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FANO FOURFOLDS OF K3 TYPE

by Marcello BERNARDARA, Enrico FATIGHENTI, Laurent MANIVEL & Fabio TANTURRI (*)

Abstract. — We produce a list of 64 families of Fano fourfolds of K3 type, extracted from our database of at least 634 Fano fourfolds constructed as zero loci of general global sections of completely reducible homogeneous vector bundles on products of flag manifolds. We study the geometry of these Fano fourfolds in some detail, and we find the origin of their K3 structure by relating most of them either to cubic fourfolds, Gushel–Mukai fourfolds, or actual K3 surfaces. Their main invariants and some information on their rationality and on possible semiorthogonal decompositions for their derived categories are provided.

Résumé. — Nous produisons une liste de 64 familles de variétés de Fano de type K3 et de dimension 4, extraite d'une base de données contenant 634 familles définies comme lieux de zéros de sections globales génériques de fibrés vectoriels homogènes complètement réductibles sur des produits de variétés de drapeaux. Nous étudions en détail la géométrie de chacune de ces variétés et nous identifions l'origine de leurs structures K3 par des correspondances avec soit des cubiques, soit des variétés de Gushel–Mukai, soit des surfaces K3. Nous donnons aussi leurs principaux invariants et quelques informations sur leur rationalité et leurs possibles décompositions semiorthogonales.

1. Introduction

In this paper we pursue the quest for Fano varieties of K3 type started in [24], as a chapter of the quest for Fano varieties of Calabi–Yau type initiated from [33]. We refer to these papers for details on the motivations, both from the side of IHS manifolds and of Fano varieties. Our main objects

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of study here are Fano fourfolds that can be obtained as zero loci of general sections of fully decomposable equivariant vector bundles on products of flag manifolds.

With the help of computer calculations, we produced a huge database containing thousands of families of Fano fourfolds. We computed the main numerical invariants thereof, yielding at least 634 distinct families. From this database we extracted those Fano fourfolds with Hodge number $h^{3,1}=1$, which is the main numerical condition for being of K3 type. This leads to a list of 64 families, and the main goal of this paper is to study these families in some detail.

Being constructed inside products, these fourfolds admit natural projection maps which are usually either Mori fibrations (often conic or quadric bundles), or birational transformations. In the latter case, they can roughly be described as blowups of rational surfaces inside cubic fourfolds or Gushel–Mukai fourfolds, or as blowups of (sometimes non-minimal, meaning blown-up at some points) K3 surfaces inside Fano fourfolds with $h^{3,1} = 0$ (often, but not always, rational varieties). In fact, there are only four families of Fano fourfolds in our list which do not admit such a birational description. We thus get a rather colorful zoo of Fano fourfolds with interesting geometric properties.

In particular, most of the K3 surfaces involved are nicely embedded into rational manifolds. We recover classical examples of degree 7 in $\mathbb{P}^4$ or degree 5 in $\mathbb{P}^5$, studied in [57, 58], and meet other ones of a similar flavor. Moreover, in many cases our fourfolds admit several birational contractions, and we obtain without much effort very nice birational transformations of simple rational manifolds, sometimes *special* in the sense of [1, 19]. For example, from case Fano K3–33, we construct a birational involution of the four-dimensional quadric $\mathbb{Q}^4$, whose exceptional locus is a K3 surface of genus 7 blown up at two points. This is very reminiscent of the famous birational involution of $\mathbb{P}^3$ defined by the cubic equations of a genus 3 curve of degree 6 [42]. Of course more examples with similar properties (but not involving K3 surfaces) are to be found in the part of the database that we temporarily left aside, and that we plan to exploit in the next future. More generally, we get many Sarkisov links in dimension 4, and we tried to present our database not only as a mere list of items, but rather as an ecosystem where these geometric objects interact through natural transformations.

Another very interesting question that we shall touch upon in this paper is the possibility to write down explicit semiorthogonal decompositions for the derived category of the Fano fourfolds presented here. Recall that such decompositions could be fundamental for example in studying birational properties, or moduli spaces of bundles, or complexes on the varieties in question.

Beside the several aspects mentioned above, the interested reader might be eager to know what else our list of Fano varieties can be used for and, more specifically, what are the different angles they can look at this paper from. Depending on their taste, the reader may find of interest, for instance: the toolbox we developed to give a systematic interpretation of the geometry of zero loci of sections of homogeneous vector bundles over products of flag varieties; the web of birational correspondences between many of the fourfolds to be met in this paper; the rationality questions we address concerning our varieties and their link with derived categories; the four Fano varieties for which the K3 structure seem to be induced by particular singular fourfolds. We refer to Section 2.6 for more details on these aspects.

An important problem that we have to mention, but that will not be visible in the sequel, is that we have been confronted to many kinds of Fano fourfolds in our database, with several different descriptions as zero loci of vector bundles. We had to find identifications that in some cases were not obvious at all (see Example 2.9 for a flavor). Even more seriously, we were not able to construct these identifications in a systematic and complete way. In particular we are not able to bound a priori the Picard ranks and the dimensions of our ambient manifolds, and even less to guarantee that our lists are complete. It would nevertheless be very interesting to be able to carry over a complete classification along the lines followed, in similar but simpler settings, in [9, 35, 44].

This is all the more tempting than we can suspect that the examples of smooth Fano fourfolds in the larger database could provide a relevant part of an eventual future full classification. For Fano threefolds this is the point made in [20]. In dimension 4 there is an overlap with the list of families of Fano subvarieties of quiver flag varieties found in [39], since the latter can be constructed as towers of Grassmann bundles; a geometric analysis of most relevant cases of [39] has been performed in [25]. This overlap could be further clarified by comparing the quantum periods, which for quiver flag zero loci were computed in the Appendix of [39]. In our setting this computation could also be carried out, using the equivariant quantum Lefschetz principle, and the results should be compared to those coming from the other main source of examples of Fano manifolds, namely from the toric world [18]. It would in particular be extremely interesting to understand whether the property of being Fano of K3 type can be read on the quantum periods.

The paper is structured as follows. In Section 2 we report on the general context and summarize our main results, giving an insight on the methods and more details on the motivations behind this work. In Section 3 we compile a few classical tools and results we take advantage of in the subsequent sections. In Sections 4, 5, 6 and 7 we pursue the investigation of our 64 families of FK3, following the convenient classification we presented above. Appendix A contains a few families of Fano fourfolds which are not FK3 themselves, but whose Hodge structure in middle dimension contains a proper K3 substructure. In Appendix B we provide some background and references for the semiorthogonal decompositions we exhibit in our examples. Finally, in Appendix C we present tables summarizing the relevant data for the FK3 under investigation in this paper.

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2. Context and results

2.1. Fano varieties of K3 type

We begin with the definition of *Fano varieties of K3 type* (abbreviated FK3), which is purely Hodge-theoretical. Recall first the very classical notion of *level* of a Hodge structure.

Definition 2.1. — *Let $H = \bigoplus_{p+q=k} H^{p,q}$ be a non-zero Hodge structure of weight k. The* level $\lambda(H)$ *of H is*

$$\lambda(H) \coloneqq \max\{q - p \mid H^{p,q} \neq 0\}.$$

The level of a Hodge structure is a measure of its complexity. It is conventionally set to $-\infty$ if $H = 0$, and it is equal to zero if and only if H is central (or of *Hodge–Tate type*), i.e. $H^{p,q} \neq 0$ only for $p = q$.

We now introduce the key notion of Hodge structure of K3 type (see [24, 33]).

Definition 2.2. — *Let H be a Hodge structure of weight k. We define H to be of K3 type if*

$$\lambda(H) = 2 \qquad \text{and} \qquad h^{\frac{k+2}{2},\frac{k-2}{2}} = 1.$$

Finally, we define a FK3 as follows.

Definition 2.3. — *Let X be a smooth Fano variety. We say that X is a Fano variety of K3 type (abbreviated FK3) if $H^*(X,\mathbb{C})$ contains at least one sub-Hodge structure of K3 type, and at most one for each given weight.*

For Fano fourfolds, this simplifies into the condition that $h^{3,1} = 1$.

2.2. Fano fourfolds of K3 type of index bigger than 1

Fano fourfolds X of index $\iota_X > 1$ have been classified, see e.g. [38, Chapter 5], or [17, 4-6] for an overview. In short, there are 35 such families: one of index 5, one of index 4, six of index 3 (out of which five with $\rho = 1$) and 27 of index 2 (out of which nine with $\rho = 1$). Among these there are three well-known families of FK3 varieties: the cubic fourfolds, the Gushel–Mukai (GM for short) fourfolds, and the Verra fourfolds (double covers of $\mathbb{P}^2 \times \mathbb{P}^2$ ramified over a divisor of bidegree $(2,2)$). Note that an important difference between the cubic or GM fourfolds and the Verra fourfolds is that for the general member of the former two families, the derived categories (or Hodge theory) cannot be equivalent to the one of a K3 surface [49, 52]. On the other hand the derived category of a Verra fourfold can be realized in two different ways as the (Brauer-twisted) derived category of a K3 surface naturally occurring in the construction [41].

We checked that among these 35 families there are no other FK3.

2.3. Fano fourfolds of K3 type of index 1

Fano fourfolds X of index $\iota_X = 1$ are of course not classified. An important source of examples is Küchle's list [44]; only three FK3 can be found in there, namely:

(c5) $$\mathscr{Z}\big(\mathrm{Gr}(3,7), \textstyle\bigwedge^2 \mathcal{U}^\vee \oplus \mathcal{Q}^\vee(1) \oplus \mathcal{O}(1)\big);$$

(c7) $$\mathscr{Z}\big(\mathrm{Gr}(3,8), \textstyle\bigwedge^3 \mathcal{Q} \oplus \mathcal{O}(1)\big);$$

(d3) $$\mathscr{Z}\big(\mathrm{Gr}(5,10), (\textstyle\bigwedge^2 \mathcal{U}^\vee)^{\oplus 2} \oplus \mathcal{O}(1)\big).$$

Convention. Here and in all the sequel the zero locus of a general section of a globally generated vector bundle $\mathcal{F}$ on a variety Y will be denoted by $\mathscr{Z}(Y, \mathcal{F})$, or simply by $\mathscr{Z}(\mathcal{F})$ when no confusion can arise. In particular $\mathscr{Z}(\mathcal{F})$ is smooth of codimension equal to the rank of $\mathcal{F}$ (or empty). Moreover its canonical bundle is given by the adjunction formula. It will always be implicit in all the examples below that we only consider general sections, and we will never discuss this genericity condition in more detail. (Note that we will not discuss the question of the completeness of our families.)

If (c5) is still mysterious, (c7) and (d3) have been studied by Kuznetsov in [51], where he showed how they are (respectively) the blowup of a cubic fourfold in a Veronese surface, and the blowup of $(\mathbb{P}^1)^4$ in a K3 surface. We do not include (c5) and (c7) in our list; we only briefly mention (d3), which we denote by Fano K3–60, since we add some extra information other than what is already contained in the two aforementioned works.

Two other examples of FK3 fourfolds were included in [24], with labels (B1) and (B2) therein. Since they were only sketched in the cited work, we include them in our analysis. They are the Fano fourfolds Fano K3–30 and Fano K3–41.

Plentiful of four-dimensional Fano toric complete intersections were found in [18] (at least 738 families), and a few dozens more (at least 141) were constructed in [39] as quiver flag zero loci. As already mentioned it would be very interesting to compare these families with ours, and find how many families have been found altogether — probably a very significant proportion of the complete, still elusive list of Fano fourfolds. Moreover, having two different descriptions for the same variety allows one to get more information, as shown e.g. in [8].

2.4. FK3 in this paper: a recap

Once again, the main goal of this paper is to analyze 64 families of Fano fourfolds of K3 type not previously considered — or only briefly mentioned — in the literature. Given their description from zero loci of bundles in products of flag manifolds, it was not a surprise to check that all of them have Picard rank greater than 1. For each case, we will provide the main numerical invariants, and we will geometrically describe the projections to each factor and deduce semiorthogonal decompositions for their derived categories. This gives access to the geometric structure of the Fano fourfolds in each family.

We subdivided our examples into four disjoint groups, namely Fano varieties of **C, GM, K3** or **R** origin, inside which the examples are listed, roughly, by increasing complexity. Fano varieties of **C** origin admit at least one birational map to a cubic fourfold, either special or general. Similarly, Fano fourfolds of **GM** origin admit at least one birational map to a Gushel–Mukai fourfold, either special or general. We remark that the two sets are disjoint; in other words, no Fano fourfold in our list has a birational morphism onto both a cubic and a GM fourfold.

Fano varieties of **K3** origin are varieties which do not belong to any of the two previous groups, but can be realized as blowups of other Fano varieties with central Hodge structure along a (sometimes non-minimal) K3 surface. Varieties in each of these first three groups share many properties among each other, and we can often understand their K3 structure from the birational maps in question.

Finally, there is a small bunch of Fano varieties (of Picard rank 2) of **R** (or *rogue*) origin that do not fall in any of the previous categories. They are in fact birational to singular fourfolds, and we can understand their K3 structures from one of their projections (either a dP_5-fibration or a conic bundle). Summing up, we have this following convenient classification:

- 17 Fano fourfolds of **C** origin;
- 6 Fano fourfolds of **GM** origin;
- 37 Fano fourfolds of **K3** origin;
- 4 Fano fourfolds of **R** origin.

However, thinking of the more ambitious task of classifying Fano fourfolds without restricting to those of K3 type, such a notation is certainly not a good idea. We introduced an identification number for each Fano fourfold that we hope can carry over to any larger database: we denoted by ($\sharp\rho$-s-v-n-?) a family of Fano fourfolds X of Picard rank ρ, with $h^0(-K_X) = s$, $(-K_X)^4 = v$ and Hodge level n. The last symbol "?" is needed only in case the first four numbers do not identify a unique deformation class of fourfolds in our list, and it is a capital latin letter.

Remark 2.4. — The study of the natural projections on the factors of the ambient variety of the Fano varieties presented here is particularly relevant for the study of extremal contractions of Fano fourfolds (cf. [54, Chapter 8]). In particular, when $h^{1,1}(X) = 2$ for such an X, the restrictions of the two projections give the two extremal contractions: indeed, if the two projections contract at least a curve, say C and D respectively, the pullbacks L and M of ample bundles verify $L.C = M.D = 0$, from which one can deduce that $\mathrm{Eff} = \langle C, D\rangle$ and $\mathrm{Nef} = \langle L, M\rangle$. The other conditions (normality of the target and connectedness of the fibers) can be checked case-by-case on all relevant examples. When $h^{1,1} > 2$, X may have other extremal contractions not induced by the projections.

Along the study of our examples, there are a few instances of birational maps which are quite special, as the contracted divisors have a curve or a point as image.

Remark 2.5. — No Fano fourfold in our list has a birational morphism to either Küchle's (c5) fourfolds or to the *Verra* fourfolds. In the (c5) case, it is likely that the low index prevents any blowup from having ample anticanonical class. For *Verra* fourfolds the situation is different. In fact, the index is 2, and it seems conceivable that some blowup could be Fano. However, the Verra fourfolds themselves can only be described, to the best of our knowledge, as zero loci of sections of a non-completely reducible vector bundle. To see this, consider the zero locus of a general section of $\mathcal{Q}_{\mathbb{P}^n}(0,1)$ on $\mathbb{P}^n \times \mathbb{Q}^n$; the second projection is an isomorphism onto $\mathbb{Q}^n$, while the first projection to $\mathbb{P}^n$ is a double cover branched over a quadric. Letting $n = 8$ and restricting to $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, we thus get the Verra fourfolds as zero loci of general sections of $\Lambda(0,0,1) \oplus \mathcal{O}(0,0,2)$ on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^9$, where the rank 8 vector bundle Λ is the restriction of $\mathcal{Q}_{\mathbb{P}^8}$ to $\mathbb{P}^2 \times \mathbb{P}^2$: a homogeneous but not completely reducible bundle.

A particularly intriguing question about FK3 varieties is about the generalization of Kuznetsov's conjecture on the cubic fourfold, which is expected to be rational whenever its Kuznetsov component is equivalent to the derived category of a K3 surface. One could ask a similar question for each of the 64 families in our list.

For Fano fourfolds of **C** and **GM** origin, there is no surprise: we either recover the rationality of known special examples, or we are unable to conclude. For almost all Fano fourfolds of **K3** origin, we can easily decide about their rationality. There are two families however, for which we cannot. In particular, Fano K3–26 is rational if and only if the prime index 2 Fano fourfold X_{16} of degree 16 is (which is currently unknown). For Fano K3–36, rationality would imply the stable rationality of a smooth cubic threefold, which is another famously open problem.

For two out of the four families of **R** origin, we are not able either to determine their rationality. The exceptions are the case Fano R–61 and Fano R–63, where we expect the Fano fourfold to contain in its derived category a copy of the derived category of a K3 surface blown up in three points. In the case of Fano R–63 and Fano R–64, we can identify a copy of the category of a smooth K3 surface in the derived category. We have only a partial understanding of the K3 structure of Fano R–62. On the other hand all these fourfolds are birational to some singular fourfolds defined explicitly. The question of their rationality could therefore turn out to be quite interesting, and a good testing ground for a generalization of Kuznetsov's conjecture.

Finally, we thought it would be useful for the reader to have a visualization of the various birational links we found among all our Fano fourfolds. The tables at the end of this paper give a recap of this and many other pieces of information on the Fano varieties analyzed in this paper.

2.5. Construction of the database

We insist that the leading role in this manuscript is played by Fano fourfolds X of K3 type satisfying the following.

Working hypothesis 2.6. — *X admits a description as the zero locus of a general global section of a globally generated completely reducible homogeneous bundle over a product of flag varieties, together with a strong amplitude condition of the anticanonical bundle, see assumption* (2.1) *below.*

These bundles can be conveniently built by taking direct sums of irreducible bundles, for which a combinatorial description is easy to handle.

More in general, for a pair $(Y, \mathcal{F})$ with Y a product of flag varieties and $\mathcal{F}$ a vector bundle on Y as above, we can consider the n-dimensional zero locus X of a general global section of $\mathcal{F}$. We can make use of classical tools such as the Hirzebruch–Riemann–Roch Theorem and the Borel–Weil–Bott Theorem, and Koszul complexes, to (at least partially) determine the following data:

$$h^0(-K_X), \quad (-K_X)^n, \quad h^{i,j} = h^i(X, \Omega_X^j), \quad \chi(T_X).$$

For a more detailed explanation of the steps to be performed for this sake, we refer to [20, Section 3]. We remark that the determination of the Hodge numbers $h^{i,j}$ is based on the computation of Euler characteristics from several cohomology sequences. If one is unfortunate, this can give only partial results, especially when $\mathcal{F}$ has several summands. We are nonetheless able to determine all the Hodge numbers for the Fano varieties we deal with in this paper.

The search for Fano fourfolds satisfying Working hypothesis 2.6 and the computation of their invariants can be efficiently implemented on computer algebra softwares. In order to construct our database we took advantage of the computer algebra software [26] and its packages [27, 66].

The reader may wonder whether a classification of all Fano fourfolds satisfying Working hypothesis 2.6 is possible. This was our initial, a posteriori very optimistic aim. However, a first serious obstacle occurs when one realizes that there exist varieties $\mathscr{Z}(Y,\mathcal{F})$ whose anticanonical bundle is ample, even if $\omega_Y \otimes \det(\mathcal{F})$ is not, as the following examples show.

Example 2.7. — Consider $Y = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and let $\mathcal{F} = \mathcal{O}(1,1,0,0,0) \oplus \mathcal{O}(1,0,1,0,0) \oplus \mathcal{O}(1,0,0,1,0) \oplus \mathcal{O}(1,0,0,0,1)$. Then

$$\omega_Y \otimes \det(\mathcal{F}) = \mathcal{O}(1,-1,-1,-1,-1),$$

which is clearly not anti-ample on Y. However, the zero locus of a general section of $\mathcal{F}$ is, by [20, Lemma 2.8], a del Pezzo surface of degree 5.

Example 2.8. — We have seen in Remark 2.5 that the zero locus of a general section of $\mathcal{Q}_{\mathbb{P}^n}(0,1)$ on $\mathbb{P}^n \times \mathbb{Q}^n$ is isomorphic to $\mathbb{Q}^n$, which we can thus also obtain as the zero locus of a general section of $\mathcal{F} = \mathcal{Q}_{\mathbb{P}^n}(0,1) \oplus \mathcal{O}(0,2)$ on $Y = \mathbb{P}^n \times \mathbb{P}^{n+1}$. But the bundle $\omega_Y \otimes \det(\mathcal{F}) = \mathcal{O}(n,0)$ is not ample, although it will of course become ample when restricted to $\mathbb{Q}^n$.

For this reason, in our search and in this paper we only deal with Fano fourfolds $X = \mathscr{Z}(Y,\mathcal{F})$ such that

$$\omega_Y \otimes \det(\mathcal{F}) \text{ is ample.} \tag{2.1}$$

Another serious problem is that a given fourfold X can have several incarnations as such zero loci, and even infinitely many as the next example shows.

Example 2.9. — Consider $X = \mathbb{P}^1 \times \mathbb{P}^{d-1}$. This is the zero locus of a general section of $\mathcal{F} = (\mathcal{Q} \boxtimes \mathcal{U}^\vee)^{\oplus 2}$ on $Y = \mathbb{P}^{m-1} \times \mathbb{P}^{m+d-1}$ for any $m \geqslant d$. Indeed, these sections are defined by two general morphisms $s,t\colon V_{m+d} \to V_m$, and with a little abuse of notation they vanish at (x,y) when $s(y),t(y) \subset x$. In particular there exist scalars σ,τ, not both zero, such that $\sigma s(y)+\tau t(y) = 0$. Moreover the point $[\sigma,\tau]$ in $\mathbb{P}^1$ is unique since when $m \geqslant d$, the kernels of s and t intersect trivially. So $\mathscr{Z}(Y,\mathcal{F})$ is the projectivization of a kernel bundle of rank d on $\mathbb{P}^1$, and this kernel bundle is trivial since $\sigma s + \tau t$ is everywhere surjective. Hence the claim.

We found many instances of such phenomena, sometimes organizing into infinite series like in the previous example, sometimes more sporadic (see Fano GM–20 for an example). A systematic understanding of these redundancies would be necessary for a classification, at least when (2.1) holds, but remains out of reach for the time being and falls outside the scope of this paper.

2.6. What can the reader do with this list?

There are several angles from which a reader could look at this paper. For example, people interested in the study of the (at the moment untamed) geography of Fano fourfolds could think of this paper as a *toolbox*. There are several attempts to classify, or at least gain a partial understanding of these varieties, including our own *vector bundle search* method as explained in the introduction, or for example via mirror symmetry. The problem often is not only to produce lists of examples, but to understand them, and in a systematic way. The tools that the second and fourth author started developing in [20] and that we broaden here in Section 3 and throughout the whole paper will be useful to describe and characterize whichever Fano (in dimension 4 and above) we will encounter from now on.

Moreover, we believe that one of the merits of the present paper is to highlight the web of birational correspondences between many of the fourfolds we present. This matches, at least in spirit, the style of Mori–Mukai original classification of Fano threefolds with Picard rank $\rho > 1$. We tried to make these links as visually clear as possible by summarizing them in the tables at the end of the paper. More information can also be extracted from the actual description of the examples.

The reader keen in understanding *rationality questions* (especially in connection with *derived categories*) could feel intrigued by some special examples in our list and their connection with a generalized version of Kuznetsov's conjecture on the rationality of the cubic fourfold. In fact, many of our examples are both (sometimes evidently) rational, and with their derived category containing (sometimes less evidently) a copy of the derived category of a K3 surface. There are however some exceptions, in which our understanding is less complete and they give rise to many interesting questions.

For example, in the cases Fano R–63 we understand the K3 category as a subcategory coming from an actual K3 surface, but we cannot conclude anything (yet) about its rationality. There are as well examples where again the K3 category is geometrical (or *commutative*), but the rationality of the examples would be equivalent to solving known open problems. For instance, determining the rationality of Fano K3–26 would imply the rationality of the Fano fourfold X_{16}, while the rationality of Fano K3–36 and Fano R–64 would yield the stable rationality of the cubic threefold. Finally, if we look at Fano R–62, there is an interesting link between our work and the recent work of Hassett–Pirutka–Tschinkel [30]. In fact, they showed how there exists a family of smooth complete intersections of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ whose very general fiber is not even stably rational, whereas some special fiber are rational. In our case Fano R–62 is in fact birational to a *singular* complete intersection of this type. It would be interesting to push this analogy further and say something similar for our family, at the rationality level.

Any Hodge theorist or categorist interested in the study of *K3 Hodge structures* or *K3 categories* may feel the urge to check the most unexpected cases, that is, those which we call *rogue* listed in Section 7. In particular, we invite the interested reader to think of how these K3 structures seem induced by singular fourfolds, and how the study of special singular fourfolds could lead to the discovery of new, unexpected, FK3 varieties in future.

Last but not least, the *elephant in the room* is the absence in this paper of any projective families of *irreducible holomorphic symplectic manifolds* (IHS, or hyperkähler) constructed from these Fano varieties of K3 type. In fact, we could speculate that to many of these Fano varieties one could associate (possibly non-locally complete) families of IHS, very likely of $\mathrm{K3}^{[n]}$ type, for example considering moduli spaces of appropriate objects contained in these FK3. This is definitely a direction worth pursuing, and which will hopefully lead to many other interesting geometric constructions in the future.

2.7. Notation

We denote by $\mathbb{P}^n$ the projective space, and by $\mathbb{Q}^n$ the smooth quadric of dimension n. (When there are several, we distinguish them by an index.) The Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$ is denoted $\mathrm{Gr}(k,n)$. More generally, $\mathrm{Fl}(k_1,k_2,n)$ denotes the flag manifold parametrizing flags of subspaces of dimensions $k_1 < k_2$ in $\mathbb{C}^n$ (and a similar notation is used for more general flags). As in the case of projective case, the projectivization of a vector bundle $\mathbb{P}(\mathcal{E})$ will denote the space of 1-dimensional subspaces in the fiber.

A fixed i-dimensional vector space is denoted V_i, and a moving subspace of dimension j is denoted U_j, except for lines which we usually denote by l, m, etc.

General vector bundles are denoted by $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, and line bundles by $\mathcal{L}$, $\mathcal{M}$, etc. The tautological and quotient vector bundles on $\mathrm{Gr}(k,n)$, whose ranks are k and $n-k$, are denoted by $\mathcal{U}$ and $\mathcal{Q}$ respectively. Over $\mathrm{Fl}(k_1,k_2,n)$, we denote by $\mathcal{Q}_2$ the pull-back of the quotient vector bundle from the projection to $\mathrm{Gr}(k_2,n)$. We will occasionally use $\mathcal{R}_i$ to denote the tautological bundle $\mathcal{U}_i/\mathcal{U}_{i-1}$. As already discussed, the zero locus of a general section of a globally generated vector bundle $\mathcal{F}$ on a variety Y will be denoted by $\mathscr{Z}(Y,\mathcal{F})$, or simply by $\mathscr{Z}(\mathcal{F})$ when no confusion can arise

The Fano fourfolds from our database are denoted X in each example, while Y,Z are usually some auxiliary fourfolds. When they are singular we add a $\circ$, typically Y°, Z°, except for a projective cone that we rather denote $C(Y)$ or $C(Z)$, with vertex v. We add a $\sim$ when we blow up a variety already encountered in a previous example. In many cases our Fano fourfold X admits a quadric bundle structure, and we denote the discriminant by Δ.

We also use the notation X_d for a Fano fourfold of degree d: typically, X_3 for a cubic fourfold and X_{10} for a Gushel–Mukai fourfold, that we freely abbreviate into GM fourfold. Notice that in the case of a prime Fano, the *classical* degree coincides with the anticanonical degree, up to $\iota^{\dim X}$.

Fano threefolds of degree d are denoted by W_d. Similarly, K3 surfaces of degree d are denoted by S_d, and a blowup in k points of a K3 surface of degree d is denoted by $S_d^{(k)}$. Also, a del Pezzo surface of degree d is denoted by dP_d, and a scroll or a Hirzebruch surface of degree d by Σ_d. Finally, a curve of degree d is denoted by C_d.

3. A few preliminaries

In this section we compile a few classical tools that will be used multiple times in the study of our examples. Unless otherwise specified, with *blowup* we mean the blowup of a smooth irreducible variety with center a smooth subvariety.

3.1. Cayley tricks

We start with some natural birational transformations naturally showing up in our settings.

LEMMA 3.1. — *Let X be a smooth projective variety with a globally generated line bundle $\mathcal{L}$, and consider $Y := \mathscr{Z}(\mathbb{P}^m \times X, \mathcal{Q}_{\mathbb{P}^m} \boxtimes \mathcal{L})$. Then $Y \cong \mathrm{Bl}_Z X$, where $Z = \mathscr{Z}(X, \mathcal{L}^{\oplus m+1})$ and Y and Z are smooth for a general choice of a section. In particular $Y \cong X$ if and only if $m \geqslant \dim X$.*

Proof. — Pick a basis of $v_0, \dots, v_m$ of $\mathbb{C}^{m+1}$. A section $\sigma \in H^0(\mathbb{P}^m \times X, \mathcal{Q}_{\mathbb{P}^m} \boxtimes \mathcal{L})$ is given by $\sigma = \sum v_i \otimes s_i$ for some sections $s_0, \dots, s_m$ of $\mathcal{L}$, and σ vanishes at $(l, x) \in \mathbb{P}^m \times X$ if and only if $\sum s_i(x) v_i$ belongs to $l \otimes \mathcal{L}_x$. This implies that the projection to X is an isomorphism outside the base locus $Z = \mathscr{Z}(s_0, \dots, s_m)$, which is smooth in general when $\mathcal{L}$ is globally generated. The special fibers over Z are isomorphic to $\mathbb{P}^m$. So we end up with a birational morphism $f\colon Y \to X$ whose irreducible exceptional divisor $f^{-1}(Z)$ is a $\mathbb{P}^m$-bundle over a smooth locus Z, and we can conclude by [23, Theorem 1.1]. □

In the above lemma, picking $m = 1$ results in a certain (codimension 2) blowup being described as the zero locus of a line bundle in a product space. There is a well-known generalization to this, under the name of *Cayley trick* (see [43, Theorem 2.4], or [33, 3.7]), which admits both a Hodge-theoretical and a derived categorical statement, and can be considered as a generalization of the formula for the Hodge structure of the blowup and Orlov's formula for the derived category of blowups.

To be explicit, assume that we have $Y = \mathscr{Z}(G, \mathcal{F})$, where $\mathcal{F}$ is ample of rank $r \geqslant 2$. The natural isomorphism $H^0(G, \mathcal{F}) \cong H^0\big(\mathbb{P}(\mathcal{F}^\vee), \mathcal{O}_{\mathbb{P}(\mathcal{F}^\vee)}(1)\big)$ yields that the section defining Y also defines a hypersurface X in $\mathbb{P}(\mathcal{F}^\vee)$. It follows that X admits a stratified projective bundle structure over G, with generic fiber $\mathbb{P}^{r-2}$, and special fibers $\mathbb{P}^{r-1}$ over Y, both locally trivial over their strata. This implies that the Hodge structure of X can be described in terms of those of Y and G, see [14, Proposition 48], and as well that the derived category of X admits a semiorthogonal decomposition containing $r-1$ copies of $D^b(Y)$. When the rank of $\mathcal{F}$ is exactly 2, this produces a generalization of the above lemma. In our paper, this phenomenon happens with $r = 2$, as in Fano C–6. In particular, we have the following lemma (where we remark that Y — and therefore X — are smooth for a general choice of a section).

Lemma 3.2 ([33, 3.7],[43, 3.2]). — *Let X be $\mathscr{Z}\big(\mathbb{P}_G(\mathcal{F}^\vee), \mathcal{O}_{\mathbb{P}}(1)\big)$, with $\mathcal{O}_{\mathbb{P}}(1)$ being the relative ample line bundle, G smooth and $\mathcal{F}^\vee$ of rank 2. Then $X \cong \operatorname{Bl}_Y G$, with $Y = \mathscr{Z}(G, \mathcal{F})$.*

3.2. Ubiquity

An important number of ambiguities in our models of Fano fourfolds come from the bundle $\mathcal{Q} \boxtimes \mathcal{U}^\vee$ on a product of Grassmannians. The next statement gives a few identifications that will be useful in the sequel.

Proposition 3.3. — *Consider* $\mathscr{Z} = \mathscr{Z}\big(\mathrm{Gr}(a,m) \times \mathrm{Gr}(b,n), \mathcal{Q} \boxtimes \mathcal{U}^\vee\big)$.

(1) *If* $m \leqslant n$, *then* $\mathscr{Z}$ *is isomorphic to the Grassmann bundle* $\mathrm{Gr}(b, \mathcal{U} \oplus \mathcal{O}^{\oplus n-m})$ *over* $\mathrm{Gr}(a,m)$. *It can also be seen as a general zero locus* $\mathscr{Z}\big(\mathrm{Fl}(b, a+n-m, n), \mathcal{Q}_2^{\oplus n-m}\big)$. *In particular* $\mathscr{Z}$ *is empty if* $a+n < b+m$.

(2) *If moreover* $a = b$, *then the second projection* $\mathscr{Z} \to \mathrm{Gr}(b,n)$ *is birational, with fiber isomorphic to* $\mathrm{Gr}(d, n-a+d)$ *over* $B \in \mathrm{Gr}(b,n)$ *meeting some fixed* K_{n-m} *in dimension* d.

(3) *If* $a = b$ *and* $m = n-1$, *then* $\mathscr{Z} \cong \mathrm{Bl}_{\mathrm{Gr}(a-1,n-1)} \mathrm{Gr}(a,n)$.

Proof. — We start with (1). A section of $\mathcal{Q}\boxtimes\mathcal{U}^\vee$ is defined by a morphism $\alpha \in \mathrm{Hom}(V_n, V_m)$. If $n \geqslant m$ this morphism is in general surjective and we can choose an isomorphism $V_n \simeq V_m \oplus K_{n-m}$ such that α is just the projection to the first factor. Then $\mathscr{Z}$ is variety of pairs (A,B) such that $\alpha(B) \subset A$, or equivalently, $B \subset \alpha^{-1}(A) = A \oplus K_{n-m}$. Hence the first claim. This is also the variety of flags $B \subset C \subset V_n$ such that $C \supset K_{n-m}$, which is exactly the zero locus of a general section of $\mathscr{H}om(K_{n-m}, \mathcal{Q}_2)$. Hence the second claim.

When $a = b$, the condition $\alpha(B) \subset A$ implies that $\alpha(B) = A$ when B is transverse to K_{n-m}, so the second projection is birational. Then (2) readily follows.

Finally (3) is [20, Lemma 2.2]. □

3.3. Torelli

In several cases we will get a fourfold X of K3 type which is a blowup in several different ways, and we will use the following statement.

Lemma 3.4. — *Let* X *be isomorphic both to the blowup of a smooth fourfold* Y *along a (possibly non-minimal) K3 surface* S *and to the blowup of a fourfold* Z *along a (possibly non-minimal) K3 surface* T. *If* $H^{2,2}(Y)$ *and* $H^{2,2}(Z)$ *have no transcendental part, then the two K3 surfaces* $S_{\min}$ *and* $T_{\min}$ *are derived equivalent.*

Proof. — Recall that two minimal K3 surfaces are derived equivalent if and only if there is a Hodge isometry between their transcendental lattices (see, e.g., [31, 16, Corollary 3.7]). Under our assumptions, the blowup formulae identify the transcendental lattices of $S_{\min}$ and $T_{\min}$ with the transcendental lattice of $H^{2,2}(X)$. □

3.4. Conic bundles

Assume that $f\colon X \to B$ is a conic bundle, given by a line bundle $\mathcal{K}$, a rank 3 bundle $\mathcal{E}$ on B and a bilinear map $\mathcal{K}^\vee \to \mathrm{S}^2\,\mathcal{E}^\vee$. This is a standard way of defining a conic bundle, see for example [63]. In this case the degeneracy loci of the conic bundle are given by the degeneracy loci of the induced morphism $\varphi\colon \mathcal{E} \to \mathcal{E}^\vee \otimes \mathcal{K}$. Since $\mathcal{E}$ has rank 3 there are (under the usual genericity conditions) only two degeneracy loci in B, namely the discriminant divisor Δ, and its codimension 2 singular locus Δ_{sing}. We can use the formula in [28, Theorems 1 and 10] to compute the cohomology classes of Δ and Δ_{sing}.

Lemma 3.5. — *In the above situation, let $k \coloneqq c_1(\mathcal{K})$ and $c_i \coloneqq c_i(\mathcal{E})$. Then*

$$[\Delta] = 2c_1 + 3k \qquad \text{and} \qquad [\Delta_{\mathrm{sing}}] = 4(k^3 + 2k^2c_1 + kc_1^2 + kc_2 + c_1c_2 - c_3).$$

These formulae will apply to many of our examples.

3.5. Eagon–Northcott complexes

We will use several times the following classical result (see, e.g., [67, (6.1.6)], [10, Example 2.8]).

Proposition 3.6. — *Let $\varphi\colon \mathcal{E} \to \mathcal{F}$ be a morphism of vector bundles on X, of respective ranks $e > f$. Suppose that its degeneracy locus Z (defined as a scheme by the f-th minors of φ) has the expected codimension $e - f + 1$. Then the $\mathcal{O}_X$-module $\mathcal{O}_Z$ admits a locally free resolution given by the Eagon–Northcott complex*

$$0 \longrightarrow \bigwedge^{e} \mathcal{E} \otimes S^{e-f}\mathcal{F}^\vee \otimes \det \mathcal{F}^\vee \longrightarrow \cdots \longrightarrow \bigwedge^{f+1} \mathcal{E} \otimes \mathcal{F}^\vee \otimes \det \mathcal{F}^\vee \longrightarrow$$
$$\longrightarrow \bigwedge^{f} \mathcal{E} \otimes \det \mathcal{F}^\vee \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

In our situations Z will be smooth (typically because the rank will not be allowed to drop more for dimensional reasons), and we will resolve its canonical bundle by dualizing the Eagon–Northcott complex. It turns out that in many cases the dual complex will have a special shape that will allow us to apply the next simple, but very useful statement.

Lemma 3.7. — *Let $Z \subset X$ be a smooth subvariety, with a line bundle $\mathcal{L}_Z$ that can be described, as an $\mathcal{O}_X$-module, as the cokernel of a morphism of vector bundles on X, say*

$$\mathcal{A} \longrightarrow \mathcal{B} \oplus \mathcal{M} \longrightarrow \mathcal{L}_Z \longrightarrow 0,$$

where $\mathcal{M}$ is a line bundle. Then the morphism $\mathcal{M} \to \mathcal{L}_Z$ defines a canonical section σ of $\mathcal{M}^\vee\big|_Z \otimes \mathcal{L}_Z$, whose zero locus is exactly the locus where the morphism $\mathcal{A} \to \mathcal{B}$ is not surjective.

Proof. — Since $\mathcal{L}_Z$ is the cokernel of the morphism $\theta\colon \mathcal{A} \to \mathcal{B} \oplus \mathcal{M}$, the section σ vanishes exactly when the fiber of $0_{\mathcal{B}} \oplus \mathcal{M}$ is contained in its image. If it does, the component $\theta_1\colon \mathcal{A} \to \mathcal{B}$ cannot be surjective, since otherwise θ would itself be surjective. Conversely, if θ_1 is not surjective at some point, the induced morphism $\operatorname{Ker}\theta_1 \to \mathcal{M}$ must be surjective, since the cokernel of θ would have rank bigger than 1 at any point where this fails. □

3.6. Derived categories

A natural question to ask about a Fano variety X is to describe, and possibly compare, semiorthogonal decompositions of $\mathrm{D}^{\mathrm{b}}(X)$. Due to the fact that $H^i(X, \mathcal{O}_X) = 0$ for $i \neq 0$, every line bundle is exceptional on a Fano variety. Moreover one can expect many exceptional vector bundles and objects in the derived category. Building exceptional sequences from such objects gives interesting semiorthogonal decompositions. In the particular case of Fano fourfolds of K3 type, there should exist at least one exceptional collection whose semiorthogonal complement is a K3 category, sometimes actually described by a K3 surface. In all the examples presented in this paper, we can use explicit geometric descriptions to provide semiorthogonal decompositions, and even several of them; we will avoid the interesting but a priori deeply technical question of comparing them.

We refrain from recalling here definitions and properties of semiorthogonal decompositions, for which the interested reader can consult [50]. We focus on the description of semiorthogonal decompositions of $\mathrm{D}^{\mathrm{b}}(X)$ for X a smooth projective variety. More generally, if X is singular, one can replace $\mathrm{D}^{\mathrm{b}}(X)$ by a so-called *(crepant) categorical resolution*, that is, a smooth and proper dg category $\mathcal{D}$ with a functor $\mathcal{D} \to \mathrm{D}^{\mathrm{b}}(X)$ whose homological properties are the ones of a functor induced by a geometric (crepant) resolution of singularities. We also refrain from introducing such topic, for which we refer again to [50].

For the sake of readability, we treat explicit examples and give tables with references in Appendix B. The main idea is to consider in each example surjective morphisms $\pi\colon X \to M$ from a Fano fourfold X to another variety M. We will face two cases: M is also a smooth fourfold, and π is a blowup; or the dimension of M is smaller, and π is a Mori fiber space, i.e. a map of relative Picard number 1 whose general fiber is a smooth Fano variety. In both cases, semiorthogonal decompositions can be constructed from the map π. We list all of them in Table B.1 (Fano and other rational varieties of dimension up to 4) and Table B.2 (Mori fibrations). In any case, given a Fano fourfold X, we describe natural exceptional collections whose complements, called *Kuznetsov components* (with respect to $\mathrm{D}^{\mathrm{b}}(X)$), contain the most interesting pieces of information on the geometry of X. Such components depend on the exceptional collection and are only conjecturally equivalent; moreover, in some cases (in particular Mori fiber spaces with high-dimensional fibers), the Kuznetsov component for the given exceptional collection is not known in general.

4. FK3 from cubic fourfolds

This section contains the list of the Fano fourfolds of K3 type that we obtained inside products of flag manifolds, where at least one projection is the blowup of a cubic fourfold. In this section as well as in the next sections, for each Fano fourfold we list its invariants, we study its geometric properties via the projections on the factors of the ambient variety, and we provide information on its rationality. Possible orthogonal decompositions are also exhibited, and we refer to Appendix B for the corresponding notation. We only list the Hodge numbers which are different from zero, of course avoiding to mention the Hodge symmetries and $h^{0,0}$.

Fano C–1

(#2-36-144-2) $\mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^5, \mathcal{O}(0,3) \oplus \mathcal{O}(1,1)\big)$

- Invariants:

$h^0(-K) = 36,\ (-K)^4 = 144,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 28,\ -\chi(T) = 28.$

- Description:
  - $\pi_1\colon X \to \mathbb{P}^1$ is a fibration in cubic threefolds.

  - $\pi_2\colon X \to \mathbb{P}^5$ is the blowup $\mathrm{Bl}_{\mathrm{dP}_3} X_3$ of a general cubic fourfold along a cubic surface.
  - Rationality: not known.
- Semiorthogonal decompositions:
  - 4 exceptional objects and an unknown category from π_1.
  - 12 exceptional objects and a K3 category from π_2.

Explanation

The description of π_1 is obvious. For π_2, simply apply Lemma 3.1.

Fano C–2

(#2-28-99-2) $\mathscr{Z}\big(\mathbb{P}^2 \times \mathbb{P}^5, \mathcal{Q}_{\mathbb{P}^2}(0,1) \oplus \mathcal{O}(0,3)\big)$

- Invariants:

$$h^0(-K) = 28,\ (-K)^4 = 99,\ h^{1,1} = 2,\ h^{1,2} = 1,\ h^{3,1} = 1,\ h^{2,2} = 23,\\ -\chi(T) = 29.$$

- Description:
  - $\pi_1\colon X \to \mathbb{P}^2$ is a dP_3 fibration.
  - $\pi_2\colon X \to \mathbb{P}^5$ is the blowup $\mathrm{Bl}_E X_3$ of a cubic fourfold along a plane elliptic curve E.
  - Rationality: unknown, since the cubic fourfold is general.
- Semiorthogonal decompositions:
  - 3 exceptional objects and an unknown category from π_1.
  - 3 exceptional objects, the K3 category of X_3, and two copies of $\mathrm{D}^{\mathrm{b}}(E)$ from π_2.

Explanation

We simply use Lemma 3.1 to identify the zero locus of the first bundle with $\mathrm{Bl}_{\mathbb{P}^2} \mathbb{P}^5$. Intersecting with the zero locus of a section of $\mathcal{O}(0,3)$ completes the identification, since it cuts the blown up plane in a cubic, hence elliptic curve.

Fano C–3

(#2-45-192-2) $\mathscr{Z}\big(\mathbb{P}^2 \times \mathbb{P}^5, \mathcal{O}(1,2) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,1)\big)$

- Invariants:

$h^0(-K) = 45,\ (-K)^4 = 192,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 22,\ -\chi(T) = 19.$

- Description:
  - $\pi_1\colon X \to \mathbb{P}^2$ is a quadric surface fibration degenerating along a smooth sextic.
  - $\pi_2\colon X \to \mathbb{P}^5$ is $\mathrm{Bl}_{\mathbb{P}^2} X_3$, the blowup of a cubic fourfold along a plane.
  - Rationality: unknown.
- Semiorthogonal decompositions:
  - 6 exceptional objects and $\mathrm{D^b}(S_2, \alpha)$, where $S_2 \to \mathbb{P}^2$ is the discriminant double cover associated to π_1 and α the Clifford invariant of π_1, a Brauer class of order 2 on S_2 (see Appendix B.2.2).

Explanation

The sections are defined by $\alpha \in V_3 \otimes V_6^\vee = \mathrm{Hom}(V_6, V_3)$ and $\beta \in V_3^\vee \otimes S^2 V_6^\vee$. They vanish at (l,m) if $\alpha(m) \subset l$ and $\beta(l,m,m) = 0$. This implies $\alpha\big(\beta(m), m, m\big) = 0$, which defines a cubic fourfold containing the plane $\mathbb{P}(\mathrm{Ker}\,\alpha)$. The fiber of the projection to $\mathbb{P}^5$ is a projective line over this plane, and a single point over the other points of the cubic fourfold.

Fano C–4

(#2-40-163-2) $\mathscr{Z}\big(\mathbb{P}^3 \times \mathbb{P}^5, \mathcal{O}(1,2) \oplus \mathcal{Q}_{\mathbb{P}^3}(0,1)\big)$

- Invariants:

$h^0(-K) = 40,\ (-K)^4 = 163,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 23,\ -\chi(T) = 24.$

- Description:
  - $\pi_1\colon X \to \mathbb{P}^3$ is a conic bundle, ramified over a singular quintic surface with 16 ordinary double points.
  - $\pi_2\colon X \to \mathbb{P}^5$ is the blowup $\mathrm{Bl}_{\mathbb{P}^1} X_3$ of a general cubic fourfold along a line.
  - Rationality: unknown.

- Semiorthogonal decompositions:
  - 4 exceptional objects and $\mathrm{D^b}(\mathbb{P}^3, \mathcal{C}_0)$ from π_1.
  - 7 exceptional objects and the K3 category from π_2.
  - We know that the K3 category sits inside $\mathrm{D^b}(\mathbb{P}^3, \mathcal{C}_0)$ as a complement to 3 exceptional objects [5].

Explanation

By [20, Corollary 2.7], the rank 3 bundle $\mathcal{Q}_{\mathbb{P}^3}(0,1)$ cuts in $\mathbb{P}^3 \times \mathbb{P}^5$ the blowup of $\mathbb{P}^5$ in a line. By construction a section of $\mathcal{O}(1,2)$ in this locus cuts a cubic in $\mathbb{P}^5$ containing this $\mathbb{P}^1$.

To understand the other projection, notice that $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^5 \cong \mathbb{P}_{\mathbb{P}^3}\big(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1)\big)$. Therefore X can be interpreted as $\mathscr{Z}\big(\mathbb{P}_{\mathbb{P}^3}\big(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1)\big), \mathcal{L}^2 \otimes \mathcal{O}(1)\big)$, which is a conic bundle over $\mathbb{P}^3$. Applying Lemma 3.5 with $\mathcal{E} = \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1)$ and $\mathcal{K} = \mathcal{O}_{\mathbb{P}^3}(1)$, we get that the discriminant is a quintic surface with 16 ordinary double points. Notice that to a quintic surface with 16 nodes one can associate a rank 23 Hodge structure of K3 type, as explained by Huybrechts in [32]. We highlight the (classical) fact, which is remarked in *op. cit.*, that if one takes a line L in a cubic fourfold X and its associated hyperkähler $F(X)$ constructed as the variety of lines in X, then the variety $F_L(X)$ of all lines intersecting the given one is a smooth surface of general type D_L with a natural involution. The quotient of D_L by this involution is exactly our 16-nodal quintic surface.

Remark 4.1. — The above nodal quintic will appear once again in our paper, in Fano R–62. Kuznetsov suggested that it would be interesting to understand whether this hints at some birational equivalence between the corresponding fourfolds. This is investigated in the recent [13].

Fano C–5

(#2-39-161-2) $\mathscr{Z}\big(\mathbb{P}^5 \times \mathrm{Gr}(2,8), \mathcal{U}^\vee_{\mathrm{Gr}(2,8)}(1,0) \oplus \mathcal{O}(1,1) \oplus \mathcal{Q}_{\mathbb{P}^5} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,8)}\big)$

- Invariants:

$$h^0(-K) = 39,\ (-K)^4 = 161,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 23,\ -\chi(T) = 21.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^5$ is a blowup $\mathrm{Bl}_{\mathbb{P}^1 \times \mathbb{P}^1} X_3$ of a (special) cubic fourfold along a quadric surface.

- $\pi_2 \colon X \to \mathrm{Gr}(2,8)$ is the small contraction of the plane Π residual to the quadric surface in X_3 onto a singular fourfold X° (with one singular point).
- Rationality: depends on the cubic fourfold. A cubic fourfold contains a quadric if and only if it contains a plane, see Appendix B.1.2.

- Semiorthogonal decompositions:
  - 7 exceptional objects and $\mathrm{D^b}(S_2,\alpha)$ from π_1.

Explanation

The section of the rank 10 bundle is given by $\alpha \in \mathrm{Hom}(V_8,V_6)$, and vanishes on pairs (l,P) such that $P \subset \alpha^{-1}(l)$. The section of the rank 2 bundle is given by $\beta \in \mathrm{Hom}(V_6,V_8^\vee)$, and vanishes when the linear form $\beta(l)$ is zero on P. For l general, P is thus uniquely defined as the intersection of $\alpha^{-1}(l)$ with the kernel of $\beta(l)$, and we get a fivefold Y birational to $\mathbb{P}^5$. More precisely, Y is the blowup of $\mathbb{P}^5$ along a quadric surface $\mathbb{Q}^2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

Now we consider X defined in Y by the additional line bundle section, which is defined by a tensor $\gamma \in \mathrm{Hom}\big(V_6,\bigwedge^2 V_8^\vee\big)$. The projection to $\mathbb{P}^5$ has a non-trivial fiber over l when $\beta(l) \in \alpha^{-1}(l)^\vee$ and $\gamma(l) \in \bigwedge^2 \alpha^{-1}(l)^\vee$ are compatible, which can be expressed by the vanishing of $\beta(l)\wedge\gamma(l) \in \bigwedge^3 \alpha^{-1}(l)^\vee \simeq l^\vee$. This defines a cubic fourfold X_3 in $\mathbb{P}^5$, containing $\mathbb{Q}^2$, and X must be the blowup of this surface in X_3. In particular it contains the residual plane Π to $\mathbb{Q}^2$.

The projection to $\mathrm{Gr}(2,8)$ contracts the plane Π to the point defined by $p_0 = \mathrm{Ker}\,\alpha$, and outside this plane it is an isomorphism to its image X°, which is therefore singular at p_0. In order to understand how singular it is, we parametrize $\mathrm{Gr}(2,8)$ locally around $p_0 = \langle e,f\rangle$ as usual, by choosing a codimension 2 subspace q_0 in V_8 orthogonal to this plane: any plane p transverse to q_0 admits a unique basis of the form $e+x, f+y$ with x,y in q_0. The map α sends q_0 isomorphically to V_6, and we use it to identify the two spaces. A point $\langle e+x,f+y\rangle$ is in X° if and only if x,y are collinear and

$$\big({}^t\beta(e+x)\big)(x) = \big({}^t\beta(e+x)\big)(y) = \big({}^t\beta(f+y)\big)(x) = \big({}^t\beta(f+y)\big)(y) = 0,$$
$$\big({}^t\gamma\big((e+x)\wedge(f+y)\big)\big)(x) = \big({}^t\gamma\big((e+x)\wedge(f+y)\big)\big)(y) = 0.$$

At first order, the latter conditions reduce to three linear conditions applied to both x and y, which are in general independent and amount to restricting them to a codimension 3 subspace of V_8. Then the collinearity condition defines a codimension 2 Schubert variety in $\mathrm{Gr}(2,5)$, which is well-known to be normal, and we can conclude that X° is normal as well. Blowing up Π, we get a fourfold $\widetilde{X} \subset \mathbb{P}^1 \times \mathbb{P}^5 \times \mathrm{Gr}(2,8)$, and the fibers of its projection to $\mathbb{P}^1$ are intersections of two quadrics in $\mathbb{P}^5$.

Fano C–6

(#2-49-211-2) $\mathscr{Z}\big(\mathbb{P}^4 \times \mathbb{P}^5, \mathcal{Q}_{\mathbb{P}^4}(0,1) \oplus \mathcal{O}(2,1)\big)$

- Invariants:

$$h^0(-K) = 49,\ (-K)^4 = 211,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 21,\ -\chi(T) = 19.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^4$ is a blowup $\mathrm{Bl}_{S_6} \mathbb{P}^4$.
  - $\pi_2 \colon X \to \mathbb{P}^5$ is a blowup $\mathrm{Bl}_p X_3^\circ$, where X_3° is a nodal cubic fourfold with a single node at p.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 5 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_6)$.

Explanation

Let us consider the fivefold $\mathscr{Z}\big(\mathbb{P}^4 \times \mathbb{P}^5, \mathcal{Q}_{\mathbb{P}^4}(0,1)\big)$. By [20, Corollary 2.7], this can be identified with $\mathrm{Bl}_p \mathbb{P}^5 \cong \mathbb{P}_{\mathbb{P}^4}\big(\mathcal{O} \oplus \mathcal{O}(-1)\big)$. Considering π_1, we can therefore describe X as $\mathscr{Z}\big(\mathbb{P}_{\mathbb{P}^4}\big(\mathcal{O} \oplus \mathcal{O}(-1)\big), \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^4}(2)\big)$. We conclude by the Cayley trick, since $\mathscr{Z}\big(\mathbb{P}^4, \mathcal{O}(2) \oplus \mathcal{O}(3)\big)$ is a K3 surface of degree 6. Considering π_2, a section of $\mathcal{O}(2,1)$ restricted to the fivefold $\mathrm{Bl}_p \mathbb{P}^5$ cuts a cubic fourfold which is nodal at p, and the claim follows.

Remark 4.2. — In [21, 1.3] it is shown that $F_1(X_3^\circ)$ and $(S_6)^{[2]}$ are birationally equivalent, and a birational equivalence is described explicitly.

Fano C–7

(#2-24-86-2) $\mathscr{Z}\big(\mathbb{P}^4 \times \mathbb{P}^5, \mathcal{O}(1,1) \oplus \bigwedge^3 \mathcal{Q}_{\mathbb{P}^4}(0,1)\big)$

- Invariants:

$$h^0(-K) = 24,\ (-K)^4 = 86,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 26,\ -\chi(T) = 24.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^4$ is $\mathrm{Bl}_{S_{14}^{(5)}} \mathbb{P}^4$.
  - $\pi_2 \colon X \to \mathbb{P}^5$ is $\mathrm{Bl}_{\mathrm{dP}_5} X_3$, where X_3 is a Pfaffian cubic fourfold.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 10 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_{14})$ from both maps.

Explanation

Let us first consider the fivefold Y defined by the general zero locus of $\bigwedge^3 \mathcal{Q}(0,1) = \mathcal{Q}^\vee(1,1)$ inside $\mathbb{P}^4 \times \mathbb{P}^5$. We denote by V_n the vector space such that $\mathbb{P}^{n-1} = \mathbb{P}V_n$. The space of sections $H^0\big(\mathbb{P}^4 \times \mathbb{P}^5, \mathcal{Q}^\vee(1,1)\big)$ is then $\bigwedge^2 V_5^\vee \otimes V_6^\vee = \mathrm{Hom}\big(V_6, \bigwedge^2 V_5^\vee\big)$.

We want to understand the projection $\pi \colon Y \to \mathbb{P}^5$. First note that since the morphism $\mathcal{O}_{\mathbb{P}^5}(-1) \to \bigwedge^2 V_5^\vee \otimes \mathcal{O}_{\mathbb{P}^5}$ is generically injective, π is birational. The first degeneracy locus of $\bigwedge^2 V_5$ is a Pfaffian locus of codimension 3, and a general Pfaffian surface in $\mathbb{P}^5$ is dP_5, a smooth del Pezzo surface of degree 5. The fiber of π over a point p of dP_5 is the projectivization of the 3-dimensional kernel of the form corresponding to p. Hence the fibers of π over dP_5 are projective planes. This implies that $Y \to \mathbb{P}^5$ is the blowup of the above dP_5.

Now notice that $c_1\big(K_{\mathbb{P}^4 \times \mathbb{P}^5} \otimes \det \mathcal{Q}^\vee(1,1)\big) = -2H_1 - 2H_2$. On the other hand, since Y is a blowup, we have $-K_Y = -K_{\mathbb{P}^5} + 2E$ for E the exceptional divisor. Therefore $\mathcal{O}_Y(1,1)$ is the line bundle associated to half-anticanonical sections of Y, hence it corresponds to a cubic fourfold (a half anticanonical divisor in $\mathbb{P}^5$) plus the exceptional divisor E. It follows that X, obtained by cutting Y with a section of $\mathcal{O}(1,1)$, is the blowup of a special cubic fourfold X_3 along the dP_5. Indeed, a cubic fourfold containing such a surface is Pfaffian, see [6]. In order to see this cubic fourfold concretely, note that there is a natural map

$$S^2\big(\textstyle\bigwedge^2 V_5^\vee \otimes V_6^\vee\big) \to \bigwedge^4 V_5^\vee \otimes S^2 V_6^\vee \simeq V_5 \otimes S^2 V_6^\vee.$$

The linear section that defines X is a tensor in $V_5^\vee \otimes V_6^\vee$. Contracting with the previous one we get a tensor in $S^3 V_6^\vee$, hence a cubic fourfold.

Now we consider the projection of Y to $\mathbb{P}^4$. The morphism $V_6 \to \bigwedge^3 V_5$ defines over $\mathbb{P}^4$ a map of vector bundles $V_6 \otimes \mathcal{O} \to \bigwedge^3 \mathcal{Q} = \mathcal{Q}^\vee(1)$, and the generic fiber is the projective line defined by the kernel of this morphism. The exceptional fibers are projective planes, and occur over its degeneracy locus C, a smooth curve whose twisted structure sheaf is resolved by the

Eagon–Northcott complex

$$0 \longrightarrow S^2\mathcal{Q}(-2) \longrightarrow V_6^\vee \otimes Q(-1) \longrightarrow \\ \longrightarrow \bigwedge^2 V_6^\vee \otimes \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(3) \longrightarrow \mathcal{O}_C(3) \longrightarrow 0.$$

The only contributions in cohomology comes from the h^0 of the third and fourth terms, so $\chi(\mathcal{O}_C(3)) = 35 - 15 = 20$. If we twist by $\mathcal{O}(-1)$, the only contribution comes from the fourth term and we deduce that $\chi(\mathcal{O}_C(2)) = 15$. After one extra twist, we get the bundle $S^2\mathcal{Q}(-4)$ which by Borel–Weil–Bott Theorem has $h^3 = 5$, and we deduce that $\chi(\mathcal{O}_C(1)) = 5 + 5 = 10$. Finally, after one more twist, we get $\mathcal{Q}(-4)$ with $h^3 = 1$, and $S^2\mathcal{Q}(-5)$ with $h^3 = 10$, therefore $\chi(\mathcal{O}_C) = 1 - 6 + 10 = 5$. These computations confirm that the Hilbert polynomial $\chi(\mathcal{O}_C(k)) = 5k + 5$, hence that C is the disjoint union of five projective lines.

When we pass to the hyperplane section X of Y, the general fiber becomes a single point, so the projection is birational. The exceptional fibers are projective lines, and occur over the smooth surface S defined as the degeneracy locus of the morphism $V_6 \otimes \mathcal{O} \to \mathcal{Q}^\vee(1) \oplus \mathcal{O}(1)$. The twisted structure sheaf of S is resolved by the Eagon–Northcott complex

$$0 \longrightarrow \mathcal{Q}(-1) \oplus \mathcal{O}_{\mathbb{P}^4}(-1) \longrightarrow V_6^\vee \otimes \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(4) \longrightarrow \mathcal{O}_S(4) \longrightarrow 0.$$

We deduce that the Hilbert polynomial of S is

$$\chi(\mathcal{O}_S(k)) = \frac{9k^2 - 5k}{2} + 2.$$

This means that $\mathcal{O}_S(1)^2 = 9$ and $\mathcal{O}_S(1).\omega_S = 5$. Moreover, dualizing the previous complex we get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-5) \longrightarrow V_6 \otimes \mathcal{O}_{\mathbb{P}^4}(-1) \longrightarrow \mathcal{Q}^\vee \oplus \mathcal{O}_{\mathbb{P}^4} \longrightarrow \omega_S \longrightarrow 0.$$

From Lemma 3.7 we deduce that ω_S admits a canonical section, vanishing exactly on the locus where the morphism $V_6 \otimes \mathcal{O}_{\mathbb{P}^4}(-1) \to \mathcal{Q}^\vee$ is not surjective. This is the same smooth curve C that we proved to be the disjoint union of five lines $l_1, \dots, l_5$. Moreover $\omega_S = \mathcal{O}(l_1 + \cdots + l_5)$. By the genus formula, each of these lines must be a (-1)-curve on S. Finally, after contracting them we get a genus K3 surface $S_{\min}$. We have proved the following.

Proposition 4.3. — *$S = S_{14}^{(5)}$ is a K3 surface of genus 8 blown up at five points.*

Note that a K3 surface of genus 8 is a section of the Grassmannian $\mathrm{Gr}(2,6)$ by a $\mathbb{P}^8$, so a naive projection from five points would send S to a $\mathbb{P}^3$. That K3 surfaces of genus 8 blown up at five points can be embedded in $\mathbb{P}^4$ is mentioned in [4]; it parametrizes a family of lines in $\mathbb{P}^5$, whose union is a ruled threefold $W \subset \mathbb{P}^5$; then the smooth sections of W by a hyperplane $\mathbb{P}(H)$ are blown up of the K3 at its five points contained in $\mathrm{Gr}(2, H)$.

Note also that, if we denote by H_1, H_2 the hyperplane bundles pulled-back via the two projections π_1, π_2, and by E_1, E_2 the two exceptional divisors, we can compute the canonical divisor of X as $\omega_X = -H_1 - H_2 = -5H_1 + E_1 = -3H_2 + E_2$. In particular $H_2 = 4H_1 - E_1$, which means that the birational map $\pi_2 \circ \pi_1^{-1} : \mathbb{P}^4 \dashrightarrow X_3$ is defined by the linear system of quartics containing the non-minimal K3 surface $S_{14}^{(5)}$. This special birational transformation already appears in the work of Fano [65, Section 6]. In fact, we have just re-discovered from this example the classical proof that Pfaffian cubic fourfolds are rational!

Fano C–8

(#2-30-115-2-A) $\mathscr{Z}\big(\mathrm{Fl}(1,3,8), \mathcal{Q}_2^{\oplus 2} \oplus \mathcal{O}(1,1) \oplus \mathcal{R}_2^\vee(1,1)\big)$

- Invariants:

$h^0(-K) = 30,\ (-K)^4 = 115,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 27,\ -\chi(T) = 26.$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^7$ is birational to its image, the intersection Y of three quadrics, singular along a line.
  - $\pi_2 \colon X \to \mathrm{Gr}(3,8)$ is $\mathrm{Bl}_{\mathrm{dP}_4} X_3$, the blowup of a cubic fourfold in a dP_4.
  - Rationality: depends on the cubic fourfold. A cubic fourfold contains a dP_4 if and only if it contains a plane, see Appendix B.1.2.
- Semiorthogonal decompositions:
  - 11 exceptional objects and $\mathrm{D^b}(S_2, \alpha)$ from π_2, see Appendix B.1.2.

Explanation

Over the flag manifold parametrizing flags $U_1 \subset U_3 \subset V_8$, of dimension 17, the bundles are two copies of the rank 5 bundle V_8/U_3, the line bundle $U_1^\vee \otimes \det(U_3)^\vee$, and the rank 2 bundle $(U_3/U_1)^\vee \otimes U_1^\vee \otimes \det(U_3)^\vee$. Let us start with the first two: their global sections are two vectors in V_8 and the locus where such sections vanish is where U_3 contains them. So we need that U_3 contains a fixed two-dimensional space V_2: it is then parametrized by $\mathbb{P} = \mathbb{P}(V_8/V_2) \simeq \mathbb{P}^5$. Taking U_1 into account we are reduced to the seven-dimensional projective bundle

$$Z = \mathbb{P}\big(V_2 \oplus \mathcal{O}_{\mathbb{P}}(-1)\big) \xrightarrow{\pi_2} \mathbb{P}.$$

Denote by $\mathcal{L}$ the relative tautological line bundle. We have an exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \pi_2^*\big(U_2 \oplus \mathcal{O}_{\mathbb{P}}(-1)\big) \longrightarrow \mathcal{K} \longrightarrow 0$$

for $\mathcal{K}$ the rank 2 quotient bundle. Note that the relative tangent bundle is $\mathscr{H}om(\mathcal{L}, \mathcal{K})$, whose determinant is $\det(\mathcal{K}) \otimes \mathcal{L}^{-2} = \mathcal{L}^{-3}(-1)$, hence $K_Z = \mathcal{L}^3(-5)$.

Then we take into account the rank 2 vector bundle. Note that restricted to Z, U_1 is just $\mathcal{L}$ and U_3 is $\pi_2^*\big(V_2 \oplus \mathcal{O}_{\mathbb{P}}(-1)\big)$, so $U_3/U_1 \simeq \mathcal{K}$ and $(U_3/U_1)^\vee \otimes U_1^\vee \otimes \det(U_3)^\vee \simeq \mathcal{K}^\vee \otimes \mathcal{L}^{-1}(1) = \mathcal{K}(2)$. In particular a general section of this bundle defines a smooth fivefold $Y \subset Z$ such that $K_Y = \mathcal{L}_Y^2(-2)$. Restricted to the fibers of π_2 this bundle is a copy of the quotient bundle on the projective plane, whose global sections vanish at a single point, or everywhere. So π_2 restricted to Y is birational. Moreover, the exceptional fibers are projective planes, that occur over the locus where the induced section of $\pi_{2,*}\big(\mathcal{K}(2)\big) = V_2 \otimes \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)$ vanishes, hence over a dP_4. We conclude that π_2 is the blowup of $\mathbb{P}$ in a dP_4.

Finally we take into account the line bundle $U_1^\vee \otimes \det(U_3)^\vee \simeq \mathcal{L}^{-1}(1)$. A general section vanishes along a fourfold $X \subset Y$ such that $K_X = \mathcal{L}_X(-1)$. In the fibers of Z over $\mathbb{P}$, note that X is defined by a section of $\mathcal{L}^{-1}(1)$ and a section of $\mathcal{K}(2)$. By the tautological exact sequence the latter vanishes on a section of the line bundle $\mathcal{L}(2)$, on which the section of $\mathcal{L}^{-1}(1)$ also vanishes if and only if the induced section of $\mathcal{L}(2) \otimes \mathcal{L}^{-1}(1) = \mathcal{O}_{\mathbb{P}}(3)$ vanishes. This means that the fiber of π_2 is non-trivial over a cubic fourfold, and that X is the blowup of this cubic fourfold in a dP_4.

Now we turn to π_1. Note that over Z we have a morphism of vector bundles from $U_1 \oplus V_2$ to U_3, hence by duality from $U_3^\vee$ to the trivial bundle $V_2^\vee$, and from $\det(U_3)^\vee$ to $U_1^\vee$. Therefore the sections of $U_1^\vee \otimes \det(U_3)^\vee$ and $(U_3/U_1)^\vee \otimes U_1^\vee \otimes \det(U_3)^\vee$ that define X induce sections of $(U_1^\vee)^2$ and $V_2^\vee \otimes (U_1^\vee)^2$ respectively. We thus get three quadrics in $\mathbb{P}^7$ whose intersection Y is the image of π_1. Obviously Y is isomorphic to X outside the projective line ℓ where $U_1 \subset V_2$. Over this line, Z restricts to $\mathbb{P}^1 \times \mathbb{P}^5$, and U_3/U_1 restricts to $\mathcal{O}(1,0) \oplus \mathcal{O}(0,-1)$. We deduce that the preimage of ℓ in X is the intersection in $\mathbb{P}^1 \times \mathbb{P}^5$ of three divisors of bidegrees $(0,1), (1,1), (1,2)$; hence the intersection in $\mathbb{P}^1 \times \mathbb{P}^4$ of two divisors of bidegrees $(1,1), (1,2)$.

The image in $\mathbb{P}^4$ of such a threefold is a cubic threefold with four singular points.

Fano C–9

(#2-27-101-2) $\mathscr{Z}\left(\mathbb{P}^5 \times \mathrm{Gr}(2,4), \left(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0)\right)^{\oplus 2} \oplus \mathcal{O}(1,1)\right)$

- Invariants:

$$h^0(-K) = 27,\ (-K)^4 = 101,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 23,\ -\chi(T) = 21.$$

- Description:
  - $\pi_1\colon X \to \mathbb{P}^5$ is $\mathrm{Bl}_{\Sigma_4} X_3$, the blowup of a cubic fourfold along a rational quartic scroll.
  - $\pi_2\colon X \to \mathrm{Gr}(2,4)$ is $\mathrm{Bl}_{S^{(1)}_{14}} \mathrm{Gr}(2,4)$, the blowup of a quadric along a K3 surface of genus 8 blown up in one point.
  - Rationality: yes, as a cubic containing a rational quartic scroll is rational.
- Semiorthogonal decompositions:
  - 7 exceptional objects and $\mathrm{D^b}(S_{14})$ from both maps. Indeed the cubic X_3 lies in $\mathcal{C}_{14}$ since it contains a rational quartic scroll, see Appendix B.1.3.

Explanation

Let us start with the fivefold Y defined by two global sections of the same two rank 2 bundles, hence a tensor $\alpha \in V_2 \otimes V_6^\vee \otimes V_4 = \mathrm{Hom}(V_2 \otimes V_6, V_4)$. So $Y \subset \mathbb{P}^5 \times \mathrm{Gr}(2,4)$ parametrizes the pairs $(l \subset V_6, P \subset V_4)$ such that $\alpha(V_2 \otimes l) \subset P$. Generically, there must be equality, so the projection to $\mathbb{P}^5$ is birational. The exceptional fibers are projective planes, and they appear above the locus where the morphism $V_2 \otimes \mathcal{O}(-1) \to V_4 \otimes \mathcal{O}$ defined by α drops rank. This locus is a smooth surface. More precisely, a rational quartic scroll Σ_4, and Y is the blowup of $\mathbb{P}^5$ along this surface.

Now let us consider the projection to $\mathrm{Gr}(2,4)$. Considering α as a morphism from $V_2 \otimes V_4^\vee$ to $V_6^\vee$, we can see that the preimage of a plane $P \in V_4$ is the (projectivized) linear space defined by the image of $V_2 \otimes P^\perp$. The generic fiber is thus a projective line. Moreover the exceptional fibers are projective planes, which occur above the locus where the morphism of vector bundles

$$V_2 \otimes \mathcal{Q}^\vee \longrightarrow V_6^\vee \otimes \mathcal{O}$$

drops rank, which is a smooth curve C. The twisted structure sheaf of this curve is resolved by the Eagon–Northcott complex on $\mathrm{Gr}(2,4)$

$$0 \longrightarrow S^2(V_2 \otimes \mathcal{Q}^\vee) \longrightarrow V_{12} \otimes \mathcal{Q}^\vee \longrightarrow V_{15} \otimes \mathcal{O}_{\mathrm{Gr}(2,4)} \longrightarrow \\ \longrightarrow \mathcal{O}_{\mathrm{Gr}(2,4)}(2) \longrightarrow \mathcal{O}_C(2) \longrightarrow 0,$$

where as usual V_k is a k-dimensional vector space. Here $S^2(V_2 \otimes \mathcal{Q}^\vee) = \mathcal{O}_{\mathrm{Gr}(2,4)}(-1) \oplus V_3 \otimes S^2\mathcal{Q}^\vee$ is acyclic by Borel–Weil–Bott Theorem, as well as $\mathcal{Q}^\vee$, so $\chi\big(\mathcal{O}_C(2)\big) = 20 - 15 = 5$. If we twist by $\mathcal{O}_{\mathrm{Gr}(2,4)}(-1)$, the bundle $S^2\mathcal{Q}^\vee(-1)$ has a one-dimensional second cohomology group, yielding $\chi\big(\mathcal{O}_C(1)\big) = 6 - 3 = 3$. After another twist, all the bundles in the complex become acyclic and therefore $\chi(\mathcal{O}_C) = 1$. This is more than enough to imply that the Hilbert polynomial of C is $\chi\big(\mathcal{O}_C(k)\big) = 2k+1$, which implies that C must be a smooth conic C_2. We end up with the following diagram:

$$\begin{tikzcd}
 & E \arrow[r, hook] \arrow[dl, "\mathbb{P}^2"'] & Y \arrow[dl, "\mathbb{P}^0"] \arrow[dr, "\mathbb{P}^1"'] & F \arrow[l, hook'] \arrow[dr, "\mathbb{P}^2"] & \\
\Sigma_4 \arrow[r, hook] & \mathbb{P}^5 & & \mathrm{Gr}(2,4) & C_2. \arrow[l, hook']
\end{tikzcd}$$

Remark 4.4. — How to obtain a smooth conic from the tensor α? First observe that its 6×6 minors defined a tensor $\alpha^{(6)} \in \mathrm{Hom}\big(\bigwedge^6(V_2\otimes V_4), \bigwedge^6 V_6\big)$. Once volume forms have been fixed, this is just $\bigwedge^2(V_2 \otimes V_4)$, which has a natural map p to $S^2V_2 \otimes \bigwedge^2 V_4 \simeq \mathrm{Hom}\big(S^2V_2^\vee, \bigwedge^2 V_4\big)$. So the image of $p(\alpha^{(6)})$ will in general define a projective plane in $\mathbb{P}\big(\bigwedge^2 V_4\big)$, whose intersection with $\mathrm{Gr}(2,4)$ is a conic canonically defined by α.

Now we turn to the fourfold X obtained as a hyperplane section of Y. This hyperplane is defined by a tensor $\beta \in V_6^\vee \otimes \bigwedge^2 V_4^\vee$. If we fix a basis of V_2 and denote by α_1, α_2 the components of α in this basis, X parametrizes the pairs (l, P) such that P contains $\alpha_1(l)$ and $\alpha_2(l)$, and $\beta(l, p_1, p_2) = 0$ for any basis (p_1, p_2) on P. Outside Σ_4, $\alpha_1(l)$ and $\alpha_2(l)$ are such a basis of P, hence the condition $\beta\big(l, \alpha_1(l), \beta_2(l)\big) = 0$ defines a cubic in $\mathbb{P}^5$. This implies that X is the blowup of a cubic fourfold X_3 along the rational quartic scroll Σ_4. In particular, X_3 is Pfaffian [6] and hence rational.

The projection of X to $\mathrm{Gr}(2,4)$ is also birational. Its exceptional locus is now the degeneracy locus of the morphism

$$V_2 \otimes \mathcal{Q}^\vee \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}(-1) \longrightarrow V_6^\vee \otimes \mathcal{O}_{\mathrm{Gr}(2,4)}$$

defined by α and β. This locus is a smooth surface S whose structure sheaf is resolved by the Eagon–Northcott complex

$$0 \longrightarrow V_2 \otimes \mathcal{Q}^\vee \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}(-1) \longrightarrow V_6^\vee \otimes \mathcal{O}_{\mathrm{Gr}(2,4)} \longrightarrow \\ \longrightarrow \mathcal{O}_{\mathrm{Gr}(2,4)}(3) \longrightarrow \mathcal{O}_S(3) \longrightarrow 0.$$

Cohomologically there is no difference with the previous case, so we can conclude in the same way that the Hilbert polynomial of S is again $\chi\big(\mathcal{O}_S(k)\big) = 5k^2 - k + 2$.

Proposition 4.5. — *$S = S_{14}^{(1)}$ is a K3 surface of genus 8 blown up at a single point.*

Remark 4.6. — Of course the cubic X should be Pfaffian and the K3 surface S_{14} should be its HP-dual.

Proof. — By dualizing the Eagon–Northcott complex, we get

$$0 \longrightarrow \mathcal{O}_{\mathrm{Gr}(2,4)}(-4) \longrightarrow V_6(-1) \longrightarrow V_2 \otimes \mathcal{Q}(-1) \oplus \mathcal{O}_{\mathrm{Gr}(2,4)} \longrightarrow \omega_S \longrightarrow 0.$$

We deduce that ω_S admits a canonical section, vanishing exactly on the curve D where the morphism $V_6(-1) \to V_2 \otimes \mathcal{Q}(-1)$ is not surjective. By the Thom–Porteous formula, the class of D in the Chow ring of $\mathrm{Gr}(2,4)$ is

$$[D] = s_3(V_2 \otimes \mathcal{Q} - V_6 \otimes \mathcal{O}_{\mathrm{Gr}(2,4)}) = s_3(V_2 \otimes \mathcal{Q}) = 2\sigma_{21},$$

since the Segre class $s(U_2 \otimes \mathcal{Q}) = s(\mathcal{Q})^2$ and $s(\mathcal{Q}) = 1 + \sigma_1 + \sigma_2$ is the sum of the special Schubert classes. In words, $[D]$ is twice the class of a line. Moreover, the structure sheaf of D is resolved by another Eagon–Northcott complex,

$$0 \longrightarrow S^2(V_2 \otimes \mathcal{Q})^\vee(-2) \longrightarrow V_{12} \otimes \mathcal{Q}^\vee(-2) \longrightarrow V_{15} \otimes \mathcal{O}_{\mathrm{Gr}(2,4)}(-2) \longrightarrow \\ \longrightarrow \mathcal{O}_{\mathrm{Gr}(2,4)} \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

Note that $S^2(V_2 \otimes \mathcal{Q})^\vee(-2) = S^2U_2 \otimes S^2\mathcal{Q}^\vee(-2) \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}(-3)$ is acyclic by Borel–Weil–Bott, as well as $\mathcal{Q}^\vee(-2)$ and of course $\mathcal{O}_{\mathrm{Gr}(2,4)}(-2)$. We deduce that $h^0(\mathcal{O}_D) = 1$, hence that D must be connected, and finally that this must be a smooth conic. By the genus formula, since $\omega_S = \mathcal{O}_S(D)$ we conclude that D is a (-1)-curve on S, after contracting which we get a genuine K3 surface $S_{\min}$ of degree 14. $\square$

Fano C–10

(#3-20-63-2) $\mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^5, \mathcal{O}(0,0,3) \oplus \mathcal{O}(0,1,1) \oplus \mathcal{O}(1,0,1)\big)$

- Invariants:

$$h^0(-K) = 20,\ (-K)^4 = 63,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 38,\ -\chi(T) = 36.$$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ is a dP_3-fibration.
  - $\pi_{13}, \pi_{23}\colon X \to \mathbb{P}^1 \times \mathbb{P}^5$ are blowups $\mathrm{Bl}_{\mathrm{dP}_0} Y$, where dP_0 is a blowup of a cubic surface at three points and Y is Fano C–1.
  - Rationality: unknown, since the cubic fourfold is general.
- Semiorthogonal decompositions:
  - 4 exceptional objects and an unknown category from π_{12}.
  - 24 exceptional objects and the K3 category of X_3 from π_{13} or π_{23}.

Explanation

One can apply Lemma 3.1 twice to understand π_{13} and π_{23} as the composition of two blowups of cubic surfaces. One checks that the strict transform of the second cubic surface is its blowup at three extra points (which are its intersection points with the line l at the intersection of the two $\mathbb{P}^3$'s).

In order to understand π_{12}, we reinterpret X as $\mathscr{Z}\big(\mathrm{Fl}(1,4,6), \mathcal{Q}_2^{\oplus 2} \oplus \mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{O}(3,0)\big)$. The latter is nothing but $\mathscr{Z}\big(\mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}\big(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)\big), \mathcal{L}^{\otimes 3}\big)$, which is obviously a dP_3-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$.

Fano C–11

(#3-24-81-2) $\mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^5, \mathcal{O}(0,0,3) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,0,1) \oplus \mathcal{O}(1,1,0)\big)$

- Invariants:

$$h^0(-K) = 24,\ (-K)^4 = 81,\ h^{1,1} = 3,\ h^{2,1} = 1,\ h^{3,1} = 1,\ h^{2,2} = 30,\ -\chi(T) = 31.$$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ factors as the blowup of a dP_3-fibration $T \to \mathbb{P}^2$ in a general fiber which is a cubic surface.
  - $\pi_{13}\colon X \to \mathbb{P}^1 \times \mathbb{P}^5$ is the blowup $\mathrm{Bl}_\Sigma Y$, where Σ is a ruled surface and Y is as in Fano C–1.
  - $\pi_{23}\colon X \to \mathbb{P}^2 \times \mathbb{P}^5$ is a blowup of Fano C–2 along a cubic surface.
  - Rationality: unknown, since the cubic fourfold is general.
- Semiorthogonal decompositions:
  - 12 exceptional objects and an unknown category from π_{12} or π_{23}.
  - 16 exceptional objects and the K3 category of X_3 from π_{13}.

Explanation

The projection to $\mathbb{P}^2\times\mathbb{P}^5$ is birational outside a point of $\mathbb{P}^2$, and blows up its image Y along a cubic surface. The projection of Y to $\mathbb{P}^5$ is the blowup of a cubic fourfold along a plane section, hence an elliptic curve contained in the cubic surface. (Notice that this Fano fourfold can be described as well by applying Proposition 3.3 to Fano C–2.)

The projection to $\mathbb{P}^1\times\mathbb{P}^2$ can be understood by rewriting $\mathrm{Bl}_{\mathbb{P}^2}\mathbb{P}^5$ as $Z := \mathbb{P}_{\mathbb{P}^2}\big(\mathcal{O}^{\oplus 3}\oplus\mathcal{O}(-1)\big)$. A section of $\mathcal{O}_Z(3)$ cuts the latter in a fibration in dP_3, which we call Z'. We thus obtain a zero locus $\mathscr{Z}\big(\mathbb{P}^1\times Z', \mathcal{O}_{\mathbb{P}^1}(1)\otimes\pi^*\mathcal{O}_{\mathbb{P}^2}(1)\big)$. By Proposition 3.3 this is the blowup of Z' in a general fiber, i.e. a cubic surface.

Finally, the fibers of the projection to $\mathbb{P}^1\times X_3$ are defined by morphisms $\mathcal{O}(0,-1)\to V_3$ and $\mathcal{O}(-1,0)\to V_3^\vee$, whose compatibility is verified over a section of $\mathcal{O}(1,1)$. So the image of π_{13} is isomorphic to the blowup of X_3 along a cubic surface Σ. Moreover, the fibers of π_{13} do not reduce to a single point when the first morphism $\mathcal{O}(0,-1)\to V_3$ vanishes, that is over a copy of $\mathbb{P}^1\times E$, where E is an elliptic curve in Σ.

Fano C–12

(#3-27-99-2) $\mathscr{Z}\big(\mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^5, \mathcal{O}(0,1,2)\oplus\mathcal{Q}_{\mathbb{P}^2}(0,0,1)\oplus\mathcal{O}(1,0,1)\big)$

- Invariants:

$$h^0(-K)=27,\ (-K)^4=99,\ h^{1,1}=3,\ h^{3,1}=1,\ h^{2,2}=30,\ -\chi(T)=27.$$

- Description:
  - $\pi_{12}\colon X\to\mathbb{P}^1\times\mathbb{P}^2$ factors as the blowup $\mathrm{Bl}_{\mathrm{dP}_2} Z$, where $Z\to\mathbb{P}^2$ is a quadric surface bundle over $\mathbb{P}^2$ ramified along a sextic from Fano C–3.
  - $\pi_{13}\colon X\to\mathbb{P}^1\times\mathbb{P}^5$ is a blowup $\mathrm{Bl}_{\mathrm{dP}_8} Y$, where Y is a blowup of a cubic surface in a cubic fourfold containing a plane, and the dP_8 is the strict transform of this plane.
  - $\pi_{23}\colon X\to\mathbb{P}^2\times\mathbb{P}^5$ is a blowup $\mathrm{Bl}_{\mathrm{dP}_2} W$, where W is the Fano fourfold Fano C–3.
  - Rationality: unknown.
- Semiorthogonal decompositions:
  - 16 exceptional objects and $\mathrm{D^b}(S_2,\alpha)$ from each map, where S_2 and α are the K3 surface and the Brauer class either from Fano C–3 or from the cubic containing a plane, see Appendix B.1.2.

Explanation

Let us start by describing π_{23}. By Lemma 3.1, X is the blowup of a fourfold, from Fano C–3, say Y, in a surface which is a complete intersection of Y with two extra $(0,1)$ divisors from $\mathbb{P}^2 \times \mathbb{P}^5$. In particular, this surface is described in $\mathbb{P}^2 \times \mathbb{P}^3$ as $\mathscr{Z}\big(\mathcal{Q}_{\mathbb{P}^2}(0,1) \oplus \mathcal{O}(1,2)\big)$. The latter is identified with the blowup of a dP_3 in a point. Note that such a cubic surface is a double hyperplane section of the cubic fourfold in $\mathbb{P}^4$, and hits the blown up plane in exactly one point. This gives the description of π_{13} as well.

In order to understand π_{12}, let us reinterpret this variety as $\mathscr{Z}\big(\mathbb{P}^1 \times \mathrm{Fl}(1,4,6), \mathcal{Q}_2^{\oplus 3} \oplus \mathcal{O}(0;2,1) \oplus \mathcal{O}(1;1,0)\big)$. The first four bundles cut $\mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}_{\mathbb{P}^2}\big(\mathcal{O}^{\oplus 3} \oplus \mathcal{O}(-1)\big), \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{L} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{L}^{\otimes 2}\big)$. If we just consider Z as $\mathscr{Z}\big(\mathbb{P}_{\mathbb{P}^2}\big(\mathcal{O}^{\oplus 3} \oplus \mathcal{O}(-1)\big), \mathcal{L}^{\otimes 2} \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)\big)$ this can be identified as in Fano C–3 with a quadric surface bundle over $\mathbb{P}^2$ with ramification divisor a smooth sextic. By Lemma 3.1 the latter is then the blowup in the complete intersection of two copies of $\mathcal{L}$, that is the double cover of $\mathbb{P}^2$ ramified over a sextic. The result follows.

Fano C–13

(#3-34-134-2) $\qquad \mathscr{Z}\big(\mathbb{P}^1 \times \mathrm{Fl}(1,2,6), \mathcal{Q}_2 \oplus \mathcal{O}(0;1,2) \oplus \mathcal{O}(1;0,1)\big)$

- Invariants:

$$h^0(-K) = 34,\ (-K)^4 = 134,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 28,\ -\chi(T) = 25.$$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^1 \times \mathbb{P}^5$ is a blowup $\mathrm{Bl}_{\mathrm{dP}_3} \mathrm{Bl}_p X_3^\circ$, where X_3° is a nodal cubic fourfold and $\mathrm{Bl}_p X_3^\circ$ is Fano C–6.
  - $\pi_{13}\colon X \to \mathbb{P}^1 \times \mathrm{Gr}(2,6)$ is a blowup $\mathrm{Bl}_{S_6^{(6)}} \mathrm{Bl}_{\mathbb{P}^2} \mathbb{P}^4$, where $S_6^{(6)}$ is a sextic K3 surface blown up in 6 points.
  - $\pi_{23}\colon X \to \mathrm{Fl}(1,2,6)$ is a blowup $\mathrm{Bl}_{\mathrm{dP}_3} \mathrm{Bl}_{S_6} \mathbb{P}^4$, with S_6 a sextic K3 from Fano C–6.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 14 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_6)$ from each map.

Explanation

In order to understand the first two projections, we can write down $\mathscr{Z}\big(\mathrm{Fl}(1,2,6),\mathcal{Q}_2\big)$ as $\mathbb{P}_{\mathbb{P}^4}\big(\mathcal{O}\oplus\mathcal{O}(-1)\big)\cong \mathrm{Bl}_p\,\mathbb{P}^5$. We deduce that the projection to $\mathbb{P}^1\times\mathbb{P}^5$ maps X to a $(1,1)$ divisor in $\mathbb{P}^1\times\mathrm{Bl}_p\,\widetilde{X}_3$, which in turn blows up a cubic surface.

Starting from the same description we can also understand the second projection: the image is a $(1,1)$-divisor in $\mathbb{P}^1\times\mathbb{P}^4$, i.e. the blowup of $\mathbb{P}^4$ in a plane. Using the Cayley trick we can see that we further blowup the strict transform of a sextic K3 surface in $\mathbb{P}^4$, which is the blowup of this sextic in six extra points.

Finally, without the first factor and the last bundle, we know that we get the blowup of $\mathbb{P}^4$ along a K3 surface of degree 6 by the Cayley trick. The extra factor yields another blowup, along the strict transform of a plane, which is a dP_3.

Fano C–14

(#3-30-113-2) $\mathscr{Z}\big(\mathbb{P}^1\times\mathbb{P}^3\times\mathbb{P}^5,\mathcal{O}(0,1,2)\oplus\mathcal{Q}_{\mathbb{P}^3}(0,0,1)\oplus\mathcal{O}(1,1,0)\big)$

- Invariants:

$$h^0(-K)=30,\ (-K)^4=113,\ h^{1,1}=3,\ h^{3,1}=1,\ h^{2,2}=30,\ -\chi(T)=28.$$

- Description:
  - $\pi_{12}\colon X\to\mathbb{P}^1\times\mathbb{P}^3$ has for image the blowup of $\mathbb{P}^3$ along a line and is a blowup of the conic bundle described in Fano C–4.
  - $\pi_{13}\colon X\to\mathbb{P}^1\times\mathbb{P}^5$ is a blowup of a general cubic fourfold along a cubic surface, i.e. Fano C–1, and then a quadratic surface.
  - $\pi_{23}\colon X\to\mathbb{P}^3\times\mathbb{P}^5$ is a blowup $\mathrm{Bl}_{\mathrm{dP}_3}Y$, where Y is Fano C–4.
  - Rationality: unknown, since the cubic fourfold is general.
- Semiorthogonal decompositions:
  - 16 exceptional objects and the K3 category of X_3 from the three maps.

Explanation

Let $Y\subset\mathbb{P}^3\times\mathbb{P}^5$ be a fourfold of type Fano C–4. To understand π_{23} we apply Lemma 3.1 to $\mathbb{P}^1\times Y$. The intersection of two divisors of degree

$(1,0)$ on $Y \subset \mathbb{P}^3 \times \mathbb{P}^5$ cuts a cubic surface that contains the line already blown up.

Alternatively, the other birational description for Y was a conic bundle over $\mathbb{P}^3$ ramified over a singular quintic. This suggests that X can be alternatively described as $\mathscr{Z}\big(\mathbb{P}^1 \times \mathrm{Fl}(1,3,6), \mathcal{Q}_2^{\oplus 2} \oplus \mathcal{O}(0;2,1) \oplus \mathcal{O}(1;1,1)\big)$. In particular thanks to Lemma 3.1 applied in this context we can see that this fourfold is the blowup of the said conic bundle in a cubic surface. This yields the description of π_{12}.

Finally, let Z denote the image of X in $\mathbb{P}^1 \times \mathbb{P}^5$. The projection of Z to $\mathbb{P}^5$ is the blowup of a general cubic fourfold along a surface section Σ. The projection of X to Z is the blowup of $\mathbb{P}^1 \times l$, where l is a line in Σ.

Fano C–15

(#3-35-141-2) $\mathscr{Z}\big(\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^5, \mathcal{Q}_{\mathbb{P}^2_1}(0,0,1) \oplus \mathcal{Q}_{\mathbb{P}^2_2}(0,0,1) \oplus \mathcal{O}(1,1,1)\big)$

- Invariants:

$$h^0(-K) = 35,\ (-K)^4 = 141,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 23,\ -\chi(T) = 18.$$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^2 \times \mathbb{P}^2$ is a blowup $\mathrm{Bl}_{S_{14}}(\mathbb{P}^2 \times \mathbb{P}^2)$, where the K3 surface S_{14} has Picard rank 2.
  - $\pi_{13}, \pi_{23}\colon X \to \mathbb{P}^2 \times \mathbb{P}^5$ is a blowup along a plane Π of a special Fano C–3, which is in turn a blowup $\mathrm{Bl}_{\Pi'} X_3$ of a cubic fourfold containing a second plane Π', skew to Π.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 9 exceptional objects and $\mathrm{D^b}(S_{14})$ from π_{12}.
  - 9 exceptional objects and $\mathrm{D^b}(S_2)$ from π_{13} or π_{23} (the Brauer class vanishes when X contains two skew planes).

Explanation

Let A_3 and B_3 be three-dimensional vector spaces such that $\mathbb{P}^2_1 = \mathbb{P}(A_3)$ and $\mathbb{P}^2_2 = \mathbb{P}(B_3)$. The first two sections are defined by morphisms $\alpha \in \mathrm{Hom}(V_6, A_3)$ and $\beta \in \mathrm{Hom}(V_6, B_3)$, and their zero locus is the set of triples (x, y, z) of lines such that $\alpha(z) \subset x$ and $\beta(z) \subset y$. Generically x and y are determined by z so the projection to $\mathbb{P}^5$ is birational: it is the blowup of the two skew planes $\mathbb{P}(\mathrm{Ker}\,\alpha)$ and $\mathbb{P}(\mathrm{Ker}\,\beta)$.

The extra section of the line bundle $\mathcal{O}(1,1,1)$ is defined by a tensor $\gamma \in A_3^\vee \otimes B_3^\vee \otimes V_6^\vee$. It induces a cubic $\gamma\big(\alpha(z),\beta(z),z\big)$ which gives an equation of the image of X in $\mathbb{P}^5$: this image is a generic cubic fourfold containing the projective planes $\mathbb{P}(\operatorname{Ker}\alpha)$ and $\mathbb{P}(\operatorname{Ker}\beta)$, and the claim follows.

Note that we can identify A_3 and B_3 with subspaces of V_6 in transverse position, in such a way that α and β are the associated projections. Given x and y, z must belong to the pencil they generate, on which γ cuts a unique point in general. It cuts out a whole projective line when the induced morphism $\mathcal{O}(-1,-1) \to \big(\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)\big)^\vee$ vanishes, which happens along a K3 surface of Picard number 2, intersection of two divisors of bidegrees $(1,2)$ and $(2,1)$ (degree 14 in the Segre embedding, and double covers of the two $\mathbb{P}^2$'s). We conclude that the projection to $\mathbb{P}^2 \times \mathbb{P}^2$ is the blowup of this K3 surface.

The rationality of this example follows from the well-known result about the rationality of a smooth cubic fourfold containing two skew planes, see e.g. [29, 1.2].

Fano C–16

(#3-42-176-2) $\mathscr{Z}\big(\mathbb{P}^2 \times \mathbb{P}^4 \times \mathbb{P}^5, \mathcal{Q}_{\mathbb{P}^4}(0,0,1) \oplus \mathcal{O}(1,1,1) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,1,0)\big)$

- Invariants:

$$h^0(-K) = 42,\ (-K)^4 = 176,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 23,\ -\chi(T) = 18.$$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^2 \times \mathbb{P}^4$ is a blowup $\operatorname{Bl}_{S_6} Y$, where Y is a blowup $\operatorname{Bl}_{\mathbb{P}^1} \mathbb{P}^4$, and S_6 is a sextic K3 surface containing this $\mathbb{P}^1$.
  - $\pi_{13}\colon X \to \mathbb{P}^2 \times \mathbb{P}^5$ is a blowup of $\operatorname{Bl}_\Pi X_3^0$, $X_3^\circ \subset \mathbb{P}^5$ is a cubic fourfold with one node, Π is a plane containing the node, and π_{13} is the blowup of the fiber over the node, which is another plane.
  - $\pi_{23}\colon X \to \mathbb{P}^4 \times \mathbb{P}^5$ is a blowup Fano C–6 of the same nodal cubic X_3° at its node, then blown up along the preimage of the plane, which is a Hirzebruch surface Σ_1.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 9 exceptional objects and $\mathrm{D^b}(S_6)$ from each map.

Explanation

The sections that cut out X are given by tensors $\alpha \in \mathrm{Hom}(V_6, V_5)$, $\beta \in \mathrm{Hom}(V_5, V_3)$, and $\gamma \in V_3^\vee \otimes V_5^\vee \otimes V_6^\vee$. The fourfold X parametrizes the triples of lines (x, y, z) in $\mathbb{P}^2 \times \mathbb{P}^4 \times \mathbb{P}^5$ such that

$$\alpha(z) \subset y, \qquad \beta(y) \subset x, \qquad \gamma(x, y, z) = 0.$$

We start with the description of π_{12}. The set of pairs (x, y) such that $\beta(y) \subset x$ is the blowup of $\mathbb{P}^4$ along the line $l = \mathbb{P}(\mathrm{Ker}\,\beta)$. The extra point z must then belong to $\alpha^{-1}(y) \simeq z_0 \oplus y$, where z_0 denotes the kernel of α; this defines a projective line on which γ cuts a unique point in general. So the projection of X to $X' = \mathrm{Bl}_l\,\mathbb{P}^4$ is birational, and its exceptional locus S is given by the vanishing of the induced morphism $\mathcal{O}_{X'}(-1,-1) \to \big(z_0 \oplus \mathcal{O}_{X'}(0,-1)\big)^\vee$. This surface S is isomorphic to its projection in $\mathbb{P}^4$, which is a general K3 surface of degree 6 containing the line l.

Now we turn to π_{13}. Let $\Pi = \mathbb{P}\big(\mathrm{Ker}(\beta \circ \alpha)\big)$, a projective plane containing z_0. The set of pairs (x, z) such that $\beta \circ \alpha(z) \subset x$ and $\gamma\big(x, \alpha(z), z\big) = 0$ is a blowup Y along Π of the cubic X_3° of equation $\gamma\big(\beta \circ \alpha(z), \alpha(z), z\big) = 0$, which is a general cubic fourfold containing Π and nodal at the point $z_0 \in \Pi$. (Note that the tangent cone to the cubic at z_0 is a quadric, in which the degree 3 part of the equation cuts the surface S, with its line l coming from Π.) The fiber Θ of z_0 in Y is $\mathbb{P}^2$, while the other non-trivial fibers are projective lines. The extra point y is uniquely determined as $\alpha(z)$, except if the latter is zero, that is, over the projective plane Θ that is contracted to z_0. The fiber over (x, z_0) is defined by the conditions that y belongs to the projective plane $\mathbb{P}\big(\beta^{-1}(x)\big)$ and $\gamma(x, y, z_0) = 0$; this defines a projective line and X must be the blowup of Θ.

Finally we describe π_{23}. The set of pairs (y, z) such that $\alpha(z) \subset y$ is the blowup of $\mathbb{P}^5$ at z_0. With the additional condition that $\gamma\big(\beta(y), y, z\big) = 0$ we get the blowup Z of X_3° at its node z_0. The exceptional divisor is the quadric $\mathbb{Q}^3 \subset \mathbb{P}^4$ with equation $\gamma\big(\beta(y), y, z_0\big) = 0$. The extra point x is uniquely determined as $\beta(y)$, except if the latter is zero, which happens if y is on the line l, and z belongs to the projective line $\alpha^{-1}(y)$; in other words, if (y, z) belongs to the blowup Σ_1 of Π at z_0, then x belongs to the projective line cut out by γ. So X must be the blowup of Z along Σ_1.

Remark 4.7. — A local computation allows us to check that the fourfold denoted by Y in the previous proof is smooth. We thus get another Fano fourfold of K3 type, which is not the type considered in this paper: it is not in our list of zero loci of sections of vector bundles.

Fano C–17

(#3-37-149-2) $\mathscr{Z}\big(\mathbb{P}^2\times\mathbb{P}^3\times\mathbb{P}^5, \mathcal{Q}_{\mathbb{P}^3}(0,0,1)\oplus\mathcal{O}(1,0,2)\oplus\mathcal{Q}_{\mathbb{P}^2}(0,1,0)\big)$

- Invariants:

$h^0(-K)=37,\ (-K)^4=149,\ h^{1,1}=3,\ h^{3,1}=1,\ h^{2,2}=24,\ -\chi(T)=21.$

- Description:
  - $\pi_{12}\colon X\to\mathbb{P}^2\times\mathbb{P}^3$ is a conic bundle over $\mathrm{Bl}_p\,\mathbb{P}^3$.
  - $\pi_{13}\colon X\to\mathbb{P}^2\times\mathbb{P}^5$ is a blowup $\mathrm{Bl}_{\mathbb{Q}^2} Y$, where Y is the Fano fourfold Fano C–3, that is $\mathrm{Bl}_{\mathbb{P}^2} X_3$.
  - $\pi_{23}\colon X\to\mathbb{P}^3\times\mathbb{P}^5$ is a blowup $\mathrm{Bl}_{\mathbb{P}^2} Z$, where $Z=\mathrm{Bl}_{\mathbb{P}^1} X_3$ is the Fano fourfold Fano C–4.
  - Rationality: not known. Same as the cubic fourfold containing a plane.
- Semiorthogonal decompositions:
  - 6 exceptional objects and $\mathrm{D^b}(\mathrm{Bl}_p\,\mathbb{P}^3,\mathcal{C}_0)$ from π_{12}.
  - 10 exceptional objects and $\mathrm{D^b}(S_2,\alpha)$ from π_{13} and π_{23}.

Explanation

The sections of the three bundles are given by tensors $\alpha\in\mathrm{Hom}(V_6,V_4)$, $\beta\in\mathrm{Hom}(V_4,V_3)$ and $\gamma\in V_3^\vee\otimes S^2V_6^\vee$. The fourfold X parametrizes the triples of lines (x,y,z) such that

$$\alpha(z)\subset y,\qquad \beta(y)\subset x,\qquad \gamma(x,z,z)=0.$$

We start with the description of π_{12}. The set of pairs (x,y) such that $\beta(y)\subset x$ is the blowup of $\mathbb{P}^3$ at the point $y_0=\mathrm{Ker}(\beta)$. The extra point z must be contained in $\alpha^{-1}(z)$, which defines a projective plane on which γ cuts out a conic. So X is a conic bundle over $X'=\mathrm{Bl}_{y_0}\mathbb{P}^3$. In other words, α can be seen as a morphism of vector bundles over $\mathbb{P}^3$, $\alpha\colon V_6\otimes\mathcal{O}_{\mathbb{P}^3}\to\mathcal{Q}_{\mathbb{P}^3}$, whose rank do not drop for dimension reasons, and whose kernel is exactly $\mathcal{O}(-1)\oplus\mathcal{O}^{\oplus 2}$. This is indeed a conic bundle defined by a morphism $\mathcal{O}_{X'}(-1,0)\to S^2\big(\mathcal{O}_{X'}^{\oplus 2}\oplus\mathcal{O}_{X'}(0,-1)\big)^\vee$, its discriminant locus Δ is given by a morphism

$$\mathcal{O}_{X'}(-3,0)\longrightarrow S_{222}\big(\mathcal{O}_{X'}^{\oplus 2}\oplus\mathcal{O}_{X'}(0,-1)\big)^\vee=\mathcal{O}_{X'}(0,2),$$

hence a section of $\mathcal{O}_{X'}(3,2)$: so Δ is a surface with nef canonical divisor $\mathcal{O}_\Delta(1,0)$. We notice that there are no higher dimensional fibers.

Now we turn to π_{13}. The set of pairs (x,z) such that $\beta\circ\alpha(z)\subset x$ is the blowup of $\mathbb{P}^5$ along the plane $\Pi=\mathbb{P}\big(\mathrm{Ker}(\beta\circ\alpha)\big)$. With the additional condition that $\gamma\big(\beta\circ\alpha(z),z,z\big)=0$ we get the blowup Y of a general cubic fourfold X_3 containing Π. The extra point y is then defined as $\alpha(z)$, except if the latter is zero, which means that z belongs to the line $l=\mathbb{P}(\operatorname{Ker}\alpha)\subset\Pi$. The preimage of this line in Y is a surface isomorphic to a $(1,2)$-divisor F in $\mathbb{P}^2\times\mathbb{P}^1$, hence is a quadratic surface $\mathbb{Q}^2=\mathbb{P}^1\times\mathbb{P}^1$ embedded by $\mathcal{O}(1,2)$. Finally X is the blowup of Y along $\mathbb{Q}^2$.

Finally we describe π_{23}. The set of pairs (y,z) such that $\alpha(z)\subset y$ is the blowup of $\mathbb{P}^5$ along the line l. With the additional condition that $\gamma\big(\beta(y),z,z\big)=0$ we get a divisor Z of bidegree $(1,2)$ in this blowup, containing the projective plane $\Theta=y_0\times\alpha^{-1}(y_0)=y_0\times\Pi$. Thus Z is the blowup of X_3 along the line $l\subset\Pi$; in particular it has to contain a copy of $\operatorname{Bl}_l\Pi\simeq\Pi$, namely Θ. The extra point x is then uniquely defined as $\beta(y)$, except if the latter is zero, that is on the plane Θ, which is blown up in X. Note that the intersection of Θ with the exceptional divisor in Y is just l.

5. FK3 from GM fourfolds

This section contains the list of the Fano fourfolds of K3 type that we obtained inside products of flag manifolds, where at least one projection is a blowup of a Gushel–Mukai fourfold.

Fano GM–18

(#2-22-74-2) $\mathscr{X}\big(\mathbb{P}^2\times\mathrm{Gr}(2,5),\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0)\oplus\mathcal{O}(0,1)\oplus\mathcal{O}(0,2)\big)$

- Invariants:

$h^0(-K)=22,\ (-K)^4=74,\ h^{1,1}=2,\ h^{3,1}=1,\ h^{2,2}=30,\ -\chi(T)=30.$

- Description:
  - $\pi_1\colon X\to\mathbb{P}^2$ is a dP_4 fibration over $\mathbb{P}^2$.
  - $\pi_2\colon X\to\mathrm{Gr}(2,5)$ is a blowup $\operatorname{Bl}_{\mathrm{dP}_2}X_{10}$, where the Gushel–Mukai fourfold X_{10} is general.
  - Rationality: not known. Birational to a general Gushel–Mukai fourfold.
- Semiorthogonal decompositions:
  - 7 exceptional objects and $\mathrm{D^b}(Z,\mathcal{C}_0)$, for $Z\to\mathbb{P}^2$ a $\mathbb{P}^1$-bundle constructed from π_1 (see Appendix B.2.3).
  - 14 exceptional objects and the K3 category of X_{10} from π_2.

Explanation

In general if X is $\mathscr{Z}\big(\mathrm{Gr}(2,n) \times \mathrm{Gr}(2,n+2), \mathcal{Q} \boxtimes \mathcal{U}^\vee\big)$, then outside a point p, X is isomorphic to $\mathrm{Bl}_Y \mathrm{Gr}(2,n+2)$, with $Y \cong \mathscr{Z}\big(\mathbb{P}^{n-1} \times \mathrm{Gr}(2,n+2), \mathcal{Q} \boxtimes \mathcal{U}^\vee\big)$.

We can see this by considering a section of $\mathcal{Q} \boxtimes \mathcal{U}^\vee$ as a morphism from $U_2 \to V_n$, where U_2 is the tautological bundle on $\mathrm{Gr}(2,n+2)$. This morphism is generically injective, and can drop rank twice. If we consider the projection $X \to \mathrm{Gr}(2,n+2)$, this is therefore birational, with fiber $\mathbb{P}^{n-2}$ over a certain Y' (which corresponds to the locus where the morphism has rank 1), and with fiber $\mathrm{Gr}(2,n)$ over a point p. This Y' is singular, being resolved by

$$Y = \big\{U_2 \in \mathrm{Gr}(2,n+2),\ P_1 \in V_n,\ U_2 \to V_n \to V_n/P_1 \to 0\big\}.$$

Therefore $Y \cong \mathscr{Z}\big(\mathbb{P}^{n-1} \times \mathrm{Gr}(2,n+2), \mathcal{Q} \boxtimes \mathcal{U}^\vee\big)$.

The discussion above yields that $\mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Gr}(2,5), \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0)\big)$ is a sixfold birational to the blowup of $\mathrm{Gr}(2,5)$ in a fourfold Y°, singular in one point, which is resolved by the $Y = \mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Gr}(2,5), \mathcal{Q} \boxtimes \mathcal{U}^\vee\big)$. Cutting with two extra $(0,1), (0,2)$ sections we avoid the singular point of Y°. Moreover $\mathscr{Z}\big(Y, \mathcal{O}(0,1) \oplus \mathcal{O}(0,2)\big)$ is a degree 2 del Pezzo surface, and the result follows.

The other projection, by Proposition 3.3, realizes the same sixfold as $\mathbb{G}\mathrm{r}_{\mathrm{Gr}(2,3)}(2, \mathcal{U} \oplus \mathcal{O}^{\oplus 2})$. The other two relative sections cut this sixfold along a dP_4 fibration over $\mathrm{Gr}(2,3) = \mathbb{P}^2$.

Fano GM–19

(#2-23-80-2-A) $\mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Gr}(2,5), \mathcal{O}(0,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,1)\big)$

- Invariants:

$$h^0(-K) = 23,\ (-K)^4 = 80,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 27,\ -\chi(T) = 26.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^2$ is a fibration in dP_5.
  - $\pi_2 \colon X \to \mathrm{Gr}(2,5)$ is a blowup $\mathrm{Bl}_{\mathrm{dP}_5} X_{10}$.
  - Rationality: yes. Birational to a special Gushel–Mukai containing a dP_5.
- Semiorthogonal decompositions:
  - 6 exceptional objects and $\mathrm{D^b}(Z)$, with $Z \to \mathbb{P}^2$ finite flat of degree 5, from π_1, see Appendix B.3.
  - 11 exceptional objects and $\mathrm{D^b}(S_{10})$ from π_2.

Explanation

A section of $\mathcal{Q}(0,1)$ is given by a morphism $\alpha \in \mathrm{Hom}\big(\bigwedge^2 V_5, V_3\big)$, a section of $\mathcal{O}(1,1)$ by a morphism $\beta \in \mathrm{Hom}\big(\bigwedge^2 V_5, V_3^\vee\big)$. A section of $\mathcal{O}(0,1)$ is defined by a skew-symmetric form $\Omega \in \bigwedge^2 U_5^\vee$. The fourfold X can then be described as the set of pairs $(l, P) \in \mathbb{P}^2 \times \mathrm{Gr}(2,5)$ such that

$$\alpha(p_1 \wedge p_2) \in l, \quad \beta(p_1 \wedge p_2)(l) = 0, \quad \Omega(p_1, p_2) = 0$$

for any basis (p_1, p_2) of P. The first two conditions imply that $\beta(p_1 \wedge p_2)\big(\alpha(p_1 \wedge p_2)\big) = 0$, which defines a quadric in $\mathrm{Gr}(2,5)$, while the last condition defines a hyperplane. Therefore the projection to $\mathrm{Gr}(2,5)$ maps X to the intersection of a quadric and a hyperplane, that is a Gushel–Mukai fourfold X_{10}. (Note that by construction the quadric has rank 6 in general.) The exceptional fibers occur over the locus where $\alpha(p_1 \wedge p_2) = 0$ and $\Omega(p_1, p_2) = 0$, which is a dP_5 contained in X_{10}.

The fibers of the projection to $\mathbb{P}^2$ are linear sections of $\mathrm{Gr}(2,5)$, of constant dimension. So this projection is a fibration in dP_5. Since del Pezzo surfaces of degree 5 over any field are rational (see, e.g., [37, p. 624]), X is $k(\mathbb{P}^2)$-rational and hence rational.

Fano GM–20

(#2-26-94-2) $\mathscr{X}\big(\mathrm{Fl}(1,2,5), \mathcal{O}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(0,2)\big)$

- Invariants:

$$h^0(-K) = 26,\ (-K)^4 = 94,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 28,\ -\chi(T) = 28.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^4$ is a conic bundle over a $\mathbb{P}^3 \subset \mathbb{P}^4$, whose discriminant is a sextic surface with 40 ODP.
  - $\pi_2 \colon X \to \mathrm{Gr}(2,5)$ is a blowup $\mathrm{Bl}_{\mathrm{dP}_4} X_{10}$, where the Gushel–Mukai fourfold X_{10} is general.
  - Rationality: unknown. Birational to a general Gushel–Mukai fourfold.
- Semiorthogonal decompositions:
  - 4 exceptional objects and $\mathrm{D^b}(\mathbb{P}^3, \mathcal{C}_0)$ from π_1.
  - 12 exceptional objects and the K3 category of X_{10} from π_2.

Explanation

The flag manifold $\mathrm{Fl}(1,2,5)$ is isomorphic to $\mathbb{P}_{\mathrm{Gr}(2,5)}(\mathcal{U})$. Under this identification $\mathcal{O}(1,0)$ becomes the relative $\mathcal{O}(1)$ of the projective bundle. We can therefore use the Cayley trick, to show that X is the blowup of a general Gushel–Mukai fourfold in the zero locus $\mathscr{Z}\big(\mathrm{Gr}(2,5), \mathcal{U}^\vee \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)\big)$, the latter being a quartic del Pezzo surface.

Remark 5.1. — The conic bundle structure given by π_1 has been described in [61, Corollary 2.3].

Another model. It is interesting to note that there exists another model of a Gushel–Mukai fourfold blown up in a dP_4. This model is embedded in $\mathbb{P}^3 \times \mathbb{P}^8$ as $\mathscr{Z}\big(\mathcal{O}(0,2) \oplus \bigwedge^2 \mathcal{Q}_{\mathbb{P}^2}(0,1) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,1)\big)$. So we start with a quadratic form $q \in S^2V_9^\vee$, and two tensors $\alpha \in \mathrm{Hom}(V_9, V_4)$ and $\beta \in \mathrm{Hom}\big(V_9, \bigwedge^2 V_4\big)$. The fourfold X parametrizes the pairs of lines (x, y) in $\mathbb{P}^3 \times \mathbb{P}^8$ such that $q(y) = 0$, $\alpha(y) \subset x$ and $\beta(y) \subset x \wedge V_4$.

If x is given in $\mathbb{P}^3$, y must belong to $\alpha^{-1}(x)$ and $\beta^{-1}(x \wedge V_4)$, which in general defines a projective plane on which q cuts out a conic. So the projection of X to $\mathbb{P}^3$ is a conic bundle.

If y is given in $\mathbb{P}^8$, its fiber in X is a unique point, or empty, except if $q(y) = 0$, $\alpha(y) = 0$ and $\beta(y)$ is degenerate, in which case we get a projective line. The degeneracy condition can be expressed as $\beta(y) \wedge \beta(y) = 0$, which defines a quadric, in general smooth when restricted to $\mathbb{P}(\mathrm{Ker}\,\alpha)$. So the projection of X to $\mathbb{P}^8$ should be the blowup of a smooth fourfold Y along a dP_4. This fourfold is defined by the condition that $q(y) = 0$ and $\alpha(y) \wedge \beta(y) = 0$. Observe that we may interpret α and β as defining a map $\gamma\colon V_9 \to \bigwedge^2 V_5$, where $V_5 = V_4 \oplus \mathbb{C}e_0$ so that $\bigwedge^2 V_5 \simeq \bigwedge^2 V_4 \oplus V_4$. Explicitly $\gamma(y) = \beta(y) + \alpha(y) \wedge e_0$. The condition $\alpha(y) \wedge \beta(y) = 0$ implies that $\beta(y) = \alpha(y) \wedge v$ for some vector $v \in V_4$ (or $\alpha(y) = 0$), in which case we deduce that $\gamma(y) = \alpha(y) \wedge (v + e_0)$ is decomposable. This means that Y is contained in a hyperplane section of $\mathrm{Gr}(2, V_5)$, and the quadric condition implies that Y is in fact a Gushel–Mukai variety.

We remark that there exists a link between this Fano variety and Fano K3–31, investigated in the recent [13].

Fano GM–21

(#2-30-114-2) $\mathscr{Z}\big(\mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5), \mathcal{O}(0,2) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(1,0)\big)$

- Invariants:

$$h^0(-K) = 30,\ (-K)^4 = 114,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 24,\ -\chi(T) = 24.$$

- Description:
  - $\pi_1 \colon X \to \mathrm{Gr}(2,4)$ is a conic bundle on $\mathbb{Q}^3$ with discriminant divisor a singular quartic surface.
  - $\pi_2 \colon X \to \mathrm{Gr}(2,5)$ is a blowup $\mathrm{Bl}_{\mathbb{P}^1 \times \mathbb{P}^1} X_{10}$ of a Gushel–Mukai fourfold along a σ-quadric surface.
  - Rationality: unknown.
- Semiorthogonal decompositions:
  - 6 exceptional objects and $\mathrm{D^b}(\mathbb{Q}^3, \mathcal{C}_0)$ from π_1.
  - 8 exceptional objects and the K3 category of X_{10} from π_2.

Explanation

By Proposition 3.3 the zero locus of $\mathscr{Z}\big(\mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5), \mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}\big)$ is the blowup $\mathrm{Bl}_{\mathbb{P}^3} \mathrm{Gr}(2,5)$. The two extra sections cut $\mathrm{Gr}(2,5)$ in a generic Gushel–Mukai fourfold X_{10}. The quadratic section cuts the $\mathbb{P}^3$ in a quadric surface Σ. In terms of the morphism $\theta \in \mathrm{Hom}(V_5, V_4)$ defining the section of $\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}$, our fourfold X parametrizes pairs of planes (P,Q) such that $\theta(Q) \subset P$. So P is uniquely determined by Q, unless the latter meets the kernel of α: this defines the $\mathbb{P}^3$ blown up inside $\mathrm{Gr}(2,5)$, and we conclude that π_2 is just the blowup of X_{10} along Σ.

The fiber of π_1 over a plane P is the intersection of the Gushel–Mukai fourfold with the variety of planes in $\theta^{-1}(P)$. This means we need to cut a projective plane with a quadric, yielding the conic bundle structure. In order to understand the discriminant locus, note that the latter projective plane is $\mathbb{P}\big(\bigwedge^2 \theta^{-1}(P)\big)$, and that since we restrict a fixed quadric to this plane, the discriminant is a section of

$$\det\big(\textstyle\bigwedge^2 \theta^{-1}(P)^\vee\big)^2 = \det\big(\theta^{-1}(P)^\vee\big)^4 = \det(P^\vee)^4.$$

So the discriminant locus is a quartic surface in $\mathbb{Q}^3$.

We remark that there exists a link between this Fano variety and Fano K3–35, investigated in the recent [13].

Fano GM–22

(#2-33-130-2) $\mathscr{Z}\big(\mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5), \mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(1,1)\big)$

- Invariants:

$h^0(-K) = 33,\ (-K)^4 = 130,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 23,\ -\chi(T) = 22.$

- Description:
  - $\pi_1 \colon X \to \mathrm{Gr}(2,4)$ is a blowup of $S_{10}^{(1)}$, a K3 surface of genus 6 blown up at one point.
  - $\pi_2 \colon X \to \mathrm{Gr}(2,5)$ is a blowup $\mathrm{Bl}_{\mathbb{P}^2} X_{10}$.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 7 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_{10})$ from both maps (a general X_3 containing a σ-plane also contains a dP_5, see Appendix B.1.4).

Explanation

We start by applying Proposition 3.3 to the first bundle, whose section cuts $\mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$ in $\mathrm{Bl}_{\mathbb{P}^3} \mathrm{Gr}(2,5)$. The other two sections are then identified with a linear section of $\mathrm{Gr}(2,5)$ cutting the $\mathbb{P}^3$ in a plane, and a quadratic section containing it. The special $\mathbb{P}^3$ is of the form $\mathrm{Gr}(1, V_4)$: the resulting plane is therefore a σ-plane in the notation of [22]. In Proposition 7.1, *loc. cit.*, it is shown that a Gushel–Mukai fourfold containing such a special plane is rational.

Now consider the projection π_1 to $\mathrm{Gr}(2,4)$. The section of $\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}$ is defined by a morphism θ from V_5 to V_4, and it vanishes on the pairs (A,B) such that $B \in \theta^{-1}(A)$. For a given A, this defines a projective plane, on which the two other line bundle sections define two linear conditions. So π_1 is birational and its exceptional locus is the degeneracy locus of the induced morphism

$$\mathcal{O}_{\mathrm{Gr}(2,4)} \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}(-1) \longrightarrow \bigwedge^2 \theta^{-1}(\mathcal{U})^\vee \simeq \mathcal{U}^\vee \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}(-1),$$

where $\mathcal{U} = \mathcal{U}_{\mathrm{Gr}(2,4)}$ is the tautological bundle. This degeneracy locus is a smooth surface S whose structure sheaf is resolved by the Eagon–Northcott complex

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathrm{Gr}(2,4)}(-2) \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}(-3) \longrightarrow \mathcal{O}_{\mathrm{Gr}(2,4)}(-1) \oplus \mathcal{U}(-1) \longrightarrow \\ \longrightarrow \mathcal{O}_{\mathrm{Gr}(2,4)}(1) \longrightarrow \mathcal{O}_S(1) \longrightarrow 0. \end{aligned}$$

Since $\mathcal{U}$ and $\mathcal{U}(-1)$ are acyclic by Borel–Weil–Bott, we deduce that $\chi(\mathcal{O}_S) = 2$, while $\chi\big(\mathcal{O}_S(1)\big) = 6$ and $\chi\big(\mathcal{O}_S(2)\big) = 20-1 = 19$, hence the Hilbert polynomial must be

$$\chi\big(\mathcal{O}_S(k)\big) = \frac{9k^2 - k}{2} + 2.$$

Dualizing the previous complex, we get

$$0 \longrightarrow \mathcal{O}_{\mathrm{Gr}(2,4)}(-4) \longrightarrow \mathcal{O}_{\mathrm{Gr}(2,4)}(-2) \oplus \mathcal{U}^\vee(-2) \longrightarrow \\ \longrightarrow \mathcal{O}_{\mathrm{Gr}(2,4)}(-1) \oplus \mathcal{O}_{\mathrm{Gr}(2,4)} \longrightarrow \omega_S \longrightarrow 0.$$

So ω_S has a canonical section, vanishing on the curve D where the morphism $\mathcal{O}_{\mathrm{Gr}(2,4)}(-2) \oplus \mathcal{U}^\vee(-2) \to \mathcal{O}_{\mathrm{Gr}(2,4)}(-1)$ is not surjective, that is, the zero locus of a section of $\mathcal{O}_{\mathrm{Gr}(2,4)}(1) \oplus \mathcal{U}^\vee$. But the zero locus of a nonzero section of $\mathcal{U}^\vee$ is a plane, so D is just a projective line. By the usual arguments we deduce that S is a K3 surface of genus 6 blown up at one point.

Fano GM–23

(#2-30-115-2-B) $\mathscr{Z}\big(\mathrm{Fl}(1,3,5), \mathcal{O}(0,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{R}_2(0,1)\big)$

- Invariants:

$$h^0(-K) = 30,\ (-K)^4 = 115,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 24,\ -\chi(T) = 23.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^4$ is birational, with base locus in $\mathbb{P}^4$ a singular surface S° obtained by identifying two points on a non-minimal K3.
  - $\pi_2 \colon X \to \mathrm{Gr}(3,5)$ is a blowup $\mathrm{Bl}_{\mathbb{Q}^2} X_{10}$ of a Gushel–Mukai fourfold along a quadratic surface.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 8 exceptional objects and the K3 category of X_{10} from π_2.

Explanation

In the variety of flags $U_1 \subset U_3 \subset V_5$, the fourfold X is defined by sections of $\det(\mathcal{U}_3)^\vee$, $\mathcal{U}_1^\vee \otimes \det(\mathcal{U}_3)^\vee$ and $\mathcal{U}_1^\vee \otimes (\mathcal{U}_3/\mathcal{U}_1)^\vee$.

Consider the projection to $\mathbb{P}^4$. The fibers are copies of $\mathrm{Gr}(2,4)$ and the fibers of the restriction to the fourfold X are given by sections of $\mathcal{U}^\vee \oplus \mathcal{O}(1)^{\oplus 2}$; hence they are points, generically. So the map $\pi_1 \colon X \to \mathbb{P}^4$ must be birational.

Where are the non-trivial fibers? There are two possible reasons that could explain that the zero locus of a section of $\mathcal{U}^* \oplus \mathcal{O}(1)^{\oplus 2}$ on $\mathrm{Gr}(2,4)$ is positive dimensional. First the component on $\mathcal{U}^\vee$ could vanish identically. Since this section is defined by a two-form ω, this will happen over the point p of $\mathbb{P}^4$ defined by the kernel of this two-form, whose fiber will be a quadratic surface. Outside this point, the section of $\mathcal{U}^\vee$ defines a $\mathbb{P}^2$, and we will have non-trivial fibers when the two sections of $\mathcal{O}(1)$ are linearly dependent. This should happen over a surface $S^\circ \subset \mathbb{P}^4$. What is this surface?

The two-form ω defines a morphism $\mathcal{O}(-1) \to \mathcal{Q}^\vee$ on $\mathbb{P}^4$, which is not injective (because of the kernel of ω). If it was, the quotient $\mathcal{K} = \mathcal{Q}^\vee(-1)/\mathcal{O}(-2)$ would be a rank 3 bundle and we could describe S° as a degeneracy locus for a morphism

$$\mathcal{K} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2) \longrightarrow \bigwedge^2 \mathcal{Q}^\vee.$$

In order to make this correct, we need to blow up p. Denote $Y = \mathrm{Bl}_p \mathbb{P}^4$, π the blow down to $\mathbb{P}^4$, E the exceptional divisor, H the pull-back of the hyperplane divisor. On Y there is an injective morphism $\mathcal{O}(-2H + E) \to \mathcal{Q}^\vee(-H)$, whose quotient is a rank 3 vector bundle $\mathcal{F}$. Then we can define a smooth surface S in Y as the degeneracy locus of the induced morphism

$$\mathcal{F}(E) \oplus \mathcal{O}(-H) \oplus \mathcal{O}(-2H) \longrightarrow \bigwedge^2 \mathcal{Q}^\vee.$$

The structure sheaf of S is resolved by the Eagon–Northcott complex

$$\begin{aligned} 0 \longrightarrow \mathcal{F}(E) \oplus \mathcal{O}(-H) \oplus \mathcal{O}(-2H) \longrightarrow \bigwedge^2 \mathcal{Q}^\vee \longrightarrow \\ \longrightarrow \mathcal{O}(3H - 2E) \longrightarrow \mathcal{O}_S(3H - 2E) \longrightarrow 0. \end{aligned}$$

We deduce that the class of S in the Chow ring of Y (where $HE = 0$) is $[S] = 8H^2 + 2E^2$. In particular the image S° of S in $\mathbb{P}^4$ has degree 8. Moreover the morphism $S \to S^\circ$ contracts the curve $C = S \cap E$, whose self intersection $C^2 = [S]E^2 = 2E^4 = -2$. This means that C is either a (-2)-curve or the disjoint union of two (-1)-curves, and we need to decide whether it is connected or not. For this we need to compute $h^1\big(\mathcal{O}_S(-E)\big)$, which we can do with a suitable twist of the Eagon–Northcott complex.

Lemma 5.2. — *The bundle $\mathcal{F}(3E - 3H)$ is acyclic, but $h^q\big(\mathcal{F}(2E - 3H)\big) = \delta_{q,3}$. As a consequence,*

$$h^1(\mathcal{O}_S) = 0, \qquad h^2(\mathcal{O}_S) = 2, \qquad h^1\big(\mathcal{O}_S(-E)\big) = 1.$$

Proof. — From the exact sequence that defines $\mathcal{F}$, we get

$$0 \longrightarrow \mathcal{O}\big((k+1)E - 5H\big) \longrightarrow \mathcal{Q}^\vee(kE - 4H) \longrightarrow \mathcal{F}(kE - 3H) \longrightarrow 0.$$

For $0 \leqslant k \leqslant 3$, $\mathcal{O}(kE)$ has no higher direct images by π, while $\pi_*\mathcal{O}(kE) = \mathcal{O}_{\mathbb{P}^4}$. So $Q^\vee(kE - 4H)$ has the same cohomology as $Q^\vee(-4)$ on $\mathbb{P}^4$, and by

Borel–Weil–Bott this bundle is acyclic. So

$$h^q\big(\mathcal{F}(kE-3H)\big) = h^{q+1}\big(\mathcal{O}\big((k+1)E-5H\big)\big) = h^{3-q}\big(\mathcal{O}\big((2-k)E\big)\big)$$

by Serre duality, since $\omega_Y = \mathcal{O}(3E-5H)$. This implies the first claim.

Now consider the resolution of $\mathcal{O}_S$ given by the Eagon–Northcott complex. The bundle $\bigwedge^2 \mathcal{Q}^\vee(2E-3H)$ has the same cohomology as $\bigwedge^2 \mathcal{Q}^\vee(-3)$ on $\mathbb{P}^4$, which is again acyclic by Borel–Weil–Bott. Moreover, since $\mathcal{F}(3E-3H)$ is also acyclic, we deduce that $h^1(\mathcal{O}_S) = 0$ while $h^2(\mathcal{O}_S) = h^4\big(\mathcal{O}(2E-5H)\big) = 1$. But when we twist the Eagon–Northcott complex by $\mathcal{O}(-E)$, we get a non-trivial contribution to $h^1\big(\mathcal{O}_S(-E)\big) = h^3\big(\mathcal{F}(2E-3H)\big) = 1$. $\square$

From $h^1(\mathcal{O}_S) = 0$ and $h^1\big(\mathcal{O}_S(-E)\big) = 1$ we can now conclude that C has two connected components, hence $C = C_1 + C_2$ is the disjoint union of two (-1)-curves on S. The projection to S° contracts these two curves to the same point p, at which S° is therefore not normal. Using Riemann–Roch we compute that $H\omega_S = 2$ and $C\omega_S = -2$. Now, dualizing the Eagon–Northcott complex we get

$$\begin{aligned} 0 \longrightarrow \mathcal{O}(3E-5H) \longrightarrow \bigwedge^2 \mathcal{Q}(E-2H) \longrightarrow& \\ \longrightarrow \mathcal{F}^\vee(-2H) \oplus \mathcal{O}(E-H) \oplus \mathcal{O}(E) &\longrightarrow \omega_S \longrightarrow 0. \end{aligned}$$

In particular $\omega_S(-C) = \mathcal{O}_S(D)$ is effective, and we get $DC = 0$ and $DH = 2$. So D is a smooth conic or the disjoint union of two lines, and in both cases we can conclude that S is once again a non-minimal K3 surface.

Remark 5.3. — Another Eagon–Northcott complex should allow us to decide whether D is connected or not.

Now consider the projection π_2 of X to $\mathrm{Gr}(3,5)$. The section of $\det(\mathcal{U}_3)^\vee$ defines a hyperplane section H of $\mathrm{Gr}(3,5)$ that contains the image. The section of $\mathcal{U}_1^\vee \otimes (\mathcal{U}_3/\mathcal{U}_1)^\vee$ is defined by a skew-symmetric two-form ω and U_1 must be contained in the kernel of this two-form restricted to U_3. This restriction is an element of $\bigwedge^2 U_3^\vee \simeq U_3 \otimes \det(U_3)^\vee$, which, when nonzero, gives a line in U_3 that coincides with the kernel. Then the section of $\mathcal{U}_1^\vee \otimes \det(\mathcal{U}_3)^\vee$, which is an element of $U_3^\vee \otimes \det(U_3)^\vee$, gives a line of linear forms that need to vanish on this kernel, yielding a section of $\det(\mathcal{U}_3^\vee)^2$. As a consequence, the image of X in $\mathrm{Gr}(3,5)$ is the intersection of a quadric and a hyperplane, hence a Gushel–Mukai fourfold.

Finally, π_2 has non-trivial fibers when ω is zero on U_3. This condition defines a quadric of 3-spaces containing $\mathrm{Ker}(\omega)$, that intersects H along a quadric surface.

6. FK3 from K3 surfaces

This section contains the list of the Fano fourfolds of K3 type that we obtained inside products of flag manifolds, where at least one projection is a blowup with center birational to a K3 surface, and no cubic or Gushel–Mukai fourfold is involved.

Fano K3–24

(#2-19-60-2) $\mathscr{Z}\big(\mathbb{P}^1 \times \mathrm{Gr}(2,5), \mathcal{O}(1,1) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1)\big)$

- Invariants:

$$h^0(-K) = 19,\ (-K)^4 = 60,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 32,\ -\chi(T) = 31.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^1$ is a fibration in W_{12}, Fano threefolds of index 1 and degree 12.
  - $\pi_2 \colon X \to \mathrm{Gr}(2,5)$ is a blowup $\mathrm{Bl}_{S_{12}} X_{12}$.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 4 exceptional objects and an unknown category from π_1.
  - 16 exceptional objects and $\mathrm{D^b}(S_{12})$ from π_2.

Explanation

It suffices to apply Lemma 3.1 to describe π_2. This proves the rationality of this fourfold by [53, Theorem 3.3]. On the other hand, $\mathbb{P}^1$ parametrizes a family of hyperplane sections of X_{12}.

Fano K3–25

(#2-21-70-2) $\mathscr{Z}\big(\mathbb{P}^1 \times \mathrm{Gr}(2,6), \mathcal{O}(1,1) \oplus \mathcal{O}(0,1)^{\oplus 4}\big)$

- Invariants:

$$h^0(-K) = 21,\ (-K)^4 = 70,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 28,\ -\chi(T) = 27.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^1$ is a fibration in W_{14}, Fano threefolds of index 1 and degree 14.

  - $\pi_2\colon X \to \mathrm{Gr}(2,6)$ is a blowup $\mathrm{Bl}_{S_{14}} X_{14}$.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 4 exceptional objects and an unknown category from π_1.
  - 12 exceptional objects and $\mathrm{D^b}(S_{14})$ from π_2.

Explanation

It suffices to apply Lemma 3.1 to describe π_2. On the other hand, π_1 is clearly a fibration in W_{14}. By [53, Theorem 3.3] W_{14} is rational, and so is our fourfold X as well.

Fano K3–26

(#2-23-80-2-B) $\mathscr{Z}\left(\mathbb{P}^1 \times \mathrm{Gr}(3,6), \mathcal{O}(1,1) \oplus \mathcal{O}(0,1)^{\oplus 2} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,6)}\right)$

- Invariants:

$$h^0(-K) = 23,\ (-K)^4 = 80,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 24,\ -\chi(T) = 23.$$

- Description:
  - $\pi_1\colon X \to \mathbb{P}^1$ is a fibration in W_{16}, Fano threefolds of index 1 and degree 16.
  - $\pi_2\colon X \to \mathrm{Gr}(3,6)$ is a blowup $\mathrm{Bl}_{S_{16}} X_{16}$ (recall that the index 2 Fano fourfold X_{16} is a codimension 2 linear section of the Lagrangian Grassmannian $\mathrm{LG}(3,6)$).
  - Rationality: unknown. Birational to a general X_{16}.
- Semiorthogonal decompositions:
  - 4 exceptional objects and an unknown category from π_1.
  - 8 exceptional objects and $\mathrm{D^b}(S_{16})$ from π_2.

Explanation

It suffices to apply Lemma 3.1 to describe π_2. On the other hand, π_1 is clearly a fibration in W_{16}, hyperplane sections of X_{16}. The rationality of X_{16} is unknown, see [53, Section 4].

Fano K3–27

(#2-32-124-2) $\mathscr{Z}\big(\mathrm{Fl}(1,2,8),\mathcal{Q}_2\oplus\mathcal{O}(1,1)\oplus\mathcal{O}(0,2)^{\oplus 2}\big)$

- Invariants:

$h^0(-K)=32,\ (-K)^4=124,\ h^{1,1}=2,\ h^{3,1}=1,\ h^{2,2}=28,\ -\chi(T)=28.$

- Description:
  - $\pi_1\colon X\to\mathbb{P}^7$ is a blowup $\mathrm{Bl}_{v_0}(Q_1^\circ\cap Q_2^\circ\cap\mathbb{Q}^6)$, where Q_i° are quadratic cones with vertex v_0.
  - $\pi_2\colon X\to\mathrm{Gr}(2,8)$ is a blowup $\mathrm{Bl}_{S_8}(\mathbb{Q}_1^5\cap\mathbb{Q}_2^5)$ of the intersection of two quadrics along a K3 surface of genus 5.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 12 exceptional objects and $\mathrm{D^b}(S_8)$ from π_2.

Explanation

By [20, Corollary 2.7], $\mathscr{Z}\big(\mathrm{Fl}(1,2,8),\mathcal{Q}_2\big)$ can be identified with $\mathbb{P}_{\mathbb{P}^6}\big(\mathcal{O}\oplus\mathcal{O}(-1)\big)$. The two $\mathcal{O}(0,2)$ sections further cut the base $\mathbb{P}^6$ in the smooth complete intersection $Y=\mathbb{Q}_1^5\cap\mathbb{Q}_2^5$ of two quadrics. By the Cayley trick, $\mathscr{Z}\big(\mathbb{P}_Y\big(\mathcal{O}\oplus\mathcal{O}(-1)\big),\mathcal{L}\otimes\mathcal{O}_Y(1)\big)$ is isomorphic to the blowup of Y in $\mathscr{Z}\big(\mathbb{P}^6,\mathcal{O}(2)^3\oplus\mathcal{O}(1)\big)$, which is a general K3 surface of degree 8. Note that Y is classically known to be rational, see e.g. [62], so X is rational as well.

Now consider the projection to $\mathbb{P}^7$. The section of $\mathcal{Q}_2$ defines a point $[v_0]\in\mathbb{P}^7$ and outside this point, the fiber of $[v]$ is the plane in V_8 generated by v and v_0. The two sections of $\mathcal{O}(0,2)$ become quadrics Q_1°,Q_2° in $\mathbb{P}^7$ defined by equations of the form $Q_i(v\wedge v_0)=0$, in particular these are quadratic cones singular at $[v_0]$. The section of $\mathcal{O}(1,1)$ becomes a non-singular quadratic form vanishing at v_0. We deduce that the projection of X to $\mathbb{P}^7$ is the blowup $\mathrm{Bl}_{v_0}(Q_1^\circ\cap Q_2^\circ\cap\mathbb{Q}^6)$, with exceptional divisor the intersection of two quadrics in $\mathbb{P}^5$.

Fano K3–28

(#2-25-90-2-A) $\mathscr{Z}\big(\mathbb{P}^1\times\mathrm{Gr}(2,7),\mathcal{O}(0,1)\oplus\mathcal{O}(1,1)\oplus\mathcal{Q}^\vee_{\mathrm{Gr}(2,7)}(0,1)\big)$

- Invariants:

$h^0(-K)=25,\ (-K)^4=90,\ h^{1,1}=2,\ h^{3,1}=1,\ h^{2,2}=22,\ -\chi(T)=21.$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^1$ is a fibration in W_{18}, Fano threefolds of index 1 and degree 18.
  - $\pi_2 \colon X \to \mathrm{Gr}(2,7)$ is a blowup $\mathrm{Bl}_{S_{18}} X_{18}$.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 4 exceptional objects and an unknown category from π_1.
  - 6 exceptional objects and $\mathrm{D^b}(S_{18})$ from π_2.

Explanation

This follows from Lemma 3.1. Rationality follows from [53, Theorem 3.3].

Fano K3–29

(#2-39-160-2-A) $\mathscr{X}\big(\mathbb{P}^2 \times \mathbb{P}^4, \mathcal{O}(1,1) \oplus \mathcal{O}(1,2)\big)$

- Invariants:

$$h^0(-K) = 39,\ (-K)^4 = 160,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 22,\ -\chi(T) = 21.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^2$ is a quadric bundle with discriminant a smooth sextic.
  - $\pi_2 \colon X \to \mathbb{P}^4$ is a blowup $\mathrm{Bl}_{S_8^{(1)}} \mathbb{P}^4$ along a K3 surface of degree 8 blown up at one point.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 6 exceptional objects and $\mathrm{D^b}(S_2, \alpha)$ from π_1.
  - 6 exceptional objects and $\mathrm{D^b}(S_8)$ from π_2.

Explanation

The fibers of the projection π_1 to $\mathbb{P}^4$ are defined by the image of the induced morphism

$$\mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4}(-2) \longrightarrow V_3^\vee \otimes \mathcal{O}_{\mathbb{P}^4}.$$

Generically this morphism has rank 2 and the fiber is one point, so π_1 is birational. The fibers are projective lines when the morphism drops rank, which happens over a smooth surface S whose ideal is resolved by an Eagon–Northcott complex

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-4) \oplus \mathcal{O}_{\mathbb{P}^4}(-5) \longrightarrow V_3^\vee \otimes \mathcal{O}_{\mathbb{P}^4}(-3) \longrightarrow \mathcal{I}_S \longrightarrow 0.$$

From this complex one deduces the Hilbert polynomial

$$\chi\big(\mathcal{O}_S(k)\big) = \frac{7k^2 - k}{2} + 2.$$

In particular S has degree 7, arithmetic genus 1, sectional genus 5. According to [57, Theorem 6], the surface S must be an inner projection of a K3 surface of degree 8 in $\mathbb{P}^5$.

On the other hand, $\mathscr{Z}\big(\mathbb{P}^2 \times \mathbb{P}^4, \mathcal{O}(1,1)\big)$ is $\mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$ with $\mathcal{E} = \Omega^1_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$ by the Euler sequence. The quadric fibration is given by a map $\mathcal{E} \to \mathcal{E}^\vee(1)$, and the degeneracy locus has degree $c_1(\mathcal{E}^\vee) + 4 = 6$.

Fano K3–30

(#2-39-160-2-B) $\mathscr{Z}\big(\mathbb{P}^1 \times \mathrm{Gr}(2,4), \mathcal{O}(1,2)\big)$

- Invariants:

$h^0(-K) = 39,\ (-K)^4 = 160,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 22,\ -\chi(T) = 21.$

- Description:
  - $\pi_1\colon X \to \mathbb{P}^1$ is a fibration whose fibers are intersections of two quadrics in $\mathbb{P}^5$.
  - $\pi_2\colon X \to \mathrm{Gr}(2,4)$ is a blowup $\mathrm{Bl}_{S_8}\mathbb{Q}^4$ of a quadric along a K3 surface of genus 5.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 4 exceptional objects and $\mathrm{D^b}(Z, \mathcal{C}_0)$, for $Z \to \mathbb{P}^1$ a Hirzebruch surface.
  - 6 exceptional objects and $\mathrm{D^b}(S_8)$ from π_2.

Explanation

We use Lemma 3.1 to determine π_2. In order to understand π_1, just observe that each fiber is given by the intersection of $\mathrm{Gr}(2,4)$ with an extra quadratic section, hence the result.

Remark 6.1. — This Fano fourfold already appeared in [24], under the name (B2).

Fano K3–31

(#2-23-80-2-C) $\mathscr{Z}\big(\mathbb{P}^3 \times \mathbb{P}^6, \mathcal{O}(0,2) \oplus \mathcal{O}(1,1) \oplus \bigwedge^2 \mathcal{Q}_{\mathbb{P}^3}(0,1)\big)$

- Invariants:

$h^0(-K) = 23,\ (-K)^4 = 80,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 28,\ -\chi(T) = 27.$

- Description:
  - $\pi_1\colon X \to \mathbb{P}^3$ is a conic bundle whose discriminant is a sextic hypersurface.
  - $\pi_2\colon X \to \mathbb{P}^6$ is a blowup $\mathrm{Bl}_{S_{10}}(\mathbb{Q}^5_1 \cap \mathbb{Q}^5_2)$ of the intersection of two quadrics, for S_{10} a K3 surface of genus 6.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 4 exceptional objects and $\mathrm{D^b}(\mathbb{P}^3, \mathcal{C}_0)$ from π_1.
  - 12 exceptional objects and $\mathrm{D^b}(S_{10})$ from π_2.

Explanation

Let us first consider the sixfold Y defined as the zero locus of a general section of $\mathcal{Q}^\vee(1,1)$ inside $\mathbb{P}^3 \times \mathbb{P}^6$. The space of sections is $\bigwedge^2 V_4 \vee \otimes V_7^\vee = \mathrm{Hom}\big(V_7, \bigwedge^2 V_4\big)$. If $\theta \in \mathrm{Hom}\big(V_7, \bigwedge^2 V_4\big)$ defines Y, the latter parametrizes the pairs $(x,y) \in \mathbb{P}^3 \times \mathbb{P}^6$ such that $\theta(y) \subset x \wedge V_4$. In particular $\theta(y)$ has rank 2 or zero, hence $\theta(y) \wedge \theta(y) = 0$. This equation defines a rank 6 quadric $Q^\circ \subset \mathbb{P}^6$, with vertex v given by the kernel of θ. Note that if y is a smooth point of Q°, $\theta(y)$ defines a plane in V_4 whose projectivization is the fiber over y of the projection $Y \to Q^\circ$; while over the vertex the fiber is the whole $\mathbb{P}^3$.

Now when we cut Y by a section of $\mathcal{O}(0,2)$ we are restricting the previous construction to the intersection of Q° with a general smooth quadric Q', hence we avoid the singularity and obtain $Y' \to Q^\circ \cap Q'$ as a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{F})$ over the smooth intersection of two quadrics in $\mathbb{P}^6$.

Finally, we cut Y' by a section of $\mathcal{O}(1,1)$ and we apply the Cayley trick to see that $\pi_2\colon X \to Q^\circ \cap Q'$ is the blowup along a surface S cut by a general section of $\mathcal{F}^\vee(1)$, which is a K3 surface. In fact $\mathcal{F}$ is the pullback of the tautological bundle $\mathcal{U}$ by the projection p to $\mathrm{Gr}(2,4)$, and the restriction of this projection to S is a double cover over a dP_5 (embedded in $\mathrm{Gr}(2,4)$ as the zero locus of a section of $\mathcal{U}^\vee(1)$), branched over a quadratic section. The embedding of S into $Q^\circ \cap Q'$ gives a complex

$$0 \longrightarrow p^*\mathcal{U}^\vee(-3) \longrightarrow p^*\,\mathrm{End}(\mathcal{U})(-1) \longrightarrow p^*\mathcal{U}^\vee \longrightarrow p^*\mathcal{U}^\vee_S \longrightarrow 0.$$

By Borel–Weil–Bott $h^0\big(\mathcal{U}^\vee(-3)\big) = 1$. This implies that $h^0(\mathcal{U}_S^\vee) = 5$ and that S is a K3 surface of genus 6.

To determine π_1, we notice that $\mathscr{Z}\big(\mathbb{P}^3 \times \mathbb{P}^6, \mathcal{O}(1,1) \oplus \bigwedge^2 \mathcal{Q}_{\mathbb{P}^3}(0,1)\big)$ is a $\mathbb{P}^2$-bundle on $\mathbb{P}^3$ and we cut it with a quadratic relative section. More precisely the $\mathbb{P}^2$-bundle is $\mathbb{P}(\mathcal{H})$ where $\mathcal{H}$ is the sub-bundle of the trivial bundle with fiber V_7 defined by the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \bigwedge^2 \mathcal{Q}_{\mathbb{P}^3}^\vee \longrightarrow \mathcal{O}_{\mathbb{P}^3} \otimes V_7^\vee \longrightarrow \mathcal{H}^\vee \longrightarrow 0.$$

So the discriminant is given by a section of $(\det \mathcal{H}^\vee)^2 = \mathcal{O}_{\mathbb{P}^3}(6)$.

We remark that there exists a link between this Fano variety and Fano GM–20, investigated in the recent [13].

Fano K3–32

(#2-29-110-2) $\qquad\qquad$ $\mathscr{Z}\big(\mathrm{Fl}(1,2,5), \mathcal{O}(1,1) \oplus \mathcal{O}(0,1)^{\oplus 2}\big)$

- Invariants:

$$h^0(-K) = 29,\ (-K)^4 = 110,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 22,\ -\chi(T) = 21.$$

- Description:
  - $\pi_1\colon X \to \mathbb{P}^4$ is a blowup $\mathrm{Bl}_{S_{12}^{(1)}} \mathbb{P}^4$, where $S_{12}^{(1)}$ is a K3 surface of degree 12 blown up in a point.
  - $\pi_2\colon X \to \mathrm{Gr}(2,5)$ is a blowup $\mathrm{Bl}_{S_{12}} X_5$ of a K3 surface of genus 7 in a Fano fourfold X_5 of degree 5 and index 3.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 6 exceptional objects and $\mathrm{D^b}(S_{12})$ from both maps. Note that the surfaces of degree 12 are derived equivalent (see Lemma 3.4) but we don't know if they are isomorphic.

Explanation

We can interpret the flag $\mathrm{Fl}(1,2,5)$ as $\mathbb{P}_{\mathrm{Gr}(2,5)}(\mathcal{U})$. The two $\mathcal{O}(0,1)$ sections cut the base Grassmannian in the fourfold X_5. We are left with $\mathscr{Z}\big(\mathbb{P}_{X_5}\big(\mathcal{U}\big|_{X_5}\big), \mathcal{L} \otimes \mathcal{O}_{X_5}(1)\big)$, where $\mathcal{L}$ denotes the relative line bundle, or equivalently with $\mathscr{Z}\big(\mathbb{P}_{X_5}\big(\mathcal{U}(-1)\big|_{X_5}\big), \mathcal{L}\big)$. We conclude by the Cayley trick, since $\mathscr{Z}\big(\mathrm{Gr}(2,5), \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{U}^\vee(1)\big)$ is a general K3 surface of degree 12.

When l is a line in V_5, the planes that contain it generate a linear space $l \wedge V_5$ in $\bigwedge^2 V_5$. This defines a vector bundle $\mathcal{K} \simeq \mathcal{Q}(-1)$, of rank 4. The fibers of the projection of X to $\mathbb{P} = \mathbb{P}^4$ are defined by the image of the induced morphism

$$\mathcal{O}_{\mathbb{P}}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \mathcal{K}^\vee,$$

which drops rank along a smooth surface S whose structure sheaf is resolved by the Eagon–Northcott complex

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-4)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \mathcal{K}^\vee(-4) \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

By Borel–Weil–Bott Theorem, $\mathcal{K}^\vee(-4)$ and $\mathcal{K}^\vee(-3)$ are acyclic, while the only non-zero cohomology of $\mathcal{K}^\vee(-2)$ is $h^1 = 1$. We deduce that $\chi(\mathcal{O}_S) = 2$, while $\chi\big(\mathcal{O}_S(1)\big) = 5$ and $\chi\big(\mathcal{O}_S(2)\big) = 15+1 = 16$. So the Hilbert polynomial must be

$$\chi\big(\mathcal{O}_S(k)\big) = 4k^2 - k + 2.$$

Proposition 6.2. — *S is a K3 surface of genus 7 blown up at one point, and embedded in $\mathbb{P}^4$ by a tangential projection from $\mathbb{P}^7$.*

This can be confirmed using [58, Proposition 2.7].

Fano K3–33

(#2-27-100-2) $\mathscr{X}\big(\mathbb{P}^5 \times \mathrm{Gr}(2,4), \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(1,1) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)\big)$

- Invariants:

$$h^0(-K) = 27,\ (-K)^4 = 100,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 24,\ -\chi(T) = 23.$$

- Description:
  - $\pi_1, \pi_2 \colon X \to \mathbb{P}^5, \mathrm{Gr}(2,4)$ are both blowups $\mathrm{Bl}_{S_{12}^{(2)}} \mathbb{Q}^4$ of a quadric along a K3 surface of genus 7 blown up in two points.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 8 exceptional objects and $\mathrm{D^b}(S_{12})$ from both maps. Note that the surfaces of degree 12 are derived equivalent (see Lemma 3.4) but we don't know if they are isomorphic.

Explanation

Let us start with the fivefold Y defined by the two rank 2 bundles and their global sections, which are tensors $\alpha \in V_6^\vee \otimes V_4 = \mathrm{Hom}(V_6, V_4)$ and $\beta \in V_6^\vee \otimes V_4^\vee = \mathrm{Hom}(V_6, V_4^\vee)$. So $Y \subset \mathbb{P}^5 \times \mathrm{Gr}(2,4)$ parametrizes pairs $(l \subset V_6, P \subset V_4)$ such that $\alpha(l) \subset P \subset \beta(l)^\perp$. In particular its image in $\mathbb{P}^5$ is the quadric Q defined by the equation $\big\langle \beta(l), \alpha(l) \big\rangle = 0$, which is smooth in general. Moreover the generic fiber is a projective line, and the exceptional fibers are projective planes over the two disjoint lines $\delta = \mathbb{P}\big(\mathrm{Ker}(\alpha)\big)$ and $\delta' = \mathbb{P}\big(\mathrm{Ker}(\beta)\big)$.

The fiber over P of the projection to $\mathrm{Gr}(2,4)$ is the intersection of the three-planes $\mathbb{P}\big(\alpha^{-1}(P)\big)$ and $\mathbb{P}\big(\alpha^{-1}(P^\perp)\big)$, hence a projective line in general. The exceptional fibers are linear spaces of bigger dimension, exactly over the degeneracy locus of the morphism

$$\mathcal{U} \oplus \mathcal{Q}^\vee \longrightarrow V_6^\vee \otimes \mathcal{O}_{\mathrm{Gr}(2,4)}$$

induced by α and β. Since the bundle of those morphisms is generated by global sections, for α, β generic this degeneracy locus is a smooth curve C, defined by the associated Eagon–Northcott complex

$$\begin{aligned} 0 \longrightarrow S^2(\mathcal{U} \oplus \mathcal{Q}^\vee) \longrightarrow L_6 \otimes (\mathcal{U} \oplus \mathcal{Q}^\vee) \longrightarrow L_{15} \longrightarrow& \\ \longrightarrow \mathcal{O}_{\mathrm{Gr}(2,4)}(2) \longrightarrow \mathcal{O}_C(2) \longrightarrow 0,& \end{aligned}$$

where L_k is trivial of rank k. The first bundle of this complex contains a factor $\mathcal{U} \otimes \mathcal{Q}^\vee$ isomorphic to the cotangent of $\mathrm{Gr}(2,4)$, which has therefore a one-dimensional first cohomology group. This is the only non-trivial contribution to the cohomology of this complex and we deduce that $\chi\big(\mathcal{O}_C(2)\big) = 20 - 15 + 1 = 6$. After a twist by $\mathcal{O}_{\mathrm{Gr}(2,4)}(-1)$, all the terms are acyclic by Borel–Weil–Bott Theorem except $S^2\mathcal{U}(-1)$ and $S^2\mathcal{Q}^\vee(-1)$ which have a one-dimensional second cohomology group, hence $\chi\big(\mathcal{O}_C(1)\big) = 6 - 2 = 4$. After another twist, all the terms are acyclic except $\mathcal{U} \otimes \mathcal{Q}^\vee(-2)$ which has a one-dimensional third cohomology group, hence $\chi(\mathcal{O}_C) = 1 + 1 = 2$. This is more than enough to deduce that the Hilbert polynomial of C is $\chi\big(\mathcal{O}_C(k)\big) = 2k + 2$, so that C must be the union of two disjoint lines d and d'. Note the symmetry of the picture:

$$\begin{tikzcd}
 & E \arrow[r, hook] \arrow[dl, "\mathbb{P}^2"'] & Y \arrow[dl, "\mathbb{P}^1"] \arrow[dr, "\mathbb{P}^1"'] & & F \arrow[ll, hook'] \arrow[dr, "\mathbb{P}^2"] & \\
\delta \cup \delta' \arrow[r, hook] & Q & & \mathrm{Gr}(2,4) & & d \cup d' . \arrow[ll, hook']
\end{tikzcd}$$

Now we turn to the fourfold X defined as the hyperplane section of Y defined by a section of $\mathcal{O}(1,1)$, hence a tensor $\gamma \in V_6^\vee \otimes \bigwedge^2 V_4^\vee$. The general fibers of the two projections are projective lines cut by a hyperplane, hence single points, so that the two projections are birational. Over $\mathrm{Gr}(2,4)$, the exceptional fibers are obtained over a locus S that can be described as before. The sections that define X induce a morphism

$$\mathcal{F} := \mathcal{U} \oplus \mathcal{Q}^\vee \oplus \mathcal{O}_{\mathrm{Gr}(2,4)}(-1) \longrightarrow V_6^\vee \otimes \mathcal{O}_{\mathrm{Gr}(2,4)}.$$

Generically, this morphism is surjective, its kernel is a line in V_6 defining the point in the preimage. The rank drops in codimension 2, over a smooth surface S over which the fibers of the projection are projective lines; so the projection to $\mathrm{Gr}(2,4)$ should be the blowup of S. Note that S has to contain the two lines d and d'.

$$\begin{tikzcd}
 & E_X \arrow[rr, hook] \arrow[dl, "\mathbb{P}^1"'] & & X \arrow[dl, "\mathbb{P}^1"] \arrow[dr, "\mathbb{P}^0"'] & & F_X \arrow[ll, hook'] \arrow[dr, "\mathbb{P}^1"] & \\
\delta \cup \delta' \subset T \arrow[rr, hook] & & Q & & \mathrm{Gr}(2,4) & & S \supset d \cup d'. \arrow[ll, hook']
\end{tikzcd}$$

Proposition 6.3. — *S is the blowup at two points of a K3 surface of genus 7, d and d' being the two exceptional curves of the surface.*

Proof. — The ideal of S in $\mathrm{Gr}(2,4)$ is resolved by the Eagon–Northcott complex

$$0 \longrightarrow \mathcal{F} \otimes \det(\mathcal{F}) \longrightarrow V_6 \otimes \det(\mathcal{F}) \longrightarrow \mathcal{O}_{\mathrm{Gr}(2,4)} \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

We have $\det(\mathcal{F}) = \mathcal{O}_{\mathrm{Gr}(2,4)}(-3)$, so $\mathcal{F} \otimes \det(\mathcal{F})$ has a factor $\mathcal{O}_{\mathrm{Gr}(2,4)}(-4) = K_{\mathrm{Gr}(2,4)}$ which implies that $h^2(\mathcal{O}_S) = 1$. We compute from the previous complex that the Hilbert polynomial of S is $\chi\big(\mathcal{O}_S(k)\big) = 5k^2 - k + 2$, so that $H^2 = 10$ and $H\omega_S = 2$. Moreover, dualizing the Eagon–Northcott complex we get

$$0 \longrightarrow \mathcal{O}_{\mathrm{Gr}(2,4)} \longrightarrow V_6(-1) \longrightarrow \mathcal{F}^\vee(-1) = \mathcal{U} \oplus \mathcal{Q}^\vee \oplus \mathcal{O}_{\mathrm{Gr}(2,4)} \longrightarrow \omega_S \longrightarrow 0.$$

In particular ω_S admits a canonical section, which vanishes exactly on the locus where the $V_6(-1) \to \mathcal{U} \oplus \mathcal{Q}^\vee$ is not surjective. But this locus is exactly $C = d \cup d'$! We conclude that $\omega_S = d + d'$. Then by the genus formula d and d' are two (-1)-curves, and after contracting them we get a surface $S_{\min}$ with trivial canonical bundle and $h^2(\mathcal{O}_{S_{\min}}) = 1$, hence a K3 surface. $\square$

Note that a K3 surface of degree 12, hence genus 7, can be embedded in $\mathbb{P}^7$; projecting from two of its points, one obtains in $\mathbb{P}^5$ the blowup of the surface at the two chosen points.

Now we turn to the other projection to the quadric Q defined by α and β. The fiber over $l \in Q$ is the set of planes P such that $\alpha(l) \subset P \subset \beta(l)^{\perp}$ and $\gamma(l, p \wedge p') = 0$ for any $p, p' \in P$, where $\gamma \in V_6^{\vee} \otimes \bigwedge^2 V_4^{\vee}$ is the tensor defining the section of $\mathcal{O}(1,1)$. For l general, the first condition defines a $\mathbb{P}^1$, on which the second condition, being linear, cuts a single point; the whole $\mathbb{P}^1$ is in the fiber exactly when $\gamma(l, \alpha(l) \wedge \beta(l)^{\perp}) = 0$, a codimension 2 condition that defines the surface T over which X has non-trivial fibers. Note that the lines δ and δ' are contained in T.

Generically, $\alpha(l) \wedge \beta(l)^{\perp}$ is a dimension 2 subspace of $\bigwedge^2 V_4$, but since the rank jumps over δ and δ' we will need to blow up these two lines in Q; denote by $\widetilde{Q}$ the resulting fourfold, with Δ and Δ' the two exceptional divisors. Let L be the pullback of the hyperplane bundle on Q. On $\widetilde{Q}$ we have a complex

$$L^{\vee}(\Delta) \hookrightarrow V_4 \otimes \mathcal{O}_{\tilde{Q}} \longrightarrow L(-\Delta'),$$

where the two morphisms have constant rank 1, so the kernel of the second one is a rank 3 bundle $\mathcal{M}$ containing the image of $L^{\vee}(\Delta)$ via the first one. As a consequence, we get a rank 2 vector sub-bundle $\mathcal{P} = L^{\vee}(\Delta) \wedge \mathcal{M}$ of $\bigwedge^2 V_4 \otimes \mathcal{O}_{\tilde{Q}}$, and then a surface $\widetilde{T}$ in $\widetilde{Q}$ defined by the vanishing of a section of $\mathscr{H}om(\mathcal{P}, L)$. By construction, $\widetilde{T}$ is a smooth surface dominating T. Using the fact that $\det(\mathcal{P}) = L^{\vee}(\Delta) \otimes \det(\mathcal{M})$ we compute, by adjunction, that $\omega_{\tilde{T}} = C + C'$ is the restriction of $\Delta + \Delta'$.

Now consider the Koszul complex that resolves the structure sheaf of $\widetilde{T}$, namely

$$0 \longrightarrow \mathcal{O}(-4L + \Delta + \Delta') \longrightarrow \mathcal{P}(-L) \longrightarrow \mathcal{O}_{\tilde{Q}} \longrightarrow \mathcal{O}_{\tilde{T}} \longrightarrow 0.$$

Using the fact that $\Delta\Delta' = 0$ in the Chow ring of $\widetilde{Q}$, since these two divisors do not intersect, we deduce that the fundamental class of $\widetilde{T}$ is given by

$$[\widetilde{T}] = 5L^2 - 3L(\Delta + \Delta') + (\Delta + \Delta')^2.$$

Denoting by λ the restriction of L, we deduce that $\lambda^2 = 5L^4 = 10$. Moreover $\lambda C = [\widetilde{T}]L\Delta = 5L^3\Delta - 3L^3\Delta^2 + L\Delta^3 = L\Delta^3$; indeed, Δ dominates a line in Q so $L^2\Delta = 0$. Of course $L\Delta^3$ can be computed by restricting to Δ: since L cuts a single point of δ over which Δ is just $\mathcal{O}_{\mathbb{P}^2}(-1)$, we get $\lambda C = L\Delta^3 = 1$. Again restricting to Δ and applying the usual projection formula for projective bundles, we get that $\Delta^4 = s_1(N_\delta) = s_1\big(TQ\big|_\delta\big) - s_1(T\delta) = 4 - 2 = 2$. This yields

$$C^2 = [\widetilde{T}]\Delta^2 = 5L^2\Delta^2 - 3L\Delta^3 + \Delta^4 = 5 \times 0 - 3 \times 1 + 2 = -1.$$

Note that $\lambda C = 1$ means that C is projecting isomorphically to δ in T, so that $\widetilde{T}$ is in fact isomorphic to T. Moreover, C and C' are two (-1)-curves and since $\omega_{\tilde{T}} = C + C'$, we get a genuine K3 surface $T_{\min}$ after contracting them. We have finally proved:

Proposition 6.4. — *The first projection of X to the smooth quadric Q is also a doubly blown up K3 surface T of genus 7.*

We finally get a non-trivial birational transformation of the quadric of dimension 4, as the composition of the blowup of S with the inverse of the blowup of T. In fact, we can compute the canonical bundle of X in terms of the two hyperplane divisors as $\omega_X = -H - H'$, but also in terms of the exceptional divisors as $\omega_X = -4H + E_X = -4H' + F_X$; in particular $H' = 3H - E_X$, which means that our birational involution of Q is defined by the linear system of cubics containing T (or S). Note the striking similarity with the birational involution of $\mathbb{P}^3$ defined by the linear system of cubics containing a curve of degree 6 and genus 3 [42]. Note also that since $\chi(T) = -23$, we expect a family with $23 = 19 + 2 + 2$ parameters, meaning that the K3 surface and its two points should be generic.

Remark 6.5. — The two K3 surfaces $S_{\min}$ and $T_{\min}$ are derived equivalent by Lemma 3.4. We cannot prove that they are not isomorphic, but note that a generic K3 surface of genus 7 has exactly one Fourier-Mukai partner [56], which can be constructed in terms of (HP) dual sections to the spinor tenfold. Presumably $S_{\min}$ and $T_{\min}$ should be non-isomorphic Fourier-Mukai partners. Moreover, our construction indicates that generically, choosing two points on $S_{\min}$ also yields two points on $T_{\min}$, in perfect agreement with the fact that the punctual Hilbert schemes $S_{\min}^{[2]}$ and $T_{\min}^{[2]}$ are birationally equivalent [55].

Fano K3–34

(#2-26-95-2) $\mathscr{Z}\big(\mathbb{P}^3 \times \mathrm{Gr}(2,5), \mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(1,1)\big)$

- Invariants:

$$h^0(-K) = 26,\ (-K)^4 = 95,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 23,\ -\chi(T) = 22.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^3$ is a conic bundle over $\mathbb{P}^3$ with discriminant locus $S_8^{(1)}$, the blowup of a degree 8 K3 surface in one point.
  - $\pi_2 \colon X \to \mathrm{Gr}(2,5)$ is a blowup $\mathrm{Bl}_{S_8^{(1)}} X_5$, with $S_8^{(1)}$ as above, and X_5 a Fano fourfold of index 3 and degree 5.
  - Rationality: yes.

- Semiorthogonal decompositions:
  - 4 exceptional objects and $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{C}_0)$ from π_1.
  - 7 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_8)$ from π_2.

Explanation

Let us start with the fivefold Y defined by two sections of $\mathcal{O}(0,1)$, that is two skew-symmetric forms on V_5, and a section of $\mathcal{U}^\vee(1,0)$, that is a tensor $\alpha \in V_4^\vee \otimes V_5^\vee$. So Y parametrizes the pairs (l, P), with P a doubly isotropic plane in V_5, and l a line in V_4 such that the linear form $\alpha(l)$ vanishes on P. For α generic, $\alpha(l)$ is never zero, and the fiber of Y over l is a codimension 2 linear section of $\mathrm{Gr}(2, \alpha(l)^\perp)$. On the other hand, if P is fixed there are two possibilities. Generically, $\alpha^t(P)$ is a plane in $V_4^\vee$ and the fiber of Y over P is a projective line. But if P contains the kernel of α^t, the fiber is a projective plane: this happens over a projective line d.

$$\begin{tikzcd}
 & Y \arrow[dl, "\text{quadric fibration}"'] \arrow[dr, "\mathbb{P}^1"] & & E \arrow[ll, hook'] \arrow[dr, "\mathbb{P}^2"] & \\
\mathbb{P}^3 & & I_2\,\mathrm{Gr}(2,5) & & d. \arrow[ll, hook']
\end{tikzcd}$$

Now we turn to the fourfold X defined in Y by a section of $\mathcal{O}(1,1)$, that is a tensor $\theta \in V_4^\vee \otimes \bigwedge^2 V_5^*$. The fibers of the projection to $Z = I_2\,\mathrm{Gr}(2,5)$ are defined by the image of the induced morphism

$$\mathcal{U}_Z \oplus \mathcal{O}_Z(-1) \longrightarrow V_4^\vee \otimes \mathcal{O}_Z,$$

which drops rank along a smooth surface S whose structure sheaf is resolved by the Eagon–Northcott complex

$$0 \longrightarrow \mathcal{U}_Z \oplus \mathcal{O}_Z(-1) \longrightarrow V_4^\vee \otimes \mathcal{O}_Z \longrightarrow \mathcal{O}_Z(2) \longrightarrow \mathcal{O}_S(2) \longrightarrow 0.$$

On the Grassmannian $\mathrm{Gr}(2,5)$, Borel–Weil–Bott Theorem implies that $\mathcal{U}(-k)$ is acyclic for $0 \leqslant k \leqslant 4$. The Koszul complex of Z in $\mathrm{Gr}(2,5)$ allows us to deduce that $\mathcal{U}_Z(-k)$ is acyclic for $0 \leqslant k \leqslant 2$. Moreover $\chi(\mathcal{O}_Z(2)) = 31$, hence $\chi(\mathcal{O}_S(2)) = 27$. Also $\chi(\mathcal{O}_S(1)) = \chi(\mathcal{O}_Z(1)) = 8$, and $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_Z) + \chi(\mathcal{O}_Z(-3)) = 1 + 1 = 2$. We deduce the Hilbert polynomial

$$\chi(\mathcal{O}_Z(k)) = \frac{13k^2 - k}{2} + 2.$$

Using the same kind of arguments as we already did many times, we get:

Proposition 6.6. — *S is a K3 surface of genus 8 blown up at a single point, and d is its exceptional curve.*

Now we turn to the conic bundle of X over $\mathbb{P}^3$.

$$\begin{tikzcd}
 & X \arrow[dl, "\text{conic fibration}"'] \arrow[dr, "\mathbb{P}^0"'] & & E_X \arrow[ll, hook'] \arrow[dr, "\mathbb{P}^1"] & \\
\mathbb{P}^3 & & I_2\,\mathrm{Gr}(2,5) & & S. \arrow[ll, hook']
\end{tikzcd}$$

To understand it better, notice how the fourfold X can be equivalently described as $\mathscr{Z}\big(\mathrm{Fl}(1,3,5), \mathcal{O}(1,0) \oplus \mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{O}(1,1)\big)$. The first bundle cuts the flag in $\mathbb{G}_{\mathbb{P}^3}\big(2, \mathcal{Q}(-1) \oplus \mathcal{O}(-1)\big)$. Therefore this becomes a quadric fibration if we cut with $\mathcal{O}(0,1)^{\oplus 2}$, and a conic bundle with discriminant locus S when we cut with the last $\mathcal{O}(1,1)$.

Fano K3–35

(#2-25-90-2-B) $\mathscr{Z}\big(\mathrm{Gr}(2,4)_1 \times \mathrm{Gr}(2,4)_2, \mathcal{O}(0,1) \oplus \mathcal{U}^{\vee}_{\mathrm{Gr}(2,4)_2}(1,0) \oplus \mathcal{O}(1,1)\big)$

- Invariants:

$$h^0(-K) = 25,\ (-K)^4 = 90,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 24,\ -\chi(T) = 23.$$

- Description:
  - $\pi_1 \colon X \to \mathrm{Gr}(2,4)_1$ is birational, with base locus a genus 6 K3 surface with two double points.
  - $\pi_2 \colon X \to \mathrm{Gr}(2,4)_2$ is a conic bundle over $\mathbb{Q}^3$, with discriminant a quartic hypersurface with 20 ODP.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 4 exceptional objects and $\mathrm{D^b}(\mathbb{Q}^3, \mathcal{C}_0)$ from π_2. There is no hint for the K3 category.

Explanation

Over $\mathrm{Gr}(2,U_4) \times \mathrm{Gr}(2,V_4)$, with tautological bundles $\mathcal{U}$ and $\mathcal{V}$, we have the line bundles $\det(\mathcal{V})^{\vee}$ and $\det(\mathcal{U})^{\vee} \otimes \det(\mathcal{V})^{\vee}$, and the rank 2 bundle $\det(\mathcal{U})^{\vee} \otimes \mathcal{V}$. Their global sections are given by tensors $\omega \in \bigwedge^2 V_4^{\vee}$, $\theta \in \bigwedge^2 U_4^{\vee} \otimes \bigwedge^2 V_4^{\vee}$ and $\alpha \in \bigwedge^2 U_4^{\vee} \otimes V_4$. The sections ω and α define a fivefold Y parametrizing the pairs U_2, V_2 of planes such that V_2 is ω-isotropic and contains $\alpha\big(\bigwedge^2 U_2\big)$. So V_2 has to belong to $IG_{\omega}(2,V_4) \simeq \mathbb{Q}^3$ and its fiber in Y is a codimension 2 linear section of $\mathrm{Gr}(2,U_4)$. So Y is a quadratic surface bundle over $\mathbb{Q}^3$, with a quadric hypersurface for discriminant.

When we take into account the remaining section θ, we get our Fano fourfold X as a conic bundle over $\mathbb{Q}^3$, with discriminant Δ.

Lemma 6.7. — *In general, the discriminant $\Delta \subset \mathbb{Q}^3$ is a quartic surface with exactly 20 ODP.*

Proof. — The conic bundle over $\mathbb{Q}^3 \subset \mathrm{Gr}(2, V_4)$ is contained in $\mathbb{P}(\mathcal{E})$, for $\mathcal{E}$ a rank 3 vector bundle defined by an exact sequence

$$0 \longrightarrow \mathcal{U} \oplus \mathcal{O}_{\mathbb{Q}^3}(-1) \longrightarrow \bigwedge^2 U_4^\vee \otimes \mathcal{O}_{\mathbb{Q}^3} \longrightarrow \mathcal{E}^\vee \longrightarrow 0.$$

The conic bundle is therefore defined by the natural morphism $S^2\mathcal{E} \to S^2\left(\bigwedge^2 U_4\right) \otimes \mathcal{O}_{\mathbb{Q}^3} \to \det(U_4) \otimes \mathcal{O}_{\mathbb{Q}^3}$. The Chern class of $\mathcal{E}$ is given in terms of the Schubert cycles σ_1 and σ_2 by

$$c(\mathcal{E}) = \frac{s(\mathcal{U}^\vee)}{1-\sigma_1} = \frac{1+\sigma_1+\sigma_2}{1-\sigma_1}.$$

We deduce that $c_1(\mathcal{E}) = 2\sigma_1$, $c_2(\mathcal{E}) = 2\sigma_1^2 + \sigma_2$, $c_3(\mathcal{E}) = 2\sigma_1^3 + \sigma_1\sigma_2$, hence $c_1(\mathcal{E})c_2(\mathcal{E}) = 2c_3(\mathcal{E})$. Applying Lemma 3.5, we compute the number N of singular points in Δ, as

$$\begin{aligned} N &= 4\int_{\mathbb{Q}^3} \big(c_1(\mathcal{E})c_2(\mathcal{E}) - c_3(\mathcal{E})\big) \\ &= 4\int_{\mathrm{Gr}(2,4)} \sigma_1(2\sigma_1^3 + \sigma_1\sigma_2) \\ &= 4(2\times 2+1) \\ &= 20. \end{aligned}$$

Moreover these must be ordinary double points in general. □

On the other hand, if $\alpha\left(\bigwedge^2 U_2\right)$ is non-zero, the ω-isotropic planes in V_4 containing this line form a projective line. Generically, $\alpha\left(\bigwedge^2 U_2\right) = 0$ for exactly two transverse planes P_1 and P_2, defining two points p_1, p_2 of $\mathrm{Gr}(2, U_4)$ whose fibers Q_1 and Q_2 are copies of $IG_\omega(2, V_4) \simeq \mathbb{Q}^3$. Hence the diagram:

$$\begin{tikzcd} & Q_1 \cup Q_2 \arrow[r, hook] \arrow[dl, "\mathbb{Q}^3"'] & Y \arrow[dl, "\mathbb{P}^1"] \arrow[dr, "\text{quadric fibration}"] & \\ p_1 \cup p_2 \arrow[r, hook] & \mathrm{Gr}(2, U_4) & & \mathbb{Q}^3. \end{tikzcd}$$

When we cut Y by the hyperplane section defined by θ, this shows that our fourfold X projects birationally to $\mathrm{Gr}(2, U_4)$. Where are the exceptional fibers? They occur exactly when the projective line in $\mathrm{Gr}(2, V_4)$ defined by the conditions that $\alpha\left(\bigwedge^2 U_2\right) \subset V_2 \subset \alpha\left(\bigwedge^2 U_2\right)^\perp$ is contained in the hyperplane defined by $\theta\left(\bigwedge^2 U_2\right)$, which means that $\theta\left(\bigwedge^2 U_2\right)$ vanishes on the linear space $P_U = \alpha\big(\bigwedge^2 U_2\big) \wedge \alpha\big(\bigwedge^2 U_2\big)^{\perp} \subset \bigwedge^2 V_4$. But this is meaningful only when $\alpha\big(\bigwedge^2 U_2\big) \neq 0$; in other words, P_U does not define a vector bundle on the Grassmannian.

In order to overcome this problem, we need to pass to the blowup $\widetilde{G}$ of $\mathrm{Gr}(2, U_4)$ at the two points $p_1 = [P_1]$ and $p_2 = [P_2]$. Denote by $E = E_1 + E_2$ the exceptional divisor, and by H the pull-back of the hyperplane divisor. Over $\widetilde{G}$, α and its contraction with ω define two embeddings of vector bundles

$$\mathcal{L} \simeq \mathcal{O}_{\tilde{G}}(E - H) \hookrightarrow V_4 \otimes \mathcal{O}_{\tilde{G}}, \qquad \mathcal{L}' \simeq \mathcal{O}_{\tilde{G}}(E - H) \hookrightarrow V_4^{\vee} \otimes \mathcal{O}_{\tilde{G}}.$$

Dualizing the second one, we get a rank 3 sub-bundle $\mathcal{M}$ of $V_4 \otimes \mathcal{O}_{\tilde{G}}$, and by the skew-symmetry of ω this bundle contains $\mathcal{L}$ as a sub-bundle. So $\mathcal{P} := \mathcal{L} \wedge \mathcal{M}$ is now a genuine rank 2 sub-bundle of $\bigwedge^2 V_4 \otimes \mathcal{O}_{\tilde{G}}$. As we observed, the exceptional locus of our projection is now described by the vanishing of the induced section of $\mathcal{P}^{\vee}(H)$. This must be a smooth surface S, whose structure sheaf is resolved by the Koszul complex

$$0 \longrightarrow \bigwedge^2 \mathcal{P}(-2H) \longrightarrow \mathcal{P}(-H) \longrightarrow \mathcal{O}_{\tilde{G}} \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

Note that $\bigwedge^2 \mathcal{P} \simeq \mathcal{L}^2$, and that $\mathcal{L}^2(-2H) = \omega_{\tilde{G}}(-E)$, which will imply that $h^2(\mathcal{O}_S) = 1$. Moreover, by adjunction $\omega_S = \mathcal{O}_S(E)$.

In order to compute the cohomology of $\mathcal{P}$, observe that it is a sub-bundle of $\mathcal{K} := \mathcal{L} \wedge V_4$, with quotient $\mathcal{L} \otimes (V_4/\mathcal{M})$, which is trivial. We thus get the exact sequence

$$0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_{\tilde{G}} \longrightarrow 0,$$

while the rank 3 vector bundle $\mathcal{K}$ is given by the simple sequence

$$0 \longrightarrow \mathcal{L}^2 \longrightarrow \mathcal{L} \otimes V_4 \longrightarrow \mathcal{K} \longrightarrow 0.$$

From the latter we readily deduce that $\mathcal{K}(-H)$ is acyclic, and then that $\mathcal{P}(-H)$ is acyclic as well, and therefore that $\chi(\mathcal{O}_S) = 2$. After a twist by H, $\chi(\mathcal{P}) = -1$ yields $\chi\big(\mathcal{O}_S(H)\big) = 6 - (-1) = 7$. After another twist, $\chi\big(\mathcal{K}(H)\big) = 4$ implies $\chi\big(\mathcal{P}(H)\big) = -2$ and then $\chi\big(\mathcal{O}_S(H)\big) = 20 - (-2) = 22$. Therefore we get the Hilbert polynomial

$$\chi\big(\mathcal{O}_S(kH)\big) = 5k^2 + 2.$$

Proposition 6.8. — *S is a K3 surface of genus six blown up at four points.*

Observe that in the Chow ring of $\widetilde{G}$ we have $HE = 0$. It is then easy to compute that the class of S is

$$[S] = c_2\big(\mathcal{P}^\vee(H)\big) = \frac{(1+2H-E)^4}{(1+H)(1+3H-2E)}\bigg|_{\deg=2} = 5H^2 + 2E^2.$$

Since $K_S = E_S$, we deduce that $K_S^2 = [S]E^2 = 2E^4 = -4$. This is explained as follows. Over p_1 or p_2, the exceptional divisor is a copy of $\mathbb{P}^3$ and the restriction of $\mathcal{L}$ is the tautological line bundle. Therefore, the restriction of $\mathcal{K}$ is the so-called *null-correlation bundle* $\mathcal{E} := \mathcal{L} \wedge \mathcal{L}^\perp$. There is an exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow T_{\mathbb{P}^3}(-2) = \mathcal{L} \wedge V_4 \longrightarrow \mathcal{L} \otimes (V_4/\mathcal{L}^\perp) = \mathcal{O}_{\mathbb{P}^3} \longrightarrow 0,$$

which implies that $h^1(\mathcal{E}) = 1$, and $h^k(\mathcal{E}) = 0$ for $k \neq 1$. Consider then a general section of the globally generated vector bundle $\mathcal{E}^\vee$. It vanishes on a smooth curve C such that $K_C = \mathcal{O}_C(-2)$ since $\det(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^3}(-2)$. Moreover the Koszul complex

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

shows that $h^0(\mathcal{O}_C) = h^0(\mathcal{O}_{\mathbb{P}^3}) + h^1(\mathcal{E}) = 2$. We conclude that the general section of the dual of the null-correlation bundle vanishes on the union of two skew lines.

This gives us four lines in S, which are four (-1)-curves on S. Contracting them we get a genuine K3 surface $S_{\min}$ of genus 6. Finally, note that these four lines are contracted in pairs to p_1 and p_2 in $\mathrm{Gr}(2, U_4)$; so the image of S, which is the base locus of the birational projection of X to $\mathrm{Gr}(2, U_4)$, is a non-normal image of $S_{\min}$ obtained by identifying two pairs of points.

We remark that there exists a link between this Fano variety and Fano GM–21, investigated in the recent [13].

Fano K3–36

(#3-19-60-2) $\mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^4, \mathcal{O}(0,0,3) \oplus \mathcal{O}(1,1,1)\big)$

- Invariants:

$$h^0(-K) = 19,\ (-K)^4 = 60,\ h^{1,1} = 3,\ h^{2,1} = 5,\ h^{3,1} = 1,\ h^{2,2} = 22,\ -\chi(T) = 23.$$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ is a dP_3-fibration.

- $\pi_{13}, \pi_{23}\colon X \to \mathbb{P}^1 \times W_3$ are blowups $\mathrm{Bl}_{S_{12}}(\mathbb{P}^1 \times W_3)$, for a general cubic threefold W_3, along a K3 surface of degree 12 and Picard number 2.
- Rationality: unknown.

• Semiorthogonal decompositions:
- 4 exceptional objects and an unknown category from π_{12}.
- 4 exceptional objects, two [5/3]-CY categories and $\mathrm{D^b}(S_{12})$ from π_{13} or π_{23}.

Explanation

We use Lemma 3.1 to get the blowup structure given by π_{23} and π_{13}. We check directly that π_{12} has general fiber a dP_3.

Fano K3–37

(#3-23-80-2-A) $\mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^5, \mathcal{O}(0,0,2)^{\oplus 2} \oplus \mathcal{O}(1,1,1)\big)$

• Invariants:

$$h^0(-K) = 23,\ (-K)^4 = 80,\ h^{1,1} = 3,\ h^{2,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 33,\ -\chi(T) = 20.$$

• Description:
- $\pi_{12}\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ is a dP_4-fibration.
- $\pi_{13}, \pi_{23}\colon X \to \mathbb{P}^1 \times \mathbb{P}^5$ are $\mathrm{Bl}_{S_{16}}(\mathbb{P}^1 \times W_4)$, where W_4 is the intersection of two quadrics in $\mathbb{P}^5$.
- Rationality: yes.

• Semiorthogonal decompositions:
- 4 exceptional objects and $\mathrm{D^b}(Z, \mathcal{C}_0)$, for $Z \to \mathbb{P}^1 \times \mathbb{P}^1$ a $\mathbb{P}^1$-bundle from π_{12}.
- 4 exceptional objects, two copies of $\mathrm{D^b}(C)$ for C a genus 2 curve, and $\mathrm{D^b}(S_{16})$ from π_{13}, π_{23}.

Explanation

Simple application of Lemma 3.1. The K3 surface in $\mathbb{P}^1 \times \mathbb{P}^5$ can be checked to have degree 16 with respect to $\mathcal{O}(1,1)$, and of course Picard rank 2.

Fano K3–38

(#3-27-100-2-A) $\mathscr{X}\big(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{Gr}(2,5), \mathcal{O}(0,0,1)^{\oplus 3} \oplus \mathcal{O}(1,1,1)\big)$

- Invariants:

$h^0(-K) = 27,\ (-K)^4 = 100,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 22,\ -\chi(T) = 18.$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ is a dP_5-fibration,
  - $\pi_{13}, \pi_{23}\colon X \to \mathbb{P}^1 \times \mathrm{Gr}(2,5)$ are blowups $\mathrm{Bl}_{S_{20}}(\mathbb{P}^1 \times W_5)$, with S_{20} a K3 surface of degree 20 and generic Picard rank 2, and W_5 a Fano threefold of index 2 and degree 5.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 8 exceptional objects and $\mathrm{D^b}(Z)$, for $Z \to \mathbb{P}^1 \times \mathbb{P}^1$ a flat degree 5 cover, from π_{12}.
  - 8 exceptional objects and $\mathrm{D^b}(S_{20})$ from π_{13} or π_{23}.
  - The two decompositions are not known to coincide but it is natural to imagine that Z should be a K3 surface, derived equivalent to S_{20}.

Explanation

It suffices to apply Lemma 3.1 to understand the second and third projections.

For the projection to $\mathbb{P}^1 \times \mathbb{P}^1$, just observe that the section of $\mathcal{O}(1,1,1)$ gives an extra linear condition on each fiber.

Fano K3–39

(#3-29-110-2-A) $\mathscr{X}\big(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathrm{Gr}(2,4), \mathcal{Q}_{\mathbb{P}^2}(0,0,1) \oplus \mathcal{O}(1,1,1)\big)$

- Invariants:

$h^0(-K) = 29,\ (-K)^4 = 110,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 24,\ -\chi(T) = 20.$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ is a conic bundle.
  - $\pi_{13}\colon X \to \mathbb{P}^1 \times \mathrm{Gr}(2,4)$ is a blowup of a conic in a hypersurface of bidegree $(1,2)$, itself Fano K3–30 a blowup of $\mathrm{Gr}(2,4)$ along a K3 surface of degree 8.

  - $\pi_{23}\colon X \to \mathbb{P}^2 \times \mathrm{Gr}(2,4)$ is a blowup $\mathrm{Bl}_{S_{22}} Y$, where $Y \subset \mathbb{P}^2 \times \mathrm{Gr}(2,4)$ is the blowup $\mathrm{Bl}_{C_2} \mathrm{Gr}(2,4)$ of a smooth conic C_2.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 6 exceptional objects and $\mathrm{D^b}(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{C}_0)$ from π_{12}.
  - 10 exceptional objects and $\mathrm{D^b}(S_8)$ from π_{13}.
  - 10 exceptional objects and $\mathrm{D^b}(S_{22})$ from π_{23}.
  - We deduce that S_8 and S_{22} are derived equivalent (see Lemma 3.4).

Explanation

First we use Lemma 3.1 to identify $\mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Gr}(2,4), \mathcal{Q}_{\mathbb{P}^2}(0,1) \oplus \mathcal{O}(1,1)\big)$ with the blowup Y of $\mathrm{Gr}(2,4)$ along a conic C_2. Then we use Lemma 3.1 again to cut a (generically of Picard rank 2) K3 surface of degree 22 with respect to $\mathcal{O}(1,1)$ in Y. We can project further to $\mathrm{Gr}(2,4)$ to obtain a K3 surface of degree 8 containing the conic C_2.

To understand better the other maps, we consider the equivalent description of this Fano fourfold as $\mathscr{Z}\big(\mathbb{P}^1 \times \mathrm{Fl}(1,4,6), \mathcal{Q}_2^{\oplus 3} \oplus \mathcal{O}(0;2,0) \oplus \mathcal{O}(1;1,1)\big)$. We first notice that $\mathscr{Z}\big(\mathrm{Fl}(1,4,6), \mathcal{Q}_2^{\oplus 3} \oplus \mathcal{O}(2,0)\big)$ is a quadric surface bundle over $\mathbb{P}^2$: by Lemma 3.5 the discriminant divisor Δ is a quartic plane curve. We then need to apply Lemma 3.1 once again. The codimension 2 blown up locus is the double cover of $\mathbb{P}^2$ branched in the said quartic, which is a degree 2 K3 surface.

Fano K3–40

(#3-22-74-2) $\mathscr{Z}\big(\mathbb{P}^1_1 \times \mathbb{P}^1_2 \times \mathbb{P}^5, \mathcal{O}(0,0,2) \oplus \mathcal{O}(0,1,1) \oplus \mathcal{O}(1,0,2)\big)$

- Invariants:

$$h^0(-K) = 22,\ (-K)^4 = 74,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 32,\ -\chi(T) = 29.$$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^1_1 \times \mathbb{P}^1_2$ is a dP_4-fibration.
  - $\pi_{13}\colon X \to \mathbb{P}^1_1 \times \mathbb{P}^5$ is a blowup $\mathrm{Bl}_{S_8^{(8)}} Y$, with $S_8^{(8)}$ a degree 8 K3 surface blown up in 8 points, and Y a blowup of $\mathbb{Q}^4$ along $\mathbb{P}^1 \times \mathbb{P}^1$.
  - $\pi_{23}\colon X \to \mathbb{P}^1_2 \times \mathbb{P}^5$ is a blowup $\mathrm{Bl}_\Sigma Z$, where Σ is the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in 8 points, and Z is Fano K3–30, the blowup of $\mathbb{Q}^4$ along a K3 surface of degree 8.
  - Rationality: yes.

- Semiorthogonal decompositions:
  - 8 exceptional objects and $\mathrm{D}^{\mathrm{b}}(T,\alpha)$, for $W \to \mathbb{P}^1 \times \mathbb{P}^1$ the discriminant double cover from π_{12}.
  - 18 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_8)$ from π_{13} and π_{23}.
  - This suggests that the T may be the blowup of a K3 surface in 10 points, and that the Brauer class may be trivial.

Explanation

We need to apply Lemma 3.1 twice. Start with $Y \subset \mathbb{P}^1 \times \mathbb{Q}^4$ defined by a section of $\mathcal{O}(1,2)$. The projection to $\mathbb{Q}^4$ is the blowup of a surface S defined as the intersection of two other quadrics, hence a K3 surface of degree 8. Then $X \subset \mathbb{P}^1 \times Y$ being defined by a section of $\mathcal{O}(1,1)$ is a blowup of Y along a surface Σ which is a codimension 2 linear section. This linear section is to be taken on the $\mathbb{P}^5$ factor, so the projection of Σ to $\mathbb{Q}^4$ is the blowup of a quadric surface along a linear section of S hence 8 points.

The projection to the other $\mathbb{P}^1 \times \mathbb{P}^5$ is dealt with by reversing the order of the blowups.

Fano K3–41

(#3-39-160-2) $\mathscr{Z}\left(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}(1,1,2)\right)$

- Invariants:

$$h^0(-K) = 39,\ (-K)^4 = 160,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 22,\ -\chi(T) = 18.$$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ is a quadric fibration with discriminant locus a divisor of bidegree $(4,4)$.
  - $\pi_{13}, \pi_{23}\colon X \to \mathbb{P}^1 \times \mathbb{P}^3$ are blowups of $\mathbb{P}^1 \times \mathbb{P}^3$ along a bielliptic K3 surface.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 8 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_8,\alpha)$ from π_{12}.
  - 8 exceptional divisors and $\mathrm{D}^{\mathrm{b}}(S)$ from π_{13} or π_{23}.

Explanation

The projection π_{23} to $\mathbb{P}^1\times\mathbb{P}^3$ is obviously birational, and its exceptional locus is defined by the vanishing of the induced morphism $\mathscr{O}(-1,-2)\to V_2^\vee\otimes\mathscr{O}$. We get a surface S which is a general complete intersection of two divisors of bidegree $(1,2)$, hence an elliptic K3 surface S. If the equations of the two divisors are $sq_0+tq_1=0$ and $sq_2+tq_3=0$, note that S projects isomorphically to the quartic surface in $\mathbb{P}^3$ with equation $q_0q_3=q_1q_2$.

Note also that by replacing π_{23} with π_{13} we get a birational isomorphism φ of $\mathbb{P}^1\times\mathbb{P}^3$ with itself. Normalizing the defining equation of X to $s\sigma q_0+t\sigma q_1+s\tau q_2+t\tau q_3=0$, this birational map is defined by

$$\phi\big([s,t],x\big)=\big([sq_0+tq_1,sq_2+tq_3],x\big),$$
$$\phi^{-1}\big([\sigma,\tau],x\big)=\big([\sigma q_3-\tau q_1,\tau q_0-\sigma q_2],x\big).$$

Note that the exceptional locus of π_{13} is the intersection in $\mathbb{P}^1\times\mathbb{P}^3$ of the two divisors of equations $\sigma q_3-\tau q_1=0$ and $\tau q_0-\sigma q_2=0$, which defines another elliptic fibration on the same quartic surface S.

Finally, π_{12} is a quadric fibration over $P=\mathbb{P}^1\times\mathbb{P}^1$, defined by a map $\mathscr{O}_P(-1,-1)\to \mathrm{S}^2V_4^\vee\otimes\mathscr{O}_P$. Its discriminant is given by a section of $\mathscr{O}_P(4,4)$.

Remark 6.9. — This Fano fourfold already appeared in [24], under the name (B1).

Fano K3–42

(#3-28-104-2) $\mathscr{Z}\big(\mathbb{P}^1\times\mathbb{P}^2\times\mathrm{Gr}(2,4),\mathscr{Q}_{\mathrm{Gr}(2,4)}(0,1,0)\oplus\mathscr{O}(1,0,2)\big)$

- Invariants:

$$h^0(-K)=28,\ (-K)^4=104,\ h^{1,1}=3,\ h^{3,1}=1,\ h^{2,2}=27,\ -\chi(T)=24.$$

- Description:
  - $\pi_{12}\colon X\to\mathbb{P}^1\times\mathbb{P}^2$ is a conic bundle with discriminant of bidegree $(3,4)$.
  - $\pi_{13}\colon X\to\mathbb{P}^1\times\mathrm{Gr}(2,4)$ is $\mathrm{Bl}_{\mathrm{dP}_5}Y$, where $Y=\mathrm{Bl}_{S_8}\mathrm{Gr}(2,4)$.
  - $\pi_{23}\colon X\to\mathbb{P}^2\times\mathrm{Gr}(2,4)$ is $\mathrm{Bl}_{S_8^{(4)}}Z$, where $Z=\mathrm{Bl}_{\mathbb{P}^2}\mathrm{Gr}(2,4)$ and $S_8^{(4)}$ is the blowup of the octic K3 surface S_8 in 4 points.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 13 exceptional objects and $\mathrm{D^b}(S_8)$ from the three maps.

Explanation

Our Fano fourfold X is defined by two tensors $\alpha \in \mathrm{Hom}(V_3, V_4)$ and $\beta \in V_2^\vee \otimes S^2(\bigwedge^2 V_4)^\vee$. It parametrizes the set of triples (x, y, P) such that $P \supset \alpha(y)$ and $\beta(x, \bullet) = 0$ on the Plücker representative of P.

In order to understand π_{12}, we observe that for (x, y) given, the set of planes P containing $\alpha(y)$ is a projective plane, on which β cuts a conic through a morphism $\mathcal{O}_P(-1,-2) \to S^2\big(V_4/\mathcal{O}_P(0,-1)\big)^\vee$ on $P = \mathbb{P}^1 \times \mathbb{P}^2$. We deduce that the discriminant is given by a map $\mathcal{O}_P(-3,-6) \to \det\big((V_4/\mathcal{O}_P(0,-1)^\vee)\big)^{\otimes 2}$, that is a section of $\mathcal{O}_P(3,4)$.

To describe π_{23} we observe that since α is the injection of a hyperplane, the condition $P \supset \alpha(y)$ determines uniquely y from P unless $P \subset \alpha(V_3)$. So the image of π_{23} is $\mathrm{Bl}_{\mathbb{P}^2} \mathrm{Gr}(2,4)$. Then we apply Lemma 3.1 to this fourfold. The complete intersection of two $\mathcal{O}(0,2)$ divisors on Z cuts a K3 surface that intersects the specified $\mathbb{P}^2$ in four points. It follows that S is isomorphic to the blowup of an octic K3 surface in these four points.

Finally consider π_{13}, whose image is obviously a divisor of bidegree $(1,2)$ in $\mathbb{P}^1 \times \mathrm{Gr}(2,4)$, hence the blowup of $\mathrm{Gr}(2,4)$ along the same octic K3 surface as before. For (x, P) given, y is uniquely determined by the condition that $\alpha(y) = P \cap \alpha(V_3)$, except when $P \subset \alpha(V_3)$, in which case we get a projective line as fiber. This implies that π_3 blows up a surface which is a $(1,2)$-divisor in $\mathbb{P}^1 \times \mathbb{P}^2$, hence a dP_5.

Fano K3–43

(#3-32-125-2) $\mathscr{Z}\big(\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^4, \mathcal{O}(1,0,1) \oplus \mathcal{O}(1,1,1) \oplus \mathcal{Q}_{\mathbb{P}^2_2}(0,0,1)\big)$

- Invariants:

$h^0(-K) = 32,\ (-K)^4 = 125,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 24,\ -\chi(T) = 20.$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^2 \times \mathbb{P}^2$ is a blowup $\mathrm{Bl}_{S^{(1)}_{16}}(\mathbb{P}^2 \times \mathbb{P}^2)$ of a K3 surface of degree 16 blown up in one point.
  - $\pi_{23}\colon X \to \mathbb{P}^2 \times \mathbb{P}^4$ is a blowup $\mathrm{Bl}_{S^{(1)}_{12}} Y$ of $Y = \mathrm{Bl}_{\mathbb{P}^1} \mathbb{P}^4$ along a K3 surface of degree 12 blown up in one point.
  - $\pi_{13}\colon X \to \mathbb{P}^2 \times \mathbb{P}^4$ is a blowup of a Hirzebruch surface Σ_1 in a Fano variety belonging to the Fano K3–29 family, a complete intersection of bidegree $(1,1), (1,2)$.
  - Rationality: yes.

- Semiorthogonal decompositions:
  - 10 exceptional objects and $\mathrm{D^b}(S_{16})$ from π_{12}.
  - 10 exceptional objects and $\mathrm{D^b}(S_{12})$ from π_{13}.
  - We deduce that S_{16} and S_{12} are derived equivalent (see Lemma 3.4).

Explanation

We first consider π_{12}. If we write $\mathscr{Z}\big(\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^4, \mathcal{Q}_{\mathbb{P}^2_2}(0,0,1)\big)$ as $\mathbb{P}^2_1 \times \mathbb{P}_{\mathbb{P}^2_2}\big(\mathcal{O}(0,-1) \oplus \mathcal{O}^{\oplus 2}\big)$, we see that X is the blowup of $P = \mathbb{P}^2_1 \times \mathbb{P}^2_2$ along a smooth surface S whose structure sheaf is resolved by the Eagon–Northcott complex

$$0 \longrightarrow \mathcal{O}_P(-3,-2) \oplus \mathcal{O}_P(-3,-3) \longrightarrow \mathcal{O}_P(-2,-1) \oplus \mathcal{O}_P(-2,-2)^{\oplus 2} \longrightarrow \\ \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

from which we calculate the Hilbert polynomial $\chi\big(\mathcal{O}_S(k)\big) = \frac{15k^2-k}{2} + 2$. Dualizing the complex, we get

$$0 \longrightarrow \mathcal{O}_P(-3,-3) \longrightarrow \mathcal{O}_P(-1,-2) \oplus \mathcal{O}_P(-1,-1)^{\oplus 2} \longrightarrow \\ \longrightarrow \mathcal{O}_P(0,-1) \oplus \mathcal{O}_P \longrightarrow \omega_S \longrightarrow 0.$$

So once again ω_S admits a canonical section, vanishing on the locus where the morphism

$$\mathcal{O}_P(-1,-2) \oplus \mathcal{O}_P(-1,-1)^{\oplus 2} \longrightarrow \mathcal{O}_P(0,-1)$$

is not surjective. This is the common zero locus of a section of $\mathcal{O}_P(1,1)$ and two sections of $\mathcal{O}_P(1,0)$, hence a line D projecting to a point on $\mathbb{P}^2_1$. Once again this line is a (-1)-curve on S, after contracting which we get a genuine K3 surface S_{16} of degree 16.

Now consider the projection $\pi_{23}\colon X \to \mathbb{P}^2_2 \times \mathbb{P}^4$. The zero locus of $\mathcal{Q}_{\mathbb{P}^2_2}(0,1)$ is $Y = \mathrm{Bl}_{\mathbb{P}^1} \mathbb{P}^4$. We identify $\mathbb{P}^2_1 \times Y$ with $p\colon \mathbb{P}(U_3 \otimes \mathcal{O}_Y) \to Y$ so that X the zero locus of a bundle $p^*\mathcal{F} \otimes \mathcal{O}_p(1)$, where $\mathcal{F} = \mathcal{O}(1,0) \oplus \mathcal{O}(1,1)$ is a vector bundle of rank 2 on Y (the restriction of $\mathcal{O}(1,0) \oplus \mathcal{O}(1,1)$ from $\mathbb{P}^2_2 \times \mathbb{P}^4$). So X is the blowup of a smooth surface $T \subset Y$ whose structure sheaf is resolved by the Eagon–Northcott complex

$$0 \longrightarrow \mathcal{O}_Y(-2,-3) \oplus \mathcal{O}_Y(-1,-3) \longrightarrow \mathcal{O}_Y(-1,-2)^{\oplus 3} \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_T \longrightarrow 0.$$

We deduce that the Hilbert polynomial is $\chi\big(\mathcal{O}_T(k)\big) = \frac{11k^2-k}{2} + 2$. Moreover, since $\mathcal{O}_Y(-2,-3) = K_Y$, the same arguments as before yield that $T = S^{(1)}_{12}$ is the blowup in one point of a K3 surface of degree 12.

Finally consider $\pi_{13}\colon X \to \mathbb{P}^2_1 \times \mathbb{P}^4$. The image is contained in a divisor of bidegree $(1,1)$, over which we get two additional morphisms $\alpha\colon \mathcal{O}(-1,-1) \to V_3^\vee$ and $\beta\colon \mathcal{O}(0,-1) \to V_3$, if $\mathbb{P}^2_2 = \mathbb{P}(V_3)$. There is a non-trivial fiber when these two morphisms are compatible, which yields another divisor of bidegree $(1,2)$. We conclude that the image of π_{13} is the intersection of these two divisors. Moreover, non-trivial fibers occur only when $\alpha = 0$, which can be identified with a complete intersection of a divisor of bidegree $(1,1)$ and three divisors of bidegree $(0,1)$ in $\mathbb{P}^2 \times \mathbb{P}^4$, which is a surface of type Σ_1 over which the fibers are projective lines, proving that π_{13} is a blowup.

Fano K3–44

(#3-25-89-2) $\mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^4, \mathcal{O}(0,1,1) \oplus \mathcal{O}(0,1,2) \oplus \mathcal{O}(1,0,1)\big)$

- Invariants:

$$h^0(-K) = 25,\ (-K)^4 = 89,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 30,\ -\chi(T) = 27.$$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ is a conic bundle.
  - $\pi_{13}\colon X \to \mathbb{P}^1 \times \mathbb{P}^4$ is a blowup $\mathrm{Bl}_{S_8^{(8)}} Y$, where Y is $\mathrm{Bl}_{\mathbb{P}^2} \mathbb{P}^4$ and $S_8^{(8)}$ is a K3 surface of degree 8 blown up in 8 points.
  - $\pi_{23}\colon X \to \mathbb{P}^2 \times \mathbb{P}^4$ is $\mathrm{Bl}_{\mathrm{dP}_2} Z$ where Z is the Fano fourfold from Fano K3–29.
  - Rationality: yes
- Semiorthogonal decompositions:
  - 6 exceptional objects and $\mathrm{D^b}(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{C}_0)$ from π_{12}.
  - 16 exceptional divisors and $\mathrm{D^b}(S_8)$ from π_{13} and π_{23}.

Explanation

Let us start with π_{23}. By Lemma 3.1, X is the blowup of Z in a surface which is a complete intersection of bidegrees $(1,1),(1,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$, that is a del Pezzo surface of degree 2. In fact, this a double cover of $\mathbb{P}^2$ ramified in 4 points, a section $\mathcal{L}^2 \boxtimes \mathcal{O}(1)$ over $\mathbb{P}_{\mathbb{P}^2}\big(\Omega^1(1)\big)$.

The other two maps are obtained by direct calculation. Consider first the projection to $Y = \mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^4, \mathcal{O}(1,1)\big)$, i.e. $\mathrm{Bl}_{\mathbb{P}^2} \mathbb{P}^4$. We have a projective bundle over Y given by $\mathbb{P}_Y(\mathcal{O}^{\oplus 3})$. The exceptional locus is a surface T whose structure sheaf is resolved by the Eagon–Northcott complex

$$0 \longrightarrow \mathcal{O}_Y(0,-1) \oplus \mathcal{O}_Y(0,-2) \longrightarrow \mathcal{O}_Y^3 \longrightarrow \mathcal{O}_Y(0,3) \longrightarrow \mathcal{O}_T(0,3) \longrightarrow 0.$$

If we restrict the projection $Y \to \mathbb{P}^4$ to T we can see T as the blowup of the surface S whose structure sheaf is resolved by another Eagon–Northcott complex:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-5) \oplus \mathcal{O}_{\mathbb{P}^4}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^4}^3(-3) \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

We readily deduce that the Hilbert polynomial of S is $\frac{7k^2-k}{2}+2$, so this is a degree 7 surface in $\mathbb{P}^4$ which by [57] must be an inner projection of a K3 of degree 8 in $\mathbb{P}^5$. It follows, that a general plane in $\mathbb{P}^4$ meets S in 7 points and that $T \to S$ is the blowup of 7 additional points. Finally, $T = S_8^{(8)}$ is a K3 surface of degree 8 blown up in 8 points.

Fano K3–45

(#3-31-119-2) $\mathscr{X}\big(\mathbb{P}_1^2 \times \mathbb{P}_2^2 \times \mathbb{P}^4, \mathcal{O}(0,1,2) \oplus \mathcal{O}(1,1,0) \oplus \mathcal{Q}_{\mathbb{P}_1^2}(0,0,1)\big)$

- Invariants:

$h^0(-K) = 31,\ (-K)^4 = 119,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 24,\ -\chi(T) = 21.$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}_1^2 \times \mathbb{P}_2^2$ is a conic bundle over a $(1,1)$ divisor $Z \subset \mathbb{P}^2 \times \mathbb{P}^2$ with discriminant a $(2,3)$-divisor in Z.
  - $\pi_{13}\colon X \to \mathbb{P}_1^2 \times \mathbb{P}^4$ is a blowup $\mathrm{Bl}_{S_8^{(1)}} Y$, where $Y = \mathrm{Bl}_{\mathbb{P}^1} \mathbb{P}^4$ and $S_8^{(1)}$ is a K3 surface of degree 8 blown up in one point.
  - $\pi_{23}\colon X \to \mathbb{P}_2^2 \times \mathbb{P}^4$ is a blowup of Fano K3–29 along a quadratic surface.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 6 exceptional objects and $\mathrm{D^b}(Z, \mathcal{C}_0)$ from π_{12}.
  - 10 exceptional objects and $\mathrm{D^b}(S_8)$ from π_{13}.
  - 10 exceptional objects and $\mathrm{D^b}(S_2, \alpha)$ or $\mathrm{D^b}(S_8)$ (see Fano K3–29) from π_{23}.

Explanation

Let α, β, γ denote the three sections defining X. In particular γ can be seen as a morphism with a two-dimensional kernel K_2.

The projection π_{12} maps X to the $(1,1)$-divisor defined by β. When y is given in this divisor, z must be contained in $\gamma^{-1}(x) \simeq K_2 \oplus x$, and

$\alpha(y,z,z)=0$ defines a conic in the corresponding projective plane. So π_{12} is a conic bundle, with discriminant given by a divisor of type $(2,3)$.

Now we describe the projection π_{13} to $Y=\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^4$. We apply Proposition 3.6 with $\mathcal{E}=\mathcal{O}^{\oplus 3}$ and $\mathcal{F}=\mathcal{O}(0,2)\oplus\mathcal{O}(1,0)$, where the bi-grading is with respect to the embedding of Y in $\mathbb{P}^2\times\mathbb{P}^4$ as zero locus of $\mathcal{Q}(0,1)$. The Eagon–Northcott complex thus reads

$$0\longrightarrow \mathcal{O}_Y(-1,-4)\oplus\mathcal{O}_Y(-2,-2)\longrightarrow \mathcal{O}_Y^3(-1,-2)\longrightarrow \mathcal{O}_Y\longrightarrow \mathcal{O}_S\longrightarrow 0.$$

In order to compute $H^i\big(\mathcal{O}_Y(a,b)\big)$ we use the Koszul complex for Y in $\mathbb{P}^2\times\mathbb{P}^4$. We get that

$$\chi\big(\mathcal{O}_S(a,b)\big)=2a^2+6ab+\frac{7b^2}{2}-a-\frac{b}{2}+2.$$

In particular $\chi\big(\mathcal{O}_S(0,b)\big)=\frac{7b^2}{2}-\frac{b}{2}+2$, so that if the projection of S to $\mathbb{P}^4$ is an isomorphism, we can conclude (by [57]) that S is a K3 surface of degree 8 blown up at one point. This is the case because S is defined by the condition that the two linear forms $\alpha(\bullet,z,z)$ and $\beta(x,\bullet)$ are proportional, where moreover $\gamma(z)$ must be contained in x. Over the blown up line where $z\subset\operatorname{Ker}\gamma$, the linear form $\alpha(\bullet,z,z)$ is non-zero, so x is uniquely determined since β has maximal rank. As a consequence, S is mapped isomorphically to a surface in $\mathbb{P}^4$ containing the blown up line, and this surface must be a non-minimal K3 of degree 7.

Finally, the projection π_{23} maps X to the intersection of the $(1,2)$-divisor defined by α, with the $(1,1)$ divisor defined by $\beta\big(\gamma(z),y\big)=0$. The latter contains $\mathbb{P}^2\times\mathbb{P}(K_2)$, which gives the locus over which π_{23} has non-trivial fibers, namely projective lines. This is actually a copy of $\mathbb{P}^1\times\mathbb{P}^1$ embedded by $\mathcal{O}(2,1)$, and π_{23} is finally the blowup of Fano K3–29 along a quadratic surface.

Fano K3–46

(#3-33-130-2) $\mathscr{Z}\big(\mathbb{P}^1\times\mathbb{P}^3\times\mathbb{P}^5,\mathcal{Q}_{\mathbb{P}^3}(0,0,1)\oplus\mathcal{O}(0,1,1)\oplus\mathcal{O}(1,1,1)\big)$

- Invariants:

$$h^0(-K)=33,\ (-K)^4=130,\ h^{1,1}=3,\ h^{3,1}=1,\ h^{2,2}=24,\ -\chi(T)=20.$$

- Description:
  - $\pi_{12}\colon X\to\mathbb{P}^1\times\mathbb{P}^3$ is a blowup of an elliptic K3 surface blown up in two points.

- $\pi_{13}\colon X \to \mathbb{P}^1 \times \mathbb{P}^5$ is a blowup along a quadratic surface of Fano K3–30, the intersection of two divisors of bidegree $(0,2)$ and $(1,2)$, which is itself a blowup of a quadric in $\mathbb{P}^5$ along a degree 8 K3 surface containing a line.
- $\pi_{23}\colon X \to \mathbb{P}^3 \times \mathbb{P}^5$ is a blowup $\mathrm{Bl}_{S_{26}} Y$, where Y is the blowup of $\mathrm{Gr}(2,4)$ in a line, and S_{26} is a K3 of degree 26 and generic Picard rank 2 containing the said line.
- Rationality: yes.

• Semiorthogonal decompositions:
  - 10 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_{16})$ from π_{12}.
  - 10 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_8)$ from π_{13}.
  - 10 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_{26})$ from π_{23}.
  - We deduce that S_{16}, S_8 and S_{26} are all derived equivalent (see Lemma 3.4).

Explanation

Our fourfold X is defined by tensors $\alpha \in \mathrm{Hom}(V_6, V_4)$, $\beta \in V_4^\vee \otimes V_6^\vee$ and $\gamma \in V_2^\vee \otimes V_4^\vee \otimes V_6^\vee$; it parametrizes the triples (x, y, z) such that $\alpha(z) \subset y$, $\beta(y, z)$ and $\gamma(x, y, z) = 0$.

Let us describe π_{12}. When x and y are fixed, z must belong to $\alpha^{-1}(y) = A_2 \oplus y$, which defines a projective plane to which β and γ impose two linear conditions. So the projection π_{12} is birational, and blows up a surface S inside $P = \mathbb{P}^1 \times \mathbb{P}^3$ whose structure sheaf is resolved by the Eagon–Northcott complex

$$0 \longrightarrow \mathcal{O}_P(0,-1) \oplus \mathcal{O}_P(-1,-1) \longrightarrow (A_2 \otimes \mathcal{O}_P) \oplus \mathcal{O}_P(0,1) \longrightarrow \\ \longrightarrow \mathcal{O}_P(1,3) \longrightarrow \mathcal{O}_S(1,3) \longrightarrow 0.$$

Dualizing this complex we get

$$0 \longrightarrow \mathcal{O}_P(-2,-4) \longrightarrow \big(A_2^\vee \otimes \mathcal{O}_P(-1,-1)\big) \oplus \mathcal{O}_P(-1,-2) \longrightarrow \\ \longrightarrow \mathcal{O}_P(-1,0) \oplus \mathcal{O}_P \longrightarrow \omega_S \longrightarrow 0.$$

So ω_S admits a canonical section whose zero locus is the curve D along which the morphism $\big(A_2^\vee \otimes \mathcal{O}_P(-1,-1)\big) \oplus \mathcal{O}_P(-1,-2) \to \mathcal{O}_P(-1,0)$ vanishes, which is the pull-back of two points from $\mathbb{P}^3$. So D is just the union of two lines, and S is an elliptic K3 surface blown up at two points. The first complex above also yields the Hilbert polynomial $\chi\big(\mathcal{O}_S(k,k)\big) = 7k^2 - t + 2$, so that the degree of S with respect to $\mathcal{O}_S(1,1)$ is 16.

Now we turn to π_{13}. When x and z are fixed, $y = \alpha(z)$ when this is non-zero, which implies that π_{13} blows up the locus where $\alpha(z) = 0$. The image is the intersection Y of the two divisors of equations $\beta\big(\alpha(z), z\big) = 0$ and $\gamma\big(x, \alpha(z), z\big) = 0$. The image of Y is $\mathbb{P}^5$ is the quadric Q of equation $\beta\big(\alpha(z), z\big) = 0$, which contains the line $l = \mathbb{P}(\operatorname{Ker}\alpha)$, and the projection of Y to Q blows up the intersection of Q with two other quadrics also containing l.

Finally we describe π_{23}. We use Lemma 3.1 twice: first on $\mathbb{P}^3 \times \mathbb{P}^5$ to get $\operatorname{Bl}_{\mathbb{P}^1} \mathbb{P}^5$, which is then cut with a quadric containing the line. The blown up locus is a K3 surface containing a line (whose projection in $\operatorname{Gr}(2,4)$ is the intersection of three quadrics). This K3 surface has Picard rank 2, and its degree with respect to $\mathcal{O}(1,1)$ can be checked to be 26.

Fano K3–47

(#3-27-100-2-B) $\mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^3, \mathcal{O}(0,1,1)^{\oplus 2} \oplus \mathcal{O}(1,1,1)\big)$

- Invariants:

$$h^0(-K) = 27,\ (-K)^4 = 100,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 26,\ -\chi(T) = 22.$$

- Description:
  - $\pi_{12}, \pi_{13} \colon X \to \mathbb{P}^1 \times \mathbb{P}^3$ are blowups $\operatorname{Bl}_{S_{20}^{(4)}}(\mathbb{P}^1 \times \mathbb{P}^3)$, where $S_{20}^{(4)}$ is a K3 surface of degree 20 and Picard number 2, blown up in 4 points.
  - $\pi_{23} \colon X \to \mathbb{P}^3 \times \mathbb{P}^3$ is a blowup $\operatorname{Bl}_{S_4} Y$, where $Y \subset \mathbb{P}^3 \times \mathbb{P}^3$ is a codimension 2 linear section, and S_4 a determinantal quartic K3 surface.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 12 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_{20})$ from π_{12} or π_{13}.
  - 12 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_4)$ from π_{23}.
  - We deduce that S_{20} and S_4 are derived equivalent (see Lemma 3.4).

Explanation

The projection π_{23} can be understood using Lemma 3.1. In fact one can check that the fourfold Y has $h^{1,1} = 2$, $h^{2,2} = 6$, and we can give an explicit interpretation. Indeed, first identify Y with $\mathscr{Z}\big(\operatorname{Fl}(1,3,4), \mathcal{O}(1,1)\big)$. This equivalent to writing $Y = \mathscr{Z}\big(\mathbb{P}_{\mathbb{P}^3}\big(\mathcal{Q}^\vee(-1)\big), \mathcal{L}\big)$, where we identified $\mathcal{U}$ on $\operatorname{Gr}(3,4)$ with $\mathcal{Q}^\vee$ on $\mathbb{P}^3$. By the Cayley trick, the projection $\pi \colon Y \to \mathbb{P}^3$

is therefore generically a $\mathbb{P}^1$-bundle, with special fibers $\mathbb{P}^2$ over 4 points. This is enough to compute the Hodge numbers for Y, for example via a Grothendieck ring calculation. This construction shows as well that X is rational, since Y is. The locus which is blown up is a K3 surface by adjunction. In [34, Theorem 1] this K3 is shown to be quartic determinantal.

The two other projections π_{12}, π_{13} are birational, with base locus inside $P = \mathbb{P}^1 \times \mathbb{P}^3$ given by the degeneracy locus of the induced morphism $\mathcal{O}_P(0,-1)^{\oplus 2} \oplus \mathcal{O}_P(-1,-1) \to V_4^\vee \otimes \mathcal{O}_P$. This is a surface S whose ideal sheaf is resolved by the Eagon–Northcott complex

$$0 \longrightarrow \mathcal{O}_P(-1,-4)^{\oplus 2} \oplus \mathcal{O}_P(-2,-4) \longrightarrow V_4^\vee \otimes \mathcal{O}_P(-1,-3) \longrightarrow \\ \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

It follows that the Hilbert polynomial of S is $8k^2 - 2k + 2$. Dualizing the previous complex we get

$$0 \longrightarrow \mathcal{O}_P(-2,-4) \longrightarrow V_4 \otimes \mathcal{O}_P(-1,-1) \longrightarrow \\ \longrightarrow \mathcal{O}_P(-1,0)^{\oplus 2} \oplus \mathcal{O}_P \longrightarrow \omega_S \longrightarrow 0.$$

So ω_S admits a canonical section, vanishing exactly along the curve D where the morphism $V_4 \otimes \mathcal{O}_P(-1,-1) \to \mathcal{O}_P(-1,0)^{\oplus 2}$ degenerates. This is just the preimage of the locus in $\mathbb{P}^3$ where the morphism $V_4 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 2}$, that is, four points; so D is the union of four lines $L_1, \dots, L_4$. Then $\omega_S = \mathcal{O}_S(L_1 + \cdots + L_4)$ and we conclude as usual that S is a degree 20 K3 surface blown up in 4 points. Moreover the class of S in the Chow ring is $[S] = 4lh + 6h^2$, if l and h denote the hyperplane classes of $\mathbb{P}^1$ and $\mathbb{P}^3$, respectively, so that S projects to an elliptic quartic surface in $\mathbb{P}^3$.

Fano K3–48

(#3-31-120-2-A) $\qquad \mathscr{Z}\big(\mathbb{P}^1 \times \mathrm{Fl}(1,2,4), \mathcal{O}(0;0,1) \oplus \mathcal{O}(1;1,1)\big)$

- Invariants:

$$h^0(-K) = 31,\ (-K)^4 = 120,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 22,\ -\chi(T) = 18.$$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^1 \times \mathbb{P}^3$ is a blowup $\mathrm{Bl}_{S_{14}}(\mathbb{P}^1 \times \mathbb{P}^3)$, where S_{14} is a K3 surface of Picard rank 2.
  - $\pi_{13}\colon X \to \mathbb{P}^1 \times \mathrm{Gr}(2,4)$ is a blowup $\mathrm{Bl}_{S_{16}}(\mathbb{P}^1 \times \mathbb{Q}_3)$, where S_{16} is a K3 surface of Picard rank 2.

  - $\pi_{23}\colon X \to \mathrm{Fl}(1,2,4)$ is a blowup $\mathrm{Bl}_{S_{24}}\mathbb{P}_{\mathbb{Q}^3}\big(\mathcal{U}\big|_{\mathbb{Q}_3}\big)$, where S_{24} is a K3 of Picard rank 2 and $\mathcal{U}\big|_{\mathbb{Q}^3}$ is the restriction of the tautological bundle from $\mathbb{Q}^4 = \mathrm{Gr}(2,4)$.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 8 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_{14})$ from π_{12}.
  - 8 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_{16})$ from π_{13}.
  - 8 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_{24})$ from π_{23}.
  - We deduce that S_{14}, S_{16} and S_{24} are all derived equivalent (see Lemma 3.4).

Explanation

We start with π_{23}. First interpret $\mathscr{Z}\big(\mathrm{Fl}(1,2,4),\mathcal{O}(0,1)\big)$ as $\mathbb{P}_{\mathbb{Q}^3}\big(\mathcal{U}\big|_{\mathbb{Q}^3}\big)$, simply because $\mathrm{Fl}(1,2,4) \cong \mathbb{P}_{\mathrm{Gr}(2,4)}(\mathcal{U})$. Then apply Lemma 3.1. Two copies of $\mathcal{O}(1,1)$ on the flag manifold cut a K3 which has degree 24 with respect to this bundle, and Picard rank 2.

We turn to π_{13}. Consider the projection $\mathrm{Fl}(1,2,4) \to \mathrm{Gr}(2,4)$ that identifies $\mathrm{Fl}(1,2,4)$ with $\mathbb{P}_{\mathrm{Gr}(2,4)}(\mathcal{U})$. Again, we interpret $\mathscr{Z}\big(\mathrm{Fl}(1,2,4),\mathcal{O}(0,1)\big)$ as $\mathbb{P}_{\mathbb{Q}^3}\big(\mathcal{U}\big|_{\mathbb{Q}^3}\big)$, so that we are left with the question of understanding the zero locus of $\mathcal{O}(1;1,0)$ inside $\mathbb{P}^1 \times \mathbb{P}_{\mathbb{Q}^3}\big(\mathcal{U}\big|_{\mathbb{Q}^3}(-1)\big)$. Since $\mathscr{Z}(1,0)$ inside $\mathbb{P}_{\mathbb{Q}^3}\big(\mathcal{U}\big|_{\mathbb{Q}^3}\big)$ is the blowup of the elliptic curve given by the zero locus $\mathscr{Z}\big(\mathbb{Q}^3,\mathcal{U}^\vee(1)\big)$, it is easy to see that X is the blowup of the zero locus of $\mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{Q}^3,\mathcal{U}^\vee(1,1)\big)$, which is a K3 surface S_{16} of Picard rank 2 and degree 16.

Finally we describe π_{12}. We consider the projection $\pi\colon \mathrm{Fl}(1,2,4) \to \mathbb{P}^3$ that identifies $\mathrm{Fl}(1,2,4)$ with $\mathbb{P}_{\mathbb{P}^3}\big(\mathcal{Q}^\vee(-1)\big)$. The relative hyperplane section $\mathcal{O}_{\mathrm{rel}}(1)$ of this projective bundle is the line bundle $\mathcal{O}(0,1)$ on $\mathrm{Fl}(1,2,4)$, that is, $\mathcal{O}(0;1,0)$ on $\mathbb{P}^1 \times \mathrm{Fl}(1,2,4)$. Setting $\mathcal{L} \coloneqq \mathcal{O}(1;1,0)$, our variety X is then given by sections of $\mathcal{O}_{\mathrm{rel}}(1) \otimes (\mathcal{O} \oplus \mathcal{L})$, so the projection is a birational map which blows up a smooth surface S, the degeneracy locus of a map $\mathcal{O}_P(-1,-1) \oplus \mathcal{O}_P \to \mathcal{Q}^\vee_{\mathbb{P}^3}(0,1)$ on $P = \mathbb{P}^1 \times \mathbb{P}^3$. The structure sheaf of S is resolved by the Eagon–Northcott complex

$$0 \longrightarrow \mathcal{O}_P(-1,-3) \oplus \mathcal{O}_P(-2,-4) \longrightarrow \mathcal{Q}^\vee_{\mathbb{P}^3}(-1,-2) \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

We deduce that the Hilbert polynomial of S is $7k^2 + 2$. Dualizing the complex, we get

$$0 \longrightarrow \mathcal{O}_P(-2,-4) \longrightarrow \mathcal{Q}_{\mathbb{P}^3}(-1,-2) \longrightarrow \mathcal{O}_P(-1,-1) \oplus \mathcal{O}_P \longrightarrow \omega_S \longrightarrow 0.$$

In particular, ω_S admits a canonical section, vanishing where the morphism $\mathcal{Q}_{\mathbb{P}^3}(-1,-2) \to \mathcal{O}(-1,-1)$ is not surjective, that is, on the pull-back of the zero locus Z of a section of $\mathcal{Q}_{\mathbb{P}^3}(-1)$ on $\mathbb{P}^3$. But a general section of this bundle is actually a two-form in four variables, and since a general such form does not vanish on any hyperplane, we can conclude that $Z = \emptyset$ and ω_S is trivial. Finally, S is a degree 14 K3 surface.

Fano K3–49

(#3-27-100-2-C) $\mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^4, \mathcal{O}(0,0,2) \oplus \mathcal{O}(0,1,1) \oplus \mathcal{O}(1,1,1)\big)$

- Invariants:

$$h^0(-K) = 27,\ (-K)^4 = 100,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 24,\ -\chi(T) = 20.$$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ is a conic bundle with discriminant a singular $(2,4)$ divisor.
  - $\pi_{13}\colon X \to \mathbb{P}^1 \times \mathbb{Q}^3$ is a blowup $\mathrm{Bl}_{S_{20}^{(2)}}(\mathbb{P}^1 \times \mathbb{Q}^3)$.
  - $\pi_{23}\colon X \to \mathbb{P}^2 \times \mathbb{Q}^3$ is a blowup $\mathrm{Bl}_{S_{20}} Y$, where Y is a $(1,1)$-divisor in $\mathbb{P}^2 \times \mathbb{Q}^3$.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 6 exceptional objects and $\mathrm{D^b}(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{C}_0)$ from π_{12}.
  - 10 exceptional objects and $\mathrm{D^b}(S_{20})$ from π_{13} and π_{23}.

Explanation

As usual the projection to $\mathbb{P}^2 \times \mathbb{Q}^3$ is the blowup of the locus in the image cut out by two linear sections: hence the intersection in $\mathbb{P}^2 \times \mathbb{Q}^3$ of three divisors of bidegree $(1,1)$. This is a K3 surface of degree 20, the degree of $\mathcal{O}(1,0)$ being two and that of $\mathcal{O}(0,1)$ being six. Note that the image Y of π_{23} is a hyperplane section with respect to the Segre-like embedding of $\mathbb{P}^2 \times \mathbb{Q}^3 \to \mathbb{P}^{14}$ given by $\mathcal{O}(1,1)$.

To construct the claimed semiorthogonal decomposition, it is enough to show that $\mathrm{D^b}(Y)$ is generated by 10 exceptional objects. The general fiber of the projection $\pi\colon Y \to \mathbb{Q}^3$ is a hyperplane section of $\mathbb{P}^2$, hence a $\mathbb{P}^1$. However, there is a line in $\mathbb{P}^4 \supset \mathbb{Q}^3$ where the corresponding linear form is degenerate, and hence there are two points p, q of $\mathbb{Q}^3$ where the fiber is the whole $\mathbb{P}^2$. Then $\mathrm{D^b}(X)$ admits a decomposition by two copies of $\mathrm{D^b}(\mathbb{Q}^3)$, and by one copy of $\mathrm{D^b}(p)$ and $\mathrm{D^b}(q)$, see [14, Appendix B].

When we project to $P = \mathbb{P}^1 \times \mathbb{Q}^3$, we get a birational map with $\mathbb{P}^1$-fibers over the degeneracy locus of a morphism $\mathcal{O}_P(0,-1) \oplus \mathcal{O}_P(-1,-1) \to V_3^\vee \otimes \mathcal{O}_P$. This is a smooth surface S whose structure sheaf is resolved by the Eagon–Northcott complex

$$0 \longrightarrow \mathcal{O}_P(-1,-3) \oplus \mathcal{O}_P(-2,-3) \longrightarrow V_3^\vee \otimes \mathcal{O}_P(-1,-2) \longrightarrow \\ \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

We deduce that S has degree 18 with respect to $\mathcal{O}_P(1,1)$. Dualizing the complex we get

$$0 \longrightarrow \mathcal{O}_P(-2,-3) \longrightarrow V_3 \otimes \mathcal{O}_P(-1,-1) \longrightarrow \mathcal{O}_P(-1,0) \oplus \mathcal{O}_P \longrightarrow \omega_S \longrightarrow 0.$$

So ω_S has a canonical section whose zero locus is the degeneracy locus D of the morphism $V_3 \otimes \mathcal{O}_P(-1,-1) \to \mathcal{O}_P(-1,0)$. This locus is the pull-back from $\mathbb{Q}^3$ of the common zero locus to three sections of $\mathcal{O}_{\mathbb{Q}^3}(1)$, hence two points. We conclude that D is the union of two disjoint lines, that $\omega_S = \mathcal{O}_S(D)$, and that S is finally a blowup in two points of a K3 surface of degree 20.

Finally, the sections of $\mathcal{O}(0,1,1)$ and $\mathcal{O}(1,1,1)$ define a $\mathbb{P}^2$-bundle over $\mathbb{P}^1 \times \mathbb{P}^2$, corresponding to a family of projective planes in $\mathbb{P}^4$, and the quadric in $\mathbb{P}^4$ cuts out a conic bundle over $\mathbb{P}^1 \times \mathbb{P}^2$. The discriminant is a divisor of bidegree $(2,4)$.

Fano K3–50

(#3-43-180-2) $\mathscr{Z}\big(\mathbb{P}^3 \times \mathbb{P}^4 \times \mathbb{P}^4, \mathcal{O}(1,1,1) \oplus \mathcal{Q}_{\mathbb{P}^3}(0,1,0) \oplus \mathcal{Q}_{\mathbb{P}^3}(0,0,1)\big)$

- Invariants:

$$h^0(-K) = 43,\ (-K)^4 = 180,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 22,\ -\chi(T) = 18.$$

- Description:
  - $\pi_{12}, \pi_{13}\colon X \to \mathbb{P}^3 \times \mathbb{P}^4$ are blowups $\mathrm{Bl}_{S_{22}} Y$, where S_{22} is a K3 surface of degree 22 and Y is a blowup of $\mathbb{P}^4$ at one point.
  - $\pi_{23}\colon X \to \mathbb{P}^4 \times \mathbb{P}^4$ has image a rational singular fourfold Y° and contracts a $\mathbb{P}^2$.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 8 exceptional objects and $\mathrm{D^b}(S_{22})$ from π_{12} or π_{13}.

Explanation

The sections defining X are given by $\alpha \in \mathrm{Hom}(U_5, U_4)$, $\beta \in \mathrm{Hom}(V_5, U_4)$ and $\gamma \in U_4^\vee \otimes U_5^\vee \otimes V_5^\vee$. The fourfold X parametrizes the triples of lines x, y, z in $\mathbb{P}^3 \times \mathbb{P}^4 \times \mathbb{P}^4 = \mathbb{P}(U_4) \times \mathbb{P}(U_5) \times \mathbb{P}(V_5)$ such that

$$\alpha(y) \subset x, \qquad \beta(z) \subset x, \qquad \gamma(x, y, z) = 0.$$

The projections to the two $\mathbb{P}^4$'s are clearly birational.

The set of pairs (x, y) such that $\alpha(y) \subset x$ is the blowup Y of $\mathbb{P}^4$ at the point $y_0 = \mathbb{P}\big(\mathrm{Ker}(\alpha)\big)$. The extra point z is then uniquely determined in the projective line $\mathbb{P}\big(\beta^{-1}(x)\big)$ by the linear condition given by γ, except if this condition is trivial. This happens on the locus defined by the vanishing of a morphism $\mathcal{O}_Y(-1,-1) \to \big(\mathcal{O}_Y \oplus \mathcal{O}_Y(-1,0)\big)^\vee$. This locus is a surface S cut out by two divisors of bidegrees $(1,1)$ and $(2,1)$ in Y, whose canonical divisor is $\mathcal{O}_P(-3,-2)$. So S is a K3 of degree 22 and X is be the blowup of Y along this K3. Note that the image of S in $\mathbb{P}^4$ is the intersection of a quadric through z_0, with a cubic which is nodal at z_0, in particular this is a nodal surface. The preimage of z_0 in S is a conic, which must be a (-2)-curve.

The set of pairs (y, z) in $\mathbb{P}^4 \times \mathbb{P}^4$ such that $\alpha(y)$ is collinear to $\beta(z)$, and $\gamma\big(\alpha(y), y, z\big) = \gamma\big(\beta(z), y, z\big) = 0$ must be a fourfold Y° singular at (y_0, z_0). The extra point x is uniquely determined as $\alpha(y)$ or $\beta(z)$, except when both vanish, that is at the unique point (y_0, z_0), over which the condition $\gamma(x, y_0, z_0) = 0$ defines a projective plane contracted by π_{23}. In particular the latter is a small contraction.

The projection of Y° to $\mathbb{P}V_5 = \mathbb{P}^4$ is birational, the fiber of $z \neq z_0$ consisting of those $y \subset \alpha^{-1}\big(\beta(z)\big)$ such that $\gamma\big(\beta(z), y, z\big) = 0$. This is a point in general, and a projective line when this linear condition is trivial on $\alpha^{-1}\big(\beta(z)\big) \simeq \mathbb{C} \oplus z$. The latter condition is satisfied over the intersection of a quadric in $\mathbb{P}^4$ containing z_0 and a cubic which is singular at z_0. Finally, the fiber of z_0 is the quadric of equation $\gamma\big(\alpha(y), y, z_0\big) = 0$.

To summarize, the birational projection of Y to $\mathbb{P}^4$ contracts two divisors: a quadric onto a point, and a (generically) $\mathbb{P}^1$-bundle to a singular K3. Blowing up $\mathbb{P}^4$ at z_0 we get a smooth K3 surface S' as the intersection of two divisors of type $2H - E$ and $3H - 2E$. This surface S' cuts the exceptional divisor along a conic, and the natural projection to $\mathbb{P}^3$ exhibits S' as a smooth quartic containing a conic.

Fano K3–51

(#3-23-80-2-B) $\mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3, \mathcal{O}(0,1,2) \oplus \mathcal{O}(1,1,1)\big)$

- Invariants:

$h^0(-K) = 23,\ (-K)^4 = 80,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 30,\ -\chi(T) = 26.$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ is a conic bundle, with discriminant a divisor of bidegree $(2,8)$.
  - $\pi_{13}\colon X \to \mathbb{P}^1 \times \mathbb{P}^3$ is a blowup of a special quartic K3 surface blown up at eight points.
  - $\pi_{23}\colon X \to \mathbb{P}^2 \times \mathbb{P}^3$ is $\mathrm{Bl}_{S_{16}} Y$ where S_{16} is a special quartic K3 surface and $Y = \mathscr{Z}\big(\mathbb{P}^2 \times \mathbb{P}^3, \mathcal{O}(1,2)\big)$ is a rational fourfold.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 6 exceptional objects and $\mathrm{D^b}(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{C}_0)$ from π_{12}.
  - 16 exceptional objects and $\mathrm{D^b}(S_4)$ from π_{13}.
  - 8 exceptional divisors, an unknown category and $\mathrm{D^b}(S_4)$ from π_{23}.

Explanation

Choose coordinates $[x_1, x_2]$ on $\mathbb{P}^1$ and $[y_1, y_2, y_3]$ on $\mathbb{P}^2$. The two defining sections of X can be written as $\sum_k y_k q_k$ and $\sum_{i,j} x_i y_j \ell_{ij}$ for some linear forms ℓ_{ij} and quadratic forms q_k on V_4.

Over $\mathbb{P}^1 \times \mathbb{P}^2$, the second section defines a projective bundle $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is given by the sequence $0 \to \mathcal{O}(-1,-1) \to V_4^\vee \otimes \mathcal{O} \to \mathcal{E}^\vee \to 0$. The first section then gives X as a conic bundle in $\mathbb{P}(\mathcal{E})$ defined by the composition $\mathcal{O}(0,-2) \to S^2 V_4^\vee \otimes \mathcal{O} \to S^2\mathcal{E}^\vee$. Its discriminant is given by the induced cubic map from $\mathcal{O}(0,-6)$ to $\det(\mathcal{E}^\vee)^2 = \mathcal{O}(2,2)$.

The image of X in $\mathbb{P}^2 \times \mathbb{P}^3$ is a $(1,2)$-divisor Y. Note that the projection of Y to $\mathbb{P}^3$ has projective lines for fibers, except eight projective planes over the eight intersection points $z_1, \dots, z_8$ of the quadrics q_1, q_2, q_3. As usual X is a blowup of Y along a codimension 2 linear section, that is a K3 surface $S \subset \mathbb{P}^2 \times \mathbb{P}^3$ defined by sections of $\mathcal{O}(1,2)$ and $\mathcal{O}(1,1)^{\oplus 2}$. The projection of S to $\mathbb{P}^3$ is an isomorphism with a quartic K3 of Picard number 2, whose equation $Q_4 = 0$ is obtained as

$$Q_4 = \det \begin{pmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ q_1 & q_2 & q_3 \end{pmatrix}.$$

In particular this quartic surface contains the twisted cubic defined by the condition

$$\operatorname{rank}\begin{pmatrix}\ell_{11} & \ell_{12} & \ell_{13}\\ \ell_{21} & \ell_{22} & \ell_{23}\end{pmatrix} \leqslant 1.$$

Note that quartic surfaces containing a twisted cubic span a Noether–Lefschetz divisor in the moduli space.

Observe that Y can be seen as a hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^3$ under the embedding $\mathbb{P}^2 \times \mathbb{P}^3 \to \mathbb{P}(V_3 \otimes S^2 V_4)$ given by the line bundle $\mathcal{O}(1,2)$. A Lefschetz decomposition of $\mathbb{P}^2 \times \mathbb{P}^3$ with respect to such a line bundle can be given by 3 copies of $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3)$. The section Y would inherit then 8 exceptional objects, and an unknown residual category. Comparing with the decomposition from π_{13} we expect the latter to be generated by 8 exceptional objects.

Finally, the projection to $\mathbb{P}^1 \times \mathbb{P}^3$ is birational, and X is the blowup of the surface $T \subset \mathbb{P}^1 \times \mathbb{P}^3$ defined by the condition

$$\operatorname{rank}\begin{pmatrix} q_1 & q_2 & q_3 \\ x_1\ell_{11} + x_2\ell_{21} & x_1\ell_{12} + x_2\ell_{22} & x_1\ell_{13} + x_2\ell_{23}\end{pmatrix} \leqslant 1.$$

The projection of T to $\mathbb{P}^3$ maps it to the quartic S, and blows up the eight points $z_1, \dots, z_8$.

Fano K3–52

(#3-29-110-2-B) $\mathscr{Z}\big(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3, \mathcal{O}(0,1,1) \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(1,1,1)\big)$

- Invariants:

$$h^0(-K) = 29,\ (-K)^4 = 110,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 24,\ -\chi(T) = 20.$$

- Description:
  - $\pi_{12} \colon X \to \mathbb{P}^2 \times \mathbb{P}^2$ is a blowup $\mathrm{Bl}_{S_{20}^{(1)}}(\mathbb{P}^2 \times \mathbb{P}^2)$ along a K3 surface of degree 20 blown up in one point.
  - $\pi_{13}, \pi_{23} \colon X \to \mathbb{P}^2 \times \mathbb{P}^3$ are blowups $\mathrm{Bl}_{S_{20}^{(1)}} Y$ of a $(1,1)$ divisor $Y \subset \mathbb{P}^2 \times \mathbb{P}^3$ along a K3 surface of degree 20 blown up in one point.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 10 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_{20})$ from all the maps.

Explanation

The situation being symmetrical, we can describe π_{23} and π_{13} using the same arguments. The line bundle $\mathcal{O}(0,1,1)$ cuts out $\mathbb{P}^2 \times Y$, for Y a $(1,1)$ divisor. Then X is the blowup of Y along the degeneracy locus T of the map of bundles $\mathcal{O}_Y(0,-1) \oplus \mathcal{O}_Y(-1,-1) \to \mathcal{O}_Y^{\oplus 3}$. The Eagon–Northcott complex on Y reads

$$0 \longrightarrow \mathcal{O}_Y(-1,-3) \oplus \mathcal{O}_Y(-2,-3) \longrightarrow \mathcal{O}_Y(-1,-2)^{\oplus 3} \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_T \longrightarrow 0$$

and can be dualized, since $\omega_Y = \mathcal{O}_Y(-2,-3)$, into

$$0 \longrightarrow \mathcal{O}_Y(-2,-3) \longrightarrow \mathcal{O}_Y(-1,-1)^{\oplus 3} \longrightarrow \mathcal{O}_Y(-1,0) \oplus \mathcal{O}_Y \longrightarrow \omega_T \longrightarrow 0.$$

So ω_T admits a canonical section, vanishing exactly on the locus where the morphism $\mathcal{O}_Y(-1,-1)^{\oplus 3} \to \mathcal{O}_Y(-1,0)$ is not surjective, hence on the common zero locus D of three sections of $\mathcal{O}_Y(0,1)$. This curve is a line D (projecting to a point in $\mathbb{P}^3$), and since $\omega_T = \mathcal{O}_T(D)$ we deduce that D is a (-1)-curve. Finally, after contracting this curve we get a genuine K3 surface of degree 20.

We turn to the description of π_{12}. On $\mathbb{P}^2 \times \mathbb{P}^2$ we have a map of bundles

$$\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(0,-1) \oplus \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-1,0) \oplus \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-1,-1) \longrightarrow \mathcal{O}^{\oplus 4}_{\mathbb{P}^2\times\mathbb{P}^2},$$

and π_{12} is the blowup of its degeneracy locus S, whose structure sheaf is resolved by the Eagon–Northcott complex

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-2,-3) \oplus \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-3,-2) \oplus \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-3,-3) \longrightarrow \\ \longrightarrow \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-2,-2)^{\oplus 4} \longrightarrow \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2} \longrightarrow \mathcal{O}_S \longrightarrow 0. \end{aligned}$$

We deduce that the Hilbert polynomial of S with respect to $\mathcal{O}(1,1)$ is $8k^2 - k + 2$. Moreover, dualizing as before, we get the complex

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-3,-3) \longrightarrow \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-1,-1)^{\oplus 4} \longrightarrow \\ \longrightarrow \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-1,0) \oplus \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(0,-1) \oplus \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2} \longrightarrow \omega_S \longrightarrow 0. \end{aligned}$$

So once again ω_S admits a canonical section, vanishing on the locus where the morphism $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-1,-1)^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-1,0) \oplus \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(0,-1)$ is not surjective. This is a curve D whose class is given, by the Thom–Porteous formula, as

$$[D] = s_3(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,0) \oplus \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(0,1) - \mathcal{O}^{\oplus 4}_{\mathbb{P}^2\times\mathbb{P}^2}) = h_1^2 h_2 + h_1 h_2^2,$$

where h_1 and h_2 denote the hyperplanes classes in the two copies of $\mathbb{P}^2$ respectively. This means that the projections of D to these two planes are two lines D_1 and D_2. Moreover, the Eagon–Northcott complex for D reads

$$0 \longrightarrow \mathcal{O}(-3,-1) \oplus \mathcal{O}(-2,-2) \oplus \mathcal{O}(-1,-3) \longrightarrow$$
$$\longrightarrow \mathcal{O}(-2,-1)^{\oplus 4} \oplus \mathcal{O}(-1,-2)^{\oplus 4} \longrightarrow \mathcal{O}(-1,-1)^{\oplus 6} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

We readily deduce that the D must be connected, from which we can finally conclude that D is a projective line mapping isomorphically to both D_1 and D_2. And once again, D is a (-1)-curve on S, after contracting which we get a genuine K3 surface.

Fano K3–53

(#3-36-145-2) $\mathscr{Z}\big(\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^4, \mathcal{Q}_{\mathbb{P}^3}(0,0,1) \oplus \mathcal{O}(1,1,0) \oplus \mathcal{O}(1,1,1)\big)$

- Invariants:

$$h^0(-K) = 36,\ (-K)^4 = 145,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 23,\ -\chi(T) = 19.$$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^2 \times \mathbb{P}^3$ is a blowup $\mathrm{Bl}_{S_{16}} Y$, where Y is a $(1,1)$ divisor in $\mathbb{P}^2 \times \mathbb{P}^3$.
  - $\pi_{13}\colon X \to \mathbb{P}^2 \times \mathbb{P}^4$ is a blowup $\mathrm{Bl}_{\mathbb{P}^2} Z$, where Z is a special FK3 Fano K3–29 given by the intersection of two divisors of bidegrees $(1,1)$ and $(1,2)$ in $\mathbb{P}^2 \times \mathbb{P}^4$.
  - $\pi_{23}\colon X \to \mathbb{P}^3 \times \mathbb{P}^4$ is a blowup $\mathrm{Bl}_{S^{(1)}_{26}} \mathrm{Bl}_p \mathbb{P}^4$, where $S^{(1)}_{26}$ is a degree 26 K3 surface blown up in one point.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 9 exceptional objects and $\mathrm{D^b}(S_{16})$ from π_{12}.
  - 9 exceptional objects and $\mathrm{D^b}(S_8)$ or $\mathrm{D^b}(S_2, \alpha)$ (see Fano K3–29) from π_{13}.
  - 9 exceptional objects and $\mathrm{D^b}(S_{26})$ from π_{23}.
  - We deduce that S_{16}, S_8 and S_{26} are all derived equivalent (see Lemma 3.4).

Explanation

The sections of the three bundles are given by tensors $\alpha \in \mathrm{Hom}(V_5, V_4)$, $\beta \in V_3^\vee \otimes V_4^\vee$ and $\gamma \in V_3^\vee \otimes V_4^\vee \otimes V_5^\vee$. The fourfold X parametrizes the triples of lines (x, y, z) such that

$$\alpha(z) \subset y, \qquad \beta(x,y) = 0, \qquad \gamma(x,y,z) = 0.$$

Let us start with π_{23}. The set of pairs (y,z) such that $\alpha(z)\subset y$ is the blowup $P:=\mathrm{Bl}_{z_0}\mathbb{P}^4$ of $\mathbb{P}^4$ at the point $z_0=\mathrm{Ker}(\alpha)$. For a general z, y is uniquely determined, and there are two linear conditions on x that also determine it uniquely in general. So the projection of X to $\mathbb{P}^4$ is birational. The exceptional locus S of the projection to P is the degeneracy locus of the morphism $\mathcal{O}_P(-1,0)\oplus\mathcal{O}_P(-1,-1)\to V_3^\vee$ induced by β and γ. The structure sheaf of this smooth surface is resolved by the Eagon–Northcott complex

$$0\longrightarrow \mathcal{O}_P(-3,-1)\oplus\mathcal{O}_P(-3,-2)\longrightarrow V_3^\vee\otimes\mathcal{O}_P(-2,-1)\longrightarrow \\ \longrightarrow\mathcal{O}_P\longrightarrow\mathcal{O}_S\longrightarrow 0.$$

Since $\mathcal{O}_P(-3,-2)$ is the canonical bundle of P, one easily deduces that $h^2(\mathcal{O}_S)=1$. Moreover, the Hilbert polynomial for S is $\chi\big(\mathcal{O}_S(t)\big)=\frac{25k^2-k}{2}+2$. Dualizing the complex we get

$$0\longrightarrow\mathcal{O}_P(-3,-2)\longrightarrow V_3\otimes\mathcal{O}_P(-1,-1)\longrightarrow\mathcal{O}_P(0,-1)\oplus\mathcal{O}_P\longrightarrow\omega_S\longrightarrow 0.$$

This implies that ω_S admits a canonical section, that vanishes on the locus where $V_3\otimes\mathcal{O}_P(-1,-1)\to\mathcal{O}_P(0,-1)$ is not surjective. This locus is the common zero locus of three general sections of $\mathcal{O}_P(1,0)$, hence the preimage of a point in $\mathbb{P}^3$, which is a line D inside P. We conclude that $\omega_S=\mathcal{O}_S(D)$, and by the genus formula D is a (-1)-curve in S; after contracting it, we get a genuine K3 surface $S_{\min}$ of degree 26. Moreover X is the blowup of P along the surface S.

We now turn to π_{12}. The set of pairs (x,y) such that $\beta(x,y)=0$ is a divisor Y of bidegree $(1,1)$ in $\mathbb{P}^2\times\mathbb{P}^3$. The extra point z has to belong to $\alpha^{-1}(y)\simeq z_0\oplus y$, which defines a projective line on which γ cuts out a unique point. The projection of X to Y is thus birational. The exceptional locus is the zero locus of the induced morphism $\mathcal{O}_Y(-1,-1)\to\big(z_0\oplus\mathcal{O}_Y(0,-1)\big)^\vee$, hence a surface T which is a general complete intersection of divisors of bidegrees $(1,1)$ and $(1,2)$ in Y. Being a complete intersection in $\mathbb{P}^2\times\mathbb{P}^3$ of three divisors of bidegrees $(1,1)$, $(1,1)$ and $(1,2)$, $T=S_{16}$ has trivial canonical bundle and is a K3 surface of degree 16 (isomorphic to $S_{\min}$). Moreover X is the blowup of Y along this surface.

Finally we describe π_{13}. Its image Z parametrizes the set of pairs (x,z) such that $\beta\big(x,\alpha(z)\big)=0$ and $\gamma\big(x,\alpha(z),z\big)=0$. It is the intersection in $\mathbb{P}^2\times\mathbb{P}^4$ of two divisors of bidegrees $(1,1)$ and $(1,2)$, both containing the plane $\Pi=\mathbb{P}^2\times\{z_0\}$. For a given (x,z) in Z, y is uniquely determined by the condition that $\alpha(z)\subset y$, except if $z=z_0$, for which we get a $\mathbb{P}^1$-fibration over Π. We conclude that the projection of X to Z is just the blowup of Π.

Fano K3–54

(#3-34-133-2) $\mathscr{Z}\big(\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^4, \mathcal{O}(0,1,2) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,0,1) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,1,0)\big)$

- Invariants:

$h^0(-K) = 34,\ (-K)^4 = 133,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 25,\ -\chi(T) = 23.$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^2 \times \mathbb{P}^3$ is a conic bundle over $\mathrm{Bl}_p \mathbb{P}^3$.
  - $\pi_{13}\colon X \to \mathbb{P}^2 \times \mathbb{P}^4$ is a blowup $\mathrm{Bl}_{S_6^{(2)}} Y$, where $S_6^{(2)}$ is a sextic K3 surface in $\mathbb{P}^4$ blown up at two points, and Y is the blowup $\mathrm{Bl}_{\mathbb{P}^1} \mathbb{P}^4$.
  - $\pi_{23}\colon X \to \mathbb{P}^3 \times \mathbb{P}^4$ is a small contraction of two projective planes to a fourfold Y°, singular at two points.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 6 exceptional objects and $\mathrm{D^b}(\mathrm{Bl}_p \mathbb{P}^3, \mathcal{C}_0)$ from π_{12}.
  - 11 exceptional objects and $\mathrm{D^b}(S_6)$ from π_{13}.

Explanation

The fourfold $X \subset \mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^4$ is the zero locus of sections of $\mathcal{O}(0,1,2)$, $\mathcal{Q}_{\mathbb{P}^2}(0,1,0)$ and $\mathcal{Q}_{\mathbb{P}^2}(0,0,1)$ defined by tensors $\alpha \in V_4^\vee \otimes S^2V_5^\vee$, $\beta \in \mathrm{Hom}(V_4, V_3)$ and $\gamma \in \mathrm{Hom}(V_5, V_3)$. The last two sections vanish on those triples (x, y, z) of lines such that $\beta(y) \subset x$ and $\gamma(z) \subset x$.

We start with π_{12}. For x, y fixed, z must belong to $\gamma^{-1}(x)$ which defines a projective plane, on which α traces a conic. Note that x is defined by y except when $\beta(y) = 0$. The kernel of β defines a point $y_0 \in \mathbb{P}^3$ and we conclude that π_{12} is a conic bundle over $\mathrm{Bl}_{y_0} \mathbb{P}^3$. To understand its discriminant locus, notice (in the flag interpretation) that the fiber of $(U_1 \subset V_2)$ in P is the set of lines $l \in \beta^{-1}(V_2)$ such that $\alpha(l)(U_1) = 0$, which defines a conic in a projective plane. So the projection $X \to P$ is a conic bundle, defined by a morphism

$$U_1 \longrightarrow S^2(V_2 \oplus \mathcal{O}_P)^\vee.$$

Its discriminant locus Δ is defined by a morphism $U_1^3 \to \big(\det(V_2)^\vee\big)^2$ so a section of $\mathcal{O}_P(3,2)$. Since $K_P = \mathcal{O}_P(-2,-3)$, we get $K_\Delta = \mathcal{O}_\Delta(1,-1)$.

Now we turn to π_{13}. The image Y of X in $\mathrm{Gr}(2,4) \times \mathbb{P}^4$ is in fact contained in $\mathbb{P}^2 \times \mathbb{P}^4$ (since $\mathscr{Z}(\mathrm{Fl}(1,2,4), \mathcal{Q}_2)$ base changes the second projection to $\mathbb{P}^2$) and is the set of pairs (V_2, l) such that V_2 contains L_1 and $\beta(l)$. The fibers of the projection $Y \to \mathbb{P}^4$ are non-trivial when $\beta(l) \subset L_1$, so we get the blowup of the line $d = \mathbb{P}\big(\beta^{-1}(L_1)\big)$. The fibers of the projection $X \to Y$ are non-trivial when $\alpha(l)$ vanishes on V_2. This condition defines a surface S whose image in $\mathbb{P}^4$ is defined by the conditions that

$$\alpha(l)(L_1) = \alpha(l)\big(\beta(l)\big) = 0.$$

We get the intersection S_6 of a quadric and a cubic, hence a K3 surface of degree 6. Note that the line d is bisecant to S_6, so $S = S_6^{(2)}$ is the blowup of S_6 at these two points.

Finally we describe π_{23}. One can understand Y° as follows. Given y, z such that $\alpha(y,z,z) = 0$ and $\beta(y)$ and $\gamma(z)$ are collinear, x is uniquely defined except if $\beta(y)$ and $\gamma(z)$ both vanish, which happens for exactly two points q_1, q_2 in $P = \mathbb{P}^3 \times \mathbb{P}^4$, over which the fiber of π_{23} is a projective plane. The collinearity condition is expressed by the degeneracy of the morphism

$$\varphi \colon \mathcal{O}_P(-1,0) \oplus \mathcal{O}_P(0,-1) \longrightarrow U_3 \otimes \mathcal{O}_P$$

defined by β and γ, restricted to the general $(1,2)$-divisor cut out by α. The fourfold $Y^\circ = D_1(\phi)$ is the image of π_{23}; its singular locus is $D_0(\phi) = \{q_1, q_2\}$. We conclude that π_{23} is a small contraction of two projective planes to a fourfold Y° with two singular points. When we blow up these two points, the exceptional divisors are copies of $\mathbb{P}^1 \times \mathbb{P}^2$ and the $\mathbb{P}^1$-factor can be contracted to the smooth fourfold X.

Finally, for a given x, y has to vary in a projective line and z in a projective plane, and α defines a divisor of bidegree $(1,2)$ in the corresponding $\mathbb{P}^1 \times \mathbb{P}^2$. This means that the projection to $\mathbb{P}^2$ is a dP_5-fibration. The projections to $\mathbb{P}^3$ and $\mathbb{P}^4$ have already been described.

Fano K3–55

(#3-37-150-2) $\quad \mathscr{X}\big(\mathbb{P}^1 \times \mathbb{P}^4 \times \mathbb{P}^5, \mathcal{Q}_{\mathbb{P}^4}(0,0,1) \oplus \mathcal{O}(0,2,0) \oplus \mathcal{O}(1,1,1)\big)$

- Invariants:

$$h^0(-K) = 37,\ (-K)^4 = 150,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,2} = 22,\ -\chi(T) = 18.$$

- Description:
  - $\pi_{12} \colon X \to \mathbb{P}^1 \times \mathbb{P}^4$ is a blowup $\mathrm{Bl}_{S_{14}}(\mathbb{P}^1 \times \mathbb{Q}^3)$ along a K3 surface S_{14} of genus 8.
  - $\pi_{13} \colon X \to \mathbb{P}^1 \times \mathbb{P}^5$ is a divisorial contraction to a complete intersection Y° of divisors of bidegrees $(0,2)$ and $(1,2)$, singular along a line.

  - $\pi_{23}\colon X \to \mathbb{P}^4 \times \mathbb{P}^5$ is $\mathrm{Bl}_{S_{30}} Y$, where S_{30} is a K3 surface of degree 30 and Y is a $\mathbb{P}^1$-bundle over $\mathbb{Q}^3$.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 8 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_{14})$ from π_{12}.
  - 8 exceptional objects and $\mathrm{D}^{\mathrm{b}}(S_{30})$ from π_{23}.
  - We deduce that S_{14} and S_{30} are derived equivalent (see Lemma 3.4).

Explanation

There are two line bundles involved, plus the rank 4 bundle $\mathcal{Q}_{\mathbb{P}^4}(0,0,1)$ whose space of global sections is $\mathrm{Hom}(V_6, V_5)$. Generically, a morphism $\theta \in \mathrm{Hom}(V_6, V_5)$ can be reduced to the projection from $V_6 = K_1 \oplus V_5$ to V_5, where K_1 denotes the kernel. Moreover the zero locus Z of the corresponding section is the blowup of $\mathbb{P}^5$ at the point $v = [K_1]$, which is also $\mathbb{P}_{\mathbb{P}^4}\big(\mathcal{O} \oplus \mathcal{O}(-1)\big)$. This implies that the image of π_{23} is just $Y = \mathbb{P}_Q\big(\mathcal{O} \oplus \mathcal{O}(-1)\big)$, for $Q \subset \mathbb{P}^4$ a smooth quadric. Moreover π_{23} blows up a smooth surface S in Y obtained as a double section of $\mathcal{O}_Y(1,1)$. Since by adjunction, $\omega_Y = \mathcal{O}_Y(-2,-2)$, this must be a K3 surface. If we denote by b, c the hyperplane classes from $\mathbb{P}^4$ and $\mathbb{P}^5$, the fundamental class of Z is $c_4\big(\mathcal{Q}(0,1)\big) = b^4 + b^3c + b^2c^2 + bc^3 + c^4$. We readily deduce that

$$\deg(S) = \int_S (b+c)^2 = \int_Z 2b(b+c)^4 = 30, \qquad \int_S b^2 = \int_Z 2b^2(b+c)^2 = 6.$$

So S is mapped to a special K3 surface of degree 6 in $\mathbb{P}^4$.

Now we turn to π_{12}. Its image is obviously $P = \mathbb{P}^1 \times \mathbb{Q}^3$, while the fibers are defined by the image of the morphism $\mathcal{Q} \oplus \mathcal{O}(-1,-1) \to V_6^\vee \otimes \mathcal{O}$ (where we simply denote by $\mathcal{Q}$ the restriction of $\mathcal{Q}_{\mathbb{P}^4}$ to the quadric $Q \subset \mathbb{P}^4$. This morphism degenerates over a smooth surface T, whose structure sheaf is resolved by the Eagon–Northcott complex

$$0 \longrightarrow \mathcal{Q}^\vee(-1,-2) \oplus \mathcal{O}_P(-2,-3) \longrightarrow V_6^\vee \otimes \mathcal{O}_P(-1,-2) \longrightarrow \\ \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_T \longrightarrow 0,$$

from which we calculate the Hilbert polynomial $\chi\big(\mathcal{O}_T(k,k)\big) = 7k^2 + 2$. Dualizing the complex, we get

$$0 \longrightarrow \mathcal{O}_P(-2,-3) \longrightarrow V_6 \otimes \mathcal{O}_P(-1,-1) \longrightarrow \mathcal{Q}(-1,-1) \oplus \mathcal{O}_P \longrightarrow \omega_T \longrightarrow 0.$$

So ω_T admits a canonical section whose zero locus is the degeneracy locus of the morphism $V_6 \otimes \mathcal{O}(-1,-1) \to \mathcal{Q}(-1,-1)$, which is just a twist of the surjection $V_6 \to V_5 \to \mathcal{Q}$. So T is a K3 surface of degree 14, projecting in $\mathbb{Q}^3$ to a special K3 of degree 6 in $\mathbb{P}^4$ (its Picard rank is 2, generically, and the projection from T to $\mathbb{P}^1$ is an elliptic fibration).

Finally, consider the projection π_{13} to $R = \mathbb{P}^1 \times \mathbb{P}^5$. The fiber is defined by a quadratic form on V_5 and two morphisms $\mathcal{O}_R(0,-1) \to V_5 \otimes \mathcal{O}_R$ and $\mathcal{O}_R(-1,-1) \to V_5^\vee \otimes \mathcal{O}_R$. When the first morphism is non-zero, it defines a point in $\mathbb{P}(V_5)$ which has to be the unique point of the fiber, in case it is compatible with the other sections. The compatibility with the quadratic form is verified on $\mathbb{P}^1 \times \mathbb{Q}^4_0$, where $\mathbb{Q}^4_0$ is a corank 1 quadric in $\mathbb{P}^5$; let v denote its vertex. The compatibility with the second morphism holds true over a divisor of type $(1,2)$ containing the line $d = \mathbb{P}^1 \times v$. The image of π_{13} is therefore a complete intersection of hypersurfaces of bidegree $(0,2)$ and $(1,2)$, singular along d.

Non-trivial fibers can only occur when the morphism $\mathcal{O}(0,-1) \to V_5 \otimes \mathcal{O}$ is zero, that is, over d. The fibers are then defined by a linear form and a quadratic form on $\mathbb{P}(V_5)$. So π_{13} is finally a divisorial contraction, the exceptional divisor being a quadric fibration over d.

Fano K3–56

(#4-35-140-2) $\mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3, \mathcal{Q}_{\mathbb{P}^2}(0,0,0,1) \oplus \mathcal{O}(1,1,1,1)\big)$

- Invariants:

$h^0(-K) = 35,\ (-K)^4 = 140,\ h^{1,1} = 4,\ h^{3,1} = 1,\ h^{2,2} = 24,\ -\chi(T) = 17.$

- Description:
  - $\pi_{123}\colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ is the blowup of a K3 surface obtained by intersecting divisors of multidegrees $(1,1,1)$ and $(1,1,2)$.
  - $\pi_{124}\colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$ is a blowup of Fano K3–41 along a line.
  - $\pi_{134} = \pi_{234}\colon X \to \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3$ is a blowup $\mathrm{Bl}_{S_{28}}(\mathbb{P}^1 \times \mathrm{Bl}_p \mathbb{P}^3)$, where S_{28} is a degree 28 K3 surface with Picard rank 3.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 12 exceptional objects and a $\mathrm{D^b}(S)$ from each map.

Explanation

In order to understand the last two projections, we first apply Lemma 3.1, identifying $\mathscr{Z}\big(\mathbb{P}^2 \times \mathbb{P}^3, \mathcal{Q}_{\mathbb{P}^2}(0,1)\big)$ with $\mathrm{Bl}_p \mathbb{P}^3$. We then apply in two different ways Lemma 3.1 again. For the second projection, the line is $\mathbb{P}^1 \times p$, with the point p contained in the K3 surface. For the third and fourth, we only need to check that the intersection of two divisors of degree $\mathcal{O}(1,1,1)$ gives a K3 surfaces S such that $p \in S$. The generic Picard rank of S is 3, with degree 28 with respect to $\mathcal{O}(1,1,1)$.

Fano K3–57

(#4-34-134-2) $\mathscr{Z}\big(\mathbb{P}^1 \times \mathrm{Fl}(1,3,6), \mathcal{Q}_2^{\oplus 2} \oplus \mathcal{O}(0;0,2) \oplus \mathcal{O}(1;2,0)\big)$

- Invariants:

$$h^0(-K) = 34,\ (-K)^4 = 134,\ h^{1,1} = 4,\ h^{3,1} = 1,\ h^{2,2} = 24,\ -\chi(T) = 18.$$

- Description:
  - $\pi_{12}\colon X \to \mathbb{P}^1 \times \mathbb{P}^5$ is a divisorial contraction, whose image is a complete intersection of bidegrees $(1,2)$ and $(0,2)$, singular along a twisted cubic.
  - $\pi_{13}\colon X \to \mathbb{P}^1 \times \mathrm{Gr}(3,6)$ is a conic bundle over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with discriminant locus a divisor of multidegree $(3,2,2)$.
  - $\pi_{23}\colon X \to \mathrm{Fl}(1,3,6)$ is a blowup $\mathrm{Bl}_{S_8^{(2)}} Y$, where $S_8^{(2)}$ is an octic K3 surface blown up at two points p, q and $Y = \mathrm{Bl}_{\mathbb{P}^1} CQ$, the latter being a four dimensional quadric cone with $\mathbb{P}^1$ as a vertex.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 8 exceptional objects and $\mathrm{D^b}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{C}_0)$ from π_{13}.
  - 12 exceptional objects and $\mathrm{D^b}(S_8)$ from π_{23}.

Explanation

To describe π_{23}, we apply Lemma 3.1. We need to understand $Y := \mathscr{Z}\big(\mathrm{Fl}(1,3,6),\ \mathcal{Q}_2^{\oplus 2} \oplus \mathcal{O}(0,2)\big)$. The latter by [20, Corollary 2.7] is a special quadratic section of $\mathbb{P}_{\mathbb{P}^3}\big(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1,-1)\big)$, which is of course the blowup of $\mathbb{P}^5$ in a line $l \cong \mathbb{P}(U_2)$. The quadratic section in this blowup is given by an element in $\mathrm{S}^2(V_6/U_2)^\vee$, hence describing a quadric cone with vertex l, of which Y is the blowup.

The above description gives us π_{13} as well. We start by identifying X with a $(1,2)$ divisor in the product $\mathbb{P}^1 \times \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}\big(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1,-1)\big)$. In turn this can be rewritten as $\mathscr{Z}\big(\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}\big(\mathcal{O}(-1,0,0)^{\oplus 2} \oplus \mathcal{O}(-1,-1,-1)\big), \mathcal{L}^2\big)$. Over $\mathbb{P}^1 \times T$ where $T = \mathbb{P}^1 \times \mathbb{P}^1$ we get a conic bundle structure given by a morphism $\mathcal{O}(-1,0) \to \mathrm{S}^2\, V_3^\vee$. So the discriminant is a map $\mathcal{O}(-3,0) \to$

$\det(V_3^\vee)^2$, and the discriminant locus is a divisor of multidegree $(3,2,2)$ in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$.

Finally consider the projection π_{12} to $\mathbb{P}^1\times\mathbb{P}^5$. When the line $U_1\subset V_6$ is fixed outside V_2, then $U_3=U_1\oplus V_2$ is uniquely determined. So the section of $\det(U_3^\vee)^{\otimes 2}$ yields a section of $(U_1^\vee)^{\otimes 2}$ defining a quadric whose singular locus is the line $\mathbb{P}(V_2)$. Cutting we the hypersurface of bidegree $(1,2)$, we get a twisted cubic for singular locus.

Fano K3–58

(#4-28-104-2) $\mathscr{Z}\big(\mathbb{P}^1_1\times\mathbb{P}^1_2\times\mathbb{P}^1_3\times\mathbb{P}^3,\mathcal{O}(0,0,1,1)\oplus\mathcal{O}(1,1,0,2)\big)$

- Invariants:

$$h^0(-K)=28,\ (-K)^4=104,\ h^{1,1}=4,\ h^{3,1}=1,\ h^{2,2}=28,\ -\chi(T)=22.$$

- Description:
  - $\pi_{123}\colon X\to\mathbb{P}^1_1\times\mathbb{P}^1_2\times\mathbb{P}^1_3$ is a conic bundle over $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, with discriminant a $(3,3,2)$-divisor.
  - $\pi_{124}\colon X\to\mathbb{P}^1_1\times\mathbb{P}^1_2\times\mathbb{P}^3$ is a blowup $\mathrm{Bl}_{\mathrm{dP}_4}Y$, where Y is a blowup of $\mathbb{P}^1\times\mathbb{P}^3$ along a bielliptic K3 surface S.
  - $\pi_{134},\pi_{234}\colon X\to\mathbb{P}^1_{1,2}\times\mathbb{P}^2_3\times\mathbb{P}^3$ are blowups $\mathrm{Bl}_S Y$, where S is the same bielliptic K3 surface and Y is $\mathbb{P}^1\times\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^3$.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 8 exceptional objects and $\mathrm{D^b}(\mathbb{P}^1_1\times\mathbb{P}^1_2\times\mathbb{P}^1_3,\mathcal{C}_0)$ from π_{123}.
  - 16 exceptional objects and $\mathrm{D^b}(S)$ from π_{124} and from π_{134} or π_{234}.

Explanation

We consider vector spaces U_2,V_2,W_2,U_4, and tensors $\alpha\in W_2^\vee\otimes U_4^\vee$ and $\beta\in U_2^\vee\otimes V_2^\vee\otimes S^2W_2^\vee$. The fourfold X parametrizes the fourtuples of lines (a,b,c,d) such that $\alpha(c,d)=0$ and $\beta(a,b,d,d)=0$.

For a given triple (a,b,c), the condition $\alpha(c,d)=0$ defines a projective plane in which β cuts a conic, so the projection of X to $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ is a conic bundle defined by a morphism $\mathcal{O}(-1,-1,0)\to S^2\big(\mathcal{O}^{\oplus 2}\oplus\mathcal{O}(0,0,-1)\big)^\vee$. Its discriminant is given by the induced morphism $\mathcal{O}(-3,-3,0)\to S_{222}\big(\mathcal{O}^{\oplus 2}\oplus\mathcal{O}(0,0,-1)\big)^\vee=\mathcal{O}(0,0,2)$, hence by a section of $\mathcal{O}(3,3,2)$, which has nef canonical divisor.

The set of triples (a,b,d) such that $\beta(a,b,d,d)=0$ is a divisor Y of multidegree $(1,1,2)$ in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^3$, whose projection to each $\mathbb{P}^1\times\mathbb{P}^3$ is the blowup of a bielliptic K3 surface S. The extra line c is then uniquely determined by the condition that $\alpha(c,d)=0$, except if $\alpha(W_2,d)=0$. This condition defines a line δ in $\mathbb{P}^3$, whose preimage in Y is a divisor of multidegree $(1,1,2)$ in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, hence a dP_4.

The set of pairs (c,d) such that $\alpha(c,d)=0$ is the blowup P of $\mathbb{P}^3$ along the line δ. The triple (a,c,d) being given in $Y=\mathbb{P}^1\times P$, the extra line b is uniquely determined except if $\beta(a,V_2,d,d)=0$, hence on a surface defined as the intersection of two divisors of multidegree $(1,0,2)$ in Y, whose image in $\mathbb{P}^3$ is again the bielliptic quartic surface S.

Fano K3–59

(#4-31-120-2) $\mathscr{Z}\big(\mathbb{P}^1_1\times\mathbb{P}^1_2\times\mathbb{P}^2_1\times\mathbb{P}^2_2,\mathcal{O}(0,0,1,1)\oplus\mathcal{O}(1,1,1,1)\big)$

- Invariants:

$$h^0(-K)=31,\ (-K)^4=120,\ h^{1,1}=4,\ h^{3,1}=1,\ h^{2,2}=24,\ -\chi(T)=17.$$

- Description:
  - $\pi_{123},\pi_{124}\colon X\to\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^2$ are blowups $\mathrm{Bl}_{S_{20}}(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^2)$, where S_{20} is a degree 20 K3 surface of Picard rank 3.
  - $\pi_{134},\pi_{234}\colon X\to\mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^2$ are blowups $\mathrm{Bl}_S(\mathbb{P}^1\times F)$, where $F\subset\mathbb{P}^2\times\mathbb{P}^2$ is the complete flag manifold (a Fano threefold of Picard rank 2, index 2 and degree 6) along a K3 surface S isomorphic to S_{20}.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 12 exceptional objects and $\mathrm{D^b}(S_{20})$ from $\pi_{123}=\pi_{124}$ and $\pi_{134}=\pi_{234}$.

Explanation

Here we consider vector spaces U_2,V_2,U_3,V_3 and tensors $\alpha\in U_3^\vee\otimes V_3^\vee$ and $\beta\in U_2^\vee\otimes V_2^\vee\otimes U_3^\vee\otimes V_3^\vee$, that define the fourfold X in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^2$. Obviously the projection to $\mathbb{P}^1\times\mathbb{P}^1$ is a dP_6-fibration, and the projection to $\mathbb{P}^2\times\mathbb{P}^2$ is a conic bundle over a hyperplane section of the latter, which is known to be isomorphic to the complete flag $F=\mathbb{P}(T_{\mathbb{P}^2})$.

The fibers of the projection π_{123} of X to $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ are non-trivial when the natural morphism $\mathcal{O}_Y(0,0,-1) \oplus \mathcal{O}_Y(-1,-1,-1) \to V_3^\vee \otimes \mathcal{O}_Y$ drops rank, which happens along a surface S whose structure sheaf is resolved by the Eagon–Northcott complex

$$0 \longrightarrow \mathcal{O}_Y(-1,-1,-3) \oplus \mathcal{O}_Y(-2,-2,-3) \longrightarrow V_3^\vee \otimes \mathcal{O}_Y(-1,-1,-2) \longrightarrow \\ \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

The Hilbert polynomial of S is $\chi\big(\mathcal{O}_S(k)\big) = 10k^2+2$. Dualizing the complex we get

$$0 \longrightarrow \mathcal{O}_Y(-2,-2,-3) \longrightarrow V_3 \otimes \mathcal{O}_Y(-1,-1,-1) \longrightarrow \\ \longrightarrow \mathcal{O}_Y(-1,-1,0) \oplus \mathcal{O}_Y \longrightarrow \omega_S \longrightarrow 0.$$

So ω_S admits a canonical section, vanishing on the locus where the morphism $V_3 \otimes \mathcal{O}_Y(-1,-1,-1) \to \mathcal{O}_Y(-1,-1,0)$ is not surjective, hence on the common zero locus of three sections of $\mathcal{O}_Y(0,0,1)$. Clearly this locus is empty, so ω_S is trivial and we can conclude that S must be a K3 surface of degree 20. Moreover X must be the blowup of Y along S. Note that the class of S in the Chow ring of Y can be obtained as the degree two part of $\mathrm{ch}(\mathcal{O}_S)$. If we denote by a, b, c the hyperplane classes of the three projective spaces, the previous complex yields $[S] = 2ab + 3ac + 3bc + 3c^2$. We deduce that the Picard group of S contains a copy of the lattice

$$\begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 2 \end{pmatrix}.$$

The projection π_{234} to $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ maps X to $\mathbb{P}^1 \times F$. The fibers are non-trivial over the intersection of two divisors of multidegree $(1,1,1)$, which is again a K3 surface T. Here the Picard group of T contains a copy of the lattice

$$\begin{pmatrix} 0 & 3 & 3 \\ 3 & 2 & 4 \\ 3 & 4 & 2 \end{pmatrix}.$$

Fano K3–60

(#5-31-120-2) $\mathscr{Z}\big((\mathbb{P}^1)^5, \mathcal{O}(1,1,1,1,1)\big)$

- Invariants:

$h^0(-K) = 31,\ (-K)^4 = 120,\ h^{1,1} = 5,\ h^{3,1} = 1,\ h^{2,2} = 26,\ -\chi(T) = 16.$

- Description:
  - $X \to (\mathbb{P}^1)^4$ is a blowup $\mathrm{Bl}_{S_{24}}(\mathbb{P}^1)^4$ in five different ways, where S_{24} is a K3 surface of degree 24 with Picard rank 4.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 16 exceptional objects and $\mathrm{D^b}(S_{24})$ from each map.

Explanation

The projection to each partial product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a blowup of a K3 surface defined as the intersection of two divisors of multidegree $(1, 1, 1, 1)$. This surface has degree 24 in the Segre embedding.

Remark 6.10. — This fourfold was already considered in [44] and labeled as (d3), and its birational structure was described in [51].

7. Rogue Fano fourfolds

This section contains the remaining four families of Fano fourfolds of K3 type that we obtained, which do not belong to the lists of the three previous sections. The origin of their K3 structures is therefore a bit more mysterious.

Fano R–61

(#2-27-99-2) $\mathscr{Z}\left(\mathbb{P}^2 \times \mathrm{Gr}(2,6), \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)}(0,1) \oplus \mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^{\vee}_{\mathrm{Gr}(2,6)}\right)$

- Invariants:

$$h^0(-K) = 27,\ (-K)^4 = 99,\ h^{1,1} = 2,\ h^{3,1} = 1,\ h^{2,2} = 25,\ -\chi(T) = 25.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^2$ is a dP_5 fibration.
  - $\pi_2 \colon X \to \mathrm{Gr}(2,6)$ is birational to a singular fourfold, contracting three projective planes.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 6 exceptional objects and $\mathrm{D^b}(Z)$, for $Z \to \mathbb{P}^2$ a degree 5 flat map, from π_1. One can expect Z to be the blowup of a K3 in 3 points.

Explanation

The fourfold X is defined by two sections $\alpha \in \mathrm{Hom}(V_6, V_3)$ and $\beta \in S_{2,1}V_6^\vee \subset V_6^\vee \otimes \bigwedge^2 V_6^\vee$. It parametrizes the pairs $(l, P) \in \mathbb{P}^2 \times \mathrm{Gr}(2,6)$ such that $\alpha(P) \subset l$ and $\beta(p, p_1 \wedge p_2) = 0$ for all $p \in P = \langle p_1, p_2\rangle$. The first condition implies that P meets the kernel K_3 of α. (Hence a projection to $\mathbb{P}(K_3)$ which will induce a rational quadric fibration on X.)

Projecting to $\mathbb{P}^2$, the fiber over l is the set of planes in $\mathrm{Gr}\big(2, \alpha^{-1}(l)\big)$ where a section of $\mathcal{Q}(1)$ vanishes, that is a dP_5. Notice that this description implies the rationality of this family, since we already mentioned that every del Pezzo of degree 5 is rational over its field of definition, hence X is $k(\mathbb{P}^2)$-rational.

Finally, projecting to $\mathrm{Gr}(2,6)$, we get non-trivial fibers when $\alpha(P) = 0$. Since a section of $\mathcal{Q}(1)$ on a projective plane vanishes at three points, we conclude that this projection is a birational morphism contracting three projective planes — hence a small contraction.

Fano R–62

(#2-33-129-2) $\mathscr{Z}\big(\mathbb{P}^3 \times \mathrm{Gr}(2,7), \mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{O}(1,1) \oplus \mathcal{Q}_{\mathbb{P}^3} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}\big)$

- Invariants:

$$h^0(-K) = 33,\ (-K)^4 = 129,\ h^{1,1} = 2,\ h^{3,1} =, 1\ h^{2,2} = 23,\ -\chi(T) = 23.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^3$ is a conic bundle with discriminant a quintic surface, singular in 16 points.
  - $\pi_2 \colon X \to \mathrm{Gr}(2,7)$ contracts a $\mathbb{P}^2$ to a point of a singular fourfold Y°.
  - Rationality: unknown. However, see Remark 4.1 for a connection with a (possibly special) family of Fano R–62.
- Semiorthogonal decompositions:
  - 4 exceptional objects and $\mathrm{D^b}(\mathbb{P}^3, \mathcal{C}_0)$ from π_1.
  - There is no obvious hint about the K3 structure (but compare with Fano K3–27 or Fano R–63 and the possible correspondence between singular intersections of three quadrics and singular $(2,2)$ divisors in $\mathbb{P}^2 \times \mathbb{P}^3$; see also Remarks 7.2 and 7.3).

Explanation

The fourfold X is defined by three line bundles and a rank 6 bundle. First consider the sevenfold Z defined by a section of the latter, which is just a morphism $\theta \in V_4 \otimes V_7^\vee = \mathrm{Hom}(V_7, V_4)$. Generically, one can suppose that $V_7 = K_3 \oplus V_4$ and that θ is the projection to V_4. Our Z is just the set of pairs (l, U_2) such that $\theta(U_2) \subset l$. In particular U_2 has to meet K_3 non-trivially. This suggests to define

$$\widetilde{Z} = \left\{U_1 \subset U_2 \subset U_4 \subset V_7,\ U_1 \subset K_3 \subset U_4\right\},$$

a $\mathbb{P}^2$-bundle over $\mathbb{P}(K_3) \times \mathbb{P}(V_7/K_3) = \mathbb{P}^2 \times \mathbb{P}^3$. Sending the flag $U_1 \subset U_2 \subset U_4$ to $(U_4/K_3, U_2)$, one obtains a birational isomorphism η onto Z, since U_1 can be recovered as $U_1 = U_2 \cap K_3$ when U_2 is not contained in K_3. The exceptional locus is $\mathrm{Fl}(1, 2, K_3) \times \mathbb{P}^3$, which is contracted to the fivefold $\mathrm{Gr}(2, K_3) \times \mathbb{P}^3 \subset \mathrm{Gr}(2, V_7) \times \mathbb{P}^3$ by forgetting U_1.

Now we take into account the sections of the line bundles, which are defined by two skew-symmetric forms ω_1, ω_2 on V_7, and a tensor $\alpha \in V_4^\vee \otimes \bigwedge^2 V_7^\vee$. They define a fourfold X in Z, whose preimage in $\widetilde{Z}$ is the subvariety $\widetilde{X}$ defined by the conditions

$$\omega_1(U_1, U_2) = \omega_2(U_1, U_2) = \alpha(U_4/K_3, U_1, U_2) = 0.$$

Note that U_4 is not completely determined by U_2 exactly when $U_2 \subset K_3$. Under this condition, U_2 is uniquely determined by ω_1 and ω_2, and α cuts out a projective plane in $\mathbb{P}^3$. We conclude that π_2 is a birational morphism that contracts a plane Π to a singular point v of the image.

Let us give a description of this image. Over $\mathbb{P}(K_3) \times \mathbb{P}(V_7/K_3)$, the conditions above cut out a linear space whose dimension is governed by the rank of the induced map

$$U_1^{\oplus 2} \oplus U_1 \otimes U_4/K_3 \longrightarrow (U_4/U_1)^\vee.$$

The projection from $\widetilde{X}$ to $\mathbb{P}(K_3) \times \mathbb{P}(V_7/K_3)$ has non-trivial fibers when this morphism degenerates, and this condition defines a determinantal hypersurface H of bidegree $(2, 2)$, birationally isomorphic to X. Moreover there exist positive dimensional fibers, parametrized by the curve where the morphism degenerates further, and the morphism from $\widetilde{X}$ to H is a small resolution of singularities. Note also that the projection $\widetilde{X} \to X$ has positive dimensional fibers over the locus of pairs (U_2, U_4) over which the conditions on U_1 are empty, which exactly means that $U_2 \subset K_3$; so $\widetilde{X}$ is the blowup of X along the projective plane Π.

Let us now turn to π_1. When U_4 is fixed, $U_2 \subset U_4$ has to verify three linear conditions, which defines a conic bundle over $\mathbb{P}^3$. More precisely, ω_1, ω_2 and α induce over $\mathbb{P}^3$ a morphism

$$\mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow \bigwedge^2 \left(K_3^\vee \oplus \mathcal{O}_{\mathbb{P}^3}(-1)\right)^\vee,$$

which degenerates in codimension $6-3+1=4$, hence nowhere. The orthogonal to the image is then a rank 3 vector bundle $\mathcal{E} \subset \bigwedge^2 \left(K_3^\vee \oplus \mathcal{O}_{\mathbb{P}^3}(-1)\right)$, with a natural map $S^2\mathcal{E} \to \mathcal{K} = \mathcal{O}_{\mathbb{P}^3}(-1)$ that defines our conic bundle inside $\mathbb{P}(\mathcal{E})$.

Lemma 7.1. — *The discriminant Δ is a quintic surface with 16 double points.*

Proof. — The discriminant is given by the induced map $(\det \mathcal{E})^2 \to \mathcal{O}_{\mathbb{P}^3}(-3)$. Since $\det \mathcal{E} = \mathcal{O}_{\mathbb{P}^3}(-4)$, we get a quintic surface in $\mathbb{P}^3$, with a number of double points given by Lemma 3.5. In order to get it, we simply compute

$$c(\mathcal{E}) = \frac{(1+h)^3}{1-h} = 1 + 4h + 7h^2 + 8h^3.$$

Since $k = c_1(\mathcal{K}) = -1$, we finally get $[\Delta_{\mathrm{sing}}] = 16h^3$. □

We recall that the link between Fano R–62 and Fano C–4, see Remark 4.1, is investigated in the recent [13].

Remark 7.2. — We point out the probable connection with the recent work by Huybrechts [32], where a link between 16-nodal quintic surfaces and cubic fourfolds is made explicit. In particular, it is explained how a double cover of a 16-nodal quintic is a surface of general type F, whose anti-invariant part $H^2(F,\mathbb{Z})^-$ of the cohomology is a K3 structure of rank 23, that in turn could explain the K3 structure in our Fano fourfold.

Remark 7.3. — At the level of rationality, there is an interesting link between this Fano variety and the recent work [30] by Hassett, Pirutka and Tschinkel. In fact, they showed how a specific family of smooth complete intersections of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ has the very general fiber which is not stably rational, whereas some special fibers are rational. In our case, we have seen how our Fano variety is birational to a *singular* complete intersection of the same type. It would be very interesting to have an analogy with [30], and produce some special subfamilies of this family of Fano, for which not all the fibers are rational. This would be a counterexample to the generalization of the Kuznetsov conjecture to all FK3 fourfolds. However, we have not been able to understand this yet.

Fano R–63

(#3-31-120-2-B) $\mathscr{X}\big(\mathbb{P}^5 \times \mathbb{P}^7, \mathcal{Q}_{\mathbb{P}^5}(0,1) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(2,0)^{\oplus 2}\big)$

- Invariants:

$$h^0(-K) = 31,\ (-K)^4 = 120,\ h^{1,1} = 3,\ h^{3,1} = 1,\ h^{2,1} = 2,\ h^{2,2} = 22,\ -\chi(T) = 20.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^5$ is a conic bundle over Z, the complete intersection of two quadrics in $\mathbb{P}^5$, with discriminant locus $\Delta = S_8$ a smooth K3 surface of genus 5.
  - $\pi_2 \colon X \to \mathbb{P}^7$ is the blowup $\mathrm{Bl}_{p,q} Y^\circ$, where $Y^\circ \subset \mathbb{P}^7$ is the singular complete intersection of three quadrics, $\mathbb{Q}_1, \mathbb{Q}_2^\circ, \mathbb{Q}_3^\circ$, where $\mathbb{Q}_1$ is of maximal rank, and $\mathbb{Q}_2^\circ$ and $\mathbb{Q}_3^\circ$ are double cones over quadrics in $\mathbb{P}^5$. In particular, $\mathrm{Sing}(\mathbb{Q}_2^\circ) = \mathrm{Sing}(\mathbb{Q}_3^\circ) \cong \mathbb{P}^1$, and $\mathrm{Sing}(Y^\circ) = \mathbb{Q}_1 \cap \mathbb{P}^1$ is made of the two points p and q.
  - Rationality: yes.
- Semiorthogonal decompositions:
  - 4 exceptional objects, 2 genus 2 curves and $\mathrm{D^b}(S_8)$ from π_1, see Appendix B.2.1.

Explanation

Here we have a quadric $\mathbb{Q}^6$ in $\mathbb{P}^7$, the intersection Z of two quadrics in $\mathbb{P}^5$, and X is defined in $Z \times \mathbb{Q}^6$ by the vanishing of a section of $\mathcal{Q}_5(0,1)$. This section is defined by a morphism $\theta \in \mathrm{Hom}(V_8, V_6)$, with a two-dimensional kernel K_2, and it vanishes at (l,m) if $m \subset \theta^{-1}(l)$. Note that if K_2 is the kernel of θ that we identify with the projection from $V_8 = K_2 \oplus V_6$ to V_6, then $\theta^{-1}(l) = K_2 \oplus l$. We get the conic bundle structure on X by restricting $\mathbb{Q}^6$ to the planes $\mathbb{P}(K_2 \oplus l)$, for $l \in Z$. The discriminant locus Δ is given by a section of $\det\big(\theta^{-1}(l)\big)^{-2} \simeq l^{-2}$, which means that Δ is a quadric section of Z, that is, a K3 surface of genus 5. Note that contrary to the general situation the discriminant is smooth: if the restriction of $\mathcal{Q}$ to K_2 is non-degenerate, the rank does never drop to 1 on $K_2 \oplus l$. The above description lets us construct a section (actually, two) of the conic bundle π_1 : just note that $\mathbb{P}(K_2)$ is a trivial $\mathbb{P}^1$-bundle over Z, and we are taking a quadratic section. This gives the required semiorthogonal decomposition.

This variety appears also in [64], where it is labeled as $X_4^{1,2}$. The given description realizes X as a conic bundle over Z factoring through the blowup of $\mathbb{P}_Z\big(\mathcal{O}_Z \oplus \mathcal{O}_Z(-1)\big)$ with Δ as center.

What is the image of X in $\mathbb{Q}^6$? It consists of the lines m such that $\theta(m)$, when non-zero, is contained in Z. This condition defines in $\mathbb{P}^7$ an intersection Z' of two quadrics, singular along the line $\mathbb{P}(K_2)$. The intersection $Z' \cap \mathbb{Q}^6$ has two singular points, and the projection from X to $Z' \cap \mathbb{Q}^6$ is a birational morphism that contracts exactly two copies of Z to these two singular points.

Fano R–64

(#3-25-90-2) $\mathscr{Z}\big(\mathbb{P}^4 \times \mathbb{P}^6, \mathcal{Q}_{\mathbb{P}^4}(0,1) \oplus \mathcal{O}(3,0) \oplus \mathcal{O}(0,2)\big)$

- Invariants:

$$h^0(-K) = 25,\ (-K)^4 = 90,\ h^{1,1} = 3,\ h^{2,1} = 5,\ h^{3,1} = 1,\ h^{2,2} = 22,\ -\chi(T) = 23.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^4$ is a conic bundle over W_3, with discriminant a K3 surface of degree 6.
  - $\pi_2 \colon X \to \mathbb{P}^6$ contracts two copies of W_3 to the two singular points of $Y^\circ = \mathcal{CC}(W_3) \cap \mathbb{Q}^5$, with $\mathcal{CC}(W_3)$ the double projective cone over a cubic threefold W_3.
  - Rationality: unknown.
- Semiorthogonal decompositions:
  - 4 exceptional objects, 2 [5/3]-CY categories and $\mathrm{D^b}(S_6)$ from π_1.

Explanation

Our fourfold X parametrizes the pairs (x, y), for $x \in W_3 \subset \mathbb{P}^4$ and $y \in \mathbb{Q}^5 \subset \mathbb{P}^6$, such that $\alpha(y) \subset x$ for some given morphism $\alpha \in \mathrm{Hom}(V_7, V_5)$. Denote by K_2 the kernel of α.

We first describe π_1. For x given, y must be contained in $\alpha^{-1}(x) \simeq K_2 \oplus y$, which gives a projective plane on which $\mathbb{Q}^5$ cuts a conic. So π_1 is a conic bundle over W_3, whose discriminant is defined by a section of $\mathcal{O}(2)$, which gives a K3 surface of degree 6. (Note that the discriminant is smooth, the quadratic form being non-degenerate on K_2). Moreover, $\mathbb{P}(K_2)$ is a trivial $\mathbb{P}^1$-bundle over W_3, hence the fibration has two sections given by the intersection of the fibers with this constant line. The above description of the semiorthogonal decomposition of $\mathrm{D^b}(X)$ follows (see Appendix B.2.1).

Now we turn to π_2. Observe that the cubic condition on x implies that the image of π_2 is $\mathcal{CC}(W_3) \cap \mathbb{Q}^5$, which is singular at the two points of $\mathbb{P}(K_2) \cap \mathbb{Q}^5$. For y given, x is uniquely defined by the condition $\alpha(y) \subset x$, except when $\alpha(y) = 0$, in which case the fiber is the whole W_3.

This variety appears also in [64], where it is labeled as $X_3^{1,2}$. The given description realizes X as a conic bundle over W_3 factoring through the blowup of $\mathbb{P}_{W_3}\big(\mathcal{O}_Z \oplus \mathcal{O}_Z(-1)\big)$ with Δ, the K3 surface of degree 6 above, as center.

Appendix A. Some blowups of FK3

In this appendix we collect a few families of Fano fourfolds which are not FK3 themselves, but whose Hodge structure in middle dimension contains a proper K3 substructure. A typical example is the blowup of a cubic fourfold in a K3 surface.

Fano A–65

(#2-12-27-2) $\mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^5, \mathcal{O}(0,3) \oplus \mathcal{O}(1,2)\big)$

- Invariants:

$$h^0(-K) = 12,\ (-K)^4 = 27,\ h^{1,1} = 2,\ h^{3,1} = 7,\ h^{2,2} = 79,\ -\chi(T) = 58.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^1$ is a fibration in W_6, the latter being a $(2,3)$ complete intersection in $\mathbb{P}^5$.
  - $\pi_2 \colon X \to \mathbb{P}^5$ is the blowup $\mathrm{Bl}_{S_{12}} X_3$ of a cubic fourfold, where S_{12} is a degree 12 canonical surface given by a $(2,2,3)$ complete intersection.
  - Rationality: unknown. Same as the general cubic fourfold.
- Semiorthogonal decompositions:
  - 2 exceptional objects and an unknown category from π_1.
  - 3 exceptional objects, the K3 category of X_3 ans $\mathrm{D^b}(S_{12})$ from π_2.

Explanation

It suffices to apply Lemma 3.1.

Fano A–66

(#2-16-45-2) $\mathscr{Z}\big(\mathbb{P}^2 \times \mathbb{P}^5, \mathcal{O}(0,3) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(1,1)\big)$

- Invariants:

$$h^0(-K) = 16,\ (-K)^4 = 45,\ h^{1,1} = 2,\ h^{3,1} = 3,\ h^{2,2} = 51,\ -\chi(T) = 44.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^2$ is a fibration in cubic surfaces.
  - $\pi_2 \colon X \to \mathbb{P}^5$ is a blowup $\mathrm{Bl}_{T_9} X_3$ of a cubic fourfold along an elliptic surface of degree 9 and Kodaira dimension 1.
  - Rationality: unknown, the cubic fourfold being general.
- Semiorthogonal decompositions:
  - 3 exceptional objects and an unknown category from π_1.
  - 3 exceptional objects, the K3 category of X_3 and $\mathrm{D^b}(T_9)$ from π_2.

Explanation

We have two sections of $\mathcal{O}(1,1)$ over $\mathbb{P}^2 \times X_3$, where X_3 is a cubic fourfold. The projection $\pi \colon X \to X_3$ has fibers given by two linear conditions, hence points in general. So π is birational and its base locus is the degeneracy locus of the induced morphism $\mathcal{O}_{X_3}(-1)^2 \to V_3^\vee \otimes \mathcal{O}_{X_3}$. This is a smooth surface S, with a morphism to $\mathbb{P}^1$ defined by the kernel of the morphism. In fact the degeneracy locus of the corresponding morphism $\mathcal{O}_{\mathbb{P}^5}(-1)^2 \to V_3^\vee \otimes \mathcal{O}_{\mathbb{P}^5}$ is a copy of $\mathbb{P}^1 \times \mathbb{P}^2$, so S is the intersection of the latter with X_3. The projection to $\mathbb{P}^1$ shows that S is an elliptic surface. Its canonical bundle is $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1,0)$, which explains that $h^2(\mathcal{O}_S) = 2$, and since it is globally generated, we must have $\kappa(S) = 1$.

Fano A–67

(#2-17-51-2) $\mathscr{Z}\big(\mathbb{P}^2 \times \mathbb{P}^5, \mathcal{O}(1,1) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,2)\big)$

- Invariants:

$$h^0(-K) = 17,\ (-K)^4 = 51,\ h^{1,1} = 2,\ h^{3,1} = 2,\ h^{2,2} = 41,\ -\chi(T) = 36.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^2$ is a fibration in dP_4.
  - $\pi_2 \colon X \to \mathbb{P}^5$ is a blowup $\mathrm{Bl}_{S_8} X_3$ with S_8 an octic K3 surface.

- Rationality: depends on the cubic fourfold. A cubic fourfold contains an S_8 if and only if it contains a plane, see Appendix B.1.2.

- Semiorthogonal decompositions:
  - 3 exceptional objects and $\mathrm{D}^{\mathrm{b}}(Z, \mathcal{C}_0)$, for $Z \to \mathbb{P}^2$ a $\mathbb{P}^1$-bundle, from π_1.
  - 3 exceptional objects, $\mathrm{D}^{\mathrm{b}}(S_2, \alpha)$ and $\mathrm{D}^{\mathrm{b}}(S_8)$ from π_2.

Explanation

The description of π_1 is obvious.

In order to describe π_2 we first apply Lemma 3.1 and identify $\mathscr{Z}\big(\mathbb{P}^2 \times \mathbb{P}^5, \mathcal{Q}_{\mathbb{P}^2}(0,2)\big)$ with the blowup of $\mathbb{P}^5$ with center the intersection S_8 of three quadrics. This is a Fano fivefold Y of index $\iota_Y = 2$. By definition our Fano fourfold X is a half-anticanonical section of Y. Via the blowup map the anticanonical class is identified as $6H - 2S_8$; therefore a $(1,1)$ section on Y cuts a (Hodge-special) cubic fourfold containing S_8, see also [51, 4.11], and the result follows.

Remark A.1. — Observe that the *double K3 structure* is in line with the prediction of [24, 2.1]. In fact Y is an index 2 Fano fivefold with $H^5(Y) = 0$, but with $H^4(Y)$ having a K3 structure coming $H^2(S_8)$, which is inherited by Z by the Lefschetz Theorem on hyperplane sections.

Fano A–68

(#2-26-93-2) $\mathscr{Z}\big(\mathbb{P}^5 \times \mathbb{P}^6, \mathcal{O}(1,1) \oplus \mathcal{Q}_{\mathbb{P}^5}(0,1) \oplus \mathcal{O}(3,0)\big)$

- Invariants:

$$h^0(-K) = 26,\ (-K)^4 = 93,\ h^{1,1} = 2,\ h^{3,1} = 2,\ h^{2,2} = 41,\ -\chi(T) = 39.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^5$ is a blowup $\mathrm{Bl}_{S_6} X_3$, where S_6 is a sextic K3 surface.
  - $\pi_2 \colon X \to \mathbb{P}^6$ is birational onto a singular fourfold, the intersection $C(X_3) \cap \mathbb{Q}^5$ of the cone over X_3 with a smooth quadric passing through; moreover it contracts a cubic threefold to the (singular) vertex.
  - Rationality: unknown. Same as the general cubic fourfold, which always contains a sextic K3 as an intersection with a quadric and a hyperplane.
- Semiorthogonal decompositions:
  - 3 exceptional objects, the K3 category of X_3 and $\mathrm{D}^{\mathrm{b}}(S_6)$ from π_1.

Explanation

The zero locus $\mathscr{Z}\big(\mathbb{P}^5 \times \mathbb{P}^6, \mathcal{Q}_{\mathbb{P}^5}(0,1)\big)$ is the blowup $\mathrm{Bl}_v \mathbb{P}^6$ of $\mathbb{P}^6$ at one point. This is enough to explain both projections. In fact if we consider the projection to $\mathbb{P}^6$ we have to consider the intersection between a general cubic equation in six variables (hence, a projective cone over a smooth cubic fourfold) and a smooth quadric, both passing through v.

On the other hand if we consider the projection to $\mathbb{P}^5$ this fourfold becomes $\mathscr{Z}\big(\mathbb{P}_{X_3}\big(\mathcal{O} \oplus \mathcal{O}(-1)\big), \mathcal{L} \boxtimes \mathcal{O}_{X_3}(1)\big)$. We conclude using the Cayley trick, with the blown up locus $\mathscr{Z}\big(X_3, \mathcal{O}(2) \oplus \mathcal{O}(1)\big)$ being a K3 of degree 6.

Fano A–69

(#2-17-50-2) $\mathscr{Z}\big(\mathbb{P}^1 \times \mathrm{Gr}(2,5), \mathcal{O}(0,1) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(1,1)\big)$

- Invariants:

$$h^0(-K) = 17,\ (-K)^4 = 50,\ h^{1,1} = 2,\ h^{3,1} = 2,\ h^{2,2} = 42,\ -\chi(T) = 38.$$

- Description:
  - $\pi_1 \colon X \to \mathbb{P}^1$ is a fibration in Gushel–Mukai threefolds.
  - $\pi_2 \colon X \to \mathrm{Gr}(2,5)$ is a blowup $\mathrm{Bl}_{S_{10}} X_{10}$ of a degree 10 K3 surface in a Gushel–Mukai fourfold.
  - Rationality: unknown. Same as the general Gushel–Mukai fourfold, since it contains degree 10 K3 surfaces as double hyperplane sections.
- Semiorthogonal decompositions:
  - 4 exceptional objects and an unknown category from π_1.
  - 4 exceptional objects, the K3 category of X_{10} and $\mathrm{D^b}(S_{10})$ from π_2.

Explanation

Simple application of Proposition 3.3.

Appendix B. Some semiorthogonal decompositions

In this appendix we explain how to obtain semiorthogonal decompositions of our examples by describing those of Fano varieties that are involved in the geometric constructions. In particular, we always used maps

$\pi\colon X \to M$ which are either smooth blowups or Mori fiber spaces. In the first case, we appeal to the blowup formula for derived categories [60]. The case of Mori fiber spaces such as projective bundles and quadric, conic, del Pezzo of degree 4 and 5 fibrations is detailed in this appendix, and resumed in Table B.2. Other cases are unknown.

We also summarize in Table B.1 semiorthogonal decompositions of all possible M and centers of blowups appearing in the examples, giving details in the case of special cubic and Gushel–Mukai fourfolds.

B.1. The case of cubic and GM fourfolds

If X is either a cubic fourfold or a Gushel–Mukai fourfold, then $\mathrm{D}^{\mathrm{b}}(X)$ is generated by a natural exceptional collection of length 3 (cubic) or 4 (GM) whose complementary Kuznetsov component is a K3 category, which is in general not equivalent to the derived category of a (twisted) K3 surface. However, this can happen for special such fourfolds containing specific surfaces, hence lying on some particular Noether–Lefschetz divisor. Here is a list of cases of interest for this paper, but more are known.

B.1.1. $\mathcal{C}_6$: Nodal cubics

See [49] for references. If $X \subset \mathbb{P}^5$ is a cubic fourfold with a single double point, then the (categorical resolution) of its Kuznetsov component is $\mathrm{D}^{\mathrm{b}}(S_6)$, where the degree 6 K3 surface S_6 is described as follows. Projecting from the node, one gets a birational map $X \dashrightarrow \mathbb{P}^4$, and blowing up the node this gives a birational morphism $\widetilde{X} \to \mathbb{P}^4$ which can be shown to be the blowup of $\mathbb{P}^4$ along S_6 ($\widetilde{X}$ is the Fano fourfold Fano C–6).

B.1.2. $\mathcal{C}_8$: Cubics containing a plane

If X is a smooth cubic hypersurface in $\mathbb{P}^5$ containing a plane, then its Kuznetsov component is $\mathrm{D}^{\mathrm{b}}(S_2, \alpha)$ [49], where the degree 2 K3 surface S_2 and the Brauer class α are described as follows. Projecting from the plane, one gets a birational map $X \dashrightarrow \mathbb{P}^2$, and blowing up the plane gives a quadric surface fibration $\widetilde{X} \to \mathbb{P}^2$ degenerating along a smooth sextic ($\widetilde{X}$ is the Fano fourfold Fano C–3). Note that if X is general in this divisor, its Kuznetsov component cannot be equivalent to the derived category of a K3 surface.

We notice that any cubic fourfold containing a plane has equation of the form $\sum l_i q_i$, where l_i are linear and q_i are quadratic polynomials for $i = 1, 2, 3$, the plane being given by the vanishing of the l_i's. In particular such a cubic fourfold also contains quadric surfaces (where one quadratic and two linear forms vanish), del Pezzo surfaces dP_4 (where one linear and two quadratic forms vanish), and an octic K3 surface S_8 (where the three quadratic forms vanish).

B.1.3. $\mathcal{C}_{14}$: Pfaffian cubics

If $X \subset \mathbb{P}^5$ is Pfaffian, then its Kuznetsov component is $\mathrm{D^b}(S_{14})$, where the degree 14 K3 surface S_{14} is the HP dual of X [49]. Note that cubic fourfolds in $\mathcal{C}_{14}$ contain quintic del Pezzo surfaces and rational quartic scrolls.

B.1.4. GM fourfolds containing a del Pezzo quintic

If $X \subset \mathrm{Gr}(2,5)$ is a GM fourfold containing a dP_5, then its Kuznetsov component is $\mathrm{D^b}(S_{10})$ [52], and X is rational. The surface S_{10} is also a GM variety.

Note moreover that any such X contains a σ-plane (i.e. a plane in $\mathrm{Gr}(2,5)$ whose Schubert class is $\sigma_{3,1}$), but the converse is true only in general [22, Section 7.5]. A general GM fourfold containing such a plane is rational.

B.1.5. GM fourfolds containing a τ-quadric

If $X \subset \mathrm{Gr}(2,5)$ is a GM fourfold containing a τ-quadric (i.e. a quadratic surface which is a linear section of some $\mathrm{Gr}(2,4)$) then X is rational and has degree 10 discriminant, but the Kuznetsov component of X has not been proven yet to be equivalent to some $\mathrm{D^b}(S_{10})$.

Table B.1. Semiorthogonal decompositions for (Fano) of dimension at most 4 that appear in the examples.

$\dim(X)$	X	Description of $\mathrm{D}^{\mathrm{b}}(X)$
any	$\mathbb{P}^n$	$n+1$ exceptional objects [7]
	$\mathrm{Gr}(k,n)$	$\binom{n}{k}$ exceptional objects [40]
odd	$\mathbb{Q}^n$	$n+1$ exceptional objects [40]
even		$n+2$ exceptional objects [40]
2	rational surface	$\rho(X)+2$ exceptional objects
3	cubic in $\mathbb{P}^4$	2 exceptional objects and a [5/3]-CY category [45]
	int. of 2 quadrics in $\mathbb{P}^5$	2 exceptional objects and a genus 2 curve [16]
	X_5^3	4 exceptional objects [59]
	$(1,1)$ divisor in $\mathbb{P}^2 \times \mathbb{P}^2$	6 exceptional objects (use HPD [12])
4	cubic in $\mathbb{P}^5$	3 exceptional objects and a K3 category [49]
	GM fourfold in $\mathrm{Gr}(2,5)$	4 exceptional objects and a K3 category [52]
	int. of 2 quadrics in $\mathbb{P}^6$	12 exceptional objects (see, e.g., [15, Section 6])
	del Pezzo fourfold of degree 5	6 exceptional objects (use HPD for $\mathrm{Gr}(2,5)$ [50])
	deg 12 index 2	16 exceptional objects [47, Section 6.2]
	deg 14 index 2	12 exceptional objects [46, Cor.10.1]
	deg 16 index 2	8 exceptional objects (use HPD [47])
	deg 18 index 2	6 exceptional objects (use HPD [47])
	codim 2 linear sect. of $\mathbb{P}^3 \times \mathbb{P}^3$	12 exceptional objects (use HPD [12])
	$(1,1)$ divisor in $\mathbb{P}^2 \times \mathbb{P}^3$	9 exceptional objects (use HPD [12])
	$(1,2)$ divisor in $\mathbb{P}^2 \times \mathbb{P}^3$	8 exceptional objects and an unknown category (not trivial!)
	$(1,1)$ divisor in $\mathbb{P}^2 \times \mathbb{Q}^3$	10 exceptional objects, see Fano K3–49
	$\mathbb{P}^2$-bundle over $\mathbb{P}^2$	9 exceptional objects [60]
	$\mathbb{P}^1$-bundle over a $\mathbb{Q}^3$	8 exceptional objects [60]
	$\mathbb{P}^1$-bundle over a cubic threefold	4 exceptional objects and 2 [5/3]-CY categories [60]

Table B.2. Semiorthogonal decompositions for fourfolds with a Mori fiber space structure. In the case marked with * *the category of a genus 2 curve comes from the base, so one should not consider it as part of the Kuznetsov component.*

Fiber of $X \to M$	M	Exceptional part	Kuznetsov component
conic	$\mathbb{P}^3$	4 exceptional objects	$\mathrm{D^b}(M, \mathcal{C}_0)$, see Appendix B.2.1
	$\mathbb{P}^1 \times \mathbb{P}^2$	6 exceptional objects	
	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	8 exceptional objects	
	$\mathrm{Bl}_p\, \mathbb{P}^3$	6 exceptional objects	
	$\mathbb{Q}^3$	4 exceptional objects	
	$(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$	6 exceptional objects	
	2 quadrics in $\mathbb{P}^5$	2 exc. objects, a genus 2 curve*	
quadric surface	$\mathbb{P}^2$ with sextic discr.	6 exceptional objects	twisted K3 of degree 2
	$\mathbb{P}^1 \times \mathbb{P}^1$ with $(4,4)$ discr.	8 exceptional objects	twisted K3 of degree 8
dP_5	$\mathbb{P}^2$	6 exceptional objects	$\mathrm{D^b}(Z)$ for $Z \to M$ of degree 5, see Appendix B.3
	$\mathbb{P}^1 \times \mathbb{P}^1$	8 exceptional objects	
dP_4	$\mathbb{P}^2$	3 exceptional objects	$\mathrm{D^b}(Z, \mathcal{C}_0)$ for $Z \to M$ a $\mathbb{P}^1$-bundle, see Appendix B.2.3
	$\mathbb{P}^1 \times \mathbb{P}^1$	4 exceptional objects	
dP_3	$\mathbb{P}^2$	3 exceptional objects	unknown
	$\mathbb{P}^1 \times \mathbb{P}^1$	4 exceptional objects	
int. of 2 quadrics	$\mathbb{P}^1$	4 exceptional objects	$\mathrm{D^b}(S, \mathcal{C}_0)$ for S a Hirz. surface, see Appendix B.2.3
Cubic threefolds			unknown
GM threefolds			
X^3_{12}			
X^3_{14}			
X^3_{18}			
X^3_{16}			
$2,3$ c. int. in $\mathbb{P}^5$		2 exceptional objects	

B.2. Quadric fibrations

Let $\pi\colon X \to M$ be a quadric bundle of relative dimension n. This means that there exists a rank $n+2$ vector bundle $\mathcal{E}$ and a line bundle $\mathcal{K}$ on M such that X is the zero locus of a quadratic form $q\colon \mathcal{K}^\vee \to S^2\mathcal{E}^\vee$. We assume that q is general so that the generic fiber of π is a smooth quadric. The locus $\Delta \subset M$ where the fiber is singular of corank 1 is a divisor, called the *discriminant divisor*, which is in general singular in codimension 2, and whose singularities correspond to corank 2 fibers.

In this case, one can associate to q the sheaf $\mathcal{C}_0$ of the even parts of the Clifford algebra, and show that the Lefschetz part of $\mathrm{D}^\mathrm{b}(X)$ is given by n copies of $\mathrm{D}^\mathrm{b}(M)$, while the relative Kuznetsov component can be described as $\mathrm{D}^\mathrm{b}(M, \mathcal{C}_0)$, the derived category of complexes of $\mathcal{C}_0$-algebras on M [48].

B.2.1. The case of conic bundles with a section

In the case of conic bundles, there is a structure of $\mathbb{Z}/2\mathbb{Z}$-root stack $\widehat{M}$ on M, and $\mathcal{C}_0$ lifts to an algebra $\mathcal{B}_0$ on $\widehat{M}$ such that $\mathrm{D}^\mathrm{b}(M, \mathcal{C}_0) \simeq \mathrm{D}^\mathrm{b}(\widehat{M}, \mathcal{B}_0)$. The algebra $\mathcal{B}_0$ is Azumaya outside the singularities of the discriminant divisor [48] and its triviality is equivalent to $\pi\colon X \to M$ having a section (this is in turn equivalent to X being $k(M)$-rational). The dg category of perfect complexes on the root stack $\widehat{M}$ admits a semiorthogonal decomposition [11, 36]:

$$\mathrm{Perf}(\widehat{M}) = \big\langle \mathrm{D}^\mathrm{b}(M), \mathrm{Perf}(\Delta) \big\rangle,$$

and the category of perfect complexes $\mathrm{Perf}(\Delta)$ is equivalent to $\mathrm{D}^\mathrm{b}(\Delta)$ if and only if Δ is smooth. It follows, that in the case where Δ is smooth and $\mathcal{B}_0$ is trivial, we have a semiorthogonal decomposition of $\mathrm{D}^\mathrm{b}(X)$ by two copies of $\mathrm{D}^\mathrm{b}(M)$ and one copy of $\mathrm{D}^\mathrm{b}(\Delta)$. This explains the K3 structure in examples Fano R–63 and Fano R–64. In the case where Δ is not smooth one can wonder whether the copy of $\mathrm{D}^\mathrm{b}(\Delta)$ can be replaced by a categorical resolution of singularities. We can ask the following question.

Question B.1. — *Let X be a fourfold and $\pi\colon X \to M$ be a conic bundle with discriminant divisor Δ with finitely many singular points. Assume that $\pi\colon X \to M$ has a section. Is there a categorical resolution of singularities $\mathcal{D}$ of Δ such that*

$$\mathrm{D}^\mathrm{b}(X) = \big\langle \mathcal{D}, \mathrm{D}^\mathrm{b}(M), \mathrm{D}^\mathrm{b}(M) \big\rangle$$

is a semiorthogonal decomposition?

Particularly relevant to our study is the case where Δ is resolved by a K3, for example, when Δ is a singular quartic in $\mathbb{P}^3$.

B.2.2. The case of quadric surface bundles

In the case where $\pi\colon X \to M$ is a quadric surface bundle, the situation is much simpler. Indeed, there is a degree 2 map $T \to M$ ramified along Δ such that $\mathcal{C}_0$ lifts to an algebra $\mathcal{B}_0$ which is Azumaya outside the singularities of Δ. In particular, if Δ is smooth (and this is in general the case when X is a fourfold), then $\mathrm{D}^{\mathrm{b}}(M, \mathcal{C}_0) \simeq \mathrm{D}^{\mathrm{b}}(T, \mathcal{B}_0)$ is the derived category of twisted sheaves on the variety T.

In the examples considered in this paper, there are two families of fourfolds X with a quadric surface bundle $\pi\colon X \to M$ and they both give rise to twisted K3 surfaces:

(1) M is $\mathbb{P}^2$ and Δ is a smooth sextic, so that T is a degree 2 K3 surface;
(2) M is $\mathbb{P}^1 \times \mathbb{P}^1$ and Δ is a $(4,4)$ divisor, so that T is a degree 8 K3 surface of Picard rank 2.

B.2.3. The case of fibrations in intersections of quadrics

Let $\pi\colon X \to M$ be a Mori fiber space whose general fiber is the intersection of two quadrics. As explained in [3], one can construct a $\mathbb{P}^1$-bundle $Z \to M$ and a quadric fibration on $Y \to Z$ with associated sheaf $\mathcal{C}_0$ of even parts of Clifford algebras, such that the Kuznetsov component of the fibration is equivalent to $\mathrm{D}^{\mathrm{b}}(Z, \mathcal{C}_0)$. The considerations on conic and quadric surface bundles may then be carried over these cases as well.

B.3. dP_5 fibrations

If $X \to M$ is a dP_5 fibration, the Kuznetsov component is equivalent to $\mathrm{D}^{\mathrm{b}}(Z)$ for $Z \to M$ a flat degree 5 cover [68]. In our cases, M is either $\mathbb{P}^2$ or a quadric, and the question of Z being (related to) a K3 surface remains open. Note that if X is relatively minimal over M, then Z cannot be disconnected (this would increase the Picard rank over $k(M)$, see [2]).

Appendix C. Tables

In the tables below we follow the notation of the paper, together with some additions. In particular, as usual X_3 denotes a cubic fourfold and X_{10} a Gushel–Mukai fourfold. More generally, X_d denotes a Fano fourfold of index 2 and degree d and Y_d denotes a Fano fourfold of index 3 and degree d. If needed, the superscript Y_d^n will denote Fano manifolds obtained as hyperplane sections of Y_d of dimension n. Recall that W_3 denotes a cubic threefold. By $D \subset \mathbb{P}^n \times \mathbb{P}^m$ (resp. D^c) we denote a divisor (resp. the complete intersection of c divisors), whose bidegree is specified in the detailed description of each Fano fourfold. Also Z° denotes a singular fourfold, whose description is expanded in the appropriate section.

In terms of invariants, $h^{3,1}$ is always 1 for fourfolds of K3 type, so it is not listed. Under the column "Bl" are listed the varieties Y from our lists such that $X \cong \mathrm{Bl}_S Y$, where X is the FK3 in question. In the last column, "+" means that every variety in the deformation class is rational, "*" that some special member are rational, "?" means that rationality properties are not known.

Table C.1. FK3 with $\rho = 1$.

ID	ρ	$h^{2,2}$	$h^{1,2}$	$h^0(-K)$	K^4	$-\chi(T)$	G	$\mathcal{F}$	Bl	Rat
X_3	1	21	0	55	243	20	$\mathbb{P}^5$	$\mathcal{O}(3)$	-	*
X_{10}	1	22	0	39	160	24	$\mathrm{Gr}(2,5)$	$\mathcal{O}(1) \oplus \mathcal{O}(2)$	-	*
Z_{66}	1	24	0	20	66	25	$\mathrm{Gr}(3,7)$	$\bigwedge^2 \mathcal{U}^\vee \oplus \bigwedge^3 \mathcal{Q} \oplus \mathcal{O}(1)$	-	?

Table C.2. FK3 from the cubic fourfold.

ID	ρ	$h^{2,2}$	$h^{1,2}$	$h^0(-K)$	K^4	$-\chi(T)$	G	$\mathcal{F}$	Bl	Rat
Fano C–1	2	28	0	36	144	28	$\mathbb{P}^1 \times \mathbb{P}^5$	$\mathcal{O}(0,3) \oplus \mathcal{O}(1,1)$	X_3	*
Fano C–2	2	23	1	28	99	29	$\mathbb{P}^2 \times \mathbb{P}^5$	$\mathcal{Q}_{\mathbb{P}^2}(0,1) \oplus \mathcal{O}(0,3)$	X_3	*
Fano C–3	2	22	0	45	192	19	$\mathbb{P}^2 \times \mathbb{P}^5$	$\mathcal{O}(1,2) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,1)$	X_3	*
Fano C–4	2	23	0	40	163	24	$\mathbb{P}^3 \times \mathbb{P}^5$	$\mathcal{O}(1,2) \oplus \mathcal{Q}_{\mathbb{P}^3}(0,1)$	X_3	*
Fano C–5	2	23	0	39	161	21	$\mathbb{P}^5 \times \mathrm{Gr}(2,8)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,8)}(1,0) \oplus$ $\mathcal{O}(1,1) \oplus \mathcal{Q}_{\mathbb{P}^5} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,8)}$	X_3, Z°	?
Fano C–6	2	21	0	49	211	19	$\mathbb{P}^4 \times \mathbb{P}^5$	$\mathcal{Q}_{\mathbb{P}^4}(0,1) \oplus \mathcal{O}(2,1)$	$X_3^\circ, \mathbb{P}^4$	+
Fano C–7	2	26	0	24	86	24	$\mathbb{P}^4 \times \mathbb{P}^5$	$\mathcal{O}(1,1) \oplus \bigwedge^3 \mathcal{Q}_{\mathbb{P}^4}(0,1)$	$\mathbb{P}^4, X_3$	+
Fano C–8	2	27	0	30	115	26	$\mathrm{Fl}(1,3,8)$	$\mathcal{Q}_2^{\oplus 2} \oplus \mathcal{O}(1,1) \oplus \mathcal{R}_2^\vee(1,1)$	X_3, Z°	?
Fano C–9	2	23	0	27	101	21	$\mathbb{P}^5 \times \mathrm{Gr}(2,4)$	$(\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0))^{\oplus 2} \oplus \mathcal{O}(1,1)$	$\mathrm{Gr}(2,4), X_3$	+
Fano C–10	3	38	0	20	63	28	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^5$	$\mathcal{O}(0,0,3) \oplus$ $\mathcal{O}(0,1,1) \oplus \mathcal{O}(1,0,1)$	Fano C–1	*
Fano C–11	3	30	1	24	81	31	$\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^5$	$\mathcal{O}(0,0,3) \oplus$ $\mathcal{Q}_{\mathbb{P}^2}(0,0,1) \oplus \mathcal{O}(1,1,0)$	Fano C–2	*
Fano C–12	3	30	0	27	99	27	$\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^5$	$\mathcal{O}(0,1,2) \oplus$ $\mathcal{Q}_{\mathbb{P}^2}(0,0,1) \oplus \mathcal{O}(1,0,1)$	Fano C–3	*

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Table C.2 continued from previous page

Fano C–13	3	28	0	34	134	25	$\mathbb{P}^1 \times \mathrm{Fl}(1,2,6)$	$\mathcal{Q}_2 \oplus$ $\mathcal{O}(0;1,2) \oplus \mathcal{O}(1;0,1)$	Fano C–6, $\mathrm{Bl}_{\mathbb{P}^2}\,\mathbb{P}^4$	+
Fano C–14	3	30	0	30	113	28	$\mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^5$	$\mathcal{O}(0,1,2) \oplus$ $\mathcal{Q}_{\mathbb{P}^3}(0,0,1) \oplus \mathcal{O}(1,1,0)$	Fano C–4, Fano C–1	*
Fano C–15	3	23	0	35	141	18	$\mathbb{P}^2_1 \times \mathbb{P}^2_2 \times \mathbb{P}^5$	$\mathcal{Q}_{\mathbb{P}^2_1}(0,0,1) \oplus$ $\mathcal{Q}_{\mathbb{P}^2_2}(0,0,1) \oplus \mathcal{O}(1,1,1)$	Fano C–3, $\mathbb{P}^2 \times \mathbb{P}^2$	+
Fano C–16	3	23	0	42	176	18	$\mathbb{P}^2 \times \mathbb{P}^4 \times \mathbb{P}^5$	$\mathcal{Q}_{\mathbb{P}^4}(0,0,1) \oplus$ $\mathcal{O}(1,1,1) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,1,0)$	Fano C–6, $\mathrm{Bl}_{\mathbb{P}^1}\,\mathbb{P}^4$	+
Fano C–17	3	24	0	37	149	21	$\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^5$	$\mathcal{Q}_{\mathbb{P}^3}(0,0,1) \oplus$ $\mathcal{O}(1,0,2) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,1,0)$	Fano C–3, Fano C–4	?

Table C.3. FK3 from the GM fourfold.

ID	ρ	$h^{2,2}$	$h^{1,2}$	$h^0(-K)$	K^4	$-\chi(T)$	G	$\mathcal{F}$	Bl	Rat
Fano GM–18	2	30	0	22	74	30	$\mathbb{P}^2 \times \mathrm{Gr}(2,5)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0) \oplus$ $\mathcal{O}(0,1) \oplus \mathcal{O}(0,2)$	X_{10}	*
Fano GM–19	2	27	0	23	80	26	$\mathbb{P}^2 \times \mathrm{Gr}(2,5)$	$\mathcal{O}(0,1) \oplus$ $\mathcal{O}(1,1) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,1)$	X_{10}	+
Fano GM–20	2	28	0	26	94	28	$\mathrm{Fl}(1,2,5)$	$\mathcal{O}(1,0) \oplus$ $\mathcal{O}(0,1) \oplus \mathcal{O}(0,2)$	X_{10}	*
Fano GM–21	2	24	0	30	114	24	$\mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus$ $\mathcal{O}(0,2) \oplus \mathcal{O}(1,0)$	X_{10}	?
Fano GM–22	2	23	0	33	130	22	$\mathrm{Gr}(2,4) \times \mathrm{Gr}(2,5)$	$\mathcal{Q}_{\mathrm{Gr}(2,4)} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,5)} \oplus$ $\mathcal{O}(0,1) \oplus \mathcal{O}(1,1)$	X_{10}, $\mathrm{Gr}(2,4)$	+

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Fano GM–23	2	24	0	30	115	23	$\mathrm{Fl}(1,3,5)$	$\mathcal{O}(0,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{R}_2(0,1)$	X_{10}, $\mathbb{P}^4$	+

Table C.4. FK3 from the K3 surfaces.

ID	ρ	$h^{2,2}$	$h^{1,2}$	$h^0(-K)$	K^4	$-\chi(T)$	G	$\mathcal{F}$	Bl	Rat
Fano K3–24	2	32	0	19	60	31	$\mathbb{P}^1 \times \mathrm{Gr}(2,5)$	$\mathcal{O}(1,1) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(0,1)$	X_{12}	+
Fano K3–25	2	28	0	21	70	27	$\mathbb{P}^1 \times \mathrm{Gr}(2,6)$	$\mathcal{O}(1,1) \oplus \mathcal{O}(0,1)^{\oplus 4}$	X_{14}	+
Fano K3–26	2	24	0	23	80	23	$\mathbb{P}^1 \times \mathrm{Gr}(3,6)$	$\mathcal{O}(1,1) \oplus \mathcal{O}(0,1)^{\oplus 2} \oplus \bigwedge^2 \mathcal{U}^\vee_{\mathrm{Gr}(3,6)}$	X_{16}	?
Fano K3–27	2	28	0	32	124	28	$\mathrm{Fl}(1,2,8)$	$\mathcal{Q}_2 \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(0,2)^{\oplus 2}$	Y_4, Z°	+
Fano K3–28	2	22	0	25	90	21	$\mathbb{P}^1 \times \mathrm{Gr}(2,7)$	$\mathcal{O}(0,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{Q}^\vee_{\mathrm{Gr}(2,7)}(0,1)$	X_{18}	+
Fano K3–29	2	22	0	39	160	21	$\mathbb{P}^2 \times \mathbb{P}^4$	$\mathcal{O}(1,1) \oplus \mathcal{O}(1,2)$	$\mathbb{P}^4$	+
Fano K3–30	2	22	0	39	160	21	$\mathbb{P}^1 \times \mathrm{Gr}(2,4)$	$\mathcal{O}(1,2)$	$\mathrm{Gr}(2,4)$	+
Fano K3–31	2	28	0	23	80	27	$\mathbb{P}^3 \times \mathbb{P}^6$	$\mathcal{O}(0,2) \oplus \mathcal{O}(1,1) \oplus \bigwedge^2 \mathcal{Q}_{\mathbb{P}^3}(0,1)$	Y_4	+
Fano K3–32	2	22	0	29	110	21	$\mathrm{Fl}(1,2,5)$	$\mathcal{O}(1,1) \oplus \mathcal{O}(0,1)^{\oplus 2}$	$Y_5, \mathbb{P}^4$	+
Fano K3–33	2	24	0	27	100	23	$\mathbb{P}^5 \times \mathrm{Gr}(2,4)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(1,1) \oplus \mathcal{Q}_{\mathrm{Gr}(2,4)}(1,0)$	$\mathrm{Gr}(2,4)$	+
Fano K3–34	2	23	0	26	95	22	$\mathbb{P}^3 \times \mathrm{Gr}(2,5)$	$\mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,5)}(1,0) \oplus \mathcal{O}(1,1)$	Y_5	+
Fano K3–35	2	24	0	25	90	23	$(\mathrm{Gr}(2,4))^2$	$\mathcal{O}(0,1) \oplus \mathcal{U}^\vee_{\mathrm{Gr}(2,4)_2}(1,0) \oplus \mathcal{O}(1,1)$	$\mathrm{Gr}(2,4)$	+
Fano K3–36	3	22	5	19	60	23	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^4$	$\mathcal{O}(0,0,3) \oplus \mathcal{O}(1,1,1)$	$\mathbb{P}^1 \times W_3$	?

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Fano K3–37	3	33	2	23	80	20	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^5$	$\mathcal{O}(0,0,2)^{\oplus 2} \oplus \mathcal{O}(1,1,1)$	$\mathbb{P}^1 \times Y_4^3$	+
Fano K3–38	3	22	0	27	100	18	$(\mathbb{P}^1)^2 \times \mathrm{Gr}(2,5)$	$\mathcal{O}(0,0,1)^{\oplus 3} \oplus \mathcal{O}(1,1,1)$	$\mathbb{P}^1 \times Y_5^3$	+
Fano K3–39	3	24	0	29	110	20	$\mathbb{P}^1 \times \mathbb{P}^2 \times \mathrm{Gr}(2,4)$	$\mathcal{Q}_{\mathbb{P}^2}(0,0,1) \oplus$ $\mathcal{O}(1,1,1)$	Fano K3–30, $\mathrm{Bl}_C \mathrm{Gr}(2,4)$	+
Fano K3–40	3	32	0	22	74	29	$\mathbb{P}^1_1 \times \mathbb{P}^1_2 \times \mathbb{P}^5$	$\mathcal{O}(0,0,2) \oplus$ $\mathcal{O}(0,1,1) \oplus \mathcal{O}(1,0,2)$	Fano K3–30, $\mathrm{Bl}_{\mathbb{P}^1 \times \mathbb{P}^1} \mathrm{Gr}(2,4)$	+
Fano K3–41	3	22	0	39	160	18	$(\mathbb{P}^1)^2 \times \mathbb{P}^3$	$\mathcal{O}(1,1,2)$	$\mathbb{P}^1 \times \mathbb{P}^3$	+
Fano K3–42	3	27	0	28	104	24	$\mathbb{P}^1 \times \mathbb{P}^2 \times \mathrm{Gr}(2,4)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(0,1,0) \oplus$ $\mathcal{O}(1,0,2)$	$\mathrm{Bl}_{\mathbb{P}^2} \mathrm{Gr}(2,4)$, $\mathbb{P}_{\mathbb{P}^2}(\mathcal{Q}^\vee \oplus \mathcal{O}(-1))$	+
Fano K3–43	3	24	0	32	125	20	$(\mathbb{P}^2)^2 \times \mathbb{P}^4$	$\mathcal{O}(1,0,1) \oplus$ $\mathcal{O}(1,1,1) \oplus \mathcal{Q}_{\mathbb{P}^2_2}(0,0,1)$	Fano K3–29 $\mathrm{Bl}_{\mathbb{P}^1} \mathbb{P}^4$	+
Fano K3–44	3	30	0	25	89	27	$\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^4$	$\mathcal{O}(0,1,1) \oplus$ $\mathcal{O}(0,1,2) \oplus \mathcal{O}(1,0,1)$	Fano K3–29, $\mathrm{Bl}_{\mathbb{P}^2} \mathbb{P}^4$	+
Fano K3–45	3	24	0	31	119	21	$(\mathbb{P}^2)^2 \times \mathbb{P}^4$	$\mathcal{O}(0,1,2) \oplus$ $\mathcal{O}(1,1,0) \oplus \mathcal{Q}_{\mathbb{P}^2_1}(0,0,1)$	Fano K3–29, $\mathrm{Bl}_{\mathbb{P}^1} \mathbb{P}^4$	+
Fano K3–46	3	24	0	33	130	20	$\mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^5$	$\mathcal{Q}_{\mathbb{P}^3}(0,0,1) \oplus$ $\mathcal{O}(0,1,1) \oplus \mathcal{O}(1,1,1)$	Fano K3–30, $\mathbb{P}^1 \times \mathbb{P}^3$ $\mathrm{Bl}_{\mathbb{P}^1} \mathrm{Gr}(2,4)$	+
Fano K3–47	3	26	0	27	100	22	$\mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^3$	$\mathcal{O}(0,1,1)^{\oplus 2} \oplus$ $\mathcal{O}(1,1,1)$	$\mathbb{P}^1 \times \mathbb{P}^3$, $D^2 \subset \mathbb{P}^3 \times \mathbb{P}^3$	+
Fano K3–48	3	22	0	31	120	18	$\mathbb{P}^1 \times \mathrm{Fl}(1,2,4)$	$\mathcal{O}(0;0,1) \oplus \mathcal{O}(1;1,1)$	$\mathbb{P}^1 \times \mathbb{Q}_3, \mathbb{P}_{\mathbb{Q}_3}(\mathcal{U})$, $\mathbb{P}^1 \times \mathbb{P}^3$	+
Fano K3–49	3	24	0	27	100	20	$\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^4$	$\mathcal{O}(0,0,2) \oplus$ $\mathcal{O}(0,1,1) \oplus \mathcal{O}(1,1,1)$	$\mathbb{P}^1 \times \mathbb{Q}^3$, $D \subset \mathbb{P}^2 \times \mathbb{Q}^3$	+
Fano K3–50	3	22	0	43	180	18	$\mathbb{P}^3 \times (\mathbb{P}^4)^2$	$\mathcal{O}(1,1,1) \oplus$ $\mathcal{Q}_{\mathbb{P}^3}(0,1,0) \oplus \mathcal{Q}_{\mathbb{P}^3}(0,0,1)$	Z°, $\mathrm{Bl}_p \mathbb{P}^4$	+
Fano K3–51	3	30	0	23	80	26	$\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3$	$\mathcal{O}(0,1,2) \oplus \mathcal{O}(1,1,1)$	$D \subset \mathbb{P}^2 \times \mathbb{P}^3$, $\mathbb{P}^1 \times \mathbb{P}^3$	+

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Table C.4 continued from previous page

Fano K3–52	3	24	0	29	110	20	$(\mathbb{P}^2)^2 \times \mathbb{P}^3$	$\mathcal{O}(0,1,1) \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(1,1,1)$	$\mathbb{P}^2 \times \mathbb{P}^2$, $D \subset \mathbb{P}^2 \times \mathbb{P}^3$	+
Fano K3–53	3	23	0	36	145	19	$\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^4$	$\mathcal{Q}_{\mathbb{P}^3}(0,0,1) \oplus \mathcal{O}(1,1,0) \oplus \mathcal{O}(1,1,1)$	Fano K3–29, $\mathrm{Bl}_p \mathbb{P}^4$, $D \subset \mathbb{P}^2 \times \mathbb{P}^3$	+
Fano K3–54	3	25	0	34	133	23	$\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^4$	$\mathcal{O}(0,1,2) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,0,1) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,1,0)$	$\mathrm{Bl}_S \mathbb{P}^4$, $\mathrm{Bl}_{\mathbb{P}^1} \mathbb{P}^4$	+
Fano K3–55	3	22	0	37	150	18	$\mathbb{P}^1 \times \mathbb{P}^4 \times \mathbb{P}^5$	$\mathcal{Q}_{\mathbb{P}^4}(0,0,1) \oplus \mathcal{O}(0,2,0) \oplus \mathcal{O}(1,1,1)$	Z° $\mathbb{P}^1 \times \mathbb{Q}_3$	+
Fano K3–56	4	24	0	35	140	17	$(\mathbb{P}^1)^2 \times \mathbb{P}^2 \times \mathbb{P}^3$	$\mathcal{Q}_{\mathbb{P}^2}(0,0,0,1) \oplus \mathcal{O}(1,1,1,1)$	Fano K3–41, $\mathbb{P}^1 \times \mathrm{Bl}_p \mathbb{P}^3$	+
Fano K3–57	4	24	0	34	134	18	$\mathbb{P}^1 \times \mathrm{Fl}(1,3,6)$	$\mathcal{Q}_2^{\oplus 2} \oplus \mathcal{O}(0;0,2) \oplus \mathcal{O}(1;2,0)$	$\mathrm{Bl}_{2p} \mathrm{Gr}(2,4)$	+
Fano K3–58	4	28	0	28	104	22	$(\mathbb{P}^1)^3 \times \mathbb{P}^3$	$\mathcal{O}(0,0,1,1) \oplus \mathcal{O}(1,1,0,2)$	$\mathbb{P}^1 \times \mathbb{P}^3$, $\mathbb{P}^1 \times \mathrm{Bl}_p \mathbb{P}^3$	+
Fano K3–59	4	24	0	31	120	17	$(\mathbb{P}^1)^2 \times (\mathbb{P}^2)^2$	$\mathcal{O}(0,0,1,1) \oplus \mathcal{O}(1,1,1,1)$	$D \subset \mathbb{P}^1 \times (\mathbb{P}^2)^2$, $(\mathbb{P}^1)^2 \times \mathbb{P}^2$	+
Fano K3–60	5	26	0	31	120	16	$(\mathbb{P}^1)^5$	$\mathcal{O}(1,1,1,1,1)$	$(\mathbb{P}^1)^4$	+

Table C.5. Rogue FK3.

ID	ρ	$h^{2,2}$	$h^{1,2}$	$h^0(-K)$	K^4	$-\chi(T)$	G	$\mathcal{F}$	Bl	Rat
Fano R–61	2	25	0	27	99	25	$\mathbb{P}^2 \times \mathrm{Gr}(2,6)$	$\mathcal{U}^\vee_{\mathrm{Gr}(2,6)}(0,1) \oplus \mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,6)}$	Z°	+
Fano R–62	2	23	0	33	129	23	$\mathbb{P}^3 \times \mathrm{Gr}(2,7)$	$\mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{O}(1,1) \oplus \mathcal{Q}_{\mathbb{P}^3} \boxtimes \mathcal{U}^\vee_{\mathrm{Gr}(2,7)}$	Z°	?
Fano R–63	3	22	2	31	120	20	$\mathbb{P}^5 \times \mathbb{P}^7$	$\mathcal{Q}_{\mathbb{P}^5}(0,1) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(2,0)^{\oplus 2}$	Z°	+

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Table C.5 continued from previous page

Fano R–64	3	22	5	25	90	23	$\mathbb{P}^4 \times \mathbb{P}^6$	$\mathcal{Q}_{\mathbb{P}^4}(0,1) \oplus \mathcal{O}(3,0) \oplus \mathcal{O}(0,2)$	Z°	?

Table C.6. Blowups of FK3.

ID	ρ	$h^{2,2}$	$h^{1,3}$	$h^0(-K)$	K^4	$-\chi(T)$	G	$\mathcal{F}$	Bl	Rat
Fano A–65	2	79	7	12	27	58	$\mathbb{P}^1 \times \mathbb{P}^5$	$\mathcal{O}(0,3) \oplus \mathcal{O}(1,2)$	X_3	*
Fano A–66	2	51	3	16	45	44	$\mathbb{P}^2 \times \mathbb{P}^5$	$\mathcal{O}(0,3) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(1,1)$	X_3	*
Fano A–67	2	41	2	17	51	36	$\mathbb{P}^2 \times \mathbb{P}^5$	$\mathcal{O}(1,1) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,2)$	X_3	?
Fano A–68	2	41	2	26	93	39	$\mathbb{P}^5 \times \mathbb{P}^6$	$\mathcal{O}(1,1) \oplus \mathcal{Q}_{\mathbb{P}^5}(0,1) \oplus \mathcal{O}(3,0)$	X_3, Z°	?
Fano A–69	2	42	2	17	50	38	$\mathbb{P}^1 \times \mathrm{Gr}(2,5)$	$\mathcal{O}(0,1) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(1,1)$	X_{10}	*

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