Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Dorothee Frey & Pierre Portal

L^p estimates for wave equations with specific $C^{0,1}$ coefficients

Article à paraître, mis en ligne le 26 mars 2026, 46 p.

CENTRE MERSENNE

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
Article à paraître
Mis en ligne le 26 mars 2026.

L^p ESTIMATES FOR WAVE EQUATIONS WITH SPECIFIC $C^{0,1}$ COEFFICIENTS

by Dorothee FREY & Pierre PORTAL (*)

Abstract. — Peral–Miyachi's celebrated theorem states that the operator $(I-\Delta)^{-\frac{\alpha}{2}}\exp(i\sqrt{-\Delta})$ is bounded on $L^p(\mathbb{R}^d)$ if and only if

$$\alpha \geqslant s_p := (d-1)\left|\frac{1}{p}-\frac{1}{2}\right|.$$

We extend this result to operators of the form $\mathcal{L} = -\sum_{j=1}^{d} a_{j+d}\partial_j a_j \partial_j$, such that, for $j=1,\dots,d$, the functions a_j and a_{j+d} only depend on x_j, are bounded above and below, but are merely Lipschitz continuous. This is below the $C^{1,1}$ regularity that is required in general situations. We construct spaces on which $\exp(i\sqrt{\mathcal{L}})$ is bounded by lifting L^p functions to tent spaces, using wave packets adapted to the coefficients. The result then follows from Sobolev embedding properties of these spaces.

Résumé. — Les célèbres théorèmes de Peral et Miyachi montrent que $(I-\Delta)^{-\frac{\alpha}{2}}\exp(i\sqrt{-\Delta})$ est borné sur $L^p(\mathbb{R}^d)$ si et seulement si

$$\alpha \geqslant s_p := (d-1)\left|\frac{1}{p}-\frac{1}{2}\right|.$$

Nous généralisons ce résultat à des opérateurs de la forme $\mathcal{L} = -\sum_{j=1}^{d} a_{j+d}\partial_j a_j \partial_j$ à coefficients a_j et a_{j+d} ne dépendant que de x_j, bornés et bornés inférieurement, mais seulement lipschitziens. Ceci nous place donc en dessous de l'hypothèse de régularité $C^{1,1}$ nécessaire dans des situations plus générales. Pour ce faire, nous construisons des espaces invariants par l'action de $\exp(i\sqrt{\mathcal{L}})$ via une analyse des fonctions L^p par plongement dans les espaces de tentes et montrons des plongements de Sobolev pour ces espaces.

Keywords: wave equation, rough coefficients, Hardy spaces, tent spaces, wave packets.
2020 *Mathematics Subject Classification:* 42B35, 35L05, 42B30, 42B37, 35S30.
(*) The research of D. Frey is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.
The research of P. Portal is partly supported by the Discovery Project DP160100941 of the Australian Research Council.

1. Introduction

In 1980, Peral [25] and Miyachi [23] proved that the operator $(I - \Delta)^{-\frac{\alpha}{2}} \exp(i\sqrt{-\Delta})$ is bounded on $L^p(\mathbb{R}^d)$ if and only if $\alpha \geqslant s_p := (d-1)\left|\frac{1}{p} - \frac{1}{2}\right|$. Their result was then extended to general Fourier integral operators (FIOs) in a celebrated theorem of Seeger, Sogge, and Stein [28], leading, in particular, to $L^p(\mathbb{R}^d)$ well-posedness results for wave equations with smooth variable coefficients on $\mathbb{R}^d$ or driven by the Laplace–Beltrami operator on a compact manifold. To establish well-posedness of wave equations in more complex geometric settings, many results have been obtained in the past 30 years, using extensions of Peral/Miyachi's fixed time estimates with loss of derivatives, Strichartz estimates, and/or local smoothing properties. This includes Smith's parametrix construction [30], Tataru's Strichartz estimates [35] for wave equations on $\mathbb{R}^d$ with $C^{1,1}$ coefficients, and Müller–Seeger's extension of Peral–Miyachi's result to the sublaplacian on Heisenberg type groups [24], as well as many other important results for specific operators, such as Laplace–Beltrami operators on symmetric spaces.

In this paper, we consider operators of the form $\mathcal{L} = -\sum_{j=1}^{d} a_{j+d}\partial_j a_j \partial_j$, such that, for $j = 1, \dots, d$, the functions a_j and a_{j+d} only depend on x_j, are bounded above and below, and are Lipschitz continuous. For these operators, we extend Peral/Miyachi's result as follows.

Theorem 1.1. — *Let $p \in (1,\infty)$ and $s_p = (d-1)\left|\frac{1}{p} - \frac{1}{2}\right|$. For each $t \in \mathbb{R}$, the operator $(I + \sqrt{\mathcal{L}})^{-s_p} \exp(it\sqrt{\mathcal{L}})$ is bounded on $L^p(\mathbb{R}^d)$. Moreover, if $s_p \leqslant 2$, the operator $\exp(it\sqrt{\mathcal{L}})$ is bounded from $W^{s_p,p}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$.*

When $s_p \leqslant 2$, we show that well-posedness for data in $W^{s_p,p}(\mathbb{R}^d)$ still holds even when $\mathcal{L}$ is perturbed by first order drift terms depending on all the variables (see Section 10). While the algebraic structure of the coefficient matrix is a serious limitation, the roughness of the coefficients is a satisfying and somewhat surprising feature of our result. Indeed, Strichartz estimates for wave equations are known to fail, in general, for coefficients rougher than $C^{1,1}$, see [31, 32].

Our proof is based on a new approach to Seeger–Sogge–Stein's L^p boundedness theorem for FIOs, initiated by Hassell, Rozendaal, and the second author in [16], building on earlier work of Smith [29]. The approach consists in developing a scale of Hardy spaces H^p_{FIO}, that are invariant under the action of FIOs. One then shows that this scale relates to the Sobolev scale through the embedding $W^{\frac{s_p}{2},p} \subset H^p_{\mathrm{FIO}} \subset W^{-\frac{s_p}{2},p}$, for $p \in (1,\infty)$. This is similar, in spirit, to the theory of Hardy spaces associated with operators, which has been extensively developed over the past 15 years, starting with [6, 13, 19] (see also the memoir [18]). In this theory, one first constructs a scale of spaces $H^p_{\mathcal{L}}$ by lifting functions from L^p to one of the tent spaces introduced by Coifman, Meyer, and Stein in [11], using the functional calculus of the operator $\mathcal{L}$ (rather than convolutions). One then shows that the spaces are invariant under the action of the functional calculus of $\mathcal{L}$. Finally, one relates these spaces to more classical ones. For instance $H^p_{\Delta}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ for all $p \in (1,\infty)$. More generally, when one considers Hodge–Dirac operators Π_B, $H^p_{\Pi_B} = L^p$ precisely for those p for which Hodge projections are L^p bounded (a result proven by McIntosh and the authors in [14]).

In the present paper, we go one step further in connecting both theories, by developing a scale of Hardy–Sobolev spaces $H^{p,s}_{\mathrm{FIO},a}$ on which $\exp(i\sqrt{\mathcal{L}})$ is bounded, and proving analogues of the embedding

$$W^{\frac{s_p}{2},p}(\mathbb{R}^d) \subset H^{p,0}_{\mathrm{FIO}}(\mathbb{R}^d) \subset W^{-\frac{s_p}{2},p}(\mathbb{R}^d)$$

such as, for $p \in (1,2)$, $H^{p,\frac{s_p}{2}}_{\mathrm{FIO},a} \subset L^p$ and $(I+\sqrt{\mathcal{L}})^{-\frac{s_p}{2}} \in B(L^p, H^{p,0}_{\mathrm{FIO},a})$. This gives our L^p boundedness with loss of derivatives result, and more. Indeed, one can apply the half wave group $\exp(i\sqrt{\mathcal{L}})$ repeatedly on $H^{p,s}_{\mathrm{FIO},a}$, and only lose derivatives when one compares $H^{p,s}_{\mathrm{FIO},a}$ to classical Sobolev spaces. This allows for iterative arguments in constructing parametrices (an idea used recently in [17]). One can also perturb the half wave group using abstract operator theory on the Banach space $H^{p,s}_{\mathrm{FIO},a}$ (see Corollary 10.3).

The paper is structured as follows. In Section 3, we treat the problem in dimension 1. In this simple situation, arguments based on bi-Lipschitz changes of variables can be used.

In Section 4 we consider the transport group generated, on $L^2(\mathbb{R}^d;\mathbb{C}^2)$, by

$$i\xi.D_a := \sum_{j=1}^{d} \xi_j \begin{pmatrix} 0 & -ia_{j+d}\partial_j \\ ia_j\partial_j & 0 \end{pmatrix},$$

for $\xi \in \mathbb{R}^d$. The dimension 1 results from Section 3 allow us to prove that the commuting one dimensional wave groups $\big(\exp\big(it\sqrt{(e_j.D_a)^2}\big)\big)_{t\in\mathbb{R}}$ are bounded in L^p for all $p \in [1,\infty)$ and $j = 1,\dots,d$. The Phillips functional calculus associated with the corresponding commutative d-parameter group can then replace convolutions/Fourier multipliers in the context of our Lipschitz metric, and includes functions of

$$L := D_a.D_a = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix},$$

where $L_1 := -\sum_{j=1}^{d} a_{j+d}\partial_j a_j \partial_j$ and $L_2 := -\sum_{j=1}^{d} a_j \partial_j a_{j+d}\partial_j$. Using this calculus, we use the approach of [4] to construct an adapted scale of Hardy–Sobolev spaces in Section 5. For all integrability parameters $p \in (1,\infty)$ and regularity parameter $s \in [0,2]$, these spaces coincide with classical Sobolev spaces, thanks to the regularity properties of the heat kernel of L arising from the Lipschitz continuity of its coefficients. To go from these spaces to $H^{p,s}_{\mathrm{FIO},a}$, one needs to directionally refine the Littlewood–Paley decomposition, as in the proof of Seeger–Sogge–Stein's theorem. This is done in [16] using a wave packet transform defined by Fourier multipliers. In Section 6 we construct a similar wave packet transform, replacing Fourier multipliers by the Phillips calculus of the transport group. This allows us to define $H^{p,s}_{\mathrm{FIO},a}$ in Section 7, and to prove its embedding properties in Section 8. In Section 9, we prove that the half wave group $\big(\exp(it\sqrt{L})\big)_{t\in\mathbb{R}}$ is bounded on $H^{p,s}_{\mathrm{FIO},a}$ for all $1 < p < \infty$ and $s \in \mathbb{R}$. To do so, we first notice that such bounds hold for the transport group. We then realise that, in a given direction ω, $\exp(i\sqrt{D_a.D_a})$ is close to $\exp\big(i\sum_{j=1}^{d}\omega_j\sqrt{(e_j.D_a)^2}\big)$, when acting on an appropriate wave packet, in the sense that operators of the form $\big(\exp(i\sqrt{D_a.D_a}) - \exp\big(i\sum_{j=1}^{d}\omega_j\sqrt{(e_j.D_a)^2}\big)\big)\varphi_\omega(D_a)$ are L^p bounded. Finally, in Section 10, we show that $\exp(it\sqrt{L})$ remains bounded if one appropriately perturbs L by first order terms. This is based on Theorem 10.1, a result about multiplication operators on $H^{p}_{\mathrm{FIO},a}$ that is of independent interest, even in the case where $a_j = 1$ for all $j = 1,\dots,2d$.

Our approach relies heavily on algebraic properties: the wave group commutes with the wave packet localisation operators, and can be expressed in the Phillips functional calculus of a commutative group. Although our coefficients are merely Lipschitz continuous, these algebraic properties match those of the standard Euclidean wave group. However, in dimension $d > 1$, the problem does not reduce to its Euclidean counterpart through a change of variables (see Remark 4.5).

In the same way as Peral–Miyachi's result for the standard half wave group is a starting point for the well-posedness theory of wave equations with coefficients that are smooth enough perturbations of constant coefficients, we expect the results proven here to provide a basis for the development of a well-posedness theory of wave equations with coefficients that are smooth enough perturbations of structured Lipschitz continuous coefficients.

Acknowledgments

We thank Andrew Hassell and Jan Rozendaal for many interesting discussions on the relations between this work and theirs. We particularly want to thank Jan Rozendaal for pointing out a mistake in a previous version of Section 10. We also thank the anonymous referee of a previous version of this paper for pointing out the change of variable approach that we now use in Section 3, before moving on to more general operators for which such an approach is not available.

2. Preliminaries

We first recall the following Banach space valued Marcinkiewicz–Lizorkin Fourier multiplier's theorem (see [34, Theorem 4.5]). We work here in the very special case where the target Banach space is $L^p(\mathbb{R}^d)$, and denote by $I_{L^p(\mathbb{R}^d)}$ the identity map on this space.

Theorem 2.1 (Fernandez/Štrkalj–Weis). — *Let $p \in (1,\infty)$. Let $m \in C^1\big(\mathbb{R}^d \backslash \{0\}\big)$ be such that, for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha|_\infty \leqslant 1$ there exists a constant $C = C(\alpha) > 0$ such that*

$$\left|\zeta^\alpha \partial_\zeta^\alpha m(\zeta)\right| \leqslant C \quad \forall\, \zeta \in \mathbb{R}^d \setminus \{0\}.$$

Let T_m denote the Fourier multiplier with symbol m. Then $T_m \otimes I_{L^p(\mathbb{R}^d)}$ extends to a bounded operator on $L^p\big(\mathbb{R}^d; L^p(\mathbb{R}^d)\big)$.

This theorem will be combined with the following version of the Coifman–Weiss transference principle (see [21, Theorem 10.7.5]). Note that the extension of this theorem from a one parameter group to a d parameter group generated by a tuple of commuting operators is straightforward.

Theorem 2.2 (Coifman–Weiss). — *Let $p \in (1,\infty)$. Let $iD_1, \dots, iD_d$ generate bounded commuting groups $\big(\exp(itD_j)\big)_{t\in\mathbb{R}}$ on $L^p(\mathbb{R}^d)$, and consider the d parameter group defined by $\exp(i\xi D) = \prod_{j=1}^d \exp(i\xi_j D_j)$ for $\xi \in \mathbb{R}^d$. Then, for all $\psi \in \mathcal{S}(\mathbb{R}^d)$, we have that*

$$\left\| \int_{\mathbb{R}^d} \widehat{\psi}(\xi) \exp(i\xi D) f \,\mathrm{d}\xi \right\|_{L^p(\mathbb{R}^d)} \lesssim \left\| T_\psi \otimes I_{L^p(\mathbb{R}^d)} \right\|_{B(L^p(\mathbb{R}^d;L^p(\mathbb{R}^d)))} \|f\|_{L^p(\mathbb{R}^d)} \quad \forall\, f \in L^p(\mathbb{R}^d).$$

To define our Hardy–Sobolev spaces, we use the tent spaces introduced by Coifman, Meyer, and Stein in [11], and used extensively in the theory of Hardy spaces associated with operators (see e.g. the memoir [18] and the references therein). These tent spaces $T^{p,2}(\mathbb{R}^d)$ are defined as follows. For $F\colon \mathbb{R}^d \times (0,\infty) \to \mathbb{C}^N$ measurable and $x \in \mathbb{R}^d$, set

$$\mathcal{A}F(x) := \left(\int_0^\infty ⨍_{B(x,\sigma)} \left|F(y,\sigma)\right|^2 \mathrm{d}y \frac{\mathrm{d}\sigma}{\sigma} \right)^{1/2} \in [0,\infty],$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{C}^N$.

Definition 2.3. — *Let $p \in [1,\infty)$. The tent space $T^{p,2}(\mathbb{R}^d)$ is defined as the space of all $F \in L^2_{\mathrm{loc}}\big(\mathbb{R}^d \times (0,\infty), \mathrm{d}x \frac{\mathrm{d}\sigma}{\sigma}\big)$ such that $\mathcal{A}F \in L^p(\mathbb{R}^d)$, endowed with the norm*

$$\|F\|_{T^{p,2}(\mathbb{R}^d)} := \|\mathcal{A}F\|_{L^p(\mathbb{R}^d)}.$$

Recall that the tent space $T^{1,2}$ admits an atomic decomposition (see [11]) in terms of atoms A supported in sets of the form $B(c_B, r) \times [0,r]$, and satisfying

$$r^d \int_0^r \int_{\mathbb{R}^d} \left|A(y,\sigma)\right|^2 \frac{\mathrm{d}y\,\mathrm{d}\sigma}{\sigma} \leqslant 1.$$

Recall also that the classical Hardy space $H^1(\mathbb{R}^d)$ norm can be obtained as

$$\|f\|_{H^1(\mathbb{R}^d)} := \left\|(t,x) \mapsto \psi(t^2\Delta)f(x)\right\|_{T^{1,2}(\mathbb{R}^d)},$$

where $\psi(t^2\Delta)$ denotes the Fourier multiplier with symbol

$$\xi \mapsto t^2|\xi|^2 \exp\big(-t^2|\xi|^2\big).$$

This is the starting point of the theory of Hardy spaces associated with operators (or equations): one replaces the Fourier multiplier by an appropriately adapted operator. To do so, one often uses the holomorphic functional calculus of a (bi)sectorial operator. The relevant theory is presented in [21]. We use it here with the following notation.

Definition 2.4. — *Let $0 < \theta < \frac{\pi}{2}$. Define the open sector in the complex plane by*

$$S^\circ_{\theta+} := \big\{z \in \mathbb{C} \setminus \{0\} \,:\, \left|\arg(z)\right| < \theta\big\},$$

as well as the bisector $S^\circ_\theta = S^\circ_{\theta+} \cup S^\circ_{\theta-}$, where $S^\circ_{\theta-} = -S^\circ_{\theta+}$. We denote by $H(S^\circ_\theta)$ the space of holomorphic functions on S°_θ, and set

$$H^\infty(S^\circ_\theta) := \big\{g \in H(S^\circ_\theta) \,:\, \|g\|_{L^\infty(S^\circ_\theta)} < \infty\big\},$$

$$\Psi^\beta_\alpha(S^0_\theta) := \left\{ \psi \in H^\infty(S^\circ_\theta) \;\middle|\; \begin{array}{r} \exists\, C > 0 \,:\, \left|\psi(z)\right| \leqslant C|z|^\alpha\big(1+|z|^{\alpha+\beta}\big)^{-1} \\ \forall\, z \in S^\circ_\theta \end{array} \right\},$$

for every $\alpha, \beta > 0$. We say that $\psi \in H^\infty(S^\circ_\theta)$ is non-degenerate if neither of its restrictions to $S^\circ_{\theta+}$ or $S^\circ_{\theta-}$ vanishes identically.

For bisectorial operators D such that iD generates a bounded group on L^p, we also use the Phillips calculus defined by

$$\psi(D)f \coloneqq \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\psi}(\xi)\exp(i\xi D)f \,\mathrm{d}\xi,$$

for $f \in L^p$ and $\psi \in \mathcal{S}(\mathbb{R})$. See [4, 22] for more information on how these two functional calculi interact in the theory of Hardy spaces associated with operators. The results in Section 5 are fundamentally inspired by these papers.

3. The one dimensional case

In dimension 1, the type of wave equations we are studying in this paper can be treated through a combination of simple changes of variables and perturbation arguments. In this section, we present this method both for pedagogical reasons, and because its results are used to set up our approach to higher dimensional problems in the next sections.

Let $a, b \in C^{0,1}(\mathbb{R})$ with $\frac{\mathrm{d}}{\mathrm{d}x}a, \frac{\mathrm{d}}{\mathrm{d}x}b \in L^\infty$, and assume that there exist $0 < \lambda \leqslant \Lambda$ such that $\lambda \leqslant a(x) \leqslant \Lambda$ and $\lambda \leqslant b(x) \leqslant \Lambda$ for all $x \in \mathbb{R}$. We consider the wave equation $\partial_t^2 u = (a\partial_x b\partial_x)u$.

Proposition 3.1. — *The operators*

$$a\frac{\mathrm{d}}{\mathrm{d}x} \qquad \text{and} \qquad i\sqrt{-a\frac{\mathrm{d}}{\mathrm{d}x}a\frac{\mathrm{d}}{\mathrm{d}x}}$$

generate bounded C_0 groups on $L^p(\mathbb{R})$ for all $p \in (1,\infty)$.

Proof. — Define $\phi\colon x \mapsto \int_0^x \frac{1}{a(y)}\,\mathrm{d}y$, and note that it is a C^1 diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. The map $\chi \in C^1(\mathbb{R}^2)$ defined by

$$\chi\colon (t,x) \longmapsto \phi^{-1}\big(t + \phi(x)\big),$$

is then a solution to

$$\partial_t\chi(t,x) = a\big(\chi(t,x)\big) \quad \forall\, t, x \in \mathbb{R}.$$

It is such that

$$t = \int_{\chi(0,x)}^{\chi(t,x)} \frac{1}{a_j(y)}\,\mathrm{d}y \quad \forall\, t, x \in \mathbb{R}. \tag{3.1}$$

and thus:

$$\frac{\mathrm{d}}{\mathrm{d}x}\chi(x,t) = \frac{a\big(\chi(x,t)\big)}{a(x)} \quad \forall\, x,t \in \mathbb{R}.$$

Therefore $x \mapsto \frac{\mathrm{d}}{\mathrm{d}x}\chi(x,t)$ is bounded above and below, uniformly in t, and χ is a thus a bi-Lipschitz flow. We now define the associated transport group by

$$T_t f(x) = f\big(\chi(t,x)\big) \quad \forall\, t,x \in \mathbb{R}$$

for $f \in C_c^\infty(\mathbb{R})$. It extends to a bounded group on $L^p(\mathbb{R}^d)$ for all $p \in [1,\infty]$, with finite speed of propagation. Strong continuity $\big\|T(t)f - f\big\|_p \underset{t\to 0}{\longrightarrow} 0$ for $p < \infty$ follows by dominated convergence for f continuous, and then density for general f. To identify the generator, let $f \in W^{1,p}$, and note that, for all $x \in \mathbb{R}^d$,

$$\begin{aligned} \frac{\partial}{\partial t}T(t)f(x)\Big|_{t=0} &= \frac{\partial}{\partial t}f\big(\chi(x,t)\big)\Big|_{t=0} \\ &= \nabla f(x)\cdot \partial_t\chi(x,t)\big|_{t=0} \\ &= a(x)\partial_x f(x). \end{aligned}$$

For $f \in C_c^\infty(\mathbb{R})$, we have that

$$T_t(f\circ\phi)(x) = f\big(t+\phi(x)\big) = \left(\exp\left(it\frac{\mathrm{d}}{\mathrm{d}x}\right)f\right)\big(\phi(x)\big) \quad \forall\, t,x \in \mathbb{R}.$$

For $f \in C_c^\infty(\mathbb{R})$, $s \in \mathbb{R}$, and $\varepsilon > 0$, we have that

$$\exp\left(-(\varepsilon+is)\sqrt{-a\frac{\mathrm{d}}{\mathrm{d}x}a\frac{\mathrm{d}}{\mathrm{d}x}}\right)f = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{\psi_s}(t)T_t f\,\mathrm{d}t$$

for $\psi_s\colon x \mapsto \exp\big(-(\varepsilon+is)|x|\big)$. We thus have that

$$\exp\left(-(\varepsilon+is)\sqrt{-a\frac{\mathrm{d}}{\mathrm{d}x}a\frac{\mathrm{d}}{\mathrm{d}x}}\right)(f\circ\phi)(x) = \left(\exp\left(-(\varepsilon+is)\frac{\mathrm{d}}{\mathrm{d}x}\right)f\right)\big(\phi(x)\big) \quad \forall\, x \in \mathbb{R},$$

for all $f \in C_c^\infty(\mathbb{R})$, $s \in \mathbb{R}$, and $\varepsilon > 0$. On $L^2(\mathbb{R})$, $i\sqrt{-a\frac{\mathrm{d}}{\mathrm{d}x}a\frac{\mathrm{d}}{\mathrm{d}x}}$ generates a bounded group and $-\sqrt{-a\frac{\mathrm{d}}{\mathrm{d}x}a\frac{\mathrm{d}}{\mathrm{d}x}}$ generates an analytic semigroup. We thus have that

$$\exp\left(is\sqrt{-a\frac{\mathrm{d}}{\mathrm{d}x}a\frac{d}{\mathrm{d}x}}\right)(f\circ\phi)(x) = \left(\exp\left(is\frac{\mathrm{d}}{\mathrm{d}x}\right)f\right)\big(\phi(x)\big) \quad \forall\, x \in \mathbb{R},$$

for all $f \in C_c^\infty(\mathbb{R})$, and $s \in \mathbb{R}$. Since ϕ is a C^1 diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, this gives that $i\sqrt{-a\frac{\mathrm{d}}{\mathrm{d}x}a\frac{\mathrm{d}}{\mathrm{d}x}}$ generates a bounded C_0 group on $L^p(\mathbb{R})$ for all $p \in [1,\infty)$. □

Corollary 3.2. — *The operators* $i\sqrt{-\frac{\mathrm{d}}{\mathrm{d}x}a^2\frac{\mathrm{d}}{\mathrm{d}x}}$ *and* $i\sqrt{-a\frac{\mathrm{d}}{\mathrm{d}x}b\frac{\mathrm{d}}{\mathrm{d}x}}$ *generate bounded* C_0 *groups on* $L^p(\mathbb{R})$ *for all* $p \in [1,\infty)$.

Proof. — We have that

$$\frac{\mathrm{d}}{\mathrm{d}x}a^2\frac{\mathrm{d}}{\mathrm{d}x} = a\frac{\mathrm{d}}{\mathrm{d}x}a\frac{\mathrm{d}}{\mathrm{d}x} + a'a\frac{\mathrm{d}}{\mathrm{d}x} \qquad \text{and} \qquad a\frac{\mathrm{d}}{\mathrm{d}x}b\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}ab\frac{\mathrm{d}}{\mathrm{d}x} - a'b\frac{\mathrm{d}}{\mathrm{d}x}.$$

For all $p\in[1,\infty)$ and all $f \in W^{1,p}(\mathbb{R})$, we have that $\|a'bf'\|_p \leqslant \|ba'\|_\infty \|f'\|_p$. The result thus follows from perturbation theory and square root reduction for cosine families, see [2, Proposition 3.16.3 and Corollary 3.14.13]. □

4. The transport groups

The method developed in this paper applies to wave equations of the form $\partial_t^2 u = -\sum_{j=1}^d D_j^2 u$. What we need from D is that iD_j and $i\sqrt{D_j^2}$ generates a bounded C_0 group on L^p for each j, the operators $D_1^2, \dots, D_d^2$ commute, and $L = -\sum_{j=1}^d D_j^2$ is such that appropriate Riesz transform bounds and Hardy space estimates hold. In this section, we consider the simplest non-trivial example of such a Dirac operator. We then use this example throughout the paper, but indicate when the results hold for more general Dirac operators, with the same proofs.

For $j \in \{1, \dots, 2d\}$, let $a_j \in C^{0,1}(\mathbb{R})$ with $\frac{\mathrm{d}}{\mathrm{d}x}a_j \in L^\infty$, and assume that there exist $0 < \lambda \leqslant \Lambda$ such that $\lambda \leqslant a_j(x) \leqslant \Lambda$ for all $x \in \mathbb{R}$. We denote by $\widetilde{a_j} \in C^{0,1}(\mathbb{R}^d)$ the map defined by $\widetilde{a_j} \colon x \mapsto a_j(x_j)$.

Definition 4.1. — *For* $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, *define*

$$\xi.D_a \coloneqq \sum_{j=1}^d \xi_j \begin{pmatrix} 0 & -\widetilde{a_{j+d}}\partial_j \\ \widetilde{a_j}\partial_j & 0 \end{pmatrix},$$

$$\xi.\sqrt{D_a^2} \coloneqq \sum_{j=1}^d \xi_j \begin{pmatrix} \sqrt{-\widetilde{a_{j+d}}\partial_j\widetilde{a_j}\partial_j} & 0 \\ 0 & \sqrt{-\widetilde{a_j}\partial_j\widetilde{a_{j+d}}\partial_j} \end{pmatrix},$$

as an unbounded operator acting on $L^2(\mathbb{R}^d;\mathbb{C}^2)$, *with domain* $W^{1,2}(\mathbb{R}^d;\mathbb{C}^2)$.

Note that $W^{1,2}(\mathbb{R}^d;\mathbb{C}^2)$ is an appropriate domain for $\xi.\sqrt{D_a^2}$ thanks to the boundedness of the relevant Riesz transforms proven in [7, Corollary 5.19].

As in [22, Section 4, Case II], $ie_j.D_a$ generates a bounded C_0 group on $L^2(\mathbb{R}^d;\mathbb{C}^2)$ for all $j = 1, \dots, d$, since $e_j.D_a$ is self-adjoint with respect to an equivalent inner product of the form $(u,v) \mapsto \langle A^{-1}u, v\rangle$, where A is a diagonal multiplication operator with $C^{0,1}$ entries.

Remark 4.2. — For $E,F\subset\mathbb{R}^d$ Borel sets and $\omega\in S^{d-1}$, we set

$$\omega.d(E,F):=\inf_{x\in E,y\in F}\big|\langle\omega,x-y\rangle\big|.$$

By [22, Remark 3.6], we have the following (strong) form of finite speed of propagation: there exists $\kappa>0$ such that for all $f\in L^2(\mathbb{R}^d;\mathbb{C}^2)$, all Borel sets $E,F\subset\mathbb{R}^d$, all $j=1,\dots,d$, all $\xi\in\mathbb{R}^d$, and all $\omega\in S^{d-1}$ we have

$$1_E\exp(i\xi_j e_j.D_a)(1_F f)=0,$$

whenever $\frac{\kappa}{\sqrt{d}}\big|\langle\omega,\xi_j e_j\rangle\big|<\omega.d(E,F)$. Consequently,

$$1_E\prod_{j=1}^{d}\exp(i\xi_j e_j.D_a)(1_F f)=0,$$

whenever $\kappa\big|\langle\omega,\xi\rangle\big|<\omega.d(E,F)$. Indeed, we have that

$$1_E\prod_{j=1}^{d}\exp(i\xi_j e_j.D_a)(1_F f)=1_E\exp(i\xi_1 e_1.D_a)1_{E_1}\prod_{j=2}^{d}\exp(i\xi_j e_j.D_a)(1_F f),$$

for $E_1=\big\{(y_1,x_2,\dots,x_d)\in\mathbb{R}^d\ ;\ (x_1,\dots,x_d)\in E \text{ and } |y_1-x_1|\leqslant\frac{\kappa}{\sqrt{d}}|\xi_1|\big\}$. Iterating this argument gives us that

$$1_E\prod_{j=1}^{d}\exp(i\xi_j e_j.D_a)(1_F f)=1_E\prod_{j=1}^{d}\exp(i\xi_j e_j.D_a)1_{\widetilde{E}}(1_F f),$$

for $\widetilde{E}=\big\{(y_1,\dots,y_d)\in\mathbb{R}^d\ ;\ (x_1,\dots,x_d)\in E \text{ and } |y_j-x_j|\leqslant\frac{\kappa}{\sqrt{d}}|\xi_j|\quad\forall\ j=1,\dots,d\big\}$. Assuming that there exists $y\in\widetilde{E}\cap F$ when $\kappa\big|\langle\omega,\xi\rangle\big|<\omega.d(E,F)$, we obtain that, for all $x\in E$,

$$\big|\langle\omega,x-y\rangle\big|\leqslant\kappa\max_{j=1,\dots,d}\big|\langle\xi,e_j\rangle\omega_j\big|<\omega.d(E,F),$$

which is a contradiction.

Proposition 4.3. — *Let $\xi\in\mathbb{R}^d$ and $p\in(1,\infty)$. The group $\big(\exp(it\xi.\sqrt{D_a^2})\big)_{t\in\mathbb{R}}$ is bounded on $L^p(\mathbb{R}^d;\mathbb{C}^2)$.*

Proof. — Let $p\in(1,\infty)$. Using linearity and freezing $d-1$ of the variables, it suffices to show that the group generated by $i\begin{pmatrix}0 & -b\frac{\mathrm{d}}{\mathrm{d}x}\\ a\frac{\mathrm{d}}{\mathrm{d}x} & 0\end{pmatrix}$ is bounded on $L^p(\mathbb{R};\mathbb{C}^2)$ for $a:=a_1$ and $b:=a_{d+1}$. For $f,g\in C_c^\infty(\mathbb{R})$, and $t\in\mathbb{R}$, let us consider

$$\begin{pmatrix}u(t,\cdot)\\ v(t,\cdot)\end{pmatrix}:=\exp\left(it\begin{pmatrix}0 & -b\frac{\mathrm{d}}{\mathrm{d}x}\\ a\frac{\mathrm{d}}{\mathrm{d}x} & 0\end{pmatrix}\right)\begin{pmatrix}f\\ g\end{pmatrix}.$$

We have that

$$\begin{pmatrix} \partial_t u(t,\cdot) \\ \partial_t v(t,\cdot) \end{pmatrix} = i \begin{pmatrix} -b\frac{\mathrm{d}}{\mathrm{d}x} v(t,\cdot) \\ a\frac{\mathrm{d}}{\mathrm{d}x} u(t,\cdot) \end{pmatrix} \quad \forall\, t, x \in \mathbb{R},$$

and

$$\begin{pmatrix} \partial_t^2 u(t,\cdot) \\ \partial_t^2 v(t,\cdot) \end{pmatrix} = \begin{pmatrix} -b\frac{\mathrm{d}}{\mathrm{d}x} a\frac{\mathrm{d}}{\mathrm{d}x} u(t,\cdot) \\ -a\frac{\mathrm{d}}{\mathrm{d}x} b\frac{\mathrm{d}}{\mathrm{d}x} v(t,\cdot) \end{pmatrix} \quad \forall\, t, x \in \mathbb{R}.$$

Using Corollary 3.2 and solving these wave equations using the relevant cosine families (see [2, Corollary 3.14.12]), this gives

$$\big\|u(t,\cdot)\big\| \lesssim \|f\|_p + \left\| \left(-b\frac{\mathrm{d}}{\mathrm{d}x} a\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-\frac{1}{2}} g' \right\|_p \lesssim \|f\|_p + \|g\|_p,$$

$$\big\|v(t,\cdot)\big\| \lesssim \|g\|_p + \left\| \left(-a\frac{\mathrm{d}}{\mathrm{d}x} b\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-\frac{1}{2}} f' \right\|_p \lesssim \|f\|_p + \|g\|_p,$$

with constants independent of t, using the boundedness of the Riesz transforms $\frac{\mathrm{d}}{\mathrm{d}x}\big(-b\frac{\mathrm{d}}{\mathrm{d}x} a\frac{\mathrm{d}}{\mathrm{d}x}\big)^{-\frac{1}{2}}$ proven in [5, 8]. □

Remark 4.4. — Given the vector-valued nature of the Dirac operator D_a, all function spaces considered in the remaining of the paper will be implicitly $\mathbb{C}^2$ valued.

Remark 4.5. — The transport group generated by iD_a is, even in dimension 1, substantially more complicated than the transport group generated by $a\frac{\mathrm{d}}{\mathrm{d}x}$ considered in Section 3. Its L^p boundedness, for instance, does not follow from the boundedness of the translation group through bi-Lipschitz changes of variables. Indeed, for non-constant coefficients $a \in C^{0,1}(\mathbb{R})$, no intertwining relation

$$U \begin{pmatrix} 0 & -\frac{\mathrm{d}}{\mathrm{d}x} \\ a\frac{\mathrm{d}}{\mathrm{d}x} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\mathrm{d}}{\mathrm{d}x} \\ \frac{\mathrm{d}}{\mathrm{d}x} & 0 \end{pmatrix} U$$

can hold for U of the form $U\colon (f,g) \mapsto (f\circ\phi, g\circ\psi)$ where $\phi,\psi\colon \mathbb{R}\to\mathbb{R}$ are bi-Lipschitz changes of variables.

5. Hardy spaces associated with the transport groups

Definition 5.1. — *Given $\Psi \in \mathcal{S}(\mathbb{R}^d)$, we define $\Psi(\sqrt{D_a^2})$ using the Phillips functional calculus associated with the commutative group $\big(\exp(i\xi.\sqrt{D_a^2})\big)_{\xi\in\mathbb{R}^d}$:*

$$\Psi(\sqrt{D_a^2}) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{\Psi}(\xi)\exp(i\xi.\sqrt{D_a^2})\,\mathrm{d}\xi.$$

We restrict our attention to functions Ψ *that satisfy* $\Psi = \Psi^s$*, where*

$$\Psi^s(x) := 2^{-d} \sum_{(\delta_j)_{j=1}^d \in \{-1,1\}^d} \Psi(\delta_1 x_1, \ldots, \delta_d x_d).$$

For such functions, we have that

$$\begin{aligned}
&\Psi^s(\sqrt{D_a^2}) \\
&\quad = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \widehat{\Psi^s}(\xi) \frac{1}{2}\big(\exp(i\xi_1 e_1 \sqrt{D_a^2}) + \exp(-i\xi_1 e_1 \sqrt{D_a^2})\big)\, \mathrm{d}\xi_1 \\
&\qquad\qquad\qquad\qquad \exp\big(i(\xi - \xi_1 e_1)\sqrt{D_a^2}\big)\, \mathrm{d}\xi_2 \cdots \mathrm{d}\xi_d \\
&\quad = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \widehat{\Psi^s}(\xi) \frac{1}{2}\big(\exp(i\xi_1 e_1 D_a) + \exp\big(-i\xi_1 e_1 D_a\big)\big)\, \mathrm{d}\xi_1 \\
&\qquad\qquad\qquad\qquad \exp\big(i(\xi - \xi_1 e_1)\sqrt{D_a^2}\big)\, \mathrm{d}\xi_2 \cdots \mathrm{d}\xi_d \\
&\quad = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\Psi^s}(\xi) \prod_{j=1}^d \exp(i\xi_j e_j.D_a)\, \mathrm{d}\xi,
\end{aligned}$$

since $e_j.D_a$ *and* $e_j.\sqrt{D_a^2}$ *generate the same cosine family. We write* $\Psi(D_a)$ *instead of* $\Psi(\sqrt{D_a^2})$ *when* $\Psi = \Psi^s$*.*

Lemma 5.2. — *There exists* $C > 0$ *such that, for all* $\Psi \in \mathcal{S}(\mathbb{R}^d)$ *such that* $\Psi = \Psi^s$*, all* $E, F \subset \mathbb{R}^d$ *Borel sets and all* $\omega \in S^{d-1}$*, we have that*

$$\big\|1_E \Psi(D_a)(1_F f)\big\|_2 \leqslant C\|1_F f\|_2 \int_{\{|\xi| \geqslant \frac{\mathrm{d}(E,F)}{\kappa}\} \cap \{|\langle \omega, \xi\rangle| \geqslant \frac{\omega.d(E,F)}{\kappa}\}} \big|\widehat{\Psi}(\xi)\big|\, \mathrm{d}\xi \qquad \forall\, f \in L^2(\mathbb{R}^d),$$

where κ *and* $\omega.d(E,F)$ *are defined as in Remark 4.2. Consequently, for every* $\Psi \in \mathcal{S}(\mathbb{R}^d)$ *and every* $M \in \mathbb{N}$*, there exists* $C_M > 0$ *such that*

$$\big\|1_E \Psi(\sigma D_a)(1_F f)\big\|_2 \leqslant C_M \left(1 + \frac{\mathrm{d}(E,F)}{\kappa\sigma}\right)^{-M} \|1_F f\|_2 \quad \forall\, f \in L^2(\mathbb{R}^d)$$

for all Borel sets $E, F \subset \mathbb{R}^d$ *and all* $\sigma > 0$*.*

Proof. — Let $f \in L^2(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$. By Remark 4.2, we have that

$$1_E \prod_{j=1}^d \exp(i\xi_j e_j.D_a)(1_F f) = 0,$$

whenever $\kappa|\xi| < d(E,F)$ or $\kappa\big|\langle\omega,\xi\rangle\big| < \omega.d(E,F)$. Therefore, using Phillips functional calculus, we have that

$$\begin{aligned}\big\|1_E\Psi(D_a)(1_Ff)\big\|_2 &\leqslant \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\big|\widehat{\Psi}(\xi)\big|\big\|1_E\textstyle\prod_{j=1}^d\exp(i\xi_je_j.D_a)(1_Ff)\big\|_2\,\mathrm{d}\xi\\ &\leqslant C\|1_Ff\|_2\int_{\{|\xi|\geqslant\frac{\mathrm{d}(E,F)}{\kappa}\}\cap\{|\langle\omega,\xi\rangle|\geqslant\frac{\omega.d(E,F)}{\kappa}\}}\big|\widehat{\Psi}(\xi)\big|\,\mathrm{d}\xi,\end{aligned}$$

where $C := \frac{1}{(2\pi)^d}\sup\big\{\big\|\prod_{j=1}^d\exp(i\xi_je_j.D_a)\big\|_{B(L^2)};\ \xi\in\mathbb{R}^d\big\}$. The last statement then follows from a change of variables and $\Psi\in\mathcal{S}(\mathbb{R}^d)$. □

We recall the following fact, which is a corollary of the results in [7], using that the coefficients a_j are Lipschitz continuous.

Theorem 5.3 (Auscher, McIntosh, Tchamitchian). — *Let $p\in(1,\infty)$. On $L^p(\mathbb{R}^d)$, the operator D_a^2, with domain $W^{2,p}(\mathbb{R}^d)$, generates an analytic semigroup, and has a bounded H^∞ calculus of angle 0. Moreover, $\big\{\exp(-tL)\ ;\ t>0\big\}$ satisfies Gaussian estimates.*

Definition 5.4. — *From now on, we denote by $L := -D_a^2$ the negative generator of the semigroup in Theorem 5.3.*

As a consequence of the Gaussian estimates for $\big\{\exp(-tL)\ ;\ t>0\big\}$, we have the key fact that applying a family of bounded operators T_σ acting on functions of the spatial variable x, and indexed by the scale variable σ, defines a bounded "multiplication" operation on $T^{p,2}(\mathbb{R}^d)$, as long as the family $(T_\sigma)_{\sigma>0}$ has appropriate off-diagonal decay. We use the notation $T_\sigma F(\sigma,\cdot\,)$ for the functions in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ defined, for a fixed $\sigma>0$, by applying the operator T_σ to $x\mapsto F(\sigma,x)$.

Corollary 5.5. — *Let $p\in(1,\infty)$, $\theta>0$, $g\in H^\infty(S^\circ_{\theta+})$, and let $\Psi\in C_c^\infty(\mathbb{R}^d)$ be supported away from 0 and such that $\Psi=\Psi^s$. Then there exists a constant $C>0$ independent of g such that, for all $F\in T^{p,2}(\mathbb{R}^d)$,*

$$\big\|(\sigma,x)\mapsto\Psi(\sigma D_a)g(L)F(\sigma,\cdot\,)(x)\big\|_{T^{p,2}(\mathbb{R}^d)}\leqslant C\|g\|_{L^\infty(S^\circ_{\theta+})}\|F\|_{T^{p,2}(\mathbb{R}^d)}.$$

Proof. — For $M\in\mathbb{N}$, set $q_M(z) := z^M(1+z)^{-2M}$, $z\in S^\circ_{\theta+}$. Note that then $q_M\in\Psi_M^M(S^\circ_{\theta+})$. The statement for $\Psi(\sigma D_a)$ replaced by $q_M(\sqrt{\sigma}L)$ for M large enough then follows from a combination of [20, Theorem 5.2] and [20, Lemma 7.3], using Lemma 5.2 and Theorem 5.3 to check the assumptions.

On the other hand, we have by assumption $\zeta\mapsto\Psi(\zeta)q_M^{-1}\big(|\zeta|^2\big)\in\mathcal{S}(\mathbb{R}^d)$, so that an application of [20, Theorem 5.2] together with Lemma 5.2 yields the assertion. □

Lemma 5.6. — *Let $\alpha \in \mathbb{R}$, and non-degenerate $\Psi, \widetilde{\Psi} \in C_c^\infty(\mathbb{R}^d)$ be supported away from 0 and such that $\Psi = \Psi^s$, $\widetilde{\Psi} = \widetilde{\Psi}^s$. Let $p \in [1,\infty)$. Then*

$$\left\|(\sigma,x) \mapsto \sigma^\alpha \Psi(\sigma D_a)f(x)\right\|_{T^{p,2}(\mathbb{R}^d)} \sim \left\|(\sigma,x) \mapsto \sigma^\alpha \widetilde{\Psi}(\sigma D_a)f(x)\right\|_{T^{p,2}(\mathbb{R}^d)},$$

for all f such that the above quantities are finite. Moreover, we have that

$$\left\|(\sigma,x) \mapsto \Psi(\sigma D_a)f(x)\right\|_{T^{p,2}(\mathbb{R}^d)} \sim \left\|(\sigma,x) \mapsto \sigma^2 L \exp(-\sigma^2 L)f(x)\right\|_{T^{p,2}(\mathbb{R}^d)}.$$

Proof. — Since

$$\begin{aligned}\left\|(\sigma,x) \mapsto \sigma^\alpha \Psi(\sigma D_a)f(x)\right\|_{T^{p,2}(\mathbb{R}^d)} \\ \sim \left\|(\sigma,x) \mapsto \int_0^\infty \sigma^\alpha \Psi(\sigma D_a)(\widetilde{\Psi})^2(\tau D_a)f(x)\frac{\mathrm{d}\tau}{\tau}\right\|_{T^{p,2}(\mathbb{R}^d)},\end{aligned}$$

by [20, Corollary 5.1], it suffices to show that, for all $\sigma, \tau > 0$,

$$\left(\frac{\sigma}{\tau}\right)^\alpha \Psi(\sigma D_a)\widetilde{\Psi}(\tau D_a) = \min\left(\frac{\sigma}{\tau}, \frac{\tau}{\sigma}\right)^N S_{\sigma,\tau}$$

for some $N > \frac{\mathrm{d}}{2}$ and a family of operators $S_{\sigma,\tau} \in B(L^2)$ such that for every $M \in \mathbb{N}$, there exists $C_M > 0$ such that

$$\left\|1_E S_{\sigma,\tau}(1_F f)\right\|_2 \leqslant C_M \left(1 + \frac{\mathrm{d}(E,F)}{\kappa \max(\sigma,\tau)}\right)^{-M} \|1_F f\|_2 \quad \forall\, f \in L^2(\mathbb{R}^d)$$

for all Borel sets $E, F \subset \mathbb{R}^d$ and all $\sigma > 0$. This follows from Lemma 5.2 using that, for all $\xi \in \mathbb{R}^d \backslash \{0\}$,

$$\left(\frac{\sigma}{\tau}\right)^\alpha \Psi(\sigma\xi)\widetilde{\Psi}(\tau\xi) = \left(\frac{\sigma}{\tau}\right)^{N'-\alpha} \overline{\Psi}(\sigma\xi)\underline{\widetilde{\Psi}}(\tau\xi) = \left(\frac{\tau}{\sigma}\right)^{N'+\alpha} \underline{\Psi}(\sigma\xi)\overline{\widetilde{\Psi}}(\tau\xi),$$

for $\overline{\Psi}\colon \xi \mapsto \frac{\Psi(\xi)}{\xi^\beta}$ and $\underline{\Psi}\colon \xi \mapsto \xi^\beta \Psi(\xi)$ with $\beta \in \mathbb{N}^d$, $|\beta|_1 = N'$, for $N' > |\alpha| + N$. For the second statement, we first show the comparison of $\Psi(\sigma D_a)$ with $(\sigma^2 L)^M \exp(-\sigma^2 L)$ for some $M \in \mathbb{N}$, $M > \frac{\mathrm{d}}{4}$ in the exact same way as above. For the comparison of $(\sigma^2 L)^M \exp(-\sigma^2 L)$ with $\sigma^2 L \exp(-\sigma^2 L)$, we use [14, Proposition 10.1] instead of [20, Corollary 5.1], together with the Gaussian estimates for $\exp(-tL)$ as stated in Theorem 5.3. □

Theorem 5.7. — *Let $s \in \mathbb{R}$, let $p \in (1,\infty)$. For all non-degenerate $\Psi \in C_c^\infty(\mathbb{R}^d)$ supported away from 0 such that $\Psi = \Psi^s$, and all $M \in \mathbb{N}$, we have that*

$$\begin{aligned}(5.1)\quad \left\|(\sigma,x) \mapsto 1_{[0,1)}(\sigma)\sigma^{-s}\Psi(\sigma D_a)f(x) + 1_{[1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)\right\|_{T^{p,2}(\mathbb{R}^d)} \\ \sim \left\|(I+\sqrt{L})^s f\right\|_p,\end{aligned}$$

for all $f \in D\big((I+\sqrt{L})^s\big)$. *Moreover, for* $s \in [0,2]$, *we have that*

$$\big\|(\sigma,x) \mapsto 1_{[0,1)}(\sigma)\sigma^{-s}\Psi(\sigma D_a)f(x) + 1_{[1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)} \sim \|f\|_{W^{s,p}} \tag{5.2}$$

for all $f \in W^{s,p}(\mathbb{R}^d)$.

Proof. — We use the Hardy space H^p_L associated with L, as defined in [12]. For all $f \in L^p \cap L^2$, we have, by Lemma 5.6,

$$\big\|(\sigma,x) \mapsto \Psi(\sigma D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)} \sim \|f\|_{H^p_L}.$$

It is a folklore fact that $H^p_L = L^p$ for $p \in (1,\infty)$, thanks to the heat kernel bounds of $(e^{tL})_{t\geqslant 0}$. This result appeared in draft form in an unpublished manuscript of Auscher, Duong, McIntosh, and inspired the proofs of many similar results. For our particular L, an appropriate version of the result does not seem to have appeared in the literature. It can however be proven as follows. By [7, Theorem 4.19], the operators $tL\exp(-tL)$ have standard kernels satisfying the assumptions of [15, Theorem 4.4]. Therefore, for all $f \in L^p \cap L^2$, $f \in H^p_L$ and

$$\|f\|_{H^p_L} \lesssim \|f\|_p.$$

The reverse inequality is proven in [12, Proposition 4.2] for $p \leqslant 2$. Given that the above reasoning also applies to L^*, we obtain the full result by duality. Combined with Lemma 5.6, this gives the result for $s = 0$. For $s \in \mathbb{N}$, using Lemma 5.6 with an appropriate $\widetilde{\Psi} \in C^\infty_c(\mathbb{R}^d)$, we then have that

$$\begin{aligned}
&\big\|(\sigma,x) \mapsto 1_{[0,1)}(\sigma)\sigma^{-s}\Psi(\sigma D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)} \\
&\qquad\lesssim \big\|(\sigma,x) \mapsto 1_{[0,1)}(\sigma)\widetilde{\Psi}(\sigma D_a)L^{\frac{s}{2}}f(x)\big\|_{T^{p,2}(\mathbb{R}^d)} \\
&\qquad\lesssim \|L^{\frac{s}{2}}f\|_p \\
&\qquad\lesssim \big\|(I+\sqrt{L})^s f\big\|_p.
\end{aligned}$$

We also have that

$$\big\|(\sigma,x) \mapsto 1_{[1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)} \lesssim \|f\|_p \lesssim \big\|(I+\sqrt{L})^s f\big\|_p.$$

For $-s \in \mathbb{N}$, we have that

$$\begin{aligned}
&\big\|(\sigma,x) \mapsto 1_{[0,1)}(\sigma)\sigma^{-s}\Psi(\sigma D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)} \\
&\quad\lesssim \sum_{k=0}^{|s|}\big\|(\sigma,x) \mapsto 1_{[0,1)}(\sigma)\sigma^{|s|}L^{\frac{k}{2}}\Psi(\sigma D_a)(I+\sqrt{L})^{-|s|}f(x)\big\|_{T^{p,2}(\mathbb{R}^d)}
\end{aligned}$$

$$\begin{aligned}&\lesssim \sum_{k=0}^{|s|}\big\|(\sigma,x)\mapsto 1_{[0,1)}(\sigma)\widetilde{\Psi}(\sigma D_a)(I+\sqrt{L})^{-|s|}f(x)\big\|_{T^{p,2}(\mathbb{R}^d)}\\&\lesssim \big\|(I+\sqrt{L})^s f\big\|_p,\end{aligned}$$

as well as

$$\begin{aligned}&\big\|(\sigma,x)\mapsto 1_{[1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)}\\&\qquad\lesssim \sum_{k=0}^{|s|}\big\|(\sigma,x)\mapsto 1_{[1,\infty)}(\sigma)\sigma^k L^{\frac{k}{2}}\Psi(\sigma D_a)(I+\sqrt{L})^{-|s|}f(x)\big\|_{T^{p,2}(\mathbb{R}^d)}\\&\qquad\lesssim \sum_{k=0}^{|s|}\big\|(\sigma,x)\mapsto 1_{[0,1)}(\sigma)\widetilde{\Psi}(\sigma D_a)(I+\sqrt{L})^{-|s|}f(x)\big\|_{T^{p,2}(\mathbb{R}^d)}\\&\qquad\lesssim \big\|(I+\sqrt{L})^s f\big\|_p.\end{aligned}$$

Reverse inequalities are proven similarly, using that, for all $s\in\mathbb{R}$,

$$\big\|(I+\sqrt{L})^s f\big\|_p \sim \big\|(\sigma,x)\mapsto (I+\sqrt{L})^s\Psi(\sigma D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)}.$$

This gives (5.1) for all $s\in\mathbb{Z}$, and the result for all $s\in\mathbb{R}$ then follows by complex interpolation of weighted tent spaces as in [1, Theorem 2.1].

To obtain (5.2) one first remarks that, for $s\in\{0,1,2\}$, the above reasoning also gives

$$\big\|(\sigma,x)\mapsto 1_{[0,1)}(\sigma)\sigma^{-s}\Psi(\sigma D_a)f(x)+1_{[1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)} \sim \sum_{m=0}^{s}\|D_a^m f\|_p,$$

for all $f\in\bigcap_{m=0}^{s}D\big(D_a^m\big)$. We then notice that, for all $j=1,\dots,d$, we have that $\|\partial_j f\|_p\sim\|\widetilde{a_j}\partial_j f\|_p\sim\|\widetilde{a_{j+d}}\partial_j f\|_p$, and thus $\|f\|_{W^{1,p}}\sim\|f\|_p+\|D_a f\|_p$, for all $f\in W^{1,p}$. Moreover,

$$\widetilde{a_{j+d}}\partial_j\widetilde{a_j}\partial_j f=\widetilde{a_j'}\widetilde{a_{j+d}}\partial_j f+\widetilde{a_j}\widetilde{a_{j+d}}\partial_j^2 f\quad \forall\, f\in W^{2,p},$$

and thus

$$\|f\|_{W^{2,p}}\sim\|f\|_p+\|D_a f\|_p+\|D_a^2 f\|_p\quad\forall\, f\in W^{2,p}.$$

□

Corollary 5.8. — *Let $\alpha\geqslant 0$, $p\in(1,\infty)$, and $q\in[p,\infty)$ be such that*

$$\alpha=\frac{\mathrm{d}}{2}\left(\frac{1}{p}-\frac{1}{q}\right).$$

Then there exists $C > 0$ such that, for all $f \in L^p(\mathbb{R}^d)$ with $L^\alpha f \in L^p(\mathbb{R}^d)$, we have that

$$\|f\|_{L^q(\mathbb{R}^d)} \leqslant C\|L^\alpha f\|_{L^p(\mathbb{R}^d)}.$$

Proof. — For $f \in L^p(\mathbb{R}^d)$ with $L^\alpha f \in L^p(\mathbb{R}^d)$, Theorem 5.7 gives that

$$\begin{aligned}\|f\|_{L^q(\mathbb{R}^d)} &\lesssim \left\|(\sigma, x) \mapsto L^{-\alpha}\Psi(\sigma D_a)L^\alpha f(x)\right\|_{T^{q,2}(\mathbb{R}^d)} \\ &\lesssim \left\|(\sigma, x) \mapsto \sigma^{2\alpha}\widetilde{\Psi}(\sigma D_a)L^\alpha f(x)\right\|_{T^{q,2}(\mathbb{R}^d)}\end{aligned}$$

for $\widetilde{\Psi}\colon \xi \mapsto |\xi|^{-\alpha}\Psi(\xi)$. Using the embedding properties of weighted tent spaces proven in [1, Theorem 2.19], we have that

$$\left\|(\sigma, x) \mapsto \sigma^{2\alpha}\widetilde{\Psi}(\sigma D_a)L^\alpha f\right\|_{T^{q,2}(\mathbb{R}^d)} \lesssim \left\|(\sigma, x) \mapsto \widetilde{\Psi}(\sigma D_a)L^\alpha f\right\|_{T^{p,2}(\mathbb{R}^d)},$$

and thus

$$\|f\|_{L^q(\mathbb{R}^d)} \lesssim \|L^\alpha f\|_{L^p(\mathbb{R}^d)},$$

by Theorem 5.7. □

Remark 5.9. — All results in this section, except (5.2), hold for a general Dirac operator D_a such that ie_jD_a and $ie_j\sqrt{D_a^2}$ generate bounded C_0 group on L^p for each j, the operators $D_1^2, \dots, D_d^2$ commute, $\big(\exp(it\xi.D_a)\big)_{t\in\mathbb{R}}$ has finite speed of propagation as in Remark 4.2, and $H^p_{D_a^2} = L^p$. Property (5.2) also holds as long as D_a has domain $W^{1,p}$ and D_a^2 has domain $W^{2,p}$ with equivalence of norms. All results in the next sections also hold for such Dirac operators.

6. Wave packet transform

We use a wave packet transform which is similar to the ones used in [16, 26], but symmetrised to ensure $\Psi_{\omega,\sigma} = \Psi^s_{\omega,\sigma}$.

Let $\Psi \in C_c^\infty(\mathbb{R}^d)$ be a non-negative radial function with $\Psi(\zeta) = 0$ for $|\zeta| \notin \left[\frac{1}{2}, 2\right]$, and

$$\int_0^\infty \Psi(\sigma\zeta)^2 \,\frac{\mathrm{d}\sigma}{\sigma} = 1 \tag{6.1}$$

for $\zeta \neq 0$. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be a radial, non-negative function with $\varphi(\zeta) = 1$ for $|\zeta| \leqslant \frac{1}{2}$ and $\varphi(\zeta) = 0$ for $|\zeta| > 1$. These functions Ψ, φ are now fixed for the remainder of the paper.

For $\omega \in S^{d-1}$, $\sigma > 0$ and $\zeta \in \mathbb{R}^d \setminus \{0\}$, we denote $\widehat{\zeta} := \zeta/|\zeta|$, and set $\underline{\varphi_{\omega,\sigma}}(\zeta) := c_\sigma\varphi\big(\frac{\hat{\zeta}-\omega}{\sqrt{\sigma}}\big)$, and $\varphi_{\omega,\sigma} = \underline{\varphi_{\omega,\sigma}}^s$, where

$$c_\sigma := \left(\int_{S^{d-1}} \varphi\left(\frac{e_1 - \nu}{\sqrt{\sigma}}\right)^2 \mathrm{d}\nu\right)^{-1/2}.$$

Set $\varphi_{\omega,\sigma}(0) := 0$. Set furthermore $\Psi_\sigma(\zeta) := \Psi(\sigma\zeta)$ and

$$\psi_{\omega,\sigma}(\zeta) := \Psi_\sigma(\zeta)\varphi_{\omega,\sigma}(\zeta)$$

for $\omega \in S^{d-1}$, $\sigma > 0$ and $\zeta \in \mathbb{R}^d$. By construction, we then have

$$\int_0^\infty \int_{S^{d-1}} \psi_{\omega,\sigma}(\zeta)^2 \, \mathrm{d}\omega \frac{\mathrm{d}\sigma}{\sigma} = 1 \tag{6.2}$$

for all $\zeta \in \mathbb{R}^d \setminus \{0\}$, see [16, Lemma 4.1]. For $\omega \in S^{d-1}$ and $\zeta \in \mathbb{R}^d$, we moreover set

$$\varphi_\omega(\zeta) := \int_0^4 \psi_{\omega,\tau}(\zeta) \, \frac{\mathrm{d}\tau}{\tau}.$$

For the convenience of the reader, we recall the following properties of $\psi_{\omega,\sigma}$ stated in [26, Lemma 3.2]. Note that the symmetrisation (using $\varphi_{\omega,\sigma}$ instead of $\underline{\varphi_{\omega,\sigma}}$) only affects (6.3). See also Remark 6.3 below.

Lemma 6.1. — *Let $\omega \in S^{d-1}$ and $\sigma \in (0,1)$. Each $\zeta \in \operatorname{supp}(\psi_{\omega,\sigma})$ satisfies*

$$\frac{1}{2\sigma} \leqslant |\zeta| \leqslant \frac{2}{\sigma}, \qquad \min_{(\varepsilon_j)_{j=1}^d \in \{-1,1\}^d} \left|(\varepsilon_1\widehat{\zeta}_1, \dots, \varepsilon_d\widehat{\zeta}_d) - \omega\right| \leqslant 2\sqrt{\sigma}. \tag{6.3}$$

For all $\alpha \in \mathbb{N}_0^d$ and $\beta \in \mathbb{N}_0$ there exists a constant $C = C(\alpha,\beta) > 0$ such that

$$\left|\langle \omega, \nabla_\zeta\rangle^\beta \partial_\zeta^\alpha \psi_{\omega,\sigma}(\zeta)\right| \leqslant C\sigma^{-\frac{\mathrm{d}-1}{4}+\frac{|\alpha|_1}{2}+\beta} \tag{6.4}$$

for all $(\zeta,\omega,\sigma) \in \mathbb{R}^d \times S^{d-1} \times (0,\infty)$. For every $N \geqslant 0$ there exists a constant $C_N > 0$ such that

$$\left|\mathcal{F}^{-1}(\psi_{\omega,\sigma})(x)\right| \leqslant C_N \sigma^{-\frac{3d+1}{4}} \left(1 + \sigma^{-1}|x|^2 + \sigma^{-2}\langle\omega, x\rangle^2\right)^{-N} \tag{6.5}$$

for all $(x,\omega,\sigma) \in \mathbb{R}^d \times S^{d-1} \times (0,\infty)$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

In particular, $\left\{\sigma^{\frac{\mathrm{d}-1}{4}}\mathcal{F}^{-1}(\psi_{\omega,\sigma}) \;\middle|\; \omega \in S^{d-1}, \sigma > 0\right\} \subseteq L^1(\mathbb{R}^d)$ is uniformly bounded.

We also recall important properties of the family $(\varphi_\omega)_{\omega \in S^{d-1}}$ from [26, Remark 3.3].

Lemma 6.2. — *Let $\omega \in S^{d-1}$. By construction, $\varphi_\omega \in C^\infty(\mathbb{R}^d)$, and for $\zeta \neq 0$, $\varphi_\omega(\zeta) = 0$ for $|\zeta| < \frac{1}{8}$ or $\min_{(\varepsilon_j)_{j=1}^d \in \{-1,1\}^d} \left|(\varepsilon_1\widehat{\zeta}_1, \dots, \varepsilon_d\widehat{\zeta}_d) - \omega\right| > 2|\zeta|^{-1/2}$. Moreover, for all $\alpha \in \mathbb{N}_0^d$ and $\beta \in \mathbb{N}_0$, there exists a constant $C = C(\alpha,\beta) > 0$ such that*

$$\left|\langle \omega, \nabla_\zeta\rangle^\beta \partial_\zeta^\alpha \varphi_\omega(\zeta)\right| \leqslant C|\zeta|^{\frac{\mathrm{d}-1}{4}-\frac{|\alpha|_1}{2}-\beta}$$

for all $\omega \in S^{d-1}$ *and* $\zeta \neq 0$, *and*

$$\left|\langle\widehat{\zeta}, \nabla_\zeta\rangle^\beta \partial_\zeta^\alpha \left(\int_{S^{d-1}} \varphi_\nu(\zeta)^2 \,\mathrm{d}\nu\right)\right| \leqslant C|\zeta|^{-\frac{|\alpha|_1}{2}-\beta} \tag{6.6}$$

for all $\zeta \in \mathbb{R}^d \setminus \{0\}$.

Remark 6.3. — For $\omega = e_1$ and ζ, σ chosen as in (6.3) with $\sigma \in (0, 2^{-8})$, we have

$$\frac{1}{4\sigma} < |\zeta_1| \leqslant \frac{2}{\sigma}, \qquad |\zeta_j| \leqslant \frac{4}{\sqrt{\sigma}}, \qquad j \in \{2, \dots, d\}. \tag{6.7}$$

This follows from

$$\begin{aligned}\left|(\varepsilon_1\widehat{\zeta}_1, \dots, \varepsilon_d\widehat{\zeta}_d) - e_1\right|^2 &= \left|e_1.\big((\varepsilon_1\widehat{\zeta}_1, \dots, \varepsilon_d\widehat{\zeta}_d) - e_1\big)\right|^2 \\ &\qquad + \sum_{j=2}^d \left|e_j.\big((\varepsilon_1\widehat{\zeta}_1, \dots, \varepsilon_d\widehat{\zeta}_d) - e_1\big)\right|^2 \\ &= \left|\frac{\varepsilon_1\zeta_1}{|\zeta|} - 1\right|^2 + \sum_{j=2}^d \left|\frac{\zeta_j}{|\zeta|}\right|^2,\end{aligned}$$

for all $(\varepsilon_j)_{j=1}^d \in \{-1, 1\}^d$. Therefore we have that, for some $\varepsilon_1 \in \{-1, 1\}$,

$$\big|\varepsilon_1\zeta_1 - |\zeta|\big|^2 + \sum_{j=2}^d |\zeta_j|^2 \leqslant 4\sigma|\zeta|^2 \leqslant \frac{16}{\sigma},$$

which directly yields (6.7) for $j \geqslant 2$. The case $j = 1$ then follows from

$$|\zeta_1| = |\varepsilon_1\zeta_1| > |\zeta| - \frac{4}{\sqrt{\sigma}} \geqslant \frac{1}{2\sigma} - \frac{4}{\sqrt{\sigma}}.$$

LEMMA 6.4. — *For all* $\sigma \in (0, 1)$, *and all* $f \in L^2(\mathbb{R}^d)$, *we have that*

$$\begin{aligned}|S^{d-1}|^{-1} \int_{S^{d-1}} \int_1^\infty \Psi(\sigma D_a)^2 f \,\frac{\mathrm{d}\sigma}{\sigma}\,\mathrm{d}\omega \\ + \int_{S^{d-1}} \int_0^1 \varphi_\omega(D_a)^2 \Psi(\sigma D_a)^2 f \,\frac{\mathrm{d}\sigma}{\sigma}\,\mathrm{d}\omega = f,\end{aligned} \tag{6.8}$$

$$\int_{S^{d-1}} \varphi_{\omega,\sigma}(D_a)^2 f \,\mathrm{d}\omega = f, \tag{6.9}$$

$$\sigma^{-\frac{\mathrm{d}-1}{4}} \int_{S^{d-1}} \varphi_{\omega,\sigma}(D_a) f \,\mathrm{d}\omega = C_\sigma f, \tag{6.10}$$

with constant C_σ *such that* $\sigma \mapsto C_\sigma$ *is bounded above and below.*

Proof. — These identities follow (respectively) from (6.2), the fact that $\int_{S^{d-1}} \varphi_{\omega,\sigma}(\xi)^2 \,\mathrm{d}\omega = 1$ for all $\xi \neq 0$, and [16, formula (7.4)], using the Philipps functional calculus of $\sqrt{D_a^2}$. □

Lemma 6.5. — *For all $\sigma \in (0,1)$, we have that*

$$\int_{S^{d-1}} \big\|\varphi_{\omega,\sigma}(D_a)f\big\|_2^2 \,\mathrm{d}\omega \lesssim \|f\|_2^2 \quad \forall\, f \in L^2(\mathbb{R}^d).$$

Moreover,

$$\int_{S^{d-1}} \int_0^\infty \big\|\psi_{\omega,\sigma}(D_a)f\big\|_2^2 \,\frac{\mathrm{d}\sigma}{\sigma}\,\mathrm{d}\omega \lesssim \|f\|_2^2 \quad \forall\, f \in L^2(\mathbb{R}^d).$$

Proof. — Let $f \in L^2(\mathbb{R}^d)$ and $\sigma \in (0,1)$. Using (6.9), and the fact that $\sqrt{D_a^2}$ is self-adjoint with respect to an equivalent inner product (see Definition 4.1), we have that

$$\int_{S^{d-1}} \big\|\varphi_{\omega,\sigma}(D_a)f\big\|_2^2 \,\mathrm{d}\omega \sim \int_{S^{d-1}} \big\langle \varphi_{\omega,\sigma}(D_a)^2 f, f\big\rangle \,\mathrm{d}\omega \lesssim \|f\|_2^2.$$

Similarly, using (6.8), we have that

$$\begin{aligned}\int_{S^{d-1}} \int_0^\infty \big\|\psi_{\omega,\sigma}(D_a)f\big\|_2^2 \,\frac{\mathrm{d}\sigma}{\sigma}\,\mathrm{d}\omega &\sim \int_{S^{d-1}} \int_0^\infty \big\langle \psi_{\omega,\sigma}(D_a)^2 f, f\big\rangle \frac{\mathrm{d}\sigma}{\sigma}\,\mathrm{d}\omega \\ &\lesssim \|f\|_2^2.\end{aligned}$$

□

Definition 6.6. — *We define a wave packet transform adapted to D_a, $W_a \in B\big(L^2(\mathbb{R}^d), L^2\big(\mathbb{R}^d \times S^{d-1} \times (0,\infty); \mathrm{d}x\,\mathrm{d}\omega\frac{\mathrm{d}\sigma}{\sigma}\big)\big)$ by*

$$\begin{aligned}W_a f(\omega,\sigma,x) := 1_{(1,\infty)}(\sigma)|S^{d-1}|^{-1/2}\Psi(\sigma D_a)f(x) \\ + 1_{[0,1]}(\sigma)\varphi_\omega(D_a)\Psi(\sigma D_a)f(x) \quad \forall\, f \in L^2(\mathbb{R}^d).\end{aligned}$$

We define $\pi_a \in B\big(L^2\big(\mathbb{R}^d \times S^{d-1} \times (0,\infty); \mathrm{d}x\,\mathrm{d}\omega\frac{\mathrm{d}\sigma}{\sigma}\big), L^2(\mathbb{R}^d)\big)$ by

$$\begin{aligned}\pi_a F(x) := |S^{d-1}|^{-1/2} \int_{S^{d-1}} \int_1^\infty \Psi(\sigma D_a)F(\omega,\sigma,\cdot\,)(x)\,\frac{\mathrm{d}\sigma}{\sigma}\,\mathrm{d}\omega \\ + \int_{S^{d-1}} \int_0^1 \varphi_\omega(D_a)\Psi(\sigma D_a)F(\omega,\sigma,\cdot\,)(x)\,\frac{\mathrm{d}\sigma}{\sigma}\,\mathrm{d}\omega\end{aligned}$$

for all $F \in L^2\big(\mathbb{R}^d \times S^{d-1} \times (0,\infty); \mathrm{d}x\,\mathrm{d}\omega\frac{\mathrm{d}\sigma}{\sigma}\big)$.

Note that W_a is well defined thanks to Lemma 6.5, and that π_a is the adjoint of the operator $\overline{W}_a$, where $\overline{W}_a$ is defined as W_a with D_a replaced by D_a^*.

Definition 6.7. — *Given $\omega \in S^{d-1}$, we fix vectors $\omega_1, \dots, \omega_{d-1}$ such that $\{\omega, \omega_1, \dots, \omega_{d-1}\}$ is an orthonormal basis of $\mathbb{R}^d$. We then define the parabolic (quasi) distance in the direction of ω by*

$$d_\omega(x,y) := \big|\langle \omega, x-y\rangle\big| + \sum_{j=1}^{d-1} \langle \omega_j, x-y\rangle^2 \quad \forall\, x,y \in \mathbb{R}^d.$$

We also define (anistropic) operators associated with this parabolic distance by

$$\Delta_{\omega^\perp} := \sum_{j=1}^{d-1} \langle \omega_j, \nabla \rangle^2, \quad L_{\omega^\perp} := -\sum_{j=1}^{d-1} \langle \omega_j, D_a \rangle^2.$$

Lemma 6.8.

(i) *Let $N \in \mathbb{N}$, $N > \frac{\mathrm{d}+1}{2}$. There exists $C > 0$ such that for all $\sigma \in (0,1)$ and $\omega \in S^{d-1}$, we have*

$$\big\| \big(1 + \sigma L_{\omega^\perp} + \sigma^2 \langle \omega, D_a \rangle^2\big)^{-N} f \big\|_{L^2(\mathbb{R}^d)} \leqslant C \sigma^{-\frac{\mathrm{d}+1}{4}} \|f\|_{L^1(\mathbb{R}^d)}$$

for all $f \in L^1(\mathbb{R}^d)$.

(ii) *For every $M \in \mathbb{N}$, there exists $C_M > 0$ such that for all $E, F \subset \mathbb{R}^d$ Borel sets, $\sigma \in (0,1)$ and $\omega \in S^{d-1}$, we have*

$$\big\| 1_E \psi_{\omega,\sigma}(D_a)(1_F f) \big\|_{L^2(\mathbb{R}^d)} \leqslant C_M \sigma^{-\frac{\mathrm{d}}{2}} \left(1 + \frac{\mathrm{d}_\omega(E,F)}{\sigma}\right)^{-M} \|1_F f\|_{L^1(\mathbb{R}^d)}$$

for all $f \in L^1(\mathbb{R}^d)$.

(iii) *Let $1 \leqslant p \leqslant r < \infty$. For every $M \in \mathbb{N}$, there exists $C_M > 0$ such that for all $E, F \subset \mathbb{R}^d$ Borel sets, $\sigma \in (0,1)$ and $\omega \in S^{d-1}$, we have*

$$\big\| 1_E \psi_{\omega,\sigma}(D_a)(1_F f) \big\|_{L^r(\mathbb{R}^d)} \leqslant C_M \sigma^{-d\left(\frac{1}{p} - \frac{1}{r}\right)} \sigma^{-\frac{\mathrm{d}-1}{4}} \left(1 + \frac{\mathrm{d}(E,F)}{\sigma}\right)^{-M} \|1_F f\|_{L^p(\mathbb{R}^d)}$$

for all $f \in L^p(\mathbb{R}^d)$.

Proof.

(i). — Part (i) follows from [7, Proposition 4.3], tracking the scaling factor σ in its proof.

(ii). — Let $\omega \in S^{d-1}$. For given Borel sets $E, F \subseteq \mathbb{R}^d$ with $d(E,F) > 0$, let $\chi_\omega \in C^\infty(\mathbb{R}^d)$ be a function with values in $[0,1]$ such that $\chi_\omega = \chi_\omega^s$, $\chi_\omega(\zeta) = 0$ for $|\zeta| \leqslant \frac{1}{2}\kappa^{-1} d_\omega(E,F)$ and $\chi_\omega(\zeta) = 1$ for $|\zeta| \geqslant \kappa^{-1} d_\omega(E,F)$, and $\big\|\langle \omega, \nabla \rangle \chi_\omega\big\|_\infty + \|\Delta_{\omega^\perp} \chi_\omega\|_\infty \lesssim \frac{1}{d_\omega(E,F)}$. Lemma 5.2 implies

$$c_d 1_E \psi_{\omega,\sigma}(D_a) 1_F f = 1_E \int_{\mathbb{R}^d} \chi_\omega(\zeta) \mathcal{F}^{-1}(\psi_{\omega,\sigma})(\zeta) e^{i\zeta D_a} 1_F f \,\mathrm{d}\zeta.$$

Now note that

$$\big(1 - \sigma \Delta_{\omega^\perp} - \sigma^2 \langle \omega, \nabla_\zeta \rangle^2\big) e^{i\zeta D_a} = \big(1 + \sigma L_{\omega^\perp} + \sigma^2 \langle \omega, D_a \rangle^2\big) e^{i\zeta D_a},$$

thus for $N \in \mathbb{N}$,

$$e^{i\zeta D_a} = \big(1 + \sigma L_{\omega^\perp} + \sigma^2 \langle \omega, D_a \rangle^2\big)^{-N} \big(1 - \sigma \Delta_{\omega^\perp} - \sigma^2 \langle \omega, \nabla_\zeta \rangle^2\big)^N e^{i\zeta D_a}.$$

From integration by parts we then get for $j \in \{0,1\}$

$$c_d 1_E \psi_{\omega,\sigma}(D_a) 1_F f = \left(1 + \sigma L_{\omega^\perp} + \sigma^2 \langle \omega, D_a\rangle^2\right)^{-N} \circ \int_{\mathbb{R}^d} \left(\left(1 - \sigma\Delta_{\omega^\perp} - \sigma^2\langle \omega, \nabla_\zeta\rangle^2\right)^N\right)^* \left(\chi_\omega^j \cdot \mathcal{F}^{-1}(\psi_{\omega,\sigma})\right)(\zeta) e^{i\zeta D_a}(1_F f)\, \mathrm{d}\zeta. \tag{6.11}$$

Consider first the case $d_\omega(E,F) \leqslant \sigma$, for which we take $j=0$. According to Lemma 6.1, we have $\left\|\mathcal{F}^{-1}(\psi_{\omega,\sigma})\right\|_{L^1(\mathbb{R}^d)} \lesssim \sigma^{-\frac{\mathrm{d}-1}{4}}$. Similarly, one can check that

$$\left\|\zeta \mapsto \left(\sigma\langle \omega, \nabla_\zeta\rangle\right)^\beta (\sigma\Delta_{\omega^\perp})^\alpha \mathcal{F}^{-1}(\psi_{\omega,\sigma})(\zeta)\right\|_{L^1(\mathbb{R}^d)} \lesssim \sigma^{-\frac{\mathrm{d}-1}{4}}$$

for all $\alpha \in \mathbb{N}_0^d$ and $\beta \in \mathbb{N}_0$. We use this estimate together with Proposition 4.3 and part (i) to obtain for $N > \frac{\mathrm{d}+1}{2}$

$$\begin{aligned}\left\|\psi_{\omega,\sigma}(D_a) f\right\|_{L^2(\mathbb{R}^d)} &\lesssim \sigma^{-\frac{\mathrm{d}-1}{4}} \left\|\left(1 + \sigma L_{\omega^\perp} + \sigma^2\langle \omega, D_a\rangle^2\right)^{-N}\right\|_{1\to 2} \|f\|_{L^1(\mathbb{R}^d)} \\ &\lesssim \sigma^{-\frac{\mathrm{d}}{2}} \|f\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

In the case $d_\omega(E,F) > \sigma$, we choose $j=1$ in (6.11). Then note that according to the choice of χ_ω, we have for $\sigma \in (0,1)$ that

$$\left\|\zeta \mapsto \left(\sigma\langle \omega, \nabla_\zeta\rangle\right)^\beta (\sigma\Delta_{\omega^\perp})^\alpha \chi_\omega(\zeta)\right\|_\infty \lesssim \left(\frac{\sigma}{d_\omega(E,F)}\right)^{|\alpha|+\beta} \lesssim 1,$$

for all $\alpha \in \mathbb{N}_0^d$, $\beta \in \mathbb{N}_0$. Using the product rule, a version of (6.5) for derivatives of $\mathcal{F}^{-1}(\psi_{\omega,\sigma})$, part (i), and an anisotropic change of variable, we obtain

$$\begin{aligned}&\left\|1_E \psi_{\omega,\sigma}(D_a)(1_F f)\right\|_2 \\ &\quad\lesssim \sigma^{-\frac{\mathrm{d}+1}{4}} \|1_F f\|_1 \sup_{\substack{\alpha\in\mathbb{N}_0^d,\, \beta\in\mathbb{N}_0 \\ |\alpha|+2\beta\leqslant N}} \int_{\{|\zeta|\geqslant \frac{\mathrm{d}(E,F)}{\kappa}\}\cap\{|\langle\omega,\zeta\rangle|\geqslant \frac{\omega.d(E,F)}{\kappa}\}} \left|\left(\sigma\langle \omega, \nabla_\zeta\rangle\right)^\beta \right. \\ &\qquad\qquad \left. (\sqrt{\sigma}\partial_\zeta)^\alpha \mathcal{F}^{-1}(\psi_{\omega,\sigma})(\zeta)\right| \mathrm{d}\zeta \\ &\quad\lesssim \sigma^{-\frac{\mathrm{d}+1}{4}} \sigma^{-\frac{3d+1}{4}} \|1_F f\|_1 \\ &\qquad\qquad \int_{\{|\zeta|\geqslant \frac{\mathrm{d}(E,F)}{\kappa}\}\cap\{|\langle\omega,\zeta\rangle|\geqslant \frac{\omega.d(E,F)}{\kappa}\}} \left(1 + \sigma^{-1}|\zeta|^2 + \sigma^{-2}\langle\omega,\zeta\rangle^2\right)^{-\tilde{N}} \mathrm{d}\zeta \\ &\quad\lesssim \sigma^{-\frac{\mathrm{d}}{2}} \left(1 + \frac{\mathrm{d}_\omega(E,F)}{\sigma}\right)^{-(2\tilde{N}-d)} \|1_F f\|_1.\end{aligned}$$

Choosing $\widetilde{N}$ large enough in (6.5) yields the result.

(iii). — This can be proven as in [6, Lemma 3.6] using that $\psi_{\omega,\sigma}(D_a) = \Psi_\sigma(D_a)\varphi_{\omega,\sigma}(D_a)$. Here we argue as in (i) and (ii), noting that the argument becomes simpler, since we can rely on the Gaussian estimates of the resolvent instead of the anisotropic bounds of the wave packets. By Theorem 5.3, we have that

$$\left\|(1+\sigma^2 L)^{-N} f\right\|_{L^r(\mathbb{R}^d)} \leqslant C\sigma^{-d\left(\frac{1}{p}-\frac{1}{r}\right)} \|f\|_{L^p(\mathbb{R}^d)},$$

for $N > \frac{\mathrm{d}+1}{2}$, and corresponding isotropic L^p-L^r off-diagonal estimates. Integrating by parts, and using Lemma 5.2 together with Proposition 4.3, we obtain that

$$\begin{aligned}
&\left\|1_E \psi_{\omega,\sigma}(D_a)(1_F f)\right\|_{L^r(\mathbb{R}^d)} \\
&\quad\lesssim \sigma^{-d\left(\frac{1}{p}-\frac{1}{r}\right)} \left(1+\frac{\mathrm{d}(E,F)}{\sigma}\right)^{-M} \int_{\mathbb{R}^d} \left|(\sigma^2\Delta)^\alpha \mathcal{F}^{-1}(\psi_{\omega,\sigma})\right| \mathrm{d}\xi \cdot \|1_F f\|_{L^p(\mathbb{R}^d)} \\
&\quad\lesssim \sigma^{-d\left(\frac{1}{p}-\frac{1}{r}\right)} \sigma^{-\frac{\mathrm{d}-1}{4}} \left(1+\frac{\mathrm{d}(E,F)}{\sigma}\right)^{-M} \|1_F f\|_{L^p(\mathbb{R}^d)},
\end{aligned}$$

using that, for all $\alpha \in \mathbb{N}$, $\left\|\zeta \mapsto (\sigma^2\Delta)^\alpha \mathcal{F}^{-1}(\psi_{\omega,\sigma})(\zeta)\right\|_{L^1(\mathbb{R}^d)} \lesssim \sigma^{-\frac{\mathrm{d}-1}{4}}$, by Lemma 6.1. □

7. The Hardy–Sobolev spaces $H^{p,s}_{\mathrm{FIO},a}(\mathbb{R}^d)$

In the following, we denote by $\Psi \in C_c^\infty(\mathbb{R}^d)$ the function defining the wave packet transforms from Section 6. We denote by $H^1_L(\mathbb{R}^d)$ the Hardy space associated with L as defined in [12]. Recall that for all $f \in H^1_L(\mathbb{R}^d)$, we have by Lemma 5.6,

$$\|f\|_{H^1_L(\mathbb{R}^d)} \sim \left\|(\sigma,x) \mapsto \Psi(\sigma D_a) f(x)\right\|_{T^{1,2}(\mathbb{R}^d)}.$$

Definition 7.1. — *Define*

$$\mathcal{S}_1 = \left\{f \in H^1_L(\mathbb{R}^d) : \exists\, g \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d),\ \exists\, \tau>0,\ f = \Psi(\tau D_a) g\right\},$$

and for $p \in (1,\infty)$

$$\mathcal{S}_p = \left\{f \in L^p(\mathbb{R}^d) : \exists\, g \in L^p(\mathbb{R}^d)\cap L^2(\mathbb{R}^d),\ \exists\, \tau>0,\ f = \Psi(\tau D_a) g\right\}.$$

Lemma 7.2. — *Let $p \in [1,\infty)$ and $f \in \mathcal{S}_p$. Then, for all $\omega \in S^{d-1}$, $\varphi_\omega(D_a) f \in L^p(\mathbb{R}^d)$, and, in the case $p=1$, $\varphi_\omega(D_a) f \in H^1_L(\mathbb{R}^d)$, each with norm independent of ω.*

Proof. — We have that $\varphi_\omega(D_a) f = \psi_{\omega,\tau}(D_a) g$ for some $g \in L^p(\mathbb{R}^d)$, up to a change of constants in the support conditions of $\psi_{\omega,\tau}$. By Lemma 6.8, we have $\psi_{\omega,\tau}(D_a) \in B\big(L^p(\mathbb{R}^d)\big)$, and thus $\left\|\varphi_\omega(D_a) f\right\|_p \lesssim_\tau \|g\|_p$. In the case $p=1$, we obtain that $\left\|\psi_{\omega,\tau}(D_a) g\right\|_{L^1} \lesssim \|g\|_{H^1_L}$ by reasoning as in the proof of 6.8(iii), using the boundedness of Riesz transforms associated with L from H^1_L to L^1 to deduce the H^1_L to L^1 uniform boundedness of the transport group $\big(\exp(i\xi D_a)\big)_{\xi\in\mathbb{R}^d}$. We moreover have that $\psi_{\omega,\tau}(D_a)g \in R(L)$, since Ψ is supported away from 0, hence $\psi_{\omega,\tau}(D_a)g \in H^1_L(\mathbb{R}^d)$. □

Corollary 7.3. — *Let $p \in [1,\infty)$, $s \in \mathbb{R}$, and $f \in \mathcal{S}_p$. Then*

$$\omega \longmapsto \big[(\sigma,x)\mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)+1_{[0,1]}(\sigma)\sigma^{-s}\varphi_\omega(D_a)\Psi(\sigma D_a)f(x)\big] \in L^p\big(S^{d-1};T^{p,2}(\mathbb{R}^d)\big).$$

Proof. — This follows from Lemma 7.2 and Theorem 5.7. □

Lemma 7.4. — *Let $\widetilde{\Psi} \in C_c^\infty(\mathbb{R}^d)$ be non-degenerate, supported away from 0 and such that $\widetilde{\Psi} = \widetilde{\Psi}^s$. Let $p \in (1,\infty)$, $s \in \mathbb{R}$, and $f \in \mathcal{S}_p$. Then, we have that*

$$\omega \longmapsto \big[(\sigma,x)\mapsto 1_{(1,\infty)}(\sigma)\widetilde{\Psi}(\sigma D_a)f(x)+1_{[0,1]}(\sigma)\sigma^{-s}\varphi_\omega(D_a)\widetilde{\Psi}(\sigma D_a)f(x)\big] \in L^p\big(S^{d-1};T^{p,2}(\mathbb{R}^d)\big),$$

with an equivalent norm to the corresponding map in Corollary 7.3, and

$$\big\|(I+\sqrt{L})^{-M}f\big\|_{L^p} \lesssim \big\|\omega\mapsto \big[(\sigma,x)\mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)f(x) + 1_{[0,1]}(\sigma)\sigma^{-s}\varphi_\omega(D_a)\Psi(\sigma D_a)f(x)\big]\big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))},$$

for all $M \in \mathbb{N}$ such that $M \geqslant \frac{d-1}{4} - s$.

Proof. — Let $M \in \mathbb{N}$ be such that $M \geqslant \frac{d-1}{4} - s$. Lemma 5.6 and Corollary 7.3 give the first part, and Corollary 5.5, Lemma 5.6 together with Theorem 5.7 give

$$\begin{aligned}\big\|(I+\sqrt{L})^{-M}f\big\|_{L^p} &\lesssim \big\|(\sigma,x)\mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)(I+\sqrt{L})^{-M}f(x)\big\|_{T^{p,2}(\mathbb{R}^d)} \\ &\quad + \big\|(\sigma,x)\mapsto 1_{[0,1]}(\sigma)(\sigma\sqrt{L})^M(I+\sqrt{L})^{-M}\Psi^2(\sigma D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)}.\end{aligned}$$

Using Corollary 5.5 again, we then have that

$$\big\|(I+\sqrt{L})^{-M}f\big\|_{L^p} \lesssim \big\|(\sigma,x)\mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)} + \big\|(\sigma,x)\mapsto 1_{[0,1]}(\sigma)\sigma^M\Psi^2(\sigma D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)}.$$

We then use the reproducing formula (6.10) to obtain that

$$\begin{aligned}
&\big\|(I+\sqrt{L})^{-M}f\big\|_{L^p}\\
&\lesssim \Big\|(\sigma,x)\mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)+1_{[0,1]}(\sigma)\\
&\qquad\qquad \int_{S^{d-1}}\sigma^{M-\frac{d-1}{4}}\varphi_{\omega,\sigma}(D_a)\Psi^2(\sigma D_a)f(x)\,\mathrm{d}\omega\Big\|_{T^{p,2}(\mathbb{R}^d)}\\
&\lesssim \big\|\omega\mapsto \big[(\sigma,x)\mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)\\
&\qquad\qquad +1_{[0,1]}(\sigma)\sigma^{-s}\varphi_\omega(D_a)\Psi(\sigma D_a)f(x)\big]\big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d)},
\end{aligned}$$

since $M\geqslant \frac{d-1}{4}-s$. □

Definition 7.5. — *Let $p\in[1,\infty)$, and $s\in\mathbb{R}$. We define the space $H^{p,s}_{\mathrm{FIO},a}(\mathbb{R}^d)$ as the completion of $\mathcal{S}_p$ for the norm defined by*

$$\begin{aligned}
\|f\|_{H^{p,s}_{\mathrm{FIO},a}(\mathbb{R}^d)} := \big\|\omega\mapsto \big[&(\sigma,x)\mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)\\
&+1_{[0,1]}(\sigma)\sigma^{-s}\varphi_\omega(D_a)\Psi(\sigma D_a)f(x)\big]\big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}.
\end{aligned}$$

We write $H^{p}_{\mathrm{FIO},a}(\mathbb{R}^d):=H^{p,0}_{\mathrm{FIO},a}(\mathbb{R}^d)$.

Remark 7.6. — By Lemma 7.4, we have that $H^{p}_{\mathrm{FIO},a}(\mathbb{R}^d)$ is a subspace of the M-th extrapolation space associated with L (i.e. the completion of the range of $(I+\sqrt{L})^M$ with respect to the norm defined by $\big\|(I+\sqrt{L})^{-M}f\big\|_{L^p}$, seen as a Sobolev space of negative order adapted to the operator L), and is independent of the choice of $\Psi\in C_c^\infty(\mathbb{R}^d)\backslash\{0\}$, supported away from 0, and such that $\widetilde{\Psi}=\widetilde{\Psi}^s$.

Remark 7.7. — By Lemma 6.4, interpolation properties of $H^{p,s}_{\mathrm{FIO},a}(\mathbb{R}^d)$ follow from the interpolation properties of weighted tent spaces (see [1]) with the same proof as in [16, Proposition 6.7].

We also have the following versions of [26, Theorem 4.1] and [26, Corollary 4.4], respectively.

Proposition 7.8. — *Let $p\in(1,\infty)$, and $s\in\mathbb{R}$. Let $q\in C_c^\infty(\mathbb{R}^d)$ radial with $q(\zeta)\equiv 1$ for $|\zeta|\leqslant\frac{1}{8}$. Then*

$$\begin{aligned}
&\|f\|_{H^{p,s}_{\mathrm{FIO},a}(\mathbb{R}^d)}\\
&\quad\simeq \big\|q(D_a)f\big\|_{L^p(\mathbb{R}^d)}+\Big(\int_{S^{d-1}}\big\|\varphi_\omega(D_a)(I+\sqrt{L})^s f\big\|^p_{L^p(\mathbb{R}^d)}\,\mathrm{d}\omega\Big)^{1/p}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, f\in\mathcal{S}_p.
\end{aligned}$$

Proof. — Let $f \in \mathcal{S}_p$. By Lemma 5.6, we can choose Ψ with an appropriate support, such that $\Psi(\sigma D_a)f = \Psi(\sigma D_a)q(D_a)f$ for all $\sigma \geqslant 1$, $\Psi(\sigma D_a)q(D_a) = 0$ for all $\sigma \leqslant \frac{1}{8}$, and $\varphi_\omega(D_a)\Psi(\sigma D_a) = 0$ for all $\sigma \geqslant 1$ and $\omega \in S^{d-1}$.

Then, by Theorem 5.7, we have that

$$\begin{aligned}
&\|f\|_{H^{p,s}_{\mathrm{FIO},a}(\mathbb{R}^d)} \\
&\lesssim \big\|(\sigma,x) \mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)q(D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)} \\
&\quad + \big\|\omega \mapsto \big[(\sigma,x) \mapsto 1_{[0,1]}(\sigma)\sigma^{-s}\varphi_\omega(D_a)\Psi(\sigma D_a)f(x)\big]\big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))} \\
&\lesssim \big\|q(D_a)f\big\|_{L^p(\mathbb{R}^d)} + \left(\int_{S^{d-1}} \big\|(I+\sqrt{L})^s\varphi_\omega(D_a)f\big\|^p_{L^p(\mathbb{R}^d)}\,\mathrm{d}\omega\right)^{1/p}.
\end{aligned}$$

In the other direction, Theorem 5.7 and the support properties of q and Ψ give us that

$$\begin{aligned}
&\big\|q(D_a)f\big\|_{L^p(\mathbb{R}^d)} \\
&\qquad \lesssim \|f\|_{H^{p,s}_{\mathrm{FIO},a}(\mathbb{R}^d)} + \big\|(\sigma,x) \mapsto 1_{[\frac{1}{8},1]}(\sigma)\Psi(\sigma D_a)q(D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)}.
\end{aligned}$$

With the same proof as in Lemma 5.6, we then have that, for all $M \geqslant \frac{\mathrm{d}-1}{4} - s$,

$$\begin{aligned}
&\big\|(\sigma,x) \mapsto 1_{[\frac{1}{8},1]}(\sigma)\Psi(\sigma D_a)q(D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)} \\
&\quad \lesssim \Big\|(\sigma,x) \mapsto 1_{[\frac{1}{8},1]}(\sigma)\int_0^\infty \Psi(\sigma D_a)q(D_a)\Psi(\tau D_a) \\
&\qquad\qquad\qquad (I+\sqrt{L})^M(I+\sqrt{L})^{-M}f(x)\frac{\mathrm{d}\tau}{\tau}\Big\|_{T^{p,2}(\mathbb{R}^d)} \\
&\quad \lesssim \big\|(I+\sqrt{L})^{-M}f\big\|_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

Therefore, using Lemma 7.4, we have that $\big\|q(D_a)f\big\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{H^{p,s}_{\mathrm{FIO},a}(\mathbb{R}^d)}$. For the second term, we use Theorem 5.7 and the support properties of Ψ again to get that

$$\begin{aligned}
&\left(\int_{S^{d-1}} \big\|\varphi_\omega(D_a)(I+\sqrt{L})^s f\big\|^p_{L^p(\mathbb{R}^d)}\,\mathrm{d}\omega\right)^{1/p} \\
&\lesssim \big\|\omega \mapsto \big[(\sigma,x) \mapsto 1_{[0,1)}(\sigma)\sigma^{-s}\varphi_\omega(D_a)\Psi(\sigma D_a)f(x)\big]\big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))} \\
&\lesssim \|f\|_{H^{p,s}_{\mathrm{FIO},a}(\mathbb{R}^d)}. \qquad \square
\end{aligned}$$

Proposition 7.9. — *Let $p \in (1,\infty)$. Let $q \in C^\infty_c(\mathbb{R}^d)$ radial with $q(\zeta) \equiv 1$ for $|\zeta| \leqslant \frac{1}{8}$, and $\Phi \in \mathcal{S}(\mathbb{R}^d)$ with $\Phi(0) = 1$ and $\Phi_\sigma(\zeta) = \Phi(\sigma\zeta)$ for*

$\sigma > 0$, $\zeta \in \mathbb{R}^d$. *Then*

$$\big\|q(D_a)f\big\|_{L^p(\mathbb{R}^d)} + \Big(\int_{S^{d-1}} \big\|(\sigma,\cdot) \mapsto \Phi_\sigma(D_a)\varphi_\omega(D_a)f\big\|^p_{T^{p,\infty}(\mathbb{R}^d)}\,\mathrm{d}\omega\Big)^{1/p} \lesssim \|f\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)} \quad \forall\, f \in \mathcal{S}_p,$$

and

$$\Big(\int_{S^{d-1}} \big\|(\sigma,\cdot) \mapsto \sigma^{\frac{d-1}{4}}\Phi_\sigma(D_a)\varphi_\omega(D_a)^2 f\big\|^p_{T^{p,\infty}(\mathbb{R}^d)}\,\mathrm{d}\omega\Big)^{1/p} \lesssim \|f\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)} \quad \forall\, f \in \mathcal{S}_p.$$

Proof. — Let $r \in [1,p)$. For the first assertion, note that Theorem 5.3 implies L^r-L^∞ off-diagonal estimates for $\Phi_\sigma(D_a)$ of the following form: for every $M \in \mathbb{N}$, there exists $C_M > 0$ such that for all $E, F \subset \mathbb{R}^d$ Borel sets, $\sigma \in (0,1)$, we have

$$\big\|1_E\Phi_\sigma(D_a)(1_F g)\big\|_{L^\infty(\mathbb{R}^d)} \leqslant C_M \sigma^{-\frac{d}{r}}\Big(1 + \frac{\mathrm{d}(E,F)}{\sigma}\Big)^{-M}\|1_F g\|_{L^r(\mathbb{R}^d)}$$

for all $g \in L^r(\mathbb{R}^d)$. This implies that for $x \in \mathbb{R}^d$,

$$\sup_{|y-x|\leqslant\sigma}\big|\Phi_\sigma(D_a)g(y)\big| \lesssim \sup_{|y-x|\leqslant\sigma}\sum_{j=0}^\infty 2^{-jM}\Big(\sigma^{-d}\int_{S_j(B_{y,\sigma})}\big|g(z)\big|^r\,\mathrm{d}z\Big)^{1/r} \lesssim M_r g(x),$$

where $M_r g = \big(M(g^r)\big)^{1/r}$, with M the Hardy–Littlewood maximal function, $S_j(B_{y,\sigma}) := \big\{z \in \mathbb{R}^d \;;\; 2^{j-1}\sigma \leqslant |y-z| < 2^j\sigma\big\}$ for $j \geqslant 1$, and $S_0(B_{y,\sigma}) = \big\{z \in \mathbb{R}^d \;;\; |y-z| < \sigma\big\}$. The conclusion follows from the $L^p(\mathbb{R}^d)$ boundedness of M_r together with Proposition 7.8.

For the second assertion, we first note that by renormalisation, we can change $\Phi_\sigma(D_a)\varphi_\omega(D_a)$ to $\Phi_\sigma(D_a)^2\varphi_\omega(D_a)$. We slightly change the above argument by noting that for $q \in (r,\infty)$, we have L^q-L^∞ off-diagonal estimates for $\Phi_\sigma(D_a)$. On the other hand, we have by Lemma 6.8 L^r-L^q off-diagonal estimates for $\Phi_\sigma(D_a)\varphi_\omega(D_a)$ of the form

$$\big\|1_E\Phi_\sigma(D_a)\varphi_\omega(D_a)(1_F g)\big\|_{L^q(\mathbb{R}^d)} \leqslant C_M\sigma^{-d(\frac{1}{r}-\frac{1}{q})}\sigma^{-\frac{d-1}{4}}\Big(1 + \frac{\mathrm{d}(E,F)}{\sigma}\Big)^{-M}\|1_F g\|_{L^r(\mathbb{R}^d)}$$

for all $g \in L^r(\mathbb{R}^d)$. We then conclude as above, using composition of off-diagonal bounds as in [3, Theorem 2.3]. □

8. Sobolev embedding properties of $H^p_{\mathrm{FIO},a}(\mathbb{R}^d)$

We use a variation of the arguments in [16, Section 7].

We let $m(D_a) = (I+\sqrt{L})^{-\frac{d-1}{4}}$.

Lemma 8.1. — *For every $0<\theta<\frac{\pi}{2}$ there exist $C_\theta, c_\theta > 0$ such that for all atoms $A \in T^{1,2}(\mathbb{R}^d)$, and all $s \in \mathbb{R}$*

$$\int_{S^{d-1}} \big\|(\sigma,x) \mapsto 1_{[0,1]}(\sigma) m(\sqrt{L})^{1+is}\psi_{\omega,\sigma}(D_a)A(\sigma,\cdot\,)(x)\big\|_{T^{1,2}(\mathbb{R}^d)}\,\mathrm{d}\omega \leqslant C_\theta e^{|s|c_\theta}. \tag{8.1}$$

Proof. — Let A be a $T^{1,2}(\mathbb{R}^d)$ atom associated with a ball $B = B(c_B, r)$. Without loss of generality, we assume that $A(\sigma,\cdot\,) = 0$ for all $\sigma \geqslant 1$.

By renormalisation, we can replace $\psi_{\omega,\sigma}(D_a)$ in (8.1) by $\Psi_\sigma(D_a)\psi_{\omega,\sigma}(D_a)$. Noting that $\|m^{is}\|_{L^\infty(S^\circ_\theta)} \leqslant ce^{|s|c_\theta}$, for $c_\theta = \frac{\theta(d-1)}{4}$, we use Corollary 5.5 to obtain for every $\omega \in S^{d-1}$ and given $\theta \in (0,\frac{\pi}{2})$

$$\begin{aligned}
&\big\|(\sigma,x)\mapsto 1_{[0,1]}(\sigma)m(D_a)^{1+is}\Psi_\sigma(D_a)\psi_{\omega,\sigma}(D_a)A(\sigma,\cdot\,)(x)\big\|_{T^{1,2}(\mathbb{R}^d)}\\
&\quad= \big\|(\sigma,x)\mapsto 1_{[0,1]}(\sigma)L^{\frac{d-1}{8}}m(D_a)^{1+is}\Psi_\sigma(D_a)\\
&\qquad\qquad L^{-\frac{d-1}{8}}\psi_{\omega,\sigma}(D_a)A(\sigma,\cdot\,)(x)\big\|_{T^{1,2}(\mathbb{R}^d)}\\
&\quad\leqslant C_\theta e^{|s|c_\theta}\big\|(\sigma,x)\mapsto 1_{[0,1]}(\sigma)L^{-\frac{d-1}{8}}\psi_{\omega,\sigma}(D_a)A(\sigma,\cdot\,)(x)\big\|_{T^{1,2}(\mathbb{R}^d)},
\end{aligned}$$

with C_θ independent of $s\in\mathbb{R}$.

For $j\in\mathbb{N}^*$, and $\omega\in S^{d-1}$, define $C_{j,\omega} := \big\{y\in\mathbb{R}^d\ ;\ 2^{j-1}r < \big|\langle\omega, c_B - y\rangle\big| + |c_B-y|^2 \leqslant 2^j r\big\}$ and $C_{0,\omega} := \big\{y\in\mathbb{R}^d\ ;\ \big|\langle\omega,c_B-y\rangle\big| + |c_B-y|^2\leqslant r\big\}$. Remark that $|C_{j,\omega}|\sim(2^jr)^{\frac{d+1}{2}}$, and that $d_\omega(C_{j,\omega},C_{0,\omega}) > 2^{j-1}r$. Using a slight generalisation of Lemma 6.5 and Corollary 5.8 for $p=\frac{4d}{3d-1}$, we have that

$$\begin{aligned}
&\left(\int_{S^{d-1}}\big\|(\sigma,x)\mapsto 1_{C_{0,\omega}}(x)1_{[0,1]}(\sigma)L^{-\frac{d-1}{8}}\psi_{\omega,\sigma}(D_a)A(\sigma,\cdot\,)(x)\big\|_{T^{1,2}(\mathbb{R}^d)}\,\mathrm{d}\omega\right)^2\\
&\quad\lesssim r^{\frac{d+1}{2}}\int_{S^{d-1}}\int_0^{\min(r,1)}\big\|L^{-\frac{d-1}{8}}\psi_{\omega,\sigma}(D_a)A(\sigma,\cdot\,)(x)\big\|^2_{L^2(\mathbb{R}^d)}\frac{\mathrm{d}\sigma}{\sigma}\,\mathrm{d}\omega\\
&\quad\lesssim r^{\frac{d+1}{2}}\int_0^{\min(r,1)}\big\|L^{-\frac{d-1}{8}}A(\sigma,\cdot\,)(x)\big\|^2_{L^2(\mathbb{R}^d)}\frac{\mathrm{d}\sigma}{\sigma}\\
&\quad\lesssim r^{\frac{d+1}{2}}\int_0^r\big\|A(\sigma,\cdot\,)(x)\big\|^2_{L^p(\mathbb{R}^d)}\frac{\mathrm{d}\sigma}{\sigma}\\
&\quad\lesssim r^{\frac{d+1}{2}}r^{\frac{d-1}{2}}\int_0^r\big\|A(\sigma,\cdot\,)(x)\big\|^2_{L^2(\mathbb{R}^d)}\frac{\mathrm{d}\sigma}{\sigma}
\end{aligned}$$

$$\begin{aligned}&\lesssim r^d \|A\|_{T^{2,2}}^2 \\ &\lesssim 1.\end{aligned}$$

Let $M > d+1$, and define

$$\widetilde{\Psi}\colon \xi \mapsto \frac{|\xi|^{-\frac{d-1}{4}}\Psi(\xi)}{\left(\int_0^\infty |\sigma\xi|^{-\frac{d-1}{2}}\left|\Psi(\sigma\xi)\right|^2 \frac{d\sigma}{\sigma}\right)^{\frac{1}{2}}}$$

and $\widetilde{\psi}_{\omega,\sigma}\colon \xi \mapsto \varphi_{\omega,\sigma}(\xi)\widetilde{\Psi}(\sigma\xi)$.

For all $j \in \mathbb{N}^*$, we obtain from Lemma 6.8 for $\widetilde{\psi_{\omega,\sigma}}$ instead of $\psi_{\omega,\sigma}$

$$\begin{aligned}&\left(\int_{S^{d-1}} \left\|(\sigma,x) \mapsto 1_{C_{j,\omega}}(x)1_{[0,1]}(\sigma)L^{-\frac{d-1}{8}}\psi_{\omega,\sigma}(D_a)A(\sigma,\cdot)(x)\right\|_{T^{1,2}(\mathbb{R}^d)} d\omega\right)^2 \\ &\lesssim (2^j r)^{\frac{d+1}{2}} \int_{S^{d-1}} \int_0^{\min(r,1)} \sigma^{\frac{d-1}{2}} \left\|\widetilde{\psi_{\omega,\sigma}}(D_a)A(\sigma,\cdot)\right\|_{L^2(C_{j,\omega})}^2 \frac{d\sigma}{\sigma}\, d\omega \\ &\lesssim (2^j r)^{\frac{d+1}{2}} \int_{S^{d-1}} \int_0^{\min(r,1)} \sigma^{\frac{d-1}{2}} \sigma^{-d}\left(\frac{\sigma}{2^j r}\right)^M \left\|A(\sigma,\cdot)\right\|_{L^1(\mathbb{R}^d)}^2 \frac{d\sigma}{\sigma}\, d\omega \\ &\lesssim r^d \int_{S^{d-1}} \int_0^{\min(r,1)} \left(\frac{2^j r}{\sigma}\right)^{\frac{d+1}{2}} \left(\frac{\sigma}{2^j r}\right)^M \left\|A(\sigma,\cdot)\right\|_{L^2(\mathbb{R}^d)}^2 \frac{d\sigma}{\sigma}\, d\omega \\ &\lesssim 2^{-j\left(M-\frac{d+1}{2}\right)} r^d \|A\|_{T^{2,2}}^2 \\ &\lesssim 2^{-j\left(M-\frac{d+1}{2}\right)}.\end{aligned}$$

Summing over j yields the conclusion. □

Remark 8.2. — Note that basically the same proof as above also yields the statement that for all $s \in \mathbb{R}$,

$$\left\|(\omega,\sigma,\cdot) \mapsto \sigma^{\frac{s_1}{2}+is}\psi_{\omega,\sigma}(D_a)F(\sigma,\cdot)\right\|_{L^1(S^{d-1};T^{1,2}(\mathbb{R}^d))} \lesssim \|F\|_{T^{1,2}(\mathbb{R}^d)}$$

for all $F \in T^{1,2}(\mathbb{R}^d)$. By a slight modification of Lemma 6.5, we obtain on the other hand $\left\|(\omega,\sigma,\cdot) \mapsto \psi_{\omega,\sigma}(D_a)F(\sigma,\cdot)\right\|_{L^2(S^{d-1};T^{2,2}(\mathbb{R}^d))} \lesssim \|F\|_{T^{2,2}(\mathbb{R}^d)}$ for all $F \in T^{2,2}(\mathbb{R}^d)$. Stein interpolation and duality then yield for all $p \in (1,\infty)$,

$$\left\|(\omega,\sigma,\cdot) \mapsto \sigma^{\frac{s_p}{2}}\psi_{\omega,\sigma}(D_a)F(\sigma,\cdot)\right\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))} \lesssim \|F\|_{T^{p,2}(\mathbb{R}^d)},$$

for all $F \in T^{p,2}(\mathbb{R}^d)$.

Lemma 8.3. — *For all $p \in [1,2]$, and $s_p = (d-1)\left(\frac{1}{p}-\frac{1}{2}\right)$, we have the continuous inclusion $H^{p,\frac{s_p}{2}}_{\mathrm{FIO},a}(\mathbb{R}^d) \subset H^p_L(\mathbb{R}^d)$, where $H^p_L(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ for*

$p>1$. For $p\in(1,\infty)$, and $b\colon \xi\mapsto |\xi|^{\frac{\mathrm{d}-1}{4}}m(\xi)$, we have that

$$\begin{aligned}\big\|(\sigma,x)\mapsto m(D_a)\Psi(\sigma D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)} &\lesssim \big\|\big(b(D_a)+m(D_a)\big)f\big\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)}\\ &\lesssim \|f\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)},\end{aligned}$$

for all $f\in\mathcal{S}_p$.

Proof. — Let f be an H^1_L atom. We have, using the reproducing formula (6.10), that

$$\begin{aligned}\|f\|_{H^1_L} &\sim \big\|(\sigma,x)\mapsto \Psi(\sigma D_a)f(x)\big\|_{T^{1,2}(\mathbb{R}^d)}\\ &\lesssim \int_{S^{d-1}}\big\|(\sigma,x)\mapsto 1_{[0,1]}(\sigma)\sigma^{-\frac{\mathrm{d}-1}{4}}\psi_{\omega,\sigma}(D_a)f(x)\\ &\qquad\qquad + 1_{[1,\infty)}(\sigma)\Psi(\sigma D_a)f(x)\big\|_{T^{1,2}(\mathbb{R}^d)}\,\mathrm{d}\omega\\ &\lesssim \|f\|_{H^{1,\frac{\mathrm{d}-1}{4}}_{\mathrm{FIO},a}(\mathbb{R}^d)},\end{aligned}$$

where the last inequality follows from the comparability of $\psi_{\omega,\sigma}$ with $\varphi_\omega\Psi_\sigma$ for $\sigma\in(0,1)$. Since $H^2_{\mathrm{FIO},a}=L^2$, the continuous inclusion $H^{p,\frac{s_p}{2}}_{\mathrm{FIO},a}(\mathbb{R}^d)\subset H^p_L(\mathbb{R}^d)$ follows by interpolation. In the same way,

$$\begin{aligned}&\big\|(\sigma,x)\mapsto 1_{[0,1]}(\sigma)m(D_a)\Psi(\sigma D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)}\\ &\quad\lesssim \int_{S^{d-1}}\big\|(\sigma,x)\mapsto 1_{[0,1]}(\sigma)b(D_a)\varphi_\omega(D_a)\widetilde{\Psi}(\sigma D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)}\,\mathrm{d}\omega,\end{aligned}$$

for $\widetilde{\Psi}$ such that $\Psi(\xi)=|\xi|^{\frac{\mathrm{d}-1}{4}}\widetilde{\Psi}(\xi)$ for all $\xi\in\mathbb{R}^d$. Turning to the low frequency term, we note that, for $\sigma>1$, we have that $\Psi(\sigma\xi)=\Psi(\sigma\xi)q(\xi)$ for all $\xi\in\mathbb{R}^d$. Therefore, by Theorem 5.7 and Proposition 7.8 we have that

$$\begin{aligned}\big\|(\sigma,x)\mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)m(D_a)f(x)\big\|_{T^{p,2}(\mathbb{R}^d)} &\lesssim \big\|m(D_a)q(D_a)f\big\|_{L^p(\mathbb{R}^d)}\\ &\lesssim \big\|m(D_a)f\big\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)}.\end{aligned}$$

To conclude the proof, we use Theorems 2.1 and 2.2, along with Proposition 4.3, to show that $b(D_a)$ and $m(D_a)$ are bounded operators on $L^p(\mathbb{R}^d)$, and thus also on $H^p_{\mathrm{FIO},a}(\mathbb{R}^d)$, thanks to Proposition 7.8. □

Corollary 8.4. — *Let $p\in(1,2]$. Then*

$$\big\|(I+\sqrt{L})^{-\frac{s_p}{2}}f\big\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)}\lesssim \|f\|_{L^p(\mathbb{R}^d)},$$

for all $f\in\mathcal{S}_p$.

Proof. — For $z\in\mathbb{C}$ such that $\mathrm{Re}(z)\in[0,1]$, we consider the operators defined by

$$T_zf(x,\omega,\sigma):=1_{[0,1]}(\sigma)(I+\sqrt{L})^{-(\frac{\mathrm{d}-1}{4})z}\psi_{\omega,\sigma}(D_a)f(x)\quad \forall\, f\in L^2(\mathbb{R}^d).$$

For $\mathrm{Re}(z) = 0$, they are well defined as operators from $L^2(\mathbb{R}^d)$ to $L^2\big(\mathbb{R}^d \times S^{d-1} \times (0,\infty); \mathrm{d}x\,\mathrm{d}\omega\frac{\mathrm{d}\sigma}{\sigma}\big)$ by Lemma 6.5, with norm independent of $\mathrm{Im}(z)$. For $\mathrm{Re}(z) = 1$, by Lemma 8.1, T_z extends to a bounded operator from $H^1(\mathbb{R}^d)$ to $L^1\big(S^{d-1}; T^{1,2}(\mathbb{R}^d)\big)$ with norm bounded by $C_\theta e^{|\mathrm{Im}(z)|c_\theta}$ for fixed $\theta > 0$. Therefore, by Stein interpolation [33] with admissible growth, $T_z \in B\big(L^p(\mathbb{R}^d), L^p\big(S^{d-1}; T^{p,2}(\mathbb{R}^d)\big)\big)$ for $\mathrm{Re}(z) = \frac{2}{p} - 1$. To conclude the proof, we thus only have to show the low frequency estimate

$$\big\|(\sigma, x) \mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)(I + \sqrt{L})^{-\frac{sp}{2}} f(x)\big\|_{T^{p,2}(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

This follows from Theorem 5.7 and the L^p boundedness of $(I+\sqrt{L})^{-\frac{sp}{2}}$. □

9. The wave group

Theorem 9.1. — *Let $p \in (1,\infty)$, and $s \in \mathbb{R}$. Then*

$$e^{it\sqrt{L}} \colon H^{p,s}_{\mathrm{FIO},a}(\mathbb{R}^d) \to H^{p,s}_{\mathrm{FIO},a}(\mathbb{R}^d)$$

is bounded for each $t > 0$.

For simplicity, we set $t = 1$ and $s = 0$. All the proofs extend verbatim to other values of t. The case $s \in \mathbb{R}$ is an immediate consequence of the case $s = 0$ by Proposition 7.8. For the transport groups, and the one dimensional wave groups, the L^p boundedness is clear.

Lemma 9.2. — *Let $p \in (1,\infty)$ and $\omega \in S^{d-1}$. Then*

$$e^{i\omega\cdot\sqrt{D_a^2}} \in B\big(L^p(\mathbb{R}^d)\big) \cap B\big(H^p_{\mathrm{FIO},a}(\mathbb{R}^d)\big).$$

Proof. — The L^p boundedness is proven in Proposition 4.3. The boundedness on $H^p_{\mathrm{FIO},a}(\mathbb{R}^d)$ is an immediate consequence of the L^p boundedness, by Proposition 7.8. □

For the low frequency estimate, we need the following lemma.

Lemma 9.3. — *Let $p \in (1,\infty)$, let $q \in C_c^\infty(\mathbb{R}^d)$ be radial. Then*

$$q(D_a)e^{i\sqrt{L}} \colon L^p(\mathbb{R}^d) \longrightarrow L^p(\mathbb{R}^d)$$

is bounded.

Proof. — Because of the compact support of q, the symbol $m \colon \zeta \mapsto q(\zeta)e^{i|\zeta|}$ clearly satisfies the Marcinkiewicz–Lizorkin multiplier condition of Theorem 2.1. The result thus follows from Theorems 2.1 and 2.2 using that $(e_j\sqrt{D_a^2})_{j=1,\dots,d}$ generates a bounded commutative d-parameter group (as shown in Proposition 4.3), along with the fact that

$$m(D_a) = m^s(D_a) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{m}(\xi)\exp(i\xi\sqrt{D_a^2})\,\mathrm{d}\xi,$$

as explained in Definition 5.1. □

Proof of Theorem 9.1. — For $f \in \mathcal{S}_p$, Proposition 7.8 yields

$$\|e^{i\sqrt{L}}f\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)} \lesssim \big\|q(D_a)e^{i\sqrt{L}}f\big\|_{L^p(\mathbb{R}^d)} + \bigg(\int_{S^{d-1}}\big\|\varphi_\omega(D_a)e^{i\sqrt{L}}f\big\|^p_{L^p(\mathbb{R}^d)}\,\mathrm{d}\omega\bigg)^{1/p}.$$

For the low frequency part, recall that $q \in C_c^\infty(\mathbb{R}^d)$ with $q(\zeta) \equiv 1$ for $|\zeta| \leqslant \frac{1}{8}$. Choose $\widetilde{q} \in C_c^\infty(\mathbb{R}^d)$ radial with $\widetilde{q}(\zeta) \equiv 1$ on $\operatorname{supp} q$. Then $q(D_a)e^{i\sqrt{L}} = \widetilde{q}(D_a)e^{i\sqrt{L}}q(D_a)$, since $\sqrt{D_a^2}$ and $\sqrt{L}$ are commuting, and $\widetilde{q}(D_a)e^{i\sqrt{L}}$ is L^p bounded according to Lemma 9.3. Thus,

$$\big\|q(D_a)e^{i\sqrt{L}}f\big\|_{L^p(\mathbb{R}^d)} = \big\|\widetilde{q}(D_a)e^{i\sqrt{L}}q(D_a)f\big\|_{L^p(\mathbb{R}^d)} \lesssim \big\|q(D_a)f\big\|_{L^p(\mathbb{R}^d)}.$$

Let us now consider the high frequency part. For fixed $\omega \in S^{d-1}$, we decompose

$$\varphi_\omega(D_a)e^{i\sqrt{L}} = \varphi_\omega(D_a)e^{i\omega\cdot\sqrt{D_a^2}} + \varphi_\omega(D_a)(e^{i\sqrt{L}} - e^{i\omega\cdot\sqrt{D_a^2}}).$$

The first part can be dealt with Lemma 9.2, which directly yields

$$\bigg(\int_{S^{d-1}}\big\|\varphi_\omega(D_a)e^{i\omega\cdot\sqrt{D_a^2}}f\big\|^p_{L^p(\mathbb{R}^d)}\,\mathrm{d}\omega\bigg)^{1/p} \lesssim \|f\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)}.$$

For the second part, we use (6.8) to write

$$\varphi_\omega(D_a)(e^{i\sqrt{L}} - e^{i\omega\cdot\sqrt{D_a^2}}) = \varphi_\omega(D_a)e^{i\omega\cdot\sqrt{D_a^2}}(e^{-i\omega\cdot\sqrt{D_a^2}}e^{i\sqrt{L}} - I)\pi_a W_a.$$

Since $e^{i\omega\cdot\sqrt{D_a^2}}$ is bounded on $L^p(\mathbb{R}^d)$ by Lemma 9.2, it suffices to show that

$$\big\|\varphi_\omega(D_a)(e^{-i\omega\cdot\sqrt{D_a^2}}e^{i\sqrt{L}} - I)\pi_a W_a f\big\|_{L^p(\mathbb{R}^d)} \lesssim \big\|\varphi_\omega(D_a)f\big\|_{L^p(\mathbb{R}^d)}.$$

We can write

$$\varphi_\omega(D_a)(e^{-i\omega\cdot\sqrt{D_a^2}}e^{i\sqrt{L}} - I)\pi_a W_a = m_\omega(D_a)\varphi_\omega(D_a) + q_\omega(D_a)\varphi_\omega(D_a)$$

for the symbols

$$m_\omega(\zeta) = \widetilde{\varphi}_\omega(\zeta)\widetilde{m}_\omega(\zeta)\int_0^1\int_{S^{d-1}}\psi_{\nu,\sigma}(\zeta)^2\,\mathrm{d}\nu\frac{\mathrm{d}\sigma}{\sigma} \tag{9.1}$$

and

$$q_\omega(\zeta) = \widetilde{\varphi}_\omega(\zeta)\widetilde{m}_\omega(\zeta)r(\zeta)^2$$

with $\widetilde{m}_\omega(\zeta) = e^{-i\sum_{j=1}^d \omega_j|\zeta_j|+i|\zeta|} - 1$, $\widetilde{\varphi}_\omega \in C_c^\infty(\mathbb{R}^d)$ a function with $\widetilde{\varphi}_\omega \equiv 1$ on $\operatorname{supp}\varphi_\omega$ and $\widetilde{\varphi}_\omega(\zeta) = 0$ for $|\zeta| < \frac{1}{16}$ or $\min_{(\varepsilon_j)_{j=1}^d \in \{-1,1\}^d} \big|(\varepsilon_1\widehat{\zeta}_1, \dots, \varepsilon_d\widehat{\zeta}_d) - \omega\big| > 4|\zeta|^{-1/2}$, and

$$r(\zeta) := \left(\int_1^\infty \Psi_\sigma(\zeta)^2 \,\frac{\mathrm{d}\sigma}{\sigma}\right)^{1/2}, \quad \zeta \neq 0,$$

and $r(0) := 1$. As noted in [16, Section 4.1], we have $r \in C_c^\infty(\mathbb{R}^d)$.

The proof will be concluded by applying Theorem 2.1, and Theorem 2.2, using Proposition 4.3. We only have to check that m_ω and q_ω satisfy the assumption of Theorem 2.1. For q_ω, this directly follows from the fact that $r \in C_c^\infty(\mathbb{R}^d)$. For m_ω, this is proven in Lemma 9.5 below. □

Remark 9.4. — Let $\omega \in S^{d-1}$. Let $\widetilde{\varphi}_\omega \in C_c^\infty(\mathbb{R}^d)$ be a function with $\widetilde{\varphi}_\omega \equiv 1$ on $\operatorname{supp}\varphi_\omega$ and $\widetilde{\varphi}_\omega(\zeta) = 0$ for $|\zeta| < \frac{1}{16}$ or $\min_{(\varepsilon_j)_{j=1}^d \in \{-1,1\}^d} \big|(\varepsilon_1\widehat{\zeta}_1, \dots, \varepsilon_d\widehat{\zeta}_d) - \omega\big| > 4|\zeta|^{-1/2}$. By the choice of the cut-off function $\widetilde{\varphi}_\omega$ and the support properties of φ_ω, we have the following: for all $\alpha \in \mathbb{N}_0^d$ and $\beta \in \mathbb{N}_0$, there exists a constant $C = C(\alpha, \beta) > 0$ such that

$$\big|\langle\omega, \nabla_\zeta\rangle^\beta \partial_\zeta^\alpha \widetilde{\varphi}_\omega(\zeta)\big| \leqslant C|\zeta|^{-\frac{|\alpha|}{2}-\beta}$$

for all $\omega \in S^{d-1}$ and $\zeta \in \mathbb{R}^d \setminus \{0\}$.

Lemma 9.5. — *Let $\omega \in S^{d-1}$, let m_ω be as defined in* (9.1)*. For all $\alpha \in \mathbb{N}_0^d$ with $|\alpha|_\infty \leqslant 1$ there exists a constant $C = C(\alpha) > 0$ such that*

$$\big|\zeta^\alpha \partial_\zeta^\alpha m_\omega(\zeta)\big| \leqslant C$$

for all $\zeta \in \mathbb{R}^d \setminus \{0\}$.

Proof. — By rotational invariance it suffices to consider the case $\omega = e_1$. Let $\zeta \in \mathbb{R}^d \setminus \{0\}$. The bound $\big|m_{e_1}(\zeta)\big| \leqslant C$ directly follows from (6.2) and the boundedness of $\widetilde{m}_{e_1}$ and $\widetilde{\varphi}_{e_1}$. Moreover, by the specific form of $\widetilde{m}_{e_1}(\zeta) = e^{ib(\zeta)} - 1$ with $b(\zeta) = -|\zeta_1| + |\zeta|$, it can easily be seen that the condition

$$\big|\zeta^\alpha \partial_\zeta^\alpha b(\zeta)\big| \leqslant c \tag{9.2}$$

for $|\alpha|_\infty \leqslant 1$ immediately implies $\big|\zeta^\alpha \partial_\zeta^\alpha \widetilde{m}_{e_1}(\zeta)\big| \leqslant c$ for $|\alpha|_\infty \leqslant 1$. We check (9.2):

$$\begin{aligned}\big|\zeta_1\partial_1 b(\zeta)\big| = \big|\zeta_1\partial_1\big(-|\zeta_1| + |\zeta|\big)\big| \leqslant |\zeta_1|\left|1 - \frac{|\zeta_1|}{|\zeta|}\right| = \left|\frac{\zeta_1}{|\zeta|}\right|\big||\zeta| - |\zeta_1|\big| \\ \leqslant \big||\zeta| - |\zeta_1|\big| = |\zeta_1|\left(\sqrt{1 + \sum_{j=2}^d \frac{\zeta_j^2}{\zeta_1^2}} - 1\right).\end{aligned}$$

According to the support properties of $\widetilde{\varphi}_{e_1}$ and $\psi_{\nu,\sigma}$, we have $|\nu - \varepsilon_1 e_1| \lesssim \sqrt{\sigma}$ for some $\varepsilon_1 \in \{-1, 1\}$. Thus a slight modification of (6.7) yields that there exist constants $c_1, c_2 > 0$ such that for $0 < \sigma \ll 1$, one has

$$|\zeta_1| > \frac{c_1}{\sigma} \qquad \text{and} \qquad |\zeta_j| \leqslant \frac{c_2}{\sqrt{\sigma}}, \quad j \in \{2, \dots, d\}, \tag{9.3}$$

on the support of m_{e_1}. Thus, for such choice of ζ,

$$\left|\zeta_1 \partial_1 b(\zeta)\right| \lesssim |\zeta_1| \left(\sqrt{1 + \frac{c}{|\zeta_1|}} - 1 \right).$$

This expression remains bounded for $|\zeta_1| \to \infty$ or equivalently $|\zeta| \to \infty$, since replacing $h = \frac{1}{|\zeta_1|}$, we see that

$$\lim_{h \to 0} \frac{\sqrt{1 + ch} - 1}{h} = \frac{c}{2}.$$

Again using (9.3) and $|\zeta| \geqslant |\zeta_1| > \frac{c_1}{\sigma}$, we obtain for $j \in \{2, \dots, d\}$ that

$$\left|\zeta_j \partial_j b(\zeta)\right| = \left|\zeta_j \partial_j \big(-|\zeta_1| + |\zeta|\big)\right| \leqslant \left| \zeta_j \frac{\zeta_j}{|\zeta|} \right| \leqslant c.$$

Concerning the mixed derivatives, one can inductively show that for $\alpha \in \mathbb{N}_0^d$ with $|\alpha|_\infty \leqslant 1$ and $\alpha_1 = 0$, $\left|\zeta^\alpha \partial_\zeta^\alpha b(\zeta)\right| = \left|\frac{\zeta^{2\alpha}}{|\zeta|^{2|\alpha|-1}}\right| \leqslant c$, for ζ as in (9.3). Finally, for $j \neq 1$,

$$\left|\zeta_1 \zeta_j \partial_1 \partial_j b(\zeta)\right| = \left|\zeta_1 \zeta_j \partial_1 \partial_j \big(-|\zeta_1| + |\zeta|\big)\right| = \left|\zeta_1 \zeta_j\right| \left| \frac{\zeta_1 \zeta_j}{|\zeta|^3} \right| \leqslant c.$$

Putting all arguments together shows (9.2). The bound $\left|\zeta^\alpha \partial_\zeta^\alpha \widetilde{\varphi}_{e_1}(\zeta)\right| \leqslant c$ follows from Remark 9.4 together with (9.3), whereas the analogous bound for the last factor in (9.1) concerning $\psi_{\nu,\sigma}$ is a consequence of (6.6) together with (9.3). □

Combining Corollary 8.4 with Theorems 9.1 and 5.7 then gives our main result, Theorem 1.1. We restate it here for the reader's convenience.

Theorem (Theorem 1.1). — *Let $p \in (1, \infty)$ and $s_p = (d-1)\left|\frac{1}{p} - \frac{1}{2}\right|$. For each $t \in \mathbb{R}$, the operator $(I + \sqrt{L})^{-s_p} \exp(it\sqrt{L})$ is bounded on $L^p(\mathbb{R}^d)$. Moreover, if $s_p \leqslant 2$, the operator $\exp(it\sqrt{L})$ is bounded from $W^{s_p,p}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$.*

Proof. — By duality, it suffices to consider the case $p \in (1, 2)$. Let $f \in \mathcal{S}_p$. By Lemma 8.3 and Theorem 9.1, we have that

$$\left\|\exp(it\sqrt{L}) f\right\|_{L^p(\mathbb{R}^d)} \lesssim \left\|\exp(it\sqrt{L}) f\right\|_{H^{p,\frac{s_p}{2}}_{\mathrm{FIO},a}(\mathbb{R}^d)} \lesssim \|f\|_{H^{p,\frac{s_p}{2}}_{\mathrm{FIO},a}(\mathbb{R}^d)}.$$

Using Proposition 7.8, and Corollary 8.4, we then have that

$$\big\|\exp(it\sqrt{L})f\big\|_{L^p(\mathbb{R}^d)} \lesssim \big\|(I+\sqrt{L})^{\frac{s_p}{2}}f\big\|_{\mathcal{H}^p_{\mathrm{FIO},a}(\mathbb{R}^d)} \lesssim \big\|(I+\sqrt{L})^{s_p}f\big\|_{L^p(\mathbb{R}^d)}.$$

For $s_p \leqslant 2$, Theorem 5.7 then gives $\|f\|_{W^{s_p,p}} \sim \big\|(I+\sqrt{L})^{s_p}f\big\|_{L^p(\mathbb{R}^d)}$. □

10. Lower order perturbations

We consider the operators

$$L_1 := -\sum_{j=1}^{d} \widetilde{a_{j+d}}\partial_j\widetilde{a_j}\partial j \qquad \text{and} \qquad L_2 := -\sum_{j=1}^{d} \widetilde{a_j}\partial_j\widetilde{a_{j+d}}\partial j.$$

For a function $g\colon \mathbb{R}^d \to \mathbb{R}$, we denote by M_g the multiplication operator $(f,F) \mapsto (gf, gF)$. We will evaluate the norm of g in Besov spaces $\dot{B}^{0,L_k}_{\infty,\infty}$ associated with the operators L_k, in the sense of [9], as well as in BMO_{L_k} spaces, in the sense of [13].

Theorem 10.1. — *Let $p \in (1,\infty)$ and $s_p = (d-1)\big|\frac{1}{p} - \frac{1}{2}\big|$. Let $g \in L^\infty$ be such that $g \in \dot{B}^{0,L_m}_{\infty,\infty}$, $\nabla L_m^{-\frac{1}{2}} g \in \dot{B}^{0,L_m}_{\infty,\infty}$ and $L_m^{s_p} g \in \mathrm{BMO}_{L_m}$ for $m = 1,2$. Then $M_g \in B\big(\mathcal{H}^p_{\mathrm{FIO},a}(\mathbb{R}^d)\big)$.*

Proof. — For $p = 2$, there is nothing to prove. For $p \neq 2$, this is a consequence of Lemmas 10.4 and 10.6 below. □

Remark 10.2. — If the coefficients $(a_j)_{j=1,\dots,2d}$ are $C^{1,\alpha}$ for some $\alpha \in (0,1]$, then [7, Theorem 4.19] implies that

$$\max_{m=1,2} \|g\|_{\dot{B}^{0,L_m}_{\infty,\infty}} + \max_{m=1,2} \|\nabla L_m^{-\frac{1}{2}} g\|_{\dot{B}^{0,L_m}_{\infty,\infty}} \lesssim \|g\|_\infty.$$

If the coefficients $(a_j)_{j=1,\dots,2d}$ are $C^{1,1}$, then, for all $t \geqslant 0$ and $m = 1,2$, $\exp(-tL_m)(1) = 1$ in L^∞ by Feynman–Kac's formula. Therefore [13, Proposition 6.7] gives that, for $m = 1,2$,

$$\|L_m^{s_p} g\|_{\mathrm{BMO}_{L_m}} \lesssim \|L_m^{s_p} g\|_{\mathrm{BMO}}.$$

If the coefficients $(a_j)_{j=1,\dots,2d}$ are constant, then the assumptions on g reduce to $g \in W^{2s_p,\infty}$. In the special case where $L_1 = L_2 = -\Delta$, a more general result for pseudo-differential operators has been proven recently in [27, Theorem 1.1] for symbols which are C^r regular in the spatial variable, with $r > s_p$. Even just for multiplication operators, we do not fully recover this result, partly because our abstract setting prevents us from using arguments about the Fourier support of products. In this section, we are merely demonstrating that adding lower perturbations with smooth enough coefficients is possible. We intend to develop a more complete perturbation theory in subsequent work.

We state our perturbation result for first order perturbations of the wave equation under consideration.

Corollary 10.3. — *Let $p \in (1,\infty)$ and $s_p = (d-1)\big|\frac{1}{p}-\frac{1}{2}\big|$. Assume that $s_p \leqslant 2$. For $j=1,\dots,d$, let $g_j \in L^\infty$ be such that $g_j \in \dot{B}^{0,L_m}_{\infty,\infty}$, $\nabla L_m^{-\frac{1}{2}} g_j \in \dot{B}^{0,L_m}_{\infty,\infty}$ and $L_m^{s_p} g_j \in \mathrm{BMO}_{L_m}$ for $m=1,2$. Consider*

$$\widetilde{L}\colon (f,F) \longmapsto (L_1 f, L_2 F) + \sum_{j=1}^{d} (g_j \partial_j f, g_j \partial_j F).$$

For each $t \in \mathbb{R}$, the operator $(I+\sqrt{\widetilde{L}})^{-s_p} \exp(it\sqrt{\widetilde{L}})$ is bounded on $L^p(\mathbb{R}^d)$.

Proof. — Without loss of generality, we assume that $p \leqslant 2$ (using duality to get the full result). By Theorem 9.1, [2, Example 3.14.15] and Proposition 7.8, the operator L generates a cosine family on $H^p_{\mathrm{FIO},a}(\mathbb{R}^d)$, with Kisyński space $D(\sqrt{L}) = H^{p,1}_{\mathrm{FIO},a}(\mathbb{R}^d)$ (see [2] for the theory of cosine families). By Theorem 10.1, boundedness of Riesz transforms [7, Corollary 5.19], and Proposition 7.8, we have, for all $j=1,\dots,d$, that

$$\begin{aligned} \big\|M_{g_j}(\partial_j f, \partial_j F)\big\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)} &\lesssim \big\|(\partial_j f, \partial_j F)\big\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)} \\ &\lesssim \big\|(f,F)\big\|_{H^{p,1}_{\mathrm{FIO},a}(\mathbb{R}^d)} \quad \forall\, (f,F) \in H^{p,1}_{\mathrm{FIO},a}(\mathbb{R}^d). \end{aligned}$$

We thus obtain from [2, Corollary 3.14.13] that

$$\exp(it\sqrt{\widetilde{L}}) \in B\big(H^p_{\mathrm{FIO},a}(\mathbb{R}^d)\big).$$

Another application of [7, Corollary 5.19], also gives that

$$\big\|(I+\sqrt{\widetilde{L}})^{-\frac{s_p}{2}}(f,F)\big\|_{L^p} \sim \big\|(I+\sqrt{L})^{-\frac{s_p}{2}}(f,F)\big\|_{L^p} \quad \forall\, f,F \in W^{1,p},$$

since $s_p \leqslant 2$. Using Lemma 8.3 and Corollary 8.4, we thus have that

$$\begin{aligned} \big\|(I+\sqrt{\widetilde{L}})^{-\frac{s_p}{2}} \exp(it\sqrt{\widetilde{L}})f\big\|_{L^p} &\lesssim \big\|(I+\sqrt{L})^{-\frac{s_p}{2}} \exp(it\sqrt{\widetilde{L}})f\big\|_{L^p} \\ &\lesssim \big\|\exp(it\sqrt{\widetilde{L}})f\big\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)} \\ &\lesssim \|f\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)} \\ &\lesssim \big\|(I+\sqrt{L})^{\frac{s_p}{2}} f\big\|_{L^p} \\ &\lesssim \big\|(I+\sqrt{\widetilde{L}})^{\frac{s_p}{2}} f\big\|_{L^p} \\ &\qquad \forall\, f \in L^p(\mathbb{R}^d;\mathbb{C}^2). \quad \square \end{aligned}$$

For the proof of Theorem 10.1, we use the following paraproduct decomposition.

Let $\Phi \in \mathcal{S}(\mathbb{R}^d), \phi \in \mathcal{S}(\mathbb{R}^d)$ with $\phi(0) = 1$ and $\Phi_\sigma(\zeta) = \phi\big(\sigma^2|\zeta|^2\big)$ for $\sigma > 0$, $\zeta \in \mathbb{R}^d$. We denote by $M_{\phi(L)g}$ the multiplication operator $(f, F) \mapsto \big(\phi(L_1)g.f, \phi(L_2)g.F\big)$. We denote by $M_{\phi(\underline{L})g}$ the multiplication operator $(f, F) \mapsto \big(\phi(L_2)g.f, \phi(L_1)g.F\big)$.

For $f \in \mathcal{S}_p$ and $g \in \mathcal{S}(\mathbb{R}^d)$, we use (6.8) to decompose the product gf as follows.

$$\begin{aligned} M_g f = \int_1^\infty M_{\phi(\tau L)g}\Psi(\tau D_a)^2 f \,\frac{\mathrm{d}\tau}{\tau} + \int_1^\infty (M_g - M_{\phi(\tau L)g})\Psi(\tau D_a)^2 f \,\frac{\mathrm{d}\tau}{\tau} \\ + \int_{S^{d-1}}\int_0^1 M_{\phi(\tau L)g}\varphi_\nu(D_a)^2\Psi(\tau D_a)^2 f \,\frac{\mathrm{d}\tau}{\tau}\,\mathrm{d}\nu \\ + \int_{S^{d-1}}\int_0^1 (M_g - M_{\phi(\tau L)g})\varphi_\nu(D_a)^2\Psi(\tau D_a)^2 f \,\frac{\mathrm{d}\tau}{\tau}\,\mathrm{d}\nu. \end{aligned}$$

Since the two low-frequency terms in the first line are similar but simpler than the two high-frequency terms, we only consider the two latter in the following. Moreover, note that we can choose Φ and Ψ such that by integration by parts, the last integral is — up to a low-frequency term — equal to

$$\int_{S^{d-1}}\int_0^1 M_{\Psi_\tau(L)g}\varphi_\nu(D_a)^2\Phi(\tau D_a) f \,\frac{\mathrm{d}\tau}{\tau}\,\mathrm{d}\nu,$$

where $\Psi(\sigma\zeta) =: \psi\big(\sigma^2|\zeta|^2\big)$ for $\sigma > 0$, $\zeta \in \mathbb{R}^d$.

Lemma 10.4. — *Let $p \in (1, \infty)$. Let $g \in L^\infty$ be such that $g \in \dot{B}^{0,L_m}_{\infty,\infty}$ and $\nabla L_m^{-\frac{1}{2}} g \in \dot{B}^{0,L_m}_{\infty,\infty}$ for $m = 1, 2$. For all $f \in H^p_{\mathrm{FIO},a}(\mathbb{R}^d)$, we have that*

$$\begin{aligned} \Bigg\| (\omega, \sigma, \cdot\,) \mapsto \psi_{\omega,\sigma}(D_a) \int_{S^{d-1}}\int_0^1 M_{\phi(\tau L)g}\varphi_\nu(D_a)^2 \\ \Psi(\tau D_a)^2 f \,\frac{\mathrm{d}\tau}{\tau}\,\mathrm{d}\nu \Bigg\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))} \\ \lesssim \big(\|g\|_\infty + \max_{m=1,2}\|g\|_{\dot{B}^{0,L_m}_{\infty,\infty}} + \max_{m=1,2}\|\nabla L_m^{-\frac{1}{2}} g\|_{\dot{B}^{0,L_m}_{\infty,\infty}}\big)\|f\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)}. \end{aligned}$$

Proof. — We split the integral in τ into two parts, corresponding to $\tau \in \big(0, \min(\sigma, 1)\big)$ and $\tau \in \big(\min(\sigma, 1), 1\big)$. We also split the integral over S^{d-1} into two parts, corresponding to $|\nu \pm \omega| \leqslant \sqrt{\tau}$ and $|\nu \pm \omega| > \sqrt{\tau}$. Consider first $\tau \in \big(0, \min(\sigma, 1)\big)$ and $|\nu \pm \omega| \leqslant \sqrt{\tau}$. Using Lemma 6.8, and [20, Theorem 5.2], we have that

$$\Big\|(\omega,\sigma,\cdot\,)\mapsto \psi_{\omega,\sigma}(D_a)\int_0^{\min(1,\sigma)}\int_{|\nu\pm\omega|\leqslant\sqrt{\tau}} M_{\phi(\tau L)g}\varphi_\nu(D_a)^2 \Psi(\tau D_a)^2 f\,\mathrm{d}\nu\,\frac{\mathrm{d}\tau}{\tau}\Big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}$$
$$\lesssim \Big\|(\omega,\sigma,\cdot\,)\mapsto \sigma^{-\frac{\mathrm{d}-1}{4}}\int_0^{\min(1,\sigma)}\int_{|\nu\pm\omega|\leqslant\sqrt{\tau}} M_{\phi(\tau L)g}\varphi_\nu(D_a)^2 \Psi(\tau D_a)^2 f\,\mathrm{d}\nu\,\frac{\mathrm{d}\tau}{\tau}\Big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}$$

On the other hand, Hardy's inequality implies that

$$(\sigma,\cdot\,)\mapsto\int_0^\sigma\Big(\frac{\tau}{\sigma}\Big)^{\frac{\mathrm{d}-1}{4}}F(\tau,\cdot\,)\,\frac{\mathrm{d}\tau}{\tau}$$

is bounded on $T^{p,2}(\mathbb{R}^d)$. We thus have that

$$\Big\|(\omega,\sigma,\cdot\,)\mapsto \sigma^{-\frac{\mathrm{d}-1}{4}}\int_0^{\min(1,\sigma)}\int_{|\nu\pm\omega|\leqslant\sqrt{\tau}} M_{\phi(\tau L)g}\varphi_\nu(D_a)^2 \Psi(\tau D_a)^2 f\,\mathrm{d}\nu\,\frac{\mathrm{d}\tau}{\tau}\Big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}$$
$$\lesssim \sup_{\tau>0}\big\|\phi(\tau L)g\big\|_\infty\Big\|(\omega,\tau,\cdot\,)\mapsto \tau^{-\frac{\mathrm{d}-1}{4}}\int_{|\nu\pm\omega|\leqslant\sqrt{\tau}}\varphi_\nu(D_a)^2 \Psi(\tau D_a)^2 f\,\mathrm{d}\nu\Big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}$$
$$\lesssim \|g\|_\infty\Big\|(\omega,\tau,\cdot\,)\mapsto \tau^{-\frac{\mathrm{d}-1}{4}}\int_{|\nu\pm\omega|\leqslant\sqrt{\tau}}\varphi_\nu(D_a) \Psi(\tau D_a)\widetilde{\psi}_{\omega,\tau}(D_a) f\,\mathrm{d}\nu\Big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))},$$

for some $\widetilde{\psi}_{\omega,\tau}$ that satisfies the same assumptions as $\psi_{\omega,\tau}$ in Section 6. Noting that

$$\tau^{-\frac{\mathrm{d}-1}{4}}\int_{|\nu\pm\omega|\leqslant\sqrt{\tau}}\big\|\mathcal{F}^{-1}(\psi_{\nu,\tau})\big\|_{L^1}\,\mathrm{d}\nu\lesssim\tau^{-\frac{\mathrm{d}-1}{2}}\int_{|\nu\pm\omega|\leqslant\sqrt{\tau}}\mathrm{d}\nu\lesssim 1, \tag{10.1}$$

uniformly in τ, we can apply a slight modification of Lemma 6.8, together with [20, Theorem 5.2], and get that

$$\Big\|(\omega,\tau,\cdot\,)\mapsto \tau^{-\frac{d-1}{4}}\int_{|\nu\pm\omega|\leqslant\sqrt{\tau}}\varphi_\nu(D_a)\Psi(\tau D_a)\widetilde{\psi}_{\omega,\tau}(D_a)f\,\mathrm{d}\nu\Big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))},$$
$$\lesssim \big\|(\omega,\tau,\cdot\,)\mapsto\widetilde{\psi}_{\omega,\tau}(D_a)f\big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}$$
$$\lesssim \|f\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)}.$$

We now turn to the part where $\tau\in\big(0,\min(\sigma,1)\big)$ and $|\nu\pm\omega|>\sqrt{\tau}$. Denoting by $(\omega,\omega_1,\dots,\omega_{d-1})$ an orthonormal basis of $\mathbb{R}^d$, we remark that, in this region,

$$\begin{aligned}\tau(\nu.D_a)\psi_{\omega,\sigma}(D_a) &= \frac{\tau}{\sigma}(\nu.\omega)\sigma(\omega.D_a)\psi_{\omega,\sigma}(D_a)\\ &\quad+\sqrt{\tau}\sqrt{\frac{\tau}{\sigma}}\sum_{j=1}^{d-1}(\nu.\omega_j)\sqrt{\sigma}(\omega_j.D_a)\psi_{\omega,\sigma}(D_a)\\ &=\sqrt{\tau}\left(\frac{\tau}{\sigma}+\sqrt{\frac{\tau}{\sigma}}\right)\widetilde{\psi_{\omega,\sigma}}(D_a),\end{aligned}$$

for some $\widetilde{\psi}_{\omega,\sigma}$ that satisfies the same assumptions as $\psi_{\omega,\sigma}$ in Section 6 (integrating by parts as in Lemma 6.8), since $|\omega.\nu|\lesssim\sqrt{\tau}\leqslant\sqrt{\sigma}$. We combine this fact with the following version of the product rule:

$$M_{\phi(\tau L)g}(e_j.D_a)=(e_j.D_a)M_{\phi(\tau\underline{L})g}-M_{(e_j.D_a)\phi(\tau\underline{L})g},$$

for $j=1,\dots,d$, where

$$M_{(e_j.D_a)\phi(\tau\underline{L})g}\colon (f,F)\mapsto\big(-\widetilde{a_{j+d}}\partial_j\phi(\tau L_1)g\cdot F,\widetilde{a_j}\partial_j\phi(\tau L_2)g\cdot f\big).$$

We obtain that, for any $M\in\mathbb{N}$,

$$\Big\|(\omega,\sigma,\cdot\,)\mapsto\psi_{\omega,\sigma}(D_a)\int_0^{\min(1,\sigma)}\int_{|\nu\pm\omega|>\sqrt{\tau}}M_{\phi(\tau L)g}\varphi_\nu(D_a)^2\Psi(\tau D_a)^2f\,\mathrm{d}\nu\,\frac{\mathrm{d}\tau}{\tau}\Big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}$$

$$\lesssim \max_{j=0,\dots,2M}\Big\|(\omega,\sigma,\cdot\,)\mapsto \tau^M\widetilde{\psi}_{\omega,\sigma}(D_a)\iint M_{(\sqrt{\tau}\nu.D_a)^j\phi(\tau L)g}\varphi_\nu(D_a)\\ \Psi(\tau D_a)\underline{\psi}_{\nu,\tau}(D_a)f\,\mathrm{d}\nu\,\frac{\mathrm{d}\tau}{\tau}\Big\|$$

$$+\max_{j=0,\dots,2M}\Big\|(\omega,\sigma,\cdot\,)\mapsto \tau^M\widetilde{\psi}_{\omega,\sigma}(D_a)\iint M_{(\sqrt{\tau}\nu.D_a)^j\phi(\tau\underline{L})g}\varphi_\nu(D_a)\\ \Psi(\tau D_a)\underline{\psi}_{\nu,\tau}(D_a)f\,\mathrm{d}\nu\,\frac{\mathrm{d}\tau}{\tau}\Big\|,$$

for some $\widetilde{\psi}_{\omega,\sigma}$ and $\underline{\psi}_{\omega,\sigma}$ that satisfy the same assumptions as $\psi_{\omega,\sigma}$ in Section 6. From Remark 8.2 we know that

$$\big\|(\omega,\sigma,\cdot\,)\mapsto \sigma^{\frac{s_p}{2}}\psi_{\omega,\sigma}(D_a)F(\sigma,\cdot\,)\big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}\lesssim\|F\|_{T^{p,2}(\mathbb{R}^d)}.$$

Picking $M>\frac{\mathrm{d}-1}{4}+\frac{s_p}{2}$, and using Hardy's inequality again, we thus get that - suppressing a similar estimate with L replaced by $\underline{L}$ -

$$\max_{j=0,\dots,2M}\Big\|(\omega,\sigma,\cdot\,)\mapsto \tau^M\widetilde{\psi}_{\omega,\sigma}(D_a)\iint M_{(\sqrt{\tau}\nu.D_a)^j\phi(\tau L)g}\varphi_\nu(D_a)\\ \Psi(\tau D_a)\underline{\psi}_{\nu,\tau}(D_a)f\,\mathrm{d}\nu\,\frac{\mathrm{d}\tau}{\tau}\Big\|$$

$$\lesssim \max_{j=0,\dots,2M}\int_{S^{d-1}}\big\|(\tau,\cdot\,)\mapsto \tau^{\frac{\mathrm{d}-1}{4}}M_{(\sqrt{\tau}\nu.D_a)^j\phi(\tau L)g}\varphi_\nu(D_a)\\ \Psi(\tau D_a)\underline{\psi}_{\nu,\tau}(D_a)f\big\|_{T^{p,2}(\mathbb{R}^d)}\,\mathrm{d}\nu$$

$$\lesssim\big(\|g\|_\infty+\max_{m=1,2}\|g\|_{\dot{B}^{0,L_m}_{\infty,\infty}}+\max_{m=1,2}\|\nabla L_m^{-\frac12}g\|_{\dot{B}^{0,L_m}_{\infty,\infty}}\big)\\ \int_{S^{d-1}}\big\|(\tau,\cdot\,)\mapsto\underline{\psi}_{\nu,\tau}(D_a)f\big\|_{T^{p,2}(\mathbb{R}^d)}\,\mathrm{d}\nu$$

$$\lesssim\big(\|g\|_\infty+\max_{m=1,2}\|g\|_{\dot{B}^{0,L_m}_{\infty,\infty}}+\max_{m=1,2}\|\nabla L_m^{-\frac12}g\|_{\dot{B}^{0,L_m}_{\infty,\infty}}\big)\|f\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)}.$$

For the integral over $\tau\in\big(\min(\sigma,1),1\big)$, we slightly rewrite the above argument, by picking $M\in\mathbb{N}$ such that $M>\frac{\mathrm{d}-1}{8}$, and using that $\widetilde{\psi}_{\omega,\sigma}(D_a):=\psi_{\omega,\sigma}(D_a)(\sigma^2L)^{-M}$ satisfies the same assumptions as $\psi_{\omega,\sigma}$ in Section 6. In the region where $|\nu\pm\omega|\leqslant\sqrt{\tau}$, we first use Lemma 6.8, [20, Theorem 5.2], and Hardy's inequality as before to obtain that

$$\Big\|(\omega,\sigma,\cdot\,)\mapsto\psi_{\omega,\sigma}(D_a)\\ \int_{\min(1,\sigma)}^1\int_{|\nu\pm\omega|\leqslant\sqrt{\tau}}M_{\phi(\tau L)g}\varphi_\nu(D_a)^2\Psi(\tau D_a)^2f\,\mathrm{d}\nu\,\frac{\mathrm{d}\tau}{\tau}\Big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}$$

$$\lesssim \Big\| (\omega,\sigma,\cdot\,) \mapsto \sigma^{2M-\frac{\mathrm{d}-1}{4}} \sigma^{\frac{\mathrm{d}-1}{4}} \widetilde{\psi}_{\omega,\sigma}(D_a) \int_{\min(1,\sigma)}^{1} \int_{|\nu\pm\omega|\leqslant\sqrt{\tau}} L^M \big[M_{\phi(\tau L)g} \varphi_\nu(D_a)^2 \Psi(\tau D_a)^2 f\big] \,\mathrm{d}\nu \,\frac{\mathrm{d}\tau}{\tau} \Big\|$$

$$\lesssim \Big\| (\omega,\tau,\cdot\,) \mapsto \tau^{2M-\frac{\mathrm{d}-1}{4}} \int_{|\nu\pm\omega|\leqslant\sqrt{\tau}} L^M \big[M_{\phi(\tau L)g} \varphi_\nu(D_a)^2 \Psi(\tau D_a)^2 f\big] \,\mathrm{d}\nu \Big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}$$

For $j = 1, \dots, d$, we now use the following version of the product rule:

$$(e_j.D_a) M_{\phi(\tau L)g} = M_{\phi(\tau \underline{L})g}(e_j.D_a) + M_{(e_j.D_a)\phi(\tau L)g}.$$

Let $k \in \{0, \dots, 2M\}$ be even, and $j = 1, \dots, d$. Letting $\phi_k \colon x \mapsto x^{\frac{k}{2}} \phi(x)$, $m = 1, 2$, and $\delta \in \{0, 1\}$, we can estimate further by multiples of terms of the form

$$\Big\| (\omega,\tau,\cdot\,) \mapsto \tau^{-\frac{\mathrm{d}-1}{4}} \int_{|\nu\pm\omega|\leqslant\sqrt{\tau}} M_{\tau^\delta (e_j.D_a)^\delta \phi_k(\tau L_m)g} (\tau D_a)^{2M-k} \varphi_\nu(D_a)^2 \Psi(\tau D_a)^2 f \,\mathrm{d}\nu \Big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}$$

$$\lesssim \sup_{\tau\in[0,1]} \big\| (\tau,\cdot\,) \mapsto (\tau\partial_j)^\delta (\tau L_m)^{\frac{k}{2}} \phi(\tau L_m) g \big\|_{L^\infty(\mathbb{R}^d)} \cdot \Big\| (\omega,\tau,\cdot\,) \mapsto \tau^{-\frac{\mathrm{d}-1}{4}} \tau^{\frac{k}{2}} \int_{|\nu\pm\omega|\leqslant\sqrt{\tau}} \varphi_\nu(D_a) \Psi(\tau D_a) \widetilde{\psi}_{\omega,\tau}(D_a) f \,\mathrm{d}\nu \Big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))},$$

for some $\widetilde{\psi}_{\omega,\tau}$ that satisfies the same assumptions as $\psi_{\omega,\tau}$ in Section 6.

For $k \in \{0, \dots, 2M-1\}$ even, $m = 1, 2$, and $j = 1, \dots, d$, we also obtain multiples of terms of the form

$$\Big\| (\omega,\tau,\cdot\,) \mapsto \tau^{-\frac{\mathrm{d}-1}{4}} \int_{|\nu\pm\omega|\leqslant\sqrt{\tau}} M_{\tau^\delta (e_j.D_a)^\delta \phi_k(\tau L_m)g} (\tau D_a)^{2M-k-1} \varphi_\nu(D_a)^2 \Psi(\tau D_a)^2 f \,\mathrm{d}\nu \Big\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}$$

$$\lesssim \sup_{\tau\in[0,1]} \big\|(\tau,\cdot\,)\mapsto (\tau\partial_j)^{\delta}(\tau L_m)^{\frac{k}{2}}\phi(\tau L_m)g\big\|_{L^\infty(\mathbb{R}^d)}$$
$$\cdot\bigg\|(\omega,\tau,\cdot\,)\mapsto \tau^{-\frac{\mathrm{d}-1}{4}}\tau^{\frac{k}{2}}\int_{|\nu\pm\omega|\leqslant\sqrt{\tau}}(\tau^2L)^{M-\frac{k+2}{2}}(\tau e_j.D_a)^{1-\delta}\tau D_a$$
$$\varphi_\nu(D_a)^2\Psi(\tau D_a)^2 f\,\mathrm{d}\nu\bigg\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}.$$

The result for the region where $\tau\in\big(\min(\sigma,1),1\big)$ and $|\nu\pm\omega|\leqslant\sqrt{\tau}$ then follows as in the case of the region where $\tau\in\big(0,\min(\sigma,1)\big)$ and $|\nu\pm\omega|\leqslant\sqrt{\tau}$. Finally, we consider the region where $\tau\in\big(\min(\sigma,1),1\big)$ and $|\nu\pm\omega|>\sqrt{\tau}$. We first apply the product rule as we did in the region where $\tau\in\big(0,\min(\sigma,1)\big)$ and $|\nu\pm\omega|>\sqrt{\tau}$ to obtain that, for any $M'\in\mathbb{N}$,

$$\bigg\|(\omega,\sigma,\cdot\,)\mapsto\psi_{\omega,\sigma}(D_a)\int_{\min(1,\sigma)}^1\int_{|\nu\pm\omega|>\sqrt{\tau}}M_{\phi(\tau L)g}\varphi_\nu(D_a)^2$$
$$\Psi(\tau D_a)^2 f\,\mathrm{d}\nu\,\frac{\mathrm{d}\tau}{\tau}\bigg\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}$$
$$\lesssim \max_{j=0,\dots,2M'}\bigg\|(\omega,\sigma,\cdot\,)\mapsto\tau^{\frac{M'}{2}}\left(\frac{\tau}{\sigma}+\sqrt{\frac{\tau}{\sigma}}\right)^{M'}\widetilde{\psi}_{\omega,\sigma}(D_a)$$
$$\int M_{(\sqrt{\tau}\nu.D_a)^j\phi(\tau L)g}\varphi_\nu(D_a)\Psi(\tau D_a)\underline{\psi}_{\nu,\tau}(D_a)f\,\mathrm{d}\nu\,\frac{\mathrm{d}\tau}{\tau}\bigg\|,$$
$$+\max_{j=0,\dots,2M'}\bigg\|(\omega,\sigma,\cdot\,)\mapsto\tau^{\frac{M'}{2}}\left(\frac{\tau}{\sigma}+\sqrt{\frac{\tau}{\sigma}}\right)^{M'}\widetilde{\psi}_{\omega,\sigma}(D_a)$$
$$\int M_{(\sqrt{\tau}\nu.D_a)^j\phi(\tau\underline{L})g}\varphi_\nu(D_a)\Psi(\tau D_a)\underline{\psi}_{\nu,\tau}(D_a)f\,\mathrm{d}\nu\,\frac{\mathrm{d}\tau}{\tau}\bigg\|,$$

for some $\widetilde{\psi}_{\omega,\sigma}$ and $\underline{\psi}_{\omega,\sigma}$ that satisfy the same assumptions as $\psi_{\omega,\sigma}$ in Section 6. We then fix $M'>s_p+\frac{\mathrm{d}-1}{2}$, and argue as we did in the region $\tau\in\big(\min(\sigma,1),1\big)$ and $|\nu\pm\omega|\leqslant\sqrt{\tau}$, to obtain that, for all $M>\frac{M'}{2}$, again suppressing similar terms with L replaced by $\underline{L}$,

$$\max_{j=0,\dots,2M'}\bigg\|(\omega,\sigma,\cdot\,)\mapsto\tau^{\frac{M'}{2}}\left(\frac{\tau}{\sigma}+\sqrt{\frac{\tau}{\sigma}}\right)^{M'}\widetilde{\psi}_{\omega,\sigma}(D_a)$$
$$\int M_{(\sqrt{\tau}\nu.D_a)^j\phi(\tau L)g}\varphi_\nu(D_a)\Psi(\tau D_a)\underline{\psi}_{\nu,\tau}(D_a)f\,\mathrm{d}\nu\,\frac{\mathrm{d}\tau}{\tau}\bigg\|,$$
$$\lesssim\max_{j=0,\dots,2M'}\bigg\|\tau^{\frac{M'}{2}}\left(\frac{\tau}{\sigma}+\sqrt{\frac{\tau}{\sigma}}\right)^{M'}\sigma^{2M}\underline{\widetilde{\psi}}_{\omega,\sigma}(D_a)$$
$$\int L^M\big[M_{(\sqrt{\tau}\nu.D_a)^j\phi(\tau L)g}\varphi_\nu(D_a)\Psi(\tau D_a)\underline{\psi}_{\nu,\tau}(D_a)f\big]\,\mathrm{d}\nu\,\frac{\mathrm{d}\tau}{\tau}\bigg\|,$$

$$\lesssim \max_{j=0,\dots,2M'} \int_{S^{d-1}} \Big\| (\tau,\cdot\,) \mapsto \tau^{2M+\frac{M'}{2}-\frac{s_p}{2}} L^M \\ \big[M_{(\sqrt{\tau}\nu.D_a)^j \phi(\tau L) g} \varphi_\nu(D_a)\Psi(\tau D_a)\underline{\psi}_{\nu,\tau}(D_a) f\big] \Big\|_{T^{p,2}(\mathbb{R}^d)} \,\mathrm{d}\nu$$

$$\lesssim \max_{j=0,\dots,2M'} \int_{S^{d-1}} \Big\| (\tau,\cdot\,) \mapsto \tau^{\frac{\mathrm{d}-1}{4}} (\tau^2 L^M) \\ \big[M_{(\sqrt{\tau}\nu.D_a)^j \phi(\tau L) g} \varphi_\nu(D_a)\Psi(\tau D_a)\underline{\psi}_{\nu,\tau}(D_a) f\big] \Big\|_{T^{p,2}(\mathbb{R}^d)} \,\mathrm{d}\nu.$$

Finally, using the product rule as we did in the region where $\tau \in \big(\min(\sigma,1),1\big)$ and $|\nu\pm\omega| \leqslant \sqrt{\tau}$, we estimate further by terms of the form

$$\int_{S^{d-1}} \big\| (\tau,\cdot\,) \mapsto \tau^{\frac{\mathrm{d}-1}{4}} \varphi_\nu(D_a)\Psi(\tau D_a)\underline{\psi}_{\nu,\tau}(D_a) f\big\|_{T^{p,2}(\mathbb{R}^d)} \,\mathrm{d}\nu \\ \lesssim \int_{S^{d-1}} \big\| (\tau,\cdot\,) \mapsto \underline{\psi}_{\nu,\tau}(D_a) f\big\|_{T^{p,2}(\mathbb{R}^d)} \,\mathrm{d}\nu \\ \lesssim \|f\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)},$$

multiplied by $\big(\|g\|_\infty + \max_{m=1,2}\|g\|_{\dot B^{0,L_m}_{\infty,\infty}} + \max_{m=1,2}\|\nabla L_m^{-\frac12} g\|_{\dot B^{0,L_m}_{\infty,\infty}}\big)$. □

For the second paraproduct, we make use of the following factorisation result for tent spaces (see [11] for the definition of the tent spaces $T^{p,q}$ when $p=\infty$ or $q\neq 2$).

Theorem 10.5 ([10, Theorem 1.1]). — *Let $p,q \in (1,\infty)$. If $F \in T^{p,\infty}(\mathbb{R}^d)$ and $G \in T^{\infty,q}(\mathbb{R}^d)$, then $FG \in T^{p,q}(\mathbb{R}^d)$ and*

$$\|F\cdot G\|_{T^{p,q}(\mathbb{R}^d)} \leqslant C\|F\|_{T^{p,\infty}(\mathbb{R}^d)}\|G\|_{T^{\infty,q}(\mathbb{R}^d)},$$

with a constant $C>0$ which is independent of F and G.

Lemma 10.6. — *Let $p \in (1,\infty)$. Let $g \in L^\infty$ be such that $L_m^{s_p} g \in \mathrm{BMO}_{L_m}$ for $m=1,2$, and let $f \in H^p_{\mathrm{FIO},a}(\mathbb{R}^d)$. Then*

$$\Big\| (\omega,\sigma,\cdot\,) \mapsto \psi_{\omega,\sigma}(D_a) \int_{S^{d-1}} \int_0^1 M_{\Psi_\tau(L) g} \cdot \varphi_\nu(D_a)^2 \Phi(\tau D_a) f \,\frac{\mathrm{d}\tau}{\tau}\,\mathrm{d}\nu \Big\|_{L^p(T^{p,2})} \\ \lesssim \max_{m=1,2} \|L_m^{s_p} g\|_{\mathrm{BMO}_{L_m}} \|f\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)}.$$

Proof. — Using Remark 8.2 and Hardy's inequality as in the proof of Lemma 10.4, we have that

$$\Big\| (\omega,\sigma,\cdot\,) \mapsto \psi_{\omega,\sigma}(D_a) \int_{S^{d-1}} \int_0^{\min(\sigma,1)} M_{\Psi_\tau(L) g} \varphi_\nu(D_a)^2 \\ \Phi(\tau D_a) f \,\frac{\mathrm{d}\tau}{\tau}\,\mathrm{d}\nu \Big\|_{L^p(T^{p,2})}$$

$$\lesssim \int_{S^{d-1}} \big\| (\tau,\cdot\,) \mapsto \tau^{-\frac{s_p}{2}} M_{\Psi_\tau(L) g} \varphi_\nu(D_a)^2 \Phi(\tau D_a) f \big\|_{T^{p,2}(\mathbb{R}^d)} \,\mathrm{d}\nu.$$

Applying Theorem 10.5, the above is bounded by a constant times

$$\big\|(\tau,\cdot\,)\mapsto \tau^{-s_p}\Psi_\tau(L)g\big\|_{T^{\infty,2}(\mathbb{R}^d)} \int_{S^{d-1}}\big\|(\tau,\cdot\,)\mapsto \tau^{\frac{s_p}{2}}\varphi_\nu(D_a)^2\Phi(\tau D_a)f\big\|_{T^{p,\infty}(\mathbb{R}^d)}\,\mathrm{d}\nu$$

$$\lesssim \max_{m=1,2}\|L_m^{s_p}g\|_{\mathrm{BMO}_{L_m}}\|f\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)},$$

where we use [13, Lemma 4.3], and a modification of Proposition 7.9 in the last line, which uses the improved bound (10.1) on the wave packets.

For the integral over $\tau\in\big(\min(\sigma,1),1\big)$, we again have to use the product rule. With the same arguments as in the proof of Lemma 10.4, we end up with terms of the form

$$\big\|(\tau,\cdot\,)\mapsto \tau^{-s_p}\Psi_\tau(L)g\big\|_{T^{\infty,2}(\mathbb{R}^d)} \int_{S^{d-1}}\big\|(\tau,\cdot\,)\mapsto \tau^{\frac{s_p}{2}}(\tau^2L)^{M-\frac{k}{2}} (\tau e_j.D_a)^{1-\delta}\varphi_\nu(D_a)^2\Phi(\tau D_a)^2f\big\|_{T^{p,\infty}(\mathbb{R}^d)}\,\mathrm{d}\nu$$

$$\lesssim \max_{m=1,2}\|L_m^{s_p}g\|_{\mathrm{BMO}_{L_m}}\|f\|_{H^p_{\mathrm{FIO},a}(\mathbb{R}^d)},$$

for $k\in\{0,\dots,2M\}$ even, and $\delta\in\{0,1\}$ (and similar terms for k odd, as in Lemma 10.4). □

BIBLIOGRAPHY

[1] A. Amenta, "Interpolation and embeddings of weighted tent spaces", *J. Fourier Anal. Appl.* **24** (2018), no. 1, p. 108-140.

[2] W. Arendt, C. J. K. Batty, M. Hieber & F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, 2nd ed., Monographs in Mathematics, vol. 96, Springer, 2011.

[3] P. Auscher & J. M. Martell, "Weighted norm inequalities, off-diagonal estimates and elliptic operators. II. Off-diagonal estimates on spaces of homogeneous type", *J. Evol. Equ.* **7** (2007), no. 2, p. 265-316.

[4] P. Auscher, A. McIntosh & A. J. Morris, "Calderón reproducing formulas and applications to Hardy spaces", *Rev. Mat. Iberoam.* **31** (2015), no. 3, p. 865-900.

[5] P. Auscher, A. McIntosh & A. Nahmod, "The square root problem of Kato in one dimension, and first order elliptic systems", *Indiana Univ. Math. J.* **46** (1997), no. 3, p. 659-695.

[6] P. Auscher, A. McIntosh & E. Russ, "Hardy spaces of differential forms on Riemannian manifolds", *J. Geom. Anal.* **18** (2008), no. 1, p. 192-248.

[7] P. Auscher, A. McIntosh & P. Tchamitchian, "Heat kernels of second order complex elliptic operators and applications", *J. Funct. Anal.* **152** (1998), no. 1, p. 22-73.

[8] P. Auscher & P. Tchamitchian, "Calcul fontionnel précisé pour des opérateurs elliptiques complexes en dimension un (et applications à certaines équations elliptiques complexes en dimension deux)", *Ann. Inst. Fourier* **45** (1995), no. 3, p. 721-778.

[9] H.-Q. Bui, X. T. Duong & L. Yan, "Calderón reproducing formulas and new Besov spaces associated with operators", *Adv. Math.* **229** (2012), no. 4, p. 2449-2502.

[10] W. S. Cohn & I. E. Verbitsky, "Factorization of tent spaces and Hankel operators", *J. Funct. Anal.* **175** (2000), no. 2, p. 308-329.

[11] R. R. Coifman, Y. Meyer & E. M. Stein, "Some new function spaces and their applications to harmonic analysis", *J. Funct. Anal.* **62** (1985), no. 2, p. 304-335.

[12] X. T. Duong & J. Li, "Hardy spaces associated to operators satisfying Davies–Gaffney estimates and bounded holomorphic functional calculus", *J. Funct. Anal.* **264** (2013), no. 6, p. 1409-1437.

[13] X. T. Duong & L. Yan, "Duality of Hardy and BMO spaces associated with operators with heat kernel bounds", *J. Am. Math. Soc.* **18** (2005), no. 4, p. 943-973.

[14] D. Frey, A. McIntosh & P. Portal, "Conical square function estimates and functional calculi for perturbed Hodge–Dirac operators in L^P", *J. Anal. Math.* **134** (2018), no. 2, p. 399-453.

[15] E. Harboure, J. L. Torrea & B. E. Viviani, "A vector-valued approach to tent spaces", *J. Anal. Math.* **56** (1991), p. 125-140.

[16] A. Hassell, P. Portal & J. Rozendaal, "Off-singularity bounds and Hardy spaces for Fourier integral operators", *Trans. Am. Math. Soc.* **373** (2020), no. 8, p. 5773-5832.

[17] A. Hassell & J. Rozendaal, "L^p and $\mathcal{H}^p_{FIO}$ regularity for wave equations with rough coefficients", 2010, `https://arxiv.org/abs/2010.13761`.

[18] S. Hofmann, G. Lu, D. Mitrea, M. Mitrea & L. Yan, *Hardy spaces associated to non-negative self-adjoint operators satisfying Davies–Gaffney estimates*, Memoirs of the American Mathematical Society, vol. 1007, American Mathematical Society, 2011.

[19] S. Hofmann & S. Mayboroda, "Hardy and BMO spaces associated to divergence form elliptic operators", *Math. Ann.* **344** (2009), no. 1, p. 37-116.

[20] T. Hytönen, J. van Neerven & P. Portal, "Conical square function estimates in UMD Banach spaces and applications to H^∞-functional calculi", *J. Anal. Math.* **106** (2008), p. 317-351.

[21] T. Hytönen, J. van Neerven, M. Veraar & L. Weis, *Analysis in Banach spaces. Vol. II. Probabilistic methods and operator theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 67, Springer, 2017.

[22] A. McIntosh & A. J. Morris, "Finite propagation speed for first order systems and Huygens' principle for hyperbolic equations", *Proc. Am. Math. Soc.* **141** (2013), no. 10, p. 3515-3527.

[23] A. Miyachi, "On some estimates for the wave equation in L^p and H^p", *J. Fac. Sci., Univ. Tokyo, Sect. I A* **27** (1980), no. 2, p. 331-354.

[24] D. Müller & A. Seeger, "Sharp L^p bounds for the wave equation on groups of Heisenberg type", *Anal. PDE* **8** (2015), no. 5, p. 1051-1100.

[25] J. C. Peral, "L^p estimates for the wave equation", *J. Funct. Anal.* **36** (1980), no. 1, p. 114-145.

[26] J. Rozendaal, "Characterizations of Hardy spaces for Fourier integral operators", *Rev. Mat. Iberoam.* **37** (2021), no. 5, p. 1717-1745.

[27] ———, "Rough pseudodifferential operators on Hardy spaces for Fourier integral operators II", 2022, `https://arxiv.org/abs/2103.13378`.

[28] A. Seeger, C. D. Sogge & E. M. Stein, "Regularity properties of Fourier integral operators", *Ann. Math. (2)* **134** (1991), no. 2, p. 231-251.

[29] H. F. Smith, "A Hardy space for Fourier integral operators", *J. Geom. Anal.* **8** (1998), no. 4, p. 629-653.

[30] ———, "A parametrix construction for wave equations with $C^{1,1}$ coefficients", *Ann. Inst. Fourier* **48** (1998), no. 3, p. 797-835.

[31] H. F. Smith & C. D. Sogge, "On Strichartz and eigenfunction estimates for low regularity metrics", *Math. Res. Lett.* **1** (1994), no. 6, p. 729-737.

[32] H. F. Smith & D. Tataru, "Sharp counterexamples for Strichartz estimates for low regularity metrics", *Math. Res. Lett.* **9** (2002), no. 2-3, p. 199-204.

[33] E. M. Stein, "Interpolation of linear operators", *Trans. Am. Math. Soc.* **83** (1956), p. 482-492.

[34] Ž. Štrkalj & L. Weis, "On operator-valued Fourier multiplier theorems", *Trans. Am. Math. Soc.* **359** (2007), no. 8, p. 3529-3547.

[35] D. Tataru, "Strichartz estimates for second order hyperbolic operators with non-smooth coefficients. II", *Am. J. Math.* **123** (2001), no. 3, p. 385-423.

Manuscrit reçu le 3 mai 2021,
révisé le 30 novembre 2023,
accepté le 29 novembre 2024.

Dorothee FREY
Karlsruhe Institute of Technology,
Department of Mathematics,
76131 Karlsruhe (Germany)
dorothee.frey@kit.edu

Pierre PORTAL
Australian National University,
Mathematical Sciences Institute,
Hanna Neumann Building,
Ngunnawal and Ngambri Country,
Canberra ACT 2601 (Australia)
pierre.portal@anu.edu.au