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ENVELOPING SPACE OF GLOBALLY HYPERBOLIC CONFORMALLY FLAT SPACETIMES

by Rym SMAÏ (*)

ABSTRACT. — We prove that any simply-connected globally hyperbolic conformally flat spacetime V can be conformally embedded in a bigger conformally flat spacetime, called *enveloping space* of V, containing all the conformally flat Cauchy extensions of V, in particular its $\mathcal{C}_0$-maximal extension. As a result, we establish a new proof of the existence and the uniqueness of the $\mathcal{C}_0$-maximal extension of a globally hyperbolic conformally flat spacetime. Furthermore, this approach allows us to prove that $\mathcal{C}_0$-maximal extensions respect inclusion.

RÉSUMÉ. — Nous prouvons que tout espace-temps conformément plat globalement hyperbolique simplement connexe V peut-être plongé conformément dans un espace-temps conformément plat plus grand, appelé *espace enveloppant* de V, qui contient toutes les extensions de Cauchy conformément plates de V, en particulier son extension $\mathcal{C}_0$-maximale. Il en découle une nouvelle preuve de l'existence et de l'unicité de l'extension $\mathcal{C}_0$-maximale d'un espace-temps conformément plat globalement hyperbolique. En outre, cette approche nous permet de montrer que les extensions $\mathcal{C}_0$-maximales respectent l'inclusion.

1. Introduction

The notion of maximal extension of a globally hyperbolic spacetime arises from the resolution of Einstein equations in general relativity. This physical theory suggests that our universe is modeled by a Lorentzian manifold (M, g) of dimension 4 where the metric g satisfies some PDEs, the so-called *Einstein equations*. One approach to solving them is to require that M is homeomorphic to $S \times \mathbb{R}$ where S is a Riemannian manifold. This allows the

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definition of a Cauchy problem where the initial data is the Riemannian manifold (S,h) equipped with a $(2,0)$-tensor II. A solution is a Lorentzian metric g on $S\times\mathbb{R}$ such that the restriction of g to $S\times\{0\}$ is h and II is the shape operator of this hypersurface. It turns out that a necessary condition to have such a solution is that h and II satisfy *the constraint equations*. Conversely, Choquet-Bruhat and Geroch [4, Section 2] proved that when the constraint equations are satisfied, a local solution exists. Two natural questions arise: is it possible to extend this solution to a maximal one? If yes, is it unique up to isometry? Choquet-Bruhat and Geroch answered positively to both questions (see [4, Theorem 3]).

The solutions of the Cauchy problem for Einstein equations turn out to be *globally hyperbolic* (abbrev. GH; see Definition 2.2). More generally, Geroch [7] proved that any GH spacetime admits an embedded Riemannian hypersurface which intersects every inextensible causal curve exactly once, called *Cauchy hypersurface*. It turns out that all smooth Cauchy hypersurfaces of a GH spacetime are diffeomorphic to one another.

There is a natural partial ordering on GH spacetimes: given two GH spacetimes, M and N, we say that N is a Cauchy extension of M if there exists an isometric embedding from M to N sending every Cauchy hypersurface of M on a Cauchy hypersurface of N. Such an embedding is called *an isometric Cauchy embedding*. In this general setting, we can ask again the questions of the existence and the uniqueness, up to isometry, of a maximal extension. The answer to both questions is yes within a *rigid category* of spacetimes.(1) Actually, the spacetimes which are solution of a Cauchy problem for Einstein equations constitute a rigid category and it turns out that the arguments of Choquet-Bruhat and Geroch could be adapted to any other rigid category.

In this paper, we are interested in the notion of maximality in the setting of *conformally flat* spacetimes of dimension $n\geqslant 3$. The morphisms preserving these structures are the *conformal* diffeomorphisms. In [11, Section 3.1], C. Rossi adapted to this setting the definition of the ordering relation on GH spacetimes by considering conformal Cauchy embeddings instead of isometric Cauchy embeddings. A GH conformally flat spacetime M is then said to be *$\mathcal{C}_0$-maximal* if any conformal Cauchy embedding from M to any GH conformally flat spacetime is surjective. C. Rossi proved that any GH conformally flat spacetime admits a $\mathcal{C}_0$-maximal extension, unique up to conformal diffeomorphism (see [11, Sections 3.2 & 3.3]). Her proof is

(1) See e.g. [11, Definitions 2 and 4]. Spacetimes of constant curvature are examples of rigid categories.

mainly based on Zorn lemma and so does not give any description of the $\mathcal{C}_0$-maximal extension. In this paper, we propose a new approach which allows us to give a constructive proof of the existence and the uniqueness of the $\mathcal{C}_0$-maximal extension. Indeed, given a simply-connected GH conformally flat spacetime M, we construct a bigger conformally flat spacetime $E(M)$ in which M and all its conformally flat Cauchy extensions embeds conformally. The $\mathcal{C}_0$-maximal extension of M turns out to be the Cauchy development of a Cauchy hypersurface of M in $E(M)$. The images in $E(M)$ of the previous embeddings satisfy the nice property of being *causally convex*. A subset U of a spacetime is causally convex if any causal curve joining two points of U is contained in U. While convexity is a metric notion, causal convexity is a conformal notion. Let us add that causal convexity is a strong property in a GH spacetime: it is a classical fact that any causally convex open subset of a GH spacetime is GH.

THEOREM 1.1. — *Let M be a simply-connected globally hyperbolic conformally flat spacetime. There exists a conformally flat spacetime $E(M)$ with the following properties:*

(1) *$E(M)$ fibers trivially over a conformally flat Riemannian manifold $\mathcal{B}$ diffeomorphic to any Cauchy hypersurface of M;*
(2) *M embeds conformally in $E(M)$ as a causally convex open subset;*
(3) *all the conformally flat Cauchy extensions of M embed conformally in $E(M)$ as causally convex open subsets. In particular, the $\mathcal{C}_0$-maximal extension of M is the Cauchy development of a Cauchy hypersurface of M in $E(M)$.*

Such a spacetime $E(M)$ is called an enveloping space *of M.*

This result still holds for the larger class of *developable* GH conformally flat spacetimes (see Definition 4.3). In Section 4.3, we describe causally convex open subsets of an enveloping space $E(M)$ then, in Section 7, we characterize those which are $\mathcal{C}_0$-maximal.

A consequence of Theorem 1.1 is the following result.

COROLLARY 1.2. — *Any globally hyperbolic conformally flat spacetime admits a $\mathcal{C}_0$-maximal extension, unique up to conformal diffeomorphism.*

Now we ask the following question. Let V be a globally hyperbolic conformally flat spacetime and let U be a causally convex open subset of V. Does the $\mathcal{C}_0$-maximal extension of U embed conformally in the $\mathcal{C}_0$-maximal extension of V?

The $\mathcal{C}_0$-maximal extensions of U and V are *a priori* abstract objects which depend on the Cauchy hypersurfaces of U and V, respectively. These last ones are completely independent so the question above is not tautological. We prove in Section 6 that the answer is yes, in other words, that $\mathcal{C}_0$-maximal extensions preserve inclusion.

Theorem 1.3. — *Let V be a globally hyperbolic conformally flat spacetime and let U be a causally convex open subset of V. Then, the $\mathcal{C}_0$-maximal extension of U is conformally equivalent to a causally convex open subset of the $\mathcal{C}_0$-maximal extension of V.*

Overview of the paper

In Section 2, we introduce the preliminary material on causality of spacetimes. We focus in particular on *globally hyperbolic* spacetimes and we recall some of their main properties. Section 3 deals with the model space of conformally flat Lorentzian structure, the so-called *Einstein universe*. After a quick description of its geometry, we characterize *causally convex open subsets* of its universal cover (see Sections 3.5 and 3.6). We devote Section 4 to the proof of Theorem 1.1: we construct an enveloping space (see Section 4.2) and we describe its causally convex globally hyperbolic open subsets (see Section 4.3). In Section 5, we propose a new proof of the existence and the uniqueness of the maximal extension of a globally hyperbolic conformally flat spacetime, using the notion of enveloping space. Section 6 is devoted to the proof of Theorem 1.3. Lastly, we establish a link between the notion of $\mathcal{C}_0$-maximality and the notion of *eikonal functions* in Section 7.

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2. Globally hyperbolic spacetimes

2.1. Preliminaries on spacetimes

The aim of this preliminary section is to introduce the concept of *causality* in a Lorentzian manifold and briefly recall some basic causal notions as *causal curves*, *future and past* of points, *lightcones*, *achronal and acausal* subsets, etc.

Throughout this paper, we denote by (p, q) the signature of a non-degenerate quadratic form Q where p and q are respectively the number of negative and positive coefficients in the polar decomposition of Q.

Spacetimes. A *Lorentzian metric* on a manifold of dimension n is a non-degenerate symmetric 2-tensor g of signature $(1, n-1)$. A manifold equipped with a Lorentzian metric is called *Lorentzian*.

In a Lorentzian manifold (M, g), we say that a non-zero tangent vector v is *timelike, lightlike, spacelike* if $g(v, v)$ is respectively negative, zero, positive. The set of timelike vectors is the union of two convex open cones. When it is possible to make a continuous choice of a connected component in each tangent space, the manifold M is said *time-orientable*. The timelike vectors in the chosen component are said *future-directed* while those in the other component are said *past-directed*. A *spacetime* is an oriented and time-oriented Lorentzian manifold.

Future, past. In a spacetime M, a differential curve is *timelike, lightlike, spacelike* if its tangent vectors are timelike, lightlike, spacelike. It is *causal* if its tangent vectors are either timelike or lightlike.

Given a point p in M, the *future* (resp. *chronological future*) of p, denoted $J^+(p)$ (resp. $I^+(p)$), is the set of endpoints of future-directed causal (resp. timelike) curves starting from p. More generally, the future (resp. chronological future) of a subset A of M, denoted $J^+(A)$ (resp. $I^+(A)$), is the union of $J^+(a)$ (resp. $I^+(a)$) where $a \in A$.

An open subset U of M is a spacetime and the intrinsic causality relations of U imply the corresponding ones in M. We denote $J^+(A, U)$ (resp. $I^+(A, U)$) the future (resp. chronological future) in the manifold U of a set $A \subset U$. Then, $I^+(A, U) \subset I^+(A) \cap U$.

Dual to the preceding definitions are corresponding *past* versions. In general, *past* definitions and proofs follow from future versions (and vice versa) by reversing time-orientation.

Diamonds. We call *diamond of M* any intersection $J^-(p) \cap J^+(q)$, where $p, q \in M$ such that $p \in J^+(q)$. We denote it $J(p, q)$. Given two points $p, q \in M$ such that $p \in I^+(q)$, the interior of the diamond $J(p, q)$ is the intersection $I^-(p) \cap I^+(q)$ and is denoted $I(p, q)$ (see [9, Lemma 6, p. 404]).

Achronal, acausal subsets. A subset A of a spacetime M is called *achronal* (resp. *acausal*) if no timelike (resp. causal) curve intersects A more than once.

Causal convexity. In Riemannian geometry, it is often useful to consider open neighborhoods which are *geodesically convex*. In Lorentzian geometry, there is, in addition, *a causal convexity* notion. A subset U of M is said *causally convex* if for every $p, q \in U$, any causal curve of M joining p to q is contained in U. Equivalently, if every diamond $J(p, q)$ of M with $p, q \in U$ is contained in U. It is easy to check that the intersection of two causally convex subsets is causally convex.

Cauchy developments. Let A be an achronal subset of M. *The future* (resp. *past*) *Cauchy development* of A, denoted $\mathcal{C}^+(A)$ (resp. $\mathcal{C}^-(A)$), is the set of points p of M such that every past-inextensible (resp. future-inextensible) causal curve through p meets A. *The Cauchy development of A* is the union of $\mathcal{C}^+(A)$ and $\mathcal{C}^-(A)$, denoted $\mathcal{C}(A)$.

2.2. Global hyperbolicity

Definition 2.1. — *A spacetime M is said* strongly causal *if for every point $p \in M$ and every neighborhood U of p, there exists a neighborhood V of p contained in U, which is causally convex in M.*

Definition 2.2. — *A spacetime M is said* globally hyperbolic *(abbrev. GH) if the two following conditions hold:*

(1) *M is strongly causal;*
(2) *all diamonds of M are compact.*

It was proved by Sanchez in [3] that the first condition can be weakened to *M is causal*, that is M contains no causal loop.

A classical result of Geroch [7], later improved by Bernal and Sanchez [2], gives a characterization of global hyperbolicity involving the notion of *Cauchy hypersurface*.

Definition 2.3. — *A* topological *(resp.* smooth*)* Cauchy hypersurface *is an achronal topological hypersurface (resp. an embedded Riemannian hypersurface) that is met exactly once by every inextensible causal curve of M.*

Theorem 2.4 ([7]). — *A spacetime M is globally hyperbolic if and only if it contains a topological Cauchy hypersurface.*

Bernal and Sanchez [2] improved this result by proving the existence of a *smooth* Cauchy hypersurface.

Topological (resp. smooth) Cauchy hypersurfaces of a globally hyperbolic spacetime are homeomorphic (resp. diffeomorphic). Therefore, one can set the following definition.

DEFINITION 2.5. — *A globally hyperbolic spacetime is said* Cauchy-compact *(or* spatially compact*) if it admits a compact Cauchy hypersurface.*

A remarkable property of globally hyperbolic spacetimes is that causal convexity implies global hyperbolicity.

PROPOSITION 2.6. — *Let M be a globally hyperbolic spacetime. Then, any causally convex open subset of M is globally hyperbolic.*

Proof. — Since M is globally hyperbolic, there is no causal loop in U. Since U is causally convex, the diamonds of U are exactly the diamonds of M contained in U. Thus, they are compact. □

2.3. Shadows

In this section, we show that the causal structure of globally hyperbolic spacetimes is encoded by compact subsets of a Cauchy hypersurface, called *shadows*.[2]

Let M be a globally hyperbolic spacetime and let $S \subset M$ be a Riemannian Cauchy hypersurface.

DEFINITION 2.7. — *Let $p \in M$. We call* shadow of p on S, *denoted by $O(p, S)$, the set of points in S which are causally related to p. When there is no confusion about the Cauchy hypersurface S, we will simply write $O(p)$ instead of $O(p, S)$.*

If $p \in I^{\pm}(S)$, then $O(p, S) = J^{\mp}(p) \cap S$; if $p \in S$, $O(p, S)$ is reduced to $\{p\}$. Thus, by [9, Lemma 40, p. 423]), shadows are *compact*.

The main interest of the notion of *shadows* is given by the following proposition proved by C. Rossi in her thesis (see [10, Proposition 2.6, Chapter 4]).

PROPOSITION 2.8. — *Suppose S is not compact. Then, two points p and q of M in the chronological future of S coincide if and only if their shadows on S are equal.*

(2) This terminology has been introduced by C. Rossi in her thesis [10, Chapitre 4].

By Proposition 2.8, the shadows on S characterize completely the points of the globally hyperbolic spacetime M. This allows to reduce, in some situations, the study of the spacetime to the study of compact subsets of a Riemannian manifold.

3. Geometry of Einstein universe

In this section, we introduce the model space of conformally flat Lorentzian structures, the so-called *Einstein universe*, and we describe its causal structure.

3.1. The Klein model

Let $\mathbb{R}^{2,n}$ be the vector space $\mathbb{R}^{n+2}$ of dimension $(n+2)$ equipped with the nondegenerate quadratic form $q_{2,n}$ of signature $(2,n)$ given by

$$q_{2,n}(u,v,x_1,\ldots,x_n) = -u^2 - v^2 + x_1^2 + \cdots + x_n^2$$

in the coordinate system $(u,v,x_1,\ldots,x_n)$ associated to the canonical basis of $\mathbb{R}^{n+2}$.

Definition 3.1. — *The* Einstein universe *of dimension n, denoted by $\mathsf{Ein}_{1,n-1}$, is the space of isotropic lines of $\mathbb{R}^{2,n}$ with respect to the quadratic form $q_{2,n}$, namely*

$$\mathsf{Ein}_{1,n-1} = \left\{[x] \in \mathbb{P}(\mathbb{R}^{2,n}) \ : \ q_{2,n}(x) = 0\right\}.$$

In practice, it is more convenient to work with the double cover of the Einstein universe, denoted by $\mathrm{Ein}_{1,n-1}$:

$$\mathrm{Ein}_{1,n-1} = \left\{[x] \in \mathbb{S}(\mathbb{R}^{2,n}) \ : \ q_{2,n}(x) = 0\right\}$$

where $\mathbb{S}(\mathbb{R}^{2,n})$ is the sphere of rays, namely the quotient of $\mathbb{R}^{2,n}\backslash\{0\}$ by positive homotheties.

3.2. Spatio-temporal decomposition of Einstein universe

The choice of a timelike plane of $\mathbb{R}^{2,n}$, i.e. a plane on which the restriction of $q_{2,n}$ is negative definite, defines a spatio-temporal decomposition of Einstein universe.

Lemma 3.2. — *Any timelike plane $P \subset \mathbb{R}^{2,n}$ defines a diffeomorphism between $\mathbb{S}^{n-1} \times \mathbb{S}^1$ and* $\mathrm{Ein}_{1,n-1}$.

Proof. — Consider the orthogonal splitting $\mathbb{R}^{2,n} = P^\perp \oplus P$ and call $q_{P^\perp}$ and q_P the positive definite quadratic form induced by $\pm q_{2,n}$ on $P^\perp$ and P respectively. The restriction of the canonical projection $\mathbb{R}^{2,n}\backslash\{0\}$ on $\mathbb{S}(\mathbb{R}^{2,n})$ to the set of points $(x,y) \in P^\perp \oplus P$ such that $q_{P^\perp}(x) = q_P(y) = 1$ defines a map from $\mathbb{S}^{n-1} \times \mathbb{S}^1$ to $\mathrm{Ein}_{1,n-1}$. It is easy to check that this map is a diffeomorphism. □

For every timelike plane $P \subset \mathbb{R}^{2,n}$, the quadratic form $q_{2,n}$ induces a Lorentzian metric g_P on $\mathbb{S}^{n-1} \times \mathbb{S}^1$ given by

$$g_P = \mathrm{d}\sigma^2(P) - \mathrm{d}\theta^2(P)$$

where $\mathrm{d}\sigma^2(P)$ is the round metric on $\mathbb{S}^{n-1} \subset (P^\perp, q_{P^\perp})$ induced by $q_{P^\perp}$ and $\mathrm{d}\theta^2(P)$ is the round metric on $\mathbb{S}^1 \subset (P, q_P)$ induced by q_P.

An easy computation shows that if $P' \subset \mathbb{R}^{2,n}$ is another timelike plane, the Lorentzian metric $g_{P'}$ is conformally equivalent to g_P, i.e. q_P and g'_P are proportional by a positive smooth function on $\mathbb{S}^{n-1} \times \mathbb{S}^1$. As a result, Einstein universe is naturally equipped with a conformal class of Lorentzian metrics. This Lorentzian conformal structure induces causality on Einstein universe. Indeed, changing the metric in the conformal class consists in multiplying by a positive function and so does not change the sign of the norm of a tangent vector. The causal structure of Einstein universe is trivial: any point is causally related to any other one (see e.g. [10, Corollary 2.10, Chapter 2]).

Let us point out that in general geodesics are not well-defined in a conformal spacetime. Indeed, a computation of the Levi-Civita connection shows that geodesics are not preserved by conformal changes of metrics. Nevertheless, lightlike geodesics are preserved as *non-parametrized* curves (see e.g. [5, Théorème 3]).

3.3. Causal structure of the universal cover

Let $\widetilde{\mathrm{Ein}}_{1,n-1}$ be the universal cover of $\mathrm{Ein}_{1,n-1}$. When $n \geqslant 3$, every diffeomorphism between $\mathrm{Ein}_{1,n-1}$ and $\mathbb{S}^{n-1} \times \mathbb{S}^1$ lifts to a diffeomorphism between $\widetilde{\mathrm{Ein}}_{1,n-1}$ and $\mathbb{S}^{n-1} \times \mathbb{R}$. The pull-back by the projection $\mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{S}^{n-1} \times \mathbb{S}^1$ of the conformal class $[\mathrm{d}\sigma^2 - \mathrm{d}\theta^2]$ on $\mathbb{S}^{n-1} \times \mathbb{S}^1$ defined previously is the conformal class of the Lorentzian metric $\mathrm{d}\sigma^2 - \mathrm{d}t^2$ where $\mathrm{d}t^2$ is the usual metric on $\mathbb{R}$. This induces a natural conformally flat Lorentzian structure on $\widetilde{\mathrm{Ein}}_{1,n-1}$.

Definition 3.3. — *We call* spatio-temporal decomposition of $\widetilde{\mathrm{Ein}}_{1,n-1}$ *any conformal diffeomorphism between* $\widetilde{\mathrm{Ein}}_{1,n-1}$ *and* $\mathbb{S}^{n-1} \times \mathbb{R}$.

In what follows, we fix a spatio-temporal decomposition and we identify $\widetilde{\mathrm{Ein}}_{1,n-1}$ to $\mathbb{S}^{n-1} \times \mathbb{R}$.

The fundamental group of $\mathrm{Ein}_{1,n-1}$ is isomorphic to $\mathbb{Z}$, generated by the transformation $\delta\colon \widetilde{\mathrm{Ein}}_{1,n-1} \to \widetilde{\mathrm{Ein}}_{1,n-1}$ defined by $\delta(x,t) = (x, t + 2\pi)$. That of $\mathsf{Ein}_{1,n-1}$ is generated by the transformation $\sigma\colon \widetilde{\mathrm{Ein}}_{1,n-1} \to \widetilde{\mathrm{Ein}}_{1,n-1}$ such that $\sigma^2 = \delta$, i.e. the map defined by $\sigma(x,t) = (-x, t + \pi)$.

Definition 3.4. — *Two points p and q of* $\widetilde{\mathrm{Ein}}_{1,n-1}$ *are said to be* conjugate *if one is the image under σ of the other.*

While the causal structure of $\mathrm{Ein}_{1,n-1}$ is trivial, the causal structure of $\widetilde{\mathrm{Ein}}_{1,n-1}$ is rich. We give a brief description below. We direct to [10, Chapter 2] for more details.

Lightlike geodesics of $\widetilde{\mathrm{Ein}}_{1,n-1}$ are the curves which can be written, up to reparametrization, as $\big(x(t), t\big)$ where $x\colon I \to \mathbb{S}^{n-1}$ is a geodesic of $\mathbb{S}^{n-1}$ defined on an interval I of $\mathbb{R}$. The inextensible ones are those for which x is defined on $\mathbb{R}$.

It turns out that the photons going through a point (x_0, t_0) have common intersections at the points $\sigma^k(x_0, t_0)$, for $k \in \mathbb{Z}$; and are pairwise disjoint outside these points. The lightcone of a point (x_0, t_0) is the set of points (x,t) such that $d(x, x_0) = |t - t_0|$ where d is the distance on the sphere $\mathbb{S}^{n-1}$ induced by the round metric. It disconnects $\widetilde{\mathrm{Ein}}_{1,n-1}$ in three connected components:

- The *chronological future* of (x_0, t_0): this is the set of points (x,t) of $\mathbb{S}^{n-1} \times \mathbb{R}$ such that $d(x, x_0) < t - t_0$.
- The *chronological past* of (x_0, t_0): this is the set of points (x,t) of $\mathbb{S}^{n-1} \times \mathbb{R}$ such that $d(x, x_0) < t_0 - t$.
- The set of points non-causally related to (x_0, t_0), i.e. the set of points (x,t) of $\mathbb{S}^{n-1} \times \mathbb{R}$ such that $d(x, x_0) > |t - t_0|$. This is exactly the interior of the diamond of vertices $\sigma(x_0, t_0)$ and $\sigma^{-1}(x_0, t_0)$. It is conformally diffeomorphic to Minkowski spacetime (see e.g. [12, Lemma 2.38 and Corollary 2.43]) and is called *affine chart*. We denote it $\mathrm{Mink}_0(x_0, t_0)$.

There are two other affine charts associated to the point (x_0, t_0), namely:

- the set of points non-causally related to $\sigma(x_0, t_0)$, contained in the chronological future of (x_0, t_0), denoted $\mathrm{Mink}_+(x_0, t_0)$;
- the set of points non-causally related to $\sigma^{-1}(x_0, t_0)$, contained in the chronological past of (x_0, t_0), denoted $\mathrm{Mink}_-(x_0, t_0)$.

The universal cover $\widetilde{\mathrm{Ein}}_{1,n-1}$ is globally hyperbolic: any sphere $\mathbb{S}^{n-1}\times\{t\}$, where $t\in\mathbb{R}$, is a Cauchy hypersurface.

3.4. Conformal group

The subgroup $O(2,n)\subset \mathrm{Gl}_{n+2}(\mathbb{R})$ preserving $q_{2,n}$, acts conformally on $\mathrm{Ein}_{1,n-1}$. When $n\geqslant 3$, the conformal group of $\mathrm{Ein}_{1,n-1}$ is *exactly* $O(2,n)$. This is a consequence of the following result, which is an extension to Einstein universe of a classical theorem of Liouville in Euclidean conformal geometry (see e.g. [5]).

THEOREM 3.5. — *Let $n\geqslant 3$. Any conformal transformation between two open subsets of $\mathrm{Ein}_{1,n-1}$ is the restriction of an element of $O(2,n)$.*

It is a classical fact that every conformal diffeomorphism of $\mathrm{Ein}_{1,n-1}$ lifts to a conformal diffeomorphism of $\widetilde{\mathrm{Ein}}_{1,n-1}$. Conversely, by Theorem 3.5, every conformal transformation of $\widetilde{\mathrm{Ein}}_{1,n-1}$ defines a unique conformal transformation of the quotient space $\mathrm{Ein}_{1,n-1}=\widetilde{\mathrm{Ein}}_{1,n-1}/\langle\delta\rangle$.

Let $\mathrm{Conf}(\widetilde{\mathrm{Ein}}_{1,n-1})$ denote the group of conformal transformations of $\widetilde{\mathrm{Ein}}_{1,n-1}$. Let $j\colon\ \mathrm{Conf}(\widetilde{\mathrm{Ein}}_{1,n-1})\to O(2,n)$ be the natural projection. This is a surjective group morphism whose kernel is generated by δ.

3.5. Causally convex open subsets of Einstein universe

In this section, we characterize causally convex open subsets of $\widetilde{\mathrm{Ein}}_{1,n-1}$ in a spatio-temporal decomposition $\mathbb{S}^{n-1}\times\mathbb{R}$. We denote by d the distance on $\mathbb{S}^{n-1}$ induced by the round metric.

PROPOSITION 3.6. — *Let Ω be a causally convex open subset of $\widetilde{\mathrm{Ein}}_{1,n-1}$. Then, there exist two 1-Lipschitz functions f^+ and f^- from an open subset U of $\mathbb{S}^{n-1}$ to $\overline{\mathbb{R}}$ such that the following hold:*

- *$f^-<f^+$ on U;*
- *the extensions of f^+ and f^- to ∂U coincide;*
- *Ω is the set of points (x,t) of $\widetilde{\mathrm{Ein}}_{1,n-1}$ such that $f^-(x)<t<f^+(x)$.*

Remark 3.7. — The functions $f^\pm$ either take value in $\mathbb{R}$ or $f^\pm\equiv\pm\infty$. In this latter case, $U=\mathbb{S}^{n-1}$.

The proof of Proposition 3.6 uses the following lemma.

Lemma 3.8. — *For every points p and q in the closure of Ω such that $p \in I^+(q)$, the intersection $I^-(p) \cap I^+(q)$ is contained in Ω.*

Proof. — Let $p, q \in \overline{\Omega}$ such that $p \in I^+(q)$. There exist two sequences $\{p_i\}$ and $\{q_i\}$ of elements of Ω such that $\lim p_i = p$ and $\lim q_i = q$. Let $r \in I(p, q)$. Then, $I^+(r)$ is an open neighborhood of p and $I^-(r)$ is an open neighborhood of q. As a result, there exists an integer i_0 such that $p_{i_0} \in I^+(r)$ and $q_{i_0} \in I^-(r)$. It follows that $r \in I(p_{i_0}, q_{i_0}) \subset \Omega$. □

Proof of Proposition 3.6. — Let U be the projection of Ω on the sphere $\mathbb{S}^{n-1}$. Since Ω is causally convex, the intersection of Ω with any timelike line $\{x\} \times \mathbb{R}$, where $x \in U$, is connected, i.e. it is a segment $\{x\} \times \left]f^-(x), f^+(x)\right[$. This defines two functions f^+ and f^- from U to $\overline{\mathbb{R}}$ such that Ω is the set of points (x, t) such that $f^-(x) < t < f^+(x)$.

Fact. — *If there exists $x \in U$ such that $f^+(x) = +\infty$ then $f^+ \equiv +\infty$.*

This is equivalent to proving that Ω is future-complete, i.e. $I^+(\Omega) \subset \Omega$, as soon as it contains a timelike half-line $\alpha = \{x\} \times [t, +\infty[$. Let $p \in \Omega$ and let $q \in I^+(p)$. Since α is future-inextensible, it intersects $I^+(q)$. Let q' be a point in this intersection. Then, $q \in J(q', p')$ where $p' \in \Omega \cap I^-(p)$. Thus, $q \in \Omega$.

Fact. — *If f^+ is finite, it is 1-Lipschitz.*

This is equivalent to proving that the graph of f^+ is achronal. Suppose there exist two distinct points p, q in the graph of f^+ such that $p \in I^+(q)$. Since $p \in \partial\Omega$, we have $I^+(q) \cap \Omega \neq \emptyset$. Then, $q \in I(p', q')$ where $p' \in I^+(q) \cap \Omega$ and $q' \in \Omega \cap I^-(q)$. Hence, $q \in \Omega$. Contradiction.

Fact. — *If f^+ and f^- are finite and ∂U is non-empty, then the extensions of f^+ and f^- to ∂U are equal.*

Let $\overline{f^+}$ (resp. $\overline{f^-}$) be the extension of f^+ (resp. f^-) to $\overline{U}$. Suppose there exists $x \in U$ such that $\overline{f^-}(x) < \overline{f^+}(x)$. Then, the timelike segment $\{x\} \times \left]\overline{f^-}(x), \overline{f^+}(x)\right[$ is contained in the boundary of Ω. This contradicts Lemma 3.8. □

Conversely, we have the following result.

Proposition 3.9. — *Let $f^+, f^- : U \subset \mathbb{S}^{n-1} \to \overline{\mathbb{R}}$ be two 1-Lipschitz functions defined on an open subset U of $\mathbb{S}^{n-1}$ such that:*

- *$f^- < f^+$ on U;*
- *the extensions of f^+ and f^- to ∂U coincide.*

Then, the set of points (x, t) of $\widetilde{\mathrm{Ein}}_{1,n-1}$ such that $f^-(x) < t < f^+(x)$, named Ω, is causally convex in $\widetilde{\mathrm{Ein}}_{1,n-1}$.

The proof of Proposition 3.9 uses the following lemma.

Lemma 3.10. — *Let $\overline{f^+}$ be the extension of f^+ to $\overline{U}$. Then, for every point p in the graph of $\overline{f^+}$, the future $J^+(p)$ of p is disjoint from Ω.*

Proof. — Let $x \in U$ and set $p = \big(x, f^+(x)\big)$. Suppose there exists $(y,s) \in \Omega \cap J^+(p)$. Then, $d(x,y) \leqslant s - f^+(x) < f^+(y) - f^+(x) \leqslant d(x,y)$. Contradiction. Then, $J^+(p) \cap \Omega = \emptyset$.

Now, let $x \in \partial U$ and let $\{x_i\}$ be a sequence of elements of U such that $x = \lim x_i$. Set $p_i = \big(x_i, f^+(x_i)\big)$. Suppose there exists $q \in J^+(p) \cap \Omega$. Since Ω is open, there exists $q' \in I^+(q) \cap \Omega$. By transitivity, $q' \in I^+(p)$. Then, $I^-(q')$ is an open neighborhood of p. Since $\lim p_i = p$, we deduce that $p_i \in I^-(q')$ for i big enough. Equivalently, $q' \in I^+(p_i)$. Thus, $I^+(p_i) \cap \Omega \neq \emptyset$. Contradiction. □

There is an analogous statement for the extension of f^- to $\overline{U}$, denoted $\overline{f^-}$, with the reverse time-orientation.

Proof of Proposition 3.9. — Since $f^\pm$ is 1-Lipschitz, if $f^\pm$ is infinite in a point of U, it is infinite on U. If $f^+ \equiv +\infty$ and $f^- \equiv -\infty$, we have $U = \mathbb{S}^{n-1}$ and $\Omega = \widetilde{\mathrm{Ein}}_{1,n-1}$.

Suppose $f^+ < +\infty$ and $f^- \equiv -\infty$. Let $p, q \in \Omega$ such that $q \in J^+(p)$. Let γ be a future causal curve of $\widetilde{\mathrm{Ein}}_{1,n-1}$ joining p to q. Suppose that $\gamma \nsubseteq \Omega$. Then, γ intersects the boundary of Ω, reduced in this case to the graph of f^+, in a point $r = \big(x, f^+(x)\big)$ where $x \in U$. By Lemma 3.10, $J^+(r)$ is disjoint from Ω. Then, the segment of γ joining r to q is contained in $J^+(r)$. Thus, $q \notin \Omega$. Contradiction.

Suppose now that f^+ and f^- are finite. If ∂U is empty, the proof is similar to the previous case. Otherwise, we call f the common extension of f^+ and f^- to ∂U. In this case, the boundary of Ω is the union of the graphs of f^+, f^- and f. By Lemma 3.10, the points of Ω are not causally related to any point in the graph of f. Therefore, the previous arguments still hold. □

Now, we describe Cauchy hypersurfaces of causally convex open subsets of $\widetilde{\mathrm{Ein}}_{1,n-1}$. Let

$$\Omega \coloneqq \big\{(x,t) \in U \times \mathbb{R} \ : \ f^-(x) < t < f^+(x)\big\}$$

be a causally convex open subset of $\widetilde{\mathrm{Ein}}_{1,n-1}$ where U is an open subset of $\mathbb{S}^{n-1}$ and f^+, f^- are the functions from U to $\overline{\mathbb{R}}$ given by Proposition 3.6.

Proposition 3.11. — *Let h be a 1-Lipschitz real-valued function defined on U such that its extension to ∂U coincides with that of f^+ and*

f^- and $f^- < h < f^+$ on U. Then, the graph of h is a topological Cauchy hypersurface of Ω.

Proposition 3.11 is a consequence of the following lemma.

Lemma 3.12. — *Suppose f^+ and f^- are finite. Then, every inextensible timelike curve of $\widetilde{\mathrm{Ein}}_{1,n-1}$ that intersects Ω meets each of the graphs of f^+ and f^-.*

Proof. — Let γ be an inextensible timelike curve of $\widetilde{\mathrm{Ein}}_{1,n-1}$ that intersects Ω. Then, γ intersects the boundary of Ω. If ∂U is empty, $\partial\Omega$ is the union of the graphs of f^+ and f^-. Otherwise, $\partial\Omega$ is the union of the graphs of f^+, f^+ and f where f is the common extension of f^+ and f^- to ∂U. Since the points of Ω are not causally related to any point of the graph of f (see Lemma 3.10), we deduce that in both cases, γ intersects the graph of f^+ or the graph of f^-.

Suppose γ meets the graph of f^+. Since Ω is not past-complete, γ leaves Ω and so intersects again the its boundary. Since the graph of f^+ is achronal, γ could not intersects the graph of f^+ a second time. Thus, γ intersects the graph of f^-. □

Proof of Proposition 3.11. — Let γ be an inextensible timelike curve of Ω. Since Ω is causally convex, γ is the intersection of Ω with an inextensible timelike curve $\widetilde{\gamma}$ of $\widetilde{\mathrm{Ein}}_{1,n-1}$. By Lemma 3.12, $\widetilde{\gamma}$ meets the graph of h. Moreover, since the graph of h is achronal, $\widetilde{\gamma}$ intersects it exactly once. The proposition follows from [9, Definition 28 and Lemma 29, p. 415]. □

3.6. Duality

In this section, we highlight a particular class of causally convex open subsets of $\widetilde{\mathrm{Ein}}_{1,n-1}$ involving a notion of *duality* in Einstein universe.

3.6.1. Duality in the Klein model

Recall that a subset of $\mathbb{S}(\mathbb{R}^{2,n})$ is said to be *convex* if it is the projectivization of a convex subset of $\mathbb{R}^{2,n}$. The convex hull of a subset A of $\mathbb{S}(\mathbb{R}^{2,n})$ is the smallest convex containing A.

Let $\Lambda \subset \mathrm{Ein}_{1,n-1}$. Let us denote $\mathrm{Conv}(\Lambda)$ the convex hull of Λ in $\mathbb{S}(\mathbb{R}^{2,n})$. The *dual convex cone* of Λ in $\mathbb{S}(\mathbb{R}^{n+2})$ is

$$\mathrm{Conv}^*(\Lambda) = \big\{ \mathrm{x} \in \mathbb{S}(\mathbb{R}^{2,n}) \ : < \mathrm{x}, \mathrm{y} >_{2,n} < 0 \ \ \forall\, \mathrm{y} \in \mathrm{Conv}(\Lambda) \big\}.$$

Definition 3.13. — *We call* dual of Λ *the intersection of* $\mathrm{Ein}_{1,n-1}$ *with the dual cone* $\mathrm{Conv}^*(\Lambda)$.

Notice that

$$\mathrm{Conv}^*(\Lambda) \cap \mathrm{Ein}_{1,n-1} = \left\{\mathrm{x} \in \mathrm{Ein}_{1,n-1} \ :< \mathrm{x}, \mathrm{y} >_{2,n} < 0 \ \ \forall\, \mathrm{y} \in \Lambda\right\}.$$

3.6.2. Duality in the universal cover

Let $\pi\colon \widetilde{\mathrm{Ein}}_{1,n-1} \to \mathrm{Ein}_{1,n-1}$ be the universal covering map.

Lemma 3.14. — *Let* $\Lambda \subset \widetilde{\mathrm{Ein}}_{1,n-1}$*. The restriction of the projection* π *to the set of points which are non-causally related to any point of* Λ *is injective. Furthermore, its image is contained in the dual of the projection of* Λ *in* $\mathrm{Ein}_{1,n-1}$*. If in addition,* Λ *is acausal, we have equality.*

Proof. — Set $\Omega := \widetilde{\mathrm{Ein}}_{1,n-1} \backslash \big(J^+(\Lambda) \cup J^-(\Lambda)\big)$. By definition, Ω is the intersection of the affine charts $\mathrm{Mink}_0(p)$ where $p \in \Lambda$. As a result, the restriction of π to Ω is injective and its image is contained in the dual of $\pi(\Lambda)$ (see [12, Corollary 2.43]).

Suppose Λ is acausal. Then, $\pi(\Lambda)$ is *negative*, i.e. for every $\mathrm{x}, \mathrm{y} \in \pi(\Lambda)$, we have $\langle x, y\rangle_{2,n} < 0$ for every representatives $x, y \in \mathbb{R}^{2,n}$ of x, y (see [1, Lemma 10.13]).

Let $\mathrm{x} \in \mathrm{Ein}_{1,n-1} \cap \mathrm{Conv}^*\big(\pi(\Lambda)\big)$. Set $\widehat{\Lambda_0} := \{x\} \cup \pi(\Lambda)$. By definition, $\widehat{\Lambda_0}$ is a negative subset of $\mathrm{Ein}_{1,n-1}$. By [12, Proposition 2.47], there exists an acausal subset Λ_0 of $\widetilde{\mathrm{Ein}}_{1,n-1}$ which projects on $\widehat{\Lambda_0}$. Furthermore, the proof of [12, Proposition 2.47] shows that we can choose such a Λ_0 such that it contains Λ. As a consequence, Λ_0 is the union of a lift p of x and Λ. Since Λ_0 is acausal, p is non-causally related to any point of Λ. The lemma follows. □

Lemma 3.14 motivates the following definition.

Definition 3.15. — *Let* Λ *be a subset of* $\widetilde{\mathrm{Ein}}_{1,n-1}$*. We call* dual *of* Λ*, denoted by* Λ°*, the set of points which are non-causally related to any point of* Λ*.*

Lemma 3.16. — *Let* Λ *be a subset of* $\widetilde{\mathrm{Ein}}_{1,n-1}$ *such that its dual is non-empty. Then, the dual of* Λ *is causally convex. If in addition* Λ *is closed, its dual is open.*

Proof. — Let p and q be two points in the dual Λ°, joined by a causal curve $\gamma\colon I \subset \mathbb{R} \to M$. Suppose there is $t \in I$ such that $\gamma(t) \notin \Lambda^\circ$, in other words $\gamma(t)$ is causally related to a point $\lambda \in \Lambda$. By transitivity, it follows that p or q is causally related to λ. Contradiction.

Suppose Λ is closed. If Λ is not compact, it would contain a causal curve inextensible in the future or in the past. Then, $J^+(\Lambda) \cup J^-(\Lambda)$ would be the whole space $\mathrm{Ein}_{1,n-1}$ and Λ° would be empty. Therefore, Λ is compact. It follows that $J^\pm(\Lambda)$ is closed. Hence, Λ° is open. ☐

Now, we characterize duals of achronal closed subsets of $\widetilde{\mathrm{Ein}}_{1,n-1}$ in a spatio-temporal decomposition $\mathbb{S}^{n-1} \times \mathbb{R}$.

Let Λ be a closed achronal subset of $\widetilde{\mathrm{Ein}}_{1,n-1}$. It is the graph of a 1-Lipschitz real-valued function f defined on a closed subset Λ_0 of the sphere $\mathbb{S}^{n-1}$. Let f^+, f^- be the real-valued functions defined for every $x \in \mathbb{S}^{n-1}$ by:

$$f^+(x) = \inf_{x_0 \in \Lambda_0} \big\{ f(x_0) + d(x, x_0) \big\},$$
$$f^-(x) = \sup_{x_0 \in \Lambda_0} \big\{ f(x_0) - d(x, x_0) \big\}.$$

Notice that $f^-(x) < f^+(x)$ for every $x \in \mathbb{S}^{n-1} \backslash \Lambda_0$ and that f^+ and f^- are equal to f on Λ_0.

Proposition 3.17. — *The dual of* Λ *is the set of points* (x, t) *of* $\widetilde{\mathrm{Ein}}_{1,n-1}$ *such that* $f^-(x) < t < f^+(x)$.

Proof. — Let $(x, t) \in \mathbb{S}^{n-1} \times \mathbb{R}$ be a point in the dual of Λ. By definition, (x, t) is non-causally related to any point $\big(x_0, f(x_0)\big)$ where $x_0 \in \Lambda_0$. In other words, for every $x_0 \in \Lambda_0$, we have $d(x, x_0) > \big|t - f(x_0)\big|$, i.e. $f(x_0) - t < d(x, x_0) < f(x_0) + t$. Hence,

$$\sup_{x_0 \in \Lambda_0} \big\{ f(x_0) - d(x, x_0) \big\} \leqslant t \leqslant \inf_{x_0 \in \Lambda_0} \big\{ f(x_0) + d(x, x_0) \big\}.$$

Since Λ_0 is compact, the supremum and infimum above are attained; the previous inequalities are then strict. Thus, we obtain $f^-(x) < t < f^+(x)$. The converse inclusion is clear. ☐

4. Enveloping space of a simply connected GH conformally flat spacetime

4.1. Conformally flat spacetimes

A spacetime is called *conformally flat* if it is locally conformal to Minkowski spacetime. Einstein universe is conformally flat since any point of Einstein universe admits a neighborhood conformally equivalent to Minkowski spacetime (see Section 3.3). It follows that any spacetime locally modeled on Einstein universe, i.e. equipped with a $\big(O(2,n), \mathrm{Ein}_{1,n-1}\big)$-structure, is conformally flat. Conversely, by Theorem 3.5, any conformally flat spacetime of dimension $n \geqslant 3$ admits a $\big(O(2,n), \mathrm{Ein}_{1,n-1}\big)$-structure. We deduce the following statement.

Proposition 4.1. — *A conformally flat Lorentzian structure on a manifold M of dimension $n \geqslant 3$ is equivalent to a $\big(O_0(2,n), \mathrm{Ein}_{1,n-1}\big)$-structure.*

From the causal point of view, it is more relevant to consider as model space the universal cover of Einstein universe with its group of conformal diffeomorphisms. As a result, a conformally flat Lorentzian structure on a manifold M of dimension $n \geqslant 3$ is encoded by the data of a development pair (D, ρ) where $D\colon \widetilde{M} \to \widetilde{\mathrm{Ein}}_{1,n-1}$ is a developing map and $\rho\colon \pi_1(M) \to \mathrm{Conf}(\widetilde{\mathrm{Ein}}_{1,n-1})$ is the associated holonomy morphism.[3]

Lemma 4.2. — *The restriction of the developing map D to a causal curve of $\widetilde{M}$ is injective.*

Proof. — Let $\gamma\colon I \subset \mathbb{R} \to \widetilde{M}$ be a causal curve. Since D is conformal, it follows that $D \circ \gamma\colon I \to \widetilde{\mathrm{Ein}}_{1,n-1}$ is a causal curve. Then, if there exist $t_0, t_1 \in I$ such that $t_0 \neq t_1$ and $D\big(\gamma(t_0)\big) = D\big(\gamma(t_1)\big)$, the curve $D \circ \gamma$ would be a causal loop. Contradiction. □

Definition 4.3. — *A conformally flat spacetime M is said* developable *if any developing map descends to the quotient, giving a local diffeomorphism from M to $\widetilde{\mathrm{Ein}}_{1,n-1}$, called again developing map.*

Lemma 4.4. — *Any developable conformally flat spacetime M is* strongly causal.

Proof. — Let $D\colon M \to \widetilde{\mathrm{Ein}}_{1,n-1}$ be a developing map. Let $p \in M$ and let U be a neighborhood of p. Without loss of generality, we suppose that the restriction of D to U is a diffeomorphism on its image. Then, $D(U)$ is a neighborhood of $D(p)$. Since $\widetilde{\mathrm{Ein}}_{1,n-1}$ is GH, it is in particular strongly causal. Thus, there exists a neighborhood V' of $D(p)$ contained in $D(U)$ and causally convex in $\widetilde{\mathrm{Ein}}_{1,n-1}$. Let V be the preimage of V' under $D\big|_U$. By definition, V is a neighborhood of p contained in U. Moreover, V is causally convex in M. Indeed, let γ be a causal curve of M joining two points $q, q' \in V$. By Lemma 4.2, the image under D of γ is a causal curve of $\widetilde{\mathrm{Ein}}_{1,n-1}$ joining $D(q)$ to $D(q')$. Since V' is causally convex, $D(\gamma)$ is contained in V. If γ is not contained in V, there exists $r \in \gamma \cap (U \backslash V)$. Hence, $D(r) \in D(\gamma) \backslash V'$. Contradiction. □

[3] We direct the reader not familiar with (G, X)-structures to [8, Chapter 5].

4.2. Construction of an enveloping space

Let V be a simply-connected globally hyperbolic conformally flat spacetime of dimension $n \geqslant 3$. In this section, we prove Theorem 1.1. Our proof still hold if we weaken the assumption *simply-connected* by *developable*.

Let (D, φ) be a pair where:

- $D\colon V \to \widetilde{\mathrm{Ein}}_{1,n-1}$ is a developing map;
- $\varphi\colon \widetilde{\mathrm{Ein}}_{1,n-1} \to \mathbb{S}^{n-1} \times \mathbb{R}$ is a spatio-temporal decomposition of $\widetilde{\mathrm{Ein}}_{1,n-1}$.

Throughout this section, we call π the projection of $\widetilde{\mathrm{Ein}}_{1,n-1}$ on $\mathbb{S}^{n-1}$ defined as $\pi_0 \circ \varphi$ where $\pi_0\colon \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{S}^{n-1}$ is the projection on the first factor.

4.2.1. Timelike foliation on V

Consider the vector field T on V defined as the pull-back by D of ∂_t. The flow of T defines a foliation of V by smooth timelike curves. Let $\mathcal{B}$ be the leaf space, namely the quotient space of V by the equivalence relation that identifies two points if they are on the same leaf. We denote by $\psi\colon V \to \mathcal{B}$ the canonical projection.

Fact 4.5. — *The leaf space $\mathcal{B}$ is homeomorphic to a Cauchy hypersurface S of V.*

Proof. — Every leaf is a timelike curve of V and so meets S in a unique point. Therefore, the restriction of ψ to S is a continuous bijection on $\mathcal{B}$. The restriction of ψ to S is open. Indeed, any open subset U of S coincides with $\psi\big|_S^{-1}\big(\psi\big|_S(U)\big)$, so $\psi\big|_S(U)$ is open in $\mathcal{B}$. Then, the restriction of ψ to S is a homeomorphism on $\mathcal{B}$. □

By definition, the map $\pi \circ D$ is constant on each leaf, therefore it induces a map d from to $\mathcal{B}$ to $\mathbb{S}^{n-1}$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{D} & \widetilde{\mathrm{Ein}}_{1,n-1} \\ \psi\big\downarrow & & \big\downarrow\pi \\ \mathcal{B} & \xrightarrow{d} & \mathbb{S}^{n-1} \end{array}$$

that is, $d \circ \psi = \pi \circ D$. Since $\pi \circ D$ and ψ are submersions, d is a local homeomorphism.

4.2.2. Fiber bundle over the leaf space

Let $E(V)$ be the fiber bundle over $\mathcal{B}$ defined as the pullback by $d\colon \mathcal{B} \to \mathbb{S}^{n-1}$ of the trivial bundle $\pi\colon \widetilde{\mathrm{Ein}}_{1,n-1} \to \mathbb{S}^{n-1}$, in other words:

$$E(V) := \left\{(p,b) \in \widetilde{\mathrm{Ein}}_{1,n-1} \times \mathcal{B} \ : \ \pi(p) = d(b)\right\}.$$

We denote by $\widehat{\pi}\colon E(V) \to \mathcal{B}$ the projection on the second factor.

Fact 4.6. — *The fiber bundle $E(V)$ is trivial.*

Proof. — Let f be the continuous map from $\mathcal{B} \times \mathbb{R}$ in $E(V)$ that sends (b,t) on (p,b) where p is the point of $\widetilde{\mathrm{Ein}}_{1,n-1}$ with coordinates $\big(d(b),t\big)$ in the decomposition $\mathbb{S}^{n-1} \times \mathbb{R}$. It is easy to see that f is bijective. Indeed, the inverse is the continuous map that sends $(p,b) \in E(V)$ on $(b,t) \in \mathcal{B} \times \mathbb{R}$ where t is the projection of $p \in \widetilde{\mathrm{Ein}}_{1,n-1} \simeq \mathbb{S}^{n-1} \times \mathbb{R}$ on $\mathbb{R}$. Therefore, f is a homeomorphism. Clearly, the following diagram commutes

$$\begin{array}{ccc} \mathbb{R} \times \mathcal{B} & \xrightarrow{\ f\ } & E(V) \\ & {\scriptstyle \widehat{\pi}_0} \searrow \quad \swarrow {\scriptstyle \widehat{\pi}} & \\ & \mathcal{B} & \end{array}$$

i.e. $\widehat{\pi} \circ f = \widehat{\pi}_0$. In other words, f is an isomorphism of fiber bundles. The lemma follows. □

The projection on the first factor $\widehat{D}\colon E(V) \to \widetilde{\mathrm{Ein}}_{1,n-1}$ is a local homeomorphism inducing a structure of conformally flat spacetime on $E(V)$. In particular, $E(V)$ is strongly causal (see Lemma 4.4).

Fact 4.7. — *The fibers of $E(V)$ are inextensible timelike curves.*

Proof. — The fiber E_b of $E(V)$ over a point $b \in \mathcal{B}$ is the set of points (p,b) such that $\pi(p) = d(b)$. It is then easy to see that the restriction of $\widehat{D}$ to E_b is a homeomorphism on the fiber $\pi^{-1}\big(d(b)\big)$ of $\pi\colon \widetilde{\mathrm{Ein}}_{1,n-1} \to \mathbb{S}^{n-1}$. The lemma follows. □

4.2.3. Conformal embedding of V in $E(V)$

Let i be the map from V to $E(V)$ defined by

$$i(p) := \big(D(p), \psi(p)\big)$$

for every $p \in V$.

Fact 4.8. — *The map i is a conformal embedding of V into $E(V)$.*

Proof. — Since the maps D and ψ are continuous, open and conformal, so does the map i. All we need to check is that i is injective. Let $p, q \in V$ such that $i(p) = i(q)$. Then, $D(p) = D(q)$ and $\psi(p) = \psi(q)$. This last equality implies that p and q belongs to the same timelike leaf. Since the restriction of D to a leaf is injective (see Lemma 4.2), it follows from $D(p) = D(q)$ that $p = q$. □

Remark 4.9. — The restriction of $\widehat{D}$ to $i(M)$ coincides with D, more precisely $\widehat{D} \circ i = D$.

Now we prove that the image of i is causally convex in $E(V)$. The proof uses the following lemma.

Lemma 4.10. — *The image under i of a Cauchy hypersurface S of V is a spacelike hypersurface of $E(V)$ which disconnects $E(V)$.*

Proof. — Since i is a conformal embedding, $i(S)$ is a spacelike embedded hypersurface of $E(V)$. Let $\psi\colon V \to \mathcal{B}$ be the canonical projection of V on the leaf space $\mathcal{B}$. Recall that the restriction of ψ to S is a homeomorphism on $\mathcal{B}$. Clearly, the map $i\big|_S \circ \psi\big|_S^{-1}\colon \mathcal{B} \to E(V)$ is a section of $\widehat{\pi}$. Hence, $i(S)$ is a global section of $E(V)$. The lemma follows. □

Fact 4.11. — *The image $i(V)$ is causally convex in $E(V)$.*

Proof. — Let $p_0, p_1 \in V$ such that there is a causal curve γ of $E(V)$ joining p_0 to p_1. Let $\widehat{\gamma}$ be an inextensible causal curve of $E(V)$ containing γ. Each connected component of the intersection of $\widehat{\gamma}$ with $i(V)$ is an inextensible causal curve of $i(V)$. We call γ_0 and γ_1 the connected components containing p_0 and p_1 respectively. To prove that γ is contained in $i(V)$ is equivalent to prove that $\gamma_0 = \gamma_1$. Suppose that γ_0 and γ_1 are disjoint. Then, each one of them meets $i(S)$ in a single point, x_0 and x_1 respectively, which are distinct. Therefore, the curve $\widehat{\gamma}$ intersects $i(S)$ in at least two distinct points. But, $i(S)$ is acausal in $E(M)$ (see Lemma 4.10 and [9, Chapter 14, Lemmas 45 and 42]). Contradiction. □

Remark 4.12. — All the results of this section stated until now are based on the existence of a developing map, and so are still valid if V is not-simply connected but developable.

4.2.4. Embedding of the conformally flat Cauchy extensions of V in $E(V)$

Proposition 4.13. — *Let W be a conformally flat Cauchy extension of V. Then, there is a conformal embedding i' of W into $E(V)$ such that the following assertions hold:*

- *The image $i'(W)$ is causally convex in $E(V)$ and contains $i(V)$;*
- *Every Cauchy hypersurface of $i(V)$ is a Cauchy hypersurface of $i'(W)$.*

The proof of Proposition 4.13 uses the following lemma.

LEMMA 4.14. — *Let W be a conformally flat Cauchy extension of V. Then, the enveloping spaces $E(V)$ and $E(W)$ are isomorphic.*

Proof. — Let $f\colon V \to W$ be a Cauchy embedding and let $D'\colon W \to \widetilde{\mathrm{Ein}}_{1,n-1}$ be a developing map such that $D' \circ f = D$. Consider the foliation of W by inextensible timelike curves induced by the pull-back by D' of the vector field ∂_t on $\widetilde{\mathrm{Ein}}_{1,n-1} \simeq \mathbb{S}^{n-1} \times \mathbb{R}$ and let $\psi'\colon W \to \mathcal{B}'$ be the canonical projection on the leaf space. Since f is a Cauchy embedding, $f(V)$ is causally convex in W (see [11, Lemma 8]). Then, the image under f of every leaf of V is the intersection of a unique leaf of W with $f(V)$. It follows that the map $\psi' \circ f$ descends to the quotient in a diffeomorphism $\bar{f}\colon \mathcal{B} \to \mathcal{B}'$. Let $d'\colon \mathcal{B}' \to \mathbb{S}^{n-1}$ be the developing map induced by D'. It is clear that $d \coloneqq d' \circ \bar{f}$ is the developing map induced by D. Therefore, the map $F\colon E(V) \to E(W)$ defined by $F(p,b) = \big(p, \bar{f}(p)\big)$ is a conformal diffeomorphism that sends every fiber of $E(V)$ on a fiber of $E(W)$. The lemma follows. ☐

Proof of Proposition 4.13. — Let $j : W \hookrightarrow E(W)$ be the conformal embedding of W into $E(W)$ and let $F\colon E(V) \to E(W)$ be the isomorphism defined in the proof of Lemma 4.14. The map $i' \coloneqq F^{-1} \circ j$ defines a conformal embedding of W into $E(V)$. By Lemma 4.11, $i(V)$ and $i'(W)$ are causally convex in $E(V)$. Moreover, according to the proof of Lemma 4.14, the following diagram commutes:

$$\begin{array}{ccc} V & \overset{i}{\hookrightarrow} & E(V) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ W & \overset{j}{\hookrightarrow} & E(W) \end{array}$$

that is $F \circ i = j \circ f$, i.e. $i = F^{-1} \circ j \circ f = i' \circ f$. It follows that:

- $i(V) = i'\big(f(V)\big) \subset i'(W)$.
- Every Cauchy hypersurface of $i(V)$ is a Cauchy hypersurface of $i'(W)$. Indeed, since f is a Cauchy embedding, if S is a Cauchy hypersurface of V, $f(S)$ is a Cauchy hypersurface of W. Then, $i(S) = i'\big(f(S)\big)$ is a Cauchy hypersurface of $i'(W)$.

The proposition follows. ☐

Remark 4.15. — If V is developable, it is easy to see that any conformally flat Cauchy extension of W is also developable. As a result, Proposition 4.13 is still true in this setting.

4.2.5. The $\mathcal{C}_0$-maximal extension of V

From now on, we identify V and the conformally flat Cauchy extensions of V with their images in $E(V)$. Let S be a Cauchy hypersurface of V.

LEMMA 4.16. — *The Cauchy development of S in $E(V)$ contains all the Cauchy extensions of V. In particular, it contains V.*

Proof. — Let W be a Cauchy extension of V. Let $x \in W$ and let $\widehat{\gamma}$ be an inextensible causal curve of $E(V)$ going through x. Since W is causally convex in $E(V)$ (see Proposition 4.13), the intersection of $\widehat{\gamma}$ with W is an inextensible causal curve γ of W. By Proposition 4.13, S is a Cauchy hypersurface of W, then γ intersects S in a single point. It follows that x belongs to the Cauchy development of S in $E(V)$. □

PROPOSITION 4.17. — *The Cauchy development $\mathcal{C}(S)$ of S in $E(V)$ is a $\mathcal{C}_0$-maximal extension of V.*

Proof. — By [9, Theorem 38, p. 421], $\mathcal{C}(S)$ is a globally hyperbolic spacetime for which S is a Cauchy hypersurface. According to Lemma 4.16, it is a Cauchy extension of V. Let W be a conformally flat Cauchy extension of $\mathcal{C}(S)$. In particular, W is a Cauchy extension of V. Then, W embeds conformally in $E(V)$ and the image is a causally convex open subset of $E(V)$ containing $\mathcal{C}(S)$. By Lemma 4.16, $\mathcal{C}(S)$ is exactly W (seen in $E(V)$). Hence, $\mathcal{C}(S)$ is $\mathcal{C}_0$-maximal. □

COROLLARY 4.18. — *The $\mathcal{C}_0$-maximal extension $\mathcal{C}(S)$ is unique up to conformal diffeomorphism.*

Proof. — Let $\widehat{V}$ another $\mathcal{C}_0$-maximal extension of V. By Lemma 4.16, $\widehat{V}$, seen in $E(V)$, is contained in $\mathcal{C}(S)$. If this inclusion is strict, $\mathcal{C}(S)$ would be a Cauchy extension of $\widehat{V}$. This contradicts the $\mathcal{C}_0$-maximality of $\widehat{V}$. □

We proved again the existence and the uniqueness of the $\mathcal{C}_0$-maximal extension for *simply-connected* conformally flat globally hyperbolic flat spacetimes.

COROLLARY 4.19. — *If V is simply-connected and Cauchy-compact then the $\mathcal{C}_0$-maximal extension of V is conformally equivalent to* $\widetilde{\mathrm{Ein}}_{1,n-1}$.

Proof. — By Lemma 4.5, the leaf space $\mathcal{B}$ is compact. Since d is a local homeomorphism, it follows that d is a covering. But $\mathbb{S}^{n-1}$ is simply connected, so d is a homeomorphism. As a result, $\widehat{D}\colon E(V) \to \widetilde{\mathrm{Ein}}_{1,n-1}$ is a conformal diffeomorphism. Indeed, let $p \in \widetilde{\mathrm{Ein}}_{1,n-1}$. There exists a unique $b \in \mathcal{B}$ such that $d(b) = \pi(p)$. Thus, (p, b) is the unique point of $E(V)$ such that $\widehat{D}(p, b) = p$. The corollary follows. □

Definition 4.20. — *The trivial fiber bundle* $\widehat{\pi}\colon E(V) \to \mathcal{B}$ *is called* enveloping space *of* V.

The construction of this fiber bundle depends on the choice of a pair (D, φ) where

- $D\colon V \to \widetilde{\mathrm{Ein}}_{1,n-1}$ is a developing map;
- $\varphi\colon \mathbb{S}^{n-1} \times \mathbb{R} \to \widetilde{\mathrm{Ein}}_{1,n-1}$ is a spatio-temporal decomposition of $\widetilde{\mathrm{Ein}}_{1,n-1}$.

Given two such pairs (D, φ), (D', φ'), there exists a conformal transformation ϕ of $\widetilde{\mathrm{Ein}}_{1,n-1}$ such that $D' = \phi \circ D$. We say that (D, φ) and (D', φ') are *equivalent* if $\varphi' = \varphi \circ \phi^{-1}$.

Lemma 4.21. — *If* (D, φ) *and* (D', φ') *are equivalent, the enveloping spaces* $E(V)$ *and* $E'(V)$ *defined by* (D, φ) *and* (D', φ') *are isomorphic, i.e. there exists a conformal diffeomorphism from* $E(V)$ *to* $E'(V)$ *which sends fiber on fiber.*

Example 4.22 (Enveloping space of Minkowski spacetime). — Minkowski spacetime $\mathbb{R}^{1,n-1}$ is conformally equivalent to the set of points of $\widetilde{\mathrm{Ein}}_{1,n-1}$ which are not causally related to a point $p \in \widetilde{\mathrm{Ein}}_{1,n-1}$. Given a spatio-temporal decomposition $\varphi\colon \widetilde{\mathrm{Ein}}_{1,n-1} \to \mathbb{S}^{n-1} \times \mathbb{R}$, the enveloping space of $\mathbb{R}^{1,n-1}$ is the complement in $\widetilde{\mathrm{Ein}}_{1,n-1}$ of the fiber going through p of the trivial bundle $\pi_0 \circ \varphi\colon \widetilde{\mathrm{Ein}}_{1,n-1} \to \mathbb{S}^{n-1}$. This is a trivial fiber bundle over $\mathbb{S}^{n-1} \backslash \{\pi_0 \circ \varphi(p)\} \simeq \mathbb{R}^{n-1}$.

Example 4.23 (Enveloping space of de Sitter spacetime). — Given a spatio-temporal decomposition $\varphi\colon \widetilde{\mathrm{Ein}}_{1,n-1} \to \mathbb{S}^{n-1} \times \mathbb{R}$, de Sitter spacetime is conformally equivalent to $\mathbb{S}^{n-1} \times]0, \pi[$. Therefore, the enveloping space of de Sitter spacetime relatively to this decomposition is $\pi_0 \circ \varphi\colon \widetilde{\mathrm{Ein}}_{1,n-1} \to \mathbb{S}^{n-1}$ where $\pi_0\colon \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{S}^{n-1}$ is the projection on the first factor.

4.3. Causally convex GH open subsets of the enveloping space

In this section, we describe causally convex open subsets of the enveloping space $E(V)$ — defined in the previous section — which are globally hyperbolic. Notice that since $E(V)$ is, *a priori*, not globally hyperbolic (see Example 4.22), causal convexity does not imply global hyperbolicity anymore.

By [11, Theorem 10], causally convex GH open subsets of $E(V)$ which contain *conjugate points*, i.e. points whose image under $\widehat{D}$ are conjugate in $\widetilde{\mathrm{Ein}}_{1,n-1}$, are conformally equivalent to causally convex open subsets of $\widetilde{\mathrm{Ein}}_{1,n-1}$. These are described in Section 3.5. This is why we only deal here with causally convex open subsets of $E(V)$ without conjugate points. We basically generalize the description of causally convex open subsets of $\widetilde{\mathrm{Ein}}_{1,n-1}$ to causally convex GH open subsets of any enveloping space $E(V)$. As a result, we obtain a description of V and its conformally flat Cauchy extensions within $E(V)$.

The following definition introduces a class of sections of the enveloping space $E(V) \to \mathcal{B}$, expressed in a global trivialization.

Definition 4.24. — *A real-valued function f defined on an open subset U of $\mathcal{B}$ is said* 1-Lipschitz *if for every $x \in U$, there exists an open neighborhood U_x of x contained in U such that the following hold:*

(1) *the restriction of d to U_x is injective;*
(2) *the map $f \circ d\big|_{U_x}^{-1} \colon d(U_x) \subset \mathbb{S}^{n-1} \to \mathbb{R}$ is 1-Lipschitz.*

This definition generalizes the (local) notion of 1-Lipschitz functions defined on an open subset of the sphere $\mathbb{S}^{n-1}$.

Remark 4.25. — The function f is 1-Lipschitz if and only if its graph is *locally achronal.*

Lemma 4.26. — *Any 1-Lipschitz real-valued function f defined on an open subset $U \subsetneq \mathcal{B}$ admits a unique extension to $\overline{U}$.*

Proposition 4.27. — *Any causally convex GH open subset Ω of $E(V)$ is the domain bounded by the graphs of two 1-Lipschitz real-valued functions f^+ and f^- defined on an open subset U of $\mathcal{B}$ such that:*

(1) *$f^- < f^+$ on U;*
(2) *the extensions of f^+ and f^- to ∂U coincide.*

Proof. — Let $\widehat{\pi}$ be the natural projection of $E(V)$ on $\mathcal{B}$. We call U the projection of Ω on $\mathcal{B}$. Since Ω is causally convex, the intersection of Ω with any fiber $\widehat{\pi}^{-1}(x)$, where $x \in U$, is connected, i.e. it is a segment $\{x\} \times \left]f^-(x), f^+(x)\right[$. Notice that $-\infty < f^-(x)$ and $f^+(x) < +\infty$, otherwise Ω would contain conjugate points. Contradiction. This defines two real-valued functions f^+, f^- defined on U such that:

$$\Omega = \left\{(x,t) \in U \times \mathbb{R} \,:\, f^-(x) < t < f^+(x)\right\}.$$

Let $x \in U$. Set $p_+ = \big(x, f^+(x)\big)$. Let $p \in I^-(p^+) \cap \Omega$. Since Ω is GH, the restriction of $\widehat{D}$ to $I^+(p,\Omega)$ is injective and its image is causally convex in $\widetilde{\mathrm{Ein}}_{1,n-1}$ (see [10, Proposition 2.7 and Corollary 2.8, p. 151]). It follows that:

- $\widehat{D}\big(I^+(p,\Omega)\big)$ is the domain bounded by the graphs of two 1-Lipschitz real-valued functions $g^- < g^+$ defined on an open subset of $\mathbb{S}^{n-1}$ (see Proposition 3.6);
- d is injective on $U_x := \widehat{\pi}\big(I^+(p,\Omega)\big)$ and $g^+ = f^+ \circ d\big|_{U_x}^{-1}$.

Thus, f^+ is 1-Lipschitz. The proof is similar for f^- with the reverse time-orientation. Lemma 3.8 is still valid for causally convex open subsets of $E(V)$. Hence, the same arguments used in the proof of Proposition 3.6 show that the extensions of f^+ and f^- to ∂U coincide. □

Corollary 4.28. — *The graphs of f^+ and f^- are achronal.*

Proof. — The proof is similar to that of the second fact in the proof of Proposition 3.6. □

It follows immediately from Proposition 4.27 that V (resp. any conformally flat Cauchy extension of V) is the domain bounded between the graphs of two real-valued 1-Lipschitz functions f^+ and f^- defined on $\mathcal{B}$ such that $f^- < f^+$.

5. $\mathcal{C}_0$-maximal extensions of conformally flat globally hyperbolic spacetimes

In this section, we give a new proof of the existence and the uniqueness of the $\mathcal{C}_0$-maximal extension of a globally hyperbolic conformally flat spacetime V of dimension $n \geqslant 3$, using the notion of enveloping space introduced in Section 4.

When V is simply-connected, the proof is given in Section 4.2 (see Proposition 4.17 and Corollary 4.18). We deal here with the case where V is not simply-connected. The proof consists to extend the action of $\pi_1 V$ on $\widetilde{V}$ to a proper action on the $\mathcal{C}_0$-maximal extension of $\widetilde{V}$. We prove then that the

$\mathcal{C}_0$-maximal extension of V is the quotient of the $\mathcal{C}_0$-maximal extension of $\widetilde{V}$ by $\pi_1 V$.

Notations. Let $D\colon \widetilde{V} \to \widetilde{\mathrm{Ein}}_{1,n-1}$ be a developing map and let $\rho\colon \Gamma \to \mathrm{Conf}(\widetilde{\mathrm{Ein}}_{1,n-1})$ be the associated holonomy representation, where $\Gamma := \pi_1(V)$.

Fix a decomposition $\varphi\colon \widetilde{\mathrm{Ein}}_{1,n-1} \to \mathbb{S}^{n-1} \times \mathbb{R}$. Let $E(\widetilde{V})$ be the enveloping space related to the pair (D, φ) (see Section 4.2). We denote by $\widehat{\pi}\colon E(\widetilde{V}) \to \mathcal{B}$ the projection on the second factor and by $\widehat{D}\colon E(\widetilde{V}) \to \widetilde{\mathrm{Ein}}_{1,n-1}$ the projection on the first factor.

Fact 5.1. — *If $\widetilde{V}$ is Cauchy-compact, the $\mathcal{C}_0$-maximal extension of V is a finite quotient of $\widetilde{\mathrm{Ein}}_{1,n-1}$.*

Proof. — By Corollary 4.19, the $\mathcal{C}_0$-maximal extension of $\widetilde{V}$ is $\widetilde{\mathrm{Ein}}_{1,n-1}$. Moreover, the proof of Corollary 4.19 shows that the developing map

$$D\colon \widetilde{V} \longrightarrow \widetilde{\mathrm{Ein}}_{1,n-1}$$

is injective. Consequently, the holonomy $\rho\colon \Gamma \to \mathrm{Conf}(\widetilde{\mathrm{Ein}}_{1,n-1})$ is also injective and V is the quotient of $\widetilde{\mathrm{Ein}}_{1,n-1}/\rho(\Gamma)$. Since $\rho(\Gamma)$ preserves a compact Cauchy hypersurface of $\widetilde{\mathrm{Ein}}_{1,n-1}$, we deduce that it is finite. The fact follows. □

Suppose now that $\widetilde{V}$ is not Cauchy-compact. Let $\widetilde{S}$ be a non-compact Cauchy hypersurface of $\widetilde{V}$. The $\mathcal{C}_0$-maximal extension of $\widetilde{V}$ is the Cauchy development $\mathcal{C}(\widetilde{S})$ of $\widetilde{S}$ in $E(\widetilde{V})$ (see Proposition 4.17). In what follows, we extend the action of Γ to $\mathcal{C}(\widetilde{S})$.

5.1. Action of Γ on $\mathcal{C}(\widetilde{S})$

The points of $\mathcal{C}(\widetilde{S})$ are characterized by their shadows on $\widetilde{S}$ (see Proposition 2.8). As a result, we show that the action of Γ on $\widetilde{S}$ induces naturally an action of Γ on $\mathcal{C}(\widetilde{S})$.

Proposition 5.2. — *Let $p \in \mathcal{C}(\widetilde{S})$ and let $\gamma \in \Gamma$. There exists a unique point $\gamma.p$ in $\mathcal{C}(\widetilde{S})$ such that its shadow on $\widetilde{S}$ is exactly the image under γ of the shadow of p on $\widetilde{S}$. This defines an action of Γ on $\mathcal{C}(\widetilde{S})$ which satisfies the following properties:*

- *the restriction of the action of Γ on $\mathcal{C}(\widetilde{S})$ to $\widetilde{V}$ coincides with the usual action of Γ on $\widetilde{V}$;*

- *the action of* Γ *on* $\mathcal{C}(\widetilde{S})$ *preserves the causality relations, i.e. for every* $p \in \mathcal{C}(\widetilde{S})$ *and for every* $\gamma \in \Gamma$, *we have:*

$$p \in J^-(q) \iff \gamma.p \in J^-(\gamma.q),$$
$$p \in I^-(q) \iff \gamma.p \in I^-(\gamma.q);$$

- *the restriction of* $\widehat{D}$ *to* $\mathcal{C}(\widetilde{S})$ *is* ρ*-equivariant, i.e. for every* $p \in \mathcal{C}(\widetilde{S})$ *and for every* $\gamma \in \Gamma$, *we have:*

$$\widehat{D}(\gamma.p) = \rho(\gamma)\widehat{D}(p).$$

We prove Proposition 5.2 by an analysis-synthesis reasoning. In the analysis, we suppose that $\gamma.p$ exists and is unique, and we look for properties satisfied by $\gamma.p$ that will characterize it. In the synthesis, we use the criteria found in the analysis to determine the point $\gamma.p$. In what follows, the shadow of a point $p \in \mathcal{C}(\widetilde{S})$ on $\widetilde{S}$ is denoted by $O(p)$.

Analysis. Suppose that for every $\gamma \in \Gamma$ and every $p \in \mathcal{C}(\widetilde{S})$, the point $\gamma.p$ exists and is unique. For every $\gamma \in \Gamma$ and every $p \in \widetilde{V}$, we denote by γp (without the dot between γ and p) the usual action of the deck transformation γ on p. Let us start with this easy remark.

Remark 5.3. — Let $p \in \mathcal{C}(\widetilde{S})$ and let $\gamma \in \Gamma$. If $p \in \widetilde{V}$, then $\gamma.p = \gamma p$. Indeed, since the action of Γ on $\widetilde{V}$ respect the causality relations, we have $J^-(\gamma p) = \gamma J^-(p)$. Therefore,

$$O(\gamma p) := J^-(\gamma p) \cap \widetilde{S} = \gamma J^-(p) \cap \widetilde{S} = \gamma\big(J^-(p) \cap \gamma^{-1}\widetilde{S}\big).$$

Since $\widetilde{S}$ is Γ-invariant, we deduce that $\gamma\big(J^-(p) \cap \gamma^{-1}\widetilde{S}\big) = \gamma O(p)$. Hence, $O(\gamma p) = \gamma O(p)$, i.e. $\gamma.p = \gamma p$.

Lemma 5.4. — *The map which associates to every* $(\gamma, p) \in \Gamma \times \mathcal{C}(\widetilde{S})$ *the point* $\gamma.p \in \mathcal{C}(\widetilde{S})$ *is a group action. Moreover, this action respects causality relations.*

Proof. — The fact that the map $(\gamma, p) \in \Gamma \times \mathcal{C}(\widetilde{S}) \mapsto \gamma.p$ is a group action follows easily from the fact that the restriction to $\Gamma \times \widetilde{S}$ is the usual group action of Γ on $\widetilde{S}$.

Let $\gamma \in \Gamma$ and let $p, q \in \mathcal{C}(\widetilde{S})$. Suppose that $p, q \in J^+(S)$. Then,

$$\begin{aligned} p \in J^-(q) &\iff O(p) \subset O(q) \\ &\iff O(\gamma.p) := \gamma O(p) \subset \gamma O(q) =: O(\gamma.q) \\ &\iff \gamma.p \in J^-(\gamma.q). \end{aligned}$$

By symmetry, the same arguments still hold if $p, q \in J^-(\widetilde{S})$. It remains the case where $q \in I^+(\widetilde{S})$ and $p \in I^-(\widetilde{S})$. In this case, we have

$$\begin{aligned} p \in J^-(q) \quad &\iff \quad O(p) \cap O(q) \neq \emptyset \\ &\iff \quad \gamma O(p) \cap \gamma O(q) \neq \emptyset \\ &\iff \quad O(\gamma.p) \cap O(\gamma.q) \neq \emptyset. \end{aligned}$$

Since Γ preserves $\widetilde{S}$, it preserves the chronological future/past of $\widetilde{S}$ in $\mathcal{C}(\widetilde{S})$. Thus, $O(\gamma.p) \cap O(\gamma.q) \neq \emptyset \Leftrightarrow \gamma.p \in J^-(\gamma.q)$. The lemma follows. □

Lemma 5.5. — *The restriction of the developing map $\widehat{D}\colon E(\widetilde{V}) \to \widetilde{\mathrm{Ein}}_{1,n-1}$ to $\mathcal{C}(\widetilde{S})$ is ρ-equivariant, i.e. for every $p \in \mathcal{C}(\widetilde{S})$ and for every $\gamma \in \Gamma$, we have $\widehat{D}(\gamma.p) = \rho(\gamma)\widehat{D}(p)$.*

Proof. — Let $\gamma \in \Gamma$ and let $p \in \mathcal{C}(\widetilde{S})$. Suppose that $p \in I^+(\widetilde{S})$. Let $q \in \mathcal{C}(\widetilde{S}) \cap I^+(p)$. Since $\mathcal{C}(\widetilde{S})$ is globally hyperbolic, the restriction of $\widehat{D}$ to $I^-\big(q, \mathcal{C}(\widetilde{S})\big)$ is injective and its image is causally convex in $\widetilde{\mathrm{Ein}}_{1,n-1}$ (see [10, Proposition 2.7 and Corollary 2.8, p. 151]). It follows that $D\big(I^-(q) \cap \widetilde{S}\big)$ is an achronal hypersurface of $\widehat{D}\big(I^-\big(q, \mathcal{C}(\widetilde{S})\big)\big)$.

Set $\Sigma := D\big(I^-(q) \cap \widetilde{S}\big)$. Since $O(p) \subset I^-\big(q, \mathcal{C}(\widetilde{S})\big)$, the restriction of D to $O(p)$ is injective and its image is exactly $O\big(\widehat{D}(p), \Sigma\big)$. Thus

$$\begin{aligned} D\big(\gamma O(p)\big) &= \rho(\gamma) D\big(O(p)\big) \\ &= \rho(\gamma) O\big(\widehat{D}(p), \Sigma\big) \\ &= O\big(\rho(\gamma)\widehat{D}(p), \rho(\gamma)\Sigma\big). \end{aligned} \tag{5.1}$$

By definition, $\gamma O(p) = O(\gamma.p)$. From Lemma 5.4, we get $O(\gamma.p) \subset I^-\big(\gamma.q, \mathcal{C}(\widetilde{S})\big)$. As above, we deduce that the restriction of D to $O(\gamma.p)$ is injective and its image is exactly $O\big(\widehat{D}(\gamma.p), D\big(I^-(\gamma.p) \cap \widetilde{S}\big)\big)$. But, $D\big(I^-(\gamma.p) \cap \widetilde{S}\big) = \rho(\gamma)\Sigma$. Indeed, since $O(\gamma.q) = \gamma O(q)$ we have $I^-(\gamma.q) \cap \widetilde{S} = \gamma\big(I^-(q) \cap \widetilde{S}\big)$. Thus, $D\big(I^-(\gamma.q) \cap \widetilde{S}\big) = \rho(\gamma)\Sigma$. Hence,

$$D\big(\gamma O(p)\big) = D\big(O(\gamma.p)\big) = O\big(\widehat{D}(\gamma.p), \rho(\gamma)\Sigma\big). \tag{5.2}$$

It follows from (5.1) and (5.2) that

$$O\big(\rho(\gamma)\widehat{D}(p), \rho(\gamma)\Sigma\big) = O\big(\widehat{D}(\gamma.p), \rho(\gamma)\Sigma\big).$$

By Proposition 2.8, we deduce that $\widehat{D}(\gamma.p) = \rho(\gamma)\widehat{D}(p)$.

If $p \in I^-(\widetilde{S})$, the same arguments hold with the reverse time-orientation. Lastly, if $p \in \widetilde{S}$, the lemma follows from Remark 5.3. □

Remark 5.6. — In the proof of Lemma 5.5, we can replace $I^-\big(q, \mathcal{C}(\widetilde{S})\big)$ by any other causally convex open neighborhood U of p in $\mathcal{C}(\widetilde{S})$ such that:

- U intersects $\widetilde{S}$;
- the restriction of $\widehat{D}$ to U is injective;
- the image $\widehat{D}(U)$ is causally convex in $\widetilde{\mathrm{Ein}}_{1,n-1}$.

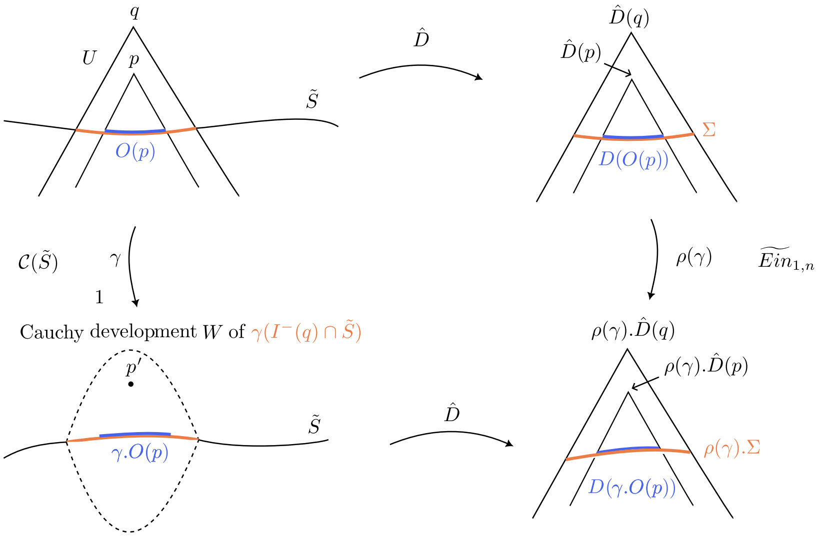

Figure 5.1. Action of Γ *on the Cauchy development of* $\widetilde{S}$ *in the enveloping space* $E(\widetilde{S})$.

Synthesis. Let $p \in \mathcal{C}(\widetilde{S})$ and let $\gamma \in \Gamma$. Without loss of generality, we suppose that $p \in I^+(\widetilde{S})$. The idea is to reconstruct the proof of Lemma 5.5 to determine $\gamma.p$. More precisely, we choose a relevant causally convex open neighborhood U of p in $\mathcal{C}(\widetilde{S})$ satisfying the properties stated in Remark 5.6 and we construct the open subset that will turn out to be $\gamma.U$ and will therefore contain $\gamma.p$ (see Figure 5.1).

Fix $q \in \mathcal{C}(\widetilde{S}) \cap I^+(p)$. Let U be the Cauchy development of $I^-(q) \cap \widetilde{S}$ in $E(\widetilde{V})$.

Lemma 5.7. — *The image under D of $I^-(q) \cap \widetilde{S}$ is achronal in* $\widetilde{\mathrm{Ein}}_{1,n-1}$.

Proof. — We have $I^-(q) \cap \widetilde{S} \subset I^-\big(q, \mathcal{C}(\widetilde{S})\big)$. Since $\mathcal{C}(\widetilde{S})$ is GH, the restriction of $\widehat{D}$ to $I^-\big(q, \mathcal{C}(\widetilde{S})\big)$ is injective (see [10, Proposition 2.7, p. 151]). Then, $D\big(I^-(p) \cap \widetilde{S}\big)$ is achronal in $\widehat{D}\big(I^-\big(q, \mathcal{C}(\widetilde{S})\big)\big)$. This last one being causally convex in $\widetilde{\mathrm{Ein}}_{1,n-1}$ (see [10, Corollary 2.8, p. 151]), we deduce easily that $D\big(I^-(p) \cap \widetilde{S}\big)$ is achronal in $\widetilde{\mathrm{Ein}}_{1,n-1}$. □

Fact 5.8. — *The restriction of $\widehat{D}$ to U is injective. Moreover, the image under $\widehat{D}$ of U is equal to the Cauchy development of $D\big(I^-(q)\cap\widetilde{S}\big)$ in $\widetilde{\mathrm{Ein}}_{1,n-1}$.*

Proof. — Clearly, $U \subset \mathcal{C}(\widetilde{S})\cap I^-(q)$. Since $\mathcal{C}(\widetilde{S})$ is causally convex in $E(\widetilde{V})$, we have $\mathcal{C}(\widetilde{S})\cap I^-(q) = I^-\big(q,\mathcal{C}(\widetilde{S})\big)$. Since the restriction of $\widehat{D}$ to $I^-\big(q,\mathcal{C}(\widetilde{S})\big)$ is injective and its image is causally convex in $\widetilde{\mathrm{Ein}}_{1,n-1}$ (see [10, Proposition 2.7 and Corollary 2.8, p. 151]), the lemma follows. □

Let W be the Cauchy development of $\gamma\big(I^-(q)\cap\widetilde{S}\big)$ in $E(\widetilde{V})$.

Remark 5.9. — The image under D of $\gamma\big(I^-(q)\cap\widetilde{S}\big)$ is achronal in $\widetilde{\mathrm{Ein}}_{1,n-1}$. Indeed, $D\big(\gamma\big(I^-(q)\cap\widetilde{S}\big)\big) = \rho(\gamma)D\big(I^-(q)\cap\widetilde{S}\big)$. The assertion follows then from Lemma 5.7.

Fact 5.10. — *The restriction of $\widehat{D}$ to W is injective. Moreover, the image under $\widehat{D}$ of W is equal to the Cauchy development of $D\big(\gamma\big(I^-(q)\cap\widetilde{S}\big)\big)$ in $\widetilde{\mathrm{Ein}}_{1,n-1}$.*

Proof. — Let $r, r' \in W$ such that $\widehat{D}(r) = \widehat{D}(r')$. Let $E_{\hat{\pi}(r)}$ and $E_{\hat{\pi}(r')}$ be the fibers going through r and r'. By definition, their images under $\widehat{D}$ are the fibers of the trivial bundle $\widetilde{\mathrm{Ein}}_{1,n-1}\to\mathbb{S}^{n-1}$ going through $\widehat{D}(r)$ and $\widehat{D}(r')$. Since $\widehat{D}(r) = \widehat{D}(r')$, these fibers coincide. We call this fiber Δ. Since $E_{\hat{\pi}(r)}$ and $E_{\hat{\pi}(r')}$ are inextensible timelike curves, they intersect $\gamma\big(I^-(q)\cap\widetilde{S}\big)$ in two points r_0 and r_0'. Hence, $D(r_0), D(r_0') \in \Delta\cap D\big(\gamma\big(I^-(q)\cap\widetilde{S}\big)\big)$. But, $D\big(\gamma\big(I^-(q)\cap\widetilde{S}\big)\big)$ is achronal in $\widetilde{\mathrm{Ein}}_{1,n-1}$ (see Remark 5.9). Thus, the timelike line Δ intersects $D\big(\gamma\big(I^-(q)\cap\widetilde{S}\big)\big)$ at most once. Hence, $D(r_0) = D(r_0')$. The restriction of D to $\gamma\big(I^-(q)\cap\widetilde{S}\big)$ being injective, we deduce that $r_0 = r_0'$. Thus, $E_{\hat{\pi}(r)} = E_{\hat{\pi}(r')}$. Since the restriction of $\widehat{D}$ to any causal curve is injective (see Lemma 4.2), we get $r = r'$.

We deduce easily from the injectivity of $\widehat{D}$ on W that $\widehat{D}(W)$ is contained in the Cauchy development of $D\big(\gamma\big(I^-(q)\cap\widetilde{S}\big)\big)$ in $\widetilde{\mathrm{Ein}}_{1,n-1}$. This inclusion is clearly a Cauchy embedding. Since W is $\mathcal{C}_0$-maximal (see Proposition 4.17), we deduce equality. □

Proof of Proposition 5.2. — Set $\Sigma := D\big(I^-(q)\cap\widetilde{S}\big)$. Since $O(p)\subset U$, it follows from Fact 5.8 that $D\big(O(p)\big)$ is equal to the shadow $O\big(\widehat{D}(p),\Sigma\big)$ of $\widehat{D}(p)$ on Σ. Hence

$$D\big(\gamma O(p)\big) = \rho(\gamma)D\big(O(p)\big) = \rho(\gamma)O\big(\widehat{D}(p),\Sigma\big) = O\big(\rho(\gamma)\widehat{D}(p),\rho(\gamma)\Sigma\big).$$

Hence, $O\big(\rho(\gamma)\widehat{D}(p),\rho(\gamma)\Sigma\big)\subset\widehat{D}(W)$. Since $\widehat{D}(W)$ is the Cauchy development of $\rho(\gamma)\Sigma$ (see Fact 5.10), it follows that $\rho(\gamma)\widehat{D}(p)\in\widehat{D}(W)$. Thus, there exists a unique point $p' \in U$ such that $\widehat{D}(p') = \rho(\gamma)\widehat{D}(p)$. It is clear that $O(p') = \gamma O(p)$. If there is another point $p'' \in \mathcal{C}(\widetilde{S})$ such that $O(p'') = \gamma O(p)$, by Proposition 2.8, we get $p' = p''$. Then, we set $\gamma.p := p'$. $\square$

5.1.1. Dynamical properties of the action of Γ on $\mathcal{C}(\widetilde{S})$

In this section, we prove that the action of Γ on $\mathcal{C}(\widetilde{S})$ is free and properly discontinuous.

Lemma 5.11. — *Let p be a point in the complement of $\widetilde{S}$ in $\mathcal{C}(\widetilde{S})$. Then, the shadow of p on $\widetilde{S}$ is a topological disk of dimension $n-1$.*

Proof. — We can suppose without loss of generality that $p \in I^+(\widetilde{S})$. Let $q \in I^+(p)$ and set $\Sigma := D\big(I^-(q) \cap \widetilde{S}\big)$. We prove that $\widehat{D}\big(I^-\big(q, \mathcal{C}(\widetilde{S})\big)\big)$ is contained in the affine chart $\mathrm{Mink}_-\big(\widehat{D}(q)\big)$ (see Section 3.3). Suppose there exists $r \in I^-\big(q, \mathcal{C}(\widetilde{S})\big)$ such that $\widehat{D}(r) \notin \mathrm{Mink}_-\big(\widehat{D}(q)\big)$. Then, $I\big(\widehat{D}(q), \widehat{D}(r)\big)$ contains conjugate points. Since the image under $\widehat{D}$ of $I^-\big(q, \mathcal{C}(\widetilde{S})\big)$ is causally convex in $\widetilde{\mathrm{Ein}}_{1,n-1}$ (see [10, Corollary 2.8, p. 151]), we deduce that $I^-\big(q, \mathcal{C}(\widetilde{S})\big)$ admits a photon whose image under $\widehat{D}$ contains conjugate points. Therefore, by [11, Theorem 10], $\mathcal{C}(\widetilde{S})$ is conformally equivalent to $\widetilde{\mathrm{Ein}}_{1,n-1}$. Contradiction. Then, $\widehat{D}(p)$ and Σ are contained in $\mathrm{Mink}_-\big(\widehat{D}(q)\big)$. As a result, the shadow $O\big(\widehat{D}(p), \Sigma\big)$ is the intersection in Minkowski spacetime of the past causal cone of $\widehat{D}(p)$ with Σ. Thus, the map which associates to every past causal direction at $\widehat{D}(p)$, the intersection of the straight line tangent to this direction with Σ, is a homeomorphism. Hence, $O\big(\widehat{D}(p), \Sigma\big)$ is a topological $(n-1)$-disk. Since the restriction of $\widehat{D}$ to $O(p)$ is a diffeomorphism on $O\big(\widehat{D}(p), \Sigma\big)$ (see [10, Proposition 2.7, p. 151]), the lemma follows. $\square$

Proposition 5.12. — *The action of Γ on $\mathcal{C}(\widetilde{S})$ is free and properly discontinous.*

Proof. — Let $p \in \mathcal{C}(\widetilde{S})$ and let $\gamma \in \Gamma$ such that $\gamma.p = p$. Thus, γ preserves $O(p)$. Since $O(p)$ is a topological disk (see Lemma 5.11), by Brouwer's theorem, γ admits a fixed point in $O(p)$. Since the action of Γ on $\widetilde{S}$ is free, we deduce that $\gamma = \mathrm{id}$. This proves that the action of Γ on $\mathcal{C}(\widetilde{S})$ is free.

Suppose the action of Γ on $\mathcal{C}(\widetilde{S})$ is not properly discontinuous. Then, by [6, Proposition 1], there exists a sequence $\{p_i\}$ of points of $\mathcal{C}(\widetilde{S})$ converging to some point $p_\infty \in \mathcal{C}(\widetilde{S})$ and a divergent sequence $\{\gamma_i\}$ of elements of Γ such that $\{\gamma_i.p_i\}$ converges to some point $q_\infty \in \mathcal{C}(\widetilde{S})$. Without loss of generality, we can suppose that $p_\infty, q_\infty \in J^+(\widetilde{S})$.

Let $p \in I^+(p_\infty)$. Since $\lim p_i = p_\infty$, all the p_i belong to $I^-(p)$ except a finite number. Hence, $O(p_i) \subset O(p)$ for every $i \geqslant i_0$ where i_0 is a natural integer. Let $x_i \in O(p_i)$. Up to extracting, $\{x_i\}$ converges to some $x_\infty \in O(p)$. Similarly, let $q \in I^+(q_\infty)$; all the $\gamma_i p_i$ belong to $O(q)$ except a finite number. Hence, $\gamma_i O(p_i) = O(\gamma_i p_i) \subset O(q)$. Therefore, up to extracting, $\{\gamma_i x_i\}$ converges to some point $y_\infty \in O(q)$. This contradicts the properness of the action of Γ on $\widetilde{S}$ (see [6, Proposition 1]). □

5.1.2. The $\mathcal{C}_0$-maximal extension of V

Proposition 5.13. — *The quotient space $\Gamma\backslash\mathcal{C}(\widetilde{S})$ is a $\mathcal{C}_0$-maximal extension of V.*

Proof. — Let $i\colon \widetilde{V} \to E(\widetilde{V})$ the embedding of $\widetilde{V}$ in $E(\widetilde{V})$ defined in Section 4. The co-restriction of i to $\mathcal{C}(\widetilde{S})$ is a Cauchy embedding, denoted by $\widetilde{f}$. Let $\pi'\colon \mathcal{C}(\widetilde{S}) \to \Gamma\backslash\mathcal{C}(\widetilde{S})$ be the canonical projection. Since the action of Γ on $i(\widetilde{V})$ coincide with the usual action of Γ on $\widetilde{V}$, the map $\widetilde{f}$ descends to quotient in a Cauchy embedding $f\colon V \to \Gamma\backslash\mathcal{C}(\widetilde{S})$. It is easy to see that since $\mathcal{C}(\widetilde{S})$ is $\mathcal{C}_0$-maximal (see Proposition 4.17), the quotient $\Gamma\backslash\mathcal{C}(\widetilde{S})$ is also $\mathcal{C}_0$-maximal. The proposition follows. □

Corollary 5.14. — *The $\mathcal{C}_0$-maximal extension $\Gamma\backslash\mathcal{C}(\widetilde{S})$ is unique up to conformal diffeomorphism.*

Proof. — Let $f\colon V \to W$ be a Cauchy embedding of V in a $\mathcal{C}_0$-maximal globally hyperbolic conformally flat spacetime W. It lifts to a Cauchy embedding $\widetilde{f}\colon \widetilde{V} \to \widetilde{W}$. By Proposition 4.13, $\widetilde{W}$ admits a conformal copy in $E(\widetilde{V})$ contained in $\mathcal{C}(\widetilde{S})$, where $\widetilde{S}$ is a Cauchy hypersurface of $\widetilde{V}$ (seen in $E(\widetilde{V})$). The inclusion of $\widetilde{W}$ in $\mathcal{C}(\widetilde{S})$ is a Cauchy embedding which descends to the quotient in a Cauchy embedding from W to $\Gamma\backslash\mathcal{C}(\widetilde{S})$. By Proposition 5.13, this last one is surjective. The corollary follows. □

The proof of Proposition 5.13 is based on the fact that if the universal covering of a globally hyperbolic spacetime is maximal then this spacetime is maximal. *A priori*, the converse assertion is not true in general. However, Proposition 5.13 allows to prove that it is true in the conformally flat setting.

Corollary 5.15. — *Let V be a $\mathcal{C}_0$-maximal spacetime. Then, the universal covering of V is $\mathcal{C}_0$-maximal.*

Proof. — Let $\widetilde{S}$ be a Cauchy hypersurface of $\widetilde{V}$. By Proposition 5.13, there is a Cauchy embedding f from V to $\Gamma\backslash\mathcal{C}(\widetilde{S})$. This last one lifts to a Cauchy embedding $\widetilde{f}\colon \widetilde{V} \to \mathcal{C}(\widetilde{S})$. Since V is maximal, f is surjective. Therefore, $\widetilde{f}$ is surjective. Since $\mathcal{C}(\widetilde{S})$ is $\mathcal{C}_0$-maximal (see Proposition 4.17), $\widetilde{V}$ is $\mathcal{C}_0$-maximal. □

6. $\mathcal{C}_0$-maximal extensions respect inclusion

In this section, we show that the *functor maximal extension* respects inclusion in the setting of conformally flat spacetimes. More precisely, we establish the following result.

THEOREM 6.1. — *Let V be a conformally flat globally hyperbolic spacetime and let U be a causally convex open subset of V. Then, there is a conformal embedding from the $\mathcal{C}_0$-maximal extension $\widehat{U}$ of U into the $\mathcal{C}_0$-maximal extension $\widehat{V}$ of V. Moreover, the image of this embedding is causally convex in $\widehat{V}$.*

We first prove this result in the case where V is simply-connected in Section 6.1 before dealing with the general case in Section 6.2.

6.1. The simply-connected case

We prove Theorem 6.1 in the case where V is simply-connected.

PROPOSITION 6.2. — *Let V be a* simply-connected *conformally flat globally hyperbolic spacetime and let U be a causally convex open subset of V. Then, there is a conformal embedding from the $\mathcal{C}_0$-maximal extension $\widehat{U}$ of U into the $\mathcal{C}_0$-maximal extension $\widehat{V}$ of V. Moreover, the image is causally convex in $\widehat{V}$.*

The key idea is to realize the $\mathcal{C}_0$-maximal extensions of U and V in the enveloping space $E(V)$ so we can compare them. The proof of Proposition 6.2 uses the following lemma.

Let $D\colon V \to \widetilde{\mathrm{Ein}}_{1,n-1}$ be a developing map.

LEMMA 6.3. — *The inclusion map $i : U \hookrightarrow V$ induces a conformal embedding of $E(U)$ into $E(V)$ that sends every fiber of $E(U)$ on a fiber of $E(V)$ and such that the restriction to every fiber of $E(U)$ is surjective.*

Proof. — Consider the foliation of V by inextensible timelike curves induced by the pull-back by the developing map D of the vector field ∂_t on $\mathrm{Ein}_{1,n} \simeq \mathbb{S}^{n-1} \times \mathbb{R}$. Since U is causally convex in V, the intersection of every leaf of V with U is an inextensible timelike curve of U. Therefore, the foliation of V induces a foliation of U by inextensible timelike curves. Notice that this foliation coincides with that induced by the pull-back of ∂_t by the restriction of D to U. Let $\psi\colon V \to \mathcal{B}$ and $\psi_U\colon U \to \mathcal{B}_U$ be the canonical projections on the leaf spaces. Then, the map $\psi \circ i$ descends to the quotient in an embedding $\bar{i}\colon \mathcal{B}_U \to \mathcal{B}$. Let $d\colon \mathcal{B} \to \mathbb{S}^{n-1}$ (resp. $d_U\colon \mathcal{B}_U \to \mathcal{S}^{n-1}$) be the developing map induced by the developing map D (resp. the restriction of D to U). It is clear that $d \circ \bar{i} = d_U$. It follows that the map from $E(U)$ to $E(V)$ that sends (p, b) on $\big(p, \bar{i}(b)\big)$ is a conformal embedding that sends every fiber of $E(U)$ on a fiber of $E(V)$. Moreover, the restriction to every fiber of $E(U)$ is clearly surjective. □

In other words, Lemma 6.3 says that $E(U)$ can be seen as the union of the fibers of $E(V)$ over some open subset of $\mathcal{B}$.

Proof of Proposition 6.2. — We identify V (resp. U) with its image in $E(V)$ (resp. $E(U)$), then we identify $E(U)$ with its image in $E(V)$. Let $\widehat{V}$ (resp. $\widehat{U}$) be the $\mathcal{C}_0$-maximal extension of V (resp. U). By Proposition 4.17, $\widehat{V}$ (resp. $\widehat{U}$) is the Cauchy development $\mathcal{C}(S)$ of a Cauchy hypersurface S of V (resp. Σ of U) in $E(V)$ (resp. $E(U)$).

We prove that the Cauchy development $\mathcal{C}(\Sigma)$ of Σ in $E(V)$ is exactly $\widehat{U}$, then we prove that $\mathcal{C}(\Sigma) \subset \mathcal{C}(S)$.

Let $x \in \widehat{U}$ and let $\widehat{\gamma}$ be an inextensible causal curve of $E(V)$ going through x. The intersection of $\widehat{\gamma}$ with $E(U)$ is a union of connected components. The component containing x is an inextensible causal curve of $E(U)$, then it intersects Σ in a single point. Hence, $x \in \mathcal{C}(\Sigma)$. This proves that $\widehat{U} \subset \mathcal{C}(\Sigma)$. Actually, this inclusion is a Cauchy embedding. Since U is $\mathcal{C}_0$-maximal, we deduce that $\widehat{U} = \mathcal{C}(\Sigma)$.

Let $x \in \mathcal{C}(\Sigma)$ and let $\widehat{\gamma}$ be an inextensible causal curve of $E(V)$ going through x. By definition, the curve $\widehat{\gamma}$ meets Σ, hence V. Since V is causally convex in $E(V)$, the intersection of $\widehat{\gamma}$ with V is an inextensible causal curve of V, thus it intersects S in a single point. It follows that $x \in \mathcal{C}(S)$. Hence, $\mathcal{C}(\Sigma) \subset \mathcal{C}(S)$. The proposition follows. □

6.2. The general case

In the previous section, we proved Theorem 6.1 in the case where V is simply-connected. In this section, we prove it for any conformally flat globally hyperbolic spacetime V. Without loss of generality, we suppose that V is $\mathcal{C}_0$-maximal. Let $\pi\colon \widetilde{V} \to V$ be the universal cover of V. Set

$\Gamma := \pi_1(V)$. Let U' be a connected component of $\pi^{-1}(U)$ and let Γ' be the stabilizer of U' in Γ.

Lemma 6.4. — *The open set U' is causally convex in $\widetilde{V}$.*

Proof. — Let $p, q \in U'$ and let γ be a causal curve in $\widetilde{V}$ joining p to q. The projection $\pi(\gamma)$ of γ in V is a causal curve joining the points $\pi(p)$ and $\pi(q)$ of U. Since U is causally convex in V, the curve $\pi(\gamma)$ is contained in U. Hence, γ is contained in U'. The lemma follows. $\square$

Let $\widehat{U'}$ be the $\mathcal{C}_0$-maximal extension of U'. By Proposition 6.2, $\widehat{U'}$ can be conformally identified to a causally convex open subset of $\widetilde{V}$.

Lemma 6.5. — *The stabilizer of $\widehat{U'}$ in Γ is equal to Γ'.*

Proof. — Since $U' \subset \widehat{U'}$, the stabilizer of U' is contained in the stabilizer of $\widehat{U'}$. Conversely, let γ be an element of Γ stabilizing $\widehat{U'}$. Let $p \in U'$. Consider a Cauchy hypersurface Σ of U' going through p. Let φ be an inextensible causal curve of $\widehat{U'}$ going through p. Then, $\gamma\varphi$ is an inextensible causal curve of $\widehat{U'}$. Therefore, $\gamma\varphi$ intersects Σ in a unique point q. We prove that $q = \gamma p$.

Since $\widehat{U'}$ is causally convex in $\widetilde{V}$, the projection $\pi(\widehat{U'})$ is causally convex in V and $\pi(\Sigma)$ is a Cauchy hypersurface of $\pi(\widehat{U'})$. Moreover, $\pi(\varphi)$ is an inextensible causal curve of $\pi(\widehat{U'})$, then it meets $\pi(\Sigma)$ in a single point. Since $\pi(p), \pi(q) \in \pi(\varphi) \cap \pi(\Sigma)$, we deduce that $\pi(p) = \pi(q)$. Then, $q = \gamma' p$ with $\gamma' \in \Gamma$. Since $\gamma p, \gamma' p \in \gamma\varphi$, if $\gamma \neq \gamma'$, the causal curve $\pi(\varphi)$ would be closed. Contradiction. Hence, $\gamma = \gamma'$, so $q = \gamma p$. This shows that $\gamma p \in U'$. Thus, $\gamma \in \Gamma'$. $\square$

The inclusion $\widehat{U'} \subset \widetilde{V}$ descends to the quotient in a conformal embedding from $\Gamma' \backslash \widehat{U'}$ to V. Since $\widehat{U'}$ is causally convex in $\widetilde{V}$, the image of this embedding is causally convex in V. It remains to prove the following assertion to conclude.

Lemma 6.6. — *The quotient space $\Gamma' \backslash \widehat{U'}$ is the $\mathcal{C}_0$-maximal extension of U.*

Proof. — The inclusion $U' \subset \widehat{U'}$ is a Cauchy embedding which descends to the quotient in a Cauchy embedding from U to $\Gamma' \backslash \widehat{U'}$. We have to prove that $\Gamma' \backslash \widehat{U'}$ is $\mathcal{C}_0$-maximal.

Let f be a Cauchy embedding from $\Gamma' \backslash \widehat{U'}$ in a globally hyperbolic conformally flat spacetime W. Then, the morphism $f_* \colon \pi_1(\Gamma' \backslash \widehat{U'}) \to \pi_1(W)$ induced by f is an isomorphism. Let $\pi' \colon \widehat{U'} \to \Gamma' \backslash \widehat{U'}$ be the canonical projection. It induces an injective morphism $\pi'_* \colon \pi_1(\widehat{U'}) \to \pi_1(\Gamma' \backslash \widehat{U'})$. Hence, the morphism $(f\circ\pi')_*\colon \pi_1(\widehat{U'}) \to \pi_1(W)$ is injective. Let $p\colon W'\to W$ the cover such that $p_*\big(\pi_1(W')\big) = (f\circ\pi')_*\big(\pi_1(\widehat{U'})\big)$. Then, f lifts to a Cauchy embedding $f'\colon \widehat{U'} \to W'$. Since $\widehat{U'}$ is $\mathcal{C}_0$-maximal, f' is surjective. Hence, f is surjective. The lemma follows. □

7. Eikonal functions and $\mathcal{C}_0$-maximality

7.1. Eikonal functions on the sphere

In this section, we characterize causally convex open subsets of $\widetilde{\mathrm{Ein}}_{1,n-1}$ which are $\mathcal{C}_0$-maximal in a spatio-temporal decomposition $\mathbb{S}^{n-1}\times\mathbb{R}$. We denote by d the distance on $\mathbb{S}^{n-1}$ induced by the round metric. By [11, Theorem 10], causally convex open subsets of $\widetilde{\mathrm{Ein}}_{1,n-1}$ with conjugate points are conformally equivalent to $\widetilde{\mathrm{Ein}}_{1,n-1}$. For this reason, we consider a causally convex open subset Ω *without conjugate points.*

Let f^+ and f^- be the two 1-Lipschitz real-valued functions defined on the projection U of Ω in $\mathbb{S}^{n-1}$ such that:

$$\Omega = \big\{(x,t)\in U\times\mathbb{R},\ f^-(x) < t < f^+(x)\big\}.$$

We call f the common extension of f^+ and f^- to ∂U. Let g^+, g^- be the real-valued functions defined for every $x\in U$ by:

$$g^+(x) = \inf_{x_0\in\partial U}\big\{f(x_0) + d(x,x_0)\big\}, \tag{7.1}$$

$$g^-(x) = \sup_{x_0\in\partial U}\big\{f(x_0) - d(x,x_0)\big\}. \tag{7.2}$$

Proposition 7.1. — *The causally convex open subset Ω of $\widetilde{\mathrm{Ein}}_{1,n-1}$ is $\mathcal{C}_0$-maximal if and only if f^+ equals g^+ and f^- equals g^-.*

The proof of Proposition 7.1 uses the following lemma.

Lemma 7.2. — *Any 1-Lipschitz real-valued function g defined on U, whose extension to ∂U equals f, is bounded by g^- and g^+.*

Proof. — Let $x\in U$ and let $x_0\in\partial U$. We denote by $\bar g$ the extension of g to ∂U. Since $\bar g$ is 1-Lipschitz, we have $\bar g(x) - \bar g(x_0) \leqslant d(x,x_0)$, i.e. $g(x) - f(x_0) \leqslant d(x,x_0)$. Hence, $g(x)\leqslant f(x_0) + d(x,x_0)$. Therefore, $g(x)\leqslant f^+(x)$. From $-d(x,x_0)\leqslant \bar g(x) - \bar g(x_0)$, we deduce similarly that $f^-(x)\leqslant g(x)$. The lemma follows. □

Proof of Proposition 7.1. — Let us denote Ω' the set of points (x, t) of $\widetilde{\mathrm{Ein}}_{1,n-1}$ such that $g^-(x) < t < g^+(x)$. By Lemma 7.2 and Proposition 3.11, Ω' is a Cauchy extension of Ω. According to Theorem 6.1, the $\mathcal{C}_0$-maximal extension of Ω is conformally equivalent to a causally convex open subset $\widehat{\Omega}$ of $\widetilde{\mathrm{Ein}}_{1,n-1}$ such that $\Omega \subset \Omega' \subset \widehat{\Omega}$ where each inclusion is a Cauchy embedding. Thus, there exist two 1-Lipschitz real-valued functions h^-, h^+ defined on U, whose extensions to ∂U equal f, such that $\widehat{\Omega}$ is the set of points (x, t) such that $h^-(x) < t < h^+(x)$. Hence, $g^-(x) \leqslant h^-(x) < h^+(x) \leqslant g^+(x)$ for every $x \in U$ (see 7.2). In other words, $\widehat{\Omega} \subset \Omega'$. The proposition follows. □

The domain $\widehat{\Omega}$ in the proof of Proposition 7.1 is a union of connected components of the dual of the graph of f (see Proposition 3.17). Therefore, Proposition 7.1 can be reformulated as: *$\mathcal{C}_0$-maximal causally convex open subsets of $\widetilde{\mathrm{Ein}}_{1,n-1}$ are exactly unions of connected components of duals of closed achronal subsets of $\widetilde{\mathrm{Ein}}_{1,n-1}$.*

Definition 7.3. — *The function g^+ (resp. g^-) is called* future eikonal *(resp.* past eikonal*).*

Remark 7.4. — This definition is motivated by the classical notion of eikonal function in analysis: a real-valued function f defined on an open subset U of $\mathbb{R}^n$ is called eikonal if f is differentiable almost everywhere and satisfies the eikonal equation $\|\nabla f\| = 1$. It turns out that if U has a piecewise smooth boundary, then the function $f(x) := d(x, \partial U)$, where d is the usual distance on $\mathbb{R}^n$, is eikonal in this sense.

The following proposition gives a geometrical characterization of eikonal functions.

Proposition 7.5 (Geometrical criterion of eikonality). — *A real-valued function f^+ defined on an open subset U of the sphere $\mathbb{S}^{n-1}$ is future eikonal if and only if for every $x \in U$, there exists a past-directed lightlike geodesic starting from $\big(x, f^+(x)\big)$, entirely contained in the graph of f^+ and with no past endpoint in the graph of f^+.*

Proof. — Suppose f^+ is eikonal. We call f the extension of f^+ to ∂U. Let $x \in U$. There exists $x_0 \in \partial U$ such that $f^+(x) = f(x_0) + d(x, x_0)$. Hence, $f^+(x) - f(x_0) = d(x, x_0)$, i.e. the points $\big(x, f^+(x)\big)$ and $\big(x_0, f(x_0)\big)$ are joined by a past lightlike geodesic φ. This last one is contained in the graph of f^+. Conversely, let us prove that f^+ equals the function g^+ defined by (7.1). By Lemma 7.2, we have $f^+ \leqslant g^+$. Let $x \in U$. There exists a past-directed lightlike geodesic φ starting from $\big(x, f^+(x)\big)$, entirely contained in the graph of f^+ and with no past endpoint in the graph of f^+. In $\widetilde{\mathrm{Ein}}_{1,n-1}$, the geodesic φ admits a past endpoint $\big(x_0, f(x_0)\big)$ with $x_0 \in \partial U$. Thus, $d(x, x_0) = f^+(x) - f(x_0)$. Hence, $f^+(x) = f(x_0) + d(x, x_0) \geqslant g^+(x)$. Then, $f^+ = g^+$. □

Now, we show that eikonal is *a local property*.

Definition 7.6. — *A real-valued function f^+ defined on an open subset U of the sphere $\mathbb{S}^{n-1}$ is* locally future eikonal *if every point x of U admits an arbitrarily small neighborhood V_x such that the restriction of f^+ to V_x is future eikonal.*

Remark 7.7. — A locally future eikonal function is locally 1-Lipschitz, thus 1-Lipschitz.

Let f^+ be a 1-Lipschitz real-valued function defined on an open subset U of $\mathbb{S}^{n-1}$. We call f its extension to ∂U.

Lemma 7.8. — *If f^+ is future eikonal then f^+ is locally future eikonal.*

Proof. — Let V be an open subset of U. We call g the extension of $f^+|_V$ to ∂V. Let g^+ (resp. g^-) be the function defined by the expression (7.1) (resp. (7.2)) after replacing U by V. By Lemma 7.2, we have $f^+|_V \leqslant g$. Let us prove that $f^+|_V \geqslant g$ so we get the equality. Let Ω (resp. W) be the $\mathcal{C}_0$-maximal causally convex open subset of $\widetilde{\mathrm{Ein}}_{1,n-1}$ bounded by the graphs of f^+ and f^- (resp. g^+ and g^-). The intersection $W \cap \Omega$ is a causally convex open subset of Ω; its $\mathcal{C}_0$-maximal extension is W. By Theorem 6.1, we get $W \subset \Omega$. Hence, $g^+ \leqslant f^+|_V$. Thus, $g^+ = f^+|_V$, then $f^+|_V$ is future eikonal. □

Lemma 7.9. — *If f^+ is locally future eikonal then f^+ is future eikonal.*

Proof. — We use the criterion given by Proposition 7.5 to prove that f^+ is eikonal. Let $x \in U$ and let $V \subset U$ be a neighborhood of x such that $f^+|_V$ is future eikonal. Then, there is a past-directed lightlike geodesic φ starting from $\big(x, f^+(x)\big)$ with the properties of Proposition 7.5. In $\widetilde{\mathrm{Ein}}_{1,n-1}$, the geodesic φ admits a past endpoint $\big(x_0, f^+(x_0)\big)$ where $x_0 \in \partial V$ (we still denote by f^+ its extension to $\overline{U}$). If $x_0 \in \partial U$, the lemma is proved. Otherwise, we choose a neighborhood $V_0 \subset U$ of x_0 such that $f^+|_{V_0}$ is future eikonal. Again, there is a past-directed lightlike geodesic φ_0 starting from $\big(x_0, f(x_0)\big)$, entirely contained in the graph of $f^+|_{V_0}$, with past endpoint $\big(x_1, f^+(x_1)\big)$ where $x_1 \in \partial V_0$. The geodesic φ_0 extends φ; indeed, otherwise $\big(x_1, f^+(x_1)\big)$ would be in the chronological past of $\big(x, f^+(x)\big)$, i.e. we would have $d(x, x_1) < f^+(x) - f^+(x_1)$. This contradicts the fact that f^+ is 1-Lipschitz. Consequently, we extend φ in a lightlike geodesic, entirely contained in the graph of f^+, with no past endpoint in the graph of f^+. The lemma follows. □

We proved the following statement.

Proposition 7.10. — *A real-valued function defined on an open subset of the sphere $\mathbb{S}^{n-1}$ is future eikonal if and only if it is* locally *future eikonal.*

7.2. Eikonal functions on a conformally flat Riemannian manifold

Let $\mathcal{B}$ be a conformally flat Riemannian manifold of dimension $(n-1) \geqslant 2$ and let $d\colon \mathcal{B} \to \mathbb{S}^{n-1}$ be a developing map. The notion of eikonal function on an open subset of the sphere $\mathbb{S}^{n-1}$ being *local*, it naturally generalizes to functions on an open subset of $\mathcal{B}$.

Definition 7.11. — *A real-valued function f^+ defined on an open subset U of $\mathcal{B}$ is said* future eikonal *if for every $x \in U$, there exists an open neighborhood U_x of x contained in U such that the following hold:*

(1) *the restriction of d to U_x is injective;*
(2) *the map $f^+ \circ d\big|_{U_x}^{-1}\colon d(U_x) \subset \mathbb{S}^{n-1} \to \mathbb{R}$ is future eikonal.*

Past eikonal functions on U are defined similarly with the reverse time orientation. The geometrical criterion of eikonality given in Proposition 7.5 is still valid in this setting.

Eikonality is closely related to $\mathcal{C}_0$-maximality. Indeed, let E be the pullback of the trivial fiber bundle $\pi\colon \widetilde{\mathrm{Ein}}_{1,n-1} \to \mathbb{S}^{n-1}$ by d:

$$E := \big\{(p,b) \in \widetilde{\mathrm{Ein}}_{1,n-1} \times \mathcal{B} \;:\; \pi(p) = d(b)\big\}.$$

We denote by $\pi\colon E \to \mathcal{B}$ the projection on the second factor. The construction of this fiber bundle is exactly the same as that of the enveloping space in Section 4.2. Then, E is a conformally flat spacetime of dimension $n \geqslant 3$ sharing the same properties as that of the enveloping space. Let $\widehat{D}\colon E \to \widetilde{\mathrm{Ein}}_{1,n-1}$ be the developing map defined as the projection on the first factor. A causally convex GH open subset Ω of E is given by

$$\Omega = \big\{(x,t) \in U \times \mathbb{R} \;:\; f^-(x) < t < f^+(x)\big\}$$

where f^+ and f^- are 1-Lipschitz functions defined on an open subset U of $\mathcal{B}$ such that their extensions to ∂U coincide (see Section 4.3).

Proposition 7.12. — *The GH conformally flat spacetime Ω is $\mathcal{C}_0$-maximal if and only if f^+ is future-eikonal and f^- past-eikonal.*

Proof. — Suppose Ω is $\mathcal{C}_0$-maximal. Let $x \in U$. Set $p_+ = \big(x, f^+(x)\big)$. Let $p \in I^-(p^+) \cap \Omega$. Since Ω is GH, the restriction of $\widehat{D}$ to $I^+(p,\Omega)$ is injective and its image is causally convex in $\widetilde{\mathrm{Ein}}_{1,n-1}$ (see [10, Proposition 2.7 and Corollary 2.8, p. 151]). The $\mathcal{C}_0$-maximality of Ω implies that $\widehat{D}\big(I^+(p,\Omega)\big)$ is also $\mathcal{C}_0$-maximal (see [10, Proposition 3.6, p. 156]). It follows that:

- $\widehat{D}\big(I^+(p,\Omega)\big)$ is the domain bounded by the graph of a future-eikonal function g^+ and the graph of a past-eikonal function g^- defined on an open subset of $\mathbb{S}^{n-1}$ (see Proposition 7.1);
- d is injective on $U_x := \widehat{\pi}\big(I^+(p,\Omega)\big)$ and $g^+ = f^+ \circ d|_{U_x}^{-1}$.

Thus, f^+ is future-eikonal. The proof is similar for f^- with the reverse time-orientation.

Conversely, suppose that f^+ is future-eikonal and f^- past-eikonal. Let S be a Cauchy hypersurface in Ω. By Proposition 4.17, the $\mathcal{C}_0$-maximal extension of Ω is conformally equivalent to a causally convex open subset $\widehat{\Omega}$ of E, containing Ω and for which S is a Cauchy hypersurface. Suppose that Ω is strictly contained in $\widehat{\Omega}$. Then, there exists $x \in U$ such that $\big(x, f^+(x)\big) \in \widehat{\Omega}$ or $\big(x, f^-(x)\big) \in \widehat{\Omega}$. Suppose that $\big(x, f^+(x)\big) \in \widehat{\Omega}$. The proof is symmetric if $\big(x, f^-(x)\big) \in \widehat{\Omega}$. Since f^+ is future-eikonal, there exists a past-directed lightlike geodesic φ starting from $\big(x, f^+(x)\big)$, entirely contained in the graph of f^+ and with no endpoint in the graph of f^+. Then, φ does not intersect S. Contradiction. Hence, $\Omega = \widehat{\Omega}$, in other words Ω is $\mathcal{C}_0$-maximal. □

BIBLIOGRAPHY

[1] L. Andersson, T. Barbot, F. Béguin & A. Zeghib, "Cosmological time versus CMC time in spacetimes of constant curvature", *Asian J. Math.* **16** (2012), no. 1, p. 37-88.

[2] A. N. Bernal & M. Sánchez, "On smooth Cauchy hypersurfaces and Geroch's splitting theorem", *Commun. Math. Phys.* **243** (2003), p. 461-470.

[3] ———, "Globally hyperbolic spacetimes can be defined as 'causal' instead of 'strongly causal'", *Class. Quant. Grav.* **24** (2007), no. 3, p. 745-749.

[4] Y. Choquet-Bruhat & R. P. Geroch, "Global aspects of the Cauchy problem in general relativity", *Commun. Math. Phys.* **14** (1969), p. 329-335.

[5] C. Frances, "Une preuve du théorème de Liouville en géométrie conforme dans le cas analytique", *Enseign. Math. (2)* **49** (2003), no. 1-2, p. 95-100.

[6] ———, "Lorentzian Kleinian groups", *Comment. Math. Helv.* **80** (2005), no. 4, p. 883-910.

[7] R. P. Geroch, "The domain of dependence", *J. Math. Phys.* **11** (1970), p. 437-449.

[8] W. Goldman, "Geometric structures on manifolds", unpublished manuscript, online at `https://www.terpconnect.umd.edu/~wmg/gstom.pdf`, 2021.

[9] B. O'neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press Inc., 1983.

[10] C. R. Salvemini, "Espace-temps globalement hyperboliques conformément plats", PhD Thesis, Université d'Avignon (France), 2012.

[11] ———, "Maximal extension of conformally flat globally hyperbolic spacetimes", *Geom. Dedicata* **174** (2013), p. 235-260.

[12] R. Smaï, "Anosov representations as holonomies of globally hyperbolic spatially compact conformally flat spacetimes", *Geom. Dedicata* **216** (2022), no. 4, article no. 45 (36 pages).

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