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PIECEWISE CIRCULAR CURVES AND POSITIVITY

by Jean-Philippe BURELLE & Ryan T. KIRK (*)

Abstract. — We introduce the moduli space of generic circular n-gons in the Riemann sphere and relate it to a moduli space of Legendrian polygons. We prove that when $n = 2k$, this moduli space contains a connected component homeomorphic to the Fock–Goncharov space of k-tuples of positive flags for $\mathsf{PSp}(4, \mathbb{R})$ and hence is a topological ball. We characterize this component geometrically as the space of simple circular n-gons with decreasing curvature.

Résumé. — Nous définissons l'espace de modules des n-gones circulaires génériques dans la sphère de Riemann et nous le relions à un espace de modules de polygones legendriens. Nous démontrons que lorsque n est pair, cet espace de modules contient une composante homéomorphe à l'espace des k-uplets positifs de drapeaux dans $\mathsf{PSp}(4, \mathbb{R})$ défini par Fock et Goncharov, et est donc une boule topologique. Nous identifions cette composante de manière géométrique en tant que l'espace des polygones circulaires simples de courbure décroissante.

1. Introduction

A *piecewise circular curve* is a curve in the plane made of finitely many arcs of circles, such that tangents agree at the intersection of pieces. We allow circular arcs with infinite radius (line segments), as well as cusps at the intersection of two arcs (see Figures 1.1, 5.1, 5.2). We will call a closed piecewise circular curve made of n circular arcs a circular n-gon. We will use the terminology of *edges* and *vertices* of a circular polygon by analogy with standard polygons. Since Möbius transformations map circular arcs

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to circular arcs, it makes sense to consider piecewise circular curves in the Riemann sphere up to Möbius transformations.

A co-orientation on a piecewise circular curve is a continuous choice of perpendicular orientation on the curve. There is a natural map on the set of co-oriented piecewise circular curves which moves each point on the curve a fixed (signed) distance d in the direction of the co-orientation. We will call this map the *radial translation* with parameter d. If the piecewise circular curve is interpreted as a wavefront, this map models wave propagation with d being the time parameter.

We consider the natural problem of classifying co-oriented piecewise circular curves up to Möbius transformations and radial translations.

Piecewise circular curves have previously been investigated, especially for the purpose of approximating other types of curves ([1, 2, 14]) but as far as we know this classification problem has not been considered in the literature.

Since the radial translation may introduce singularities by collapsing a circular arc to a point, we allow such singularities in our piecewise circular curves. We can think of such a collapsed arc as an arc of a circle of radius 0. Keeping track of the co-orientation, collapsed circular arcs are not degenerate as curves in the space of oriented contact elements of the sphere, and the action of radial translations and Möbius transformation is well-defined.

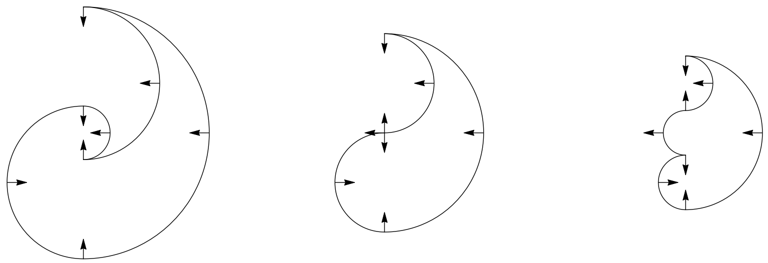

Figure 1.1. A simple co-oriented piecewise circular quadrilateral and its images under radial translation. The central figure shows a collapsed circular arc.

Even though we cannot compare the radii of any two co-oriented circles in the Riemann sphere, there is a partial cyclic order coming from the Maslov index which detects when three co-oriented circles are nested, and is invariant under Möbius transformations and radial translations. When the collection of circles extending the edges of a co-oriented piecewise circular curve is cyclically ordered, we will say that this curve has monotone curvature. This property means that piecewise circular curves with monotone curvature are locally spirals.

We say that a co-oriented piecewise circular curve is *generic* if non-adjacent vertices are never part of a common co-oriented circle, and non-adjacent edges are never arcs of tangent co-oriented circles. We say it is *simple* if *all* of its radial translates have no self-intersections. This is equivalent to there being no co-oriented circle which is tangent to two non-adjacent edges (see Proposition 4.11).

In Sections 3 and 4, we define the moduli spaces rigorously and make the following simple observations.

PROPOSITION 1.1. — *The moduli space of generic circular n-gons with $n \geqslant 5$ is a smooth $2(n-5)$-dimensional manifold.*

PROPOSITION 1.2. — *The moduli space of generic circular* 4*-gons (quadrilaterals) consists of* 4 *points. The moduli space of generic circular* 5*-gons (pentagons) consists of* 64 *points.*

PROPOSITION 1.3. — *The moduli space of simple circular* 4*-gons consists of two points.*

PROPOSITION 1.4. — *As a curve in the space of oriented contact elements to the* 2*-sphere, a simple piecewise circular polygon is non-contractible.*

A related moduli space, the space of $(2,n)$-*Lagrangians configurations* in $\mathbb{R}^4$ was investigated in [5] and [12]. We show :

PROPOSITION 1.5. — *The moduli space of simple circular n-gons embeds as an open subset in the space $\mathcal{L}_{2,n}$ of generic configurations of n Lagrangians in $\mathbb{R}^4$.*

Among generic circular $2k$-gons, there is a particular class consisting of curves which are simple and have decreasing curvature. Our main result is:

THEOREM 1.6. — *Let $k \geqslant 1$. In the moduli space of generic circular $2k$-gons, the subspace of simple curves with decreasing curvature is a connected component homeomorphic to an open ball.*

We also show that for hexagons, the curvature condition is automatically satisfied by all simple curves, a result which was proved in the second author's Ph.D thesis [9]:

PROPOSITION 1.7. — *A simple circular hexagon has monotone curvature.*

The strategy of the proof of Theorem 1.6 is to construct a homeomorphism between the moduli space of pairwise transverse k-tuples of $\mathsf{PSp}(4,\mathbb{R})$ flags and the moduli space of generic circular $2k$-gons. Under this homeomorphism, positive k-tuples of flags are in bijection with simple circular $2k$-gons with decreasing curvature. Fock and Goncharov showed in [6] that the subspace of positive k-tuples of flags is a connected component homeomorphic to a ball, and so the theorem follows.

Positive k-tuples of flags for the group $\mathsf{PSL}(3,\mathbb{R})$ also parameterize a natural moduli space of geometric objects: pairs of nested convex k-gons in the projective plane [7]. Our result provides a similar interpretation for positive k-tuples of flags in $\mathsf{PSp}(4,\mathbb{R})$. Both can be considered toy models of higher Teichmüller spaces.

In Section 5.3, we adapt some of the tools developed in the paper to the setting of continuous curves, for instance we define the notion of monotone curvature for a sufficiently regular curve in analogy with the discrete case by using osculating circles. We prove the following result about positive curves and Hitchin representations of surface groups into $\mathsf{PSp}(4,\mathbb{R})$:

Theorem 1.8. — *Let ξ be a positive Frenet $\mathsf{PSp}(4,\mathbb{R})$ flag curve. Then ξ is the tangent flag curve to a contact lift in $\mathbb{RP}^3$ of a simple closed curve with decreasing curvature on the sphere.*

Corollary 1.9. — *Let Σ be a closed surface and $\rho : \pi_1(\Sigma) \to \mathsf{PSp}(4,\mathbb{R})$ a Hitchin representation. Then, the $\mathbb{RP}^3$ limit curve of ρ is the contact lift of a simple closed curve with decreasing curvature on the sphere.*

This corollary gives an alternative geometric interpretation of the $\mathsf{PSp}(4,\mathbb{R})$ Hitchin component to the one presented in [8]. Guichard and Wienhard interpret the Hitchin component as the moduli space of convex-foliated contact projective structures on the tangent bundle of Σ. Corollary 1.9 gives an interpretation of this component as the moduli space of $\pi_1(\Sigma)$-equivariant simple closed curves on the 2-sphere with decreasing curvature.

We now describe the structure of the paper. After establishing some notation, in Section 3 we introduce Legendrian polygons and their moduli spaces. In Section 4 we develop the dictionary between the contact geometry of projective 3-space and the geometry of circles on the 2-sphere. Most of these results are known and have been described using the perspective of the Lie group $\mathsf{SO}^0(3,2)$ (for instance in [4]). For our purposes, the group $\mathsf{PSp}(4,\mathbb{R})$ which is isomorphic to $\mathsf{SO}^0(3,2)$ is more convenient and so we develop *Lie circle geometry* from that point of view in a coordinate-free way. Finally, in Section 5 we recall the notion of positivity of a configuration of flags and prove the main theorem and describe the applications to higher Teichmüller theory.

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2. Notation

When working with indexed objects A_i with a natural cyclic structure (for instance vertices of a polygon) we will consider indices from the set $\{1, 2, \ldots, n\}$ *modulo* n, meaning for instance that $A_{n+1} = A_1$ and $A_{n+2} = A_2$. When writing inequalities of the form $i < j$ for those indices, we use the unique representatives in $\{1, 2, \ldots, n\}$.

If V is a vector space over a field k, we denote by $\mathbb{P}(\mathsf{V})$ the associated projective space. If $\mathsf{U} \subset \mathsf{V}$ is a vector subspace, we also denote by $\mathbb{P}(\mathsf{U}) \subset \mathbb{P}(\mathsf{V})$ the corresponding projective subspace. Over $\mathbb{R}$, we also define the *sphere of directions* $\mathbb{S}(\mathsf{V}) := (\mathsf{V} \smallsetminus \{0\})/\mathbb{R}_{>0}$ and we use the same notation $\mathbb{S}(E)$ for the sphere bundle associated to a real vector bundle $E \to M$ over a manifold M. If V is a vector space over two different fields, we will use $\mathbb{P}_k(\mathsf{V})$ to specify which field we are considering. We use bold variables $\mathbf{v}$ to denote elements of V and brackets $[\mathbf{v}]_F \in \mathbb{P}_F(\mathsf{V})$ to denote projective equivalence classes. Similarly, the notation $[\mathbf{v}]_{\mathbb{R}_{>0}}$ will refer to the equivalence class of $\mathbf{v}$ in $\mathbb{S}(\mathsf{V})$.

A *symplectic form* ω on V is a skew-symmetric nondegenerate bilinear form. When V is equipped with a symplectic form, we will call it a symplectic vector space. Let $S \subset \mathsf{V}$ be a subset. Then,

$$S^\perp = \{\mathbf{v} \in \mathsf{V} \mid \omega(\mathbf{v}, \mathbf{u}) = 0,\ \forall\, \mathbf{u} \in S\}$$

denotes the orthogonal subspace to S.

A Lagrangian subspace $L \subset V$ is a subspace such that $L^\perp = L$. The Lagrangian Grassmannian $\mathsf{Lag}(\mathsf{V})$ is the space of Lagrangian subspaces in V. When a basis of V is fixed and $\dim(\mathsf{V}) = 2n$, we will use $2n \times n$ matrices between brackets $[\ ]$ to represent elements of $\mathsf{Lag}(\mathsf{V})$, understood as maps $k^n \to \mathsf{V}$. This notation is unique up to right-multiplication by an invertible $n \times n$ matrix. If the symplectic form is represented by the matrix Ω in the fixed basis, then a $2n \times n$ matrix M represents a Lagrangian if and only if M is of full rank and $M^t \Omega M = 0$.

3. Legendrian Polygons

Let (V,ω) be a 4-dimensional symplectic vector space over $\mathbb{R}$. A *Lagrangian* in V is a 2-dimensional subspace on which the symplectic form ω vanishes.

Orthogonality with respect to ω defines a hyperplane distribution $\mathbf{v}^\perp \subset T_\mathbf{v}\mathsf{V}$ on $V \smallsetminus \{0\}$. As $\mathbf{v} \in \mathbf{v}^\perp$, this distribution descends to a contact structure on the projective space $\mathbb{P}(\mathsf{V})$, making it a contact manifold.

Definition 3.1. — *Two points $p, q \in \mathbb{P}(\mathsf{V})$ will be called* incident *whenever $p \in \mathbb{P}(q^\perp)$, or equivalently $q \in \mathbb{P}(p^\perp)$.*

Two points $p, q \in \mathbb{P}(\mathsf{V})$ are contained in a common Lagrangian if and only if they are incident.

A path in $\mathbb{P}(\mathsf{V})$ which is always tangent to the contact distribution is called *Legendrian*.

Definition 3.2. — *A* line segment *in $\mathbb{P}(\mathsf{V})$ is a closed, connected proper subset of a projective line. A* polygon *in $\mathbb{P}(\mathsf{V})$ is an unparametrized closed curve consisting of finitely many line segments, together with a fixed labeling of the vertices $p_1, \dots, p_n$.*

Note that since projective lines are topological circles, there are *two* line segments joining a pair of distinct points $p, q \in \mathbb{P}(\mathsf{V})$. If $p = [\mathbf{u}]$ and $q = [\mathbf{v}]$, then the two segments are the images of the parametrized paths $[(1-t)\mathbf{u} + t\mathbf{v}]$ and $[(1-t)\mathbf{u} + t(-\mathbf{v})]$ for $0 \leqslant t \leqslant 1$.

To shorten the notation for such projective segments, we introduce for a pair of linearly independent vectors $\mathbf{u}, \mathbf{v}$ the parametrized segment

$$s_{\mathbf{u},\mathbf{v}}(t) \coloneqq [(1-t)\mathbf{u} + t\mathbf{v}]$$

with $t \in [0,1]$.

Definition 3.3. — *A* Legendrian polygon *in $\mathbb{P}(\mathsf{V})$ is a polygon where each segment is Legendrian.*

We will assume that all Legendrian polygons are nondegenerate in the following sense: no two adjacent segments are part of the same projective line and no segment is degenerated to a point.

Let $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ be a tuple of vectors in V such that $\mathbf{v}_k$ and $\mathbf{v}_{k+1}$ are linearly independent for $1 \leqslant k \leqslant n$. We define two polygons $P_\pm(\mathbf{v}_1, \dots, \mathbf{v}_n)$ consisting of the n following segments: $s_{\mathbf{v}_k,\mathbf{v}_{k+1}}(t)$ for $0 \leqslant t \leqslant 1$ and $1 \leqslant k \leqslant n-1$, and $s_{\mathbf{v}_n,\pm\mathbf{v}_1}(t)$.

Any polygon can be parametrized this way, and the ambiguity in the choice of representatives $\mathbf{v}_1, \dots, \mathbf{v}_n$ is multiplication of each $\mathbf{v}_j$ by a positive scalar and simultaneously multiplying all $\mathbf{v}_j$ by -1. The polygon $P_\pm(\mathbf{v}_1, \dots, \mathbf{v}_n)$ is Legendrian and nondegenerate when $\mathbf{v}_k, \mathbf{v}_{k+1}, \mathbf{v}_{k+2}$ are linearly independent, and $\omega(\mathbf{v}_k, \mathbf{v}_{k+1}) = 0$ for $1 \leqslant k \leqslant n$.

We will make frequent use of the following observation: the signs of the symplectic products $\omega(\mathbf{v}_i, \mathbf{v}_j)$ for a transverse Legendrian polygon $P_\pm(\mathbf{v}_1, \dots, \mathbf{v}_n)$ do not depend on the choice of representatives for the vertices $\mathbf{v}_1, \dots, \mathbf{v}_n$. Indeed, multiplication by a positive scalar does not change this sign, and simultaneously changing the sign of all $\mathbf{v}_j$ also does not change this sign.

PROPOSITION 3.4. — *Let $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathsf{V}$ and suppose that $\mathbf{v}_k$ and $\mathbf{v}_{k+1}$ are linearly independent for $1 \leqslant k \leqslant n$. Then, the polygon $P_+(\mathbf{v}_1, \dots, \mathbf{v}_n)$ is contractible (trivial in $\pi_1(\mathbb{P}(\mathsf{V}))$), and the polygon $P_-(\mathbf{v}_1, \dots, \mathbf{v}_n)$ generates $\pi_1(\mathbb{P}(\mathsf{V}))$.*

Proof. — Let $[\mathbf{u}_1], \dots, [\mathbf{u}_n]$ be n pairwise distinct points on a projective line $\ell \subset \mathbb{P}(\mathsf{V})$, placed in that order. Multiplying certain $\mathbf{u}_k$ by -1 if needed, we may assume that the full line is parametrized by $P_-(\mathbf{u}_1, \dots, \mathbf{u}_n)$.

For each vector $\mathbf{v}_i$, choose a path $\mathbf{v}_i(s) \in \mathsf{V} \smallsetminus \{0\}$ such that $\mathbf{v}_i(0) = \mathbf{v}_i$ and $\mathbf{v}_i(1) = \mathbf{u}_i$. Perturbing the paths while fixing the endpoints, we may assume $\mathbf{v}_k(s)$ and $\mathbf{v}_{k+1}(s)$ are linearly independent for all s (this is always possible as $\dim(\mathbb{P}(\mathsf{V})) = 3$). Then, $P_\pm(\mathbf{v}_1(s), \dots, \mathbf{v}_n(s))$ is a well-defined homotopy between $P_\pm(\mathbf{v}_1, \dots, \mathbf{v}_n)$ and $P_\pm(\mathbf{u}_1, \dots, \mathbf{u}_n)$.

The path $P_+(\mathbf{u}_1, \dots, \mathbf{u}_n)$ is contained in a line segment, hence contractible, and the path $P_-(\mathbf{u}_1, \dots, \mathbf{u}_n)$ is a full projective line, hence generates $\pi_1(\mathbb{P}(\mathsf{V}))$. □

DEFINITION 3.5. — *A Legendrian polygon P is* generic *if its non-adjacent vertices are non-incident and its non-adjacent edges are parts of non-intersecting projective lines.*

The moduli space of generic Legendrian n-gons up to the action of $\mathsf{PSp}(\mathsf{V})$ will be denoted by $\mathscr{P}_n$.

As a consequence of Proposition 3.4, $\mathscr{P}_n$ naturally separates into the disjoint union of the space of generic contractible n-gons $\mathscr{P}_n^+$ and the space of generic non-contractible n-gons $\mathscr{P}_n^-$, and each subspace $\mathscr{P}_n^+, \mathscr{P}_n^-$ is a union of connected components.

As a first result, we count the number of generic Legendrian quadrilaterals and pentagons.

Proposition 3.6. — *The moduli space $\mathscr{P}_4$ consists of* 8 *points,* 4 *of which are contractible and* 4 *which are not.*

The moduli space $\mathscr{P}_5$ consists of $2^6 = 64$ points, 32 of which are contractible and 32 which are not.

Proof. — We give the full proof for $\mathscr{P}_5$, and the claim for $\mathscr{P}_4$ can be proven in a similar way.

We first choose a basis in which the symplectic form ω has matrix expression

$$\Omega = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

We claim that all generic Legendrian polygons are in the $\mathsf{PSp}(\mathsf{V})$ orbit of exactly one of the polygons $P_\pm(\mathbf{v}_1^\varepsilon, \mathbf{v}_2^\varepsilon, \mathbf{v}_3^\varepsilon, \mathbf{v}_4^\varepsilon, \mathbf{v}_5^\varepsilon)$, with the vectors $\mathbf{v}_i^\varepsilon$ given by the columns of the matrix

$$\begin{pmatrix} 1 & 0 & -\varepsilon_{2,4}\varepsilon_{3,5}\varepsilon_{1,4}\varepsilon_{2,5} & 0 & 0 \\ 0 & 1 & \varepsilon_{3,5}\varepsilon_{2,5} & 0 & 0 \\ 0 & 0 & \varepsilon_{1,3} & \varepsilon_{1,4} & 0 \\ 0 & 0 & 0 & \varepsilon_{2,4} & \varepsilon_{2,5} \end{pmatrix},$$

where $\varepsilon_{i,j} \in \{-1, 1\}$ and $\varepsilon = (\varepsilon_{1,3}, \varepsilon_{1,4}, \varepsilon_{2,4}, \varepsilon_{2,5}, \varepsilon_{3,5})$. Since there are 5 choices of sign, this gives 2^5 contractible polygons and 2^5 non-contractible polygons. A simple calculation shows that they are all Legendrian and generic.

First we note that no two of these polygons can be in the same orbit since $\omega(\mathbf{v}_i^\varepsilon, \mathbf{v}_j^\varepsilon) = \varepsilon_{i,j}$ for $2 \leqslant |i-j| \leqslant 3$, and the signs of these symplectic products are invariant under the $\mathsf{PSp}(4,\mathbb{R})$ action, change of representatives $\mathbf{v}_i^\varepsilon$ by positive scalars, and simultaneously multiplying all representatives by -1.

Now let $\mathbf{u}_i \in \mathsf{V}$ for $1 \leqslant i \leqslant 5$ and assume P is a generic Legendrian pentagon, so that it can be written either $P_+(\mathbf{u}_1, \dots, \mathbf{u}_5)$ or $P_-(\mathbf{u}_1, \dots, \mathbf{u}_5)$. Denote $\omega(\mathbf{u}_i, \mathbf{u}_j) = \lambda_{i,j}$. The only non-vanishing symplectic products $\lambda_{i,j}$ with $i \leqslant j$ are $\lambda_{1,3}$, $\lambda_{1,4}$, $\lambda_{2,4}$, $\lambda_{2,5}$, $\lambda_{3,5}$.

Consider the positive rescaling constants

$$c_i = \sqrt{\left| \frac{\lambda_{i+1,i+3}\lambda_{i+2,i+4}}{\lambda_{i,i+2}\lambda_{i,i+3}\lambda_{i+1,i+4}} \right|},$$

with the indices taken modulo 5 as indicated in the notation section. Then, $\omega(c_i\mathbf{u}_i, c_j\mathbf{u}_j) = \frac{\lambda_{i,j}}{|\lambda_{i,j}|}$ so we have simultaneously normalized all symplectic products to 1 or -1, and we let $\varepsilon_{i,j} := \frac{\lambda_{i,j}}{|\lambda_{i,j}|}$. The vectors $c_i\mathbf{u}_i$ for $i = 1,2,3,4$ form a basis of V in which the symplectic form has matrix

$$\begin{pmatrix} 0 & 0 & \varepsilon_{1,3} & \varepsilon_{1,4} \\ 0 & 0 & 0 & \varepsilon_{2,4} \\ -\varepsilon_{1,3} & 0 & 0 & 0 \\ -\varepsilon_{1,4} & -\varepsilon_{2,4} & 0 & 0 \end{pmatrix}.$$

Therefore, there exists $g \in \mathsf{Sp}(\mathsf{V})$ such that $gc_i\mathbf{u}_i = \mathbf{v}_i^{\varepsilon}$, for $1 \leqslant i \leqslant 4$. Moreover, since the coordinates of $c_5\mathbf{u}_5$ in the basis $c_i\mathbf{u}_i$ can be recovered from the symplectic products with these vectors, we must have $gc_5\mathbf{u}_5 = \mathbf{v}_5^{\varepsilon}$. We conclude that P is in the orbit of one of the 2^6 pentagons $P_{\pm}(\mathbf{v}_1^{\varepsilon}, \mathbf{v}_2^{\varepsilon}, \mathbf{v}_3^{\varepsilon}, \mathbf{v}_4^{\varepsilon}, \mathbf{v}_5^{\varepsilon})$, completing the proof. □

Definition 3.7. — *A Legendrian polygon P is* transverse *if whenever two points p, q of the polygon are incident, they lie in the same closed edge of the polygon.*

We first observe that there are no nondegenerate Legendrian triangles. This is because the vertices of a Legendrian triangle would span a 3-dimensional totally isotropic subspace, which is impossible by nondegeneracy of ω. The first interesting case of Legendrian polygon is therefore $n = 4$.

For a first example of transverse Legendrian polygon, we show that in a basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ in which the symplectic form ω is represented by the matrix Ω (Equation (3.1)), the Legendrian quadrilateral $P_-(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ is generic and transverse. This is a consequence of the following lemma:

Lemma 3.8. — *Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in \mathsf{V}$ such that $\omega(\mathbf{u}_1, \mathbf{u}_2) = \omega(\mathbf{v}_1, \mathbf{v}_2) = 0$ and $\omega(\mathbf{u}_1, \mathbf{v}_2) \neq 0$. Then, the points $[(1-t)\mathbf{u}_1 + t\mathbf{u}_2]$ and $[(1-s)\mathbf{v}_1 + s\mathbf{v}_2]$ are non-incident for all $t, s \in (0,1)$ if and only if the four symplectic products $\omega(\mathbf{u}_1, \mathbf{v}_1)$, $\omega(\mathbf{u}_1, \mathbf{v}_2)$, $\omega(\mathbf{u}_2, \mathbf{v}_1)$, and $\omega(\mathbf{u}_2, \mathbf{v}_2)$ are all nonpositive or all nonnegative.*

Proof. — The points $[(1-t)\mathbf{u}_1 + t\mathbf{u}_2]$ and $[(1-s)\mathbf{v}_1 + s\mathbf{v}_2]$ are incident if and only if $\omega((1-t)\mathbf{u}_1 + t\mathbf{u}_2, (1-s)\mathbf{v}_1 + s\mathbf{v}_2) = 0$, which expands to

$$(1-t)(1-s)\,\omega(\mathbf{u}_1, \mathbf{v}_1) + (1-t)s\;\omega(\mathbf{u}_1, \mathbf{v}_2) + t(1-s)\,\omega(\mathbf{u}_2, \mathbf{v}_1) + ts\,\omega(\mathbf{u}_2, \mathbf{v}_2) = 0.$$

Suppose that the four symplectic products $\omega(\mathbf{u}_i, \mathbf{v}_j)$ are all nonpositive or all nonnegative. Then, since one of them is nonzero the above equation has no solution for $0 < t, s < 1$, and so $[(1-t)\mathbf{u}_1 + t\mathbf{u}_2]$ and $[(1-s)\mathbf{v}_1 + s\mathbf{v}_2]$ are non-incident.

Conversely, note that for fixed values of a, b, c, d the function

$$\phi(t,s) = (1-t)(1-s)a + (1-t)s\,b + t(1-s)c + ts\,d$$

satisfies $\phi(0,0) = a$, $\phi(0,1) = b$, $\phi(1,0) = c$, and $\phi(1,1) = d$.

If any two distinct vertices x, y of the square $[0,1] \times [0,1]$ have different signs, say $\phi(x) > 0$ and $\phi(y) < 0$, we can apply the intermediate value theorem to a path joining x and y whose relative interior is contained in the interior of the square to find $(t,s) \in (0,1) \times (0,1)$ such that $\phi(t,s) = 0$.

Setting

$$(a,b,c,d) = (\omega(\mathbf{u}_1, \mathbf{v}_1),\ \omega(\mathbf{u}_1, \mathbf{v}_2),\ \omega(\mathbf{u}_2, \mathbf{v}_1),\ \omega(\mathbf{u}_2, \mathbf{v}_2))$$

and using the argument above yields a pair (t,s) such that $[(1-t)\mathbf{u}_1 + t\mathbf{u}_2]$ and $[(1-s)\mathbf{v}_1 + s\mathbf{v}_2]$ are incident. □

Proposition 3.9. — *The Legendrian polygon $P_+(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is not transverse for $n \geqslant 4$.*

Proof. — Assume $P_+(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is transverse. In particular, non-adjacent vertices must be non-incident so $\omega(\mathbf{v}_i, \mathbf{v}_j) \neq 0$ whenever $|i - j| > 1$. Then, applying Lemma 3.8 to $\mathbf{v}_i, \mathbf{v}_{i+1}, \mathbf{v}_j, \mathbf{v}_{j+1}$ the symplectic products $\omega(\mathbf{v}_i, \mathbf{v}_j)$ have the same sign for all $i < j - 1$. Moreover, if $n > 4$ the segment $s_{\mathbf{v}_3,\mathbf{v}_4}$ is transverse to the segment $s_{\mathbf{v}_n,\mathbf{v}_1}$, so $\omega(\mathbf{v}_3, \mathbf{v}_n)$ has the same sign as $\omega(\mathbf{v}_3, \mathbf{v}_1) = -\omega(\mathbf{v}_1, \mathbf{v}_3)$, a contradiction. The case $n = 4$ can be normalized to have the matrix of symplectic products Ω (Equation (3.1)) and we check by hand that the finitely many sign choices which can be made for the vectors $\mathbf{v}_i$ are all non-transverse. □

Combining Propositions 3.9 and 3.4, we obtain

Corollary 3.10. — *A transverse Legendrian polygon generates $\pi_1(\mathbb{P}(V))$.*

For non-contractible Legendrian polygons, we now give a simple criterion for transversality.

Proposition 3.11. — *For $n \geqslant 4$, the Legendrian polygon $P_-(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is transverse if and only if $\omega(\mathbf{v}_i, \mathbf{v}_j) > 0$ for all $i < j - 1$, or $\omega(\mathbf{v}_i, \mathbf{v}_j) < 0$ for all $i < j - 1$.*

Proof. — We start with the reverse direction. The non-incidence for segments $s_{\mathbf{v}_i,\mathbf{v}_{i+1}}$ and $s_{\mathbf{v}_j,\mathbf{v}_{j+1}}$ with $i < j - 1$ follows at once from Lemma 3.8 applied to the quadruple $\mathbf{v}_i, \mathbf{v}_{i+1}, \mathbf{v}_j, \mathbf{v}_{j+1}$. It remains to show that segments $s_{\mathbf{v}_i,\mathbf{v}_{i+1}}$ are non-incident to the final segment $s_{\mathbf{v}_n,-\mathbf{v}_1}$. This follows from another application of Lemma 3.8 since $\omega(\mathbf{v}_i, -\mathbf{v}_1) = \omega(\mathbf{v}_1, \mathbf{v}_i)$ and $\omega(\mathbf{v}_{i+1}, -\mathbf{v}_1) = \omega(\mathbf{v}_1, \mathbf{v}_{i+1})$.

For the forward direction, if the polygon is transverse then the non-adjacent vertices are non-incident and so $\omega(\mathbf{v}_i, \mathbf{v}_j) \neq 0$ for $i < j-1$. Using Lemma 3.8 we find that $\omega(\mathbf{v}_i, \mathbf{v}_j)$ all have the same sign. □

As noted before, the common sign of $\omega(\mathbf{v}_i, \mathbf{v}_j)$ for a transverse Legendrian polygon $P_-(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is well-defined (independent of the representatives $\mathbf{v}_i$). We will call the corresponding classes of Legendrian polygons *positive-transverse* and *negative-transverse* according to this sign. Reversing the cyclic order exchanges positive-transverse and negative-transverse polygons.

We now recall the definition of the space of generic Lagrangian configurations from [5] and relate it to the space of transverse Legendrian polygons.

Definition 3.12. — *A generic $(2, n)$-Lagrangian configuration is an n-tuple $(p_1, \ldots, p_n)$ in $\mathbb{P}(\mathsf{V})$ such that a pair p_i, p_j is incident if and only if it is cyclically adjacent.*

The moduli space of generic Lagrangian configurations $\mathcal{L}_{2,n}$ is the space of generic $(2, n)$-Lagrangian configurations, up to the action of $\mathsf{PSp}(\mathsf{V})$.

When $n \geqslant 5$, the map sending a transverse Legendrian n-gon to its tuple of vertices is injective, because if a tuple $p_1, \ldots, p_n$ admits representatives $\mathbf{v}_1, \ldots, \mathbf{v}_n$ such that $\omega(\mathbf{v}_i, \mathbf{v}_j)$ all have the same sign for $j > i+1$, then that choice is unique up to multiplying each $\mathbf{v}_i$ by a positive scalar or reversing all signs. Therefore, these representatives determine a unique transverse Legendrian n-gon $P_-(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. The image of this vertex map in $\mathcal{L}_{2,n}$ is an open subset, and the vertex map is a homeomorphism onto its image since locally we can keep the same signs for the representatives.

Similarly, generic Legendrian n-gons (not necessarily transverse) form a 2^n-sheeted cover of an open subset of $\mathcal{L}_{2,n}$: there are 2^n choices of sign for the representative vectors, up to switching all of them, so 2^{n-1} distinct options; each of these 2^{n-1} choices defines a pair of generic Legendrian n-gons, one via P_+ and one via P_-, for a total of 2^n points in each fiber.

As a consequence of Proposition 2.7 of [5], these three moduli spaces are smooth manifolds of dimension $2(n-5)$ when $n \geqslant 5$.

Proposition 3.13. — *The moduli space of transverse Legendrian 4-gons is a pair of points, represented by the polygons $P_-(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ and $P_-(\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_3, -\mathbf{e}_4)$, where $\mathbf{e}_i$ are vectors forming a basis in which the symplectic form has matrix Ω (Equation (3.1)).*

Proof. — Let P be a Legendrian 4-gon. Then, up to rescaling, its vertices form a basis in which the symplectic form is the standard symplectic form of Equation (3.1).

In this basis, the pointwise stabilizer of the four projective vertices in $\mathsf{PSp}(V)$ is the collection of diagonal matrices. Naming the four basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, the eight possible Legendrian quadrilaterals are

$$P_{\pm}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4), \quad P_{\pm}(\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_3, \mathbf{e}_4),$$
$$P_{\pm}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, -\mathbf{e}_4), \quad P_{\pm}(\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_3, -\mathbf{e}_4),$$

where we normalized the first two basis vectors by applying diagonal matrices with diagonal entries $(-1, 1, -1, 1)$ and $(1, -1, 1, -1)$. By Lemma 3.8, we conclude that out of these only $P_{-}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ and $P_{-}(\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_3, -\mathbf{e}_4)$ are transverse. □

We now give a definition of the *Maslov index*, a classical invariant of triples of real Lagrangians taking values in the integers. Let L_1, L_2 be transverse Lagrangians. Define a linear map $\sigma_{L_1,L_2} \coloneqq I \oplus -I$ according to the splitting $\mathsf{V} = L_1 \oplus L_2$. It is easy to check that this map is an anti-symplectic involution.

Definition 3.14. — *Let L_1, L_2, L_3 be Lagrangians in V and assume that L_1 and L_3 are transverse. The* Maslov form *is the symmetric nondegenerate bilinear form on L_2 defined by*

$$b_{L_1,L_2,L_3}(\mathbf{u}, \mathbf{v}) = \omega(\mathbf{u}, \sigma_{L_1,L_3}\mathbf{v})|_{L_2}.$$

The Maslov index $\mathcal{M}(L_1, L_2, L_3)$ *is the signature of b_{L_1,L_2,L_3}, that is,*

$$\mathcal{M}(L_1, L_2, L_3) = k_+ - k_-$$

where k_+ is the number of positive eigenvalues and k_- is the number of negative eigenvalues of b_{L_1,L_2,L_3}.

The Maslov index takes its values in the set $\{-2, 0, 2\}$ and is a complete $\mathsf{PSp}(4, \mathbb{R})$ invariant of triples of pairwise transverse Lagrangians.

4. Legendrian polygons and piecewise circular curves

In this section, we establish a dictionary between the contact projective geometry of $\mathbb{P}(\mathsf{V})$ and the geometry of co-oriented circles in the 2-sphere.

Let (V, ω) be a 4-dimensional real symplectic vector space as before. Let J be a complex structure on V which is anti-symplectic, meaning J is a linear automorphism of V such that $J^2 = -I$ and $J^*\omega = -\omega$.

As the name suggests, J endows V with the structure of a complex vector space where multiplication by i is given by the automorphism J. Moreover, since $J^2 = -I$ and J is anti-symplectic, the bilinear form

$$\omega_J(\mathbf{u}, \mathbf{v}) \coloneqq \omega(\mathbf{u}, J\mathbf{v})$$

is also symplectic. Together, ω and ω_J define a $\mathbb{C}$-linear symplectic form $\omega_{\mathbb{C}}$ on V:

$$\omega_{\mathbb{C}} := \omega - i\omega_J.$$

Remark 4.1. — This is somewhat analogous to the standard Kähler geometry construction of Hermitian form from a symplectic form and an almost-complex structure, but the goal here is to obtain a complex symplectic form in order to relate the real and complex symplectic groups.

Consider the projection

$$\pi : \mathbb{P}_{\mathbb{R}}(\mathsf{V}) \longrightarrow \mathbb{P}_{\mathbb{C}}(\mathsf{V})$$

which maps the real projectivization of a vector $\mathbf{v} \in \mathsf{V}$, denoted by $[\mathbf{v}]_{\mathbb{R}}$, to its complex projectivization $[\mathbf{v}]_{\mathbb{C}}$. This projection π is a circle bundle, and it is related to the Hopf fibration $p : \mathbb{S}^3 \to \mathbb{S}^2 \cong \mathbb{CP}^1$ by $p = \pi \circ \iota$, where ι is the covering map $\mathbb{S}^3 \to \mathbb{RP}^3$.

DEFINITION 4.2. — *The* space of oriented contact elements *of* $\mathbb{P}_{\mathbb{C}}(\mathsf{V})$ *is the spherical cotangent bundle* $\mathbb{S}(T^*\mathbb{P}_{\mathbb{C}}(\mathsf{V}))$*. It is naturally a contact manifold with the contact distribution given by the tautological* 1*-form (or Liouville* 1*-form).*

The tangent space at a point $[\mathbf{v}]_{\mathbb{C}} \in \mathbb{P}_{\mathbb{C}}(\mathsf{V})$ identifies with the space of $\mathbb{C}$-linear maps $\mathrm{Hom}_{\mathbb{C}}([\mathbf{v}]_{\mathbb{C}}, \mathsf{V}/[\mathbf{v}]_{\mathbb{C}})$. Similarly, the cotangent space at a point $[\mathbf{v}]_{\mathbb{C}} \in \mathbb{P}_{\mathbb{C}}(\mathsf{V})$ identifies with the space of $\mathbb{C}$-linear maps $\mathrm{Hom}_{\mathbb{C}}(\mathsf{V}/[\mathbf{v}]_{\mathbb{C}}, [\mathbf{v}]_{\mathbb{C}})$.

With these identifications, the evaluation of a covector $\alpha \in T^*_{[\mathbf{v}]_{\mathbb{C}}}\mathbb{P}_{\mathbb{C}}(\mathsf{V})$ on a vector $X \in T_{[\mathbf{v}]_{\mathbb{C}}}\mathbb{P}_{\mathbb{C}}(\mathsf{V})$ is given by the composition

$$\alpha \circ X \in \mathrm{Hom}([\mathbf{v}]_{\mathbb{C}}, [\mathbf{v}]_{\mathbb{C}}) \cong \mathbb{C}.$$

Since we are mostly concerned with real manifolds, when U is a complex vector space we use the $\mathbb{C}$-linear isomorphism $\alpha \mapsto \Re(\alpha)$ between $\mathrm{Hom}_{\mathbb{C}}(U, \mathbb{C})$ and $\mathrm{Hom}_{\mathbb{R}}(U, \mathbb{R})$ to identify complex covectors with real covectors.

In order to identify $\mathbb{S}\mathrm{Hom}_{\mathbb{C}}(\mathsf{V}/[\mathbf{v}]_{\mathbb{C}}, [\mathbf{v}]_{\mathbb{C}})$ with the space of oriented contact elements at $[\mathbf{v}]_{\mathbb{C}}$, we note that an element $\alpha \in \mathrm{Hom}_{\mathbb{C}}(\mathsf{V}/[\mathbf{v}]_{\mathbb{C}}, [\mathbf{v}]_{\mathbb{C}})$ defines a halfspace of the tangent space at $[\mathbf{v}]_{\mathbb{C}}$ by $\Re(\alpha(X)) > 0$, and this is invariant under multiplication of α by a positive real. In figures we depict an oriented contact element as an arrow perpendicular to the boundary of this halfspace and pointing to its interior (Figures 1.1, 5.1, and 5.2).

The following proposition will allow us to interpret the points of $\mathbb{P}_{\mathbb{R}}(\mathsf{V})$ as oriented contact elements of $\mathbb{P}_{\mathbb{C}}(\mathsf{V})$.

Proposition 4.3. — *The map*

$$\begin{aligned} F : \mathbb{P}_{\mathbb{R}}(\mathsf{V}) &\longrightarrow \mathbb{S}(T^*\mathbb{P}_{\mathbb{C}}(\mathsf{V})) \\ [\mathbf{v}]_{\mathbb{R}} &\longmapsto ([\mathbf{v}]_{\mathbb{C}}, [(\mathbf{u} \mapsto \omega_{\mathbb{C}}(\mathbf{u},\mathbf{v})\mathbf{v})]_{\mathbb{R}_{>0}}), \end{aligned}$$

where the right hand side is viewed as an element of $\mathrm{Hom}_{\mathbb{C}}(\mathsf{V}/[\mathbf{v}]_{\mathbb{C}}, [\mathbf{v}]_{\mathbb{C}})$, *is an isomorphism of circle bundles over* $\mathbb{P}_{\mathbb{C}}(\mathsf{V})$.

Proof. — We first verify that the formula $(\mathbf{u} \mapsto \omega_{\mathbb{C}}(\mathbf{u},\mathbf{v})\mathbf{v})$ defines an element of $\mathrm{Hom}_{\mathbb{C}}(\mathsf{V}/[\mathbf{v}]_{\mathbb{C}}, [\mathbf{v}]_{\mathbb{C}})$.

Since $\omega_{\mathbb{C}}$ is $\mathbb{C}$-bilinear, the map $\mathbf{u} \to \omega_{\mathbb{C}}(\mathbf{u},\mathbf{v})\mathbf{v}$ is $\mathbb{C}$-linear. Moreover, it descends to the quotient $\mathsf{V}/[\mathbf{v}]_{\mathbb{C}}$ since for all $\lambda \in \mathbb{C}$ we have

$$\omega_{\mathbb{C}}(\mathbf{u}+\lambda\mathbf{v},\mathbf{v}) = \omega_{\mathbb{C}}(\mathbf{u},\mathbf{v}) + \lambda\omega_{\mathbb{C}}(\mathbf{v},\mathbf{v}) = \omega_{\mathbb{C}}(\mathbf{u},\mathbf{v}).$$

So we obtain a map from $\mathsf{V} \smallsetminus \{0\}$ to $\mathbb{S}(T^*\mathbb{P}_{\mathbb{C}}(\mathsf{V}))$. We now show that it induces a map from the projective space to the sphere of directions. For any $t \in \mathbb{R}^*$, we have

$$\omega_{\mathbb{C}}(\mathbf{u},t\mathbf{v})t\mathbf{v} = t^2\omega_{\mathbb{C}}(\mathbf{u},\mathbf{v})\mathbf{v},$$

so the map F is well-defined on the quotients.

Identifying $\mathsf{V} \cong \mathbb{C}^2$, we can trivialize the bundle $\mathbb{P}_{\mathbb{R}}(\mathsf{V})$ over the affine patch $\left\{[\begin{smallmatrix} z \\ 1 \end{smallmatrix}]_{\mathbb{C}} \in \mathbb{CP}^1\right\} \subset \mathbb{P}_{\mathbb{C}}(\mathsf{V})$ by

$$\begin{aligned} \varphi : \mathbb{C} \times S^1 &\longrightarrow \mathbb{P}_{\mathbb{R}}(\mathbb{C}^2) \\ (z, \mathrm{e}^{i\theta}) &\longmapsto \begin{bmatrix} z\,\mathrm{e}^{\frac{i\theta}{2}} \\ \mathrm{e}^{\frac{i\theta}{2}} \end{bmatrix}_{\mathbb{R}}. \end{aligned}$$

Note that $\mathrm{e}^{(i\theta+2\pi)/2} = -\,\mathrm{e}^{i\theta/2}$ and so the right-hand side is well-defined as a projective equivalence class, only depending on $\mathrm{e}^{i\theta} \in S^1$.

Similarly, we can trivialize $\mathrm{Hom}_{\mathbb{C}}(\mathsf{V}/[\mathbf{v}]_{\mathbb{C}}, [\mathbf{v}]_{\mathbb{C}})$ over the same affine patch by

$$\begin{aligned} \psi : \mathbb{C} \times S^1 &\longrightarrow \mathbb{S}\left(\mathrm{Hom}_{\mathbb{C}}(\mathsf{V}/[\mathbf{v}]_{\mathbb{C}}, [\mathbf{v}]_{\mathbb{C}})\right) \\ (z, \mathrm{e}^{i\theta}) &\longmapsto \left(\begin{bmatrix} z \\ 1 \end{bmatrix}_{\mathbb{C}}, \left[\mathbf{u} \mapsto \mathrm{e}^{i\theta}\,\omega_{\mathbb{C}}\left(\mathbf{u}, \begin{pmatrix} z \\ 1 \end{pmatrix}\right)\begin{pmatrix} z \\ 1 \end{pmatrix}\right]_{\mathbb{R}_{>0}} \right). \end{aligned}$$

In this pair of trivializations, the map F is the identity and hence is continuous and an isomorphism on each fiber. Similar trivilizations on the affine patch $\left\{[\begin{smallmatrix} 1 \\ z \end{smallmatrix}]_{\mathbb{C}} \in \mathbb{CP}^1\right\}$ yield the same result, and we conclude that F is a bundle isomorphism. □

The group

$$\mathsf{PSp}(2,\mathbb{C}) \coloneqq \{M \in \mathsf{PSp}(\mathsf{V}) \mid MJ = JM\}$$

is isomorphic to $\mathsf{PSL}(2,\mathbb{C})$ and acts transitively on the base $\mathbb{P}_{\mathbb{C}}(\mathsf{V})$ and also acts on the total spaces $\mathbb{P}_{\mathbb{R}}(\mathsf{V})$ and $\mathbb{S}(T^*\mathbb{P}_{\mathbb{C}}(\mathsf{V}))$ of the two circle bundles. The action on $\mathbb{S}(T^*\mathbb{P}_{\mathbb{C}}(\mathsf{V}))$ is as follows : let $f \in \mathbb{S}\mathrm{Hom}_{\mathbb{C}}(\mathsf{V}/[\mathbf{v}]_{\mathbb{C}}, [\mathbf{v}]_{\mathbb{C}})$, then

$$M \cdot f = MfM^{-1} \in \mathbb{S}\mathrm{Hom}_{\mathbb{C}}(\mathsf{V}/[M\mathbf{v}]_{\mathbb{C}}, [M\mathbf{v}]_{\mathbb{C}})$$

is the image of f by the action of $M \in \mathsf{PSp}(2,\mathbb{C})$. The isomorphism in Proposition 4.3 intertwines the two actions on the total spaces.

Proposition 4.4. — *The map* $F : \mathbb{P}_{\mathbb{R}}(\mathsf{V}) \to \mathbb{S}(T^*\mathbb{P}_{\mathbb{C}}(\mathsf{V}))$ *is an isomorphism of contact manifolds.*

Proof. — We will use the trivializations φ, ψ as in the proof of Proposition 4.3.

In the coordinate chart φ, a path $\gamma(t) = (z(t), \mathrm{e}^{i\theta(t)})$ is tangent to the contact distribution of $\mathbb{P}_{\mathbb{R}}\mathbb{C}^2$ if and only if

$$\omega\left(\begin{pmatrix} z(t)\,\mathrm{e}^{\frac{i\theta(t)}{2}} \\ \mathrm{e}^{\frac{i\theta(t)}{2}} \end{pmatrix}, \begin{pmatrix} z'(t)\,\mathrm{e}^{\frac{i\theta(t)}{2}} + z(t)\frac{i\theta'(t)}{2}\,\mathrm{e}^{\frac{i\theta(t)}{2}} \\ \frac{i\theta'(t)}{2}\,\mathrm{e}^{\frac{i\theta(t)}{2}} \end{pmatrix}\right) = 0,$$

which simplifies to

$$\Re\left(\mathrm{e}^{i\theta(t)}\,\omega_{\mathbb{C}}\left(\begin{pmatrix} z(t) \\ 1 \end{pmatrix}, \begin{pmatrix} z'(t) \\ 0 \end{pmatrix}\right)\right) = 0.$$

In the coordinate chart ψ for $\mathbb{S}(T^*\mathbb{CP}^1)$, the path $\gamma(t) = (z(t), \mathrm{e}^{i\theta(t)})$ is in the contact distribution of if and only if

$$\psi(\gamma(t))\left(d\pi\left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t}\psi(\gamma(s))\right)\right) = \Re\left(\mathrm{e}^{i\theta(t)}\,\omega_{\mathbb{C}}\left(\begin{pmatrix} z'(t) \\ 0 \end{pmatrix}, \begin{pmatrix} z(t) \\ 1 \end{pmatrix}\right)\right) = 0.$$

Since these equations define the same plane, and the map $\psi^{-1} \circ F \circ \varphi$ is the identity, we conclude that F is a contactomorphism. ☐

Now that we have established the contact isomorphism between $\mathbb{P}_{\mathbb{R}}(\mathsf{V})$ and the space of oriented contact elements to the sphere, we can begin to explore the geometric interpretations. The object which corresponds to a Legendrian line (of Lagrangian subspace) is a co-oriented circle.

Definition 4.5. — *A* co-oriented circle *in* $\mathbb{P}_{\mathbb{C}}(\mathsf{V})$ *is a round circle* C *of* $\mathbb{P}_{\mathbb{C}}(\mathsf{V})$*, possibly degenerated to a point, together with a choice of disk bounded by* C *when nondegenerate.*

The choice of disk provides a transverse orientation at each point of a co-oriented circle, pointing towards the interior of the disk. We say that two co-oriented circles are tangent if the circles are tangent and the orientations match at the tangency point. If one of the circles is degenerate, they are considered tangent if the point circle is contained in the other circle.

Every complex line $\ell \in \mathbb{P}_{\mathbb{C}}(\mathsf{V})$, considered as a 2-dimensional real subspace of V, defines a real Lagrangian so we get an inclusion $\mathbb{P}_{\mathbb{C}}(\mathsf{V}) \subset \mathsf{Lag}(\mathsf{V})$. We interpret this inclusion as saying that a Lagrangian which is a complex subspace is mapped to a degenerate circle by the projection π. For the other Lagrangians, we have the following :

Lemma 4.6. — *Let $L \subset \mathsf{V}$ be a Lagrangian subspace for ω. If L is not a complex subspace, then $\pi(\mathbb{P}_{\mathbb{R}}L)$ is a nondegenerate circle. The two connected components of $\mathbb{P}_{\mathbb{C}}(\mathsf{V}) \smallsetminus \pi(\mathbb{P}_{\mathbb{R}}L)$ are distinguished by the Maslov index :*

$$\{\ell \in \mathbb{P}_{\mathbb{C}}(\mathsf{V}) \mid \mathcal{M}(L, \ell, JL) = 2\}$$

and

$$\{\ell \in \mathbb{P}_{\mathbb{C}}(\mathsf{V}) \mid \mathcal{M}(L, \ell, JL) = -2\}.$$

Proof. — Since L is not a complex subspace, $L \oplus JL = \mathsf{V}$ defines a splitting of the vector space V into a pair of Lagrangians: the sum must be direct when $L \neq JL$ because $L + JL$ is invariant under J, hence is a complex subspace which must have even real dimension and therefore be equal to V.

The anti-symplectic involution $\sigma_{L,JL} = I \oplus -I$ is anti-linear ($\sigma_{L,JL} J = -J\sigma_{L,JL}$) and therefore defines a *real structure* on the complex vector space V. The subspace fixed by $\sigma_{L,JL}$ is the Lagrangian L. When endowed with this real structure, the complex projectivization $\pi(\mathbb{P}_{\mathbb{R}}L)$ therefore identifies with the real projective line $\mathbb{P}_{\mathbb{R}}(L) \subset \mathbb{P}_{\mathbb{C}}(\mathsf{V})$, which is a (generalized) circle.

The involution $\sigma_{L,JL}$ exchanges the two connected components of the complement $\mathbb{P}_{\mathbb{C}}(\mathsf{V}) \smallsetminus \pi(\mathbb{P}_{\mathbb{R}}L)$. The Maslov index $\mathcal{M}(L, \ell, JL)$ is the signature of the symmetric bilinear form $b_{L,\ell,JL} = \omega(\,\cdot\,, \sigma_{L,JL}(\,\cdot\,))$ restricted to ℓ.

Note that J is an automorphism of the bilinear form $b_{L,\ell,JL}$:

$$\omega(J\mathbf{u}, \sigma_{L,JL}J\mathbf{v}) = \omega(J\mathbf{u}, -J\sigma_{L,JL}\mathbf{v}) = \omega(\mathbf{u}, \sigma_{L,JL}\mathbf{v}).$$

Moreover, $\mathbf{v}$ is always orthogonal to $J\mathbf{v}$ for the form b_{L,J_L}. Hence, if ℓ is the complex line spanned by the vector $\mathbf{v}$, $b_{L,\ell,JL}$ is either definite or identically zero. If it is identically zero, then $\omega(\mathbf{v}, \sigma_{L,JL}\mathbf{v}) = 0$ and $\omega(J\mathbf{v}, \sigma_{L,JL}\mathbf{v}) = 0$ and the fact that $\ell = \mathrm{span}_{\mathbb{R}}(\mathbf{v}, J\mathbf{v})$ is a Lagrangian imply that $\sigma_{L,JL}(\mathbf{v}) \in \ell$, and hence $\ell \in \pi(\mathbb{P}_{\mathbb{R}}(L))$.

By continuity of the map $\ell \mapsto b_{L,\ell,JL}$ we conclude that the two complementary components of $\pi(\mathbb{P}_{\mathbb{R}}L)$ consist of where $b_{L,\ell,JL}$ is positive definite and where it is negative definite. □

Let $L \subset \mathsf{V}$ be a Lagrangian subspace for the symplectic form ω. If $L \neq JL$, let C_L denote the co-oriented circle $\pi(\mathbb{P}_{\mathbb{R}}L)$ with the co-orientation given by the disk of complex lines $\ell \subset \mathsf{V}$ such that $\mathcal{M}(L, \ell, JL) = 2$. If $L = JL$, then L is a complex subspace and we let $C_L = \pi(\mathbb{P}_{\mathbb{R}}L)$ which we interpret as a zero radius circle. Then, $L \mapsto C_L$ is a bijection between the set of Lagrangians in V and the set of co-oriented circles in $\mathbb{P}_{\mathbb{C}}(\mathsf{V})$.

4.1. In coordinates

Concretely, we fix a basis $E = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ in which the symplectic form is given by

$$\Omega' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{4.1}$$

We call such a basis E a *symplectic basis*. We can further specify the basis so that the compatible complex structure has matrix expression

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In such a basis, the affine patch consisting of the Lagrangians

$$L(a, b, c) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a & b + c \\ b - c & -a \end{bmatrix}$$

is an open and dense subset. Recall from Section 2 that 4×2 matrices between square brackets are to be interpreted as the Lagrangians spanned by their columns in the fixed basis.

The complex structure acts on Lagrangians $L(a, b, c)$ by reversing the sign of c, and so the Lagrangians in the affine patch which are complex lines are precisely those of the form $L(a, b, 0)$. The pair $(\mathbf{e}_1, \mathbf{e}_3)$ forms a complex basis for V in which the complex symplectic form $\omega_{\mathbb{C}}$ has matrix $\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$. In this basis, we have

$$L(x, y, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ x & y \\ y & -x \end{bmatrix} = \begin{bmatrix} 1 \\ x + iy \end{bmatrix},$$

where again the 2×1 matrix represents the complex subspace spanned by its single column.

Suppose $c \neq 0$ so that $L(a,b,c)$ is not a complex subspace. The Maslov index $\mathcal{M}(L(a,b,c), L(x,y,0), JL(a,b,c))$ is the index of the symmetric matrix

$$\begin{pmatrix} \frac{(x-a)^2(y-b)^2-c^2}{c} & 0 \\ 0 & \frac{(x-a)^2+(y-b)^2-c^2}{c} \end{pmatrix},$$

which is positive definite if and only if either $c > 0$ and (x,y) is outside of the circle of radius $|c|$ centered at $a+bi$, or $c < 0$ and (x,y) is inside of that circle. Thus, $L(a,b,c)$ represents the circle of radius $|c|$ centered at $a+bi$ co-oriented towards the outside if $c > 0$ and towards the inside if $c < 0$.

Lemma 4.7. — *Let $[\mathbf{v}]_{\mathbb{R}} \in \mathbb{P}_{\mathbb{R}}(\mathsf{V})$ and $L \subset \mathsf{V}$ be a Lagrangian. Then, $\mathbf{v} \in L$ if and only if $\pi([\mathbf{v}]_{\mathbb{R}}) \in C_L$ and the oriented contact element to C_L at $\pi([\mathbf{v}]_{\mathbb{R}})$ is given by $[\mathbf{v}]_{\mathbb{R}}$ through the isomorphism F of Proposition 4.3.*

Proof. — By applying an element of $\mathsf{Sp}(2,\mathbb{C})$ if needed, we may assume that the circle C_L is contained in the standard affine patch of $\mathbb{CP}^1$. This means that the Lagrangian L does not contain any vector of the form $\begin{pmatrix} 0 \\ 0 \\ x \\ y \end{pmatrix}$ and hence that it lies in the affine patch of Lagrangians of the form $L(a,b,c)$.

We consider

$$[\mathbf{v}]_{\mathbb{R}} = \begin{bmatrix} k \\ l \\ ka + l(b+c) \\ k(b-c) - la \end{bmatrix}_{\mathbb{R}} \subset L(a,b,c).$$

This point projects to

$$\begin{aligned} \pi([\mathbf{v}]_{\mathbb{R}}) = [\mathbf{v}]_{\mathbb{C}} &= \begin{bmatrix} k - il \\ k(a + (b-c)i) + l(b + c - ai) \end{bmatrix}_{\mathbb{C}} \\ &= \begin{bmatrix} 1 \\ a + bi - ic\frac{k+il}{k-il} \end{bmatrix}_{\mathbb{C}} \in \mathbb{CP}^1. \end{aligned}$$

This is a parametrization by $[k,l] \in \mathbb{RP}^1$ of the circle corresponding to $L(a,b,c)$ where $[\mathbf{v}]_{\mathbb{C}}$ is the unique point on the circle of radius $|c|$ centered at $a+bi$ which is in the direction of $-i\frac{(k+il)^2}{|k+il|^2}$ if $c > 0$, and in the opposite direction if $c < 0$.

The tangent vector at $[\mathbf{v}]_{\mathbb{C}}$ corresponding to $z \in \mathbb{C}$, as a linear map, is represented by

$$X_z = \mathbf{v}_{\mathbb{C}}^* \otimes \begin{pmatrix} 0 \\ (k-il)z \end{pmatrix} \in \mathrm{Hom}_{\mathbb{C}}([\mathbf{v}]_{\mathbb{C}}, \mathbb{C}^2/[\mathbf{v}]_{\mathbb{C}})$$

where $\mathbf{v}_{\mathbb{C}}^*$ is the unique $\mathbb{C}$-linear form mapping $\mathbf{v}$ to 1.

Evaluating the covector $F([\mathbf{v}]_{\mathbb{R}})$ at X_z yields

$$\omega_{\mathbb{C}}\left(\begin{pmatrix} 0 \\ (k-il)z \end{pmatrix}, \mathbf{v}\right) = zi(k-il)^2.$$

We conclude that the halfspace of the tangent space at $[\mathbf{v}]_{\mathbb{C}}$ defined by the covector $F([\mathbf{v}]_{\mathbb{R}})$ is

$$\{z \in \mathbb{C} \mid \Re(zi(k-il)^2) > 0\} = \{z \in \mathbb{C} \mid \Re(z\,\overline{(-i(k+il)^2)}) > 0\}.$$

This halfspace is bounded by the line spanned by $(k+il)^2 = |k+il|^2 \frac{k+il}{k-il}$ and contains $-i(k+il)^2$.

Therefore, in the affine chart where the first coordinate is 1, if $c > 0$ the circle corresponding to $L(a,b,c)$ is oriented outwards and the halfspace of the tangent space at $a + bi - ic\frac{k+il}{k-il}$ determined by $F([\mathbf{v}]_{\mathbb{R}})$ is also oriented outwards. If $c < 0$, the circle corresponding to $L(a,b,c)$ is oriented inwards and the halfspace of the tangent space is also oriented inwards (see Figure 4.1).

We conclude that the oriented contact elements represented by lines in $\mathbb{R}^4$ contained in the Lagrangian $L(a,b,c)$ are precisely the oriented contact elements of the co-oriented circle corresponding to $L(a,b,c)$, proving the proposition. □

Corollary 4.8. — *Let $[\mathbf{u}]_{\mathbb{R}}, [\mathbf{v}]_{\mathbb{R}} \in \mathbb{P}_{\mathbb{R}}(\mathsf{V})$. Then, $\omega(\mathbf{u},\mathbf{v}) = 0$ if and only if $F([\mathbf{u}]), F([\mathbf{v}])$ are oriented contact elements tangent to a common co-oriented circle.*

In the coordinates we are using, the embedding $\mathsf{SL}(2,\mathbb{C}) \cong \mathsf{Sp}(2,\mathbb{C}) \hookrightarrow \mathsf{Sp}(4,\mathbb{R})$ induced by forgetting the imaginary part of the complex symplectic form $\omega_{\mathbb{C}}$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} \Re(a) & \Im(a) & \Re(b) & -\Im(b) \\ -\Im(a) & \Re(a) & -\Im(b) & -\Re(b) \\ \Re(c) & \Im(c) & \Re(d) & -\Im(d) \\ \Im(c) & -\Re(c) & \Im(d) & \Re(d) \end{pmatrix}.$$

The form of this embedding is slightly unusual because of our choice of normalization for the complex structure J. These matrices act by Möbius transformations on $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^2)$, inducing an action on the unit tangent bundle of $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^2)$.

Let $r \in \mathbb{R}$ and define

$$T_r := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & r & 1 & 0 \\ -r & 0 & 0 & 1 \end{pmatrix}.$$

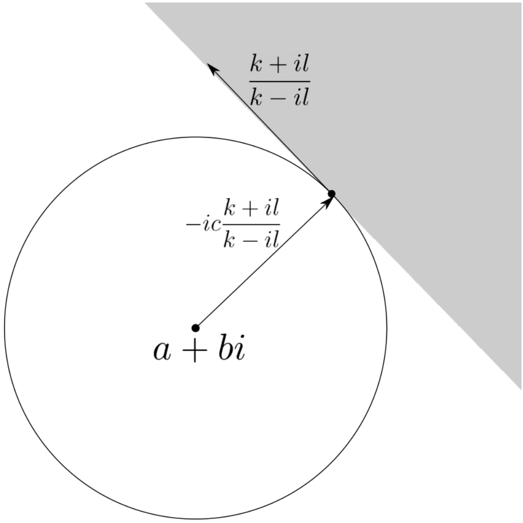


Figure 4.1. An oriented contact element to the circle corresponding to $L(a,b,c)$ with $c>0$, in local coordinates.

Then,

$$T_r(L(a,b,c)) = L(a,b,c+r).$$

In geometric terms, the transformation $T_r \in \mathsf{Sp}(4,\mathbb{R})$ maps any oriented circle in the affine patch $\{L(a,b,c) \mid a,b,c \in \mathbb{R}\}$ to the circle with the same center and (signed) radius increased by r. Its action on piecewise circular curves is therefore the radial translation of parameter r.

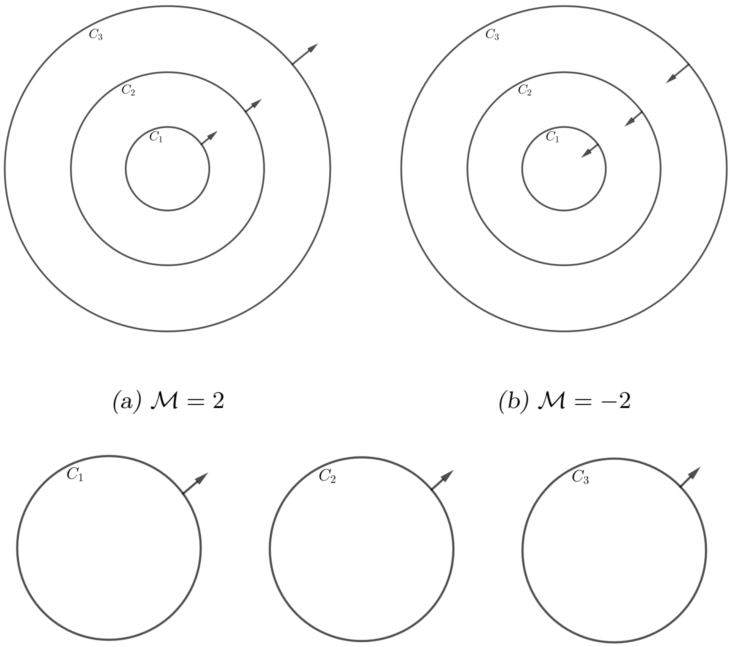
Proposition 4.9. — *Möbius transformations and radial translations T_r generate the Lie group $\mathsf{PSp}(4,\mathbb{R})$.*

Proof. — The proposition follows by a Lie algebra computation using the matrix expressions of the two subgroups and the fact that $\mathsf{PSp}(4,\mathbb{R})$ is connected. □

Therefore, the classification problem of piecewise circular n-gons up to Möbius transformations and radial translations reduces to the classification of Legendrian n-gons in $\mathbb{RP}^3$ modulo the action of $\mathsf{PSp}(4,\mathbb{R})$.

We can now justify the interpretation of the Maslov index as a generalization of nestedness of circles mentioned in the introduction. The group $\mathsf{PSp}(4,\mathbb{R})$ acts transitively on pairs of transverse Lagrangians, and transitively on pairwise transverse triples of Lagrangians which have the same Maslov index. Therefore, any triple with Maslov index $+2$ is equivalent to the triple $(L(0,0,1), L(0,0,2), L(0,0,3))$ corresponding to three nested circles.

In other words, any configuration of three co-oriented circles which are pairwise non tangent can be brought by a sequence of Möbius transformations and radial translations to exactly one of the configurations depicted in Figure 4.2, and Maslov index 2 corresponds to nested circles with increasing radius (or equivalently, decreasing curvature).



(c) $\mathcal{M} = 0$

Figure 4.2. The Maslov index of a triple of co-oriented circles.

Definition 4.10. — *We say that a transverse Legendrian polygon and its corresponding piecewise circular curve have* decreasing curvature *if every cyclically ordered triple of pairwise non-adjacent segments is a triple of Lagrangians with Maslov index* $+2$ *(the order of the vertices induces an order on the edges).*

Recall that we defined a *simple* piecewise circular curve by the property that every radial translate is a curve without self-intersection.

Table 4.1. Dictionary between Legendrian polygons and piecewise circular curves

(V,ω)	$\mathbb{P}_{\mathbb{C}}(\mathsf{V})$
Point in $\mathbb{P}_{\mathbb{R}}(\mathsf{V})$	Oriented contact element
Lagrangian subspace of V	Co-oriented circle
Isotropic flag in V	Pointed co-oriented circle
Non-transverse Lagrangians	Tangent co-oriented circles
Incident points in $\mathbb{P}_{\mathbb{R}}(\mathsf{V})$	Oriented contact elements to a common co-oriented circle
Legendrian polygon	Co-oriented piecewise circular curve
Moduli space of generic Legendrian n-gons	Moduli space of generic circular n-gons
Transverse Legendrian n-gon	Simple circular n-gon

Proposition 4.11. — *A piecewise circular curve is simple if and only if the corresponding Legendrian polygon is transverse.*

Proof. — A self-intersection of a piecewise circular curve can be interpreted as a zero-radius circle which is tangent to two oriented contact elements of the curve lying on non-adjacent pieces. This means that there exists a Lagrangian containing two points in $\mathbb{P}(\mathsf{V})$ lying on non-adjacent edges of the corresponding Legendrian polygon. In other words, those two points are incident, so the Legendrian polygon is not transverse.

Conversely, if a Legendrian polygon is non-transverse, then there exist two points on non-adjacent edges which are incident. The Lagrangian spanned by these two points corresponds to a circle tangent to two non-adjacent circular pieces of the piecewise circular curve. Applying a Möbius transformation if needed, we may assume that this circle is not a line. Then, applying a radial translation by the opposite of its radius brings that circle to a zero radius circle, producing a translate with a self-intersection. □

4.2. The dictionary

From the above analysis, we deduce the dictionary in Table 4.1 between the geometry of the symplectic vector space (V,ω) and the geometry of circles in the Riemann sphere $\mathbb{P}_{\mathbb{C}}(\mathsf{V})$.

We can translate the results of the previous sections with this dictionary as follows.

Corollary 4.12. — *The moduli space of simple circular n-gons and the moduli space of generic circular n-gons are smooth manifolds of dimension $2(n-5)$ when $n \geqslant 5$.*

Corollary 4.13. — *The moduli space of simple circular quadrilaterals is a pair of points (Proposition 3.13), and the moduli space of generic circular 5-gons consists of 64 points (Proposition 3.6).*

5. Positivity

5.1. Flag positivity

In this section, we describe positivity in the oriented flag manifold of the group $G = \mathsf{Sp}(\mathsf{V}, \omega) \cong \mathsf{Sp}(4, \mathbb{R})$. For a more general perspective on positivity in oriented flag manifolds see [3]. Many of the proofs in this section are taken from this source, but adapted to the symplectic setting.

Let F be a flag in V. We denote by $F^{(k)}$ the k-dimensional part of F.

Note that the symplectic vector space V is canonically oriented by the volume form $\omega \wedge \omega$. Whenever $U, W \subset \mathsf{V}$ are oriented vector subspaces of V, the equality $U = W$ will mean that the subspaces are equal and the orientations agree.

Definition 5.1. — *An* isotropic flag *in V is a full flag F such that $F^{(4-k)} = F^{(k)\perp}$ for all k.*

Let F be an isotropic flag and choose orientations on each $F^{(k)}$. The orientations on $F^{(1)}$ and $F^{(3)}$ induce an orientation on $F^{(3)}/F^{(1)}$ in the following way: if $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an oriented basis of $F^{(3)}$ such that $\mathbf{e}_1$ is an oriented basis of $F^{(1)}$, then $(\mathbf{e}_2, \mathbf{e}_3)$ gives a well-defined orientation on the quotient. We say that the orientations on $F^{(1)}$ and $F^{(3)}$ are *compatible* if this orientation on $F^{(3)}/F^{(1)}$ matches the orientation induced on this quotient by the symplectic form $-\omega$, that is, when $\omega(\mathbf{e}_2, \mathbf{e}_3) < 0$.

Definition 5.2. — *An* oriented isotropic flag *in V is an isotropic flag F together with a choice of orientations satisfying:*

- *The orientations on $F^{(1)}$ and $F^{(3)} = F^{(1)\perp}$ are compatible;*
- *The orientation on $F^{(4)}$ is the same as that of V.*

Remark 5.3. — Because of the condition on compatible orientations, an oriented isotropic flag F is uniquely determined by $F^{(1)}$ and $F^{(2)}$.

A symplectic basis E determines an oriented isotropic flag

$$F_E = \operatorname{span}(\mathbf{e}_1) \subset \operatorname{span}(\mathbf{e}_1, \mathbf{e}_2) \subset \operatorname{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \subset \mathsf{V}.$$

We now fix such a symplectic basis E and we denote an oriented isotropic flag F by a 4×4 matrix with vertical lines dividing the columns, such that the first k columns form an oriented basis of $F^{(k)}$. To shorten notation we will also sometimes use the 4×2 matrix consisting of only the first two columns, which provide the same data by Remark 5.3 (the vertical line distinguishes flags from Lagrangians).

Given a symplectic basis $E = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$, we define the *opposite symplectic basis* $\widehat{E}$ to be $(\mathbf{e}_4, -\mathbf{e}_3, \mathbf{e}_2, -\mathbf{e}_1)$.

Definition 5.4. — *A pair of oriented isotropic flags F_1, F_2 is called* oriented-transverse *if there exists a symplectic basis E such that $F_1 = F_E$ and $F_2 = F_{\hat{E}}$. Note that this basis is unique up to the action of $\mathbb{R}^2_{>0}$ given by $(\lambda, \mu) \cdot (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4) = (\lambda \mathbf{e}_1, \mu \mathbf{e}_2, \frac{1}{\mu}\mathbf{e}_3, \frac{1}{\lambda}\mathbf{e}_4)$.*

The group $G = \mathsf{Sp}(\mathsf{V}, \omega)$ acts transitively on oriented isotropic flags and the stabilizer of the flag F_E is the subgroup B^0_+ of upper triangular matrices with positive entries on the diagonal (in the basis E). The space of oriented isotropic flags then identifies with the homogeneous space G/B^0_+.

The group G also acts transitively on pairs of oriented-transverse oriented isotropic flags $F_E, F_{\hat{E}}$, with stabilizer the subgroup of diagonal matrices with positive entries in the basis E.

Proposition 5.5. — *A pair of oriented isotropic flags (F_1, F_2) is oriented-transverse if and only if for every $0 \leqslant i \leqslant 4$ we have*

$$F_1^{(i)} \oplus F_2^{(4-i)} = \mathbb{R}^4,$$

where the orientation on the direct sum is given by concatenating oriented bases of the oriented subspaces.

Proof. — For the forward direction, it suffices to check that for any positively oriented symplectic basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$, the bases

$$\mathbf{e}_4, -\mathbf{e}_3, \mathbf{e}_2, -\mathbf{e}_1$$
$$\mathbf{e}_1, \mathbf{e}_4, -\mathbf{e}_3, \mathbf{e}_2$$
$$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, -\mathbf{e}_3$$

are all positively oriented.

For the reverse direction, we first note that one can choose an orientation on an intersection $A \cap B$ of oriented subspaces such that $A + B = \mathbb{R}^4$ by requiring that there exist choices of complements $A' \subset A$, $B' \subset B$ with

$A' \oplus (A \cap B) = A$, $B' \oplus (A \cap B) = B$, and $A' \oplus (A \cap B) \oplus B' = \mathbb{R}^4$ as oriented subspaces. With this convention, we pick vectors $\mathbf{v}_i \in F_1^{(i)} \cap F_2^{(5-i)}$ for $1 \leqslant i \leqslant 4$ which are positively oriented in each intersection.

By compatibility of orientations of $F_1^{(1)}$ and $F_3^{(1)}$, we get that $\omega(\mathbf{v}_2, \mathbf{v}_3) < 0$. Moreover, since $\mathbf{v}_i$ form an oriented basis of $F^{(4)}$ (whose orientation is compatible with $\omega \wedge \omega$), we get that $\omega(\mathbf{v}_1, \mathbf{v}_4)\omega(\mathbf{v}_2, \mathbf{v}_3) < 0$ implying that $\omega(\mathbf{v}_1, \mathbf{v}_4) > 0$. Hence, after a positive rescaling, the vectors $\mathbf{v}_i$ form an oriented symplectic basis E in which $F_1 = F_E$ and $F_2 = F_{\hat{E}}$. □

Remark 5.6. — If (F_1, F_2) is an oriented-transverse pair, then (F_2, F_1) is *not* oriented-transverse, but $(F_2, -F_1)$ is, where $-F_1$ denotes the flag obtained by applying $-I \in G$ to the flag F_1.

Lemma 5.7. — *Given an oriented-transverse pair of flags (F_1, F_2), no other choice of orientations on the subspaces $F_2^{(i)}$ gives an oriented-transverse pair.*

Proof. — Let E be a symplectic basis such that $F_1 = F_E$ and $F_2 = F_{\hat{E}}$. The stabilizer of the oriented flag F_E, in the basis E, is given by upper triangular matrices with positive entries on the diagonal. Among those matrices, the only ones which preserve the (unoriented) subspaces $F_2^{(i)}$ are diagonal, and they preserve the orientations on those subspaces. Since the action of G is transitive on oriented-transverse pairs, we conclude that the other choices of orientation on F_2 are not oriented-transverse. □

The unipotent radical of the Borel subgroup B_+ is the subgroup:

$$U_+ = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & ad-b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}.$$

The stabilizer of $F_{\hat{E}}$ is the subgroup B_-^0 of lower triangular matrices with positive entries on the diagonal. The Borel subgroup B_- is opposite to B_+, and its unipotent radical is

$$U_- = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & d & 1 & 0 \\ c & ad-b & a & 1 \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}.$$

The *positive semigroup* $U_+^{>0}$ is the following subsemigroup of U_+:

$$U_+^{>0} = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & ad-b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; \begin{matrix} a > 0,\ b > 0,\ c > 0 \\ ad-b > 0,\ -b^2 + abd - cd > 0 \end{matrix} \right\},$$

and similarly

$$U_-^{>0} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & d & 1 & 0 \\ c & ad-b & a & 1 \end{pmatrix} \;\middle|\; \begin{matrix} a > 0,\ b > 0,\ c > 0 \\ ad-b > 0,\ -b^2 + abd - cd > 0 \end{matrix} \right\}.$$

The inequalities defining the positive semigroups are equivalent to the statement that all minors of the matrix which aren't zero by triangularity are positive ([13, Theorem 2.8]).

The construction of the semigroups $U_\pm^{>0}$ seems to depend on the choice of basis E, but in fact only depends on the pair of oriented-transverse isotropic flags $(F_E, F_{\hat{E}})$, since conjugating by a positive diagonal matrix leaves $U_+^{>0}$ and $U_-^{>0}$ invariant. In other words, we have associated to each oriented-transverse pair of flags $(F_E, F_{\hat{E}})$ a pair of subsemigroups, one in the stabilizer of F_E and one in the stabilizer of $F_{\hat{E}}$.

Moreover, a simple computation shows that these semigroups satisfy $\left(U_\pm^{>0}\right)^{-1} = K U_\pm^{>0} K$, where K is the diagonal matrix $K = \operatorname{diag}(1,-1,1,-1)$ in the basis E.

Definition 5.8. — *Let (F_1, F_2, F_3) be a triple of pairwise oriented-transverse isotropic flags, and let E be a basis such that $F_1 = F_E$ and $F_3 = F_{\hat{E}}$. The triple is called* positive *if $F_2 = uF_E$ with $u \in U_-^{>0}$.*

A tuple of flags $(F_1, \dots, F_n)$ is positive *if every ordered sub-triple F_i, F_j, F_k with $1 \leqslant i < j < k \leqslant n$ is positive.*

Remark 5.9. — The element $u \in U_-^{>0}$ in the definition is the unique element in the subgroup U_- such that $F_2 = uF_E$.

Remark 5.10. — In the more well-known unoriented setting, positivity of triples is not enough to guarantee positivity of a tuple of flags, for instance there exist quadruples of flags which are not positive but for which every sub-triple is positive. Indeed, in $\mathbb{RP}^1$ every triple of distinct points is positive, but a quadruple of distinct points is only positive if it is cyclically ordered in one direction or the other. The fact that triples are sufficient in the oriented setting follows from property (3) of Proposition 5.12 below.

We next give a useful characterization of positive triples, similar to Proposition 5.5, which is more obviously G-invariant.

Proposition 5.11. — *A triple of oriented isotropic flags (F_1, F_2, F_3) is positive if and only if for every $0 \leqslant i, j, k \leqslant 4$ with $i + j + k = 4$ we have*

$$F_1^{(i)} \oplus F_2^{(j)} \oplus F_3^{(k)} = \mathbb{R}^4.$$

Proof. — Since F_1, F_3 are oriented-transverse, we can work in a basis where $F_1 = F_E$ and $F_3 = F_{\hat{E}}$. In that basis, since F_2 is transverse to F_3, we can write F_2 as a lower triangular matrix with ones on the diagonal. Then, the oriented direct sum conditions directly translate to positivity of minors for this matrix :

$$\begin{aligned}
&F_1^{(1)} \oplus F_2^{(1)} \oplus F_3^{(2)} = \mathbb{R}^4 \text{ gives } a > 0,\\
&F_1^{(2)} \oplus F_2^{(1)} \oplus F_3^{(1)} = \mathbb{R}^4 \text{ gives } b > 0,\\
&F_1^{(1)} \oplus F_2^{(3)} = \mathbb{R}^4 \text{ gives } c > 0,\\
&F_1^{(1)} \oplus F_2^{(2)} \oplus F_3^{(1)} = \mathbb{R}^4 \text{ gives } ad - b > 0, \text{ and}\\
&F_1^{(2)} \oplus F_2^{(2)} = \mathbb{R}^4 \text{ gives } -b^2 + abd - cd > 0.
\end{aligned}$$

Since these five conditions are equivalent to positivity of all minors of the matrix for F_2, and all oriented sum conditions correspond to the positivity of some minor, both implications of the proposition follow. □

The following proposition gives the key properties that positivity satisfies (recall that $-F$ denotes the action of the element $-I \in \mathsf{Sp}(4, \mathbb{R})$ on an oriented flag F).

Proposition 5.12. — *Let F_1, F_2, F_3, F_4 be oriented isotropic flags.*

(1) *If (F_1, F_2, F_3) is positive, then $(F_2, F_3, -F_1)$ is positive.*
(2) *If (F_1, F_2, F_3) is positive, then (F_3, F_2, F_1) is not positive.*
(3) *If (F_1, F_2, F_3) and (F_1, F_3, F_4) are positive, then (F_1, F_2, F_4) is positive.*

Proof.

Claim (1). — We use the characterization of Proposition 5.11. Whenever $i + j + k = 4$, the oriented direct sum $F_1^{(i)} \oplus F_2^{(j)} \oplus F_3^{(k)}$ has the same orientation as $F_2^{(j)} \oplus F_3^{(k)} \oplus (-F_1^{(i)})$, so the claim follows.

Claim (2). — Is also direct from Proposition 5.11 since swapping F_1 and F_3 will already violate oriented-transversality, which is included in the definition (when $j = 0$).

Claim (3). — Is a consequence of the fact that $U_-^{\geq 0}$ is a semigroup. We choose a basis E such that $F_1 = -F_{\hat{E}}$ and $F_2 = F_E$. This basis exists since $(F_2, -F_1)$ is an oriented-transverse pair. Since (F_1, F_2, F_3) is positive, $(F_2, F_3, -F_1)$ also is hence there exists (in the basis E) a lower triangular, totally positive symplectic matrix u with $F_3 = uF_E$. Similarly, since (F_1, F_3, F_4) is positive, $(F_3, F_4, -F_1)$ is also positive by (1) and by G-invariance we obtain that $(F_E, u^{-1}F_4, F_{\hat{E}})$ is positive. This implies that $u^{-1}F_4 = vF_E$ for some $v \in U_-^{\geq 0}$, and so $F_4 = (uv)F_E$. Since $U_-^{\geq 0}$ is a semigroup, $(uv) \in U_-^{\geq 0}$ and we conclude that $(F_E, (uv)F_E, F_{\hat{E}}) = (F_2, F_4, -F_1)$ is positive. Using (1) again, we deduce that (F_1, F_2, F_4) is positive. □

The first property implies that if $(F_1, \ldots, F_n)$ is positive, then $(-F_1, \ldots, -F_n)$ also is. This motivates the introduction of the following moduli space.

Definition 5.13. — *The space $\mathscr{F}^{(n)}$ is the space of $\mathsf{PSp(V)}$-orbits of n-tuples of pairwise transverse oriented isotropic flags up to the diagonal action of $-I$. That is, in $\mathscr{F}^{(n)}$, $(F_1, \ldots, F_n) = (-F_1, \ldots, -F_n)$. The subspace consisting of (orbits of) positive n-tuples is denoted by $\mathscr{F}^{(n)}_{>0}$.*

We can deduce the following characterizations for positive quadruples from the definition :

Proposition 5.14. — *A quadruple of oriented isotropic flags $(F_E, F_1, F_2, F_{\hat{E}})$ is positive if and only if $F_1 = u_1F_E$ and $F_2 = u_1u_2F_E$ with $u_1, u_2 \in U_-^{\geq 0}$.*

A quadruple of oriented isotropic flags $(F_E, F_+, F_{\hat{E}}, F_-)$ is positive if and only if $F_+ = u_+F_E$ and $F_- = -u_-F_E$, where $u_+ \in U_-^{\geq 0}$ and $u_- \in KU_-^{\geq 0}K$.

Proof. — For the forwards implication of the first statement, we first note that positivity of the triple $(F_E, F_i, F_{\hat{E}})$ implies that $F_i = u_iF_E$ with $u_i \in U_-^{\geq 0}$ for $i = 1, 2$. Since $(F_1, F_2, F_{\hat{E}})$ is positive, by G-invariance we get that $(F_E, u_1^{-1}u_2F_E, F_{\hat{E}})$ is positive and therefore $u_1^{-1}u_2 \in U_-^{\geq 0}$ (by Remark 5.9). Setting $u_2' = (u_1^{-1}u_2)$ we therefore have $F_2 = u_1u_2'F_E$ with $u_1, u_2' \in U_-^{\geq 0}$.

For the reverse implication, we get positivity of the triples $(F_E, F_1, F_{\hat{E}})$ and $(F_E, F_2, F_{\hat{E}})$ by definition and the fact that $U_-^{\geq 0}$ is a semigroup. Moreover, by G-invariance, positivity of $(F_2, F_3, F_{\hat{E}})$ is equivalent to positivity of $(F_E, u_2F_E, F_{\hat{E}})$ which is obtained directly from the definition. Positivity of the last triple (F_1, F_3, F_4) follows from the other three and the symmetries in Proposition 5.12.

Next, we prove the second statement. For the forward implication, we get $F_+ = u_+F_E$ with $u_+ \in U_-^{\geq 0}$ directly from the definition. By oriented transversality, we can write $F_- = -u_- F_E$ for some $u_- \in U_-$. The minus sign comes from the fact that the pair $(u_- F_E, F_{\hat{E}})$ is oriented transverse, so to reverse the order we have to add a minus sign as in Remark 5.6. Positivity of $(F_E, F_{\hat{E}}, F_-)$ is equivalent to that of $(u_-^{-1} F_E, F_{\hat{E}}, -F_E)$, which in turn is equivalent to positivity of $(F_E, u_-^{-1} F_E, F_{\hat{E}})$. From the fact that $(U_-^{\geq 0})^{-1} = K U_-^{\geq 0} K$, we get $u_-^{-1} = K u'_- K$ for some $u'_- \in U_-^{\geq 0}$.

For the reverse implication, we get positivity of $(F_E, F_+, F_{\hat{E}})$ immediately and positivity of $(F_E, F_{\hat{E}}, F_-)$ by applying the same trick as in the forwards implication of the claim. Together with the symmetries of positivity, these two imply that (F_E, F_+, F_-) is positive. To show that $(F_+, F_{\hat{E}}, F_-)$ is positive, we apply u_+^{-1} to get $(F_E, F_{\hat{E}}, -u_+^{-1} u_- F_E)$. Since $u_+^{-1}, u_- \in (U_-^{\geq 0})^{-1}$, their product is also an element of this inverse semigroup, and hence the triple is positive (again using the argument of the previous paragraph). □

The following simple lemma will be useful in the proof of Proposition 5.23.

Lemma 5.15. — *Let $(F_1, \ldots, F_n) \in \mathscr{F}^{(n)}$. If (F_i, F_{i+1}, F_n) is positive for every $1 < i+1 \leqslant n$, then $(F_1, \ldots, F_n)$ is positive.*

Proof. — Let $1 \leqslant i < j < k \leqslant n$. Since (F_i, F_{i+1}, F_n) and (F_{i+1}, F_{i+2}, F_n) are positive by hypothesis, by property (1) above we find that $(F_n, -F_i, -F_{i+1})$ and $(F_n, -F_{i+1}, -F_{i+2})$ are positive. By properties (3) and (1), we conclude that (F_i, F_{i+2}, F_n) is positive. Repeating this argument we find that (F_i, F_{i+m}, F_n) is positive for all $0 < m < n-i$ and so (F_i, F_j, F_n) is positive. Repeating the argument we get that (F_j, F_k, F_n) is positive. One last application of properties (1) and (3) to the triples (F_i, F_j, F_n) and (F_j, F_k, F_n) yields that (F_i, F_j, F_k) is positive. □

Positivity was originally defined for unoriented flags. Let us denote by $\overline{\mathscr{F}}$ the space of unoriented isotropic flags. An n-tuple of isotropic flags is called positive if it can be written

$$(\overline{F}_E, u_1 \overline{F}_E, u_1 u_2 \overline{F}_E, \ldots, u_1 u_2 \ldots u_{n-2} \overline{F}_E, \overline{F}_{\hat{E}})$$

for some basis E, where $u_i \in U_-^{\geq 0}$, and $\overline{F}_E$ denote the projection of F_E to unoriented flags. The unoriented setting is slightly more subtle because in this case, the subgroup $U_-^{\geq 0}$ is not uniquely determined by the pair of flags $\overline{F}_E, \overline{F}_{\hat{E}}$. Nevertheless, it is uniquely determined by a choice of symplectic basis E, and the definition makes sense if we allow all possible choices of E.

Let $\overline{\mathscr{F}}^{(n)}$ denote the moduli space of n-tuples of pairwise transverse flags up to the action of $\mathsf{PSp}(4, \mathbb{R})$. We will use the following special case of a theorem of Fock and Goncharov to prove our main result.

Theorem 5.16 ([6, Theorem 1.5]). — *Positivity of an n-tuple of flags is invariant under the action of $\mathsf{PSp}(V)$, and the subset $\overline{\mathscr{F}}^{(n)}_{>0} \subset \overline{\mathscr{F}}^{(n)}$ consisting of orbits of positive n-tuples is a connected component homeomorphic to a ball of dimension $4n-10$, for all $n \geqslant 3$.*

For our oriented isotropic flag setting, we have the same result.

Proposition 5.17. — *The subset $\mathscr{F}^{(n)}_{>0} \subset \mathscr{F}^{(n)}$ consisting of positive n-tuples of oriented isotropic flags is a connected component homeomorphic to a ball of dimension $4n-10$.*

Proof. — The projection $\mathscr{F}^{(n)}_{>0} \to \overline{\mathscr{F}}^{(n)}$ corresponding to removing the orientations is a covering map. By Theorem 5.16, the preimage of the positive component, which is contractible, is a disjoint union of homeomorphic copies.

Let

$$(\overline{F}_E, u_1\overline{F}_E, u_1u_2\overline{F}_E, \dots, (u_1u_2\dots u_{n-2})\overline{F}_E, \overline{F}_{\hat{E}}),$$

be a positive n-tuple of unoriented flags.

There are four lifts of $\overline{F}_E$ to oriented isotropic flags, and they are all equivalent under the $\mathsf{Sp}(4,\mathbb{R})$ action, so we can make the choice of lift F_E. By definition,

$$(F_E, u_1F_E, u_1u_2F_E, \dots, (u_1u_2\dots u_{n-2})F_E, F_{\hat{E}})$$

is a positive lift of this tuple. By Lemma 5.7, this choice of lift, among those whose first element is F_E, is the only one such that every ordered sub-pair is oriented-transverse, and therefore it is also the only positive lift. This shows that the preimage of the positive component is a single homeomorphic copy. □

The cited Fock–Goncharov result shows that positive n-tuples form a connected component in the space of *generic* n-tuples, rather than pairwise-transverse n-tuples. However, since positivity is characterized by triples, and a positive triple can only degenerate to a triple containing a non-transverse pair ([11, Proposition 8.14]), the theorem is true as stated.

Example 5.18. — The following map is a parametrization of $\mathscr{F}^{(3)}_{>0}$:

$$\begin{aligned} \mathbb{R}_{>0} \times \mathbb{R}_{>0} &\longrightarrow \mathscr{F}^{(3)}_{>0} \\ (x,y) &\longmapsto \left(F_E, \left[\begin{array}{c|c} 1 & 0 \\ x+y+\frac{1}{y} & 1 \\ y & 1 \\ 1 & x+\frac{1}{y} \end{array}\right], F_{\hat{E}} \right). \end{aligned}$$

The two-dimensional parts of a triple of oriented isotropic flags are Lagrangians, and so given a triple of flags we obtain a triple of Lagrangians. If the triple is positive, their Maslov index is always +2:

Lemma 5.19. — *Let (F_1, F_2, F_3) be a positive triple of oriented isotropic flags. Then, the Maslov index $\mathcal{M}\big(F_1^{(2)}, F_2^{(2)}, F_3^{(2)}\big)$ is +2.*

Proof. — After acting by an element of $\mathsf{Sp}(4,\mathbb{R})$, we may assume

$$F_1 = \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \quad F_2 = \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & d & 1 & 0 \\ c & ad-b & a & 1 \end{array}\right], \quad F_3 = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

with the matrix for F_2 in the semigroup $U_-^{\geq 0}$.

The matrix of the Maslov bilinear form of the three Lagrangians $F_1^{(2)}, F_2^{(2)}, F_3^{(2)}$, in the basis given by the first two columns of F_2, is then

$$2\begin{pmatrix} ab-c & b \\ b & d \end{pmatrix}.$$

Since $d > 0$ and the determinant $(ab-c)d - b^2 = b(ad-b) - cd > 0$, this matrix is positive-definite. □

5.2. Positivity and Legendrian polygons

Given a generic, non-contractible Legendrian polygon P with $2k$ vertices $p_1, \ldots, p_{2k}$, we can associate to it a k-tuple of oriented isotropic flags as follows. Choose representatives $\mathbf{v}_1, \ldots, \mathbf{v}_{2k}$ of $p_1, \ldots, p_{2k}$ such that $P = P_-(\mathbf{v}_1, \ldots, \mathbf{v}_{2k})$, and let $F_1 = (\mathrm{span}(\mathbf{v}_1) \subset \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2))$, $F_2 = (\mathrm{span}(\mathbf{v}_3) \subset \mathrm{span}(\mathbf{v}_3, \mathbf{v}_4))$, $\ldots$, $F_k = (\mathrm{span}(\mathbf{v}_{2k-1}) \subset \mathrm{span}(\mathbf{v}_{2k-1}, \mathbf{v}_{2k}))$, where the orientation on each subspace is the one given by the basis specified. Since the representatives $\mathbf{v}_i$ are unique up to positive scalar multiplication or replacing all $\mathbf{v}_i$ with $-\mathbf{v}_i$, this procedure gives a well-defined element of $\mathscr{F}^{(k)}$ for each generic non-contractible polygon P. We denote the induced map by

$$\mathcal{H} : \mathscr{P}_{2k}^- \longrightarrow \mathscr{F}^{(k)}.$$

Remark 5.20. — The configuration of oriented flags $\mathcal{H}(P)$ depends on the labeling of the vertices of P. If we re-label the vertices starting at any odd vertex p_{2k+1} we obtain a cyclic permutation of the flags, but if we start at an even vertex or reverse the cyclic ordering we obtain different flags. In total, up to cyclic permutations and choice of cyclic ordering, there are four different (unlabeled) k-tuples of flags that could be associated to the Legendrian polygon P.

Proposition 5.21. — *Let $P = P_-(\mathbf{v}_1, \ldots, \mathbf{v}_6)$ be a positive-transverse Legendrian polygon. Then, the associated triple $\mathcal{H}(P) = (F_1, F_2, F_3)$ is positive. (Recall the definition of positive-transverse Legendrian polygon below Proposition 3.11)*

Proof. — We fix a basis E in which the symplectic form is given by Ω'. Applying an element of $\mathsf{Sp}(\mathsf{V}, \omega)$, we may assume that $F_1 = F_E$ and $F_3 = F_{\hat{E}}$.

Then, we may scale by positive scalars so that $\mathbf{v}_1 = \mathbf{e}_1$ and $\mathbf{v}_5 = \mathbf{e}_4$. Moreover, since $F_1^{(2)} = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$, we have $\mathbf{v}_2 = a\mathbf{e}_1 + \mathbf{e}_2$ for some $a \in \mathbb{R}$. Since $F_3^{(2)} = \operatorname{span}(\mathbf{v}_5, \mathbf{v}_6)$ and $\omega(\mathbf{v}_6, \mathbf{v}_1) = 0$ we have $\mathbf{v}_6 = -\mathbf{e}_3$ (with sign chosen so that $\omega(\mathbf{v}_2, \mathbf{v}_6) > 0$, as required by the positive-transversality hypothesis).

Using the relations $\omega(\mathbf{v}_3, \mathbf{v}_5) > 0$ and $\omega(\mathbf{v}_4, \mathbf{v}_6) > 0$ we can normalize $\mathbf{v}_3$ and $\mathbf{v}_4$ and represent the flags F_1, F_2, F_3 by the following three matrices:

$$\left[\begin{array}{c|c} 1 & a \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array}\right], \quad \left[\begin{array}{c|c} 1 & 0 \\ b & 1 \\ ad & c \\ d & e \end{array}\right], \quad \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{array}\right],$$

where $a, b, c, d, e \in \mathbb{R}$, and $e - bc + ad = 0$ since $\omega(\mathbf{v}_3, \mathbf{v}_4) = 0$. We must show that the matrix representing F_2 is totally positive. It suffices to show the positivity of b, ad, d, $bc - ad$, and $(ae - c)d$.

But $b = \omega(\mathbf{v}_3, \mathbf{v}_6) > 0$, $a = \omega(\mathbf{v}_2, \mathbf{v}_5) > 0$, $d = \omega(\mathbf{v}_1, \mathbf{v}_3) > 0$, $e = bc - ad = \omega(\mathbf{v}_1, \mathbf{v}_4) > 0$, and $ae - c = \omega(\mathbf{v}_2, \mathbf{v}_4) > 0$, proving the claim. □

We get the following consequence for piecewise circular hexagons.

Corollary 5.22. — *A simple circular hexagon either has decreasing curvature or it has increasing curvature.*

For Legendrian polygons with more than 6 vertices however, transversality does not suffice to guarantee the positivity of the associated tuple of flags.

An explicit example is the transverse Legendrian octagon with vertices $(1,0,0,0)$, $(1,1,0,0)$, $(1,1,1,1)$, $(4,2,-1,1)$, $(4,8,-4,1)$, $(0,1,1,3)$, $(0,0,0,1)$, and $(0,0,-1,0)$ in a symplectic basis. We show its image in the space of oriented contact elements to S^2 in Figure 5.1.

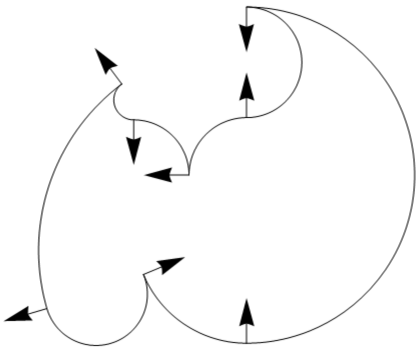


Figure 5.1. A simple circular octagon which is not positive.

Proposition 5.23. — *Suppose $P = P_-(\mathbf{v}_1, \dots, \mathbf{v}_{2N})$ is positive-transverse and has decreasing curvature. Then, the associated N-tuple $\mathcal{H}(P) = (F_1, \dots, F_N)$ is positive.*

Proof. — We first show that F_1, F_2, F_k is positive for $k = 4, \dots, N$.

We use the symplectic basis $\mathbf{e}_1 = \mathbf{v}_1$, $\mathbf{e}_2 = \mathbf{v}_2 - a\mathbf{v}_1$, $\mathbf{e}_3 = b\mathbf{v}_{2k-1} - \mathbf{v}_{2k}$ and $\mathbf{e}_4 = \mathbf{v}_{2k-1}$, where $a = \omega(\mathbf{v}_2, \mathbf{v}_{2k-1})$ and $b = \omega(\mathbf{v}_1, \mathbf{v}_{2k})$. Then, normalizing each vector we may write $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_{2k-1}, \mathbf{v}_{2k}$ as the columns of the following matrix:

$$\begin{pmatrix} 1 & a & 1 & 1 & 0 & 0 \\ 0 & 1 & c & e & 0 & 0 \\ 0 & 0 & ad & f & 0 & -1 \\ 0 & 0 & d & d + cf - ade & 1 & b \end{pmatrix}.$$

So we must show that the triple of flags

$$\left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array}\right], \left[\begin{array}{c|c} 1 & 0 \\ c & 1 \\ ad & \frac{f-ad}{e-c} \\ d & \frac{cf-ade}{e-c} \end{array}\right], \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{array}\right] \tag{5.1}$$

is positive, and for this it suffices to show that the second matrix is totally positive. For this, we need the following five inequalities coming from minors:

$$c > 0,\ ad > 0,\ d > 0$$
$$\frac{cf - ade}{e - c} > 0$$
$$\frac{ad(d + cf - ade) - df}{e - c} > 0.$$

From positive-transversality applied to the pairs $(\mathbf{v}_1, \mathbf{v}_3)$, $(\mathbf{v}_2, \mathbf{v}_{2k-1})$, $(\mathbf{v}_1, \mathbf{v}_4)$ respectively we obtain the inequalities: $d > 0$, $a > 0$, $d + cf - ade > 0$. From the pair $(\mathbf{v}_2, \mathbf{v}_4)$ we get

$$a(d + cf - ade) - f > 0. \tag{5.2}$$

The decreasing curvature condition means that the Maslov index of the three Lagrangians (5.1) is 2. We find that the Maslov bilinear form, computed in the column basis given for F_2, is

$$2\begin{pmatrix} (ac-1)d & d(ae-1) \\ d(ae-1) & d(ae-1) + (e-c)f \end{pmatrix}$$

and so from Sylvester's positivity criterion we obtain $(ac - 1)d > 0$, which together with $d > 0$ implies

$$ac - 1 > 0, \tag{5.3}$$

and

$$d(e - c)(a(d + cf - ade) - f) > 0. \tag{5.4}$$

Inequality (5.3) together with $a > 0$ immediately implies $c > 0$.

Inequality (5.4), together with $d > 0$ and inequality (5.2) implies that $e - c > 0$, so it only remains to show that the numerators of the last two minors are positive.

Moreover, inequalities (5.2) and (5.3) as well as $d > 0$ imply

$$f > \frac{ad(ae - 1)}{ac - 1}.$$

It follows that

$$cf - ade > \frac{c(ad)(ae - 1) - ade(ac - 1)}{ac - 1} = \frac{ad(e - c)}{ac - 1} > 0.$$

Finally, using equation (5.2)

$$ad(d + cf - ade) - df > df - df = 0.$$

This shows that the triple F_1, F_2, F_k is positive for $k \geqslant 4$.

If $k = 3$, then using the same basis as in the previous case we normalize the vectors $\mathbf{v}_1, \dots, \mathbf{v}_6$ to the columns of the matrix

$$\begin{pmatrix} 1 & a & 1 & 0 & 0 & 0 \\ 0 & 1 & c & 1 & 0 & 0 \\ 0 & 0 & ad & e & 0 & -1 \\ 0 & 0 & d & ce - ad & 1 & b \end{pmatrix}.$$

We must then show that the triple of flags

$$\left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array}\right], \left[\begin{array}{c|c} 1 & 0 \\ c & 1 \\ ad & e \\ d & ce - ad \end{array}\right], \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{array}\right] \tag{5.5}$$

is positive. From positive-transversality we have $a = \omega(\mathbf{v}_2, \mathbf{v}_5) > 0$, $d = \omega(\mathbf{v}_1, \mathbf{v}_3) > 0$, $ce - ad = \omega(\mathbf{v}_1, \mathbf{v}_4)$ and $a(ce - ad) - e = \omega(\mathbf{v}_2, \mathbf{v}_4) > 0$.

From decreasing curvature, we know that $\mathbf{v}_4$ must be positive for the Maslov quadratic form which gives $2e > 0$. Combining this with $a > 0, d > 0$ and $ce - ad > 0$ we deduce $c > ad/e > 0$. The only remaining inequality required for total positivity is $ad(ce - ad) - ed > 0$ but this factorizes as $d(a(ce - ad) - e)$ and we already showed $d > 0$ and $a(ce - ad) - e > 0$.

Since transversality and monotonicity are invariant under cyclic permutations, we might have chosen any odd index for $\mathbf{v}_1$ and so all triples of the form F_i, F_{i+1}, F_k with $i + 1 \neq k$ are positive. This implies, by Lemma 5.15, that the N-tuple of flags is positive. □

We illustrate examples of simple circular $2k$-gons with decreasing curvature in Figures 1.1 and 5.2.

Next we will define a partial inverse for $\mathcal{H}$, using the orientation on intersections of subspaces from Proposition 5.5.

Define a map $\mathcal{C} : \mathcal{H}(\mathscr{P}_{2k}^-) \to \mathscr{P}_{2k}^-$ by

$$\begin{aligned} &\mathcal{C}(F_1, \dots, F_k) \\ &\quad = P_-(F_1^{(1)},\ F_1^{(2)} \cap F_2^{(3)}, F_2^{(1)}, \dots, F_{k-1}^{(2)} \cap F_k^{(3)}, F_k^{(1)}, F_k^{(2)} \cap F_1^{(3)}), \end{aligned}$$

where we slightly abuse notation since the image of P_- does not depend on the choice of representative for each oriented 1-dimensional subspace.

Remark 5.24. — We restrict to the image of $\mathcal{H}$ for this definition to make sense, because the definition of generic polygon is slightly stronger than transversality for tuples of flags. More precisely, the formula for $\mathcal{C}$ always defines a Legendrian polygon, but it might not be generic and so might not lie in the space $\mathscr{P}_{2k}^-$.

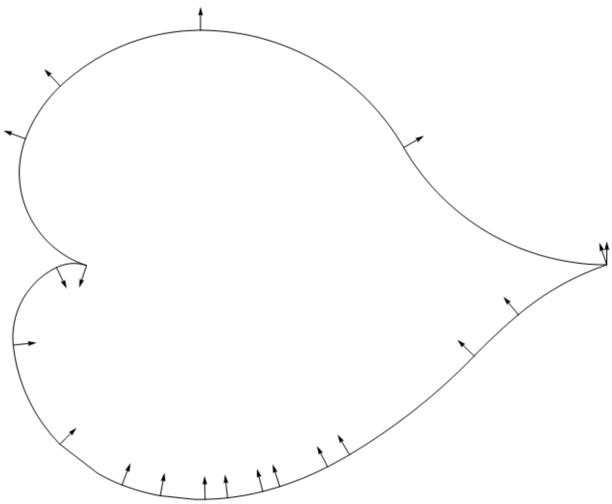


Figure 5.2. A simple circular 20-gon with decreasing curvature. Oriented contact elements are indicated by arrows at the vertices.

Example 5.25. — The image by $\mathcal{C}$ of the parametrized triple of flags from Example 5.18 is the Legendrian polygon $P_-(\mathbf{v}_1, \dots, \mathbf{v}_6)$, where

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix}y\\1\\0\\0\end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix}1\\x+y+\frac{1}{y}\\y\\1\end{pmatrix},$$

$$\mathbf{v}_4 = \begin{pmatrix}0\\1\\1\\x+\frac{1}{y}\end{pmatrix}, \ \mathbf{v}_5 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \ \mathbf{v}_6 = \begin{pmatrix}0\\0\\-1\\0\end{pmatrix}.$$

It is simple to check that it is positive-transverse and has decreasing curvature.

Proposition 5.26. — *The map $\mathcal{H}$ is a homeomorphism onto its image.*

Proof. — Let $P = P_-(\mathbf{v}_1, \dots, \mathbf{v}_{2k})$ be a generic, noncontractible Legendrian polygon let $F_1, \dots, F_k = \mathcal{H}(P)$. Since $\omega(\mathbf{v}_{2j-1}, \mathbf{v}_{2k-1}) \neq 0$ for all $j \neq k$, we have that $F_j^{(1)}$ is transverse to $F_k^{(3)}$ for all $j \neq k$. Moreover, since the edges spanned by $\mathbf{v}_{2j-1}, \mathbf{v}_{2j}$ and $\mathbf{v}_{2k-1}, \mathbf{v}_{2k}$ are linearly independent,

$F_j^{(2)}$ is transverse to $F_k^{(2)}$ for all $j \neq k$. We conclude that the image of P by $\mathcal{H}$ is an n-tuple of pairwise transverse flags.

The map $\mathcal{C}$, when restricted to the image $\mathcal{H}(\mathscr{P}_{2k}^{-})$, is an inverse of $\mathcal{H}$. We only need to show that both maps are continuous.

The topology on $\mathscr{P}_{2k}^{-}$ can be seen as the subspace topology in $(\mathbb{S}(\mathsf{V}))^{2k}/\sim$, where the equivalence relation is given by $(\mathbf{v}_1, \dots, \mathbf{v}_{2k}) \sim (-\mathbf{v}_1, \dots, -\mathbf{v}_{2k})$ and the subspace is given by the conditions which insure that the polygon $P_-(\mathbf{v}_1, \dots, \mathbf{v}_{2k})$ is Legendrian and generic. The flags

$$F_i = (\operatorname{span}(\mathbf{v}_{2i-1}) \subset \operatorname{span}(\mathbf{v}_{2i-1}, \mathbf{v}_{2i}))$$

depend continuously on $\mathbf{v}_i$ and the map $\mathcal{H}$ intertwines the actions of $\mathsf{PSp}(4, \mathbb{R})$ on the space of Legendrian polygons and the space of flags. We conclude that $\mathcal{H}$ is continuous.

For the inverse map $\mathcal{C}$, we can see continuity directly in coordinates. The points $[\mathbf{v}_{2i-1}]$ are simply the 1-dimensional part of the flags F_i, so they depend continuously on the tuple of flags. The vectors $\mathbf{v}_{2i}$ depend on F_i and F_{i+1}. Writing

$$F_i = (\operatorname{span}(\mathbf{u}_{2i-1}) \subset \operatorname{span}(\mathbf{u}_{2i-1}, \mathbf{u}_{2i}))$$

and

$$F_{i+1} = (\operatorname{span}(\mathbf{u}_{2i+1}) \subset \operatorname{span}(\mathbf{u}_{2i+1}, \mathbf{u}_{2i+2})),$$

we get $\mathbf{v}_{2i} = \pm\left(\mathbf{u}_{2i-1} - \frac{\omega(\mathbf{u}_{2i-1}, \mathbf{u}_{2i+1})}{\omega(\mathbf{u}_{2i}, \mathbf{u}_{2i+1})}\mathbf{u}_{2i}\right)$, where the sign is determined by the orientations but does not matter for continuity. This is well-defined and continuous since $\omega(\mathbf{u}_{2i}, \mathbf{u}_{2i+1}) \neq 0$ by transversality of the flags F_i and F_{i+1}. □

Proposition 5.27. — *Let $F_1, \dots, F_k$ be a positive k-tuple of oriented isotropic flags. Then, $\mathcal{C}(F_1, \dots, F_k)$ is a positive-transverse Legendrian $2k$-gon with decreasing curvature.*

Proof. — Positivity of the tuple $F_1, \dots, F_k$ means positivity of every cyclically ordered sub-triple. Consider a sub-triple of the form F_i, F_{i+1}, F_j with $j > i + 1$ and apply an element of $\mathsf{Sp}(4, \mathbb{R})$ so that

$$F_i = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array}\right], \quad F_{i+1} = \left[\begin{array}{c|c} 1 & 0 \\ a & 1 \\ b & d \\ c & ad-b \end{array}\right], \quad F_j = \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{array}\right].$$

By positivity, $a, b, c > 0$, $ad - b > 0$, and $-b^2 + abd - cd > 0$. Then, the vertices $\mathbf{v}_{2i-1}, \mathbf{v}_{2i}, \mathbf{v}_{2i+1}, \mathbf{v}_{2i+2}, \mathbf{v}_{2j-1}, \mathbf{v}_{2j}$ of $\mathcal{C}(F_1, \dots, F_k)$ have coordinates of the form given in the columns of the following matrix:

$$\begin{pmatrix} 1 & \frac{b}{c} & 1 & k_1 & 0 & 0 \\ 0 & 1 & a & 1+k_1a & 0 & 0 \\ 0 & 0 & b & d+k_1b & 0 & -1 \\ 0 & 0 & c & ad-b+k_1c & 1 & k_2 \end{pmatrix}$$

for some $k_1, k_2 \in \mathbb{R}$.

We first show that $k_1, k_2 \geqslant 0$.

If $j = i+2$, then $k_1 = 0$. Otherwise, note that the flag F_{i+2} has the form

$$F_{i+2} = \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & w & 1 & 0 \\ z & xw-y & x & 1 \end{array}\right],$$

and by positivity of the quadruple $F_1, F_{i+1}, F_{i+2}, F_j$ and Proposition 5.14,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & d & 1 & 0 \\ c & ad-b & a & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & w & 1 & 0 \\ z & xw-y & x & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ x-a & 1 & 0 & 0 \\ ad-b+y-dx & w-d & 1 & 0 \\ z-c-ay+bx & b-y+w(x-a) & x-a & 1 \end{pmatrix} \in U_-^{>0}.$$

The fact that $\omega(\mathbf{v}_{2i+2}, \mathbf{v}_{2i+3}) = 0$ implies that $k_1 = \frac{ad-b+y-dx}{-ay+bx+z-c}$, which is positive since the numerator and denominator appear as entries in the first column of the above totally positive lower triangular matrix.

Similarly, if $j = k$ and $i = 1$ so that F_j and F_i are adjacent flags in the cyclic tuple, then $k_2 = 0$. Otherwise, the flag F_{j+1} is given by the columns of a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha & 1 & 0 & 0 \\ \beta & -\delta & 1 & 0 \\ -\gamma & \alpha\delta-\beta & -\alpha & 1 \end{pmatrix} \in KU_-^{>0}K,$$

and $k_2 = \alpha > 0$.

Now we compute the symplectic products:

$$\omega(\mathbf{v}_{2i-1},\mathbf{v}_{2i+1}) = c > 0; \quad \omega(\mathbf{v}_{2i-1},\mathbf{v}_{2i+2}) = ad - b + k_1 c > 0;$$
$$\omega(\mathbf{v}_{2i-1},\mathbf{v}_{2j-1}) = 1 > 0; \quad \omega(\mathbf{v}_{2i-1},\mathbf{v}_{2j}) = k_2 \geqslant 0;$$
$$\omega(\mathbf{v}_{2i},\mathbf{v}_{2i+2}) = \frac{-b^2 + abd - cd}{c} > 0; \quad \omega(\mathbf{v}_{2i},\mathbf{v}_{2j-1}) = \frac{b}{c} > 0;$$
$$\omega(\mathbf{v}_{2i},\mathbf{v}_{2j}) = \frac{bk_2}{c} + 1 > 0; \quad \omega(\mathbf{v}_{2i+1},\mathbf{v}_{2j-1}) = 1 > 0;$$
$$\omega(\mathbf{v}_{2i+1},\mathbf{v}_{2j}) = k_2 + a > 0; \quad \omega(\mathbf{v}_{2i+2},\mathbf{v}_{2j-1}) = k_1 \geqslant 0;$$
$$\omega(\mathbf{v}_{2i+2},\mathbf{v}_{2j}) = k_1 k_2 + 1 + k_1 a > 0.$$

The two non-strict inequalities are strict as soon as they are not forced to be zero which occurs precisely when F_j is adjacent to F_i or F_{i+1}, in which case the vectors in the inequality represent adjacent vertices.

Since i, j are arbitrary, we conclude that $\omega(\mathbf{v}_i,\mathbf{v}_j) > 0$ whenever $i+1 < j$ and $\mathcal{C}(F_1,\ldots,F_k)$ is positive-transverse.

Finally, we conclude from Lemma 5.19 that the Maslov indices are $+2$, and so the Legendrian polygon has decreasing curvature. □

Theorem 5.28. — *The set of positive-transverse Legendrian $2k$-gons with decreasing curvature is a connected component of $\mathscr{P}_{2k}$ homeomorphic to a ball.*

Proof. — By Proposition 5.26, the map $\mathcal{H} : \mathscr{P}^-_{2k} \to \mathscr{F}^{(k)}$ is a homeomorphism onto its image. By Propositions 5.23 and 5.27, the set of positive-transverse polygons with decreasing curvature is mapped onto the set of positive tuples of flags by $\mathcal{H}$. We conclude, by the Fock–Goncharov theorem (Theorem 5.16, Proposition 5.17), that the former is a connected component homeomorphic to a ball. □

Translating with the dictionary of Section 4, we find

Corollary 5.29. — *The subspace of simple circular $2k$-gons with decreasing curvature is a connected component homeomorphic to a ball in the moduli space of generic circular $2k$-gons.*

5.3. Flag curves

Let $\gamma : \mathbb{R} \to \mathbb{P}_{\mathbb{C}}(\mathsf{V})$ be a smooth curve. A *Legendrian lift* (or *contact lift*) of γ to $\mathbb{P}_{\mathbb{R}}(\mathsf{V}) \cong \mathbb{S}(T^*\mathbb{P}_{\mathbb{C}}(\mathsf{V}))$ is a curve $\widehat{\gamma} : \mathbb{R} \to \mathbb{P}_{\mathbb{R}}(\mathsf{V})$ such that $\pi \circ \widehat{\gamma} = \gamma$. There are exactly two contact lifts of γ, given by a choice of transverse orientation.

Proposition 5.30. — *Let $\gamma : \mathbb{R} \to \mathbb{P}_{\mathbb{C}}(\mathsf{V})$ be a C^2 curve with nowhere vanishing curvature. Let $\widehat{\gamma}$ be one of its two contact lifts to $\mathbb{P}_{\mathbb{R}}(\mathsf{V})$. Then, the tangent lines to $\widehat{\gamma}$ are Lagrangians which project to the osculating circles to γ.*

Proof. — For this proof we use coordinates as in Section 4.1. Assume without loss of generality that γ is parametrized by arc length and write, in an affine chart,

$$\gamma(s) = \begin{bmatrix} 1 \\ z(s) \end{bmatrix}.$$

Consider the following pair of lifts of γ to $\mathbb{P}_{\mathbb{R}}(\mathsf{V})$:

$$\widehat{\gamma}(s) = \sqrt{\pm\overline{T(s)}} \begin{pmatrix} 1 \\ z(s) \end{pmatrix},$$

where $T(s) := z'(s)$ is the unit tangent vector to $z(s)$. Denote $T'(s) = k(s)N(s)$, where $N = iT$ is the unit normal vector and $k(s)$ the signed curvature function. We have

$$\widehat{\gamma}'(s) = \begin{pmatrix} \frac{\pm k(s)\overline{N(s)}}{2\sqrt{\pm\overline{T(s)}}} \\ \frac{\pm k(s)\overline{N(s)}z(s)}{2\sqrt{\pm\overline{T(s)}}} + \sqrt{\pm\overline{T(s)}}T(s) \end{pmatrix},$$

and a simple computation shows that $\omega_{\mathbb{C}}(\widehat{\gamma}(s), \widehat{\gamma}'(s)) = \pm i$. This implies that $\widehat{\gamma}$ is Legendrian for $\omega = \Re(\omega_{\mathbb{C}})$, so these two lifts are the contact lifts of γ.

We also compute, choosing the positive sign in the choice of contact lift:

$$\begin{aligned} \pi(a\widehat{\gamma}(s) + b\widehat{\gamma}'(s)) &= \begin{bmatrix} 2a\overline{T(s)} + bk(s)\overline{N(s)} \\ z(s)(2a\overline{T(s)} + bk(s)\overline{N(s)}) + 2b \end{bmatrix}_{\mathbb{C}} \\ &= \begin{bmatrix} 1 \\ z(s) + \frac{2b}{2a\overline{T(s)}+bk(s)\overline{N(s)}} \end{bmatrix}_{\mathbb{C}} \\ &= \begin{bmatrix} 1 \\ z(s) + \frac{1}{k(s)}N(s) - \frac{1}{k(s)}N(s) + \frac{4abT(s)+2b^2k(s)N(s)}{4a^2+b^2k(s)^2} \end{bmatrix}_{\mathbb{C}} \\ &= \begin{bmatrix} 1 \\ z(s) + \frac{1}{k(s)}N(s) + \frac{4abk(s)T(s)+(b^2k(s)^2-4a^2)N(s)}{k(s)(4a^2+b^2k(s)^2)} \end{bmatrix}_{\mathbb{C}} \\ &= \begin{bmatrix} 1 \\ z(s) + \frac{1}{k(s)}N(s) + \frac{1}{k(s)}\frac{(bk(s)-2ai)^2N(s)}{(4a^2+b^2k(s)^2)} \end{bmatrix}_{\mathbb{C}} \end{aligned}$$

which, since $\left|\frac{(bk(s)-2ai)^2}{(4a^2+b^2k(s)^2)}\right| = 1$, is a parametrization of the circle of radius $\frac{1}{k(s)}$ centered at $z(s) + \frac{1}{k(s)}N(s)$ in the affine patch. This circle is the osculating circle to the curve z at $z(s)$. A similar computation with the negative sign yields the same circle (but corresponds to the opposite co-orientation). □

Let $\gamma : S^1 \to \mathbb{P}_{\mathbb{R}}(\mathsf{V})$ be a C^1 Legendrian curve. Motivated by the previous proposition, we call the projection to $\mathbb{P}_{\mathbb{C}}(\mathsf{V})$ of the tangent line at $\gamma(t)$ the *osculating circle* to $\pi \circ \gamma$ at $\pi(\gamma(t))$. This defines osculating circles even for curves with potentially less regularity than the usual C^2 assumption, instead assuming that their Legendrian lifts are C^1.

Accordingly, we say that γ has *decreasing curvature* if for every cyclically ordered $t_1, t_2, t_3 \in S^1$, the triple of Lagrangians given by the tangent lines to γ at $\gamma(t_i)$ have Maslov index $+2$. See Figure 5.5 for an example of osculating circles to a decreasing curvature curve.

Now consider a curve $\xi : S^1 \to \mathscr{F}$ in the space of isotropic flags. The curve ξ is *Frenet* if whenever $n_1 + n_2 + n_3 = p \leqslant 3$, $0 \leqslant n_i \leqslant 2$ we have

$$\lim_{\substack{t_1,t_2,t_3 \text{ distinct} \\ t_i \to t}} \xi(t_1)^{(n_1)} \oplus \xi(t_2)^{(n_2)} \oplus \xi(t_3)^{(n_3)} = \xi(t)^{(p)}.$$

Proposition 5.31. — *If ξ is Frenet, the projection $\pi(\xi(t)^{(2)})$ is the osculating circle to the curve $\gamma = \pi \circ \gamma^{(1)}$ at $\gamma(t)$.*

Proof. — By definition of a Frenet flag curve, $\xi(t)^{(2)}$ is the tangent line to $\xi^{(1)}$ at $\xi(t)$, so its projection to $\mathbb{P}_{\mathbb{C}}(\mathsf{V})$ is the osculating circle to the projection $\pi \circ \xi^{(1)}$. □

A closed curve $\xi : S^1 \to \mathscr{F}$ is *positive* if for every cyclically ordered triple $(t_1, t_2, t_3) \in S^1$, the triple of flags $(\xi(t_1), \xi(t_2), \xi(t_3))$ is positive. We obtain:

Theorem 5.32. — *Let $\xi : S^1 \to \mathscr{F}$ be a positive Frenet curve in $\mathscr{F}$. Then, ξ is the tangent curve to a contact lift of a decreasing curvature simple closed curve in $\mathbb{P}_{\mathbb{C}}(\mathsf{V})$.*

By theorems of Labourie [10, Theorem 1.4] and Fock–Goncharov [6, Theorems 1.14 and 1.15], we know that the limit curve of a Hitchin representation is a positive Frenet curve.

We conclude

Corollary 5.33. — *Let ξ be the limit curve of some Hitchin representation in $\mathsf{PSp}(\mathsf{V})$. Then, ξ is a contact lift of a simple closed curve in $\mathbb{P}_{\mathbb{C}}(\mathsf{V})$ with decreasing curvature.*

For example, for a *Fuchsian* representation, that is, one which factors through the irreducible representation $\mathsf{PSL}(2,\mathbb{R}) \to \mathsf{PSp}(\mathsf{V})$, the limit curve is the Veronese curve (or *twisted cubic*) and its projection to $\mathbb{P}_{\mathbb{C}}(\mathsf{V})$ is depicted in Figure 5.3. More precisely, the Veronese curve with equation

$$v(t) = [1, \sqrt{3}t, \sqrt{3}t^2, t^3]$$

in homogeneous coordinates for $\mathbb{P}(\mathbb{R}^4)$ projects to

$$\frac{1 - i\sqrt{3}t}{\sqrt{3}t^2 + it^3}$$

in the standard affine chart of $\mathbb{P}_{\mathbb{C}}(\mathbb{R}^4)$ and we have applied the Möbius transformation $z \mapsto \frac{iz}{z+1}$ in order to see the whole curve.

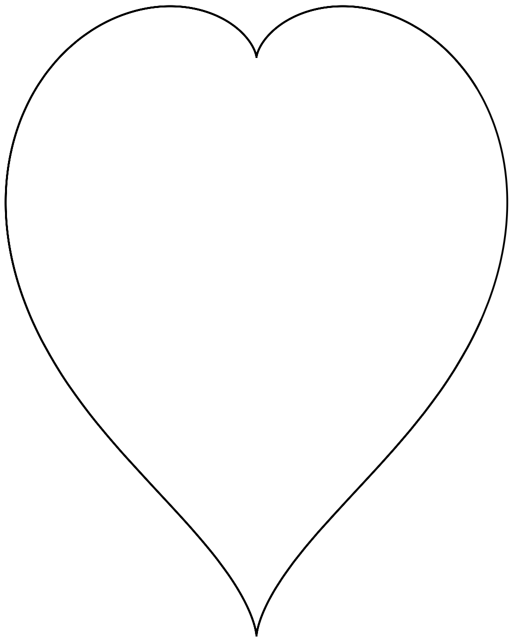


Figure 5.3. The projection to $\mathbb{P}_{\mathbb{C}}(\mathsf{V})$ of the Veronese curve.

We show the collection of osculating circles to the Veronese curve in Figure 5.4, and highlight the decreasing curvature by showing only half of them in Figure 5.5.

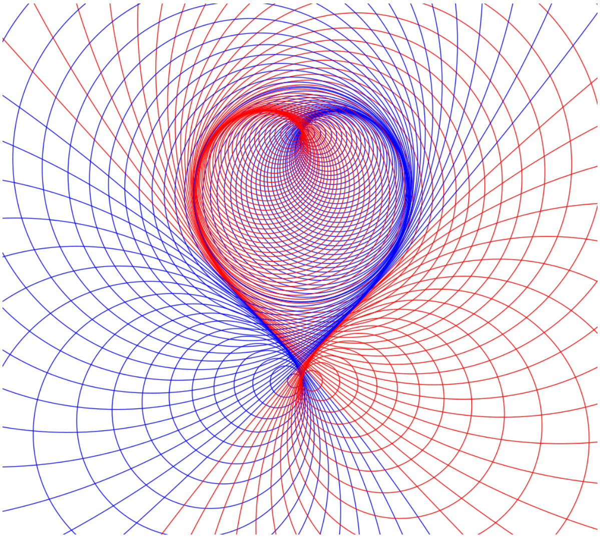


Figure 5.4. The osculating circles to the projected Veronese curve. Colors represent co-orientations. There are four points where the co-orientation changes: the two singularities (circles of radius zero) and the two inflection points (circles of infinite radius).

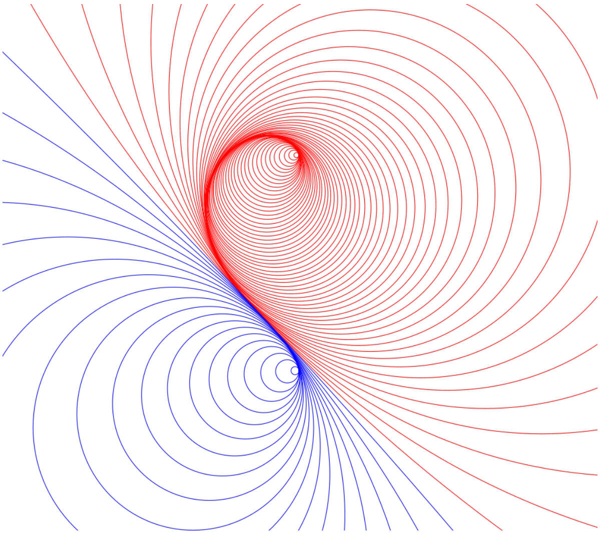


Figure 5.5. The osculating circles to half of the projected Veronese curve.

BIBLIOGRAPHY

[1] T. Banchoff & P. Giblin, "Symmetry sets of piecewise-circular curves", *Proc. R. Soc. Edinb., Sect. A, Math.* **123** (1993), no. 6, p. 1135-1149.

[2] ———, "On the geometry of piecewise circular curves", *Am. Math. Mon.* **101** (1994), no. 5, p. 403-416.

[3] J.-P. Burelle & N. Treib, "Schottky presentations of positive representations", *Math. Ann.* **382** (2022), no. 3-4, p. 1705-1744.

[4] T. E. Cecil, *Lie sphere geometry. With applications to submanifolds*, second ed., Universitext, Springer, 2008, xii+208 pages.

[5] C. H. Conley & V. Ovsienko, "Lagrangian configurations and symplectic cross-ratios", *Math. Ann.* **375** (2019), no. 3-4, p. 1105-1145.

[6] V. Fock & A. Goncharov, "Moduli spaces of local systems and higher Teichmüller theory", *Publ. Math., Inst. Hautes Étud. Sci.* **103** (2006), p. 1-211.

[7] ———, "Moduli spaces of convex projective structures on surfaces", *Adv. Math.* **208** (2007), no. 1, p. 249-273.

[8] O. Guichard & A. Wienhard, "Convex foliated projective structures and the Hitchin component for $\mathrm{PSL}_4(\mathbf{R})$", *Duke Math. J.* **144** (2008), no. 3, p. 381-445.

[9] R. T. Kirk, "Positive Tuples of Flags, Piecewise Circular Wavefronts, and the 3-Dimensional Einstein Universe", PhD Thesis, University of Maryland, College Park, 2019, 157 pages.

[10] F. Labourie, "Anosov flows, surface groups and curves in projective space", *Invent. Math.* **165** (2006), no. 1, p. 51-114.

[11] G. Lusztig, "Total positivity in reductive groups", in *Lie theory and geometry*, Progress in Mathematics, vol. 123, Birkhäuser, 1994, p. 531-568.

[12] S. Morier-Genoud, "Symplectic frieze patterns", *SIGMA, Symmetry Integrability Geom. Methods Appl.* **15** (2019), article no. 089 (36 pages).

[13] A. Pinkus, *Totally positive matrices*, Cambridge Tracts in Mathematics, vol. 181, Cambridge University Press, 2010, xii+182 pages.

[14] P. L. Rosin, "A survey and comparison of traditional piecewise circular approximations to the ellipse", *Comput.-Aided Geom. Des.* **16** (1999), no. 4, p. 269-286.

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