for this space. The space Y is a spherical variety, and hence the summation formula confirms a special case of conjectures of Braverman and Kazhdan, later investigated by L. Lafforgue and Ngô, and extended to spherical varieties by Sakellaridis [3, 4, 5, 21, 26, 29]. It is the first summation formula for a spherical variety that is not a Braverman–Kazhdan space. Here a Braverman–Kazhdan space is the affine closure of $[P, P] \backslash G$ where G is a reductive group and $P < G$ is a parabolic subgroup.

In this paper we prove that the Fourier transform on Y , originally defined as a correspondence, descends to an automorphism of the Schwartz space. Let us be more precise. Let $X^\circ := [P, P] \backslash \mathrm{Sp}_6$, where $P < \mathrm{Sp}_6$ is the Siegel parabolic, and let $X := \overline{\mathrm{PI}(X^\circ)}$ be the corresponding Braverman–Kazhdan space (see (3.6)). As explained in Section 3, the Schwartz space $\mathcal{S}(X(\mathbb{A}_F))$ is defined and comes equipped with a Fourier transform $\mathcal{F}_X : \mathcal{S}(X(\mathbb{A}_F)) \rightarrow \mathcal{S}(X(\mathbb{A}_F))$. We define $\mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F))$ using the conventions in Section 2. For notational simplicity, we let

$$\mathcal{F}_X : \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F)) \longrightarrow \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F))$$

be the automorphism given on pure tensors by $\mathcal{F}_X(f_1 \otimes f_2) = \mathcal{F}_X(f_1) \otimes f_2$. Let $Y^{\mathrm{sm}} \subset Y$ be the smooth locus. In this paper we introduce the *Schwartz space*

$$\mathcal{S}(Y(\mathbb{A}_F)) := \mathrm{Im}(I : \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F)) \longrightarrow C^\infty(Y^{\mathrm{sm}}(\mathbb{A}_F)))$$

where I is defined as in (1.4) below. The Poisson summation formula in [14] relies not on a Fourier transform from $\mathcal{S}(Y(\mathbb{A}_F))$ to itself, but a correspondence

$$\begin{array}{ccc} \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F)) & \xrightarrow{\mathcal{F}_X} & \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F)) \\ \downarrow I & & \downarrow I \\ \mathcal{S}(Y(\mathbb{A}_F)) & \cdots \cdots \cdots \rightarrow & \mathcal{S}(Y(\mathbb{A}_F)). \end{array}$$

In the current paper we prove the following theorem:

THEOREM 1.1. — *Assume $Y^{\text{sm}}(\mathbb{A}_F)$ is nonempty. There is a unique \mathbb{C} -linear isomorphism $\mathcal{F}_Y : \mathcal{S}(Y(\mathbb{A}_F)) \rightarrow \mathcal{S}(Y(\mathbb{A}_F))$ such that $I \circ \mathcal{F}_X = \mathcal{F}_Y \circ I$.*

In other words, the dotted arrow in the diagram above can be replaced by \mathcal{F}_Y and the resulting diagram is commutative. Theorem 1.1 follows from Theorem 12.1 below. We prove in Proposition 11.2 below that $\mathcal{S}(Y(F_v))$ is contained in $L^2(Y(F_v))$ (with respect to an appropriate measure) for all places v of F . As an application of Theorem 1.1, in a follow-up paper [13] with S. Leslie, we give an explicit formula for \mathcal{F}_Y and prove that it extends to a unitary operator in the non-Archimedean case. This will constitute a sound setup for harmonic analysis on $Y(F_v)$. We refer to [12, 16, 20] for analogous work when Y is replaced by the zero locus of a single quadratic form. We would like to emphasize again that the setting of the current paper is a significant leap from the setting of these other papers because our space is not a Braverman–Kazhdan space.

We prove Theorem 1.1 via global-to-local argument. Though global-to-local arguments using summation formulae such as the trace formula are common in the literature, the authors do not know of another example where such a technique is used to define a unitary operator. The argument is contained in Section 12. To execute it, we prove a more flexible version of the Poisson summation formula of [14] that involves boundary terms. We also develop Fourier and harmonic analysis on $\mathcal{S}(Y(\mathbb{A}_F))$ to the point that we can take limits of functions. This work is of independent interest.

In remainder of the introduction, we state the Poisson summation formula we prove in this paper. Before stating it in full generality, we highlight a special case. Fix a nontrivial additive character $\psi : F \backslash \mathbb{A}_F \rightarrow \mathbb{C}^\times$. For $1 \leq i \leq 3$, let

$$\rho_i := \rho_{i,\psi} : \text{SL}_2(\mathbb{A}_F) \times \mathcal{S}(V_i(\mathbb{A}_F)) \longrightarrow \mathcal{S}(V_i(\mathbb{A}_F))$$

be the Weil representation attached to ψ and the \mathcal{Q}_i . For a place v of F , let

$$(1.1) \quad \begin{aligned} \mathcal{S}_{iv} &:= \{f \in \mathcal{S}(V_i(F_v)) : \rho_i(g)f(0) = 0 \text{ for all } g \in \text{SL}_2(F_v)\}, \\ \mathcal{S}_{0v} &:= \mathcal{S}_{1v} \otimes \mathcal{S}_{2v} \otimes \mathcal{S}_{3v}. \end{aligned}$$

From the definition of the Weil representation and the Bruhat decomposition of $\text{SL}_2(F_v)$, we have \mathcal{S}_{iv} consists of functions

$$\left\{ f \in \mathcal{S}(V_i(F_v)) : f(0) = 0, \int_{V_i(F_v)} \psi(tQ_i(w))f(w) dw = 0 \text{ for all } t \in F_v \right\}.$$

Restrictions of elements of \mathcal{S}_{0v} to $Y^{\text{sm}}(F_v)$ are elements of $\mathcal{S}(Y(F_v))$ by Lemma 5.3. We check in Lemma 12.4 below that \mathcal{S}_{0v} is nonzero for finite $v \nmid 2$ (in fact, infinite-dimensional).

THEOREM 1.2. — *Let $f \in \mathcal{S}(Y(\mathbb{A}_F))$. Assume that there are finite places v_1, v_2 of F such that $f = f_{v_1} f_{v_2} f^{v_1 v_2}$ where f_{v_1} and $\mathcal{F}_Y(f_{v_2})$ are restrictions of elements of \mathcal{S}_{0v_1} and \mathcal{S}_{0v_2} , respectively. Then*

$$\sum_{y \in Y^{\text{sm}}(F)} f(y) = \sum_{y \in Y^{\text{sm}}(F)} \mathcal{F}_Y(f)(y).$$

This is similar to the main theorem of [14], but Theorem 1.2 has the additional benefit that the hypotheses and conclusion are given intrinsically in terms of the space $\mathcal{S}(Y(\mathbb{A}_F))$ and not extrinsically in terms of the map $I : \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F)) \rightarrow \mathcal{S}(Y(\mathbb{A}_F))$.

1.1. The boundary terms

We now describe our main summation formula more precisely. Possibly confusing (but useful) notational conventions on Schwartz spaces are given in Section 2. Let G be the image of SL_2^3 under a natural embedding $\text{SL}_2^3 \rightarrow \text{Sp}_6$ (see (4.11)). The quasi-affine scheme $X^\circ = [P, P] \backslash \text{Sp}_6$ admits a natural right G -action.

Over a field of characteristic zero, there are five orbits in X° under the action of G . We fix representatives $\gamma_b, \gamma_0 := \text{Id}, \gamma_1, \gamma_2, \gamma_3$ as in Section 4 and let G_γ be the stabilizer of γ in G . The subscript b stands for basepoint; γ_b is a representative for the open orbit.

We have a Weil representation

$$\rho := \rho_1 \otimes \rho_2 \otimes \rho_3 : G(\mathbb{A}_F) \times \mathcal{S}(V(\mathbb{A}_F)) \longrightarrow \mathcal{S}(V(\mathbb{A}_F)).$$

We will require the following assumption on $f = f_1 \otimes f_2 \in \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F))$: There are finite places v_1, v_2 of F such that

$$(1.2) \quad f_1 = f_{v_1} f_{v_2} f^{v_1 v_2} \text{ and } f_{v_1} \in C_c^\infty(X^\circ(F_{v_1})), \mathcal{F}_X(f_{v_2}) \in C_c^\infty(X^\circ(F_{v_2}))$$

$$(1.3) \quad \rho(g) f_2(v) = 0 \text{ for } v \notin V^\circ(F), \text{ for all } g \in G(\mathbb{A}_F).$$

Here V° is the open subscheme of V consisting of triples (v_1, v_2, v_3) with each $v_i \neq 0$. The origin of condition (1.3) is explained below (1.5). We point out that if $f_2 = f_{2v} f_2^v$ for some place v of F with $f_{2v} \in \mathcal{S}_{0v}$ then f_2 satisfies (1.3).

Remark 1.3. — A similar condition on f_2 was assumed in [14]. We warn the reader that in loc. cit. the assertion that (5.0.7) implies (5.0.5) is false. Fortunately, this claim is never used in loc. cit.

Let $\Phi \in \mathcal{S}(\mathbb{A}_F^2)$ and let $N_2 < \mathrm{SL}_2$ be the subgroup of upper triangular unipotent matrices. For

$$f = f_1 \otimes f_2 \in \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F))$$

we define

$$(1.4) \quad \begin{aligned} I(f)(\xi) &:= \int_{G_{\gamma_b}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(\gamma_b g) \rho(g) f_2(\xi) dg \quad \text{for } \xi \in Y^{\mathrm{sm}}(\mathbb{A}_F), \\ I_0(f)(\xi) &:= \int_{N_2^3(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(g) \rho(g) f_2(\xi) dg \quad \text{for } \xi \in \tilde{Y}_0(\mathbb{A}_F). \end{aligned}$$

Here \tilde{Y}_0 (and \tilde{Y}_i for $1 \leq i \leq 3$) is defined as in Section 4. For $\xi \in \tilde{Y}_i(\mathbb{A}_F)$ and $s \in \mathbb{C}$, we set

$$\begin{aligned} I_i(f \otimes \Phi)(\xi, s) &:= \int_{G_{\gamma_i}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(\gamma_i g) \\ &\times \left(\int_{N_2(\mathbb{A}_F) \backslash \mathrm{SL}_2(\mathbb{A}_F)} \int_{\mathbb{A}_F^\times} \rho(\Delta_i(h)g) f_2(\xi) \Phi(x(0, 1)hp_i(g)) |x|^{2s} d^\times x dh \right) dg \end{aligned}$$

where Δ_i is defined as in (5.2) and p_i is defined as in (5.3). In Section 4, certain quotient schemes Y_i of \tilde{Y}_i are also defined. Roughly, $Y_0(F)$ is a quotient of

$$\tilde{Y}_0(F) := \{(v_1, v_2, v_3) \in V^\circ(F) : \mathcal{Q}_1(v_1) = \mathcal{Q}_2(v_2) = \mathcal{Q}_3(v_3) = 0\}$$

by an action of $(F^\times)^2$ and Y_i is the product of the zero locus of \mathcal{Q}_i in V_i° and the quasi-projective scheme cut out of $\mathbb{P}(V_{i-1}^\circ \times V_{i+1}^\circ)$ by $\mathcal{Q}_{i-1} - \mathcal{Q}_{i+1}$, where the indices are understood “modulo 3” in the obvious sense. Here $\mathbb{P}(V_{i-1}^\circ \times V_{i+1}^\circ)$ is the image of $V_{i-1}^\circ \times V_{i+1}^\circ$ in $\mathbb{P}(V_{i-1} \times V_{i+1})$.

The main summation formula proved in this paper is the following theorem:

THEOREM 1.4. — *Assume that*

$$(f = f_1 \otimes f_2, \Phi) \in \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F)) \times \mathcal{S}(\mathbb{A}_F^2)$$

where f satisfies (1.2) and (1.3), and $\widehat{\Phi}(0) \neq 0$. Let

$$\kappa_F = \frac{2}{\text{Vol}(F^\times \backslash (\mathbb{A}_F^\times)^1)}.$$

One has

$$\begin{aligned} & \sum_{\xi \in Y^{\text{sm}}(F)} I(f)(\xi) + \sum_{\xi \in Y_0(F)} I_0(f)(\xi) + \frac{\kappa_F}{\widehat{\Phi}(0)} \text{Res}_{s=1} \sum_{i=1}^3 \sum_{\xi \in Y_i(F)} I_i(f \otimes \Phi)(\xi, s) \\ &= \sum_{\xi \in Y^{\text{sm}}(F)} I(\mathcal{F}_X(f))(\xi) + \sum_{\xi \in Y_0(F)} I_0(\mathcal{F}_X(f))(\xi) \\ & \quad + \frac{\kappa_F}{\widehat{\Phi}(0)} \text{Res}_{s=1} \sum_{i=1}^3 \sum_{\xi \in Y_i(F)} I_i(\mathcal{F}_X(f) \otimes \Phi)(\xi, s). \end{aligned}$$

Here $\widehat{\Phi}$ denotes the Fourier transform of Φ normalized as in (6.1), and $(\mathbb{A}_F^\times)^1 < \mathbb{A}_F^\times$ is the subgroup of ideles of norm 1. When we speak of *boundary terms* in the paper, we mean the summands in Theorem 1.4 involving I_0 and I_i . The proof of Theorem 1.4 is a refinement of the proof of the main theorem of [14]. Briefly, one substitutes a triple of Θ -functions into the integral representation of the triple product L -function due to Garrett [11]. We make use of the adelic reformulation of Piatetski–Shapiro and Rallis [27]. The boundary terms correspond to the SL_2^3 -orbits in X° that are not open. In [14] assumptions were made to eliminate these terms. At the suggestion of the referee, we point out that Theorem 1.4 implies that the same formula is valid if f is replaced by a \mathbb{C} -linear combination of functions f satisfying conditions (1.2) and (1.3); in particular, one can allow the places v_1, v_2 to vary.

Of course it would be desirable to remove assumptions (1.2) and (1.3). To obtain an identity without these assumptions will require the addition of boundary terms on both sides of the formula. Thus the statement of the Poisson summation formula for general test functions will be more intricate.

The proof would also be far more technical as we now explain. There is no assumption on support in the main theorem of [15], but the boundary terms (i.e. those not given by the evaluation of a Schwartz function on a point of $X^\circ(F)$) are given in terms of residues of Eisenstein series. It seems wise to wait until there is an explicit geometric understanding of these terms before attempting to remove (1.2).

On the other hand, assumption (1.3) implies that the Θ -function

$$(1.5) \quad \Theta_f(g) := \sum_{\xi \in V(F)} \rho(g)f(\xi)$$

is cuspidal as a function of $g \in \mathrm{SL}_2^3(F) \backslash \mathrm{SL}_2^3(\mathbb{A}_F)$. Removing this assumption would probably require using Arthur truncation as in [12]. Even in the simpler situation of the zero locus of a single quadratic form in loc. cit., one has to come to grips with a host of additional complications. Explicitly, truncation introduces new analytic difficulties, the lack of invariance of the truncation requires attention, and at the end one has to introduce a whole family of boundary terms indexed by all lower dimensional quadratic forms in the Witt class of the original quadratic form. We also suspect that using Arthur truncation it might be possible to rewrite the terms involving residues at $s = 1$.

In any case, all of the possible refinements above, though interesting, are not necessary to define the Schwartz space of Y and prove that the Fourier transform \mathcal{F}_Y is well-defined. For this purpose, we use the fact that the summation formula above allows us to treat f_1 such that f_1 and $\mathcal{F}_X(f_1)$ are not supported on the open $\mathrm{SL}_2^3(F)$ -orbit in $X^\circ(F)$. The corresponding summation formula in [14] was limited to functions satisfying this additional assumption.

1.2. Outline

We now outline the sections of this paper. We state conventions for Schwartz spaces in Section 2. In Section 3 we recall and refine certain results from harmonic analysis on Braverman–Kazhdan spaces. In particular we prove a Plancherel formula for $\mathcal{S}(X(F_v))$ (see Proposition 3.9).

After this, we discuss the geometric preliminaries necessary for the study of Y in Section 4. We turn in Section 5 to the definition of the local integrals necessary to prove our main summation formula, Theorem 1.4. We give a definition of the Schwartz space of $Y(F_v)$ in Section 5.1.

Theorem 1.4 is proved in Section 6. This summation formula is given in terms of infinite sums of Eulerian integrals, that is, integrals that factor along the places of F (or residues of such expressions). The local integrals are computed in the unramified case in Section 8. The proof of Theorem 1.4 depends on bounds on local integrals that are deferred to Section 7 and Section 9. We discuss the L^2 -theory in Section 11, and prove in particular that $\mathcal{S}(Y(F_v)) \subset L^1(Y(F_v)) \cap L^2(Y(F_v))$ with respect to a natural measure. In Section 12 we construct the isomorphism \mathcal{F}_Y as described above and prove Theorem 1.2. We have appended a list of symbols for the reader's convenience.

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In response to the paper [14] several people including Y. Sakellaridis and Z. Yun asked the first author about whether it was possible to define \mathcal{F}_Y ; he thanks them for this question. He also thanks D. Kazhdan for many interesting conversations on material related to the topic of this paper, and H. Hahn for her help with editing and her constant encouragement. The authors thank D. Kazhdan for suggesting that a discussion of the boundary terms be added to the introduction, T. Ikeda for answering a question on his paper [18], and J.-L. Colliot-Thélène for pointing out his result with Sansuc in [8] (see Theorem 12.3). We also acknowledge the comments of the anonymous referee; they helped improve the exposition.

2. Schwartz spaces

This work involves several Schwartz spaces. Let F be a local field. For a quasi-affine scheme X of finite type over F , let $X^{\text{sm}} \subset X$ be the smooth locus. Any Schwartz space $\mathcal{S}(X(F))$ we discuss will be a space of functions on $X^{\text{sm}}(F)$. Functions in the Schwartz space need not be defined on all of $X(F)$ if X is not smooth. We will not define Schwartz spaces of general quasi-affine schemes of finite type over F . In fact obtaining a good definition for general spherical varieties is an important open problem [29]. In this subsection we explain the definition for smooth quasi-affine schemes and how to form Schwartz spaces of products $X(F) \times Y(F)$ given that the Schwartz spaces of each factor have been defined. We have modeled our approach on the treatment of the smooth case in [1].

Suppose F is non-Archimedean. If X is smooth, we set $\mathcal{S}(X(F)) := C_c^\infty(X(F))$. More generally, if we have already defined $\mathcal{S}(X(F))$ and $\mathcal{S}(Y(F))$, we set $\mathcal{S}(X(F) \times Y(F)) := \mathcal{S}(X(F)) \otimes \mathcal{S}(Y(F))$ (the algebraic tensor product).

Now assume that F is Archimedean. If X is smooth, we define $\mathcal{S}(X(F)) = \mathcal{S}(\text{Res}_{F/\mathbb{R}}X(\mathbb{R}))$ as in [10, Remark 3.2]. By [10, Section 2.2], the Schwartz space of a real algebraic variety and the Schwartz space of its underlying Nash manifold defined in [1] may be canonically identified. In any case $\mathcal{S}(X(F))$ is a Fréchet space. It is defined as a quotient of a nuclear space by a closed subspace, and hence is nuclear. In general, suppose that we have defined Schwartz spaces $\mathcal{S}(X(F))$ and $\mathcal{S}(Y(F))$ that are additionally Fréchet spaces. We then define $\mathcal{S}(X(F) \times Y(F)) := \mathcal{S}(X(F)) \widehat{\otimes} \mathcal{S}(Y(F))$

where the hat denotes the (completed) projective topological tensor product. It is also a Fréchet space.

We warn the reader that in [10] there is a definition of a Schwartz space for any quasi-affine scheme of finite type over the real numbers. In the smooth case their definition coincides with ours. In the limited situations in which we define Schwartz spaces of nonsmooth schemes, our definitions do not coincide with theirs because functions in our Schwartz spaces need not extend to the singular set.

Finally we discuss the adelic setting. Let X be a quasi-affine scheme of finite type over a number field F . Then for all finite sets S of places of F , $X(\mathbb{A}_F^S)$ is defined as a topological space [9]. Assume $\mathcal{S}(X(F_v))$ is defined for all places v and a *basic function* $b_{X,v} \in \mathcal{S}(X(F_v))$ is chosen for almost all v . If S contains all infinite places, $\mathcal{S}(X(\mathbb{A}_F^S))$ will always be a restricted tensor product $\bigotimes'_{v \notin S} \mathcal{S}(X(F_v))$ with respect to $b_{X,v}$; if S is a set of infinite places of F , then $\mathcal{S}(X(F_S)) := \widehat{\bigotimes}_{v \in S} \mathcal{S}(X(F_v))$ is the (completed) projective topological tensor product. For general S , we put

$$\mathcal{S}(X(\mathbb{A}_F^S)) := \mathcal{S}(X(F_{\infty-S})) \otimes \mathcal{S}(X(\mathbb{A}_F^{\infty \cup S})).$$

3. Braverman–Kazhdan spaces

In this section we study Schwartz spaces of certain Braverman–Kazhdan spaces. In particular we endow these Schwartz spaces with a Fréchet structure in the Archimedean case. This refinement is necessary for continuity arguments.

Let Sp_{2n} denote the symplectic group on a $2n$ -dimensional vector space, and let $P < \mathrm{Sp}_{2n}$, $M < P$ denote the usual Siegel parabolic and Levi subgroup. More specifically, for \mathbb{Z} -algebras R , set

$$\begin{aligned} \mathrm{Sp}_{2n}(R) &:= \left\{ g \in \mathrm{GL}_{2n}(R) : g^t \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix} g := \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix} \right\}, \\ (3.1) \quad M(R) &:= \left\{ \begin{pmatrix} A & \\ & A^{-t} \end{pmatrix} : A \in \mathrm{GL}_n(R) \right\}, \\ N(R) &:= \left\{ \begin{pmatrix} I_n & Z \\ & I_n \end{pmatrix} : Z \in \mathfrak{gl}_n(R), Z^t = Z \right\}, \end{aligned}$$

and $P = MN$. Apart from this section, we will only use the $n = 3$ case, but since it is no more difficult to treat the general case, we include it. We

define a character

$$(3.2) \quad \begin{aligned} \omega : M(R) &\longrightarrow R^\times \\ \begin{pmatrix} A & \\ & A^{-t} \end{pmatrix} &\longmapsto \det A. \end{aligned}$$

Let

$$(3.3) \quad X^\circ := [P, P] \backslash \mathrm{Sp}_{2n}.$$

Let GSp_{2n} denote the group of similitudes and let

$$\nu : \mathrm{GSp}_{2n} \longrightarrow \mathbb{G}_m$$

denote the similitude norm. We note that there is a left action

$$(3.4) \quad \begin{aligned} M^{\mathrm{ab}}(R) \times \mathrm{GSp}_{2n}(R) \times X^\circ(R) &\longrightarrow X^\circ(R) \\ (m, g, x) &\longmapsto m \begin{pmatrix} I_n & \\ & \nu(g)I_n \end{pmatrix} xg^{-1}. \end{aligned}$$

One has the Plücker embedding

$$(3.5) \quad \mathrm{Pl} : X^\circ \longrightarrow \wedge^n \mathbb{G}_a^{2n}$$

given by taking the wedge product of the last n rows of a representative $g \in \mathrm{Sp}_{2n}(R)$ for $[P, P](R)g$. We denote by X the closure of $\mathrm{Pl}(X^\circ)$:

$$(3.6) \quad X := \overline{\mathrm{Pl}(X^\circ)}.$$

It is an affine variety (in fact a spherical variety, for many more details see [23, Section 7.2]). As explained in loc. cit., X is the affine closure of X° .

3.1. The local Schwartz spaces

Let F be a local field. Let

$$\delta_P(p) := |\det(\mathrm{Ad}(p) : \mathrm{Lie} P \longrightarrow \mathrm{Lie} P)|$$

be the modular quasi-character of $P(F)$. The Schwartz space $\mathcal{S}(X(F))$ of $X(F)$ was defined in [13, Section 5], where it was denoted $\mathcal{S}(X_P(F))$. For $f \in C^\infty(X^\circ(F))$ and $g \in \mathrm{Sp}_{2n}(F)$, and a quasi-character $\chi : F^\times \rightarrow \mathbb{C}$, let

$$(3.7) \quad f_{\chi_s}(g) := \int_{M^{\mathrm{ab}}(F)} \delta_P(m)^{1/2} \chi_s(\omega(m)) f(m^{-1}g) dm$$

be its Mellin-transform. Here $\chi_s := \chi|\cdot|^s$ and ω is defined as in (3.2). When this integral is well-defined, either because it converges absolutely or

is meromorphically continued from a half-plane of absolute convergence, it is a section of

$$I(\chi_s) := \text{Ind}_P^{\text{Sp}_{2n}}(\chi_s \circ \omega).$$

Here the induction is normalized and taken in the category of smooth representations.

Let $\psi : F \rightarrow \mathbb{C}^\times$ be a nontrivial additive character.

THEOREM 3.1 ([15, Theorem 4.4], [13, Section 5.3]). — *There is a linear automorphism*

$$\mathcal{F}_X := \mathcal{F}_{X,\psi} : \mathcal{S}(X(F)) \longrightarrow \mathcal{S}(X(F)).$$

For $f \in \mathcal{S}(X(F))$, the Fourier transform $\mathcal{F}_X(f)$ is the unique function in $\mathcal{S}(X(F))$ such that

$$\mathcal{F}_X(f)_{\chi_s} = M_{w_0}^*(f_{\bar{\chi}_{-s}})$$

for all (unitary) characters χ and all $s \in \mathbb{C}$.

Here

$$\begin{aligned} M_{w_0}^* &:= M_{w_0,\psi}^* \\ &:= \gamma\left(s + \frac{1-n}{2}, \chi, \psi\right) \\ &\quad \times \prod_{r=1}^{\lfloor n/2 \rfloor} \gamma(2s - n + 2r, \chi^2, \psi) M_{w_0} : I(\chi_s) \longrightarrow I(\bar{\chi}_{-s}) \end{aligned} \tag{3.8}$$

is the normalized intertwining operator of [15, (3.5)]. The Tate γ -factors depend on a choice of Haar measure on F which we always take to be the self-dual measure with respect to ψ . In [13, Corollary 6.10] one finds an explicit formula for \mathcal{F}_X . We point out that $M_{w_0} = \iota_{w_0} \circ \mathcal{R}_{P|P \circ P}$ in the notation of loc. cit.

Let F be an Archimedean local field. For real numbers $A < B$, $p(x) \in \mathbb{C}[x]$ and meromorphic functions $f : \mathbb{C} \rightarrow \mathbb{C}$, let

$$\begin{aligned} V_{A,B} &:= \{s \in \mathbb{C} : A \leq \text{Re}(s) \leq B\}, \\ |f|_{A,B,p} &:= \sup_{s \in V_{A,B}} |p(s)f(s)|. \end{aligned} \tag{3.9}$$

To complete our discussion of $\mathcal{S}(X(F))$ we must endow it with a topology. Recall the L -factors $a_w(s, \eta)$ indexed by $w \in \{\text{Id}, w_0\}$ from [15]. Consider the Lie algebra

$$\mathfrak{g} := \text{Lie}(M^{\text{ab}}(F) \times \text{Sp}_{2n}(F)),$$

viewed as a real Lie algebra. It acts on $C^\infty(X^\circ(F))$ via the differential of the action (3.4) and hence we obtain an action of $U(\mathfrak{g})$, the universal enveloping algebra of the complexification of \mathfrak{g} . Let $\widehat{K}_{\mathbb{G}_m}$ be a set of representatives for

the (unitary) characters of F^\times modulo equivalence, where χ is equivalent to χ' if and only if $\chi = |\cdot|^{it}\chi'$ for some $t \in \mathbb{R}$. For all real numbers $A < B$, $w \in \{\text{Id}, w_0\}$, $D \in U(\mathfrak{g})$, any polynomials $p_w \in \mathbb{C}[s]$ such that $p_w(s)a_w(s, \eta)$ has no poles for all $(s, \eta) \in V_{A,B} \times \widehat{K}_{\mathbb{G}_m}$ and compact subsets $\Omega \subset X^\circ(F)$, let

$$(3.10) \quad |f|_{A,B,w,p_w,\Omega,D} := \sum_{\eta \in \widehat{K}_{\mathbb{G}_m}} \sup_{g \in \Omega} |M_w(D.f)_{\eta_s}(g)|_{A,B,p_w}.$$

By definition of the Schwartz space [13, Section 5] this is a seminorm on $\mathcal{S}(X(F))$ and the collection of these seminorms as A, B, p_w, Ω, D vary gives $\mathcal{S}(X(F))$ the structure of a locally convex space.

One could probably rewrite the seminorm in (3.10) in terms of asymptotics toward the origin in place of Mellin transforms using work of Igusa (see [17, 19] for non-Archimedean analogues). The expressions one would likely obtain are not any simpler or more conceptual from our point of view. We do not know whether it is possible to remove dependence on the intertwining operator M_{w_0} in the definition of the topology.

LEMMA 3.2. — *Assume F is Archimedean. The space $\mathcal{S}(X(F))$ is a Fréchet space.*

Proof. — We first observe that we can replace the family of seminorms with a countable subfamily inducing the same topology. More specifically we can choose a countable basis of $U(\mathfrak{g})$, and restrict the (A, B) to lie in the set $\{(-N, N) : N \in \mathbb{Z}_{>0}\}$. Since the poles of $a_w(s, \eta)$ can only occur at points in $\frac{1}{2}\mathbb{Z}$ (see [15, (3.4)]), we can similarly restrict our attention to a countable family of p_w . Finally we can restrict attention to a countable family of Ω by simply choosing a countable family of compact subsets of $X^\circ(F)$ whose union is $X^\circ(F)$.

The countable family of seminorms described above is separating by Mellin inversion (see [15, Lemma 4.3]). It follows that $\mathcal{S}(X(F))$ is Hausdorff and metrizable. It is also clear that it is complete. \square

We point out that when F is non-Archimedean we do not endow $\mathcal{S}(X(F))$ with a topology.

Recall that we have a left action (3.4) of $M^{\text{ab}} \times \text{GSp}_{2n}$ on X° . This yields an action on functions: for a function f on $X^\circ(F)$ and $(m, g, x) \in M^{\text{ab}}(F) \times \text{GSp}_{2n}(F) \times X^\circ(F)$

$$(3.11) \quad L(m)R(g)f(x) := f\left(m^{-1}\begin{pmatrix} I_n & \\ & \nu(g)^{-1}I_n \end{pmatrix}xg\right).$$

Using the formula for \mathcal{F}_X from [13, Corollary 6.10] one deduces the following lemma:

LEMMA 3.3. — *If $(m, g) \in M^{\text{ab}}(F) \times \text{GSp}_{2n}(F)$ and $f \in \mathcal{S}(X(F))$, then $L(m)R(g)f \in \mathcal{S}(X(F))$. Moreover,*

$$\mathcal{F}_X(L(m)R(g)f) = |\nu(g)|^{n(n+1)/2} \delta_P^{-1}(m) L(m^{-1}) R(\nu(g)^{-1}g) \mathcal{F}_X(f).$$

One checks the following lemma:

LEMMA 3.4. — *The action of $M^{\text{ab}}(F) \times \text{GSp}_{2n}(F)$ on $\mathcal{S}(X(F))$ is smooth. When F is Archimedean, it is continuous with respect to the Fréchet topology on $\mathcal{S}(X(F))$.*

We observe that the inclusions (see Section 3.3)

$$C_c^\infty(X^\circ(F)) \longrightarrow \mathcal{S}(X^\circ(F)) \longrightarrow \mathcal{S}(X(F))$$

are continuous in the Archimedean case, where we give $C_c^\infty(X^\circ(F))$ the usual topology for compactly supported smooth functions on a real manifold and $\mathcal{S}(X^\circ(F))$ the topology explained in Section 2.

It is useful to explicitly state and prove a refinement of [15, Lemmas 5.1 and 5.7]. The group Sp_{2n} acts on $\wedge^n \mathbb{G}_a^{2n}$ via its action on \mathbb{G}_a^{2n} . Let $K < \text{Sp}_{2n}(F)$ be a maximal compact subgroup. We assume $K = \text{Sp}_{2n}(\mathcal{O})$ when F is non-Archimedean. Here \mathcal{O} is the ring of integers of F . For F Archimedean, choose a K -invariant bilinear form (\cdot, \cdot) on $\wedge^n F^{2n}$ and set $|x| := (x, x)^{[F:\mathbb{R}]/2}$. For F non-Archimedean, the standard basis of F^{2n} induces a canonical basis on $\wedge^n F^{2n}$ (given by wedge products of the standard basis of F^{2n} in increasing order). Define the norm $|x|$ on $\wedge^n F^{2n}$ to be the maximum norm with respect to the induced basis. We claim that this norm is invariant under $\text{Sp}_{2n}(\mathcal{O})$. Indeed, for $x \in \wedge^n F^{2n} - \{0\}$ we have $|x| = q^{-k}$ where k is the unique integer such that $\varpi^{-k}x \in \wedge^n \mathcal{O}^{2n} - \varpi \wedge^n \mathcal{O}^{2n}$. It follows that $|x|$ is invariant under $\text{GL}(\wedge^n \mathcal{O}^{2n})$, and the action of $\text{Sp}_{2n}(\mathcal{O})$ on $\wedge^n F^{2n}$ is the restriction of the action of $\text{GL}(\wedge^n \mathcal{O}^{2n})$ on $\wedge^n F^{2n}$.

In either the Archimedean or non-Archimedean case, we set

$$(3.12) \quad |g| := |\text{Pl}(g)|$$

where $\text{Pl} : X^\circ \rightarrow \wedge^n \mathbb{G}_a^{2n}$ is the Plücker embedding. Note that writing $g = mk$ where $(m, k) \in M^{\text{ab}}(F) \times K$, one has $|g| = |mk| = |\omega(m)|^{-1}$.

LEMMA 3.5. — *Let $0 \leq \beta < \frac{1}{2}$. If F is non-Archimedean, then any $f \in \mathcal{S}(X(F))$ satisfies*

$$|f(g)| \ll_{f,\beta} |g|^{-\frac{n+1}{2} + \beta}.$$

Moreover $f(g) = 0$ for $|g|$ sufficiently large in a sense depending on f . If F is Archimedean, then for each $N \in \mathbb{Z}_{\geq 0}$ and $D \in U(\mathfrak{g})$ there is a continuous seminorm $\nu_{D,N,\beta}$ on $\mathcal{S}(X(F))$ such that for $f \in \mathcal{S}(X(F))$ one has

$$|D \cdot f(g)| \leq \nu_{D,N,\beta}(f) |g|^{-N - \frac{n+1}{2} + \beta}.$$

Proof. — Assume first that F is non-Archimedean. When $\beta = 0$ the lemma is just [15, Lemma 5.1] and by inspecting the proof one obtains the refined estimate stated in the current lemma.

For the Archimedean assertion, using Mellin inversion [15, Lemma 4.3], we write

$$(3.13) \quad (\omega\bar{\omega}(m))^{-N} D \cdot f(mk) \\ = \delta_P^{1/2}(m) \sum_{\eta \in \hat{K}_{G_m}} \int_{i\mathbb{R} + \sigma} (D \cdot f)_{\eta_{s+2N[F:\mathbb{R}]-1}}(k) \eta_s(\omega(m)) \frac{ds}{4\pi[F:\mathbb{R}]i}$$

for σ sufficiently large. The factor $a_{\text{Id}}(s, \chi)$ is holomorphic in the half plane $\text{Re}(s) > -\frac{1}{2}$ and hence so is $a_{\text{Id}}(s + \frac{2N}{[F:\mathbb{R}]}, \chi)$. Thus using the fact that the seminorms (3.10) are finite, we can shift the contour to $\sigma = -\beta$ in (3.13) to see that it is bounded by

$$\delta_P^{1/2}(m) \left| \sum_{\eta \in \hat{K}_{G_m}} \int_{i\mathbb{R} - \beta} (D \cdot f)_{\eta_{s+2N[F:\mathbb{R}]-1}}(k) \eta_s(\omega(m)) \frac{ds}{4\pi[F:\mathbb{R}]i} \right| \\ \leq \frac{\delta_P^{1/2}(m) |\omega(m)|^{-\beta}}{4\pi[F:\mathbb{R}]} 2 (|f|_{A,B,\text{Id},1,K,D} + |f|_{A,B,\text{Id},s^2,K,D})$$

where $A := -\beta - \varepsilon + \frac{2N}{[F:\mathbb{R}]}$ and $B := -\beta + \varepsilon + \frac{2N}{[F:\mathbb{R}]}$ for some $\varepsilon < \frac{1}{2} - \beta$. Since $|mk| = |\omega(m)|^{-1}$ and $\delta_P(m) = |m|^{-(n+1)}$, we deduce the lemma in the Archimedean case. \square

To prove Proposition 3.7 below, we require a more precise version of [15, Lemma 3.3]. Assume for the moment that F is Archimedean and let

$$(3.14) \quad \mu(z) := \frac{z}{(z\bar{z})^{1/2}}$$

where in the denominator we mean the positive square root. Then any character of F^\times can be written uniquely in the form $\chi = |\cdot|^{it} \mu^\alpha$ where $t \in \mathbb{R}$, $\alpha \in \{0, 1\}$ when F is real, and $\alpha \in \mathbb{Z}$ when F is complex.

LEMMA 3.6. — Assume F is Archimedean. Let $A < B$ be real numbers, and for $w \in \{\text{Id}, w_0\}$ let $p_w, p'_w \in \mathbb{C}[x]$ be polynomials such that

$p_w(s)a_w(s, \mu^\alpha)$ and $p'_w(s)a_w(-s, \overline{\mu^\alpha})$ are holomorphic and nonvanishing for all α as above and all $s \in V_{A,B}$. Then the quotients

$$\frac{p'_{\text{Id}}(s)a_{\text{Id}}(-s, \overline{\mu^\alpha})}{p_{w_0}(s)a_{w_0}(s, \mu^\alpha)}, \quad \frac{p_{w_0}(s)a_{w_0}(s, \mu^\alpha)}{p'_{\text{Id}}(s)a_{\text{Id}}(-s, \overline{\mu^\alpha})},$$

$$\frac{p_{\text{Id}}(s)a_{\text{Id}}(s, \mu^\alpha)}{p'_{w_0}(s)a_{w_0}(-s, \overline{\mu^\alpha})}, \quad \frac{p'_{w_0}(s)a_{w_0}(-s, \overline{\mu^\alpha})}{p_{\text{Id}}(s)a_{\text{Id}}(s, \mu^\alpha)}$$

are bounded in $V_{A,B}$ by polynomials in s .

The key difference between this lemma and [15, Lemma 3.3] is the uniformity in α of the bound. However, the proof of [15, Lemma 3.3] actually yields the stronger assertion of Lemma 3.6.

Let

$$\mathcal{S}(X(F), K) < \mathcal{S}(X(F))$$

be the subspace of K -finite vectors. It is dense [31, Section 4.4.3.1]. We prove the following lemma for use in the proof of Theorem 3.13:

PROPOSITION 3.7. — *If F is Archimedean then the Fourier transform*

$$\mathcal{F}_X : \mathcal{S}(X(F)) \longrightarrow \mathcal{S}(X(F))$$

is continuous.

Proof. — To prove the proposition it suffices to show that the seminorms

$$(3.15) \quad f \longmapsto |\mathcal{F}_X(f)|_{A,B,w,p_w,\Omega,D} = \sum_{\eta \in \tilde{K}_{\mathbb{C}_m}} \sup_{g \in \Omega} |M_w(D.\mathcal{F}_X(f))_{\eta_s}(g)|_{A,B,p_w}$$

are continuous on $\mathcal{S}(X(F))$. Let $w \in \{\text{Id}, w_0\}$. Assume $D = D_1 \otimes D_2$ where $D_1 \in U(\text{Lie}(M^{\text{ab}}(F)))$ and $D_2 \in U(\mathfrak{sp}_{2n}(F))$. By [15, Lemma 5.9] and Theorem 3.1,

$$(3.16) \quad \begin{aligned} |M_w(D.\mathcal{F}_X(f))_{\mu_s^\alpha}(k)|_{A,B,p_w} &= |M_w \mathcal{F}_X(D_2.f)_{\mu_s^\alpha}(g)|_{A,B,p_{\alpha,w}} \\ &= |M_w M_{w_0}^*(D_2.f_{(\overline{\mu^\alpha})_{-s}})(g)|_{A,B,p_{\alpha,w}}. \end{aligned}$$

Here $p_{\alpha,w}(s)$ is a polynomial function of s and α divisible by $p_w(s)$ that depends on D_1 . In (3.16) we used the fact that the Mellin transform commutes with D_2 since D_2 is induced by the action of $\text{Sp}_{2n}(F)$ on $X^\circ(F)$ on the right, whereas the Mellin transform is given by integrating over the action of $M^{\text{ab}}(F)$ on $X^\circ(F)$ on the left.

Using the argument of [15, Lemma 3.4], but with Lemma 3.6 replacing [15, Lemma 3.3], the quantity (3.16) is bounded by a constant depending on A, B times

$$\max(1, |\alpha|)^N |M_{w'}(D_2 f_{(\overline{\mu^\alpha})_{-s}})(g)|_{A,B,p_{w'}}$$

for some N and an appropriate $p_{w'}$ independent of α , where

$$w' = \begin{cases} w_0 & \text{if } w = \text{Id}, \\ \text{Id} & \text{if } w = w_0. \end{cases}$$

This in turn is dominated by

$$|M_{w'}(D'D_2f_{(\bar{\mu}^\alpha)_{-s}})(g)|_{A,B,p_{w'}}$$

for an appropriate differential operator D' (see [15, Lemma 5.9]). Thus the seminorm (3.15) is dominated by

$$f \mapsto \sum_{\eta \in \hat{K}_{G,m}} \sup_{g \in \Omega} |M_{w'}(D'D_2f_{(\bar{\mu}^\alpha)_{-s}})(g)|_{A,B,p_{w'}}.$$

But this is a continuous seminorm on $\mathcal{S}(X(F))$ by definition. \square

Let

$$(3.17) \quad m(x) := \begin{pmatrix} x^{-1} & & & \\ & I_{n-1} & & \\ & & x & \\ & & & I_{n-1} \end{pmatrix}.$$

Assume now that F is non-Archimedean. Then $|m(x)| = |x|$. By the Iwasawa decomposition, a \mathbb{C} -vector space basis for $C_c^\infty(X^\circ(F))^{\text{Sp}_{2n}(\mathcal{O})}$ is given by the functions

$$(3.18) \quad \mathbb{1}_k := \mathbb{1}_{[P,P](F)m(\varpi^k)\text{Sp}_{2n}(\mathcal{O})}$$

for $k \in \mathbb{Z}$. The space $\mathcal{S}(X(F))^{\text{Sp}_{2n}(\mathcal{O})}$ contains $C_c^\infty(X^\circ(F))^{\text{Sp}_{2n}(\mathcal{O})}$ but it is larger. It contains, for example, the basic function

$$(3.19) \quad b_X := \sum_{(j_1, \dots, j_{\lfloor n/2 \rfloor}, k) \in \mathbb{Z}_{\geq 0}^{\lfloor n/2 \rfloor + 1}} q^{2j_1 + 4j_2 + \dots + 2\lfloor n/2 \rfloor j_{\lfloor n/2 \rfloor}} \mathbb{1}_{k+2j_1 + \dots + 2j_{\lfloor n/2 \rfloor}}.$$

One has $\mathcal{F}_X(b_X) = b_X$ [15, Lemma 5.4] provided that ψ is unramified.

It will be convenient to isolate another family of functions in this space. For $c \in \mathbb{Z}$, let

$$\mathbb{1}_{\geq c} := \sum_{k \geq c} \mathbb{1}_k = \mathbb{1}_{\varpi^c X(\mathcal{O})}.$$

LEMMA 3.8. — *One has $\mathbb{1}_{\geq c} \in \mathcal{S}(X(F))^{\text{Sp}_{2n}(\mathcal{O})}$.*

Proof. — Recall the action L defined in (3.11). One has $L(m(\varpi)^c)\mathbb{1}_{\geq 0} = \mathbb{1}_{\geq c}$, so by Lemma 3.3 it suffices to show $\mathbb{1}_{\geq 0} \in \mathcal{S}(X(F))^{\text{Sp}_{2n}(\mathcal{O})}$. Since

$$\left(\prod_{j=1}^{\lfloor n/2 \rfloor} (1 - q^{2j} L(m(\varpi)^2)) \right) b_X = \mathbb{1}_{\geq 0},$$

we can apply Lemma 3.3 again to deduce the result. \square

The usual Schwartz space of F is dense in $L^2(F, dx)$ and the Fourier transform extends to an isometry of $L^2(F, dx)$. We now prove analogues of these statements in the current setting. We choose a positive right $\mathrm{Sp}_{2n}(F)$ -invariant Radon measure on $X^\circ(F)$ (it is unique up to scaling). Since $X^\circ(F) \subset X(F)$ is open and dense we extend by zero to obtain a measure on $X(F)$ and we can speak of $L^2(X(F))$.

PROPOSITION 3.9. — *One has $\mathcal{S}(X(F)) < L^2(X(F))$. The Fourier transform \mathcal{F}_X extends to an isometry of $L^2(X(F))$. For $f, f_1, f_2 \in L^2(X(F))$ one has*

$$(3.20) \quad \mathcal{F}_X \circ \mathcal{F}_X = L(m(-1)^{n+1}),$$

$$(3.21) \quad \overline{\mathcal{F}_X(f)} = \mathcal{F}_X(L(m(-1))\bar{f}),$$

$$(3.22) \quad \int_{X(F)} \overline{f_1(x)} \mathcal{F}_X(f_2)(x) \, dx = \int_{X(F)} \overline{\mathcal{F}_X(f_1)(x)} (L(m(-1)^{n+1})f_2)(x) \, dx.$$

Before giving the proof we recall two lemmas. The first is an identity that was stated with a typo in [18, (1.2.3)]:

LEMMA 3.10. — *Assume that $\chi : F^\times \rightarrow \mathbb{C}^\times$ is a character and that $n = 1$. The operator*

$$M_{w_0}^* \circ M_{w_0}^* : I(\chi_s) \longrightarrow I(\chi_s)$$

is the identity.

This is well-known and may be obtained via a standard argument.

LEMMA 3.11. — *For any n and any character $\chi : F^\times \rightarrow \mathbb{C}^\times$, the operator*

$$M_{w_0}^* \circ M_{w_0}^* : I(\chi_s) \longrightarrow I(\chi_s)$$

is multiplication by $\chi(-1)^{n+1}$.

Proof. — This was stated incorrectly in [18, Lemma 1.1]. The source of the error is the typo in [18, (1.2.3)]. Upon correcting the typo using Lemma 3.10, the same argument proves the current lemma. \square

Remark 3.12. — The typos in [18] corrected in Lemma 3.10 and 3.11 do not affect the main results of [18] or their proofs. Moreover they do not affect [15], which makes use of results in [18], except for the statement of [15, Lemma 4.6]. The correct statement is in Proposition 3.9 above.

Proof of Proposition 3.9. — The inclusion $\mathcal{S}(X(F)) < L^2(X(F))$ is an easy consequence of the Iwasawa decomposition and Lemma 3.5.

For $f \in \mathcal{S}(X(F))$, assertion (3.20) is a consequence of Theorem 3.1 and Lemma 3.11. Taking the complex conjugate of the identity of Theorem 3.1, for $\sigma \geq 0$ one has

$$\begin{aligned} & \overline{\mathcal{F}_X(f)}_{\chi_{\sigma+it}} \\ &= \overline{\mathcal{F}_X(f)}_{\bar{\chi}_{\sigma-it}} = \overline{M_{w_0}^*(f_{\chi_{-\sigma+it}})} \\ &= \overline{\gamma\left(-\sigma+it-\frac{n-1}{2}, \chi, \psi\right)} \prod_{r=1}^{\lfloor n/2 \rfloor} \overline{\gamma(2(-\sigma+it)-n+2r, \chi^2, \psi) M_{w_0}(f_{\chi_{-\sigma+it}})} \\ &= \gamma\left(-\sigma-it-\frac{n-1}{2}, \bar{\chi}, \bar{\psi}\right) \prod_{r=1}^{\lfloor n/2 \rfloor} \gamma(2(-\sigma-it)-n+2r, \bar{\chi}^2, \bar{\psi}) M_{w_0}(\bar{f}_{\bar{\chi}_{-\sigma-it}}) \\ &= \chi(-1) M_{w_0}^*(\bar{f}_{\bar{\chi}_{-\sigma-it}}). \end{aligned}$$

Thus by Theorem 3.1, we deduce assertion (3.21).

Let $P^{\text{op}} < \text{Sp}_{2n}$ be the parabolic subgroup opposite to P with respect to M . In [5] Braverman and Kazhdan defined an isometry

$$\mathcal{F}_{P|P^{\text{op}}} : L^2(X(F)) \longrightarrow L^2([P^{\text{op}}, P^{\text{op}}] \backslash \text{Sp}_{2n}(F)).$$

One has $\mathcal{F}_X = \iota_{w_0} \circ \mathcal{F}_{P|P^{\text{op}}}$ by [13, (5.24)], where ι_{w_0} is the isometry

$$\begin{aligned} \iota_{w_0} : L^2([P^{\text{op}}, P^{\text{op}}] \backslash \text{Sp}_{2n}(F)) &\xrightarrow{\sim} L^2(X(F)) \\ f &\longmapsto (x \mapsto f(w_0^{-1}x)). \end{aligned}$$

It follows that \mathcal{F}_X is an isometry. The Plancherel formula (3.22) follows from (3.20), the unitarity of \mathcal{F}_X , and a standard argument using a polarization identity. \square

3.2. The summation formula

We now revert to the global setting. Let F be a number field. Recall that $\mathcal{S}(X(\mathbb{A}_F))$ is defined in Section 2 as the restricted tensor product of $\mathcal{S}(X(F_v))$ with respect to the basic functions $b_{X,v}$. Let $\psi : F \backslash \mathbb{A}_F \rightarrow \mathbb{C}^\times$ be a nontrivial character. We have $\mathcal{F}_{X,\psi_v}(b_{X,v}) = b_{X,v}$ if F_v is non-Archimedean and ψ_v is unramified. Thus we have a global Fourier transform

$$\mathcal{F}_X := \mathcal{F}_{X,\psi} := \bigotimes_v \mathcal{F}_{X,\psi_v} : \mathcal{S}(X(\mathbb{A}_F)) \longrightarrow \mathcal{S}(X(\mathbb{A}_F)).$$

By Proposition 3.14 below $\mathcal{S}(X^\circ(F_v)) < \mathcal{S}(X(F_v))$ for all places v .

THEOREM 3.13. — *Assume that for some finite places v_1, v_2 (not necessarily distinct) one has $f = f_{v_1} f^{v_1}$ and $\mathcal{F}_X(f) = \mathcal{F}_X(f_{v_2})\mathcal{F}_X(f^{v_2})$ with $f_{v_1} \in C_c^\infty(X^\circ(F_{v_1}))$ and $\mathcal{F}_X(f_{v_2}) \in C_c^\infty(X^\circ(F_{v_2}))$. Then*

$$\sum_{\gamma \in X^\circ(F)} f(\gamma) = \sum_{\gamma \in X^\circ(F)} \mathcal{F}_X(f)(\gamma).$$

Proof. — We may assume $f = f_\infty f^\infty$ with $f_\infty \in \mathcal{S}(X(F_\infty))$ and $f^\infty \in \mathcal{S}(X(\mathbb{A}_F^\infty))$. Let $K_\infty < \mathrm{Sp}_{2n}(F_\infty)$ be a maximal compact subgroup and let $\mathcal{S}(X(F_\infty), K_\infty) < \mathcal{S}(X(F_\infty))$ be the space of K_∞ -finite functions. Assume first that $f_\infty \in \mathcal{S}(X(F_\infty), K_\infty)$. Then the stated identity follows from [15, Theorem 1.1] and [14, Theorem 10.1].

We now argue by continuity to deduce the identity in general. Consider the linear form

$$(3.23) \quad \begin{aligned} &\mathcal{S}(X(F_\infty)) \longrightarrow \mathbb{C} \\ f_\infty &\longmapsto \sum_{\gamma \in X^\circ(F)} f_\infty(\gamma) f^\infty(\gamma) - \sum_{\gamma \in X^\circ(F)} \mathcal{F}_X(f_\infty)(\gamma) \mathcal{F}_X(f^\infty)(\gamma). \end{aligned}$$

The Fourier transform is continuous by Proposition 3.7. Thus following the proof of [15, Lemma 6.4], by replacing Lemma 5.7 in loc. cit. with Lemma 3.5, we see that the sums defining (3.23) are absolutely convergent and (3.23) is continuous. It vanishes on the dense subspace $\mathcal{S}(X(F_\infty), K_\infty) < \mathcal{S}(X(F_\infty))$ and hence is identically zero. \square

We remark that Theorem 3.13 was already proved in [5], but with a different definition of the Schwartz space. At the non-Archimedean places the two definitions yield the same space of functions [17]. At the Archimedean places this is less clear. In any case, it is easier to just prove the theorem directly than to rigorously check the compatibility of the two definitions.

3.3. Containment of Schwartz spaces

In this subsection we prove the following proposition:

PROPOSITION 3.14. — *One has $\mathcal{S}(X^\circ(F)) < \mathcal{S}(X(F))$. When F is Archimedean, the inclusion is continuous.*

In the non-Archimedean case this is [15, Proposition 4.7].

Remark 3.15. — In the Archimedean case the weaker statement that K_∞ -finite functions in $C_c^\infty(X^\circ(F))$ are contained in $\mathcal{S}(X(F))$ was asserted in [15, Proposition 4.7]. This is true, but the proof is incomplete. It relies on [15, Lemma A.2], which is false. Happily, this does not affect the rest

of the results in [15] because the false assertion in [15, Lemma A.2] is not used elsewhere in the paper.

Let

$$w_k := \begin{pmatrix} I_k & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_{n-k} \\ 0 & 0 & I_k & 0 \\ 0 & \beta_{n-k} & 0 & 0 \end{pmatrix}$$

where $\beta_{n-k} \in \mathrm{GL}_{n-k}(\mathbb{Z})$ is the antidiagonal matrix.

LEMMA 3.16. — *Let $C = [P, P] \backslash Pw_0Pw_0^{-1} \subset X^\circ$. Then Cg is open in X° for all $g \in \mathrm{Sp}_{2n}(F)$ and*

$$\bigcup_{0 \leq k \leq n} \bigcup_{\tau \in \mathfrak{S}_n} C(F)w_k \begin{pmatrix} \tau & \\ & \tau \end{pmatrix} = X^\circ(F).$$

Here the inner union is over the group of permutation matrices in $\mathrm{GL}_n(\mathbb{Z})$.

Proof. — It is well known that the big Bruhat cell Pw_0P is open in Sp_{2n} , hence the same is true of Cg in X° for all $g \in \mathrm{Sp}_{2n}(F)$. We have an isomorphism $\phi : \mathrm{GL}_n \times \mathrm{Sym}^2(\mathbb{G}_a^n) \rightarrow N \backslash Pw_0Pw_0^{-1}$ given on points in an F -algebra R by

$$\begin{aligned} \phi : \mathrm{GL}_n(R) \times \mathrm{Sym}^2(R^n) &\longrightarrow (N \backslash Pw_0Pw_0^{-1})(R) \\ (A, Z) &\longmapsto \begin{pmatrix} A^{-t} & \\ & A \end{pmatrix} w_0 \begin{pmatrix} I_n & -\beta_n Z \beta_n \\ & I_n \end{pmatrix} w_0^{-1} \\ &= \begin{pmatrix} * & * \\ AZ & A \end{pmatrix}. \end{aligned}$$

It follows that $C(F)$ may be characterized as the subset of $X^\circ(F)$ consisting of all classes of the form

$$(3.24) \quad \begin{pmatrix} * & * \\ B & A \end{pmatrix}$$

where $(B \ A) \in M_{n,2n}(F)$ and $\det A \neq 0$. This in turn implies that

$$C(F)w_k$$

is the subset of $X^\circ(F)$ consisting of classes of the form (3.24) where the $n \times n$ matrix formed by the columns corresponding to $\{i : k+1 \leq i \leq n+k\}$ is invertible.

To complete the proof, we claim that after multiplying by an element of $M(F)$ on the left (i.e., performing Gauss–Jordan elimination on A) and $(\tau \ \tau)$ with $\tau \in \mathfrak{S}_n$ on the right, a general class (3.24) lies in $C(F)w_k$ for

some k . Indeed, after multiplying on the left and right as just explained, we may assume

$$\begin{pmatrix} B & A \end{pmatrix} = \begin{pmatrix} u & D & 0 & 0 \\ * & * & I_k & v \end{pmatrix}$$

for some $0 \leq k \leq n$. Here $u, v^t \in M_{n-k,k}(F)$ and $D \in M_{n-k,n-k}(F)$. We claim D is invertible so that $\begin{pmatrix} B & A \end{pmatrix}$ lies in $C(F)w_k$. Since the rows of $\begin{pmatrix} B & A \end{pmatrix}$ span a Lagrangian subspace, the (i, j) th entry of u is -1 times the dot product of the i th row of D and the j th row of v . In other words, $u = -Dv^t$. Thus if D is singular, then the rows of $\begin{pmatrix} u & D \end{pmatrix}$ are linearly dependent, which is a contradiction. \square

Proof of Proposition 3.14. — We can and do assume F is Archimedean. If $f \in \mathcal{S}(X^\circ(F))$, it is easy to see that the integral defining f_{χ_s} is absolutely convergent for all $\chi : F^\times \rightarrow \mathbb{C}^\times$ and $s \in \mathbb{C}$. Thus f_{χ_s} is a good section by [18, Lemma 1.3]. We have to verify that for all real numbers $A < B$, $w \in \{\text{Id}, w_0\}$, $D \in U(\mathfrak{g})$, any polynomials $p_w \in \mathbb{C}[s]$ such that

$$(3.25) \quad p_w(s)a_w(s, \eta) \text{ has no poles for all } (s, \eta) \in V_{A,B} \times \widehat{K}_{\mathbb{G}_m}$$

and compact subsets $\Omega \subset X^\circ(F)$, one has $|f|_{A,B,w,p_w,\Omega,D} < \infty$ and $|f|_{A,B,w,p_w,\Omega,D}$ is continuous with respect to the topology on $\mathcal{S}(X^\circ(F))$. Since $U(\mathfrak{g})$ acts continuously on $\mathcal{S}(X^\circ(F))$, it suffices to verify this for $D = \text{Id}$. Note that we do not require $p_w(s)$ to depend on η . In fact by the explicit description of Archimedean local L -functions we see that $a_w(s, \eta)$ can only have a pole in $V_{A,B}$ for finitely many η .

We start by reducing to an estimate involving a single η . Let D_1 (and \bar{D}_1 when F is complex) be the generators of $U(\text{Lie}(M^{\text{ab}}(F)))$ given in [15, (4.2) and (4.3)], respectively. Every element of $\widehat{K}_{\mathbb{G}_m}$ is in the equivalence class of μ^α for $\alpha = 0, 1$ when F is real and $\alpha \in \mathbb{Z}$ when F is complex. Here μ is defined as in (3.14). If F is complex,

$$\begin{aligned} & |M_w(D_1^N \bar{D}_1^{N'} \cdot f)_{\mu_s^\alpha}(g)|_{A,B,p_w} \\ &= \left| \left(\frac{\alpha}{2} + it + s + \frac{n+1}{2} \right)^N \left(-\frac{\alpha}{2} + it + s + \frac{n+1}{2} \right)^{N'} M_w(f)_{\mu_s^\alpha}(g) \right|_{A,B,p_w} \end{aligned}$$

by [15, Lemma 5.9]. This provides us with an estimate on $M_w(f)_{\mu_s^\alpha}(g)$ as a function of α . Using this estimate, we see that to prove $|f|_{A,B,w,p_w,\Omega,1}$ is finite for all $f \in \mathcal{S}(X^\circ(F))$, it suffices to prove that for each p_w satisfying (3.25) there is a continuous seminorm ν on $\mathcal{S}(X^\circ(F))$ such that

$$\sup_{g \in \Omega} |M_w f_{\mu_s^\alpha}(g)|_{A,B,p_w} \leq \nu(f)$$

for all $f \in \mathcal{S}(X^\circ(F))$ and α . Here and below the seminorm ν is allowed to depend on A, B, w, p_w, Ω . In fact, it is enough to show that there is a continuous seminorm ν on $\mathcal{S}(X^\circ(F))$ that

$$(3.26) \quad |M_w f_{\mu_s^\alpha}(w_0)|_{A, B, p_w} \leq \nu(f).$$

Indeed, let $\Omega' \subset \mathrm{Sp}_{2n}(F)$ be a compact set whose projection to $X^\circ(F)$ is Ω . Then assuming we have a seminorm ν as just described, we have

$$\begin{aligned} \sup_{g \in \Omega} |M_w f_{\mu_s^\alpha}(g)|_{A, B, p_w} &= \sup_{g \in w_0^{-1}\Omega'} |M_w(R(g)f)_{\mu_s^\alpha}(w_0)|_{A, B, p_w} \\ &\leq \sup_{g \in w_0^{-1}\Omega'} \nu(R(g)f) \end{aligned}$$

and $\sup_{g \in w_0^{-1}\Omega'} \nu(R(g)f)$ is another continuous seminorm.

Since (3.26) is clear for $w = \mathrm{Id}$, we are left with the $w = w_0$ case. We will roughly follow the strategy of [27, Section 4], but we cannot immediately reduce to functions having support in the big cell as in [27, Lemma 4.1]. Indeed, the proof of loc. cit. uses the irreducibility of certain principal series, and the space $\mathcal{S}(X(F))$ is reducible as a representation of $\mathrm{Sp}_6(F)$.

Let C be the image of $Pw_0Pw_0^{-1}$ in X° . Choose a tempered partition of unity subordinate to the cover of $X^\circ(F)$ given by Lemma 3.16, that is, choose tempered functions $t_{k, \tau}$ supported in $C(F)w_k \begin{pmatrix} \tau & \\ & \tau \end{pmatrix}$ such that

$$\sum_{k=0}^n \sum_{\tau \in \mathfrak{S}_n} t_{k, \tau} = 1$$

and

$$t_{k, \tau} f \in \mathcal{S}\left(C(F)w_k \begin{pmatrix} \tau & \\ & \tau \end{pmatrix}\right)$$

for all $f \in \mathcal{S}(X^\circ(F))$ [10, Proposition 3.14]. Then

$$|M_{w_0} f_{\mu_s^\alpha}(w_0)|_{A, B, p_w} \leq \sum_{k=0}^n \sum_{\tau \in \mathfrak{S}_n} |M_{w_0}(t_{k, \tau} f)_{\mu_s^\alpha}(w_0)|_{A, B, p_w}.$$

Hence we can and do assume that f is supported in $C(F)w_k \begin{pmatrix} \tau & \\ & \tau \end{pmatrix}$ for some fixed $0 \leq k \leq n$ and some $\tau \in \mathfrak{S}_n$. Now

$$\begin{aligned} M_{w_0} f_{\mu_s^\alpha}(w_0) &= \int_{N(F)} \int_{M^{\mathrm{ab}}(F)} \delta_P(m)^{1/2} \mu_s^\alpha(\omega(m)) f(m^{-1}w_0^{-1}nw_0) dm dn \\ &= \int_{\mathrm{Sym}^2(F^n)} \int_{(\mathrm{SL}_n \backslash \mathrm{GL}_n)(F)} \mu_{s+(n+1)/2}^\alpha(\det A) f \begin{pmatrix} * & * \\ AZ & A \end{pmatrix} dA dZ. \end{aligned}$$

The notation is a reminder that the image of an element of $\mathrm{Sp}_{2n}(F)$ in $X^\circ(F)$ depends only on the bottom n rows of the matrix.

Write $Z = \tau^{-1} \begin{pmatrix} u & x \\ x^t & y \end{pmatrix} \tau$ where $(u, x, y) \in \text{Sym}^2(F^k) \times M_{k \times (n-k)}(F) \times \text{Sym}^2(F^{n-k})$, and let

$$A' = A'(x, y) := \begin{pmatrix} I_k & x\beta_{n-k} \\ 0 & y\beta_{n-k} \end{pmatrix}.$$

Then if A' is invertible, we have

$$\begin{aligned} \begin{pmatrix} * & * \\ AZ & A \end{pmatrix} &= \begin{pmatrix} (A\tau^{-1})^{-t} & \\ & A\tau^{-1} \end{pmatrix} \begin{pmatrix} * & * & * & * \\ u & 0 & I_k & x\beta_{n-k} \\ x^t & -\beta_{n-k} & 0 & y\beta_{n-k} \end{pmatrix} w_k \begin{pmatrix} \tau & \\ & \tau \end{pmatrix} \\ &= \begin{pmatrix} (A\tau^{-1}A')^{-t} & \\ & A\tau^{-1}A' \end{pmatrix} \\ &\quad \times \begin{pmatrix} * & * & * & * \\ u - xy^{-1}x^t & xy^{-1}\beta_{n-k} & I_k & 0 \\ \beta_{n-k}y^{-1}x^t & -\beta_{n-k}y^{-1}\beta_{n-k} & 0 & I_{n-k} \end{pmatrix} w_k \begin{pmatrix} \tau & \\ & \tau \end{pmatrix}. \end{aligned}$$

Thus $M_{w_0} f \mu_s^\alpha(w_0)$ equals

$$\begin{aligned} &\int f \left(\begin{pmatrix} (A\tau^{-1}A')^{-t} & \\ & A\tau^{-1}A' \end{pmatrix} \right. \\ &\quad \times \begin{pmatrix} * & * & * & * \\ u - xy^{-1}x^t & xy^{-1}\beta_{n-k} & I_k & 0 \\ \beta_{n-k}y^{-1}x^t & -\beta_{n-k}y^{-1}\beta_{n-k} & 0 & I_{n-k} \end{pmatrix} w_k \begin{pmatrix} \tau & \\ & \tau \end{pmatrix} \Big) \\ &\quad \times \mu_{s+(n+1)/2}^\alpha(\det A) \, dA \, du \, dx \, dy \\ &= \int \mu_{s+(n+1)/2}^\alpha(\det A\tau) \\ &\quad \times f \left(\begin{pmatrix} A^{-t} & \\ & A \end{pmatrix} \begin{pmatrix} * & * & * & * \\ u & x & I_k & 0 \\ x^t & y^{-1} & 0 & I_{n-k} \end{pmatrix} w_k \begin{pmatrix} \tau & \\ & \tau \end{pmatrix} \right) \\ &\quad \times \frac{|y|^k \, dA \, du \, dx \, dy}{\mu_{s+(n+1)/2}^\alpha(\det \beta_{n-k}y)} \\ &= \int \mu_{s+(n+1)/2}^\alpha(\det A\tau) f \left(\begin{pmatrix} A^{-t} & \\ & A \end{pmatrix} \begin{pmatrix} * & * & * & * \\ u & x & I_k & 0 \\ x^t & y & 0 & I_{n-k} \end{pmatrix} w_k \begin{pmatrix} \tau & \\ & \tau \end{pmatrix} \right) \\ &\quad \times \mu_{s-(n+1)/2}^\alpha(\det \beta_{n-k}y) \, dA \, du \, dx \, dy, \end{aligned}$$

where the integrals are over $(\text{SL}_n \backslash \text{GL}_n)(F) \times \text{Sym}^2(F^k) \times M_{k \times (n-k)}(F) \times \text{Sym}^2(F^{n-k})$. Here we have used that $d(y^{-1}) = |\det y|^{-n+k-1} dy$. Now

consider the differential operator

$$\partial := \det(\partial_{ij})$$

where (∂_{ij}) is the unique symmetric $(n-k) \times (n-k)$ matrix of partial differential operators satisfying

$$\partial_{ij} = \begin{cases} \frac{\partial}{\partial z_{ii}} & \text{if } i = j, \\ \frac{1}{2} \frac{\partial}{\partial z_{ij}} & \text{if } i > j. \end{cases}$$

When F is complex, we view these as holomorphic differential operators. Then for $y \in \text{Sym}^2(F^{n-k})$ we have

$$\partial(\det y)^s = \prod_{i=0}^{n-k-1} \left(s + \frac{i}{2}\right) (\det y)^{s-1}$$

(see, e.g. [6, Theorem 2.2]).

Applying integration by parts m times we have

$$\begin{aligned} p_{m,\alpha}(s) & \int \mu_{s-(n+1)/2}^\alpha(\det y) \mu_{s+(n+1)/2}^\alpha(\det A) \\ & \times f \left(\begin{pmatrix} A^{-t} & & & \\ & A & & \\ & & * & * \\ & & u & x \\ & & x & y \end{pmatrix} \begin{pmatrix} * & * & * & * \\ & I_k & & 0 \\ & & 0 & I_{n-k} \end{pmatrix} w_k \begin{pmatrix} \tau & \\ & \tau \end{pmatrix} \right) dA du dx dy \\ & = \int \mu_{s-(n+1)/2+m}^\alpha(\det y) \mu_{s+(n+1)/2}^\alpha(\det A) (\partial \bar{\partial})^m \\ & \times f \left(\begin{pmatrix} A^{-t} & & & \\ & A & & \\ & & * & * \\ & & u & x \\ & & x & y \end{pmatrix} \begin{pmatrix} * & * & * & * \\ & I_k & & 0 \\ & & 0 & I_{n-k} \end{pmatrix} w_k \begin{pmatrix} \tau & \\ & \tau \end{pmatrix} \right) dA du dx dy \end{aligned}$$

where $p_{m,\alpha}(s) \in \mathbb{C}[s]$ has zeros only in $\frac{1}{2}\mathbb{Z}$. Here by convention $\bar{\partial}$ is the identity operator when F is real, and we are letting ∂ and $\bar{\partial}$ act on f viewed as a function of $y \in \text{Sym}^{n-k}(F)$. We observe that the bottom integral converges absolutely for $\text{Re}(s) + m > \frac{n+1}{2}$, and thus provides us with a holomorphic continuation of $p_{m,\alpha}(s) M_{w_0} f_{\mu_s^\alpha}(w_0)$ to this range. Moreover, if $A + m > \frac{n+1}{2}$, then for all $p \in \mathbb{C}[s]$ one has

$$(3.27) \quad |M_{w_0} f_{\mu_s^\alpha}(w_0)|_{A,B,pp_{m,\alpha}} \leq \nu(f)$$

for some continuous seminorm ν on $\mathcal{S}(X^\circ(F))$ depending on p, m, A, B .

Assume henceforth that $A + m > \frac{n+1}{2}$. Since the zeros of $p_{m,\alpha}$ are located in $\frac{1}{2}\mathbb{Z}$, by slightly decreasing A and increasing B if necessary, we are free to assume that no zeros of $p_{m,\alpha}$ are on the lines $\text{Re}(s) = A$ or $\text{Re}(s) = B$ for all α . Let

$$\Omega := \{s \in V_{A,B} : |\text{Im}(s)| < 1, A < \text{Re}(s) < B\}.$$

Again using the fact that the zeros of $p_{m,\alpha}$ are located in $\frac{1}{2}\mathbb{Z}$, we have

$$(3.28) \quad \max_{s \in V_{A,B} - \Omega} \frac{1}{p_{m,\alpha}(s)} \ll_m 1$$

where the implied constant is independent of α . Assume now that p_{w_0} satisfies (3.25). Since $M_{w_0} f_{\mu_s^\alpha}$ is a good section [18, Lemmas 1.2 and 1.3], the maximum modulus principle implies

$$\begin{aligned} \sup_{s \in V_{A,B}} |p_{w_0}(s) M_{w_0} f_{\mu_s^\alpha}(w_0)| &\leq \sup_{s \in V_{A,B} - \Omega} |p_{w_0}(s) M_{w_0} f_{\mu_s^\alpha}(w_0)| \\ &\leq |M_{w_0} f_{\mu_s^\alpha}(w_0)|_{A,B,p_{w_0} p_{m,\alpha}} \max_{s \in V_{A,B} - \Omega} \frac{1}{p_{m,\alpha}(s)} \\ &\leq \nu(f) \end{aligned}$$

for some continuous seminorm ν on $\mathcal{S}(X^\circ(F))$ depending on p_{w_0}, m, A, B . Here in the last inequality we have used (3.27) and (3.28). This implies (3.26). \square

4. Groups and orbits

For this section, F is a field of characteristic zero. For $1 \leq i \leq 3$, let $V_i = \mathbb{G}_a^{d_i}$ where d_i is even and let \mathcal{Q}_i be a nondegenerate quadratic form on $V_i(F)$. Let $\mathcal{Q} := \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3$. We put

$$(4.1) \quad V_i^\circ := V_i - \{0\} \quad \text{and} \quad V^\circ := V_1^\circ \times V_2^\circ \times V_3^\circ$$

and we let $V' \subset V$ be the open subscheme consisting of (v_1, v_2, v_3) such that no two v_i are zero. For an F -algebra R , recall that

$$(4.2) \quad Y(R) = \{(y_1, y_2, y_3) \in V(R) : \mathcal{Q}_1(y_1) = \mathcal{Q}_2(y_2) = \mathcal{Q}_3(y_3)\}.$$

We observe that $Y^{\text{sm}} = Y \cap V'$. We let

$$(4.3) \quad Y^{\text{ani}} \subset Y$$

be the open complement of the vanishing locus of \mathcal{Q}_i (it is independent of i).

We let $\text{GO}_{\mathcal{Q}_i}$ be the similitude group of (V_i, \mathcal{Q}_i) and let $\nu : \text{GO}_{\mathcal{Q}_i} \rightarrow \mathbb{G}_m$ be the similitude norm. We then set

$$(4.4) \quad H(R) := \left\{ (h_1, h_2, h_3) \in \prod_{i=1}^3 \text{GO}_{\mathcal{Q}_i}(R) : \nu(h_1) = \nu(h_2) = \nu(h_3) \right\},$$

and define

$$(4.5) \quad \begin{aligned} \lambda : H(R) &\longrightarrow R^\times \\ (h_1, h_2, h_3) &\longmapsto \nu(h_1). \end{aligned}$$

Let

$$(4.6) \quad \tilde{Y}_0(R) := \{(y_1, y_2, y_3) \in V^\circ(R) : \mathcal{Q}_1(y_1) = \mathcal{Q}_2(y_2) = \mathcal{Q}_3(y_3) = 0\}$$

and let Y_0 be the (quasi-affine) quotient of \tilde{Y}_0 by $\mathbb{G}_m \times \mathbb{G}_m$, acting via the restriction of the action

$$(4.7) \quad \begin{aligned} \mathbb{G}_m(R) \times \mathbb{G}_m(R) \times V(R) &\longrightarrow V(R) \\ (a_1, a_2, (v_1, v_2, v_3)) &\longmapsto (a_1 v_1, a_2 v_2, (a_1 a_2)^{-1} v_3). \end{aligned}$$

This quotient can be constructed by taking the affine closure \bar{Y}_0 of \tilde{Y}_0 in V and viewing Y_0 as an open subscheme of the GIT quotient of \bar{Y}_0 by $\mathbb{G}_m \times \mathbb{G}_m$. We observe that Y_0 is a geometric quotient of \tilde{Y}_0 .

For $1 \leq i \leq 3$, we define the scheme

$$(4.8) \quad \tilde{Y}_i(R) := \left\{ (y_1, y_2, y_3) \in V^\circ(R) : \begin{array}{l} \mathcal{Q}_{i-1}(y_{i-1}) = \mathcal{Q}_{i+1}(y_{i+1}), \\ \mathcal{Q}_i(y_i) = 0 \end{array} \right\}.$$

Here the indices are taken modulo 3 in the obvious sense. Let Y_1 be the quotient of \tilde{Y}_1 by \mathbb{G}_m acting via the restriction of the action

$$(4.9) \quad \begin{aligned} \mathbb{G}_m(R) \times V(R) &\longrightarrow V(R) \\ (a, (v_1, v_2, v_3)) &\longmapsto (v_1, a v_2, a v_3). \end{aligned}$$

This is nothing but the product over F of the quasi-affine scheme cut out by \mathcal{Q}_1 in V_1° and the quasi-projective scheme cut out of $\mathbb{P}(V_2^\circ \times V_3^\circ)$ by $\mathcal{Q}_2 = \mathcal{Q}_3$. The schemes Y_2 and Y_3 are defined similarly. Thus

$$(4.10) \quad Y_0 := \tilde{Y}_0 / \mathbb{G}_m^2 \quad \text{and} \quad Y_i := \tilde{Y}_i / \mathbb{G}_m$$

where the quotients are defined as above. Using Hilbert's theorem 90, we deduce the following lemma:

LEMMA 4.1. — *The maps $\tilde{Y}_0(F)/(F^\times)^2 \rightarrow Y_0(F)$ and $\tilde{Y}_i(F)/F^\times \rightarrow Y_i(F)$ are bijective.*

We often identify $SL_2^3(R)$ with the subgroup $G(R) < Sp_6(R)$ defined as follows:

$$(4.11) \quad G(R) = \left\{ \begin{pmatrix} a_1 & & & b_1 & & \\ & a_2 & & & b_2 & \\ & & a_3 & & & b_3 \\ c_1 & & & d_1 & & \\ & c_2 & & & d_2 & \\ & & c_3 & & & d_3 \end{pmatrix} \in GL_6(R) : a_i d_i - b_i c_i = 1 \text{ for } 1 \leq i \leq 3 \right\}.$$

We give a set of representatives for

$$X^\circ(F)/G(F)$$

and the corresponding stabilizers. Let

$$(4.12) \quad \begin{aligned} \gamma_b &:= \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}, \\ \gamma_i &:= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \\ &\quad \times \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}^{i-1} \quad \text{for } 1 \leq i \leq 3. \end{aligned}$$

All four matrices are in $Sp_6(\mathbb{Z})$. By [14, Lemmas 2.1 and 2.2], the matrices γ_i together with the identity matrix, denoted by $\gamma_0 = \text{Id}$, form a complete set of representatives of $X^\circ(F)/G(F)$ (strictly speaking, we have chosen different representatives for the γ_i orbits than in [15], but this does not affect the validity of [14, Lemmas 2.1 and 2.2]). For $\gamma \in X^\circ(F)$, let $G_\gamma \leq G$ be the stabilizer of γ under the right action.

LEMMA 4.2 ([14, Lemma 2.3]). — *One has*

$$G_{\gamma_b}(R) := \left\{ \left(\begin{pmatrix} 1 & t_1 \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & t_2 \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & t_3 \\ & 1 \end{pmatrix} \right) : t_1, t_2, t_3 \in R, t_1 + t_2 + t_3 = 0 \right\},$$

$$G_{\text{Id}}(R) := \left\{ \left(\begin{pmatrix} b_i^{-1} & t_i \\ & b_i \end{pmatrix} \right) : t_i \in R, b_i \in R^\times, b_1 b_2 b_3 = 1 \right\},$$

$$G_{\gamma_1}(R) := \left\{ \left(\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}, g, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right) : t \in R, g \in \text{SL}_2(R) \right\},$$

$$G_{\gamma_2}(R) := \left\{ \left(\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}, g \right) : t \in R, g \in \text{SL}_2(R) \right\},$$

$$G_{\gamma_3}(R) := \left\{ \left(g, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \right) : t \in R, g \in \text{SL}_2(R) \right\}.$$

5. Local functions

In this section, we define the local integrals required to state our summation formula and prove some of their basic properties. Let F be a local field of characteristic zero. We use the conventions on Schwartz spaces explained in Section 2. For each of the 5 orbits of $G(F)$ in $X^\circ(F)$ given in Lemma 4.2, we will define a family of integrals.

For $f = f_1 \otimes f_2 \in \mathcal{S}(X(F)) \otimes \mathcal{S}(V(F))$, let

$$(5.1) \quad \begin{aligned} I(f)(y) &:= \int_{G_{\gamma_b}(F) \backslash G(F)} f_1(\gamma_b g) \rho(g) f_2(y) dg, \quad y \in Y^{\text{sm}}(F), \\ I_0(f)(y) &:= \int_{N_2^3(F) \backslash G(F)} f_1(g) \rho(g) f_2(y) dg, \quad y \in \tilde{Y}_0(F). \end{aligned}$$

Here the stabilizer G_{γ_b} is computed in Lemma 4.2. These are integrals attached to the $G(F)$ -orbit of γ_b and $\gamma_0 = \text{Id}$, respectively.

Let

$$\Delta_i : \text{SL}_2 \longrightarrow G$$

be defined by

$$(5.2) \quad \Delta_i(h) := \begin{cases} (I_2, h, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} h \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}) & \text{for } i = 1, \\ (\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} h \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, I_2, h) & \text{for } i = 2, \\ (h, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} h \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, I_2) & \text{for } i = 3. \end{cases}$$

Moreover let

$$(5.3) \quad \begin{aligned} p_i : G(R) &\longrightarrow \text{SL}_2(R) \\ (g_1, g_2, g_3) &\longmapsto g_{i+1} \end{aligned}$$

where the indices are taken modulo 3 in the obvious sense.

We need one more piece of data to define the integrals attached to the other orbits. Let $\Phi \in \mathcal{S}(F^2)$. For $y \in \widetilde{Y}_i(F)$, $1 \leq i \leq 3$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$, we define

$$(5.4) \quad I_i(f \otimes \Phi)(\xi, s) := \int_{G_{\gamma_i}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(\gamma_i g) \\ \times \left(\int_{N_2(\mathbb{A}_F) \backslash \operatorname{SL}_2(\mathbb{A}_F)} \int_{\mathbb{A}_F^\times} \rho(\Delta_i(h)g) f_2(\xi) \Phi(x(0, 1)hp_i(g)) |x|^{2s} d^\times x dh \right) dg.$$

We point out that all of the integrals above can be defined directly for a general $f \in \mathcal{S}(X(F) \times \mathcal{S}(V(F)))$ (not just a pure tensor) but the notation is more confusing. One can also define them indirectly for all $f \in \mathcal{S}(X(F) \times V(F))$ using the definition for pure tensors give above. Indeed, in the non-Archimedean case $I(f)$, $I_0(f)$, and $I_i(f \otimes \Phi)$ are defined for all $f \in \mathcal{S}(X(F) \times V(F))$ by bilinearity. In the Archimedean case, the estimates in Section 9 imply that for a given $f \in \mathcal{S}(X(F) \times V(F))$ and any sequence $f_n \in \mathcal{S}(X(F)) \otimes \mathcal{S}(V(F))$ converging to f , the functions

$$I(f) := \lim_{n \rightarrow \infty} I(f_n), \quad I_0(f) := \lim_{n \rightarrow \infty} I_0(f_n), \quad I_i(f \otimes \Phi) := \lim_{n \rightarrow \infty} I_i(f_n \otimes \Phi)$$

are well-defined (via pointwise convergence).

In Section 8 we will compute the integrals defined in this section in the unramified setting. We prove that these integrals are absolutely convergent and bound them in the non-Archimedean case in Section 7 and in the Archimedean case in Section 9.

5.1. The Schwartz space of Y

In Propositions 7.1 and 9.3, we will show that $I(f)$ is a smooth function on $Y^{\text{sm}}(F)$. With this in mind, we define

$$(5.5) \quad \mathcal{S}(Y(F)) := \operatorname{Im}(I : \mathcal{S}(X(F) \times V(F)) \longrightarrow C^\infty(Y^{\text{sm}}(F))).$$

This is the *Schwartz space* of $Y(F)$. We observe that [14, Lemma 4.3] implies in particular that the natural action of $H(F)$ on $C^\infty(Y^{\text{sm}}(F))$ preserves $\mathcal{S}(Y(F))$.

LEMMA 5.1. — *Let F be an Archimedean local field. The kernel of the map*

$$I : \mathcal{S}(X(F) \times V(F)) \longrightarrow C^\infty(Y^{\text{sm}}(F))$$

is closed.

Proof. — For any $N \geq 0$, the Cauchy–Schwarz inequality implies that $|I(f_1 \otimes f_2)|(y)$ is bounded by the product of the square-roots of the following two integrals:

$$(5.6) \quad \int_{G_{\gamma_b}(F) \backslash G(F)} |f_1|^2(\gamma_b g) \max(|\gamma_b g|, 1)^{2N} dg,$$

$$(5.7) \quad \int_{G_{\gamma_b}(F) \backslash G(F)} \max(|\gamma_b g|, 1)^{-2N} |\rho(g) f_2|^2(y) dg.$$

Now $G_{\gamma_b}(F) \backslash G(F)$ is dense in $X^\circ(F)$ and hence the right $\mathrm{Sp}_6(F)$ -invariant positive Radon measure on $X^\circ(F)$ agrees with dg , at least after scaling by a positive real constant. We continue to denote this measure on $X^\circ(F)$ by dg . Thus (5.6) is equal to

$$(5.8) \quad \int_{X^\circ(F)} |f_1|^2(g) \max(|g|, 1)^{2N} dg.$$

Let $\nu_{\mathrm{Id}, N+1, 0}$ be defined as Lemma 3.5, where Id is the identity in $U(\mathfrak{g})$. Using the decomposition of the measure dg afforded by the Iwasawa decomposition and Lemma 3.5, we see that (5.8) is bounded by $\|f_1\|_2^2 + c^2 \nu_{\mathrm{Id}, N+1, 0}(f_1)^2$, where c is a positive constant independent of f_1 .

On the other hand, by the Iwasawa decomposition (5.7) equals

$$\int_{(F^\times)^3 \times F} \max(m(t, a), 1)^{-2N} \int_{K^3} |\rho(k) f_2|^2(a^{-1}y) dk \left(\prod_{i=1}^3 |a_i|^{2-d_i} \right) d^\times a dt.$$

Here

$$(5.9) \quad m(t, a) := \max(|ta_1 a_2 a_3|, |a_1|, |a_2|, |a_3|, |a_1^{-1} a_2 a_3|, |a_2^{-1} a_1 a_3|, |a_3^{-1} a_1 a_2|)$$

(see the proof of [14, Proposition 7.1] for more details). Taking N sufficiently large and applying Lemma 9.2 in the special case $D = \mathrm{Id}$, $r = e_i = 0$ we deduce that the linear form $f \mapsto I(f)(y)$ is continuous for every $y \in Y^{\mathrm{sm}}(F)$. The kernel in the statement of the lemma is the intersection of the kernels of these continuous linear forms. \square

We endow $\mathcal{S}(Y(F)) = \mathcal{S}(X(F) \times V(F)) / \ker I$ with the quotient topology (which is Fréchet).

In this Archimedean setting, there is a family of seminorms $\{\nu\}$ such that $\mathcal{S}(X(F) \times V(F))$ is the set of smooth functions $f : X^\circ(F) \times V(F) \rightarrow \mathbb{C}$ satisfying $\nu(f) < \infty$ for all ν . The seminorms ν we have in mind are tensor products of the seminorms (3.10) and the usual seminorms on $\mathcal{S}(V(F))$. Since $\mathcal{S}(Y(F))$ is a topological quotient space of $\mathcal{S}(X(F) \times V(F))$, it is then also a space of smooth functions on $Y^\circ(F)$ on which a family of seminorms are finite.

The integrals $I(f)$ depend on the choice of additive character ψ used to define the Weil representation ρ_ψ . We write $I_\psi(f)$ for $I(f)$ defined using the Weil representation ρ_ψ . Let $\gamma(\mathcal{Q}, \psi) := \prod_{i=1}^3 \gamma(\mathcal{Q}_i, \psi)$ be the product of the Weil indices.

LEMMA 5.2. — *Let $c \in F^\times$ and $\psi_c(x) := \psi(cx)$. Then*

$$\begin{aligned} & I_{\psi_c}(f_1 \otimes f_2)(y) \\ &= \frac{\gamma(\mathcal{Q}, \psi_c)}{\gamma(\mathcal{Q}, \psi)} |c|^{-2 + \sum_{i=1}^3 d_i/2} I_\psi \left(L(m(c^{-1})) R \begin{pmatrix} cI_3 & \\ & I_3 \end{pmatrix} f_1 \otimes f_2 \right) (cy). \end{aligned}$$

In particular, the Schwartz space $\mathcal{S}(Y(F))$ is independent of the choice of ψ .

Proof. — Let $B_2 < \mathrm{SL}_2$ be the Borel subgroup of upper triangular matrices and let

$$w_0 = \left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right) \in \mathrm{SL}_2^3(F).$$

Since $N_2^3(F)w_0B_2^3(F)$ is dense in $\mathrm{SL}_2^3(F)$, we have

$$\begin{aligned} I_{\psi_c}(f_1 \otimes f_2)(y) &= \int_{F \times F^3 \times (F^\times)^3} f_1 \left(\gamma_b \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right) \\ &\quad \times \rho_{\psi_c} \left(\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right) f_2(y) \, dt \, dx \, d^\times a. \end{aligned}$$

Observe that

$$\begin{aligned} & \rho_{\psi_c} \left(\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right) f_2(y) \\ &= \psi(ct\mathcal{Q}(y)) \gamma(\mathcal{Q}, \psi_c) \int_{V(F)} \rho_{\psi_c} \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right) \\ &\quad \times f_2(u) \prod_{i=1}^3 \psi(cu_i^t J_i y_i) \, du_i \\ &= \psi(c^{-1}t\mathcal{Q}(cy)) \gamma(\mathcal{Q}, \psi_c) \\ &\quad \times \int_{V(F)} \rho_\psi \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} f_2(u) \prod_{i=1}^3 \psi(cx_i \mathcal{Q}_i(u_i)) \psi(u_i^t J_i cy_i) \, du_i \\ &= |c|^{\sum_{i=1}^3 d_i/2} \gamma(\mathcal{Q}, \psi_c) \gamma(\mathcal{Q}, \psi)^{-1} \\ &\quad \times \rho_\psi \left(\begin{pmatrix} 1 & c^{-1}t \\ & 1 \end{pmatrix} w_0 \begin{pmatrix} 1 & cx \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right) f_2(cy). \end{aligned}$$

Here J_i is the matrix of \mathcal{Q}_i . The factor of $|c|^{\sum_{i=1}^3 d_i/2}$ appears because we have to renormalize the self-dual Haar measures with respect to ψ_c so that they are self-dual with respect to ψ . Taking a change of variables $t \mapsto ct$, $x_i \mapsto c^{-1}x_i$, we see that $I_{\psi_c}(f_1 \otimes f_2)(y)$ is $|c|^{\sum_{i=1}^3 d_i/2} \gamma(\mathcal{Q}, \psi_c) \gamma(\mathcal{Q}, \psi)^{-1}$ times

$$\begin{aligned} & \int_{F \times F^3 \times (F^\times)^3} f_1 \left(\gamma_b \begin{pmatrix} 1 & ct \\ & 1 \end{pmatrix} w_0 \begin{pmatrix} 1 & c^{-1}x \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right) \\ & \quad \times \rho_\psi \left(\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right) f_2(cy) \frac{dt \, dx \, d^\times a}{|c|^2} \\ &= \int_{F \times F^3 \times (F^\times)^3} f_1 \left(\gamma_b \begin{pmatrix} 1 & \\ & c^{-1} \end{pmatrix} \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \begin{pmatrix} c & \\ & 1 \end{pmatrix} \right) \\ & \quad \times \rho_\psi \left(\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right) f_2(cy) \frac{dt \, dx \, d^\times a}{|c|^2} \\ &= |c|^{-2} I_\psi \left(L(m(c^{-1})) R \begin{pmatrix} cI_3 & \\ & I_3 \end{pmatrix} f_1 \otimes f_2 \right) (cy). \end{aligned}$$

The fact that the Schwartz space is preserved now follows from Lemma 3.3 and [14, Lemma 4.3]. \square

For F Archimedean or non-Archimedean, let

$$\mathcal{S} := \text{Im}(\mathcal{S}(V(F)) \longrightarrow C^\infty(Y^{\text{sm}}(F)))$$

where the implicit map is restriction of functions. We observe that $C_c^\infty(Y^{\text{sm}}(F)) < \mathcal{S}$. Moreover, we have the following result:

LEMMA 5.3. — *One has*

$$\begin{aligned} \mathcal{S} &= \text{Im} (I : C_c^\infty(\gamma_b G(F)) \otimes \mathcal{S}(V(F)) \longrightarrow C^\infty(Y^{\text{sm}}(F))) \\ &= \text{Im} (I : \mathcal{S}(\gamma_b G(F) \times V(F)) \longrightarrow C^\infty(Y^{\text{sm}}(F))), \end{aligned}$$

where the tensor product is algebraic. In particular, $\mathcal{S} < \mathcal{S}(Y(F))$.

Proof. — Let $\gamma_b G(F) \cong G_{\gamma_b}(F) \backslash G(F)$ be the orbit of γ_b in $X^\circ(F)$; it is open and of full measure in $X^\circ(F)$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{S}(G(F) \times V(F)) & \longrightarrow & \mathcal{S}(V(F)) \\ \downarrow & & \downarrow |_{Y^{\text{sm}}(F)} \\ \mathcal{S}(\gamma_b G(F) \times V(F)) & \xrightarrow{I} & \mathcal{S}(Y(F)) \end{array}$$

where the top horizontal arrow is the unique (continuous) linear map sending a pure tensor $\Phi \otimes f \in \mathcal{S}(G(F)) \otimes \mathcal{S}(V(F))$ to $\int_{G(F)} \Phi(g) \rho(g) f \, dg$ and

the left vertical arrow sends $f(g, v)$ to $\int_{G_{\gamma_b}(F)} f(ng, v) \, dn$. The right vertical arrow is given by restriction, and hence its image is \mathcal{S} by definition.

We claim the upper horizontal map is surjective. In the non-Archimedean case, this follows easily from the smoothness of the Weil representation. In the Archimedean case, it is a consequence of a well-known theorem of Dixmier–Malliavin, that the map is surjective even if the domain is restricted to $C_c^\infty(G(F)) \otimes \mathcal{S}(V(F))$. Therefore, I is surjective even if the domain is restricted to $C_c^\infty(\gamma_b G(F)) \otimes \mathcal{S}(V(F))$ and the lemma follows. \square

This lemma provides a robust supply of elements of the Schwartz space $\mathcal{S}(Y(F))$.

We now revert to the adelic setting, bearing in mind the conventions on Schwartz spaces explained in Section 2. Let F be a number field. The obvious global analogue of (5.1) yields a map $I : \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F)) \rightarrow C^\infty(Y^{\text{sm}}(\mathbb{A}_F))$ and we set

$$\mathcal{S}(Y(\mathbb{A}_F)) := \text{Im}(I : \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F)) \longrightarrow C^\infty(Y^{\text{sm}}(\mathbb{A}_F))).$$

To check that this is well-defined, one uses the computation of the *basic function* for $Y(F_v)$

$$b_{Y,v} := I(b_{X,v} \otimes \mathbb{1}_{V(\mathcal{O}_v)})$$

in Proposition 8.1 below. We define

$$\mathcal{S}(Y(F_\infty)) := \text{Im}(I : \mathcal{S}(X(F_\infty) \times V(F_\infty)) \longrightarrow C^\infty(Y^{\text{sm}}(F_\infty))).$$

The map I has closed kernel by a trivial modification of the proof of Lemma 5.1 and we give $\mathcal{S}(Y(F_\infty))$ the quotient topology. Hence $\mathcal{S}(Y(F_\infty))$ is the (completed) projective topological tensor product $\widehat{\otimes}_v \mathcal{S}(Y(F_v))$. We then have

$$\mathcal{S}(Y(\mathbb{A}_F)) = \widehat{\bigotimes}_{v|\infty} \mathcal{S}(Y(F_v)) \otimes \bigotimes_{v \nmid \infty} \mathcal{S}(Y(F_v))$$

where the restricted tensor product is taken with respect to the $b_{Y,v}$. Indeed, we have

$$\begin{aligned} \mathcal{S}(Y(\mathbb{A}_F)) &= I(\mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F))) \\ &= \bigcup_{\infty \subset S} I \left(\mathcal{S}(X(F_S) \times V(F_S)) \otimes \bigotimes_{v \notin S} (b_{X,v} \otimes \mathbb{1}_{V(\mathcal{O}_v)}) \right) \\ &= \bigcup_{\infty \subset S} \mathcal{S}(Y(F_S)) \otimes \bigotimes_{v \notin S} b_{Y,v} \\ &= \widehat{\bigotimes}_{v|\infty} \mathcal{S}(Y(F_v)) \otimes \bigotimes_{v \nmid \infty} \mathcal{S}(Y(F_v)). \end{aligned}$$

6. The summation formula

Let F be a number field. Our goal in this section is to prove our main summation formula, Theorem 1.4, modulo some convergence statements that we prove later in the paper. We require the following assumptions on $f = f_1 \otimes f_2 \in \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F))$: There are finite places v_1, v_2 of F (not necessarily distinct) such that

$$f_1 = f_{v_1} f_{v_2} f^{v_1 v_2} \text{ and } f_{v_1} \in C_c^\infty(X^\circ(F_{v_1})), \mathcal{F}_X(f_{v_2}) \in C_c^\infty(X^\circ(F_{v_2})), \\ \rho(g)f_2(v) = 0 \text{ for } v \notin V^\circ(F), \text{ for all } g \in G(\mathbb{A}_F).$$

We will also require that $\Phi \in \mathcal{S}(\mathbb{A}_F^2)$ satisfies $\widehat{\Phi}(0) \neq 0$, where

$$(6.1) \quad \widehat{\Phi}(x, y) = \int_{\mathbb{A}_F^2} \Phi(t_1, t_2) \psi(xt_1 + yt_2) dt_1 dt_2.$$

We prove Theorem 1.4 in this section assuming the absolute convergence statements given in Propositions 10.2 and 10.4. We will indicate precisely when these propositions are used below. After this section, much of the remainder of the paper is devoted to proving these convergence statements.

Computing formally one has

$$(6.2) \quad \int_{G(F) \backslash G(\mathbb{A}_F)} \sum_{\gamma \in X^\circ(F)} f_1(\gamma g) \Theta_{f_2}(g) dg \\ = \sum_{\gamma \in X^\circ(F)/G(F)} \int_{G_\gamma(F) \backslash G(\mathbb{A}_F)} f_1(\gamma g) \Theta_{f_2}(g) dg \\ = \sum_{\gamma \in X^\circ(F)/G(F)} \int_{G_\gamma(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(\gamma g) \int_{[G_\gamma]} \Theta_{f_2}(g_1 g) dg_1 dg.$$

The set $X^\circ(F)/G(F)$ has 5 elements represented by $\gamma_b, \gamma_i, 1 \leq i \leq 3$ and $\gamma_0 = \text{Id}$ in the notation of (4.12). The stabilizers are given explicitly by Lemma 4.2, and we will use this lemma without further comment below.

We start with the γ_b contribution. It is computed as in the proof of [14, Theorem 5.3]:

$$\int_{G_{\gamma_b}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(\gamma_b g) \int_{[G_{\gamma_b}]} \Theta_{f_2}(g_1 g) dg_1 dg = \sum_{\xi \in Y^{\text{sm}}(F)} I(f)(\xi).$$

Strictly speaking, the proof of [14, Theorem 5.3] assumed f_1 was finite under a maximal compact subgroup of $\text{Sp}_6(F_\infty)$, but the same proof is valid given our work in Section 3.

We now turn to the Id term. Using the definition of the Weil representation, we have that this term is

$$\begin{aligned} & \int_{G_{\text{Id}}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(g) \int_{[\mathbb{G}_m \times \mathbb{G}_m]} \sum_{\substack{\xi \in V(F) \\ \mathcal{Q}_1(\xi_1) = \mathcal{Q}_2(\xi_2) = \mathcal{Q}_3(\xi_3) = 0}} \\ & \quad \rho \left(\left(\begin{pmatrix} a_1 & \\ & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & \\ & a_2^{-1} \end{pmatrix}, \begin{pmatrix} (a_1 a_2)^{-1} & \\ & a_1 a_2 \end{pmatrix} \right) g \right) f_2(\xi) da_1^\times da_2^\times dg \\ & = \int_{G_{\text{Id}}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(g) \int_{\mathbb{A}_F^\times \times \mathbb{A}_F^\times} \sum_{\xi \in \tilde{Y}_0(F)/(F^\times)^2} \\ & \quad \rho \left(\left(\begin{pmatrix} a_1 & \\ & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & \\ & a_2^{-1} \end{pmatrix}, \begin{pmatrix} (a_1 a_2)^{-1} & \\ & a_1 a_2 \end{pmatrix} \right) g \right) f_2(\xi) da_1^\times da_2^\times dg. \end{aligned}$$

Here $(F^\times)^2$ acts as in (4.7). Thus using Lemma 4.1 we conclude that the above is equal to

$$\begin{aligned} & \int_{G_{\text{Id}}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(g) \int_{\mathbb{A}_F^\times \times \mathbb{A}_F^\times} \sum_{\xi \in Y_0(F)} \\ & \quad \rho \left(\left(\begin{pmatrix} a_1 & \\ & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & \\ & a_2^{-1} \end{pmatrix}, \begin{pmatrix} (a_1 a_2)^{-1} & \\ & a_1 a_2 \end{pmatrix} \right) g \right) f_2(\xi) da_1^\times da_2^\times dg \\ & = \int_{N_2^3(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(g) \sum_{\xi \in Y_0(F)} \rho(g) f_2(\xi) dg \\ & = \sum_{\xi \in Y_0(F)} I_0(f)(\xi). \end{aligned}$$

This formal computation is justified by Proposition 10.2.

We finally turn to the γ_i , $1 \leq i \leq 3$, terms. Let $\Phi \in \mathcal{S}(\mathbb{A}_F^2)$ be a function satisfying $\widehat{\Phi}(0) \neq 0$. We prove in Proposition 10.4 below that the sum

$$\sum_{\xi \in Y_i(F)} I_i(f \otimes \Phi)(\xi, s)$$

converges absolutely and defines a holomorphic function of s for $\text{Re}(s)$ sufficiently large. Moreover, it admits a meromorphic continuation to the s plane and its residue at $s = 1$ is

$$\frac{\widehat{\Phi}(0)}{\kappa_F} \int_{G_{\gamma_i}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(\gamma_i g) \int_{[G_{\gamma_i}]} \sum_{\xi \in V(F)} \rho(hg) f_2(\xi) dh dg.$$

where $\kappa_F := \frac{2}{\text{Vol}(F^\times \backslash (\mathbb{A}_F^\times)^1)}$.

Thus altogether we have shown that

$$\begin{aligned} \int_{G(F)\backslash G(\mathbb{A}_F)} \sum_{\gamma \in X^\circ(F)} f_1(\gamma g) \Theta_{f_2}(g) \, dg \\ = \sum_{\xi \in Y^{\text{sm}}(F)} I(f)(\xi) + \sum_{\xi \in Y_0(F)} I_0(f)(\xi) \\ + \frac{\kappa_F}{\widehat{\Phi}(0)} \text{Res}_{s=1} \sum_{i=1}^3 \sum_{\xi \in Y_i(F)} I_i(f \otimes \Phi)(\xi, s). \end{aligned}$$

On the other hand by Theorem 3.13

$$\begin{aligned} \int_{G(F)\backslash G(\mathbb{A}_F)} \sum_{\gamma \in X^\circ(F)} f_1(\gamma g) \Theta_{f_2}(g) \, dg \\ = \int_{G(F)\backslash G(\mathbb{A}_F)} \sum_{\gamma \in X^\circ(F)} \mathcal{F}_X(f_1)(\gamma g) \Theta_{f_2}(g) \, dg. \end{aligned}$$

Replacing f_1 by $\mathcal{F}_X(f_1)$ in the argument above we see that this is

$$\begin{aligned} \sum_{\xi \in Y^{\text{sm}}(F)} I(\mathcal{F}_X(f))(\xi) + \sum_{\xi \in Y_0(F)} I_0(\mathcal{F}_X(f))(\xi) \\ + \frac{\kappa_F}{\widehat{\Phi}(0)} \text{Res}_{s=1} \sum_{i=1}^3 \sum_{\xi \in Y_i(F)} I_i(\mathcal{F}_X(f) \otimes \Phi)(\xi, s). \end{aligned}$$

Thus assuming the absolute convergence statements in Propositions 10.2 and 10.4 we have proved Theorem 1.4.

7. Bounds on integrals in the non-Archimedean case

Throughout this section F is a non-Archimedean local field of characteristic zero. For $v_i \in V_i(F) = F^{d_i}$, we let

$$(7.1) \quad \text{ord}(v_i) \quad (\text{resp. } |v_i|)$$

be the minimum of the orders (resp. maximum of the norms) of the entries of v_i with respect to the standard basis on $V_i(F)$. Thus $|v_i| = q^{-\text{ord}(v_i)}$. We have natural induced bases on $V_i(F) \otimes V_j(F)$ and $V_1(F) \otimes V_2(F) \otimes V_3(F)$ and we define $|v_i \otimes v_j|$, etc., similarly.

Fix functions

$$(f = f_1 \otimes f_2, \Phi) \in \mathcal{S}(X(F) \times V(F)) \times \mathcal{S}(F^2).$$

We bound the integrals attached to these functions that appeared in the proof of Theorem 1.4. These bounds will be used to deduce the absolute convergence statements of Propositions 10.2 and 10.4 below. All implicit constants in this section are allowed to depend on $f \otimes \Phi$.

First we pause to justify an assertion made in Section 5.1:

PROPOSITION 7.1. — *We have $\mathcal{S}(Y(F)) < C^\infty(Y^{\text{sm}}(F))$.*

Proof. — Fix $v = (v_1, v_2, v_3) \in Y^{\text{sm}}(F)$ and let $v' \in Y^{\text{sm}}(F)$. By symmetry we can assume that $|v_2||v_3| \neq 0$. It suffices to show

$$\int_{G_{\gamma_b}(F) \backslash G(F)} |f_1(\gamma_b g)| |\rho(g)f_2(v) - \rho(g)f_2(v')| dg$$

is zero for $|v - v'|$ sufficiently small. We can choose $\kappa_v \in \mathbb{R}_{>0}$ such that if $|v - v'| < \kappa_v$ then $|v'_i| = |v_i|$ for $2 \leq i \leq 3$, and $|v_1| = |v'_1|$ if $v_1 \neq 0$. For the remainder of the proof we assume $|v - v'| < \kappa_v$. By the Cauchy-Schwarz inequality and Lemma 3.5, the integral above is bounded by

$$\|f_1\|_2 \left(\int_{G_{\gamma_b}(F) \backslash G(F)} \mathbb{1}_{\geq c}(\gamma_b g) |\rho(g)f_2(v) - \rho(g)f_2(v')|^2 dg \right)^{1/2}$$

for some $c \in \mathbb{Z}$.

By the Iwasawa decomposition, it suffices to show that for all $c \in \mathbb{Z}$ the integral

$$(7.2) \quad \int_{m(t,a) \leq q^{-c}} \int_{K^3} |\rho(k)f_2(a^{-1}v) - \psi(-t\mathcal{Q}(v) + t\mathcal{Q}(v'))\rho(k)f_2(a^{-1}v')|^2 \\ \times dk \frac{d^\times a dt}{\prod_{i=1}^3 |a_i|^{d_i-2}}$$

is zero for $|v - v'|$ sufficiently small, where $m(t, a)$ is defined as in (5.9). The integral (7.2) is supported in the set of a_i such that $|v_i|q^{-N} \leq |a_i| \leq q^{-c}$ for each i for some N depending on f_2 . Since $m(t, a) \leq q^{-c}$, we have additionally $q^{-2N+c}|v_2||v_3| \leq q^c|a_2||a_3| \leq |a_1| \leq q^{-c}$. We have assumed that $|v_2||v_3| \neq 0$; hence the support of the integral, as a function of a , lies in a compact subset of $(F^\times)^3$ independent of v' (since $|v - v'| < \kappa_v$). Thus the integral over t has support in a set that is independent of v' . In particular, if $|v - v'|$ is sufficiently small, then $\psi(-t\mathcal{Q}(v) + t\mathcal{Q}(v')) = 1$, so (7.2) becomes

$$\int_{m(t,a) \leq q^{-c}} \left(\int_{K^3} |\rho(k)f_2(a^{-1}v) - \rho(k)f_2(a^{-1}v')|^2 dk \right) \left(\prod_{i=1}^3 |a_i|^{2-d_i} \right) d^\times a dt.$$

Since the vector space $\langle \rho(k)f_2 \rangle_{k \in K}$ is finite dimensional, and the integral over a is supported in a compact set independent of v' (since $|v - v'| < \kappa_v$), for v' close enough to v , the integral above vanishes. \square

PROPOSITION 7.2. — For $v = (v_1, v_2, v_3) \in V^\circ(F)$, one has

$$\int_{N_2^3(F) \backslash G(F)} |f_1(g)\rho(g)f_2(v)| dg \ll \prod_{i=1}^3 |v_i|^{-d_i/2}.$$

The integral is supported in the set of v satisfying $|v_1 \otimes v_2 \otimes v_3| \ll 1$. The function $I_0(f)(v)$ satisfies the same bounds on its magnitude and support.

Proof. — We decompose the Haar measure using the Iwasawa decomposition to see that the integral in the proposition is equal to

$$\begin{aligned} (7.3) \quad & \int_{N_2^3(F) \backslash G(F)} |f_1(g)\rho(g)f_2(v)| dg \\ &= \int_{(F^\times)^3 \times K^3} \left| f_1 \left(\begin{pmatrix} a^{-1} & \\ & a \end{pmatrix} k \right) \rho \left(\begin{pmatrix} a^{-1} & \\ & a \end{pmatrix} k \right) f_2(v) \right| \left(\prod_{i=1}^3 |a_i|^2 d^\times a_i \right) dk \\ &= \int_{(F^\times)^3 \times K^3} \left| f_1 \left(\begin{pmatrix} a^{-1} & \\ & a \end{pmatrix} k \right) \rho(k) f_2(a^{-1}v) \right| \left(\prod_{i=1}^3 |a_i|^{2-d_i/2} d^\times a_i \right) dk. \end{aligned}$$

Now

$$\left| \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix} k \right| = |a_1 a_2 a_3|.$$

By Lemma 3.5, (7.3) is bounded by a constant times

$$(7.4) \quad \int_{|a_1 a_2 a_3| \ll 1} \tilde{f}_2(a^{-1}v) \prod_{i=1}^3 |a_i|^{-d_i/2} d^\times a_i,$$

where

$$(7.5) \quad \tilde{f}_2(v) := \int_{K^3} |\rho(k)f_2(v)| dk.$$

Since \tilde{f}_2 is compactly supported, we have that $|v_i| \ll_{f_2} |a_i|$ for $1 \leq i \leq 3$. Therefore (7.4) is bounded by a constant times

$$\prod_{i=1}^3 \int_{|v_i| \ll |a_i|} |a_i|^{-d_i/2} d^\times a_i \ll \prod_{i=1}^3 |v_i|^{-d_i/2}.$$

Moreover, the support of (7.4) as a function of v satisfies $|v_1||v_2||v_3| \ll |a_1 a_2 a_3| \ll 1$, as claimed. \square

PROPOSITION 7.3. — For $r > 0$, as a function of $v \in V^\circ(F)$, the integral

$$\int_{(N_2(F) \times \Delta_1(\mathrm{SL}_2(F)) \backslash G(F))} |f_1(\gamma_1 g)| \int_{N_2(F) \backslash \mathrm{SL}_2(F)} \int_{F^\times} \left| \rho \left(\left(I_2, h, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} h \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right) g \right) f_2(v) \Phi(x(0, 1) h p_1(g)) \right| |x|^{2r} d^\times x dh dg$$

has support in $|v_1| \ll 1$. It is bounded by a constant times

$$C'^r \zeta(2r) \zeta(2r + d_2/2 - 1) |v_1|^{-d_1/2} |v_3|^{1-d_3/2} \max(|v_1| |v_3|, |v_2|)^{1-d_2/2-2r} \times (C + \max(0, \mathrm{ord}(v_1 \otimes v_3)) - \mathrm{ord}(v_2)) + \zeta(2r + d_2/2 - 1).$$

for some constant $C, C' > 0$. If $r = \mathrm{Re}(s)$, the function $I_1(f \otimes \Phi)(v, s)$ satisfies the same bounds on its magnitude and support.

Proof. — The integral in the proposition is equal to

$$\int_{N_2(F) \backslash \mathrm{SL}_2(F)} \int_{N_2(F) \backslash \mathrm{SL}_2(F) \times \mathrm{SL}_2(F)} |f_1(\gamma_1(g_1, g_2, I_2))| \times \int_{F^\times} \left| \rho \left(\left(g_1, h g_2, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} h \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right) \right) f_2(v) \right| \times |\Phi|(x(0, 1) h g_2) |x|^{2r} d^\times x dg_1 dg_2 dh.$$

We change variables $g_2 \mapsto h^{-1}g_2$ to see that this is

$$\int_{N_2(F) \backslash \mathrm{SL}_2(F)} \int_{N_2(F) \backslash \mathrm{SL}_2(F) \times \mathrm{SL}_2(F)} |f_1(\gamma_1(g_1, h^{-1}g_2, I_2))| \times \int_{F^\times} \left| \rho \left(\left(g_1, g_2, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} h \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right) \right) f_2(v) \Phi(x(0, 1)g_2) \right| \times |x|^{2r} d^\times x dg_1 dg_2 dh.$$

Since $\Delta_1(\mathrm{SL}_2(F))$ is in the stabilizer of γ_1 , this is

$$\int_{N_2(F) \backslash \mathrm{SL}_2(F)} \int_{N_2(F) \backslash \mathrm{SL}_2(F) \times \mathrm{SL}_2(F)} \left| f_1 \left(\gamma_1(g_1, g_2, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} h \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right) \right| \times \int_{F^\times} \left| \rho \left(\left(g_1, g_2, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} h \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right) \right) f_2(v) \Phi(x(0, 1)g_2) \right| \times |x|^{2r} d^\times x dg_1 dg_2 dh.$$

Now decomposing the Haar measure using the Iwasawa decomposition, we see that this is

$$\int_{(F^\times)^3 \times F \times K^3 \times F^\times} \left| f_1 \left(\gamma_1 \left(\begin{pmatrix} a_1^{-1} & \\ & a_1 \end{pmatrix} k_1, \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \begin{pmatrix} a_2^{-1} & \\ & a_2 \end{pmatrix} k_2, \begin{pmatrix} a_3^{-1} & \\ & a_3 \end{pmatrix} k_3 \right) \right) \right| \\ \times \left| \rho \left(\begin{pmatrix} a_1^{-1} & \\ & a_1 \end{pmatrix} k_1, \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \begin{pmatrix} a_2^{-1} & \\ & a_2 \end{pmatrix} k_2, \begin{pmatrix} a_3^{-1} & \\ & a_3 \end{pmatrix} k_3 \right) f_2(v) \Phi((0, xa_2)k_2) \right| \\ \times |x|^{2r} |a_1 a_2 a_3|^2 d^\times x dt da_1^\times da_2^\times da_3^\times dk_1 dk_2 dk_3.$$

We have

$$\left| \gamma_1 \left(\begin{pmatrix} a_1^{-1} & \\ & a_1 \end{pmatrix} k_1, \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \begin{pmatrix} a_2^{-1} & \\ & a_2 \end{pmatrix} k_2, \begin{pmatrix} a_3^{-1} & \\ & a_3 \end{pmatrix} k_3 \right) \right| \\ = \max(|a_1|, |a_1 a_2 a_3^{-1}|, |a_1 a_2^{-1} a_3|, |a_1 a_2 a_3 t|) \\ =: m'(t, a).$$

Taking a change of variable $x \mapsto xa_2^{-1}$, by Lemma 3.5 the integral above is bounded by a constant times

$$q^{2nr} \zeta(2r) \int_{\substack{(F^\times)^3 \times F \\ m'(t, a) \ll 1}} m'(t, a)^{-2} \tilde{f}_2(a^{-1}v) dt |a_2|^{-2r} \prod_{i=1}^3 |a_i|^{2-d_i/2} d^\times a_i$$

for some $n \in \mathbb{Z}$, where \tilde{f}_2 is the nonnegative function defined in (7.5). For some $N \in \mathbb{Z}_{\geq 0}$ sufficiently large, we can write the integral here as

$$\sum_{k=-N}^{\infty} q^{2k} \int \tilde{f}_2(a^{-1}v) dt |a_2|^{-2r} \prod_{i=1}^3 |a_i|^{2-d_i/2} d^\times a_i$$

where the integral is over t, a such that $m'(t, a) = q^{-k}$. This is bounded by

$$\sum_{k=-N}^{\infty} \int q^k \tilde{f}_2(a^{-1}v) |a_2|^{-2r} \prod_{i=1}^3 |a_i|^{1-d_i/2} d^\times a_i,$$

where the integral is now over a such that

$$m'(a) := \max(|a_1|, |a_1 a_2 a_3^{-1}|, |a_1 a_2^{-1} a_3|) \leq q^{-k}.$$

Taking a change of variables $a_1 \mapsto \varpi^k a_1$, one arrives at

$$(7.6) \quad \sum_{k=-N}^{\infty} \int q^{k d_1/2} \tilde{f}_2 \left(\frac{v_1}{\varpi^k a_1}, \frac{v_2}{a_2}, \frac{v_3}{a_3} \right) |a_2|^{-2r} \prod_{i=1}^3 |a_i|^{1-d_i/2} d^\times a_i$$

where the integral is now over a_1, a_2, a_3 such that

$$1 \geq \max(|a_1|, |a_1 a_2 a_3^{-1}|, |a_1 a_2^{-1} a_3|).$$

The bound on the support as a function of v_1 is now obvious. We also deduce that if a is in the support of the integral for a given v , then

$$|v_3| \ll |a_3| \leq |a_1|^{-1}|a_2| \ll |v_1|^{-1}|a_2|.$$

Thus for some $C, C' > 0$ depending on f_2 , (7.6) is bounded by a constant times

$$\begin{aligned} & |v_1|^{-d_1/2} \int_{|v_3| \ll |a_3| \ll |v_1|^{-1}|a_2|, |v_2| \ll |a_2|} |a_3|^{1-d_3/2} |a_2|^{1-d_2/2-2r} d^\times a_3 d^\times a_2 \\ & \ll_{d_3} |v_1|^{-d_1/2} |v_3|^{1-d_3/2} \\ & \quad \times \int_{\max(|v_1||v_3|, |v_2|) \ll |a_2|} (C + \text{ord}(v_1 \otimes v_3) - \text{ord}(a_2)) |a_2|^{1-d_2/2-2r} d^\times a_2 \\ & \leq |v_1|^{-d_1/2} |v_3|^{1-d_3/2} C'^{1-d_2/2-2r} \max(|v_1||v_3|, |v_2|)^{1-d_2/2-2r} \\ & \quad \times \int_{1 \leq |a_2|} (C + \text{ord}(v_1 \otimes v_3) - \min(\text{ord}(v_1 \otimes v_3), \text{ord}(v_2)) - \text{ord}(a_2)) \\ & \quad \times |a_2|^{1-d_2/2-2r} d^\times a_2. \end{aligned}$$

The integral over a_2 is bounded above by

$$\zeta(2r + d_2/2 - 1) (C + \max(0, \text{ord}(v_1 \otimes v_3) - \text{ord}(v_2)) + \zeta(2r + d_2/2 - 1)).$$

The proposition follows. \square

8. The unramified calculation

For this section, F is a local field unramified over \mathbb{Q}_p and $\psi : F \rightarrow \mathbb{C}^\times$ is an unramified nontrivial character. Let

$$(8.1) \quad \chi_{\mathcal{Q}}(a_1, a_2, a_3) := \chi_{\mathcal{Q}_1}(a_1) \chi_{\mathcal{Q}_2}(a_2) \chi_{\mathcal{Q}_3}(a_3)$$

where $\chi_{\mathcal{Q}_i}$ is the (quadratic) character attached to \mathcal{Q}_i as in [14, Section 3.1]. We assume that $\chi_{\mathcal{Q}}$ is unramified and $\mathbb{1}_{V(\mathcal{O})}$ is fixed under $\rho(K^3)$ where $K = \text{SL}_2(\mathcal{O})$. Recall $\mathbb{1}_c$ defined in (3.18) and the basic function

$$b_X := \sum_{j,k=0}^{\infty} q^{2j} \mathbb{1}_{k+2j}.$$

In this section we give formulae for the unramified functions $I(b_X \otimes \mathbb{1}_{V(\mathcal{O})})(v)$, $I_0(b_X \otimes \mathbb{1}_{V(\mathcal{O})})(v)$, $I_i(b_X \otimes \mathbb{1}_{V(\mathcal{O})} \otimes \mathbb{1}_{\mathcal{O}^2})(v, s)$ for $1 \leq i \leq 3$.

8.1. The open orbit

For the reader's convenience, we state the formula for

$$(8.2) \quad b_Y := I(b_X \otimes \mathbb{1}_{V(\mathcal{O})})$$

given by [14, Proposition 6.3]:

PROPOSITION 8.1. — For $v \in Y^{\text{sm}}(F)$, $b_Y(v)$ equals

$$\sum_{j=0}^{\infty} \int \mathbb{1}_{\mathcal{O}} \left(\frac{\mathcal{Q}(v)}{a_1 a_2 a_3 \varpi^{4j}} \right) \mathbb{1}_{V(\mathcal{O})} \left(\frac{v}{a \varpi^{2j}} \right) \chi_{\mathcal{Q}}(a) \prod_{i=1}^3 \left(\frac{|a_i|}{q^{2j}} \right)^{1-d_i/2} d^\times a$$

where the integral is over $a_1, a_2, a_3 \in \mathcal{O}$ satisfying

$$\max(|a_1^{-1} a_2 a_3|, |a_2^{-1} a_1 a_3|, |a_3^{-1} a_1 a_2|) \leq 1.$$

One can also write the basic function $b_Y(v)$ as $\mathbb{1}_{Y(\mathcal{O})}(v)$ times

$$\sum_{\substack{0 \leq k_i \leq \text{ord}(v_i) \\ k_i \leq k_{i+1} + k_{i-1} \\ k_1 + k_2 + k_3 \leq \text{ord} \mathcal{Q}(v)}} \frac{1 - q \left(\min_i \left(\left\lfloor \frac{\text{ord} \mathcal{Q}(v) - \sum_{i=1}^3 k_i}{4} \right\rfloor, \left\lfloor \frac{\text{ord}(v_i) - k_i}{2} \right\rfloor \right) + 1 \right) \sum_{i=1}^3 (d_i - 2)}{(1 - q^{d_1 + d_2 + d_3 - 6}) \prod_{i=1}^3 \chi_{\mathcal{Q}_i}(\varpi^{k_i})} \times q^{\sum_{i=1}^3 k_i (d_i/2 - 1)}.$$

8.2. The identity orbit

By a minor modification of the proof of Proposition 7.2 above, we obtain the following proposition:

PROPOSITION 8.2. — Suppose $v = (v_1, v_2, v_3) \in \widetilde{Y}_0(F)$. Assume moreover that $|v_1| = |v_2| = 1$. One has

$$I_0(b_X \otimes \mathbb{1}_{V(\mathcal{O})})(v) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} q^{2j} \int \mathbb{1}_{V(\mathcal{O})}(a^{-1}v) \chi_{\mathcal{Q}}(a) \prod_{i=1}^3 |a_i|^{2-d_i/2} d^\times a_i$$

where the integral is over those $a_1^{-1}, a_2^{-1}, a_3 \in \mathcal{O}$ such that $|a_1 a_2 a_3| = q^{-k-2j}$. As a function of v_3 , the integral is supported in $V_3(\mathcal{O})$. Let $\epsilon > 0$. For $v \in V^\circ(F)$ with $|v_1| = |v_2| = 1$, we have that

$$\int_{N_2^3(F) \backslash G(F)} |b_X(g) \rho(g) \mathbb{1}_{V(\mathcal{O})}(v)| dg \leq C |v_3|^{-d_3/2 - \epsilon} \mathbb{1}_{V_3(\mathcal{O})}(v_3)$$

for some constant $C > 0$ depending on ϵ , which equals 1 for q sufficiently large.

Evaluating at $v \in \widetilde{Y}_0(F)$ with $|v_1| = |v_2| = 1$, one can rewrite $I_0(b_X \otimes \mathbb{1}_{V(\mathcal{O})})(v)$ as

$$\mathbb{1}_{V_3(\mathcal{O})}(v_3) \sum_{\substack{0 \leq k_i \leq \text{ord}(v_3) \\ k_1 + k_2 \leq k_3}} \frac{1 - q^{2+2\lfloor \frac{k_3 - k_1 - k_2}{2} \rfloor}}{1 - q^2} \times q^{k_1(2-d_1/2) + k_2(2-d_2/2) - k_3(2-d_3/2)} \prod_{i=1}^3 \chi_{\mathcal{Q}_i}(\varpi^{k_i}).$$

8.3. The other orbits

The following assertions can be proved by an easy refinement of the argument proving Proposition 7.3:

PROPOSITION 8.3. — Suppose $v = (v_1, v_2, v_3) \in \widetilde{Y}_1(F)$. For $\text{Re}(s) > 0$ one has

$$\begin{aligned} I_1(b_X \otimes \mathbb{1}_{V(\mathcal{O})} \otimes \mathbb{1}_{\mathcal{O}^2})(v, s) &= \zeta(2s) \sum_{j=0}^{\infty} \int \mathbb{1}_{\mathcal{O}} \left(\frac{\mathcal{Q}_2(v_2)}{a_1 a_2 a_3} \right) \mathbb{1}_{V(\mathcal{O})} \left(\frac{v_1}{\varpi^{2j} a_1}, \frac{v_2}{a_2}, \frac{v_3}{a_3} \right) \\ &\quad \times \frac{\chi_{\mathcal{Q}}(a)}{|a_2|^{2s} q^{2j(1-d_1/2)}} \prod_{i=1}^3 |a_i|^{1-d_i/2} d^\times a_i \end{aligned}$$

where the integral is over $a_1 \in F^\times \cap \mathcal{O}, a_2, a_3 \in F^\times$ such that $|a_1|^{-1} \geq \max(|a_2 a_3^{-1}|, |a_2^{-1} a_3|)$.

One can alternatively write $I_1(b_X \otimes \mathbb{1}_{V(\mathcal{O})} \otimes \mathbb{1}_{\mathcal{O}^2})(v, s)$ as

$$\begin{aligned} \mathbb{1}_{V_1(\mathcal{O})}(v_1) \zeta(2s) \sum_{\substack{k_i \leq \text{ord}(v_i) \\ k_1 \geq |k_2 - k_3| \\ k_1 + k_2 + k_3 \leq \text{ord}(\mathcal{Q}_2(v_2))}} q^{2k_2 s + \sum_{i=1}^3 k_i (d_i/2 - 1)} \\ \times \left(\sum_{j=0}^{\lfloor \frac{\text{ord}(v_1) - k_1}{2} \rfloor} q^{j(d_1 - 2)} \right) \prod_{i=1}^3 \chi_{\mathcal{Q}_i}(\varpi^{k_i}). \end{aligned}$$

LEMMA 8.4. — For $v = (v_1, v_2, v_3) \in V^\circ(F)$ and $r > 0$, the integral

$$\begin{aligned} & \int_{(N_2(F) \times \Delta_1(\mathrm{SL}_2(F))) \backslash G(F)} |b_X(\gamma_1 g)| \\ & \quad \times \int_{N_2(F) \backslash \mathrm{SL}_2(F)} |\rho \left(\left(I_2, h, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} h \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right) g \right) \mathbb{1}_{V(\mathcal{O})}(v)| \\ & \quad \times \mathbb{1}_{\mathcal{O}^2}((0, x) h p_1(g)) |x|^{2r} d^\times x dh dg \end{aligned}$$

vanishes unless $v_1 \in V_1(\mathcal{O})$. It is bounded by

$$\begin{aligned} & C \zeta(2r) \zeta(2r + d_2/2 - 1) |v_1|^{-d_1/2} |v_3|^{1-d_3/2} \max(|v_1 \otimes v_3|, |v_2|)^{1-d_2/2-2r} \\ & \quad \times (\max(0, \mathrm{ord}(v_1 \otimes v_3) - \mathrm{ord}(v_2)) + \zeta(2r + d_2/2 - 1)) \end{aligned}$$

for some constant $C > 0$ which equals 1 for q sufficiently large. Thus if $\mathrm{Re}(s) = r$, the function $I_1(b_X \otimes \mathbb{1}_{V(\mathcal{O})} \otimes \mathbb{1}_{\mathcal{O}^2})(v, s)$ admits the same bounds on its magnitude and support.

An expression for the integrals I_2 and I_3 and corresponding bounds and supports can be obtained by symmetry.

9. Bounds on integrals in the Archimedean case

In this section F is an Archimedean local field. We estimate the local integrals defined in Section 5. The bounds obtained in this section will be used to prove Propositions 10.2 and 10.4, the absolute convergence statements used in the proof of Theorem 1.4. As usual, the bounds in the Archimedean case are slightly harder to prove than in the non-Archimedean case, but the basic outline of the proofs is the same. We let

$$(9.1) \quad |(a_1, \dots, a_{d_i})| := \max\{|a_j| : 1 \leq j \leq d_i\}$$

for $(a_1, \dots, a_{d_i}) \in V_i(F)$. We moreover fix

$$(f_1 \otimes f_2, \Phi) \in \mathcal{S}(X(F) \times V(F)) \times \mathcal{S}(F^2).$$

We will bound integrals involving the pure tensor $f_1 \otimes f_2$ in this section. In each case, the bounds will be continuous in f_1 and f_2 with respect to the Fréchet topologies on $\mathcal{S}(X(F))$ and $\mathcal{S}(V(F))$. Thus the bounds extend by continuity to all $f \in \mathcal{S}(X(F) \times V(F))$.

The following is a rephrasing of [14, Lemma 8.1]:

LEMMA 9.1. — *Let $A, B \in \mathbb{R}_{>0}$, $C \in \mathbb{R}_{\geq 0}$ and let $x \in F^\times$. Assume $A > B$ and $A \neq B + C$. One has*

$$\int_{F^\times} \max(|a^{-1}x|, 1)^{-A} |a|^{-B} \max(|a|, 1)^{-C} da^\times \ll_{A,B,C} \max(|x|, 1)^{-\min(A-B,C)} |x|^{-B}.$$

This will be used several times below.

9.1. The open orbit

Recall that $V'(F) \subset V(F)$ is the subset of vectors (v_1, v_2, v_3) such that no two v_i are zero. In order to bound the function $I(f_1 \otimes f_2) \in \mathcal{S}(Y(F))$ and its various derivatives it is convenient to first prove the following bound:

LEMMA 9.2. — *Given $r, e_i, N \in \mathbb{R}_{\geq 0}$, $D \in U(\text{Lie}(V(F)))$, let $M : (F^\times)^3 \times F \times K^3 \times V(F) \rightarrow \mathbb{R}$ be the function*

$$(9.2) \quad M(a, t, k, v) := \max(m(t, a), 1)^{-2N} |t|^r |D\rho(k) f_2|^2 (a^{-1}v) \left(\prod_{i=1}^3 |a_i|^{2-d_i-e_i} \right),$$

where $m(t, a)$ is defined as (5.9). For $v \in V'(F)$, there is a compact neighborhood U of v , and a continuous integrable function M' on $(F^\times)^3 \times F \times K^3$ such that for all $v' \in U$

$$M(a, t, k, v') \leq M'(a, t, k).$$

Moreover, given $N_i \in \mathbb{Z}_{\geq 0}$ there exists a continuous seminorm ν' on $\mathcal{S}(V(F))$ such that

$$\int_{(F^\times)^3 \times F \times K^3} M(a, t, k, v) dk d^\times a dt \leq \nu'(f_2)^2 \begin{cases} \prod_{i=1}^3 \max(|v_i|, 1)^{-2N_i} |v_i|^{1-d_i-e_i-r} & \text{if } v \in V^\circ(F), \\ \prod_{i \neq j} \max(|v_i|, 1)^{-2N_i} |v_i|^{2-d_i-e_i-2r-d_j-e_j} & \text{if } v_j = 0, \end{cases}$$

provided $N \geq 5 \max_i \{N_i, d_i + e_i + r\}$.

This bound will be used in the proof of Proposition 9.3 below.

Proof. — By the continuity of the Weil representation and compactness of K , for any $C_1, C_2, C_3 \in \mathbb{Z}_{\geq 0}$, there exists a continuous seminorm ν_{D, C_1, C_2, C_3} on $\mathcal{S}(V(F))$ such that for all $(k, v) \in K^3 \times V(F)$ we have

$$|D\rho(k)f_2|(v) \leq \nu_{D, C_1, C_2, C_3}(f_2) \left(\prod_{i=1}^3 \max(|v_i|, 1)^{-C_i} \right).$$

Let U be a compact neighborhood of v such that for $v' \in U$, if $v_i \neq 0$ then $v'_i \neq 0$. Choose $v' \in U$ with minimum norm. Put

$$(9.3) \quad M'(a, t, k) := \nu_{D, C_1, C_2, C_3}(f_2)^2 \max(m(t, a), 1)^{-2N} |t|^r \\ \times \prod_{i=1}^3 \max(|a_i^{-1} v'_i|, 1)^{-2C_i} |a_i|^{2-d_i-e_i}.$$

Then $M(a, t, k, v) \leq M'(a, t, k)$ for all $v \in U$. Thus to prove the lemma it suffices to show that for all $v \in V'(F)$ one has

$$(9.4) \quad \int_{(F^\times)^3 \times F} \max(m(t, a), 1)^{-2N} |t|^r \\ \times \left(\prod_{i=1}^3 \max(|a_i^{-1} v_i|, 1)^{-2C_i} |a_i|^{2-d_i-e_i} d^\times a_i \right) dt \\ \leq \begin{cases} \prod_{i=1}^3 \max(|v_i|, 1)^{-2N_i} |v_i|^{1-d_i-e_i-r} & \text{if } v \in V^\circ(F), \\ \prod_{i \neq j} \max(|v_i|, 1)^{-2N_i} |v_i|^{2-d_i-e_i-2r-d_j-e_j} & \text{if } v_j = 0, \end{cases}$$

provided that $N \geq 5 \max_i \{N_i, d_i + e_i + r\}$. We break the integral into $m(t, a) \leq 1$ and $m(t, a) > 1$. Suppose $v \in V^\circ(F)$. In the range $m(t, a) \leq 1$, we have $|t| \leq |a_1 a_2 a_3|^{-1}$ and $|a_i| \leq 1$ for all i . Therefore the integral is bounded by

$$(9.5) \quad \int_{|a| \leq 1} \prod_{i=1}^3 \max(|a_i^{-1} v_i|, 1)^{-2C_i} |a_i|^{1-d_i-e_i-r} d^\times a \\ \leq \prod_{i=1}^3 \int_{F^\times} \max(|a_i|, 1)^{-2N_i} \max(|a_i^{-1} v_i|, 1)^{-2C_i} |a_i|^{1-d_i-e_i-r} d^\times a_i.$$

In the range $m(t, a) > 1$, applying the inequality

$$m(t, a)^4 \geq \max(|ta_1 a_2 a_3|, 1) \max(|a_1|, 1) \max(|a_2|, 1) \max(|a_3|, 1),$$

the contribution of this part of the integral is bounded by

$$\begin{aligned} & \int_{(F^\times)^3 \times F} \max(|ta_1a_2a_3|, 1)^{-N/2} |ta_1a_2a_3|^r \\ & \times \left(\prod_{i=1}^3 \max(|a_i|, 1)^{-N/2} \max(|a_i^{-1}v_i|, 1)^{-2C_i} |a_i|^{2-d_i-e_i-r} \right) d^\times a dt \\ & \ll_{N,r} \prod_{i=1}^3 \int_{F^\times} \max(|a_i|, 1)^{-N/2} \max(|a_i^{-1}v_i|, 1)^{-2C_i} |a_i|^{1-d_i-e_i-r} d^\times a_i, \end{aligned}$$

provided $N > 2r + 2$. Since $N \geq 4 \max_i N_i$ by assumption the integral is bounded by (9.5). The assertion then follows from Lemma 9.1 by setting $A = 2C_i, B = r + d_i + e_i - 1, C = 2N_i$ and choosing C_i so that $A - B > C$ for each i .

Now assume $v \in V'(F) - V^\circ(F)$. By symmetry we may assume $v_1 = 0$. Let

$$|a| := \max_{1 \leq i \leq 3} (|a_i|).$$

If $m(t, a) \leq 1$ then $|t| \leq |a_1a_2a_3|^{-1}, |a| \leq 1, |a_2a_3| \leq |a_1|$. Therefore the contribution of $|m(t, a)| \leq 1$ to the integral (9.4) is bounded by

$$\begin{aligned} (9.6) \quad & \int_{\substack{|a| \leq 1, \\ |a_2a_3| \leq |a_1|}} |a_1|^{1-d_1-e_1-r} \prod_{i=2}^3 \max(|a_i^{-1}v_i|, 1)^{-2C_i} |a_i|^{1-d_i-e_i-r} d^\times a \\ & \leq \prod_{i=2}^3 \int_{F^\times} \max(|a_i|, 1)^{-2N_i} \max(|a_i^{-1}v_i|, 1)^{-2C_i} |a_i|^{2-d_i-e_i-2r-d_1-e_1} d^\times a_i. \end{aligned}$$

For $m(t, a) > 1$ we have the inequality

$$m(t, a)^5 \geq \max(|ta_1a_2a_3|, 1) \left(\prod_{i=1}^3 \max(|a_i|, 1) \right) \max(|a_1^{-1}a_2a_3|, 1).$$

Thus the contribution of $m(t, a) > 1$ to (9.4) is bounded by a constant depending on N times

$$\begin{aligned} (9.7) \quad & \int_{(F^\times)^3} \max(|a_1^{-1}a_2a_3|, 1)^{-2N/5} \\ & \times \left(\prod_{i=1}^3 \max(|a_i|, 1)^{-2N/5} \max(|a_i^{-1}v_i|, 1)^{-2C_i} |a_i|^{1-d_i-e_i-r} \right) d^\times a \end{aligned}$$

since $N > 5r + 5$. The contribution of $|a_2 a_3| \leq |a_1|$ to (9.7) is dominated by (9.6) since $N \geq 5 \max_i N_i$. In the range $|a_2 a_3| \geq |a_1|$, one has that

$$\int_{|a_1| \leq |a_2 a_3|} \max(|a_1|, 1)^{-2N/5} |a_1|^{1-d_1-e_1-r+2N/5} d^\times a_1 \\ \ll_{N, e_1, r} \min(|a_2 a_3|, 1)^{1-d_1-e_1-r+2N/5}.$$

Since $2N/5 \geq d_1 + e_1 + r$, we deduce that the integral (9.7) is also dominated by (9.6). The assertion now follows from Lemma 9.1 by setting $A = 2C_i$, $B = 2r + d_1 + e_1 + d_i + e_i - 2$, $C = 2N_i$ and choosing C_i so that $A - B > C$ for $i = 2, 3$. \square

PROPOSITION 9.3. — We have $\mathcal{S}(Y(F)) < C^\infty(Y^{\text{sm}}(F))$. Moreover, for $f \in \mathcal{S}(Y(F))$ and $D \in U(\text{Lie}(V(F)))$,

$$|Df(v)| \left(\prod_{i=1}^3 \max(|v_i|, 1)^{N_i} |v_i|^{(d_i + [F:\mathbb{R}]^{-1} \deg D - 1)/2} \right)$$

is bounded on $Y^{\text{ani}}(F)$ for all $N_i \in \mathbb{Z}$.

Proof. — Let $v_0 \in Y^{\text{sm}}(F)$ and $D \in U(\text{Lie}(V(F)))$. Let $\Delta : F \rightarrow F^3$ be the diagonal embedding. Using the notation of Lemma 9.2 there is a neighborhood U of v_0 such that for $v \in U$ the expression

$$\left| f_1 \left(\gamma_b \begin{pmatrix} 1 & \Delta(t) \\ & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix} k \right) D\rho \left(\begin{pmatrix} 1 & \Delta(t) \\ & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix} k \right) f_2(v) \right|$$

is dominated by a finite sum of functions of the form

$$\left| f_1 \left(\gamma_b \begin{pmatrix} 1 & \Delta(t) \\ & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix} k \right) \right| \max(m(t, a), 1)^N M(a, t, k, v)^{1/2} |a|^{-1}$$

where $M(a, t, k, v)$ is defined using various parameters f_2 , e_i , r depending on D . We recall that

$$m(t, a) = \left| \gamma_b \begin{pmatrix} 1 & \Delta(t) \\ & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix} k \right|$$

by (5.9). Thus applying the Cauchy–Schwarz inequality we have

$$\begin{aligned} \int_{F \times (F^\times)^3 \times K} & \left| f_1 \left(\gamma_b \begin{pmatrix} 1 & \Delta(t) \\ & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix} k \right) \max(m(t, a), 1)^N \right| \\ & \times M(a, t, k, v)^{1/2} |a|^{-1} |a|^2 d^\times a dt dk \\ & \leq \left(\int_{G_{\gamma_b}(F) \backslash G(F)} |f_1(\gamma_b g) \max(|\gamma_b g|, 1)^N|^2 dg \right)^{1/2} \\ & \times \left(\int_{F \times (F^\times)^3 \times K} M(a, t, k, v) d^\times a dt dk \right)^{1/2}. \end{aligned}$$

The left integral converges by the argument in the proof of Lemma 5.1, and the right converges by Lemma 9.2. To obtain the bound in the lemma one simply keeps track of which parameters r and e_i are required in the argument above in terms of $\deg D$.

To prove the inclusion $\mathcal{S}(Y(F)) < C^\infty(Y^{\text{sm}}(F))$, we apply the Leibniz integral rule. To justify its application, we require a bound on $\int_{G_{\gamma_b}(F) \backslash G(F)} |f_1(\gamma_b g) D\rho(g) f_2(v)| dg$ that is uniform in a small neighborhood of a given $v \in Y^{\text{sm}}(F)$. Choose a compact neighborhood U of v as in the proof of Lemma 9.2. Then by Lemma 9.2 for $v \in U$ one has $M(a, t, k, v) \leq M'(a, t, k)$, defined as in (9.3). It suffices to show

$$\begin{aligned} \int_{F \times (F^\times)^3 \times K} & |f_1(\gamma_b \begin{pmatrix} 1 & \Delta(t) \\ & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix} k) \max(m(t, a), 1)^N| \\ & \times M'(a, t, k)^{1/2} |a|^{-1} |a|^2 d^\times a dt dk \end{aligned}$$

is finite. This follows from (9.4) and the argument above. □

Remark 9.4. — By mimicking the proof above one can also bound $Df(v)$ when $v_i = 0$ for some i .

9.2. The identity orbit

PROPOSITION 9.5. — *Let $v \in V^\circ(F)$. Given a positive integer N' and $\epsilon > 0$, there are continuous seminorms ν on $\mathcal{S}(X(F))$ and ν' (depending*

on N', ϵ) on $\mathcal{S}(V(F))$ such that one has the bound

$$\int_{N_2^3(F) \backslash G(F)} |f_1(g)\rho(g)f(v)| dg \leq \nu(f_1)\nu'(f_2) \max\{|v_1||v_2||v_3|, 1\}^{-N'} \prod_{i=1}^3 |v_i|^{-d_i/2-\epsilon}.$$

The function $I_0(f)(v)$ admits the same bound.

Proof. — By symmetry, we may assume $d_1 \geq d_2 \geq d_3$. Recall the seminorms $\nu_{D,N,\beta}$ mentioned in Lemma 3.5. Arguing as in the proof of Proposition 7.2, we see that the integral in the proposition is bounded by $\max(\nu_{1d,N,0}(f_1), \nu_{1d,0,0}(f_1))$ times

$$\int_{(F^\times)^3} |a_1 a_2 a_3|^{-2} \max(|a_1 a_2 a_3|, 1)^{-N} \tilde{f}_2(a^{-1}v) \prod_{i=1}^3 |a_i|^{2-d_i/2} d^\times a_i,$$

where \tilde{f}_2 is defined as in (7.5). By the continuity of Weil representation and compactness of K , for any $N_1, N_2, N_3 \in \mathbb{Z}_{\geq 0}$, there exists a continuous seminorm ν' depending on N_1, N_2, N_3 such that the integral above is bounded by $\nu'(f_2)$ times

$$\begin{aligned} & \int_{(F^\times)^3} \max(|a_1 a_2 a_3|, 1)^{-N} \prod_{i=1}^3 \max(|a_i^{-1} v_i|, 1)^{-N_i} |a_i|^{-d_i/2} da_i^\times \\ &= \int_{(F^\times)^3} \max(|a_1|, 1)^{-N} \max(|a_1^{-1}(a_2 a_3)v_1|, 1)^{-N_1} |a_1|^{-d_1/2} |a_2 a_3|^{d_1/2} \\ & \quad \times \prod_{i=2}^3 \max(|a_i^{-1} v_i|, 1)^{-N_i} |a_i|^{-d_i/2} da_i^\times. \end{aligned}$$

Here we have taken a change of variables $a_1 \mapsto (a_2 a_3)^{-1} a_1$. For the remainder of the proof all implicit constants are allowed to depend on N_1, N_2, N_3, N , and we assume

$$N_i - d_i/2 > N_{i-1} - d_{i-1}/2$$

for each i where $N_0 = d_0 = 0$.

Taking $N > N_1 + d_1/2$ and applying Lemma 9.1 with $A = N_1$, $B = d_1/2$ and $C = N$ to the a_1 integral we see that the above is bounded by

$$\begin{aligned}
 (9.8) \quad & \int_{(F^\times)^2} \max(|a_2 a_3 v_1|, 1)^{-N_1 + d_1/2} |a_2 a_3 v_1|^{-d_1/2} \\
 & \times \prod_{i=2}^3 \max(|a_i^{-1} v_i|, 1)^{-N_i} |a_i|^{(d_1 - d_i)/2} d^\times a_i \\
 & = |v_1|^{-d_1/2} \int_{(F^\times)^2} \max(|a_2 v_1|, 1)^{-N_1 + d_1/2} |a_2|^{-d_2/2} \\
 & \quad \times \max(|a_2^{-1} a_3 v_2|, 1)^{-N_2} \max(|a_3^{-1} v_3|, 1)^{-N_3} |a_3|^{(d_2 - d_3)/2} da_2^\times da_3^\times.
 \end{aligned}$$

Here we have taken a change of variables $a_2 \mapsto a_3^{-1} a_2$. The integral

$$(9.9) \quad \int_{F^\times} \max(|a_2^{-1} a_3 v_2|, 1)^{-N_2} \max(|a_3^{-1} v_3|, 1)^{-N_3} |a_3|^{(d_2 - d_3)/2} da_3^\times$$

breaks into the sum of four integrals

$$\begin{aligned}
 & \int_{|a_3| \leq \min\left(\frac{|a_2|}{|v_2|}, |v_3|\right)} + \int_{\min\left(\frac{|a_2|}{|v_2|}, |v_3|\right) < |a_3| \leq \frac{|a_2|}{|v_2|}} \\
 & \quad + \int_{\frac{|a_2|}{|v_2|} < |a_3| \leq \max\left(\frac{|a_2|}{|v_2|}, |v_3|\right)} + \int_{\max\left(\frac{|a_2|}{|v_2|}, |v_3|\right) \leq |a_3|}
 \end{aligned}$$

and this is bounded by a constant (depending only on N_2, N_3) times

$$\begin{aligned}
 & |v_3|^{-N_3} \min\left(\frac{|a_2|}{|v_2|}, |v_3|\right)^{N_3 + \frac{d_2 - d_3}{2}} + G_{d_2, d_3}(a_2, v_2, v_3) + \left(\frac{|a_2|}{|v_2|}\right)^{N_2} \\
 & \times \left(|v_3|^{-N_3} \left(\max\left(\frac{|a_2|}{|v_2|}, |v_3|\right)^{N_3 - N_2 + \frac{d_2 - d_3}{2}} - \left(\frac{|a_2|}{|v_2|}\right)^{N_3 - N_2 + \frac{d_2 - d_3}{2}} \right) \right. \\
 & \quad \left. + \max\left(\frac{|a_2|}{|v_2|}, |v_3|\right)^{-N_2 + \frac{d_2 - d_3}{2}} \right)
 \end{aligned}$$

where

$$G_{d_2, d_3}(a_2, v_2, v_3) = \begin{cases} \left(\frac{|a_2|}{|v_2|}\right)^{\frac{d_2 - d_3}{2}} - \min\left(\frac{|a_2|}{|v_2|}, |v_3|\right)^{\frac{d_2 - d_3}{2}} & \text{if } d_2 \neq d_3, \\ \log\left(\frac{|a_2|}{|v_2|}\right) - \log\min\left(\frac{|a_2|}{|v_2|}, |v_3|\right) & \text{if } d_2 = d_3. \end{cases}$$

Thus (9.9) is bounded by a constant times

$$F(a_2, v_2, v_3) := \begin{cases} |v_3|^{(d_2-d_3)/2} \left(\frac{|a_2|}{|v_2||v_3|}\right)^{N_2} & \text{if } |a_2| < |v_2||v_3|, \\ |v_3|^{(d_2-d_3)/2} \left(\frac{|a_2|}{|v_2||v_3|}\right)^{(d_2-d_3)/2} & \text{if } |a_2| \geq |v_2||v_3|, \\ 1 + \log\left(\frac{|a_2|}{|v_2||v_3|}\right) & \text{if } |a_2| \geq |v_2||v_3|, \\ & \text{if } d_2 = d_3. \end{cases}$$

Thus the original integral (9.8) is bounded by a constant times

$$\begin{aligned} & |v_1|^{-d_1/2} \int_{F^\times} \max(|a_2 v_1|, 1)^{-N_1+d_1/2} |a_2|^{-d_2/2} F(a_2, v_2, v_3) da_2^\times \\ &= \left(\prod_{i=1}^3 |v_i|^{-d_i/2} \right) \int_{F^\times} \max(|a_2|c, 1)^{-N_1+d_1/2} |a_2|^{-d_2/2} \\ & \quad \times \min(|a_2|, 1)^{N_2} F'(a_2) da_2^\times, \end{aligned}$$

where $c = |v_1||v_2||v_3|$, and

$$F'(a_2) := \begin{cases} \max(|a_2|, 1)^{(d_2-d_3)/2} & \text{if } d_2 \neq d_3, \\ \log(\max(|a_2|, 1)) + 1 & \text{if } d_2 = d_3. \end{cases}$$

Here we have changed variables $a_2 \mapsto a_2(|v_2||v_3|)$. The assertion of the proposition now follows from taking a change of variable $a_2 \mapsto a_2^{-1}$ and applying Lemma 9.1 with $A = N_1 - d_1/2$, $B = \epsilon < 1/2$, $C = N_2 - d_2/2 - \epsilon$. \square

9.3. The other orbits

PROPOSITION 9.6. — *Let $r = \operatorname{Re}(s) > 0$ and $N \in \mathbb{Z}_{>0}$, and assume $N > \max(2r + d_2/2 - 2, d_3/2 - 2)$. For $v \in V^\circ(F)$, there are continuous seminorms ν on $\mathcal{S}(X(F))$ and ν' on $\mathcal{S}(V(F))$, depending on N , such that*

$$\begin{aligned} & \int_{(N_2(F) \times \Delta_1(\operatorname{SL}_2(F))) \backslash G(F)} |f_1(\gamma_1 g)| \\ & \times \int_{N_2(F) \backslash \operatorname{SL}_2(F)} \int_{F^\times} \left| \rho \left(\left(I_2, h, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} h \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right) g \right) f_2(v) \right| \\ & \quad \times |\Phi((0, x)h p_1(g))| x|^{2r} d^\times x dh dg \\ & \leq \Psi(2r) \nu(f_1) \nu'(f_2) |v_1|^{-N-d_1/2} \max(|v_2|, |v_3|)^{-2r-d_2/2-d_3/2+2} \end{aligned}$$

where $\Psi : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ is an analytic function. The function $I_1(f \otimes \Phi)(v, s)$ admits the same bound.

Proof. — By Lemma 3.5 one has $|f_1(g)| \leq \nu_{\text{Id},N,0}(f_1)|g|^{-2-N}$ for any $N \geq 0$. Thus arguing as in Proposition 7.3, we see that the integral is bounded by

$$(9.10) \quad \nu_{\text{Id},N,0}(f_1) \int_{(F^\times)^3 \times F} m'(t, a)^{-2-N} \int_{K^3 \times F^\times} |\rho(k_1, k_2, k_3) f_2(a^{-1}v)| \\ \times |\Phi((0, x)k_2)| |x|^{2r} d^\times x dk_1 dk_2 dk_3 dt |a_2|^{-2r} \prod_{i=1}^3 |a_i|^{2-d_i/2} d^\times a_i,$$

where $m'(t, a) = \max(|a_1|, |a_1 a_2 a_3^{-1}|, |a_1 a_2^{-1} a_3|, |a_1 a_2 a_3 t|)$. For simplicity we assume $f_2 = \otimes_{i=1}^3 f_{2i}$. The general case merely requires more annoying notation. Applying the Cauchy–Schwarz inequality on the second copy of K , the inner integral is bounded by

$$\left(\prod_{i=1,3} \int_K |\rho_i(k_i) f_{2i}(a_i^{-1}v_i)| dk_i \right) \left(\int_K |\rho(k_2) f_{22}(a_2^{-1}v_2)|^2 dk_2 \right)^{1/2} \\ \times \int_{F^\times} \left(\int_K |\Phi((0, x)k_2)|^2 dk_2 \right)^{1/2} |x|^{2r} d^\times x.$$

The last factor is $\Psi(2r)$ for an appropriate analytic function $\Psi : \mathbb{R}_{>0} \rightarrow \mathbb{R}$. By the continuity of the Weil representation and compactness of K , for any $N_1, N_2, N_3 \in \mathbb{Z}_{>0}$, there exists a continuous seminorm ν'_{N_1, N_2, N_3} on $\mathcal{S}(V(F))$ such that the integral in (9.10) is bounded by $\Psi(2r)$ times

$$\nu'_{N_1, N_2, N_3}(f_2) \int_{(F^\times)^3 \times F} m'(t, a)^{-N-2} |a_2|^{-2r} \\ \times \prod_{i=1}^3 \max(|a_i^{-1}v_i|, 1)^{-N_i} |a_i|^{2-d_i/2} da^\times dt.$$

From now on all implicit constants are allowed to depend on N_1, N_2, N_3, N . Let

$$m''(t, a) := \max(1, |a_2 a_3^{-1}|, |a_2^{-1} a_3|, |a_2 a_3 t|) \\ = \max(|a_2 a_3^{-1}|, |a_2^{-1} a_3|, |a_2 a_3 t|).$$

We assume without loss of generality that $N_1 > N + d_1/2$. We then write the integral above as the product of

$$\int_{F^\times} |a_1|^{-N-d_1/2} \max(|a_1^{-1}v_1|, 1)^{-N_1} da_1^\times \ll |v_1|^{-N-d_1/2}$$

and

$$(9.11) \quad \int_{(F^\times)^2 \times F} m''(t, a)^{-N-2} |a_2|^{-2r} \\ \times \prod_{i=2}^3 \max(|a_i^{-1} v_i|, 1)^{-N_i} |a_i|^{2-d_i/2} da_2^\times da_3^\times dt.$$

Now consider (9.11). We write it as the sum of

$$(9.12) \quad \int_{|a_2| \geq |a_3|} |a_2|^{-N-d_2/2-2r} \max(|a_2^{-1} v_2|, 1)^{-N_2} \\ \times \int_F \max(|a_3^{-1}|, |a_3 t|)^{-N-2} \max(|a_3^{-1} v_3|, 1)^{-N_3} |a_3|^{2-d_3/2} da_3^\times da_2^\times dt$$

and

$$(9.13) \quad \int_{|a_3| > |a_2|} |a_3|^{-N-d_3/2} \max(|a_3^{-1} v_3|, 1)^{-N_3} \\ \times \int_F \max(|a_2^{-1}|, |a_2 t|)^{-N-2} \max(|a_2^{-1} v_2|, 1)^{-N_2} |a_2|^{2-d_2/2-2r} da_3^\times da_2^\times dt.$$

Executing the t integral in (9.12), we see that it is bounded by a constant times

$$(9.14) \quad \int_{|a_2| \geq |a_3|} |a_2|^{-N-d_2/2-2r} \max(|a_2^{-1} v_2|, 1)^{-N_2} \max(|a_3^{-1} v_3|, 1)^{-N_3} \\ \times |a_3|^{2+N-d_3/2} da_3^\times da_2^\times \\ = \int_{1 \geq |a_3|} |a_2|^{2-d_2/2-d_3/2-2r} \max(|a_2^{-1} v_2|, 1)^{-N_2} \max(|a_3^{-1} a_2^{-1} v_3|, 1)^{-N_3} \\ \times |a_3|^{2+N-d_3/2} da_3^\times da_2^\times,$$

where the latter equation is obtained by taking a change of variables $a_3 \mapsto a_2 a_3$. Similarly (9.13) is bounded by a constant times

$$(9.15) \quad \int_{|a_3| > 1} |a_3|^{-N-d_3/2} \max(|a_2^{-1} a_3^{-1} v_3|, 1)^{-N_3} \max(|a_2^{-1} v_2|, 1)^{-N_2} \\ \times |a_2|^{2-d_2/2-d_3/2-2r} da_3^\times da_2^\times.$$

Carrying out the integral over a_3 directly in (9.14) we see that it is bounded by a constant times

$$\int_{F^\times} |a_2|^{2-d_2/2-d_3/2-2r} \max(|a_2^{-1} v_2|, 1)^{-N_2} \max(|a_2^{-1} v_3|, 1)^{-N_3} da_2^\times$$

provided $N - d_3/2 + 2 > 0$. This bound is also valid for (9.15) provided $N + d_3/2 > N_3$. The integral above is bounded by a constant times

$$\max(|v_2|, |v_3|)^{-2r - d_2/2 - d_3/2 + 2}$$

provided $N_i > 2r + d_2/2 + d_3/2 - 2$ for $i = 2, 3$. □

10. Absolute convergence

In this section, we prove the absolute convergence statements that make the proof of the summation formula in Section 6 rigorous. Fix a number field F . For the remainder of the section, we fix

$$(f = f_1 \otimes f_2, \Phi) \in \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F)) \times \mathcal{S}(\mathbb{A}_F^2).$$

All implicit constants are allowed to depend on $f \otimes \Phi$. For $y_i \in V_i^\circ(\mathbb{A}_F)$, we let

$$|y_i| := \prod_v |y_i|_v.$$

LEMMA 10.1. — *Let $1/2 > \epsilon > 0$ and a finite set of places S containing the infinite places be given. For $y \in V^\circ(\mathbb{A}_F)$ such that $|y_1|_v = |y_2|_v = 1$ for all $v \notin S$, there exists a Schwartz function $\Psi \in \mathcal{S}((V_1 \otimes V_2 \otimes V_3)(\mathbb{A}_F))$ (depending on S, ϵ) such that*

$$\int_{(N_2)^3(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} |f_1(g)\rho(g) f_2(y)| dg \leq \Psi(y_1 \otimes y_2 \otimes y_3) \prod_{i=1}^3 |y_i|^{-d_i/2 - \epsilon}.$$

The function $I_0(f)$ satisfies the same bound.

Proof. — This follows from the local bounds in Propositions 7.2, 8.2, and 9.5. □

Let \mathbb{G}_m^2 act on V° via the restriction of the action (4.7).

PROPOSITION 10.2. — *The sum*

$$\sum_{\xi \in V^\circ(F)/(F^\times)^2} \int_{N_2^3(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} |f_1(g)\rho(g) f_2(\xi)| dg$$

is finite.

Proof. — Let

$$\{\mathfrak{a}_j \subset \mathcal{O} : 1 \leq j \leq k\}$$

be a set of representatives for the ideal classes of \mathcal{O} , the ring of integers of F . For every $\xi_i \in V_i(F)$, we can choose an $\alpha \in F^\times$ such that $\alpha\xi_i \in V_i(\mathcal{O})$ and the greatest common denominator $\gcd(\alpha\xi_i)$ of the coefficients of $\alpha\xi_i$ is

\mathfrak{a}_j for some $1 \leq j \leq k$. Using this observation, we see that the sum in the proposition is bounded by a constant times

$$\sum_{j_1, j_2=1}^k \sum_{\substack{\xi \in (V_1(\mathcal{O}) \times V_2(\mathcal{O}) \times V_3^\circ(F)) / (\mathcal{O}^\times)^2 \\ \gcd(v_1) = \mathfrak{a}_{j_1}, \gcd(v_2) = \mathfrak{a}_{j_2}}} \int_{N_2^3(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} |f_1(g)\rho(g) f_2(\xi)| dg.$$

Here $(\mathcal{O}^\times)^2 < (F^\times)^2$ acts via the action (4.7). Thus it suffices to fix a pair of ideals \mathfrak{b}_1 and \mathfrak{b}_2 and prove convergence of the sum

$$\sum_{\substack{\xi \in (V_1(\mathcal{O}) \times V_2(\mathcal{O}) \times V_3^\circ(F)) / (\mathcal{O}^\times)^2 \\ \gcd(\xi_1) = \mathfrak{b}_1, \gcd(\xi_2) = \mathfrak{b}_2}} \int_{N_2^3(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} |f_1(g)\rho(g) f_2(\xi)| dg.$$

Let S be a finite set of places including the infinite places such that $\mathfrak{b}_i \widehat{\mathcal{O}}^S = \widehat{\mathcal{O}}^S$ for each i . Then by Lemma 10.1, there exists $\Psi \in \mathcal{S}((V_1 \otimes V_2 \otimes V_3)(\mathbb{A}_F))$ such that the sum above is bounded by

$$\begin{aligned} \sum_{\substack{\xi \in (V_1(\mathcal{O}) \times V_2(\mathcal{O}) \times V_3^\circ(F)) / (\mathcal{O}^\times)^2 \\ \gcd(\xi_1) = \mathfrak{b}_1, \gcd(\xi_2) = \mathfrak{b}_2}} \Psi(\xi_1 \otimes \xi_2 \otimes \xi_3) \prod_{i=1}^3 |\xi_i|^{-d_i/2-\epsilon} \\ \leq \sum_{\xi \in (V_1 \otimes V_2 \otimes V_3)(F)} \Psi(\xi). \end{aligned}$$

Here we have used the fact that, by the product rule,

$$(10.1) \quad |\xi_i| \geq 1.$$

for $\xi_i \in V_i^\circ(F)$. \square

LEMMA 10.3. — *Let constants $c > 1/2 > \epsilon > 0$ be given. For $1/2 + \epsilon < r < c$ and an integer*

$$N > \max(d_1/2 + d_3/2 - 2, 2r + d_1/2 + d_2/2 - 2),$$

there exists a Schwartz function $\Psi \in \mathcal{S}(V_1(\mathbb{A}_F))$ (depending on ϵ, c) such that

$$\begin{aligned} \int_{G_{\gamma_1}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} |f_1(\gamma_1 g)| \\ \times \int_{N_2(\mathbb{A}_F) \backslash \mathrm{SL}_2(\mathbb{A}_F)} \left| \rho \left(\left(I_2, h, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} h \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right) g \right) f_2(y) \right| \\ \times |\Phi((0, x)hp_1(g))| x^{2r} d^\times x dh dg \\ \leq \Psi(y_1) |y_1|^{-N} |y_3|^{1-d_3/2-\epsilon} \prod_v \max(|y_2|_v, |y_3|_v)^{1-2r-d_2/2+\epsilon}. \end{aligned}$$

The function $I_1(f \otimes \Phi)(y, s)$ defines a holomorphic function of s in the strip $\frac{1}{2} + \epsilon < \operatorname{Re}(s) < c$ for each y and admits the same bound with $r = \operatorname{Re}(s)$.

Proof. — Let S be a finite set of places including the infinite places such that $f_1^S = b_X^S$, $f_2^S = \mathbb{1}_{V(\hat{\mathcal{O}}^S)}$ is fixed by $\rho(\operatorname{SL}_2^3(\hat{\mathcal{O}}^S))$ and $\Phi^S = \mathbb{1}_{(\hat{\mathcal{O}}^S)^2}$. Assume moreover that ψ_v is unramified for $v \notin S$ and F/\mathbb{Q} is unramified at places of \mathbb{Q} not dividing places of S . Using Lemma 8.4 and Propositions 7.3 and 9.6, for any given integers $N_1 > 0$, $N > \max(d_1/2 + d_3/2 - 2, 2r + d_2/2 + d_1/2 - 2)$, there exist $\Psi^\infty \in \mathcal{S}(V_1(\mathbb{A}_F^\infty))$ and a positive constant C depending on ϵ and c , such that the integral is bounded by a constant depending on N_1, N, ϵ, c times

$$\begin{aligned}
 (10.2) \quad & \Psi^\infty(y_1) \prod_{v|\infty} \max(|y_1|_v, 1)^{-N_1} |y_1|_v^{-N} \max(|y_2|_v, |y_3|_v)^{-2r-d_2/2-d_3/2+2} \\
 & \times \prod_{v \nmid \infty} \zeta_v(2r) \zeta_v(2r + d_2/2 - 1) \\
 & \times |y_1|_v^{-d_1/2} |y_3|_v^{1-d_3/2} \max(|y_1 \otimes y_3|_v, |y_2|_v)^{1-d_2/2-2r} \\
 & \times (C_v + \max(0, \operatorname{ord}_v(y_1 \otimes y_3) - \operatorname{ord}_v(y_2)) + \zeta_v(2r + d_2/2 - 1))
 \end{aligned}$$

where $C_v \in \{C, 0\}$ and $C_v = 0$ for almost all v . Since $1/2 + \epsilon < r$, we have

$$\begin{aligned}
 & C_v + \max(0, \operatorname{ord}_v(y_1 \otimes y_3) - \operatorname{ord}_v(y_2)) + \zeta_v(2r + d_2/2 - 1) \\
 & \ll \min(|y_1 \otimes y_3|_v / |y_2|_v, 1)^{-\epsilon} \zeta_v(2r + d_2/2 - 1) \\
 & = |y_1|_v^{-\epsilon} |y_3|_v^{-\epsilon} \max(|y_1 \otimes y_3|_v, |y_2|_v)^\epsilon \zeta_v(2r + d_2/2 - 1)
 \end{aligned}$$

for all finite v . Here the implied constant is equal to 1 for q_v sufficiently large in a sense independent of y . Thus (10.2) is bounded by a constant depending on c and ϵ times

$$\begin{aligned}
 (10.3) \quad & \Psi^\infty(y_1) \prod_{v|\infty} \max(|y_1|_v, 1)^{-N_1} |y_1|_v^{-N} \max(|y_2|_v, |y_3|_v)^{-2r-d_2/2-d_3/2+2} \\
 & \times \prod_{v \nmid \infty} |y_1|_v^{-d_1/2-\epsilon} |y_3|_v^{1-d_3/2-\epsilon} \max(|y_1 \otimes y_2|_v, |y_3|_v)^{1-d_2/2-2r+\epsilon}.
 \end{aligned}$$

For a finite place v , if $\Psi^\infty(y_1) \neq 0$ then $|y_1|_v \leq C'_v$ for some constant $C'_v \geq 1$, which is 1 for almost all v , and hence

$$\max(|y_1 \otimes y_3|_v, |y_2|_v) \geq C'^{-1}_v |y_1|_v \max(|y_3|_v, |y_2|_v).$$

Thus (10.3) is bounded by a constant times

$$\begin{aligned} & \Psi^\infty(y_1) \prod_{v|\infty} \max(|y_1|_v, 1)^{-N_1} |y_1|_v^{-N} \max(|y_2|_v, |y_3|_v)^{-2r-d_2/2-d_3/2+2} \\ & \times \prod_{v \nmid \infty} |y_1|_v^{1-d_1/2-d_2/2-2r} |y_3|_v^{1-d_3/2-\epsilon} \max(|y_2|_v, |y_3|_v)^{1-d_2/2-2r+\epsilon}. \end{aligned}$$

The desired inequality follows from $\max(|y_2|_v, |y_3|_v) \geq |y_3|_v$. □

PROPOSITION 10.4. — *If f_2 satisfies (1.3) and $r \gg 1$, the sum*

$$\begin{aligned} & \sum_{\xi \in V_1^\circ(F) \times \mathbb{P}(V_2 \times V_3)(F)} \int_{G_{\gamma_1}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} |f_1(\gamma_1 g)| \int_{N_2(\mathbb{A}_F) \backslash \text{SL}_2(\mathbb{A}_F)} \\ & \int_{\mathbb{A}_F^\times} \left| \rho \left(\left(I_2, h, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} h \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right) g \right) f_2(\xi) \Phi((0, x) h p_1(g)) \right| \\ & \times |x|^{2r} d^\times x dh dg \end{aligned}$$

is finite. Therefore,

$$\sum_{\xi \in Y_1(F)} I_1(f \otimes \Phi)(\xi, s)$$

defines a holomorphic function for $\text{Re}(s) \gg 1$. Moreover, it extends to a meromorphic function of \mathbb{C} , holomorphic except for possible simple poles at $s = 0$ and $s = 1$. The residue $\text{Res}_{s=1} \sum_{\xi \in Y_1(F)} I_1(f \otimes \Phi)(\xi, s)$ equals

$$\frac{\text{Vol}(F^\times \backslash (\mathbb{A}_F^\times)^1) \widehat{\Phi}(0)}{2} \int_{G_{\gamma_1}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(\gamma_1 g) \int_{[G_{\gamma_1}]_{\xi \in V(F)}} \sum \rho(hg) f_2(\xi) dh dg,$$

and

$$(10.4) \quad \int_{G_{\gamma_1}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} |f_1(\gamma_1 g)| \int_{[G_{\gamma_1}]} \left| \sum_{\xi \in V(F)} \rho(hg) f_2(\xi) dh \right| dg$$

is absolutely convergent.

The corresponding assertions for the integrals $I_2(f \otimes \Phi)$ and $I_3(f \otimes \Phi)$ are valid by symmetry.

Proof. — Choose ϵ sufficiently small. By Lemma 10.3 and (10.1), there exists $\Psi \in \mathcal{S}(V_1(\mathbb{A}_F))$ such that the sum is bounded by a constant times

$$\sum_{(\xi_1, \xi) \in V_1(F) \times \mathbb{P}(V_2 \times V_3)(F)} \Psi(\xi_1) |\xi|^{1-\frac{d_2}{2}-2r+\epsilon} \ll_\Psi \sum_{\xi \in \mathbb{P}(V_2 \times V_3)(F)} |\xi|^{1-\frac{d_2}{2}-2r+\epsilon}.$$

The right hand side is the height zeta function of $\mathbb{P}(V_2 \times V_3)(F)$. It converges for r sufficiently large [7, Section 3]. This completes the proof of the first

claim of the proposition, and we deduce that $\sum_{\xi \in Y_1(F)} I_1(f \otimes \Phi)(\xi, s)$ is holomorphic for $\text{Re}(s) \gg 1$.

To obtain the meromorphic continuation, we break down the integral $\sum_{\xi \in Y_1(F)} I_1(f \otimes \Phi)(\xi, s)$ into two sums of the form

$$(10.5) \quad \int_{G_{\gamma_1}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(\gamma_1 g) \times \int_{[\text{SL}_2]} \sum_{\substack{\xi \in V(F) \\ \mathcal{Q}_1(\xi_1)=0}} \rho \left(\left(I_2, h, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} h \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right) g \right) f_2(\xi) \times \int \sum_{\delta \in B_2(F) \backslash \text{SL}_2(F)} \Phi(x(0, 1) \delta h p_1(g)) |x|^{2s} d^\times x dh dg$$

where the unspecified integral is over $|x| \geq 1$ or $|x| \leq 1$. The contribution of $|x| \geq 1$ converges for $\text{Re}(s)$ large and hence converges for all s . Using the Poisson summation formula on F^2 , the contribution of $|x| \leq 1$ equals

$$\int_{G_{\gamma_1}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(\gamma_1 g) \times \int_{[\text{SL}_2]} \sum_{\substack{\xi \in V(F) \\ \mathcal{Q}_1(\xi_1)=0}} \rho \left(\left(I_2, h, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} h \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right) g \right) f_2(\xi) \times \int_{|x| \leq 1} \left(\sum_{\delta \in B_2(F) \backslash \text{SL}_2(F)} \widehat{\Phi}(x^{-1}(1, 0) \delta^{-t} h^{-t} p_1(g)^{-t}) |x|^{2s-2} d^\times x + \widehat{\Phi}(0) |x|^{2s-2} - \Phi(0) |x|^{2s} d^\times x \right) dh dg.$$

An argument similar to the argument proving the holomorphy of the $|x| > 1$ contribution implies that the contribution of the sum over δ defines an entire function of s . For $\text{Re}(s) \gg 1$, the remaining contribution is

$$\frac{\text{Vol}(F^\times \backslash (\mathbb{A}_F^\times)^1)}{2} \left(\frac{\widehat{\Phi}(0)}{s-1} - \frac{\Phi(0)}{s} \right) \times \int_{G_{\gamma_1}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(\gamma_1 g) \int_{[G_{\gamma_1}]} \sum_{\xi \in V(F)} \rho(hg) f_2(\xi) dh dg.$$

Assuming that (10.4) is convergent, this term admits a meromorphic continuation to the s plane, holomorphic except at $s \in \{0, 1\}$ with poles and

residues as specified. To obtain the convergence of (10.4) one begins with

$$\int_{G_{\gamma_1}(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} |f_1(\gamma_1 g)| \int_{[G_{\gamma_1}]} \left| \sum_{\xi \in V(F)} \rho(hg) f_2(\xi) \right| \\ \times \int_{\delta \in B_2(F)\backslash \mathrm{SL}_2(F)} \sum \Phi(x(0,1)\delta h p_1(g)) |x|^{2s} d^\times x dh dg$$

instead of (10.5), argues as before, and then observes that one obtains an equality between (10.4) times $\frac{\mathrm{Vol}(F^\times \backslash (\mathbb{A}_F^\times)^1)}{2} \left(\frac{\hat{\Phi}(0)}{s-1} - \frac{\Phi(0)}{s} \right)$ and a sum that converges for $\mathrm{Re}(s)$ large. The absolute convergence statement follows. \square

11. The L^2 -theory

We now discuss the L^2 -theory. Let F be a local field of characteristic zero. We assume throughout this section that $Y^{\mathrm{sm}}(F) \subset Y(F)$ is nonempty, and hence is dense in the Hausdorff topology.

We first improve the bound in [14, Propositions 7.1 and 8.2]:

PROPOSITION 11.1. — Assume $\frac{1}{2} > \beta \geq 0$ and $v \in V^\circ(F)$. Let

$$f = f_1 \otimes f_2 \in \mathcal{S}(X(F)) \otimes \mathcal{S}(V(F)).$$

If F is non-Archimedean then

$$\int_{G_{\gamma_b}(F)\backslash G(F)} |f_1(\gamma_b g) \rho(g) f_2(v)| dg \ll_{f_1, f_2} \prod_{i=1}^3 |v_i|^{\beta/3 - d_i/2 + 2/3}.$$

The integral as a function of v has support in $\omega^{-N}V(\mathcal{O})$ for some $N \in \mathbb{Z}$. If F is Archimedean then, given $N > 0$, there is a continuous seminorm $\nu_{\beta, N}$ on $\mathcal{S}(X(F) \times V(F))$ such that

$$\int_{G_{\gamma_b}(F)\backslash G(F)} |f_1(\gamma_b g) \rho(g) f_2(v)| dg \\ \leq \nu_{\beta, N}(f_1 \otimes f_2) \prod_{i=1}^3 \max(|v_i|, 1)^{-N} |v_i|^{\beta/3 - d_i/2 + 2/3}.$$

The function $I(f)$ satisfies the same bound and support constraint.

Proof. — Assume for the moment that F is non-Archimedean. The bound on the support of the integral is part of [14, Proposition 7.1], so we only require the bound on the magnitude. By Lemma 3.5 and Iwasawa

decomposition, the integral is bounded by a constant depending on β and f_1 times

$$(11.1) \quad \int_{m(t,a) \leq c} m(t,a)^{-2+\beta} \tilde{f}_2(a^{-1}v) \left(\prod_{i=1}^3 |a_i|^{2-d_i/2} \right) d^\times a dt$$

for some constant $c > 0$, where \tilde{f}_2 and $m(t,a)$ are defined as in (7.5) and (5.9) respectively. Observe that

$$m(t,a) = |a_1 a_2 a_3| \max(|t|, |a_1|^{-2}, |a_2|^{-2}, |a_3|^{-2}).$$

Thus (11.1) is equal to

$$(11.2) \quad \int_{m(t,a) \leq c} \max(|t|, |a_1|^{-2}, |a_2|^{-2}, |a_3|^{-2})^{\beta-2} \times \tilde{f}_2(a^{-1}v) \left(\prod_{i=1}^3 |a_i|^{\beta-d_i/2} \right) d^\times a dt.$$

Since

$$(11.3) \quad \int_F \max(|t|, |a_1|^{-2}, |a_2|^{-2}, |a_3|^{-2})^{-2+\beta} dt \ll_\beta \min(|a_1|, |a_2|, |a_3|)^{2-2\beta} \leq |a_1 a_2 a_3|^{2/3-2\beta/3},$$

the integral (11.2) is bounded by a constant times

$$\int_{(F^\times)^3} \tilde{f}_2(a^{-1}v) \left(\prod_{i=1}^3 |a_i|^{\beta/3-d_i/2+2/3} \right) d^\times a.$$

Taking a change of variables $a_i \mapsto a_i \varpi^{\text{ord}(v_i)}$, the above is bounded by a constant times

$$\prod_{i=1}^3 |v_i|^{\beta/3-d_i/2+2/3} \int_{q^{-N} \leq |a_i|} \left(\prod_{i=1}^3 |a_i|^{\beta/3-d_i/2+2/3} \right) d^\times a$$

for some N depending on f_2 . The integral converges as $d_i \geq 2$ for all i .

Now assume F is Archimedean. By Lemma 3.5 and the argument above, for any $N' \in \mathbb{R}$ there is a continuous seminorm $\nu_{\beta, N'}$ on $\mathcal{S}(X(F))$ such that the integral is bounded by $\nu_{\beta, N'}(f_1)$ times

$$(11.4) \quad \int_{(F^\times)^3 \times F} \max(|t|, |a_1|^{-2}, |a_2|^{-2}, |a_3|^{-2})^{\beta-2} \max(m(t,a), 1)^{-N'} \times \tilde{f}_2(a^{-1}v) \left(\prod_{i=1}^3 |a_i|^{\beta-d_i/2} \right) dt d^\times a.$$

Choose N' so that $N' > 3N$, and observe

$$m(t, a)^3 \geq \max(|a_1|, 1) \max(|a_2|, 1) \max(|a_3|, 1).$$

Then for any $M > 0$, there is a continuous seminorm ν'_M on $\mathcal{S}(V(F))$ such that (11.4) is bounded by $\nu'_M(f_2)$ times

$$\int_{(F^\times)^3 \times F} \max(|t|, |a_1|^{-2}, |a_2|^{-2}, |a_3|^{-2})^{\beta-2} \\ \times \left(\prod_{i=1}^3 \max(|a_i|, 1)^{-N} \max(|a_i^{-1}v_i|, 1)^{-M} |a_i|^{\beta-d_i/2} \right) dt d^\times a.$$

The proposition then follows from (11.3) (which is still valid for F Archimedean) and Lemma 9.1. \square

Let Ω_V be a top degree form on $V(F)$ such that $|\Omega_V|$ is the Haar measure. We endow $Y^{\text{sm}}(F)$ with the unique positive measure $dy = |\Omega_Y|$ such that $d(\mathcal{Q}_1 - \mathcal{Q}_2) \wedge d(\mathcal{Q}_2 - \mathcal{Q}_3) \wedge \Omega_Y$ is Ω_V on $V(F)$. Since $Y^{\text{sm}}(F) \subset Y(F)$ is dense, we can consider the L^p space

$$L^p(Y(F)) := L^p(Y(F), dy) := L^p(Y^{\text{sm}}(F), dy).$$

We observe that for $r \in F^\times$, one has

$$\begin{aligned} r^{d_1+d_2+d_3}\Omega_V &= \Omega_V(rv) \\ &= d(\mathcal{Q}_1 - \mathcal{Q}_2)(rv) \wedge d(\mathcal{Q}_2 - \mathcal{Q}_3)(rv) \wedge \Omega_Y(rv) \\ &= r^4 d(\mathcal{Q}_1 - \mathcal{Q}_2) \wedge d(\mathcal{Q}_2 - \mathcal{Q}_3) \wedge \Omega_Y(rv). \end{aligned}$$

Thus

$$(11.5) \quad d(rv) = |r|^{d_1+d_2+d_3-4} dy.$$

PROPOSITION 11.2. — *Let $0 < p \leq 2$. One has $\mathcal{S}(Y(F)) \subset L^p(Y(F))$ and the inclusion is continuous if F is Archimedean.*

Proof. — Let $f \in \mathcal{S}(Y(F))$. Since $\mathcal{S}(Y(F)) \subset C^\infty(Y^{\text{sm}}(F))$ by Propositions 7.1 and 9.3, we have

$$\int_{Y^{\text{sm}}(F)} |f(y)|^p dy = \int_{Y^{\text{ani}}(F)} |f(y)|^p dy.$$

We will therefore bound the integral on the right.

Fix $\frac{1}{2} > \beta > 0$. Assume first that F is non-Archimedean. By Proposition 11.1, for some $c \in \mathbb{Z}$ one has

$$\int_{Y^{\text{ani}}(F)} |f(y)|^p dy \ll_f \int_{Y^{\text{ani}}(F) \cap \varpi^{-c}V(\mathcal{O})} \left(\prod_{i=1}^3 |y_i|^{p(\beta/3-d_i/2+2/3)} \right) dy.$$

Let $\alpha := p\beta + 2p - 4 + \sum_{i=1}^3 (1 - \frac{p}{2})d_i \geq 2\beta > 0$. Using the homogeneity property (11.5), the integral above is bounded by a constant depending on c times

$$\zeta(\alpha) \int_{\{y \in Y^{\text{ani}}(F) : 1 \leq |y| < 2\}} \left(\prod_{i=1}^3 |y_i|^{p(\beta/3 - d_i/2 + 2/3)} \right) dy.$$

Here we could just write $|y| = 1$, but we have written $1 \leq |y| < 2$ so that we can use the same formula in both Archimedean and non-Archimedean cases.

Now assume that F is Archimedean. Fix $N > \alpha$. Then by Proposition 11.1 there is a continuous seminorm ν_β on $\mathcal{S}(Y(F))$ such that

$$\begin{aligned} & \int_{Y^{\text{ani}}(F)} |f(y)|^p dy \\ & \leq \nu_\beta(f)^p \sum_{j=1}^\infty \int_{\{y \in Y^{\text{ani}}(F) : 2^{-j} \leq |y| < 2^{1-j}\}} \left(\prod_{i=1}^3 |y_i|^{p(\beta/3 - d_i/2 + 2/3)} \right) dy \\ & \quad + \nu_\beta(f)^p \sum_{j=0}^\infty \int_{\{y \in Y^{\text{ani}}(F) : 2^j \leq |y| < 2^{j+1}\}} |y|^{-N} \left(\prod_{i=1}^3 |y_i|^{p(\beta/3 - d_i/2 + 2/3)} \right) dy. \end{aligned}$$

Using the homogeneity property (11.5) again, we see that this is

$$\begin{aligned} & \nu_\beta(f)^p \left(\sum_{j=1}^\infty 2^{-\alpha j} + \sum_{j=0}^\infty 2^{(\alpha - N)j} \right) \\ & \quad \times \int_{\{y \in Y^{\text{ani}}(F) : 1 \leq |y| < 2\}} \left(\prod_{i=1}^3 |y_i|^{p(\beta/3 - d_i/2 + 2/3)} \right) dy. \end{aligned}$$

Thus for any F , we are reduced to showing that

$$\int_{\{y \in Y^{\text{ani}}(F) : 1 \leq |y| < 2\}} \left(\prod_{i=1}^3 |y_i|^{p(\beta/3 - d_i/2 + 2/3)} \right) dy$$

is finite. By symmetry, it suffices to show that the integral

$$(11.6) \quad \int_{\{y \in Y^{\text{ani}}(F) : \max(|y_1|, |y_2|) \leq |y_3|, 1 \leq |y_3| < 2\}} \left(\prod_{i=1}^3 |y_i|^{p(\beta/3 - d_i/2 + 2/3)} \right) dy$$

is finite. After a change of variables, we can assume that $\mathcal{Q}_i(v_i) = v^t c_i v$ where

$$c_i = \begin{pmatrix} c_{i1} & & \\ & \ddots & \\ & & c_i d_i \end{pmatrix}$$

is diagonal. Write $v_i = (v_{i1}, \dots, v_{id_i})$. For any $1 \leq j \leq d_1, 1 \leq k \leq d_2$, we have

$$\begin{aligned} \left| \det \begin{pmatrix} \partial_{v_{1j}}(\mathcal{Q}_1 - \mathcal{Q}_2) & \partial_{v_{2k}}(\mathcal{Q}_1 - \mathcal{Q}_2) \\ \partial_{v_{1j}}(\mathcal{Q}_2 - \mathcal{Q}_3) & \partial_{v_{2k}}(\mathcal{Q}_2 - \mathcal{Q}_3) \end{pmatrix} \right| &= \left| \det \begin{pmatrix} 2c_{1j}v_{1j} & -2c_{2k}v_{2k} \\ & 2c_{2k}v_{2k} \end{pmatrix} \right| \\ &= |4c_{1j}c_{2k}||v_{1j}||v_{2k}|. \end{aligned}$$

Then for any j and k as above we have

$$\left(\prod_{i=1}^3 |y_i|^{p(\beta/3 - d_i/2 + 2/3)} \right) dy = \frac{\left(\prod_{i=1}^3 |y_i|^{p(\beta/3 - d_i/2 + 2/3)} \right) dy_1 dy_2 dy_3}{|4c_{1j}c_{2k}||y_{1j}y_{2k}|} \frac{dy_1 dy_2 dy_3}{dy_{1j} dy_{2k}}$$

outside a set of measure zero with respect to dy . Here $\frac{dy_i}{dy_{i1}} := dy_{i2} \dots dy_{id_i}$, etc. and the values of $|y_{1j}|$ and $|y_{2k}|$ are given implicitly in terms of the other entries of y . We can assume that j and k are chosen so that $|y_{1j}| = |y_1|$ and $|y_{2k}| = |y_2|$. Therefore, setting $t_1 = |y_1|$ and $t_2 = |y_2|$, we see that (11.6) is bounded by a constant times

$$\int_0^2 \int_0^2 \left(\int_{\substack{(x_1, x_2) \in F^{d_1-1} \times F^{d_2-1} \\ |x_1| \leq t_1, |x_2| \leq t_2}} t_1^{p(\frac{\beta+2}{3} - \frac{d_1}{2})-1} t_2^{p(\frac{\beta+2}{3} - \frac{d_2}{2})-1} dx_1 dx_2 \right) dt_1 dt_2,$$

which is finite. □

Suppose F is non-Archimedean. Let $K \leq \mathrm{Sp}_6(\mathcal{O})$ be a compact open subgroup and let $L^2(X^\circ(F))^K$ denote the space of functions on $X^\circ(F)$ that are right K -invariant and square-integrable.

LEMMA 11.3. — For $f \in L^2(X^\circ(F))^K$ we have

$$|f(x)| \leq \frac{\|f\|_2}{|x|^2 \mathrm{meas}(K)^{1/2}}$$

for any $x \in X^\circ(F)$.

Proof. — For $f \in L^2(X^\circ(F))^K$ we have

$$\|f\|_2^2 = \sum_{\gamma \in X^\circ(F)/K} \frac{|f(\gamma)|^2 \mathrm{meas}(K)}{\delta_P(\gamma)} = \sum_{\gamma \in X^\circ(F)/K} |f(\gamma)|^2 \mathrm{meas}(K) |\gamma|^4. \quad \square$$

PROPOSITION 11.4. — For F non-Archimedean and $f_2 \in \mathcal{S}(V(F))$, the map $I(\cdot \otimes f_2)$ extends to a continuous map

$$I(\cdot \otimes f_2) : L^2(X^\circ(F))^K \longrightarrow L^2_{\mathrm{loc}}(Y^{\mathrm{ani}}(F)).$$

Proof. — Assume $y \in Y^{\text{ani}}(F)$. By Lemma 11.3, for any $f_1 \in \mathcal{S}(X(F))$ the integral $I(f_1 \otimes f_2)(y)$ is bounded by $\text{meas}(K)^{-1/2}$ times

$$\|f_1\|_2 \int_{G_{\gamma_b}(F) \backslash G(F)} |\gamma_b g|^{-2} |\rho(g) f_2|(y) \, dg.$$

The proposition thus follows from the argument proving Proposition 11.1 in the special case $\beta = 0$. \square

12. The Fourier transform

Let F be a number field.

THEOREM 12.1. — *Let v be a place of F and assume that $Y^{\text{sm}}(F_v)$ is nonempty. There is a unique \mathbb{C} -linear isomorphism $\mathcal{F}_Y : \mathcal{S}(Y(F_v)) \rightarrow \mathcal{S}(Y(F_v))$ such that $\mathcal{F}_Y \circ I = I \circ \mathcal{F}_X$. It is continuous if v is Archimedean. In particular there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{S}(X(F_v) \times V(F_v)) & \xrightarrow{\mathcal{F}_X} & \mathcal{S}(X(F_v) \times V(F_v)) \\ \downarrow I & & \downarrow I \\ \mathcal{S}(Y(F_v)) & \xrightarrow{\mathcal{F}_Y} & \mathcal{S}(Y(F_v)). \end{array}$$

We should pause to explain why this theorem is not obvious. Let

$$C := \mathcal{S}(X(F_v) \times V(F_v))_{\text{SL}_2^3(F_v)}$$

denote the space of coinvariants. It is clear that the map I factors through C and yields a surjection $C \rightarrow \mathcal{S}(Y(F_v))$. Since \mathcal{F}_X is equivariant under the action of $\text{SL}_2^3(F_v) < \text{Sp}_6(F_v)$, it is clear that \mathcal{F}_X descends to define an automorphism of C . However, it is not clear that the map $C \rightarrow \mathcal{S}(Y(F_v))$ is injective. For instance, there are several orbits of $\text{SL}_2^3(F_v)$ on $X(F_v)$, but the map I depends only on the restriction of a function in $\mathcal{S}(X(F_v) \times V(F_v))$ to one of these orbits. Moreover, $\mathcal{S}(X(F_v))/\mathcal{S}(X^\circ(F_v))$ is infinite-dimensional as a representation of $\text{Sp}_6(F_v)$ and not even of finite length in the Archimedean case. The situation is even more complicated when we restrict to $\text{SL}_2^3(F_v)$. Finally, based on the example of [12] and its appendix, we expect that there are more complicated subquotients of C that are the local analogues of the hypothetical global boundary terms that we have excluded from our treatment using our assumption (1.3) (see the paragraph containing (1.5)). The injectivity of the map $C \rightarrow \mathcal{S}(Y(F_v))$ is more or less equivalent to the assertion that all of these complicated subquotients can be recovered from $\mathcal{S}(Y(F_v))$. Fortunately, with the global-to-local proof we give below, we can completely avoid the issue of describing the subquotients of C .

Using Theorem 12.1, many prior results can be stated more transparently. For example, by [14, Lemma 4.3] we have

COROLLARY 12.2. — *For any place v of F , $f \in \mathcal{S}(Y(F_v))$, and $h \in H(F_v)$ one has that*

$$\mathcal{F}_Y(L(h)f) = |\lambda(h)| \sum_{i=1}^3 d_i/2 - 2 L\left(\frac{h}{\lambda(h)}\right) \mathcal{F}_Y(f).$$

Unlike in other sections in this paper, we have not abbreviated F_v by F . We really require both F and F_v in this section because we will use a global-to-local argument to prove Theorem 12.1. The global-to-local argument is fairly simple, and we invite the reader to skip to the proof of Theorem 12.1 to see the basic idea.

There is a somewhat hidden assumption on the base change Y_{F_v} of Y to F_v in the statement of Theorem 12.1. Namely, we are assuming that Y_{F_v} is the base change to F_v of the scheme cut out of a triple of quadratic spaces over the number field F by the simultaneous values of three quadratic forms. This is no loss of generality since every characteristic zero local field is a localization of a number field [25, Section 25, Theorem 2], and every quadratic form over a local field is equivalent to the localization of a quadratic form over the corresponding number field. Indeed, the latter assertion follows from the fact that every quadratic form over F_v may be diagonalized [22, Section I.2] together with the fact that the natural map $F^\times \rightarrow F_v^\times / (F_v^\times)^2$ is surjective since F^\times is dense in F_v^\times .

We claim, moreover, that upon replacing F by another number field if necessary, we can assume that $Y^{\text{sm}}(F) \neq \emptyset$. By a change of basis, we may assume \mathcal{Q}_i is associated to the diagonal matrix $\text{diag}(c_{i1}, \dots, c_{id_i})$. In the Archimedean case, we can assume $c_{ij} \in \{\pm 1\}$. If $F_v = \mathbb{C}$, take $F = \mathbb{Q}(i)$ and if $F_v = \mathbb{R}$, take $F = \mathbb{Q}$; in either case one checks that $Y^{\text{sm}}(F) \neq \emptyset$. Now suppose F_v is non-Archimedean. Let \mathfrak{p} be the prime ideal corresponding to v . We may assume $c_{ij} \in \mathcal{O}_v$, the ring of integers of F_v , for all i, j . As $Y^{\text{sm}}(F_v)$ is nonempty, $Y^{\text{sm}}(F_v) \cap V(\mathcal{O}_v)$ is nonempty. Note that $A := F^{\text{sep}} \cap \mathcal{O}_v$ is the henselization of an excellent discrete valuation ring $\mathcal{O}_{\mathfrak{p}}$ whose completion is \mathcal{O}_v (see e.g., [30, tag 07QS], [24, Example 8.3.34]). Thus A has the approximation property by [2, Theorem 1.10]. In particular, there exists $(y_1, y_2, y_3) \in Y^{\text{sm}}(F_v)$ such that each coordinate of y_i is algebraic over F . Let E be the field extension of F obtained by adjoining the coordinates of the y_i . Then $E_w = F_v$ for some $w|v$ and $Y^{\text{sm}}(E) \neq \emptyset$. This justifies our claim.

Thus in proving Theorem 12.1, we can and do assume $Y^{\text{sm}}(F) \neq \emptyset$.

THEOREM 12.3. — *If $Y^{\text{sm}}(F_v) \neq \emptyset$ for all v , then $Y^{\text{sm}}(F)$ is nonempty and has dense image in $Y^{\text{sm}}(F_v)$ for all v .*

Proof. — Since we have assumed $\dim V_i \geq 2$ and \mathcal{Q}_i is nondegenerate for each i , this is a direct consequence of [8, Corollaire in Section 4]. \square

For a place v of F , consider the linear map

$$T : \mathcal{S}(V_i(F_v)) \longrightarrow C^\infty(F_v)$$

$$f \longmapsto T(f)(\alpha) := \int_{V_i(F_v)} f(v)\psi(\alpha\mathcal{Q}_i(v)) \, dv.$$

Let

$$(12.1) \quad \mathcal{S}_{iv} = \{f \in \mathcal{S}(V_i(F_v)) \mid T(f) = 0 \text{ and } f(0) = 0\}.$$

Note that (12.1) coincides with (1.1) by the definition of the Weil representation. We set $\mathcal{S}_{0v} = \mathcal{S}_{1v} \otimes \mathcal{S}_{2v} \otimes \mathcal{S}_{3v}$. Clearly, $\text{supp}(\rho(g)f) < V^\circ(F_v)$ for all $f \in \mathcal{S}_{0v}$ and $g \in \text{SL}_2^3(F_v)$. The following lemma implies, in particular, \mathcal{S}_{0v} is nontrivial when v is a finite place above an odd prime.

LEMMA 12.4. — *If v is a finite place lying above an odd prime then the kernel of T is infinite dimensional.*

Proof. — By diagonalizing the quadratic form \mathcal{Q}_i [22, Corollary 2.4], we see it suffices to show the kernel of the linear map

$$T' : \mathcal{S}(F_v) \longrightarrow C^\infty(F_v)$$

$$f \longmapsto T'(f)(\alpha) := \int_{F_v} f(x)\psi_v(\alpha x^2) \, dx$$

is infinite dimensional. Observe that if a nonzero f lies in the kernel of T' , then so does the infinite dimensional vector space spanned by

$$\{x \mapsto f(x\varpi_v^n) : n \in \mathbb{Z}\}.$$

Therefore, it suffices to show T' has nontrivial kernel. For $a \in \mathcal{O}_v^\times$, let $U_a := a + \varpi_v \mathcal{O}_v$. Since 2 does not divide the residual characteristic of F_v , the map $x \mapsto x^2$ induces a bijection $U_a \rightarrow U_{a^2}$. Therefore,

$$T'(\mathbb{1}_{U_a})(\alpha) = \frac{dx(\mathcal{O}_v)}{q_v} \psi_v(a^2\alpha) \mathbb{1}_{\varpi_v^{N-1}\mathcal{O}_v}(\alpha),$$

where N is the smallest integer such that ψ_v is trivial on $\varpi_v^N \mathcal{O}_v$. In particular, $T'(\mathbb{1}_{U_1}) = T'(\mathbb{1}_{U_{-1}})$. Since U_1 and U_{-1} are disjoint, the function $\mathbb{1}_{U_1} - \mathbb{1}_{U_{-1}}$ is nonzero and lies in the kernel of T' . \square

Recall $Y^{\text{ami}} \subset Y$ defined in (4.3).

LEMMA 12.5. — *Let v be a place where \mathcal{Q}_{iv} splits for all $1 \leq i \leq 3$. Suppose there exists $y_0 \in Y^{\text{ani}}(F_v)$ such that $I(\mathcal{F}_X(f_1) \otimes f_2)(y_0) = 0$ for all $f_1 \otimes f_2 \in C_c^\infty(\gamma_b G(F_v)) \otimes \mathcal{S}_{0v}$. Then*

$$I(\mathcal{F}_X(f_1) \otimes f_2) = 0$$

for any $f_1 \otimes f_2 \in C_c^\infty(\gamma_b G(F_v)) \otimes \mathcal{S}_{0v}$.

Proof. — Given $y \in Y^{\text{ani}}(F_v)$, choose $h \in H(F_v)$ such that $\lambda(h)h^{-1}y_0 = y$. Then for $f_1 \otimes f_2 \in C_c^\infty(\gamma_b G(F_v)) \otimes \mathcal{S}_{0v}$,

$$\begin{aligned} I(\mathcal{F}_X(f_1) \otimes f_2)(y) &= L\left(\frac{h}{\lambda(h)}\right) I(\mathcal{F}_X(f_1) \otimes f_2)(y_0) \\ &= |\lambda(h)|^{-\sum_{i=1}^2 d_i/2} I(\mathcal{F}_X(f))(y_0) \end{aligned}$$

for some $f \in C_c^\infty(\gamma_b G(F_v)) \otimes \mathcal{S}_{0v}$ by [14, Lemma 4.3]. Thus our hypothesis implies $I(\mathcal{F}_X(f_1) \otimes f_2)(y) = 0$. Since $I(\mathcal{F}_X(f_1) \otimes f_2)$ is continuous on $Y^{\text{sm}}(F_v)$ by Propositions 7.1 and 9.3, and $Y^{\text{ani}}(F_v) \subset Y^{\text{sm}}(F_v)$ is dense by [28, Remark 3.5.76], we deduce the lemma. \square

LEMMA 12.6. — *Let v be a finite place where \mathcal{Q}_{iv} splits for $1 \leq i \leq 3$ and \mathcal{S}_{0v} is nontrivial. For a given $y \in Y^{\text{ani}}(F_v)$, there exists $f_1 \otimes f_2 \in C_c^\infty(\gamma_b G(F_v)) \otimes \mathcal{S}_{0v}$ such that $I(\mathcal{F}_X(f_1) \otimes f_2)(y) \neq 0$.*

Proof. — Choose $f'_1 \otimes f_2 \in \mathcal{S}(X(F_v)) \otimes \mathcal{S}_{0v}$ such that $I(f'_1 \otimes f_2) \neq 0$. For example, we could take f'_1 to be the characteristic function of a sufficiently small neighborhood of γ_b in $\gamma_b G(F_v)$. Choose a compact open subgroup $K \leq \text{Sp}_6(\mathcal{O}_v)$ such that f'_1 is fixed by K . Finally, choose $f_{1n} \in C_c^\infty(\gamma_b G(F_v))^K$ indexed by $n \in \mathbb{Z}_{>0}$ such that

$$\lim_{n \rightarrow \infty} f_{1n} = \mathcal{F}_X^{-1}(f'_1)$$

in $L^2(X^\circ(F_v))^K$. Then since \mathcal{F}_X is an isometry of $L^2(X^\circ(F_v))^K$, we have $\mathcal{F}_X(f_{1n}) \rightarrow f'_1$ in $L^2(X^\circ(F_v))^K$. Since

$$I(\cdot \otimes f_2) : L^2(X^\circ(F_v))^K \longrightarrow L_{\text{loc}}^2(Y^{\text{ani}}(F_v))$$

is well-defined and continuous by Proposition 11.4, we deduce that

$$I(\mathcal{F}_X(f_{1n}) \otimes f_2) \longrightarrow I(f'_1 \otimes f_2)$$

in $L_{\text{loc}}^2(Y^{\text{ani}}(F_v))$ and hence $I(\mathcal{F}_X(f_{1n}) \otimes f_2) \neq 0$ for n large enough. The statement thus follows from Lemma 12.5. \square

Proof of Theorem 12.1. — We first prove that if $I(f_v) = 0$ then $I(\mathcal{F}_X(f_v)) = 0$. Choose finite places v_1 and $2 \nmid v_2$ distinct from v such that \mathcal{Q}_{iv_2} splits for $1 \leq i \leq 3$. Suppose that $f_{v_1} \in \mathcal{S}(X(F_{v_1}) \times V(F_{v_1}))$ is chosen so that $\mathcal{F}_X(f_{v_1}) \in C_c^\infty(\gamma_b G(F_{v_1}) \times V(F_{v_1}))$ and $I(\mathcal{F}_X(f_{v_1})) \in$

$C_c^\infty(Y^{\text{sm}}(F_{v_1}))$ and that $f_{v_2} \in C_c^\infty(\gamma_b G(F_{v_2})) \otimes \mathcal{S}_{0v_2}$. Moreover, choose $f^{v_1 v_2 v} \in \mathcal{S}(X(\mathbb{A}_F^{v_1 v_2 v}) \times V(\mathbb{A}_F^{v_1 v_2 v}))$. Then applying Theorem 1.4, we obtain

$$0 = \sum_{y \in Y^{\text{sm}}(F)} I(\mathcal{F}_X(f_v f_{v_1} f_{v_2} f^{v v_1 v_2}))(y).$$

In particular, since $\mathcal{F}_X(f_{v_1}) \in C_c^\infty(\gamma_b G(F_{v_1}) \times V(F_{v_1}))$ and $f_{v_2} \in C_c^\infty(\gamma_b G(F_{v_2})) \otimes \mathcal{S}_{0v_2}$, all of the boundary terms in the formula vanish.

We observe that $Y(F)$ is discrete in $Y(\mathbb{A}_F)$. Let $y_0 \in Y^{\text{ani}}(F)$. We claim that we can choose $f_{v_1} f_{v_2} f^{v v_1 v_2}$ so that the right hand side is equal to $I(\mathcal{F}_X(f_{v_1} f_{v_2} f^{v v_1 v_2}))(y_0)$ where $I(\mathcal{F}_X(f_{v_1} f_{v_2} f^{v v_1 v_2}))(y_0) \neq 0$. Indeed, by Lemma 5.3, we can choose f_{v_1} so that $I(\mathcal{F}_X(f_{v_1}))$ is any function in $C_c^\infty(Y^{\text{sm}}(F_{v_1}))$. Combining this with Lemma 5.3, the computation of the basic function in Proposition 8.1, and Lemma 12.6, we deduce the claim.

The claim implies that $I(\mathcal{F}_X(f_v))(y) = 0$ for all $y \in Y^{\text{ani}}(F)$. Since $Y^{\text{ani}}(F)$ is dense in $Y^{\text{sm}}(F_v)$ by [28, Remark 3.5.76] and Theorem 12.3, we can use the continuity of $I(\mathcal{F}_X(f_v))$ (Propositions 7.1 and 9.3) to deduce that $I(\mathcal{F}_X(f_v)) = 0$.

We have shown that $\mathcal{F}_X(\ker I) \leq \ker I$. On the other hand $\mathcal{F}_X \circ \mathcal{F}_X = \text{Id}$ by Proposition 3.9, so $\ker I = \mathcal{F}_X \circ \mathcal{F}_X(\ker I) \leq \mathcal{F}_X(\ker I)$, hence $\mathcal{F}_X(\ker I) = \ker I$. This implies the theorem. \square

The Fourier transform $\mathcal{F}_{X,\psi}$ and $I := I_\psi$ depend on a choice of additive character ψ . The dependence of I on ψ is through its dependence on the Weil representation $\rho = \rho_\psi$. Thus \mathcal{F}_Y also depends on ψ . We write $\mathcal{F}_{Y,\psi}$ when we need to indicate this dependence. Thus $\mathcal{F}_{Y,\psi}$ is determined by the relation

$$(12.2) \quad \mathcal{F}_{Y,\psi} \circ I_\psi = I_\psi \circ \mathcal{F}_{X,\psi}.$$

COROLLARY 12.7. — For $f \in \mathcal{S}(Y(F_v))$, we have

$$(12.3) \quad \mathcal{F}_Y^2(f)(v) = f(v),$$

$$(12.4) \quad \overline{\mathcal{F}_{Y,\psi}(f)} = \mathcal{F}_{Y,\bar{\psi}}(\bar{f}),$$

$$(12.5) \quad \mathcal{F}_{Y,\psi} = \mathcal{F}_{Y,\bar{\psi}}.$$

Proof. — The first equation (12.3) is immediate from Proposition 3.9. As for (12.4), by the explicit formula for $\mathcal{F}_{X,\psi}$ given in [13, Corollary 6.11], for any $f_1 \in \mathcal{S}(X(F_v))$ one has

$$(12.6) \quad \overline{\mathcal{F}_{X,\psi}(f_1)} = \mathcal{F}_{X,\bar{\psi}}(\bar{f}_1).$$

Moreover, we claim that for $f_2 \in \mathcal{S}(V(F_v))$ one has

$$(12.7) \quad \overline{\rho_\psi(g) f_2} = \rho_{\bar{\psi}}(g) \bar{f}_2$$

for all $g \in \mathrm{SL}_2^3(F_v)$. By the second corollary to [32, Théorème 2], the Weil index $\gamma(\mathcal{Q}_i, \psi)$ satisfies the relation $\overline{\gamma(\mathcal{Q}_i, \psi)} = \gamma(\mathcal{Q}_i, \bar{\psi})$. Using this fact, one checks (12.7) by checking it on the same set of generators for $\mathrm{SL}_2^3(F_v)$ traditionally used to define the Weil representation (see [14, Section 3.1], for example). Thus (12.7) is valid. Hence for $f_1 \otimes f_2 \in \mathcal{S}(X(F_v) \times V(F_v))$ one has

$$\overline{I_\psi(\mathcal{F}_{X,\psi}(f_1) \otimes f_2)} = I_{\bar{\psi}}(\overline{\mathcal{F}_{X,\psi}(f_1) \otimes f_2}) = I_{\bar{\psi}}(\mathcal{F}_{X,\bar{\psi}}(\bar{f}_1) \otimes \bar{f}_2).$$

This implies (12.4). The space $\mathcal{S}(Y(F_v))$ is independent of the character ψ by Lemma 5.2. Thus to show (12.5), by (12.3) it suffices to show $\mathcal{F}_{Y,\psi} \circ \mathcal{F}_{Y,\bar{\psi}}(f) = f$ for functions f of the form $I_{\bar{\psi}}(f_1 \otimes f_2)$. We compute

$$\begin{aligned} & \mathcal{F}_{Y,\psi} \circ \mathcal{F}_{Y,\bar{\psi}}(I_{\bar{\psi}}(f_1 \otimes f_2)) \\ &= \mathcal{F}_{Y,\psi}(I_{\bar{\psi}}(\mathcal{F}_{X,\bar{\psi}}(f_1) \otimes f_2)) \\ &= \frac{\gamma(\mathcal{Q}, \bar{\psi})}{\gamma(\mathcal{Q}, \psi)} \mathcal{F}_{Y,\psi} \circ L(-1) I_\psi(L(m(-1))R \begin{pmatrix} -I_3 & \\ & I_3 \end{pmatrix} \mathcal{F}_{X,\bar{\psi}}(f_1) \otimes f_2) \end{aligned}$$

by Lemma 5.2. This equals

$$\begin{aligned} & \frac{\gamma(\mathcal{Q}, \bar{\psi})}{\gamma(\mathcal{Q}, \psi)} \mathcal{F}_{Y,\psi} \circ I_\psi \left(L(m(-1))R \begin{pmatrix} -I_3 & \\ & I_3 \end{pmatrix} \mathcal{F}_{X,\bar{\psi}}(f_1) \otimes L(-1)f_2 \right) \\ &= \frac{\gamma(\mathcal{Q}, \bar{\psi})}{\gamma(\mathcal{Q}, \psi)} I_\psi \circ \mathcal{F}_{X,\psi} \left(L(m(-1))R \begin{pmatrix} -I_3 & \\ & I_3 \end{pmatrix} \mathcal{F}_{X,\bar{\psi}}(f_1) \otimes L(-1)f_2 \right) \\ &= \frac{\gamma(\mathcal{Q}, \bar{\psi})}{\gamma(\mathcal{Q}, \psi)} I_\psi \left(R \begin{pmatrix} I_3 & \\ & -I_3 \end{pmatrix} \mathcal{F}_{X,\psi}(L(m(-1))\mathcal{F}_{X,\bar{\psi}}(f_1) \otimes L(-1)f_2) \right) \end{aligned}$$

by Lemma 3.3. By Proposition 3.9 and (12.6), this is

$$\begin{aligned} & \frac{\gamma(\mathcal{Q}, \bar{\psi})}{\gamma(\mathcal{Q}, \psi)} I_\psi \left(R \begin{pmatrix} I_3 & \\ & -I_3 \end{pmatrix} \mathcal{F}_{X,\bar{\psi}} \circ \mathcal{F}_{X,\bar{\psi}}(f_1) \otimes L(-1)f_2 \right) \\ &= \frac{\gamma(\mathcal{Q}, \bar{\psi})}{\gamma(\mathcal{Q}, \psi)} I_\psi \left(R \begin{pmatrix} I_3 & \\ & -I_3 \end{pmatrix} f_1 \otimes L(-1)f_2 \right) \quad (\text{Proposition 3.9}) \\ &= I_{\bar{\psi}}(L(m(-1))R(-I_6)f_1 \otimes f_2) \quad (\text{Lemma 5.2}) \\ &= I_{\bar{\psi}}(f_1 \otimes f_2). \end{aligned}$$

Here the last equality follows from the fact that $m(-1)[P, P](F_v) = (-I_6)[P, P](F_v)$. \square

We now explain how to deduce Theorem 1.2 from Theorem 1.4 and Theorem 12.1.

Proof of Theorem 1.2. — Given such f , we can choose $f_{1v_i} \in C_c^\infty(\gamma_b G(F_{v_i}))$ for $i = 1, 2$ such that $I(f_{1v_1} \otimes f_{v_1}) = f_{v_1}$ and $I(f_{1v_2} \otimes f_{2v_2}) = \mathcal{F}_Y(f_{v_2})$ where $f_{2v_2}|_{Y^{\text{sm}}(F_{v_2})} = \mathcal{F}_Y(f_{v_2})$. Indeed, we can take f_{1v_i} to be a scalar multiple of the characteristic function of a sufficiently small neighborhood of γ_b in $\gamma_b G(F_{v_i})$.

Moreover, choose $f'^{v_1 v_2} \in \mathcal{S}(X(\mathbb{A}_F^{v_1 v_2}) \times V(\mathbb{A}_F^{v_1 v_2}))$ such that $I(f'^{v_1 v_2}) = f'^{v_1 v_2}$. To deduce the theorem, we now apply Theorem 1.4 to $f' = (f_{1v_1} \otimes f_{v_1})(\mathcal{F}_X^{-1}(f_{1v_2}) \otimes f_{v_2})f'^{v_1 v_2}$. Assumption (1.2) is clearly valid, and (1.3) is valid by our hypotheses on f_{v_1} and $\mathcal{F}_Y(f_{v_2})$. By construction, the boundary terms vanish and the theorem is proved. \square

List of symbols

b_X	basic function on X	(3.19)
b_Y	basic function on Y	(8.2)
f_{χ_s}	local Mellin transform	(3.7)
$ f _{A,B,p}$	seminorm	(3.9)
$ f _{A,B,w,p_w,\Omega,D}$	seminorm	(3.10)
\mathcal{F}_X	Fourier transform on $\mathcal{S}(X(F))$	Section 3.1
\mathcal{F}_Y	Fourier transform on $\mathcal{S}(Y(F))$	Section 12
G	SL_2^3	(4.11)
\mathfrak{g}	Lie algebra of $M^{\text{ab}} \times \text{Sp}_{2n}$	Section 3.1
γ_i	representatives of $X^\circ(F)/G(F)$	(4.12)
G_{γ_i}	stabilizer of γ_i in G	Section 4
$ g $	the norm of g under the Plücker embedding	(3.12)
H	subgroup of a similitude group on V	(4.4)
I	integral operator attached to the representative γ_b	(5.1)
I_0	integral operator attached to the representative $\text{Id} = I_6$	(5.1)
I_i	integral operator attached to the representative γ_i	(5.4)
$L(m)R(g)$	action of $M^{\text{ab}}(F) \times \text{GSp}_6(F)$ on $\mathcal{S}(X(F))$	(3.11)
$\lambda(h)$	similitude norm of h	(4.5)

m	an isomorphism $M^{\text{ab}}(F) \rightarrow F^\times$	(3.17)
M	Levi subgroup of P	(3.1)
N	unipotent radical of P	(3.1)
N_2	standard maximal unipotent subgroup in SL_2	Section 1.1
ω	character of M	(3.2)
$\mathbb{1}_k$	characteristic function of $\mathbb{1}_{[P,P](F)m(\varpi^k)\text{Sp}_6(\mathcal{O})}$	(3.18)
P	Siegel parabolic	Section 3
Pl	Plücker embedding	(3.5)
(V, \mathcal{Q})	$\Pi_{i=1}^3(V_i, \mathcal{Q}_i)$	Section 1
V°	$\Pi_{i=1}^3(V_i - \{0\})$	(4.1)
(V_i, \mathcal{Q}_i)	quadratic space of even dimension	Section 1
X	affine closure of X°	(3.6)
X°	Braverman–Kazhdan space	(3.3)
Y	$\{(y_1, y_2, y_3) \in V : \mathcal{Q}_1(y_1) = \mathcal{Q}_2(y_2) = \mathcal{Q}_3(y_3)\}$	(4.2)
Y^{ani}	anisotropic vectors in Y	Section 4
Y^{sm}	smooth locus of Y	Section 4
Y_0	$\widetilde{Y}_0/\mathbb{G}_m^2$	(4.10)
Y_i	$\widetilde{Y}_i/\mathbb{G}_m$	(4.10)
\widetilde{Y}_0	vanishing locus of $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ in V°	(4.6)
\widetilde{Y}_i	$\{(y_1, y_2, y_3) \in V^\circ : \mathcal{Q}_{i-1}(y_{i-1}) = \mathcal{Q}_{i+1}(y_{i+1}) \text{ and } \mathcal{Q}_i(y_i) = 0\}$	(4.8)

BIBLIOGRAPHY

- [1] A. AIZENBUD & D. GOUREVITCH, “Schwartz functions on Nash manifolds”, *Int. Math. Res. Not.* **2008** (2008), no. 5, article no. rnm155 (37 pages).
- [2] M. ARTIN, “Algebraic approximation of structures over complete local rings”, *Publ. Math., Inst. Hautes Étud. Sci.* **36** (1969), p. 23–58.
- [3] A. BRAVERMAN & D. KAZHDAN, “On the Schwartz space of the basic affine space”, *Sel. Math., New Ser.* **5** (1999), no. 1, p. 1–28.
- [4] ———, “ γ -functions of representations and lifting”, in *GAGA 2000. Visions in mathematics—Towards 2000 (Tel Aviv, 1999)*, Geometrical and Functional Analysis, Birkhäuser, 2000, with an appendix by V. Vologodsky, p. 237–278.

- [5] ———, “Normalized intertwining operators and nilpotent elements in the Langlands dual group. Dedicated to Yuri I. Manin on the occasion of his 65th birthday”, *Mosc. Math. J.* **2** (2002), no. 3, p. 533-553.
- [6] S. CARACCILOLO, A. D. SOKAL & A. SPORTIELLO, “Algebraic/combinatorial proofs of Cayley-type identities for derivatives of determinants and Pfaffians”, *Adv. Appl. Math.* **50** (2013), no. 4, p. 474-594.
- [7] A. CHAMBERT-LOIR, “Lectures on height zeta functions: at the confluence of algebraic geometry, algebraic number theory, and analysis”, in *Algebraic and analytic aspects of zeta functions and L-functions*, MSJ Memoirs, vol. 21, Mathematical Society of Japan, 2010, p. 17-49.
- [8] J.-L. COLLIOT-THÉLÈNE & J.-J. SANSUC, “Sur le principe de Hasse et l’approximation faible, et sur une hypothèse de Schinzel”, *Acta Arith.* **41** (1982), no. 1, p. 33-53.
- [9] B. CONRAD, “Weil and Grothendieck approaches to adelic points”, *Enseign. Math.* (2) **58** (2012), no. 1-2, p. 61-97.
- [10] B. ELAZAR & A. SHAVIV, “Schwartz functions on real algebraic varieties”, *Can. J. Math.* **70** (2018), no. 5, p. 1008-1037.
- [11] P. B. GARRETT, “Decomposition of Eisenstein series: Rankin triple products”, *Ann. Math.* (2) **125** (1987), no. 2, p. 209-235.
- [12] J. R. GETZ, “Summation formulae for quadrics”, *Sel. Math., New Ser.* **31** (2025), no. 2, article no. 41 (42 pages).
- [13] J. R. GETZ, C.-H. HSU & S. LESLIE, “Harmonic analysis on certain spherical varieties”, *J. Eur. Math. Soc.* **27** (2025), no. 2, p. 433-541.
- [14] J. R. GETZ & B. LIU, “A summation formula for triples of quadratic spaces”, *Adv. Math.* **347** (2019), p. 150-191.
- [15] ———, “A refined Poisson summation formula for certain Braverman-Kazhdan spaces”, *Sci. China, Math.* **64** (2021), no. 6, p. 1127-1156.
- [16] N. GUREVICH & D. KAZHDAN, “Fourier transform on a cone and the minimal representation of even orthogonal group”, *Isr. J. Math.* **266** (2025), no. 1, p. 99-130.
- [17] C.-H. HSU, “Asymptotics of Schwartz functions: nonarchimedean”, 2021, <https://arxiv.org/abs/2112.02403>.
- [18] T. IKEDA, “On the location of poles of the triple L -functions”, *Compos. Math.* **83** (1992), no. 2, p. 187-237.
- [19] D. JIANG, Z. LUO & L. ZHANG, *Harmonic analysis and gamma functions on symplectic groups*, vol. 295, Memoirs of the American Mathematical Society, vol. 1473, American Mathematical Society, 2024, vii+89 pages.
- [20] T. KOBAYASHI & G. MANO, *The Schrödinger model for the minimal representation of the indefinite orthogonal group $O(p, q)$* , vol. 213, Memoirs of the American Mathematical Society, vol. 1000, American Mathematical Society, 2011, vi+132 pages.
- [21] L. LAFFORGUE, “Noyaux du transfert automorphe de Langlands et formules de Poisson non linéaires”, *Jpn. J. Math.* **9** (2014), no. 1, p. 1-68.
- [22] T. Y. LAM, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, 2005, xxii+550 pages.
- [23] W.-W. LI, *Zeta integrals, Schwartz spaces and local functional equations*, Lecture Notes in Mathematics, vol. 2228, Springer, 2018, viii+139 pages.
- [24] Q. LIU, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, 2002, xvi+576 pages.
- [25] F. LORENZ, *Algebra. Vol. II. Fields with structure, algebras and advanced topics*, Universitext, Springer, 2008, translated from the German by Silvio Levy, with the collaboration of Levy, x+336 pages.

- [26] B. C. NGÔ, “Hankel transform, Langlands functoriality and functional equation of automorphic L -functions”, *Jpn. J. Math.* **15** (2020), no. 1, p. 121-167.
- [27] I. PIATETSKI-SHAPIRO & S. RALLIS, “Rankin triple L functions”, *Compos. Math.* **64** (1987), no. 1, p. 31-115.
- [28] B. POONEN, *Rational points on varieties*, Graduate Studies in Mathematics, vol. 186, American Mathematical Society, 2017, xv+337 pages.
- [29] Y. SAKELARIDIS, “Spherical varieties and integral representations of L -functions”, *Algebra Number Theory* **6** (2012), no. 4, p. 611-667.
- [30] THE STACKS PROJECT AUTHORS, “The Stacks Project”, <https://stacks.math.columbia.edu>, 2018.
- [31] G. WARNER, *Harmonic analysis on semi-simple Lie groups. I*, Grundlehren der Mathematischen Wissenschaften, vol. 188, Springer, 1972, xvi+529 pages.
- [32] A. WEIL, “Sur certains groupes d’opérateurs unitaires”, *Acta Math.* **111** (1964), p. 143-211.

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