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ON THE BEHAVIOR OF FORMAL NEIGHBORHOODS IN THE NASH SETS ASSOCIATED WITH TORIC VALUATIONS: A COMPARISON THEOREM

**by David BOURQUI,
Mario MORÁN CAÑÓN & Julien SEBAG**

ABSTRACT. — We show that there exists a strong connection between the generic formal neighborhood at a rational arc lying in the Nash set associated with a toric divisorial valuation on a toric variety and the formal neighborhood at the generic point of the same Nash set. This may be interpreted as the fact that, analytically along such a Nash set, the arc scheme of a toric variety is a product of a finite dimensional singularity and an infinite dimensional affine space.

RÉSUMÉ. — Nous montrons qu'il existe un lien étroit entre le voisinage formel générique d'un arc rationnel situé sur l'ensemble de Nash associé à une valuation divisorielle torique et le voisinage formel du point générique de cet ensemble de Nash. Cela peut être interprété comme le fait que, analytiquement le long de cet ensemble de Nash, le schéma des arcs d'une variété torique est le produit d'une singularité de dimension finie et d'un espace affine de dimension infinie.

1. Introduction

1.1.

In [24], Nash pointed out an original connection between the geometry of the arc scheme associated with a surface and the resolutions of the singularities of this surface in characteristic zero. This seminal work has deeply motivated the development of the study of arc scheme in particular following the basic idea that this natural object, defined as the space of formal germs of curves lying on the considered variety, encodes in many ways the information on the singularities of the algebraic varieties. This topic has

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become currently prominent in the broad field of singularity theory and has put forward several notions that turned out to be relevant objects of study, such as the *Nash sets* (or *maximal divisorial sets*), which are subsets of the arc scheme naturally associated with divisorial valuations.

1.2.

On the other hand, formal neighborhoods in algebraic geometry are classical tools which reflect the local structure of schemes. In the context of arc schemes, the study of the formal neighborhoods probably began with [26] (see also [17] for a characteristic-free generalization), where it is in particular shown that the formal neighborhood of the arc scheme at the generic point of any Nash set is Noetherian. Independently, Drinfeld after Grinberg–Kazhdan (see [12, 13, 18], and also [3]) showed that the formal neighborhoods at non-degenerate (i.e., not entirely contained in the singular locus of the variety) *rational* arcs are infinite dimensional, but that their singularities are entirely described by the formal neighborhood of a rational point of a Noetherian scheme, that we can consider as a finite formal model of the analytic type of the singularity of the arc, and which can also be connected with the singularities of the variety (see [6] as well as [3, 4, 5]).

1.3.

The present work proves, in the case of toric varieties, a comparison result between the two aforementioned classes of formal neighborhoods. Until now, these two classes had been studied independently, and this is the first time that a strong direct connection between them is established. Indeed, we mainly obtain the following result (see Theorem 6.11 for a more precise statement).

Theorem. — *Let k be a field of characteristic zero. Let V be an affine normal toric k-variety. Let v be a toric divisorial valuation on V and $\mathcal{N}_v$ be the associated Nash set. Let η_v be the generic point of $\mathcal{N}_v$ and κ_v be the residue field of η_v. For a general k-rational arc $\alpha \in \mathcal{N}_v$ there exists an isomorphism of κ_v-formal schemes between $\widehat{\mathcal{O}}_{\mathscr{L}_\infty(V),\eta_v}\widehat{\otimes}_{\kappa_v}\kappa_v\left[\!\left[(T_i)_{i\in\mathbb{N}}\right]\!\right]$ and $\widehat{\mathcal{O}}_{\mathscr{L}_\infty(V),\alpha}\widehat{\otimes}_k\kappa_v$.*

In [6] it was observed that the formal neighborhood of a sufficiently generic k-rational arc of the Nash set associated with a toric valuation is constant (see Theorem 1.3 of op. cit.). We stress that Theorem 6.11 (which provides a positive answer to [7, Question 7.20]) is a much stronger statement, that might geometrically be interpreted as the fact that $\mathscr{L}_\infty(V)$ is analytically along $\mathcal{N}_v$ a product of a finite dimensional singularity and an infinite dimensional affine space.

1.4.

Theorem 6.11 actually gives a more precise information, namely an explicit description of the formal neighborhood of the generic point of the Nash set associated with a toric valuation in terms of a Noetherian formal scheme associated with the same valuation, which had been introduced in our previous work [6] on the finite formal models of toric singularities. In this sense, our arguments for proving the above theorem are based on a direct comparison between the expressions of both formal neighborhoods. That being said, Theorem 6.11 is by no means a straightforward consequence of the results and techniques in [6], which cannot be directly imported; the main issue being that the interpretation of the formal neighborhood of an arc as a parameter space for the infinitesimal deformations, which provides a meaningful and very efficient tool in the context of rational arcs, has no sensible analog in the context of generic points of Nash sets. We need new general technical ingredients established in preliminary Sections 4 and 5. In Section 6 the proof of Theorem 6.11, which is our main result, is given. Some geometric consequences of our main statement are given in Section 6.9. At the end of Section 6 we provide an explicit example.

1.5.

It is natural to ask whether there exist other classes of varieties for which the formal neighborhood of a rational non degenerate arc is generically constant on Nash sets, and, if it is so, whether a comparison result akin to Theorem 6.11 still holds. In [7], it is observed that for curve singularities the genericity property holds. The first and second authors of the present work have obtained subsequently to the present article, in [1], that a comparison theorem also holds in this case.

It is unclear for us whether the genericity property holds in full generality. For normal varieties equipped with a "big" action of an algebraic group (typically for spherical varieties), it seems very probable that using the induced action on the level of arc spaces, one should be able to establish the genericity property. But even in such cases, we don't know whether one may reasonably hope that a comparison theorem holds, since the techniques used in the present paper are not likely to extend easily. It certainly deserves further investigation.

2. General conventions and notation

2.1.

Throughout the whole article, we designate by k a field of characteristic zero (as suggested by the referee, most of the techniques and results of the present paper, including Theorem 6.11, should extend rather straightforwardly over a field of arbitrary characteristic; however, Corollary 6.12 is deduced from Theorem 6.11 by using a part of [6] where the arguments use characteristic zero in an essential way). The category of k-algebras (resp. of k-schemes) is denoted by Alg_k (resp. Sch_k). If K is a field extension of k, the category $\mathfrak{LcCpl}_K$ is formed by the complete local k-algebras with residue field k-isomorphic to K. For any category $\mathcal{C}$ and any objects $A, B \in \mathcal{C}$, we denote by $\mathrm{Hom}_{\mathcal{C}}(A, B)$ the set of morphisms from A to B in the category.

2.2.

A *k-variety* is a k-scheme of finite type. The non-smooth locus of the structural morphism of a k-variety V is the *singular locus of V* and its associated reduced k-scheme is denoted by $\mathrm{nSm}(V)$. If V is an affine k-variety and f is a regular function on V, we denote by $\{f \neq 0\}$ the distinguished open subset of V where f does not vanish and by $\{f = 0\}$ the closed set $V \setminus \{f \neq 0\}$.

2.3.

Let R be a ring, let $\mathfrak{i}$ be an ideal of R and $f \in R$. We denote by R_f the localization of R with respect to the multiplicative subset $\{f^r;\ r \in \mathbb{N}\}$.

We denote by $\mathfrak{i} : f^\infty$ the ideal $\{g \in R : f^r g \in \mathfrak{i} \text{ for some } r \in \mathbb{N}\}$. Let R' be another ring and $\vartheta\colon R \to R'$ a morphism of rings. For the sake of easy reading and abusing notation, the extension ideal of $\mathfrak{i}$ in R' via the morphism ϑ is denoted by $\vartheta(\mathfrak{i})$, or even by $\mathfrak{i}$ if the involved morphism ϑ is clear from the context (for example if R is a subring of R').

2.4.

Let $R[X_\omega;\ \omega \in \Omega]$ be a polynomial ring and $f \in R$. Let S be an R-algebra and $\{s_\omega\}_{\omega\in\Omega}$ a collection of elements in S. Then we denote by $f\big|_{X_\omega = s_\omega} \in S$ the image of f by the unique morphism of R-algebras $R[X_\omega] \to S$ mapping X_ω to s_ω for each $\omega \in \Omega$.

2.5.

Let $(\mathcal{A}, \mathfrak{M}_{\mathcal{A}})$ be a complete local ring. An element $f = \sum_{i\in\mathbb{N}} f_i t^i \in \mathcal{A}[\![t]\!]$ is *regular* if $f \notin \mathfrak{M}_{\mathcal{A}}[\![t]\!]$. Its *order* is $\inf\{i \in \mathbb{N},\ f_i \notin \mathfrak{M}_{\mathcal{A}}\}$. Let $d \in \mathbb{N}$. A *Weierstrass polynomial* of order d is a monic polynomial of degree d, whose order as a regular element of $\mathcal{A}[\![t]\!]$ is d. We shall make a crucial use of the following classical results (the Weierstrass division and preparation theorems, see e.g. [21, Theorems 9.1 and 9.2]). Let $f \in \mathcal{A}[\![t]\!]$ be a regular element of order d. Then:

(i) there exists a unique pair $\big(p(t), u(t)\big) \in \mathcal{A}[\![t]\!]^2$ such that $f(t) = p(t)u(t)$, $p(t)$ is a Weierstrass polynomial of degree d and $u(t)$ is a unit in $\mathcal{A}[\![t]\!]$;

(ii) let $g \in \mathcal{A}[\![t]\!]$; then there exists a unique pair $\big(q(t), r(t)\big) \in \mathcal{A}[\![t]\!]^2$ such that $g(t) = f(t)q(t) + r(t)$ and $r(t)$ is a polynomial of degree $< d$.

Note that in particular any regular element of $\mathcal{A}[\![t]\!]$ is not a zero divisor in $\mathcal{A}[\![t]\!]$.

3. Recollection on arc scheme and toric varieties

The crucial objects of our study are arc schemes and toric varieties. For the convenience of the reader, we give in this section an overview of the main definitions and properties that we will use in the article. Along the way, we fix some notation and state and prove some technical lemmas useful for the sequel.

3.1.

Since we are only interested in local properties of arc schemes, we limit ourselves to the case of arc schemes associated with affine varieties. Proofs as well as more details on the general theory of arc schemes are to be found e.g., in [2, 10].

To every affine k-variety V one attaches its *arc scheme* $\mathscr{L}_\infty(V)$ which is an affine k-scheme characterized by the fact that for every k-algebra R one has a functorial bijection

$$\mathrm{Hom}_{\mathrm{Sch}_k}\big(\mathrm{Spec}(R), \mathscr{L}_\infty(V)\big) \cong \mathrm{Hom}_{\mathrm{Sch}_k}\big(\mathrm{Spec}\big(R[\![t]\!]\big), V\big). \tag{3.1}$$

A point of $\mathscr{L}_\infty(V)$ is called an *arc*. The above functorial bijection and the k-algebra morphism $R[\![t]\!] \to R$ mapping t to 0 induces a morphism of V-schemes $\mathscr{L}_\infty(V) \to V$ which sends an arc α to its *base-point* $\alpha(0)$.

We will need explicit equations of the affine scheme $\mathscr{L}_\infty(V)$ in terms of equations of the affine k-variety V. We begin with the case of the affine space. Let $\boldsymbol{Z} = \{Z_1, \dots, Z_h\}$ be a finite set of indeterminates. Consider the ring $k[\boldsymbol{Z}_\infty] := k\big[Z_{i,j} : i \in \{1, \dots, h\},\ j \in \mathbb{N}\big]$, and the k-algebra morphism $\varphi\colon k[\boldsymbol{Z}] \to k[\boldsymbol{Z}_\infty]$ mapping Z_i to $Z_{i,0}$. Then the affine k-scheme $\mathscr{L}_\infty(\mathbb{A}^h_k)$ is isomorphic to $\mathrm{Spec}\big(k[\boldsymbol{Z}_\infty]\big)$. The morphism $\varphi\colon k[\boldsymbol{Z}] \to k[\boldsymbol{Z}_\infty]$ induces a morphism $\mathcal{O}(\mathbb{A}^h_k) \to \mathcal{O}\big(\mathscr{L}_\infty(\mathbb{A}^h_k)\big)$ dual to the morphism $\alpha \mapsto \alpha(0)$.

For every $F \in k[\boldsymbol{Z}]$, define $\{F_s\}_{s\in\mathbb{N}} \in k[\boldsymbol{Z}_\infty]^{\mathbb{N}}$ by the following identity in $k[\boldsymbol{Z}_\infty][\![t]\!]$:

$$F\big|_{Z_i = \sum_{s\in\mathbb{N}} Z_{i,s} t^s} = \sum_{s\in\mathbb{N}} F_s t^s. \tag{3.2}$$

Note that $\mathrm{HS}\colon F \mapsto \sum_{s\in\mathbb{N}} F_s t^s$ is a morphism of k-algebras $k[\boldsymbol{Z}] \to k[\boldsymbol{Z}_\infty][\![t]\!]$. If $\mathfrak{i}$ is an ideal of $k[\boldsymbol{Z}]$, generated by a family $\{F_\delta : \delta \in \Delta\}$, the ideal $\langle F_{\delta,s} : \delta \in \Delta,\ s \in \mathbb{N}\rangle$ does not depend on the choice of the generating family. We denote it by $[\mathfrak{i}]$. (This notation is borrowed from differential algebra; for more information on the link between differential algebra and arc schemes, see e.g., [2].) The following lemma will be useful.

Lemma 3.1. — *Let $\mathfrak{i}$ be an ideal of $k[\boldsymbol{Z}]$. Let $d \leqslant h$ and $F \in k[\boldsymbol{Z}]$ such that F lies in the ideal quotient $\mathfrak{i} : \big(\prod_{i=1}^d Z_i\big)^\infty$. Let $(k_i) \in \mathbb{N}^d$ and $\mathfrak{a}$ be the ideal $\langle Z_{i,s} : 1 \leqslant i \leqslant h,\ 0 \leqslant s \leqslant k_i\rangle$ of $k[\boldsymbol{Z}_\infty]$. Let $G = \prod_{i=1}^d Z_{i,k_i}$. Then in the localization $k[\boldsymbol{Z}_\infty]_G$, the ideal $[\langle F\rangle]$ is contained in the ideal $[\mathfrak{i}] + \mathfrak{a}$.*

Proof. — Let $H \in \mathfrak{i}$ and $N \in \mathbb{N}$ such that $\left(\prod_{i=1}^{d} Z_i\right)^N F = H$. Applying HS and using the very definition of $\mathfrak{a}$, one obtains the relation

$$\prod_{i=1}^{d}\left(t^{k_i}\left[Z_{i,k_i} + t(\dots)\right]\right)^N \mathrm{HS}(F) = \mathrm{HS}(H) \pmod{\mathfrak{a}[\![t]\!]}.$$

Thus, setting $K := \sum_{i=1}^{d} Nk_i$, for $s < K$ one has $H_s \in \mathfrak{a}[\![t]\!]$ and one may write

$$\prod_{i=1}^{d}\left[Z_{i,k_i} + t(\dots)\right]^N \mathrm{HS}(F) = \sum_{s\geqslant 0} H_{s+K} t^s \pmod{\mathfrak{a}[\![t]\!]}.$$

By the definition of G, the series $\prod_{i=1}^{d}\left[Z_{i,k_i} + t(\dots)\right]^N$ is invertible in $k[\boldsymbol{Z}_\infty]_G[\![t]\!]$. That concludes the proof. □

Now if $\mathfrak{i}$ is an ideal of $k[\boldsymbol{Z}]$ and the affine k-scheme V is presented as $\mathrm{Spec}\big(k[\boldsymbol{Z}]/\mathfrak{i}\big)$, then the affine k-scheme $\mathscr{L}_\infty(V)$ is isomorphic to

$$\mathrm{Spec}\big(k[\boldsymbol{Z}_\infty]/[\mathfrak{i}]\big).$$

The morphism $\varphi\colon k[\boldsymbol{Z}] \to k[\boldsymbol{Z}_\infty]$ induces a morphism $\mathcal{O}(V) \to \mathcal{O}\big(\mathscr{L}_\infty(V)\big)$ dual to the morphism $\alpha \mapsto \alpha(0)$.

The morphism HS induces a morphism $\mathrm{HS}_V\colon \mathcal{O}(V) \to \mathcal{O}\big(\mathscr{L}_\infty(V)\big)[\![t]\!]$, dual to the so-called *universal arc on* V. If R is a k-algebra and

$$\alpha^*\colon \mathcal{O}\big(\mathscr{L}_\infty(V)\big) \longrightarrow R$$

is an R-point of $\mathscr{L}_\infty(V)$, inducing a k-algebra morphism

$$\alpha^*[\![t]\!]\colon \mathcal{O}\big(\mathscr{L}_\infty(V)\big)[\![t]\!] \longrightarrow R[\![t]\!],$$

the corresponding $R[\![t]\!]$-point of $\mathcal{O}(V)$ by bijection (3.1) is $\alpha^*[\![t]\!] \circ \mathrm{HS}_V$.

Let W be a closed k-subscheme of V and $\mathfrak{j} = \langle G_\gamma : \gamma \in \Gamma\rangle$ be an ideal of $k[\boldsymbol{Z}]$ such that $W \cong \mathrm{Spec}\big(k[\boldsymbol{Z}]/\mathfrak{i} + \mathfrak{j}\big)$. Then

$$\mathscr{L}_\infty(W) \cong \mathrm{Spec}\big(k[\boldsymbol{Z}_\infty]/[\mathfrak{i}] + [\mathfrak{j}]\big)$$

identifies with a closed subscheme of $\mathscr{L}_\infty(V)$ and the open subset $\mathscr{L}_\infty(V) \setminus \mathscr{L}_\infty(W)$ of $\mathscr{L}_\infty(V)$ is the union of the distinguished open subsets $\{G_{\gamma,s} \neq 0\}$ for $\gamma \in \Gamma$ and $s \in \mathbb{N}$.

An element of $\mathscr{L}_\infty(V) \setminus \mathscr{L}_\infty\big(\mathrm{nSm}(V)\big)$ is called a *non-degenerate arc.*

3.2.

(See, e.g., [14, 19, 20].) Let α be an arc of V with residue field $\kappa(\alpha)$, inducing a $\kappa(\alpha)[\![t]\!]$-point of V. Composing the morphism $\mathcal{O}(V) \to \kappa(\alpha)[\![t]\!]$ with the t-valuation defines a semivaluation $\mathrm{ord}_\alpha\colon \mathcal{O}(V) \to \mathbb{N}\cup\{+\infty\}$. Now let v be a divisorial valuation over V. The associated *Nash set*, or *maximal divisorial set*, is the closure in $\mathscr{L}_\infty(V)$ of the set $\{\alpha \in \mathscr{L}_\infty(V),\ \mathrm{ord}_\alpha = v\}$. It is an irreducible subset of $\mathscr{L}_\infty(V)$, denoted by $\mathcal{N}_v$.

3.3.

From now on, we introduce some notation and basic facts on normal toric varieties. (For further details, e.g., see [11, Sections 1.1 and 1.2].) Since we are studying local properties, in this article we can restrict ourselves to the case of affine normal toric varieties.

Let d be a positive integer and $\mathcal{T}$ a split algebraic k-torus of dimension d. Let $N \coloneqq \mathrm{Hom}(\mathbb{G}_{m,k}, \mathcal{T})$ be the group of its cocharacters which is a free $\mathbb{Z}$-module of rank d (i.e., a lattice isomorphic to $\mathbb{Z}^d$) and $M \coloneqq \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual $\mathbb{Z}$-module (i.e., the group of characters of $\mathcal{T}$). We denote by $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$) the $\mathbb{R}$-vector space of dimension d associated with N (resp. M). We have an $\mathbb{R}$-bilinear canonical map given by the usual pairing:

$$\langle \cdot, \cdot \rangle \colon M_{\mathbb{R}} \times N_{\mathbb{R}} \longrightarrow \mathbb{R}. \tag{3.3}$$

The points of the lattices N and M, considered as points of the associated vector spaces, are called their integral points. We will simply call a *cone* of $N_{\mathbb{R}}$ a strongly convex rational polyhedral cone of the vector space $N_{\mathbb{R}}$ (i.e., a convex cone generated by finitely many elements of N, which moreover does not contain any line).

3.4.

Let σ be a cone of $N_{\mathbb{R}}$. By Gordan's lemma (e.g., [11, Proposition 1.2.17]), the semigroup $S_\sigma \coloneqq \sigma^\vee \cap M$ is finitely generated. The spectrum of the k-algebra $k[S_\sigma]$ associated with the semigroup S_σ then defines a normal affine toric variety V_σ with torus $\mathcal{T}$. Note that every affine normal toric variety with torus $\mathcal{T}$ is of the form V_σ for σ a cone of N (see e.g., [11, Theorem 1.3.5]). For every $\boldsymbol{m} \in S_\sigma$ we denote by $\chi^{\boldsymbol{m}}$ the regular function on V_σ defined by $\boldsymbol{m}$. Recall that for every k-algebra A, the set $\mathrm{Hom}_{\mathrm{Alg}_k}\big(k[S_\sigma], A\big)$ is in natural bijection with the set of semigroup morphisms $S_\sigma \to A$, where the semigroup structure on A is induced by the multiplication.

Let $\{\boldsymbol{m}_1, \dots, \boldsymbol{m}_h\}$ be the minimal set of generators of the semigroup S_σ. We may and shall assume in the sequel that the set $\{\boldsymbol{m}_1, \dots, \boldsymbol{m}_d\}$ is a $\mathbb{Z}$-basis of M. If we call $z_1, \dots, z_h$ respectively $\chi^{\boldsymbol{m}_1}, \dots, \chi^{\boldsymbol{m}_h}$, we deduce that

$\mathcal{O}(V_\sigma) := k[S_\sigma] = k[z_1, \dots, z_h]$. Moreover, the closed subscheme defined by the ideal $\langle \prod_{1 \leqslant i \leqslant h} z_i \rangle$ has for support the closed set $V_\sigma \setminus \mathcal{T}$, and the same holds for the ideal $\langle \prod_{1 \leqslant i \leqslant d} z_i \rangle$.

Now let $\mathscr{L}_\infty^\circ(V_\sigma)$ be the open subset of $\mathscr{L}_\infty(V_\sigma)$ defined by $\mathscr{L}_\infty^\circ(V_\sigma) := \mathscr{L}_\infty(V_\sigma) \setminus \mathscr{L}_\infty(V_\sigma \setminus \mathcal{T})$. Thus, by Section 3.1, for any $\alpha \in \mathscr{L}_\infty(V_\sigma)$, one has $\alpha \in \mathscr{L}_\infty^\circ(V_\sigma)$ if and only if for every $\boldsymbol{m} \in S_\sigma$, $\alpha^*(\chi^{\boldsymbol{m}}) \neq 0$ if and only if for every $1 \leqslant i \leqslant d$ one has $\alpha^*(z_i) \in \kappa(\alpha)[\![t]\!] \setminus \{0\}$. Therefore one has

$$\mathscr{L}_\infty^\circ(V_\sigma) = \bigcap_{i=1}^{d} \bigcup_{s \in \mathbb{N}} \{z_{i,s} \neq 0\}.$$

3.5.

Let $\boldsymbol{n}$ be a point of $\sigma \cap N$. For $f \in \mathcal{O}(V_\sigma)$, set

$$\operatorname{ord}_{\boldsymbol{n}}(f) = \langle \boldsymbol{n}, f \rangle := \inf_{\chi^{\boldsymbol{m}} \in f} \langle \boldsymbol{m}, \boldsymbol{n} \rangle$$

(here $\chi^{\boldsymbol{m}} \in f$ means that the coefficient of $\chi^{\boldsymbol{m}}$ in the decomposition of f is not zero). Then $\operatorname{ord}_{\boldsymbol{n}}$ is a divisorial toric valuation on V_σ, and one easily sees that $\boldsymbol{n} \mapsto \operatorname{ord}_{\boldsymbol{n}}$ is a bijection between $\sigma \cap N$ and the set of divisorial toric valuations on V_σ. From now on we shall identify the later set with $N \cap \sigma$.

Let $\alpha \in \mathscr{L}_\infty^\circ(V_\sigma)$. The semigroup morphism $\boldsymbol{m} \in S_\sigma \mapsto \operatorname{ord}_t\big(\alpha^*(\chi^{\boldsymbol{m}})\big)$ extends uniquely to a group morphism $\boldsymbol{n}_\alpha \colon M \to \mathbb{Z}$ which is nonnegative on S_σ. In other words, $\boldsymbol{n}_\alpha$ is the unique element of $N \cap \sigma$ satisfying: for every $1 \leqslant i \leqslant h$, $\operatorname{ord}_t\big(\alpha^*(z_i)\big) = \langle \boldsymbol{m}_i, \boldsymbol{n}_\alpha \rangle$. For every $\boldsymbol{n} \in \sigma \cap N$, we set

$$\mathscr{L}_\infty^\circ(V_\sigma)_{\boldsymbol{n}} := \big\{\alpha \in \mathscr{L}_\infty^\circ(V_\sigma);\ \boldsymbol{n}_\alpha = \boldsymbol{n}\big\}$$
$$\text{and} \quad \mathscr{L}_\infty^\circ(V_\sigma)_{\geqslant \boldsymbol{n}} := \big\{\alpha \in \mathscr{L}_\infty^\circ(V_\sigma);\ \boldsymbol{n}_\alpha \in \boldsymbol{n} + \sigma\big\}.$$

Thus $\alpha \in \mathscr{L}_\infty^\circ(V_\sigma)_{\geqslant \boldsymbol{n}}$ if and only if for every $\boldsymbol{m} \in S_\sigma$ one has $\operatorname{ord}_t\big(\alpha^*(\chi^{\boldsymbol{m}})\big) \geqslant \langle \boldsymbol{m}, \boldsymbol{n} \rangle$ if and only if for every $1 \leqslant i \leqslant h$ one has $\operatorname{ord}_t\big(\alpha^*(\chi^{\boldsymbol{m}_i})\big) \geqslant \langle \boldsymbol{m}_i, \boldsymbol{n} \rangle$. If $\alpha \in \mathscr{L}_\infty^\circ(V_\sigma)_{\geqslant \boldsymbol{n}}$ and $\varphi_\alpha \colon S_\sigma \to \kappa(\alpha)[\![t]\!]$ is the associated semigroup morphism, then $\boldsymbol{m} \mapsto t^{-\langle \boldsymbol{m}, \boldsymbol{n} \rangle} \varphi_\alpha$ defines a semigroup morphism $\psi_\alpha \colon S_\sigma \to \kappa(\alpha)[\![t]\!]$, and $\boldsymbol{n}_\alpha = \boldsymbol{n}$ if and only if $\psi_\alpha(S_\sigma) \subset \big(\kappa(\alpha)[\![t]\!]\big)^\times$ if and only if for $1 \leqslant i \leqslant d$ one has $\operatorname{ord}_t\big(\alpha^*(\chi^{\boldsymbol{m}_i})\big) = \langle \boldsymbol{m}_i, \boldsymbol{n} \rangle$. Note also that the element of $\mathscr{L}_\infty(V_\sigma)(k)$ corresponding to the semigroup morphism $S_\alpha \to k[\![t]\!]$, $\boldsymbol{m} \mapsto t^{\langle \boldsymbol{m}, \boldsymbol{n} \rangle}$ lies in $\mathscr{L}_\infty^\circ(V_\sigma)_{\boldsymbol{n}}$, which is therefore nonempty.

The following lemma will be useful for describing the generic points of the Nash sets associated with toric valuations.

Lemma 3.2. — *Let* $\boldsymbol{n} \in \sigma \cap N$*.*

(i) *One has*

$$\mathscr{L}^{\circ}_{\infty}(V_{\sigma})_{\geqslant \boldsymbol{n}} = \mathscr{L}^{\circ}_{\infty}(V_{\sigma}) \cap \bigcap_{i=1}^{h} \bigcap_{s=0}^{\langle \boldsymbol{m}_i, \boldsymbol{n}\rangle - 1} \{z_{i,s} = 0\}.$$

(ii) *One has*

$$\mathscr{L}^{\circ}_{\infty}(V_{\sigma})_{\boldsymbol{n}} = \mathscr{L}^{\circ}_{\infty}(V_{\sigma}) \cap \left(\bigcap_{i=1}^{h} \bigcap_{s=0}^{\langle \boldsymbol{m}_i, \boldsymbol{n}\rangle - 1} \{z_{i,s} = 0\} \right) \cap \bigcap_{i=1}^{d} \{z_{i,\langle \boldsymbol{m}_i, \boldsymbol{n}\rangle} \neq 0\}.$$

(iii) *The closure of* $\mathscr{L}^{\circ}_{\infty}(V_{\sigma})_{\boldsymbol{n}}$ *coincides with the Nash set* $\mathcal{N}_{\boldsymbol{n}} = \mathcal{N}_{\mathrm{ord}_{\boldsymbol{n}}}$ *(see Section 3.2) associated with the toric valuation* $\boldsymbol{n}$*.*

(iv) *One has*

$$\mathscr{L}^{\circ}_{\infty}(V_{\sigma})_{\boldsymbol{n}} = \mathcal{N}_{\boldsymbol{n}} \cap \mathscr{L}^{\circ}_{\infty}(V_{\sigma}) \cap \bigcap_{i=1}^{d} \{z_{i,\langle \boldsymbol{m}_i, \boldsymbol{n}\rangle} \neq 0\}.$$

In particular, $\mathscr{L}^{\circ}_{\infty}(V_{\sigma})_{\boldsymbol{n}}$ *is a nonempty open subset of* $\mathcal{N}_{\boldsymbol{n}}$*.*

Proof. — Assertions (i) and (ii) are nothing but a reformulation of the above descriptions of $\mathscr{L}^{\circ}_{\infty}(V_{\sigma})_{\boldsymbol{n}}$ and $\mathscr{L}^{\circ}_{\infty}(V_{\sigma})_{\geqslant \boldsymbol{n}}$.

For a proof of (iii), see [20, Example 2.10].

Finally, assertion (iv) is a straightforward topological consequence of (ii) and (iii). □

3.6.

An explicit description of V_{σ} as a closed subscheme of the affine space $\mathbb{A}^h_k$ will be useful in the sequel.

Recall that $\{\boldsymbol{m}_1, \dots, \boldsymbol{m}_h\}$ is the minimal set of generators of S_{σ}, and that we may and shall assume that $\{\boldsymbol{m}_1, \dots, \boldsymbol{m}_d\}$ is a $\mathbb{Z}$-basis of M. Let $\big\{\boldsymbol{e}_i;\ i \in \{1, \dots, h\}\big\}$ be the canonical basis of $\mathbb{Z}^h$. Being given $\boldsymbol{\ell} = (\ell_1, \dots, \ell_h) \in \mathbb{Z}^h$, we set

$$\boldsymbol{\ell}^{+} = \sum_{\ell_i \geqslant 0} \ell_i \boldsymbol{e}_i \qquad \text{and} \qquad \boldsymbol{\ell}^{-} = -\sum_{\ell_i < 0} \ell_i \boldsymbol{e}_i,$$

which are both elements of $\mathbb{N}^h$. Note that $\boldsymbol{\ell} = \boldsymbol{\ell}^{+} - \boldsymbol{\ell}^{-}$. On the other hand, for $\boldsymbol{\ell} \in \mathbb{N}^h$, set $\boldsymbol{Z}^{\boldsymbol{\ell}} := \prod_{i=1}^{h} Z_i^{\ell_i}$ and $F_{\boldsymbol{\ell}} := \boldsymbol{Z}^{\boldsymbol{\ell}^{+}} - \boldsymbol{Z}^{\boldsymbol{\ell}^{-}}$.

Mapping $\boldsymbol{e}_i$ to $\pi(\boldsymbol{e}_i) := \boldsymbol{m}_i$ induces an exact sequence of groups

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^h \xrightarrow{\pi} M \longrightarrow 0, \tag{3.4}$$

where L is a subgroup of $\mathbb{Z}^h$. For $\boldsymbol{\ell} \in L$ and $\boldsymbol{n} \in N$, we introduce the following notation:

$$\boldsymbol{n} * \boldsymbol{\ell} := \underbrace{\Big\langle \boldsymbol{n}, \sum_{\substack{i=1 \\ \ell_i > 0}}^{h} \ell_i \boldsymbol{m}_i \Big\rangle}_{(3.3)} = -\Big\langle \boldsymbol{n}, \sum_{\substack{i=1 \\ \ell_i < 0}}^{h} \ell_i \boldsymbol{m}_i \Big\rangle.$$

By [11, Proposition 1.1.9], the ideal of $k[\boldsymbol{Z}]$ defining V_σ is

$$\mathfrak{i}_\sigma := \langle F_{\boldsymbol{\ell}} : \boldsymbol{\ell} \in L \rangle. \tag{3.5}$$

The set $\{\boldsymbol{m}_1, \ldots, \boldsymbol{m}_d\}$ being a $\mathbb{Z}$-basis of M, for $q \in \{d+1, \ldots, h\}$, we can write the element $\boldsymbol{m}_q$ as a linear combination with (possibly negative) integer coefficients of $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_d$. Thus we have in L an element $\boldsymbol{\ell}_q = (\ell_{q,1}, \ldots, \ell_{q,h})$ such that $\ell_{q,q} = 1$ and $\ell_{q,q'} = 0$ for every $q' \in \{d+1, \ldots, h\} \setminus \{q\}$. The element $\boldsymbol{\ell}_q \in L$ induces an element

$$F_{\boldsymbol{\ell}_q} = Z_q \prod_{\substack{i=1 \\ \ell_{q,i} \geqslant 0}}^{d} Z_i^{\ell_{q,i}} - \prod_{\substack{i=1 \\ \ell_{q,i} < 0}}^{d} Z_i^{-\ell_{q,i}} \tag{3.6}$$

in the ideal $\mathfrak{i}_\sigma$. We observe that in the binomial $F_{\boldsymbol{\ell}_q}$ none of the variables $Z_{d+1}, \ldots, Z_h$ appears, excepting Z_q.

LEMMA 3.3. — *Let* $\mathfrak{j} := \langle F_{\boldsymbol{\ell}_q} : d+1 \leqslant q \leqslant h \rangle$.

- (i) *Set* $G_d := \prod_{i=1}^{d} Z_i$. *For every* $\boldsymbol{\ell} \in L$, $F_{\boldsymbol{\ell}}$ *lies in the ideal quotient* $\mathfrak{j} : G_d^\infty$. *In other words, the ideal* $\mathfrak{i}_\sigma$ *vanishes in* $k[\boldsymbol{Z}]_{G_d}/\mathfrak{j}$.
- (ii) *Let* $\boldsymbol{n} \in \sigma \cap N$ *and* $\mathfrak{a}_{\boldsymbol{n}}$ *be the ideal* $\big\langle \big\{ Z_{i,s_i} : 1 \leqslant i \leqslant h, 0 \leqslant s_i < \langle \boldsymbol{n}, \boldsymbol{m}_i \rangle \big\} \big\rangle$ *of* $k[\boldsymbol{Z}_\infty]$. *Let* $G_{\boldsymbol{n}} := \prod_{i=1}^{d} Z_{i,\langle \boldsymbol{n}, \boldsymbol{m}_i \rangle}$. *Then the ideals* $[\mathfrak{i}_\sigma] + \mathfrak{a}_{\boldsymbol{n}}$ *and* $[\mathfrak{j}] + \mathfrak{a}_{\boldsymbol{n}}$ *coincide in the localization* $k[\boldsymbol{Z}_\infty]_{G_{\boldsymbol{n}}}$.

Proof. — Set $G_h := \prod_{i=1}^{h} Z_i$. Since $\{\boldsymbol{\ell}_q : d+1 \leqslant q \leqslant h\}$ spans the lattice L, [28, Lemma 12.2] shows that $\mathfrak{i}_\sigma$ vanishes in $k[\boldsymbol{Z}]_{G_h}/\mathfrak{j}$. But (3.6) shows that the natural morphism $k[\boldsymbol{Z}]_{G_d} \to k[\boldsymbol{Z}]_{G_h}$ induces an isomorphism $k[\boldsymbol{Z}]_{G_d}/\mathfrak{j} \cong k[\boldsymbol{Z}]_{G_h}/\mathfrak{j}$. This shows (i).

By (i) and Lemma 3.1, in the localization $k[\boldsymbol{Z}_\infty]_{G_{\boldsymbol{n}}}$, the ideal $[\mathfrak{i}_\sigma]$ is contained in $[\mathfrak{j}] + \mathfrak{a}_{\boldsymbol{n}}$. Since the inclusion $[\mathfrak{j}] \subset [\mathfrak{i}_\sigma]$ holds by definition, one deduces that (ii) also holds. □

4. Technical machinery for computing the formal neighborhood at the generic point of the Nash set

In this section we develop the technical results which we will use in Section 6 to obtain a convenient presentation of the formal neighborhood of the generic point of the Nash set associated with a divisorial toric valuation. The main result of this section is Theorem 4.4, whose hypotheses are formulated in a somewhat abstract form. In Section 6 we will verify that these hypotheses hold in the toric setting.

4.1.

We first state a version of Hensel's lemma for an arbitrary set of variables. The proof is basically the same as in the case of a finite set of variables. Since we have not been able to find a convenient reference, we include it.

PROPOSITION 4.1. — *Let $(\mathcal{A},\mathfrak{M}_{\mathcal{A}})$ be a complete local ring with residue field κ. Let I be a set and $\boldsymbol{Y}=\{Y_i\}_{i\in I}$ be a collection of indeterminates. Let J be a set and $\{F_j;\ j\in J\}$ be a collection of elements in $\mathcal{A}[\boldsymbol{Y}]$. For $\boldsymbol{y}\in\mathcal{A}^I$, we denote by $\boldsymbol{J}_{\boldsymbol{y}}$ the $\mathcal{A}$-linear map $\mathcal{A}^I\to\mathcal{A}^J$ induced by the Jacobian matrix $[\partial_{Y_i}F_j]\big|_{\boldsymbol{Y}=\boldsymbol{y}}$, and by $\boldsymbol{F}\big|_{\boldsymbol{Y}=\boldsymbol{y}}\in\mathcal{A}^J$ the J-tuple $\big(F_j\big|_{\boldsymbol{Y}=\boldsymbol{y}};\ j\in J\big)$.*

We assume that there exists $\boldsymbol{y}^{(0)}\in\mathcal{A}^I$ such that:

(1) *one has $\boldsymbol{F}\big|_{\boldsymbol{Y}=\boldsymbol{y}^{(0)}}=0 \pmod{\mathfrak{M}_{\mathcal{A}}}$;*
(2) *the κ-linear map $\kappa^I\to\kappa^J$ deduced from $\boldsymbol{J}_{\boldsymbol{y}^{(0)}}$ by reduction modulo $\mathfrak{M}_{\mathcal{A}}$ is invertible.*

Then there exists a unique element $\boldsymbol{\mathcal{Y}}=(\mathcal{Y}_i)\in\mathcal{A}^I$ such that:

(1) *one has $\boldsymbol{F}\big|_{\boldsymbol{Y}=\boldsymbol{\mathcal{Y}}}=0$;*
(2) *for every $i\in I$, one has $\mathcal{Y}_i=y_i^{(0)} \pmod{\mathfrak{M}_{\mathcal{A}}}$.*

Proof. — We begin with two remarks.

First, note that though in this context the Jacobian matrix may have an infinite number of rows and columns, each row has only a finite number of nonzero entries, thus $\boldsymbol{J}_{\boldsymbol{y}}$ is well defined as a map $\mathcal{A}^I\to\mathcal{A}^J$ for any $\boldsymbol{y}$ in $\mathcal{A}^I$. Also, by assumption, there exists an $\mathcal{A}$-linear map

$$\boldsymbol{K}_{\boldsymbol{y}^{(0)}}:\mathcal{A}^J\longrightarrow\mathcal{A}^I$$

such that $\boldsymbol{K}_{\boldsymbol{y}^{(0)}}\boldsymbol{J}_{\boldsymbol{y}^{(0)}}=\mathrm{Id}_{\mathcal{A}^I} \pmod{\mathfrak{M}_{\mathcal{A}}}$ and $\boldsymbol{J}_{\boldsymbol{y}^{(0)}}\boldsymbol{K}_{\boldsymbol{y}^{(0)}}=\mathrm{Id}_{\mathcal{A}^J} \pmod{\mathfrak{M}_{\mathcal{A}}}$.

Second, note that by the Taylor formula, for $\boldsymbol{y}\in\mathcal{A}^I$, there exists a family $\{\boldsymbol{H}_{i_1,i_2}:i_1,i_2\in I\}$ of elements of $\mathcal{A}[\boldsymbol{Y}]^J$, depending on $\boldsymbol{y}$ and the F_j's, such that for every $j \in J$, $H_{i_1,i_2,j} = 0$ for all but finitely many (i_1, i_2) and for every $\boldsymbol{z} \in \mathcal{A}^I$ one has

$$\boldsymbol{F}\big|_{\boldsymbol{Y}=\boldsymbol{y}+\boldsymbol{z}} = \boldsymbol{F}\big|_{\boldsymbol{Y}=\boldsymbol{y}} + \boldsymbol{J}_{\boldsymbol{y}}(\boldsymbol{z}) + \sum_{i_1,i_2\in I} z_{i_1} z_{i_2} \boldsymbol{H}_{i_1,i_2}\big|_{\boldsymbol{Y}=\boldsymbol{y}+\boldsymbol{z}}. \tag{4.1}$$

Note that here and elsewhere the notation we use is a condensed form for writing a possibly infinite number of relations, each of them being easily verified.

We show by induction that for every $e \geqslant 0$, there exists a family $\boldsymbol{y}^{(e)} = (y_i^{(e)};\ i \in I)$ of elements of $\mathcal{A}$, unique modulo $\mathfrak{M}_{\mathcal{A}}^{e+1}$, such that $\boldsymbol{y}^{(e)} = \boldsymbol{y}^{(0)} \pmod{\mathfrak{M}_{\mathcal{A}}}$ and $\boldsymbol{F}\big|_{\boldsymbol{Y}=\boldsymbol{y}^{(e)}} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$. The case $e = 0$ is given by our assumptions.

Now take $e \in \mathbb{N}$ and assume that our induction statement holds for e. Consider the equation

$$\boldsymbol{F}\big|_{\boldsymbol{Y}=\boldsymbol{y}^{(e)}+\boldsymbol{z}} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+2}} \tag{4.2}$$

with unknown $\boldsymbol{z} = (z_i) \in \mathcal{A}^I$ such that $\boldsymbol{z} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$. Since $\boldsymbol{y}^{(e)} = \boldsymbol{y}^{(0)} \pmod{\mathfrak{M}_{\mathcal{A}}}$, the Jacobian matrices $[\partial_{Y_i} F_j]\big|_{\boldsymbol{Y}=\boldsymbol{y}^{(0)}}$ and $[\partial_{Y_i} F_j]\big|_{\boldsymbol{Y}=\boldsymbol{y}^{(e)}}$ are equal modulo $\mathfrak{M}_{\mathcal{A}}$. Since $\boldsymbol{z} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$, one thus has

$$\boldsymbol{J}_{\boldsymbol{y}^{(e)}}(\boldsymbol{z}) = \boldsymbol{J}_{\boldsymbol{y}^{(0)}}(\boldsymbol{z}) \pmod{\mathfrak{M}_{\mathcal{A}}^{e+2}}.$$

Thus by (4.1) and using again $\boldsymbol{z} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$, equation (4.2) is equivalent to

$$\boldsymbol{J}_{\boldsymbol{y}^{(0)}}(\boldsymbol{z}) = -\boldsymbol{F}\big|_{\boldsymbol{Y}=\boldsymbol{y}^{(e)}} \pmod{\mathfrak{M}_{\mathcal{A}}^{e+2}}. \tag{4.3}$$

By assumption, $\boldsymbol{F}\big|_{\boldsymbol{Y}=\boldsymbol{y}^{(e)}} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$. Thus by the first remark above, and using $\boldsymbol{z} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$ one more time, (4.3) is equivalent to

$$\boldsymbol{z} = -\boldsymbol{K}_{\boldsymbol{y}^{(0)}}\big(\boldsymbol{F}\big|_{\boldsymbol{Y}=\boldsymbol{y}^{(e)}}\big) \pmod{\mathfrak{M}_{\mathcal{A}}^{e+2}}.$$

Since $\boldsymbol{F}\big|_{\boldsymbol{Y}=\boldsymbol{y}^{(e)}} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$, the latter expression gives indeed a solution $\boldsymbol{z}$ such that $\boldsymbol{z} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$.

In order to show the uniqueness of the solution modulo $\mathfrak{M}_{\mathcal{A}}^{e+2}$, note that if $\boldsymbol{w} \in \mathcal{A}^I$ is such that $\boldsymbol{w} = 0 \pmod{\mathfrak{M}_{\mathcal{A}}^{e+1}}$, one has by (4.1)

$$\boldsymbol{F}\big|_{\boldsymbol{Y}=\boldsymbol{y}^{(e)}+\boldsymbol{w}} = \boldsymbol{F}\big|_{\boldsymbol{Y}=\boldsymbol{y}^{(e)}} + \boldsymbol{J}_{\boldsymbol{y}^{(e)}}(\boldsymbol{w}) \pmod{\mathfrak{M}_{\mathcal{A}}^{e+2}}$$

thus

$$\boldsymbol{K}_{\boldsymbol{y}^{(0)}}\big(\boldsymbol{F}\big|_{\boldsymbol{Y}=\boldsymbol{y}^{(e)}+\boldsymbol{w}}\big) = \boldsymbol{K}_{\boldsymbol{y}^{(0)}}\big(\boldsymbol{F}\big|_{\boldsymbol{Y}=\boldsymbol{y}^{(e)}}\big) + \boldsymbol{K}_{\boldsymbol{y}^{(0)}}\big(\boldsymbol{J}_{\boldsymbol{y}^{(e)}}(\boldsymbol{w})\big) \pmod{\mathfrak{M}_{\mathcal{A}}^{e+2}}$$

and finally

$$\boldsymbol{K}_{\boldsymbol{y}^{(0)}}\big(\boldsymbol{F}\big|_{\boldsymbol{Y}=\boldsymbol{y}^{(e)}+\boldsymbol{w}}\big) = \boldsymbol{K}_{\boldsymbol{y}^{(0)}}\big(\boldsymbol{F}\big|_{\boldsymbol{Y}=\boldsymbol{y}^{(e)}}\big) + \boldsymbol{w} \pmod{\mathfrak{M}_{\mathcal{A}}^{e+2}}. \qquad \square$$

4.2.

We consider the following general setting and notation for the rest of this section. Let A be a k-algebra which is a domain. Let Ω be a finite set, I be a set, $\boldsymbol{X} = \{X_\omega\}_{\omega\in\Omega}$ and $\boldsymbol{Y} = \{Y_i;\ i \in I\}$ be collections of indeterminates. Set

$$A[\boldsymbol{X}] := A\big[\{X_\omega\}_{\omega\in\Omega}\big] \qquad \text{and} \qquad A[\boldsymbol{X},\boldsymbol{Y}] := A\big[\{X_\omega\}_{\omega\in\Omega}, \{Y_i;\ i\in I\}\big].$$

We denote by $\langle \boldsymbol{X}\rangle$ the prime ideal $\langle X_\omega;\ \omega \in \Omega\rangle$ of $A[\boldsymbol{X}]$. In accordance with Section 2.3, for any $A[\boldsymbol{X}]$-algebra B, we often still denote by $\langle \boldsymbol{X}\rangle$ the extension of the ideal $\langle \boldsymbol{X}\rangle$ to B.

4.3.

The following lemma will be useful in the proof of Theorem 4.4.

Lemma 4.2. — *Assume that we are in the setting described in Section 4.2. Let $\mathfrak{h}$ be an ideal of $A[\boldsymbol{X},\boldsymbol{Y}]$ such that:*

(i) *one has $\langle \boldsymbol{X}\rangle + \mathfrak{h} = \langle \boldsymbol{X},\boldsymbol{Y}\rangle$.*

Assume moreover that there exists an $A[\boldsymbol{X}]$-algebra morphism

$$\widehat{\varepsilon}\colon A[\boldsymbol{X},\boldsymbol{Y}] \longrightarrow \operatorname{Frac}(A)[\![\boldsymbol{X}]\!]$$

such that:

(ii) *for every $i \in I$ one has $\widehat{\varepsilon}(Y_i) = Y_i \pmod{\mathfrak{h}}$ in the ring*

$$\operatorname{Frac}(A)[\![\boldsymbol{X}]\!][\boldsymbol{Y}];$$

(iii) *for every $i \in I$, one has $\widehat{\varepsilon}(Y_i) \in \langle \boldsymbol{X}\rangle$.*

Then the $\langle \boldsymbol{X}\rangle$-adic completion of the localization $\big(A[\boldsymbol{X},\boldsymbol{Y}]/\mathfrak{h}\big)_{\langle \boldsymbol{X}\rangle}$ is isomorphic to $\operatorname{Frac}(A)[\![\boldsymbol{X}]\!]/\widehat{\varepsilon}(\mathfrak{h})$.

Remark 4.3. — Assume that the hypotheses of the lemma hold. Let $\mathfrak{g}$ be any ideal of $A[\boldsymbol{X},\boldsymbol{Y}]$ containing $\mathfrak{h}$ such that $\langle \boldsymbol{X}\rangle + \mathfrak{h} = \langle \boldsymbol{X}\rangle + \mathfrak{g}$ and $\widehat{\varepsilon}(\mathfrak{h}) = \widehat{\varepsilon}(\mathfrak{g})$. Then $\mathfrak{g}$ also satisfies the hypotheses of the lemma, with the same morphism $\widehat{\varepsilon}$. In particular, the lemma shows that the $\langle \boldsymbol{X}\rangle$-adic completions of $\big(A[\boldsymbol{X},\boldsymbol{Y}]/\mathfrak{h}\big)_{\langle \boldsymbol{X}\rangle}$ and $\big(A[\boldsymbol{X},\boldsymbol{Y}]/\mathfrak{g}\big)_{\langle \boldsymbol{X}\rangle}$ are isomorphic.

Proof. — Note that, by (i), we have $\mathfrak{h} \in \langle \boldsymbol{X},\boldsymbol{Y}\rangle$; then, by (iii), we can deduce that $\widehat{\varepsilon}(\mathfrak{h})$ is contained in $\langle \boldsymbol{X}\rangle$. Thus, we can observe that $\operatorname{Frac}(A)[\![\boldsymbol{X}]\!]/\widehat{\varepsilon}(\mathfrak{h})$ is a complete Noetherian local ring with maximal ideal

$\langle \boldsymbol{X} \rangle$. Moreover, (i) and the fact that A is a domain show that $\langle \boldsymbol{X} \rangle$ is indeed a prime ideal of $A[\boldsymbol{X}, \boldsymbol{Y}]/\mathfrak{h}$.

Let $e \geqslant 1$. Let π_e be the composition of $\widehat{\varepsilon}$ with the quotient morphism

$$\mathrm{Frac}(A)[\![\boldsymbol{X}]\!]/\widehat{\varepsilon}(\mathfrak{h}) \longrightarrow \mathrm{Frac}(A)[\boldsymbol{X}]/\big(\widehat{\varepsilon}(\mathfrak{h}) + \langle \boldsymbol{X} \rangle^e\big).$$

Thanks to (iii), any element of $A[\boldsymbol{X}, \boldsymbol{Y}]$ whose constant term is not zero is sent by $\widehat{\varepsilon}$ to an invertible element of $\mathrm{Frac}(A)[\![\boldsymbol{X}]\!]$. Thus π_e induces a morphism

$$A[\boldsymbol{X}, \boldsymbol{Y}]_{\langle \boldsymbol{X}, \boldsymbol{Y} \rangle} \longrightarrow \mathrm{Frac}(A)[\boldsymbol{X}]/\big(\widehat{\varepsilon}(\mathfrak{h}) + \langle \boldsymbol{X} \rangle^e\big)$$

which in turn induces a morphism

$$\widetilde{\pi}_e \colon A[\boldsymbol{X}, \boldsymbol{Y}]_{\langle \boldsymbol{X}, \boldsymbol{Y} \rangle}/\big(\mathfrak{h} + \langle \boldsymbol{X} \rangle^e\big) \longrightarrow \mathrm{Frac}(A)[\boldsymbol{X}]/\big(\widehat{\varepsilon}(\mathfrak{h}) + \langle \boldsymbol{X} \rangle^e\big).$$

Note that since $\mathfrak{h} + \langle \boldsymbol{X} \rangle = \langle \boldsymbol{X}, \boldsymbol{Y} \rangle$, one has $\mathfrak{h} + \langle \boldsymbol{X} \rangle^e = \mathfrak{h} + \langle \boldsymbol{X}, \boldsymbol{Y} \rangle^e$. Thus in order to obtain the claimed isomorphism, it suffices to show that $\widetilde{\pi}_e$ is an isomorphism for any $e \geqslant 1$. Since the natural inclusion $A[\boldsymbol{X}, \boldsymbol{Y}]_{\langle \boldsymbol{X}, \boldsymbol{Y} \rangle} \subset \mathrm{Frac}(A)[\boldsymbol{X}, \boldsymbol{Y}]_{\langle \boldsymbol{X}, \boldsymbol{Y} \rangle}$ is an isomorphism, surjectivity is clear.

Let us show injectivity. This amounts to showing that if

$$P \in \mathrm{Frac}(A)[\boldsymbol{X}, \boldsymbol{Y}]$$

lies in $\mathrm{Ker}(\pi_e)$, then $P \in \mathfrak{h} + \langle \boldsymbol{X} \rangle^e$. By assumption (ii), for any $P \in A[\boldsymbol{X}, \boldsymbol{Y}]$, one has $\widehat{\varepsilon}(P) = P \pmod{\mathfrak{h}}$ in the ring $\mathrm{Frac}(A)[\![\boldsymbol{X}]\!][\boldsymbol{Y}]$. In particular in the ring $\mathrm{Frac}(A)[\boldsymbol{X}, \boldsymbol{Y}]$ one has

$$\widehat{\varepsilon}(P) + \langle \boldsymbol{X} \rangle^e = P + \mathfrak{h} + \langle \boldsymbol{X} \rangle^e \qquad \text{and} \qquad \widehat{\varepsilon}(\mathfrak{h}) + \langle \boldsymbol{X} \rangle^e \subset \mathfrak{h} + \langle \boldsymbol{X} \rangle^e.$$

Now if $P \in \mathrm{Frac}(A)[\boldsymbol{X}, \boldsymbol{Y}]$ lies in $\mathrm{Ker}(\pi_e)$, then one has $\widehat{\varepsilon}(P) + \langle \boldsymbol{X} \rangle^e \subset \widehat{\varepsilon}(\mathfrak{h}) + \langle \boldsymbol{X} \rangle^e$. Therefore, by the above properties, one has $P + \mathfrak{h} + \langle \boldsymbol{X} \rangle^e \subset \mathfrak{h} + \langle \boldsymbol{X} \rangle^e$. Thus $P \in \mathfrak{h} + \langle \boldsymbol{X} \rangle^e$. That concludes the proof. □

Now we can state and prove the main result of the section.

Theorem 4.4. — *Assume that we are in the setting described in Section 4.2; we assume moreover that the set I is of the shape $\Gamma \times \mathbb{N}$ where Γ is a finite set.*

Let $\mathfrak{h}$ be an ideal of $A[\boldsymbol{X}, \boldsymbol{Y}]$ such that:

(A) *the ideal $\mathfrak{h}$ contains a collection of elements $\{H_{\gamma,s},\ \gamma \in \Gamma,\ s \in \mathbb{N}\}$ of the form $H_{\gamma,s} = Y_{\gamma,s} U_{\gamma,s} + E_{\gamma,s}$ such that for every $\gamma \in \Gamma$ and every $s \in \mathbb{N}$:*

(A1) *$U_{\gamma,s}$ is a unit in A;*

(A2) *there exists a family* $(E_{\gamma,s,r}) \in A[\boldsymbol{X}]^{\mathbb{N}\cup\{-1\}}$ *such that* $E_{\gamma,s,r} \in \langle \boldsymbol{X} \rangle$ *for* $r \geqslant s$, $E_{\gamma,s,r} = 0$ *for all but a finite number of* r, *and one has*

$$E_{\gamma,s} = E_{\gamma,s,-1} + \sum_{r\in\mathbb{N}} E_{\gamma,s,r} \cdot Y_{\gamma,r};$$

(B) *let* $(y_{\gamma,s}) \in A^{\Gamma\times\mathbb{N}}$ *be the unique family of elements of* A *such that for every* $\gamma \in \Gamma$ *and* $s \in \mathbb{N}$, *one has* $H_{\gamma,s}\big|_{Y_{\gamma,r}=y_{\gamma,r}} = 0 \pmod{\langle \boldsymbol{X} \rangle}$; *then the ideal* $\langle \boldsymbol{X} \rangle + \mathfrak{h}$ *is contained in the ideal* $\langle \boldsymbol{X} \rangle + \big\langle Y_{\gamma,s} - y_{\gamma,s};\ (\gamma, s) \in \Gamma \times \mathbb{N} \big\rangle$.

Then there exists an $A[\boldsymbol{X}]$*-algebra morphism* $\widehat{\varepsilon}\colon A[\boldsymbol{X}, \boldsymbol{Y}] \to \operatorname{Frac}(A)[\![\boldsymbol{X}]\!]$ *such that:*

(i) *for every* $(\gamma, s) \in \Gamma \times \mathbb{N}$ *one has* $\widehat{\varepsilon}(H_{\gamma,s}) = 0$;

(ii) *for every* $(\gamma, s) \in \Gamma \times \mathbb{N}$, *one has* $\widehat{\varepsilon}(Y_{\gamma,s}) = Y_{\gamma,s} \pmod{\mathfrak{h}}$ *in the ring* $\operatorname{Frac}(A)[\![\boldsymbol{X}]\!][\boldsymbol{Y}]$;

(iii) *for every ideal* $\mathfrak{g}$ *containing* $\mathfrak{h}$ *such that* $\langle \boldsymbol{X} \rangle + \mathfrak{h} = \langle \boldsymbol{X} \rangle + \mathfrak{g}$ *and* $\widehat{\varepsilon}(\mathfrak{h}) = \widehat{\varepsilon}(\mathfrak{g})$, *the* $\langle \boldsymbol{X} \rangle$*-adic completion of the localization*

$$\big(A[\boldsymbol{X}, \boldsymbol{Y}]/\mathfrak{g}\big)_{\langle \boldsymbol{X} \rangle}$$

is $\langle \boldsymbol{X} \rangle$*-adically isomorphic to* $\operatorname{Frac}(A)[\![\boldsymbol{X}]\!]/\widehat{\varepsilon}(\mathfrak{h})$.

Assume moreover that:

(C) *for every* $\gamma \in \Gamma$, *one has* $E_{\gamma,0,-1} \in A[\boldsymbol{X}] \setminus \langle \boldsymbol{X} \rangle$.

Then one has in addition:

(iv) *for every* $\gamma \in \Gamma$, $\widehat{\varepsilon}(Y_{\gamma,0})$ *is a unit in* $\operatorname{Frac}(A)[\![\boldsymbol{X}]\!]$.

Proof. — First note that for each $\gamma \in \Gamma$, the reduction of the $H_{\gamma,s}$'s modulo $\langle \boldsymbol{X} \rangle$ gives a triangular and invertible A-linear system in the $Y_{\gamma,s}$'s. Thus the existence and uniqueness of $(y_{\gamma,s})$ in assumption (B) is a straightforward consequence of assumption (A). In fact, up to dividing $H_{\gamma,s}$ by $U_{\gamma,s}$ and modifying the $E_{\gamma,s,r}$'s, one may assume that for every γ, s, r one has $E_{\gamma,s,r} \in \langle \boldsymbol{X} \rangle$ and that for every γ, s one has

$$H_{\gamma,s} = Y_{\gamma,s} - y_{\gamma,s} + \sum_{r=0}^{s-1} \alpha_{\gamma,r}(Y_{\gamma,r} - y_{\gamma,r}) + E_{\gamma,s,-1} + \sum_{r\in\mathbb{N}} E_{\gamma,s,r}(Y_{\gamma,r} - y_{\gamma,r}), \tag{4.4}$$

where the $\alpha_{\gamma,r}$'s are elements of A.

Applying Proposition 4.1 with $\mathcal{A} = \operatorname{Frac}(A)[\![\boldsymbol{X}]\!]$ and $\{F_j;\ j \in J\} = \big\{H_{\gamma,s};\ (\gamma, s) \in \Gamma \times \mathbb{N}\big\}$, this shows the existence of a family $\{\mathcal{Y}_{\gamma,s};\ \gamma \in$

$\Gamma,\ s\in\mathbb{N}\}$ of elements of $\mathrm{Frac}(A)[\![\boldsymbol{X}]\!]$ such that for every $(\gamma,s)\in\Gamma\times\mathbb{N}$ one has $\mathcal{Y}_{\gamma,s}=y_{\gamma,s}\pmod{\langle\boldsymbol{X}\rangle}$ and $H_{\gamma,s}\big|_{Y_{\gamma,r}=\mathcal{Y}_{\gamma,r}}=0$. Thus mapping $Y_{\gamma,s}$ to $\mathcal{Y}_{\gamma,s}$ defines an $A[\boldsymbol{X}]$-algebra morphism $\widehat{\varepsilon}\colon A[\boldsymbol{X},\boldsymbol{Y}]\to\mathrm{Frac}(A)[\![\boldsymbol{X}]\!]$ such that (i) holds.

For every $\gamma\in\Gamma$, (4.4) and an induction on s show that for every s one has $Y_{\gamma,s}-y_{\gamma,s}\in\langle\boldsymbol{X}\rangle+\mathfrak{h}$. By assumption (B), one then has $\langle\boldsymbol{X}\rangle+\mathfrak{h}=\langle\boldsymbol{X}\rangle+\big\langle Y_{\gamma,s}-y_{\gamma,s};\ (\gamma,s)\in\Gamma\times\mathbb{N}\big\rangle$. Thus $\mathrm{Frac}(A)[\![\boldsymbol{X}]\!][\boldsymbol{Y}]/\mathfrak{h}$ is a Noetherian local ring with maximal ideal $\langle\boldsymbol{X}\rangle$.

On the other hand, (4.4) shows that for every $(\gamma,s)\in\Gamma\times\mathbb{N}$, since $H_{\gamma,s}\in\mathfrak{h}$ and $\widehat{\varepsilon}(H_{\gamma,s})=0$, one has in the ring $\mathrm{Frac}(A)[\![\boldsymbol{X}]\!][\boldsymbol{Y}]$ the relation

$$\mathcal{Y}_{\gamma,s}-Y_{\gamma,s}=-\sum_{r=0}^{s-1}\alpha_{\gamma,r}(\mathcal{Y}_{\gamma,r}-Y_{\gamma,r})-\sum_{r\geqslant 0}E_{\gamma,s,r}(\mathcal{Y}_{\gamma,r}-Y_{\gamma,r})\pmod{\mathfrak{h}}.$$

Thus by a straightforward induction one gets that $\mathcal{Y}_{\gamma,s}-Y_{\gamma,s}\in\langle\boldsymbol{X}\rangle^e+\mathfrak{h}$ for every γ,s and $e\geqslant 1$, and finally by Krull's intersection theorem $\mathcal{Y}_{\gamma,s}-Y_{\gamma,s}\in\mathfrak{h}$ for every γ,s. Thus (ii) holds. Recalling that $\mathcal{Y}_{\gamma,s}-y_{\gamma,s}\in\langle\boldsymbol{X}\rangle$, (iii) then follows from an application of Lemma 4.2 (replacing $Y_{\gamma,s}$ with $Y_{\gamma,s}-y_{\gamma,s}$) and Remark 4.3.

Assumption (C) is equivalent to the property $y_{\gamma,0}\in A\setminus\{0\}$. Then $y_{\gamma,0}$ is a unit in $\mathrm{Frac}(A)[\![\boldsymbol{X}]\!]$, and since $\mathcal{Y}_{\gamma,0}=y_{\gamma,0}\pmod{\langle\boldsymbol{X}\rangle}$, $\mathcal{Y}_{\gamma,0}=\widehat{\varepsilon}(Y_{\gamma,0})$ also is a unit, and (iv) holds. ☐

Remark 4.5. — In the statement of the theorem, if one assumes that (A) holds and that moreover $\mathfrak{h}$ is generated by the $H_{\gamma,s}$'s and some elements of $\langle\boldsymbol{X}\rangle$, then (B) automatically holds. Indeed, the above proof shows that without changing the ideal generated by the $H_{\gamma,s}$'s, one may assume that for every γ,s one has $H_{\gamma,s}=Y_{\gamma,s}-y_{\gamma,s}\pmod{\langle\boldsymbol{X}\rangle}$.

5. Technical machinery for the comparison theorem

In this section we will obtain the crucial technical result (Theorem 5.4) allowing us to establish our comparison theorem in Section 6. As for Theorem 4.4 the hypotheses are formulated in a somewhat abstract form, and in Section 6 we will verify that these hypotheses hold in the toric setting.

5.1.

We begin with an elementary yet useful lemma.

LEMMA 5.1. — *Let K be a field, $\mathcal{A}$ be an object of $\mathfrak{LcCpl}_K$, $\mathfrak{a}$ and $\mathfrak{b}$ two ideals of $\mathcal{A}$ such that for every object $\mathcal{B}$ of $\mathfrak{LcCpl}_K$ one has the inclusion*

$$\left\{\varphi \in \mathrm{Hom}_{\mathfrak{LcCpl}_K}(\mathcal{A},\mathcal{B}),\ \mathfrak{b} \subset \mathrm{Ker}(\varphi)\right\} \subset \left\{\varphi \in \mathrm{Hom}_{\mathfrak{LcCpl}_K}(\mathcal{A},\mathcal{B}),\ \mathfrak{a} \subset \mathrm{Ker}(\varphi)\right\}.$$

Then one has the inclusion $\mathfrak{a} \subset \mathfrak{b}$.

Proof. — We apply the assumption with φ the quotient morphism $\mathcal{A} \to \mathcal{A}/\mathfrak{b}$. □

Notation 5.2. — Let Δ be a finite set and $\boldsymbol{Y}$ be the set of indeterminates $\{Y_\delta;\ \delta \in \Delta\}$. Let R be a ring. Let $\boldsymbol{\mathcal{Y}}(t) := \left\{\mathcal{Y}_\delta(t) : \delta \in \Delta\right\}$ be a family of elements in the power series ring $R[\![t]\!]$. Let $P \in R[\boldsymbol{Y}]$. Then we define the family $\left\{P_{s,\boldsymbol{\mathcal{Y}}(t)} : s \in \mathbb{N}\right\}$ of elements of R by the following equality in $R[\![t]\!]$:

$$P\big|_{Y_\delta=\mathcal{Y}_\delta(t)} = \sum_{s\in\mathbb{N}} P_{s,\boldsymbol{\mathcal{Y}}(t)} t^s. \tag{5.1}$$

Remark 5.3. — Keep the same notation as before. Let S be another ring, and $\varphi\colon R \to S$ a ring morphism. We also denote by φ the induced morphisms $R[\boldsymbol{Y}] \to S[\boldsymbol{Y}]$ and $R[\![t]\!] \to S[\![t]\!]$ obtained by applying φ coefficientwise. Then for every $s \in \mathbb{N}$ one has $\varphi(P_{s,\boldsymbol{\mathcal{Y}}(t)}) = \varphi(P)_{s,\varphi(\boldsymbol{\mathcal{Y}}(t))}$.

5.2.

Now we can state and prove the main result of the section.

THEOREM 5.4. — *Let K be a field extension of k, Δ be a finite set and $\boldsymbol{Y}$ be the set of indeterminates $\{Y_\delta; \delta \in \Delta\}$. Let $(d_\delta) \in \mathbb{N}^\Delta$ be a family of nonnegative integers. Let $\boldsymbol{X}$ be the set of variables $\{X_{\delta,j}; \delta \in \Delta,\ 0 \leqslant j < d_\delta\}$. We denote by $\langle \boldsymbol{X} \rangle$ the maximal ideal of the power series ring $K[\![\boldsymbol{X}]\!]$.*

Let Ω be a (possibly infinite) set, and let $\{P_\omega\}_{\omega\in\Omega}$ be a family of elements in the polynomial ring $K[\boldsymbol{Y}]$ such that for every $\omega \in \Omega$:

(I) *one may write $P_\omega = \prod_{\delta\in\Delta} Y_\delta^{u^+_{\omega,\delta}} - \prod_{\delta\in\Delta} Y_\delta^{u^-_{\omega,\delta}}$, where $u^+_{\omega,\delta}, u^-_{\omega,\delta} \in \mathbb{N}$;*

(II) *one has $P_\omega\big|_{Y_\delta=t^{d_\delta}} = 0$ in $K[t]$, in other words*

$$\sum_{\delta\in\Delta} d_\delta u^+_{\omega,\delta} = \sum_{\delta\in\Delta} d_\delta u^-_{\omega,\delta} =: c_\omega.$$

Let $\{x_{\delta,j} : \delta \in \Delta,\ j \geqslant d_\delta\}$ be a family of elements in $K[\![\boldsymbol{X}]\!]$. For $\delta \in \Delta$, set

$$\mathcal{Y}_\delta(t) := \sum_{j=0}^{d_\delta-1} X_{\delta,j} t^j + \sum_{j\geqslant d_\delta} x_{\delta,j} t^j \in K[\![\boldsymbol{X}]\!][\![t]\!]$$

and

$$\widetilde{\mathcal{Y}_\delta}(t) \coloneqq \sum_{j=0}^{d_\delta - 1} X_{\delta,j} t^j + t^{d_\delta} \in K[\![\boldsymbol{X}]\!][t].$$

We assume:

(a) *for every $\delta \in \Delta$, x_{δ,d_δ} is a unit;*

(b) *for every $\omega \in \Omega$ and every $s \geqslant c_\omega$, one has $P_{\omega,s,\boldsymbol{\mathcal{Y}}(t)} = 0$.*

We consider the following ideals of $K[\![\boldsymbol{X}]\!]$: $\mathfrak{a} \coloneqq \big\langle \{P_{\omega,s,\boldsymbol{\mathcal{Y}}(t)} : \omega \in \Omega,\ s \in \mathbb{N}\} \big\rangle$ and $\mathfrak{b} \coloneqq \big\langle \{P_{\omega,s,\widetilde{\boldsymbol{\mathcal{Y}}}(t)} : \omega \in \Omega,\ s \in \mathbb{N}\} \big\rangle$.

Then $K[\![\boldsymbol{X}]\!]/\mathfrak{a}$ and $K[\![\boldsymbol{X}]\!]/\mathfrak{b}$ are isomorphic objects of $\mathfrak{LcCpl}_K$.

Proof. — By assumption (a), for every $\delta \in \Delta$, the series $\mathcal{Y}_\delta(t)$ is a d_δ-regular element of $K[\![\boldsymbol{X}]\!][\![t]\!]$. Thus by the Weierstrass preparation theorem (see Section 2.5), there exists a family $\{\mathfrak{X}_{\delta,j} : \delta \in \Delta,\ 0 \leqslant j < d_\delta\}$ of elements of the maximal ideal $\langle \boldsymbol{X} \rangle$ of $K[\![\boldsymbol{X}]\!]$ and a family $\{U_{\delta,r} : \delta \in \Delta,\ r \in \mathbb{N}\}$ of elements of $K[\![\boldsymbol{X}]\!]$ with $U_{\delta,0}$ a unit, such that, setting

$$W_\delta(t) \coloneqq t^{d_\delta} + \sum_{j=0}^{d_\delta - 1} \mathfrak{X}_{\delta,j} t^j \qquad \text{and} \qquad U_\delta(t) \coloneqq \sum_{r \in \mathbb{N}} U_{\delta,r} t^r,$$

one has

$$\mathcal{Y}_\delta(t) = W_\delta(t) U_\delta(t). \tag{5.2}$$

Identifying the t-coefficients in the latter equation yields the following relations in $K[\![\boldsymbol{X}]\!]$:

$$X_{\delta,j} = \mathfrak{X}_{\delta,j} U_{\delta,0} + \sum_{r=0}^{j-1} \mathfrak{X}_{\delta,r} U_{\delta,j-r}, \quad 0 \leqslant j < d_\delta.$$

Since $U_{\delta,0}$ is a unit, we deduce that the element of $\operatorname{Hom}_{\mathfrak{LcCpl}_K}\big(K[\![\boldsymbol{X}]\!], K[\![\boldsymbol{X}]\!]\big)$ sending $X_{\delta,j}$ to $\mathfrak{X}_{\delta,j}$ for $\delta \in \Delta$ and $0 \leqslant j < d_\delta$ is an isomorphism.

Setting

$$\mathfrak{c} \coloneqq \big\langle \big\{ P_{\omega,s,\{W_\delta(t)\}} : \omega \in \Omega,\ s \in \mathbb{N} \big\} \big\rangle,$$

the above isomorphism shows that $K[\![\boldsymbol{X}]\!][\![t]\!]/\mathfrak{b}$ and $K[\![\boldsymbol{X}]\!][\![t]\!]/\mathfrak{c}$ are isomorphic objects in $\mathfrak{LcCpl}_K$. To conclude the proof, we show that $\mathfrak{a} = \mathfrak{c}$, using Lemma 5.1.

Let $(\mathcal{B}, \mathfrak{m}_{\mathcal{B}})$ be an object in $\mathfrak{LcCpl}_K$, and let φ be an element of

$$\operatorname{Hom}_{\mathfrak{LcCpl}_K}\big(K[\![\boldsymbol{X}]\!], \mathcal{B}\big).$$

We still denote by φ the induced morphism $K[\![\boldsymbol{X}]\!][\![t]\!] \to \mathcal{B}[\![t]\!]$ obtained by applying φ coefficientwise.

Let us assume that for every $\omega \in \Omega$ one has $P_\omega\big|_{Y_\delta=\varphi(\mathcal{Y}_\delta(t))} = 0$. One has to show that for every $\omega \in \Omega$ one has $P_\omega\big|_{Y_\delta=\varphi(W_\delta(t))} = 0$.

From our assumption and hypothesis (I) we deduce the following equality in $\mathcal{B}[\![t]\!]$:

$$\prod_{\delta\in\Delta} \varphi\big(\mathcal{Y}_\delta(t)\big)^{u^+_{\omega,\delta}} = \prod_{\delta\in\Delta} \varphi\big(\mathcal{Y}_\delta(t)\big)^{u^-_{\omega,\delta}}$$

which can be rewritten, using equation (5.2), as

$$(5.3)\quad \prod_{\delta\in\Delta} \varphi\big(W_\delta(t)\big)^{u^+_{\omega,\delta}} \prod_{\delta\in\Delta} \varphi\big(U_\delta(t)\big)^{u^+_{\omega,\delta}} = \prod_{\delta\in\Delta} \varphi\big(W_\delta(t)\big)^{u^-_{\omega,\delta}} \prod_{\delta\in\Delta} \varphi\big(U_\delta(t)\big)^{u^-_{\omega,\delta}}.$$

Note that for every $\delta \in \Delta$, $\varphi\big(W_\delta(t)\big)$ is a Weierstrass polynomial of degree d_δ in $\mathcal{B}[\![t]\!]$ and $\varphi\big(U_\delta(t)\big)$ is a unit in $\mathcal{B}[\![t]\!]$, since $\varphi(U_{\delta,0})$ is.

By uniqueness of the Weierstrass factorization in $\mathcal{B}[\![t]\!]$, one gets the equality

$$(5.4)\quad \prod_{\delta\in\Delta} \varphi\big(W_\delta(t)\big)^{u^+_{\omega,\delta}} = \prod_{\delta\in\Delta} \varphi\big(W_\delta(t)\big)^{u^-_{\omega,\delta}}$$

which means exactly that $P_\omega\big|_{Y_\delta=\varphi(W_\delta(t))} = 0$.

Conversely, assume that for every $\omega \in \Omega$ one has $P_\omega\big|_{Y_\delta=\varphi(W_\delta(t))} = 0$, in other words, that (5.4) holds, and let us show that for every $\omega \in \Omega$ one has $P_\omega\big|_{Y_\delta=\varphi(\mathcal{Y}_\delta(t))} = 0$. Let $\widetilde{W}_\omega(t) \in \mathcal{B}[\![t]\!]$ be the common value of both members of (5.4). Note that $\widetilde{W}_\omega(t)$ is a Weierstrass polynomial of degree c_ω. On the other hand, one has

$$P_\omega\big|_{Y_\delta=\varphi(\mathcal{Y}_\delta(t))} = \widetilde{W}_\omega(t)\left(\prod_{\delta\in\Delta} \varphi\big(U_\delta(t)\big)^{u^+_{\omega,\delta}} - \prod_{\delta\in\Delta} \varphi\big(U_\delta(t)\big)^{u^-_{\omega,\delta}}\right).$$

By assumption (b), $P_\omega\big|_{Y_\delta=\varphi(\mathcal{Y}_\delta(t))}$ is an element of the polynomial ring $\mathcal{B}[t]$ with degree less than c_ω. By the uniqueness of the Weierstrass division by $\widetilde{W}_\omega(t)$ in $\mathcal{B}[\![t]\!]$ one concludes that $P_\omega\big|_{Y_\delta=\varphi(\mathcal{Y}_\delta(t))} = 0$. □

6. A comparison theorem between formal neighborhoods

In this section we will make use of the results in Sections 4 and 5 in order to obtain the main comparison theorem as an application of the results in those sections to the toric setting. It should be noted that our results provide basically two approaches for computing effectively the formal neighborhood of the generic point of the Nash set associated with a toric valuation. The first one is based on an effective implementation of Hensel's lemma crucially used in Section 4. The second one takes advantage of the comparison theorem in order to use exactly the same techniques as in the case of rational arcs described in [6]. The latter seems to be much more efficient in practice. See Section 6.10 below for an explicit example of computation, as well as [22] for more details and explicit examples.

6.1.

We retain the notation introduced in Section 3. In particular, V_σ is the affine toric k-variety of dimension d associated with a cone σ and presented as $k[\boldsymbol{Z}]/\mathfrak{i}_\sigma$, where $\boldsymbol{Z} = \big\{Z_i : i \in \{1,\dots,h\}\big\}$ and $\mathfrak{i}_\sigma$ is generated by the binomial elements $\big\{F_{\boldsymbol{\ell}} = \boldsymbol{Z}^{\ell^+} - \boldsymbol{Z}^{\ell^-}\,;\ \boldsymbol{\ell} \in L\big\}$, L being a subgroup of $\mathbb{Z}^h$. Moreover, denoting by $\boldsymbol{Z}_\infty$ the set of variables $\big\{Z_{i,s} : i \in \{1,\dots,h\},\ s \in \mathbb{N}\big\}$, the arc scheme $\mathscr{L}_\infty(V_\sigma)$ associated with the affine toric variety V_σ may be identified with the affine scheme $\operatorname{Spec}\big(k[\boldsymbol{Z}_\infty]/[\mathfrak{i}_\sigma]\big)$; the ideal $[\mathfrak{i}_\sigma]$ is generated by the elements $\{F_{\boldsymbol{\ell},s};\ \boldsymbol{\ell} \in L,\ s \in \mathbb{N}\}$ where for every $\boldsymbol{\ell} \in L$ the elements $F_{\boldsymbol{\ell},s} \in k[\boldsymbol{Z}_\infty]$ may be characterized by the following equality in $k[\boldsymbol{Z}_\infty][\![t]\!]$:

$$F_{\boldsymbol{\ell}}\big|_{Z_i=\sum_{s\in\mathbb{N}} Z_{i,s}t^s} = \sum_{s\in\mathbb{N}} F_{\boldsymbol{\ell},s}t^s.$$

6.2.

The following proposition gives an explicit description of the generic point of the Nash set associated with a toric valuation.

Proposition 6.1. — *Let $\boldsymbol{n} \in \sigma \cap N$ be a toric valuation, $\mathcal{N}_{\boldsymbol{n}}$ be the associated Nash set and $\eta_{\boldsymbol{n}}$ be the generic point of $\mathcal{N}_{\boldsymbol{n}}$. Let $\mathfrak{a}_{\boldsymbol{n}}$ be the ideal $\big\langle\big\{Z_{i,s_i} : 1 \leqslant i \leqslant h,\ 0 \leqslant s_i < \langle \boldsymbol{n}, \boldsymbol{m}_i\rangle\big\}\big\rangle$ of $k[\boldsymbol{Z}_\infty]$. Let $G_{\boldsymbol{n}} := \prod_{i=1}^d Z_{i,\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}$ and $g_{\boldsymbol{n}}$ be the image of $G_{\boldsymbol{n}}$ in $\mathcal{O}\big(\mathscr{L}_\infty(V_\sigma)\big)$. Then:*

(i) *the prime ideal of $\mathcal{O}\big(\mathscr{L}_\infty(V_\sigma)\big)$ corresponding with $\eta_{\boldsymbol{n}}$ is the radical of the image of $\mathfrak{a}_{\boldsymbol{n}}$ in $\mathcal{O}\big(\mathscr{L}_\infty(V_\sigma)\big)$;*

(ii) *the point $\eta_{\boldsymbol{n}}$ belongs to the distinguished open subset $\{g_{\boldsymbol{n}} \neq 0\}$ of $\mathscr{L}_\infty(V_\sigma)$. The prime ideal of $\mathcal{O}\big(\mathscr{L}_\infty(V_\sigma)\big)_{g_{\boldsymbol{n}}}$ corresponding with $\eta_{\boldsymbol{n}}$ is the extension of $\mathfrak{a}_{\boldsymbol{n}}$ to $\mathcal{O}\big(\mathscr{L}_\infty(V_\sigma)\big)_{g_{\boldsymbol{n}}}$.*

Proof. — Assertion (i) follows from Lemma 3.2.

Let us now prove assertion (ii). By (i), it is enough to show that the k-algebra $R := k[\boldsymbol{Z}_\infty]_{G_{\boldsymbol{n}}}/\big([\mathfrak{i}_\sigma]+\mathfrak{a}_{\boldsymbol{n}}\big)$ is a domain. Let us show that its functor of points is isomorphic to the functor of points of the k-algebra $\mathcal{O}\big(\mathscr{L}_\infty(\mathcal{T})\big)$, the latter being a domain since $\mathcal{T}$ is a smooth irreducible variety.

Let A be a k-algebra. By the very definition of R and Sections 3.1 and 3.4, the set $\mathrm{Hom}_{\mathrm{Alg}_k}(R, A)$ is in natural bijection with the set of semigroup morphisms $\varphi\colon S_\sigma \to A[\![t]\!]$ such that for $1 \leqslant i \leqslant h$ one has $\mathrm{ord}_t\big(\varphi(\boldsymbol{m}_i)\big) \geqslant \langle \boldsymbol{m}_i, \boldsymbol{n}\rangle$ and for $1 \leqslant i \leqslant d$ one has $t^{-\langle \boldsymbol{m}_i,\boldsymbol{n}\rangle}\varphi(\boldsymbol{m}_i) \in \big(A[\![t]\!]\big)^\times$. The latter property also holds for $d+1 \leqslant i \leqslant h$ since $\boldsymbol{m}_i$ is a $\mathbb{Z}$-basis of M. In particular $\boldsymbol{m} \mapsto t^{-\langle \boldsymbol{m},\boldsymbol{n}\rangle}\varphi(\boldsymbol{m})$ is a semigroup morphism $S_\sigma \mapsto \big(A[\![t]\!]\big)^\times$. Thus the set $\mathrm{Hom}_{\mathrm{Alg}_k}(R, A)$ is in natural bijection with the set of group morphism $M \to \big(A[\![t]\!]\big)^\times$, which in turn is in natural bijection with $\mathrm{Hom}_{\mathrm{Alg}_k}\big(\mathcal{O}\big(\mathscr{L}_\infty(\mathcal{T})\big), A\big)$. □

6.3.

Recall that $\mathfrak{j}$ is the ideal $\big\langle F_{\boldsymbol{\ell}_q};\ d+1 \leqslant q \leqslant h\big\rangle$ of the ring $k[\boldsymbol{Z}]$. It defines an affine k-scheme $W := \mathrm{Spec}\big(k[\boldsymbol{Z}]/\mathfrak{j}\big)$ which contains V_σ as a closed subscheme. Recall also that $\mathscr{L}_\infty(W)$ may be identified with $\mathrm{Spec}\big(k[\boldsymbol{Z}_\infty]/[\mathfrak{j}]\big)$ and that $[\mathfrak{j}] = \big\langle F_{\boldsymbol{\ell}_q,s};\ d+1 \leqslant q \leqslant h,\ s \in \mathbb{N}\big\rangle$. The closed immersion $V_\sigma \to W$ induces a closed immersion $\mathscr{L}_\infty(V_\sigma) \to \mathscr{L}_\infty(W)$ between the corresponding arc schemes. For $\boldsymbol{n} \in \sigma \cap N$, let $\eta'_{\boldsymbol{n}}$ be the image of $\eta_{\boldsymbol{n}}$ by this closed immersion. We shall reduce the computation of the formal neighborhood of $\mathscr{L}_\infty(V_\sigma)$ at $\eta_{\boldsymbol{n}}$ to that of the formal neighborhood of $\mathscr{L}_\infty(W)$ at $\eta'_{\boldsymbol{n}}$. We will say that we are in the toric setting in the former situation and (abusing terminology) in the complete intersection setting in the latter.

The following lemma is a straightforward consequence of the definition, Lemma 3.3(ii) and Proposition 6.1.

Lemma 6.2. — *Retain the notation and hypotheses of Proposition 6.1. Let $g'_{\boldsymbol{n}}$ be the image of $G_{\boldsymbol{n}}$ in $\mathcal{O}\big(\mathscr{L}_\infty(W)\big)$.*

Then the point $\eta'_{\boldsymbol{n}}$ belongs to the distinguished open subset $\{g'_{\boldsymbol{n}} \neq 0\}$ of $\mathscr{L}_\infty(W)$, and the prime ideal of $\mathcal{O}\big(\mathscr{L}_\infty(W)\big)_{g'_{\boldsymbol{n}}}$ corresponding with $\eta'_{\boldsymbol{n}}$ is the extension of $\mathfrak{a}_{\boldsymbol{n}}$ to $\mathcal{O}\big(\mathscr{L}_\infty(W)\big)_{g'_{\boldsymbol{n}}}$.

Notation 6.3. — For $q \in \{1, \dots, h\}$ and $\boldsymbol{r} \in \mathbb{N}^q$ we denote by $\boldsymbol{Z}_{\leqslant q,\leqslant r_i}$ the set of variables $\{Z_{i,s_i};\ 1 \leqslant i \leqslant q,\ 0 \leqslant s_i \leqslant r_i\}$. If $q = h$ we write $\boldsymbol{Z}_{\bullet,\leqslant r_i}$ instead of $\boldsymbol{Z}_{\leqslant h,\leqslant r_i}$. We define similarly $\boldsymbol{Z}_{\leqslant q,\geqslant r_i}$, $\boldsymbol{Z}_{\geqslant q,\leqslant r_i}$ and so on.

6.4.

The following lemma shows that we can apply Theorem 4.4 in the complete intersection setting.

LEMMA 6.4. — *Let $\boldsymbol{n} \in \sigma \cap N$ be a toric valuation.*

Let $G_{\boldsymbol{n}} \coloneqq \prod_{i=1}^{d} Z_{i,\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}$ and A be the k-algebra $k[\boldsymbol{Z}_{\leqslant d,\geqslant\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}]_{G_{\boldsymbol{n}}}$.

Let Ω be the finite set $\big\{(i,s_i); i \in \{1,\dots,h\},\ 0 \leqslant s_i < \langle \boldsymbol{n},\boldsymbol{m}_i\rangle\big\}$. For $\omega \in \Omega$, set $X_\omega \coloneqq Z_\omega$. Set $\Gamma = \{d+1,\dots,h\}$. For $q \in \Gamma$ and $s \in \mathbb{N}$, set $Y_{q,s} \coloneqq Z_{q,\langle \boldsymbol{n},\boldsymbol{m}_q\rangle+s}$.

Let $\mathfrak{h}$ be the extension of the ideal $[\mathfrak{j}]$ in $k[\boldsymbol{Z}_\infty]_{G_{\boldsymbol{n}}}$. For $s \in \mathbb{N}$ and $q \in \Gamma$, set $H_{q,s} \coloneqq F_{\boldsymbol{\ell}_q,\boldsymbol{n}*\boldsymbol{\ell}_q+s}$ (recall from Section 3.6 the definition of $\boldsymbol{n} * \boldsymbol{\ell}_q$). Then, with this notation, the hypotheses in Theorem 4.4 hold true.

Proof. — Note that with the notation of the statement, one has in particular $A[\boldsymbol{X},\boldsymbol{Y}] = k[\boldsymbol{Z}_\infty]_{G_{\boldsymbol{n}}}$ and the ideal $\langle \boldsymbol{X}\rangle$ corresponds to

$$\mathfrak{a}_{\boldsymbol{n}} = \langle \boldsymbol{Z}_{\bullet,<\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}\rangle.$$

Let us show that assumption (A) in Theorem 4.4 holds. Pick $q \in \{d+1, \dots,h\}$. Set

$$\Lambda_q^+ \coloneqq \big\{i \in \{1,\dots,d\}; \ell_{q,i} > 0\big\},$$
$$\Lambda_q^- \coloneqq \big\{i \in \{1,\dots,d\}; \ell_{q,i} < 0\big\},$$
$$\Theta_{q,s}^+ \coloneqq \Big\{\big(r_q,(r_{i,k};\ i \in \Lambda_q^+,\ 1 \leqslant k \leqslant \ell_{q,i})\big);\ r_q, r_{i,k} \in \mathbb{N},\ r_q + \textstyle\sum_{i\in\Lambda_q^+}\sum_{k=1}^{\ell_{q,i}} r_{i,k} = s\Big\},$$

and

$$\Theta_{q,s}^- \coloneqq \Big\{(r_{i,k};\ i \in \Lambda_q^-,\ 1 \leqslant k \leqslant -\ell_{q,i});\ r_{i,k} \in \mathbb{N},\ \textstyle\sum_{i\in\Lambda_q^-}\sum_{k=1}^{-\ell_{q,i}} r_{i,k} = s\Big\}.$$

Then by (3.6) and (3.2), the polynomial $F_{\boldsymbol{\ell}_q,s}$ has the following form:

$$F_{\boldsymbol{\ell}_q,s} = \sum_{(r_q,r_{i,k})\in\Theta_{q,s}^+} Z_{q,r_q} \prod_{i\in\Lambda_q^+}\prod_{k=1}^{\ell_{q,i}} Z_{i,r_{i,k}} - \sum_{(r_{i,k})\in\Theta_{q,s}^-} \prod_{i\in\Lambda_q^-}\prod_{k=1}^{-\ell_{q,i}} Z_{i,r_{i,k}}. \tag{6.1}$$

Note that setting $r_q = \langle \boldsymbol{n},\boldsymbol{m}_q\rangle + s$ and $r_{i,k} = \langle \boldsymbol{n},\boldsymbol{m}_i\rangle$ for $1 \leqslant k \leqslant \ell_{q,i}$ defines an element of $\Theta_{q,\boldsymbol{n}*\boldsymbol{\ell}_q+s}^+$. Set

$$U_q \coloneqq \prod_{i\in\Lambda_q^+}\prod_{k=1}^{\ell_{q,i}} Z_{i,\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}.$$

By the definition of $G_{\boldsymbol{n}}$, U_q is an invertible element of $k[\boldsymbol{Z}_{\leqslant d,\geqslant\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}]_{G_{\boldsymbol{n}}}$.

Set

$$E_{q,s,-1} \coloneqq \sum_{\substack{(r_q, r_{i,k}) \in \Theta^+_{q,\boldsymbol{n}*\boldsymbol{\ell}_q+s} \\ r_q < \langle \boldsymbol{n}, \boldsymbol{m}_q \rangle}} Z_{q,r_q} \prod_{i \in \Lambda^+_q} \prod_{k=1}^{\ell_{q,i}} Z_{i,r_{i,k}} - \sum_{(r_{i,k}) \in \Theta^-_{q,\boldsymbol{n}*\boldsymbol{\ell}_q+s}} \prod_{i \in \Lambda^-_q} \prod_{k=1}^{-\ell_{q,i}} Z_{i,r_{i,k}}.$$

For $r \in \mathbb{N}$, set $\delta_{r,s} = 1$ if $r = s$ and 0 otherwise, and

$$E_{q,s,r} \eqqcolon -\delta_{s,r} U_q + \sum_{\substack{(r_{i,k};\ i \in \Lambda^+_q,\ 1 \leqslant k \leqslant \ell_{q,i});\ r_{i,k} \in \mathbb{N} \\ (\langle \boldsymbol{n}, \boldsymbol{m}_q \rangle + r, (r_{i,k})) \in \Theta^+_{q,\boldsymbol{n}*\boldsymbol{\ell}_q+s}}} \prod_{i \in \Lambda^+_q} \prod_{k=1}^{\ell_{q,i}} Z_{i,r_{i,k}}.$$

Thus by (6.1), one has

$$F_{\boldsymbol{\ell}_q, \boldsymbol{n}*\boldsymbol{\ell}_q+s} = U_q Z_{q,\langle \boldsymbol{n}, \boldsymbol{m}_q \rangle + s} + E_{q,s,-1} + \sum_{r \in \mathbb{N}} E_{q,s,r} Z_{q,\langle \boldsymbol{n}, \boldsymbol{m}_q \rangle + r}.$$

Since $\Lambda^\pm_q \subset \{1, \dots, d\}$, it is clear that for $r \in \mathbb{N}$ one has $E_{q,s,r} \in k[\boldsymbol{Z}_{\leqslant d,\bullet}]$, and that $E_{q,s,-1} \in k[\boldsymbol{Z}_{\leqslant d,\bullet} \cup \boldsymbol{Z}_{\bullet,<\langle \boldsymbol{n}, \boldsymbol{m}_i \rangle}]$.

Thus (AA1) is satisfied, and in order to show that (AA2) also holds, it remains to prove that for any $r \geqslant s$, each monomial of $E_{q,s,r}$ contains a variable Z_{i,r_i} with $i \in \{1, \dots, d\}$ and $r_i < \langle \boldsymbol{n}, \boldsymbol{m}_i \rangle$. Take $\big(r_{i,k};\ i \in \Lambda^+_q,\ 1 \leqslant k \leqslant \ell_{q,i})\big)$ a family of nonnegative integers such that $\big(\langle \boldsymbol{n}, \boldsymbol{m}_q \rangle + r, (r_{i,k})\big) \in \Theta^+_{q,\boldsymbol{n}*\boldsymbol{\ell}_q+s}$, that is

$$\langle \boldsymbol{n}, \boldsymbol{m}_q \rangle + r + \sum_{i \in \Lambda^+_q} \sum_{k=1}^{\ell_{q,i}} r_{i,k} = \boldsymbol{n} * \boldsymbol{\ell}_q + s.$$

We have to show that either at least one of the $r_{i,k}$'s is strictly smaller than $\langle \boldsymbol{n}, \boldsymbol{m}_i \rangle$ or $r = s$ and $r_{i,k} = \langle \boldsymbol{n}, \boldsymbol{m}_i \rangle$ for every i, k. (The latter case corresponds to the monomial $U_q Z_{q,\langle \boldsymbol{n}, \boldsymbol{m}_q \rangle + s}$.) Assume $r_{i,k} \geqslant \langle \boldsymbol{n}, \boldsymbol{m}_i \rangle$ for every i, k. Then

$$\begin{aligned} \boldsymbol{n} * \boldsymbol{\ell}_q + s &= \langle \boldsymbol{n}, \boldsymbol{m}_q \rangle + r + \sum_{i \in \Lambda^+_q} \sum_{k=1}^{\ell_{q,i}} r_{i,k} \\ &\geqslant r + \big\langle \boldsymbol{n}, \boldsymbol{m}_q + \textstyle\sum_{i \in \Lambda^+_q} \ell_{k,i} \boldsymbol{m}_i \big\rangle \\ &= r + \boldsymbol{n} * \boldsymbol{\ell}_q. \end{aligned}$$

If $r > s$ this is a contradiction. If $r = s$, the first minoration must be an equality, which imposes $r_{i,k} = \langle \boldsymbol{n}, \boldsymbol{m}_i \rangle$ for every i, k.

Let us prove that (C) holds. We have to show that $E_{q,0,-1}$ does not belong to the ideal $\langle \boldsymbol{Z}_{\bullet,<\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}\rangle$. By the definition of $E_{q,0,-1}$ it is enough to show that

$$\widetilde{E_{q,0,-1}} := -\sum_{(r_{i,k})\in\Theta^{-}_{q,\boldsymbol{n}*\boldsymbol{\ell}_q}} \prod_{i\in\Lambda^{-}_q} \prod_{k=1}^{-\ell_{q,i}} Z_{i,r_{i,k}}$$

does not belong to the ideal $\langle \boldsymbol{Z}_{\bullet,<\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}\rangle$. But arguing similarly as above, one sees that the only monomial in $\widetilde{E_{q,0,-1}}$ not belonging to the above ideal corresponds to $r_{i,k} = \langle \boldsymbol{n},\boldsymbol{m}_i\rangle$. Thus one has $\widetilde{E_{q,0,-1}} = \prod_{i\in\Lambda^{-}_q} Z^{-\ell_{q,i}}_{i,\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}$ (mod $\langle \boldsymbol{Z}_{\bullet,<\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}\rangle$) which allows us to conclude.

Let us show that (B) holds. Since $\mathfrak{h}$ is the extension of the ideal $[\mathfrak{j}]$ in $k[\boldsymbol{Z}_\infty]_{G_{\boldsymbol{n}}}$, it is generated by the union of the families $\{H_{q,s};\ q\in\Gamma,\ s\in\mathbb{N}\}$ and $\left\{F_{\boldsymbol{\ell}_q,s};\ q\in\Gamma,\ s\in\mathbb{N},\ s<\boldsymbol{n}*\boldsymbol{\ell}_q\right\}$.

Arguing similarly as above, one sees using (6.1) that in case $s<\boldsymbol{n}*\boldsymbol{\ell}_q$ every monomial of $F_{\boldsymbol{\ell}_q,s}$ must contain a variable $Z_{i,r}$ with $r<\langle \boldsymbol{n},\boldsymbol{m}_i\rangle$. Thus $\mathfrak{h}$ is generated by some elements of $\langle \boldsymbol{X}\rangle$ and the $H_{\gamma,s}$'s. By Remark 4.5, assumption (B) holds in this case. □

6.5.

Thanks to Lemma 6.4, we can apply Theorem 4.4 in the complete intersection setting. In the proof of the following corollary, we shall see that this also holds in the toric setting.

Corollary 6.5. — *Let $\boldsymbol{n}$ be a toric valuation of $\sigma\cap N$. There exists a $k[\boldsymbol{Z}_{\leqslant d,\geqslant\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}]$-algebra morphism $\widehat{\varepsilon}\colon k[\boldsymbol{Z}_\infty]\to k(\boldsymbol{Z}_{\leqslant d,\geqslant\langle \boldsymbol{n},\boldsymbol{m}_i\rangle})[\![\boldsymbol{Z}_{\bullet,<\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}]\!]$ such that :*

(i) *the completed local ring of the formal neighborhood of $\mathscr{L}_\infty(V_\sigma)$ at $\eta_{\boldsymbol{n}}$ (resp. of $\mathscr{L}_\infty(W)$ at $\eta'_{\boldsymbol{n}}$) are both isomorphic to the complete Noetherian local ring*

$$k(\boldsymbol{Z}_{\leqslant d,\geqslant\langle \boldsymbol{n},\boldsymbol{m}_i\rangle})[\![\boldsymbol{Z}_{\bullet,<\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}]\!]/\langle\widehat{\varepsilon}([\mathfrak{j}])\rangle;$$

(ii) *for every $i\in\{1,\dots,h\}$, $\widehat{\varepsilon}(Z_{i,\langle \boldsymbol{n},\boldsymbol{m}_i\rangle})$ is invertible;*

(iii) *for every $q\in\{d+1,\dots,h\}$ and $s\in\mathbb{N}$, we have $\widehat{\varepsilon}(F_{\boldsymbol{\ell}_q,\boldsymbol{n}*\boldsymbol{\ell}_q+s})=0$.*

Proof of Corollary 6.5. — By Lemma 3.3(ii) and Theorem 4.4(iii), it only remains to show that $\widehat{\varepsilon}([\mathfrak{j}]) = \widehat{\varepsilon}([\mathfrak{i}_\sigma])$. Recall from Section 3.1 the definition of HS. It is enough to show that for every element F of $\mathfrak{i}_\sigma$, one has $\widehat{\varepsilon}\big(\mathrm{HS}(F)\big) \in \widehat{\varepsilon}([\mathfrak{j}])[\![t]\!]$. By Lemma 3.3, there exists a positive integer N such that $\left(\prod_{i=1}^{d} Z_i\right)^N F \in \mathfrak{j}$. Thus

$$\widehat{\varepsilon}\Big(\mathrm{HS}\Big(\big(\textstyle\prod_{i=1}^{d} Z_i\big)^N\Big)\Big)\ \widehat{\varepsilon}\big(\mathrm{HS}(F)\big) \in \widehat{\varepsilon}([\mathfrak{j}])[\![t]\!].$$

Since for every $i \in \{1, \ldots, d\}$, $\widehat{\varepsilon}(Z_{i,\langle \boldsymbol{n},\boldsymbol{m}_i\rangle})$ is a unit, $\widehat{\varepsilon}\Big(\mathrm{HS}\Big(\big(\prod_{i=1}^{d} Z_i\big)^N\Big)\Big)$ is a regular element of $k(\boldsymbol{Z}_{\leqslant d,\geqslant\langle \boldsymbol{n},\boldsymbol{m}_i\rangle})[\![\boldsymbol{Z}_{\bullet,<\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}]\!][\![t]\!]$, as well as its projection to

$$k(\boldsymbol{Z}_{\leqslant d,\geqslant\langle \boldsymbol{n},\boldsymbol{m}_i\rangle})[\![\boldsymbol{Z}_{\bullet,<\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}]\!]/\widehat{\varepsilon}([\mathfrak{j}])[\![t]\!].$$

Since a regular element is not a zero divisor, we infer that $\widehat{\varepsilon}\big(\mathrm{HS}(F)\big) \in \widehat{\varepsilon}([\mathfrak{j}])[\![t]\!]$. □

6.6.

Let us recall the definition of some objects in [6, Section 5.1], adapted to the notation in the present section. Denote by $\widetilde{\varepsilon}\colon k[\boldsymbol{Z}_\infty] \to k[\![\boldsymbol{Z}_{\bullet,<\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}]\!]$ the unique k-algebra morphism mapping, for every $i \in \{1, \ldots, h\}$, $Z_{i,s}$ to $Z_{i,s}$ for $s < \langle \boldsymbol{n},\boldsymbol{m}_i\rangle$, $Z_{i,\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}$ to 1 and $Z_{i,s}$ to 0 for $s > \langle \boldsymbol{n},\boldsymbol{m}_i\rangle$.

For $L' \subseteq L$, let $W(\boldsymbol{n}, L')$ be the affine closed k-subscheme of the affine space $\mathrm{Spec}\big(k[\boldsymbol{Z}_{\bullet,<\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}]\big)$ defined by the ideal $\big\langle \widetilde{\varepsilon}(F_{\boldsymbol{\ell},s});\ \boldsymbol{\ell} \in L',\ s \in \mathbb{N}\big\rangle$ and $\mathcal{W}(\boldsymbol{n}, L')$ the formal completion of $W(\boldsymbol{n}, L')$ along the origin of

$$\mathrm{Spec}\big(k[\boldsymbol{Z}_{\bullet,<\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}]\big).$$

Remark 6.6. — Let $(\mathcal{A}, \mathfrak{M}_{\mathcal{A}})$ be an object of $\mathfrak{LcCpl}_k$. Then

$$\mathrm{Hom}_{\mathfrak{LcCpl}_k}\big(\mathcal{W}(\boldsymbol{n}, L'), \mathcal{A}\big)$$

is in natural bijection with the set of families $\big\{z_{i,s};\ i \in \{1, \ldots, h\},\ 0 \leqslant s < \langle \boldsymbol{n},\boldsymbol{m}_i\rangle\big\}$ of elements of $\mathfrak{M}_{\mathcal{A}}$ such that for every element $\boldsymbol{\ell} \in L'$ one has

$$F_{\boldsymbol{\ell}}\big|_{Z_i=\sum_{s=0}^{\langle \boldsymbol{n},\boldsymbol{m}_i\rangle-1} z_{i,s}t^i+t^{\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}} = 0.$$

The following result follows from [6, Theorem 5.2].

THEOREM 6.7. — *For an appropriate choice of $L' \subseteq L$ such that $\{\boldsymbol{\ell}_q;\ d+1 \leqslant q \leqslant h\} \subseteq L'$, for every toric valuation $\boldsymbol{n} \in N \cap \sigma$ and every arc $\alpha \in \mathscr{L}_\infty(V_\sigma)^\circ_{\boldsymbol{n}}(k)$, the formal neighborhood of $\mathscr{L}_\infty(V_\sigma)$ at α is isomorphic to $\mathcal{W}(\boldsymbol{n}, L')\widehat{\otimes}_k k\big[\!\big[(T_i)_{i\in\mathbb{N}}\big]\!\big]$.*

The following lemma shows that for the computation of formal neighborhoods of k-rational arcs on $\mathscr{L}_\infty(V_\sigma)$, one may also reduce to the complete intersection setting.

Lemma 6.8. — *Let* $\boldsymbol{n} \in \sigma \cap N$ *be a toric valuation and* L' *be a subset of* L *such that* $\{\boldsymbol{\ell}_q;\ d+1 \leqslant q \leqslant h\} \subseteq L'$. *Then* $\mathcal{W}(\boldsymbol{n}, L')$ *is isomorphic, as a formal* k*-scheme, to* $\mathcal{W}\big(\boldsymbol{n}, \{\boldsymbol{\ell}_q : d+1 \leqslant q \leqslant h\}\big)$.

Remark 6.9. — Thanks to this lemma, for any $L' \subseteq L$ such that $\{\boldsymbol{\ell}_q;\ d+1 \leqslant q \leqslant h\} \subseteq L'$ and any $\boldsymbol{n} \in \sigma \cap N$, one may denote $\mathcal{W}(\boldsymbol{n}, L')$ by $\mathcal{W}(\boldsymbol{n})$.

Proof. — By Remark 6.6, there is, for every object $(\mathcal{A}, \mathfrak{M}_{\mathcal{A}})$ of $\mathfrak{LcCpl}_k$, a natural inclusion $\mathrm{Hom}_{\mathfrak{LcCpl}_k}\big(\mathcal{W}(\boldsymbol{n}, L'), \mathcal{A}\big) \subset \mathrm{Hom}_{\mathfrak{LcCpl}_k}\big(\mathcal{W}\big(\boldsymbol{n}, \{\boldsymbol{\ell}_q;\ d+1 \leqslant q \leqslant h\}\big), \mathcal{A}\big)$. To conclude, it suffices to show that this is an equality. Let $\big\{z_{i,s};\ i \in \{1, \dots, h\},\ 0 \leqslant s < \langle \boldsymbol{n}, \boldsymbol{m}_i\rangle\big\}$ be a family of elements of $\mathfrak{M}_{\mathcal{A}}$ such that, setting

$$z_i(t) \coloneqq \sum_{s=0}^{\langle \boldsymbol{n}, \boldsymbol{m}_i\rangle - 1} z_{i,s} t^i + t^{\langle \boldsymbol{n}, \boldsymbol{m}_i\rangle},$$

one has, for every $d+1 \leqslant q \leqslant h$, $F_{\boldsymbol{\ell}_q}\big|_{Z_i = z_i(t)} = 0$. Let $\boldsymbol{\ell} \in L'$. By Lemma 3.3 there exists a positive integer N such that $\big(\prod_{i=1}^h Z_i\big)^N F_{\boldsymbol{\ell}} \in \big\langle F_{\boldsymbol{\ell}_q};\ d+1 \leqslant q \leqslant h\big\rangle$. Thus

$$\left(\prod_{i=1}^h z_i(t)\right)^N F_{\boldsymbol{\ell}}\big|_{Z_i = z_i(t)} = 0.$$

Since $z_i(t)$ is a Weierstrass polynomial in $\mathcal{A}[\![t]\!]$, it is a non zero divisor (see 2.5). Thus one infers that $F_{\boldsymbol{\ell}}\big|_{Z_i = z_i(t)} = 0$.

That concludes the proof. □

The following proposition performs the aimed comparison in the complete intersection setting.

Proposition 6.10. — *Let* $\boldsymbol{n} \in \sigma \cap N$ *be a toric valuation. Let* $K \coloneqq k(\boldsymbol{Z}_{\leqslant d, \geqslant \langle \boldsymbol{n}, \boldsymbol{m}_i\rangle})$. *Then the residue field of* $\eta'_{\boldsymbol{n}}$ *is isomorphic to* K *and the formal neighborhood of* $\mathscr{L}_\infty(W)$ *at the point* $\eta'_{\boldsymbol{n}}$ *is isomorphic, as a formal* K*-scheme, to* $K \widehat{\otimes}_k \mathcal{W}\big(\boldsymbol{n}, \{\boldsymbol{\ell}_q : d+1 \leqslant q \leqslant h\}\big)$.

Proof. — We still denote by $\widetilde{\varepsilon}$ the composition of the morphism defined in Section 6.6 with the natural inclusion morphism $k[\![\boldsymbol{Z}_{\bullet, <\langle \boldsymbol{n}, \boldsymbol{m}_i\rangle}]\!] \to K[\![\boldsymbol{Z}_{\bullet, <\langle \boldsymbol{n}, \boldsymbol{m}_i\rangle}]\!]$.

By Corollary 6.5 and the very definition of $\mathcal{W}\big(\boldsymbol{n}, \{\boldsymbol{\ell}_q : d+1 \leqslant q \leqslant h\}\big)$, it is enough to show that the quotients of $K[\![\boldsymbol{Z}_{\bullet, <\langle \boldsymbol{n}, \boldsymbol{m}_i\rangle}]\!]$ by the ideals $\big\langle \widehat{\varepsilon}([\mathrm{j}])\big\rangle = \big\langle \widehat{\varepsilon}(F_{\boldsymbol{\ell}_q, s}) : d+1 \leqslant q \leqslant h,\ s \in \mathbb{N}\big\rangle$ on one hand, $\big\langle \widetilde{\varepsilon}([\mathrm{j}])\big\rangle = \big\langle \widetilde{\varepsilon}(F_{\boldsymbol{\ell}_q, s}) : d+1 \leqslant q \leqslant h,\ s \in \mathbb{N}\big\rangle$ on the other hand, are isomorphic.

We aim to apply Theorem 5.4. Set $\Delta \coloneqq \{1, \dots, h\}$. For $i \in \Delta$, set $d_i \coloneqq \langle \boldsymbol{n}, \boldsymbol{m}_i\rangle$; for $0 \leqslant s < \langle \boldsymbol{n}, \boldsymbol{m}_i\rangle$ set $X_{i,s} \coloneqq Z_{i,s}$ and for $s \geqslant \langle \boldsymbol{n}, \boldsymbol{m}_i\rangle$ set

$x_{i,s} \coloneqq \widehat{\varepsilon}(Z_{i,s})$. For $i \in \Delta$, set

$$\mathcal{Y}_i(t) \coloneqq \sum_{s\in\mathbb{N}} \widehat{\varepsilon}(Z_{i,s})t^s = \sum_{s=0}^{\langle \boldsymbol{n},\boldsymbol{m}_i\rangle -1} X_{i,s}t^s + \sum_{s\geqslant \langle \boldsymbol{n},\boldsymbol{m}_i\rangle} x_{i,s}t^s$$

$$\text{and} \quad \widetilde{\mathcal{Y}}_i(t) \coloneqq \sum_{s=0}^{\langle \boldsymbol{n},\boldsymbol{m}_i\rangle -1} X_{i,s}t^s + t^{\langle \boldsymbol{n},\boldsymbol{m}_i\rangle}.$$

For every $P \in k[\boldsymbol{Z}]$, we then have (see Notation 5.2 and Remark 5.3) $P_{s,\boldsymbol{\mathcal{Y}}(t)} = \widehat{\varepsilon}(P_s)$ and $P_{s,\widetilde{\boldsymbol{\mathcal{Y}}}(t)} = \widetilde{\varepsilon}(P_s)$.

Set $\Omega \coloneqq \{d+1,\dots,h\}$ and for $q \in \Omega$ set $P_q \coloneqq F_{\boldsymbol{\ell}_q}$.

Assumption (a) is a consequence of Corollary 6.5.

With our identifications, the nonzero integer c_q defined in the statement of Theorem 5.4 is

$$c_q = \sum_{i=1}^{h} (\boldsymbol{\ell}_q^+)_i \langle \boldsymbol{n}, \boldsymbol{m}_i\rangle = \left\langle \boldsymbol{n}, \sum_{\substack{i=1\\ \ell_{q,i}\geqslant 0}}^{h} \ell_{q,i}\boldsymbol{m}_i \right\rangle = \boldsymbol{n} * \boldsymbol{\ell}_q.$$

Then still by Corollary 6.5, for every $s \in \mathbb{N}$ we have $\widehat{\varepsilon}(F_{\boldsymbol{\ell}_q,\boldsymbol{n}*\boldsymbol{\ell}_q+s}) = 0$. Thus $F_{\boldsymbol{\ell}_q,\boldsymbol{n}*\boldsymbol{\ell}_q+s,\boldsymbol{\mathcal{Y}}(t)} = 0$ and assumption (b) holds. That concludes the proof. □

6.7.

Now one can state the main theorem of the article. It illustrates the striking fact that not only the isomorphism class of the formal neighborhood of a generic k-rational arc of the Nash set associated with a toric valuation is constant (as observed in [6]) but moreover the involved isomorphism class is encoded in some sense in the formal neighborhood of the generic point of the Nash set. This could be interpreted as the fact that the arc scheme of a toric variety is analytically a product along the Nash set associated with the toric valuation.

Theorem 6.11. — *Let $\boldsymbol{n} \in \sigma \cap N$ be a toric valuation. Let $\eta_{\boldsymbol{n}}$ be the generic point of the Nash set $\mathcal{N}_{\boldsymbol{n}}$. Let $\mathcal{W}(\boldsymbol{n})$ be the Noetherian formal k-scheme defined in Remark 6.9.*

Then there exists a nonempty open subset $U_{\boldsymbol{n}}$ of the Nash set $\mathcal{N}_{\boldsymbol{n}}$ such that:

(i) *the formal neighborhood of $\mathscr{L}_\infty(V_\sigma)$ at $\eta_{\boldsymbol{n}}$ is isomorphic, as a formal $\kappa(\eta_{\boldsymbol{n}})$-scheme, to $\kappa(\eta_{\boldsymbol{n}})\widehat{\otimes}_k\mathcal{W}(\boldsymbol{n})$; in particular, it is isomorphic to* *the formal spectrum of the completion of an essentially of finite type local $\kappa(\eta_{\boldsymbol{n}})$-algebra;*

(ii) *for any arc $\alpha \in U_{\boldsymbol{n}}(k)$, the formal neighborhood of $\mathscr{L}_\infty(V_\sigma)$ at α is isomorphic to $\mathcal{W}(\boldsymbol{n})\widehat{\otimes}_k k[\![(T_i)_{i\in\mathbb{N}}]\!]$.*

Proof. — One takes $U_{\boldsymbol{n}} := \mathscr{L}_\infty(V_\sigma)^\circ_{\boldsymbol{n}}$ and one combines Proposition 6.10, Theorem 6.7, Lemma 6.8, and Corollary 6.5(i). □

6.8.

An element $\boldsymbol{n} \in N \setminus \{0\}$ is said to be *primitive* if it cannot be written as $d\boldsymbol{n}'$ where $\boldsymbol{n}' \in N$ and d is an integer > 1. An element $\boldsymbol{n} \in N \cap \sigma \setminus \{0\}$ is said to be *indecomposable* if it cannot be written $\boldsymbol{n} = \boldsymbol{n}_1 + \boldsymbol{n}_2$ with $\boldsymbol{n}_1, \boldsymbol{n}_2 \in N \cap \sigma \setminus \{0\}$. A decomposition of $\boldsymbol{n}$ into indecomposable elements is a decomposition $\boldsymbol{n} = \sum_{i=1}^r \boldsymbol{n}_i$ where r is a positive integer and the $\boldsymbol{n}_i$'s are indecomposable elements in $N \cap \sigma \setminus \{0\}$; the *length* of such a decomposition is r.

6.9.

Using results of [6], we deduce, as a straightforward by-product of Theorem 6.11, the following corollary. The result has been obtained independently by Reguera using a broadly different approach (see [27]).

Corollary 6.12. — *Let $\boldsymbol{n} \in \sigma \cap N$ be a toric valuation of V_σ and $\eta_{\boldsymbol{n}}$ be the generic point of the Nash set $\mathcal{N}_{\boldsymbol{n}}$.*

Then there is a natural bijection between the set of irreducible components of the formal neighborhood $\widehat{\mathscr{L}_\infty(V_\sigma)}_{\eta_{\boldsymbol{n}}}$ and the set of decompositions of $\boldsymbol{n}$ into a sum of indecomposable elements of the semigroup $N \cap \sigma$. The dimension of the component corresponding to a given decomposition of $\boldsymbol{n}$ is the length of the decomposition. In particular the dimension of $\widehat{\mathscr{L}_\infty(V_\sigma)}_{\eta_{\boldsymbol{n}}}$ is equal to the maximal length of such a decomposition of $\boldsymbol{n}$.

Proof. — The fact that the conclusion holds for $\mathcal{W}(\boldsymbol{n})$ is shown in [6, section (3) of the proof of Theorem 6.3]. In the latter, $\mathcal{W}(\boldsymbol{n})$ is denoted by $\widetilde{\mathscr{Y}}_n$. Though it is assumed in the statement of the theorem that $\boldsymbol{n}$ is primitive, this is not used in the aforementioned section of the proof. The corollary then follows from Theorem 6.11. □

This in turn implies the following statement. Recall that a divisorial valuation v on an algebraic variety V is *essential* if for every resolution $W \to V$ of the singularities of V, the center of v on W is an irreducible component of the exceptional locus of the resolution. Here we say that such a divisorial valuation is *strongly essential* if for every resolution $W \to V$ of the singularities of V, the center of v on W is an irreducible component *of codimension* 1 of the exceptional locus of the resolution (N.B.: the terminology is used in [15, Definition 6.22] with a different meaning).

Corollary 6.13. — *Let V be a toric variety and v a divisorial toric valuation on V, centered in the singular locus of V. Let η_v be the generic point of the Nash set associated with v. Then the formal neighborhood $\widehat{\mathscr{L}_\infty(V)_{\eta_v}}$ of η_v in the arc scheme $\mathscr{L}_\infty(V)$ associated with v is of dimension* 1 *if and only if v is strongly essential. Moreover, in this case, $\widehat{\mathscr{L}_\infty(V)_{\eta_v}}$ is irreducible and the associated reduced formal scheme is a formal disk.*

Proof. — Let $\boldsymbol{n} \in N \cap \sigma$ be a primitive integral point representing a toric valuation of multiplicity 1 and assume that v is centered in the singular locus of V_σ. Then, by [9, Theorem 1.10] and [8, Theorem 1.2], $\boldsymbol{n}$ is indecomposable if and only if v is a strongly essential valuation, in the sense given in the introduction. Thus Corollary 6.13 is indeed a consequence of Corollary 6.12 (using again the proof of [6, Theorem 6.3]). □

Recall that Reguera showed that in general, if η_v is the generic point of the Nash set associated with an essential divisorial valuation v on an algebraic variety V, then $\widehat{\mathscr{L}_\infty(V)_{\eta_v}}$ is irreducible of dimension 1 if and only if V satisfies a property of lifting wedges centered at η_v, and that this condition implies that v is a Nash valuation (see [26, Corollary 5.12]). Note that the latter property of lifting wedges is stronger than the one considered in [25, Section 5], which was shown to be equivalent to the fact that v is a Nash valuation.

In view of Corollary 6.13 and of the results of [6], the following question seems natural.

Question 6.14. — *Let η_v be the generic point of the Nash set associated with an essential divisorial valuation v on a an algebraic variety V. Are the following conditions equivalent?*

(1) *The formal neighborhood $\widehat{\mathscr{L}_\infty(V)_{\eta_v}}$ is irreducible of dimension* 1.
(2) *The valuation v is Nash and strongly essential.*
(3) *The minimal formal model of a generic rational arc in the Nash set associated with v is irreducible of dimension* 0.

Our results show that the answer is positive in case v is a toric valuation on a toric variety. Note also that if v is a terminal valuation (hence strongly essential), v is a Nash valuation and it is known that (1) holds (see [16] and [23, Corollary 4.3]). In the case of surfaces, an affirmative answer to the above question would imply that for any Nash (equivalently, essential) valuation, property (3) holds, which seems to be open.

6.10.

We end this section with an explicit example of computation of the formal neighborhood of the generic point of the Nash set associated with a toric valuation. See [22] for more details.

Let $N = M = \mathbb{Z}^2$, σ be the cone of $\mathbb{R}^2$ generated by $(1,0)$ and $(1,2)$, and V_σ be the associated affine toric variety. The semigroup S_σ is minimally generated by $m_1 = (0,1)$, $m_2 = (1,0)$ and $m_3 = (2,-1)$. We observe that m_1 and m_2 form a $\mathbb{Z}$-basis of M and the relation $m_1 + m_3 = 2m_2$ generates all the relations between elements of S_σ. Thus, setting $F := Z_1 Z_3 - Z_2^2$, the ideal of V_σ in $k[Z_1, Z_2, Z_3]$ is the ideal generated by F. The ideal of $\mathscr{L}_\infty(V_\sigma)$ in the ring $k[\boldsymbol{Z}_\infty] = k[Z_{1,s}, Z_{2,s}, Z_{3,s};\ s \in \mathbb{N}]$ is generated by $\{F_s;\ s \in \mathbb{N}\}$ where $F_s = \sum_{r=0}^{s} (Z_{1,s-r} Z_{3,r} - Z_{2,s-r} Z_{2,r})$.

We now consider the toric valuation $\operatorname{ord}_{\boldsymbol{n}}$ of V_σ corresponding to $\boldsymbol{n} = (1,1) \in \sigma \cap N$. The prime ideal of $k[\boldsymbol{Z}_\infty]$ corresponding to the generic point $\eta_{\boldsymbol{n}}$ of the Nash set associated with $\operatorname{ord}_{\boldsymbol{n}}$ is the radical of the ideal $\langle Z_{1,0}, Z_{2,0}, Z_{3,0} \rangle$. The residue field of $\eta_{\boldsymbol{n}}$ is isomorphic to $K := k(Z_{1,s}, Z_{2,s};\ s \geqslant 1)$.

Denote by $\{z_{3,s};\ s \geqslant 1\}$ the unique family of elements of K such that for every $s \geqslant 2$, one has

$$\sum_{r=1}^{s-1} \bigl(Z_{1,s-r} \cdot z_{3,r} - Z_{2,s-r} Z_{2,r} \bigr) = 0.$$

Note that the latter is a triangular invertible K-linear system in the $z_{3,s}$'s.

Now let $\{\mathcal{Z}_{3,r};\ r \geqslant 1\}$ be the unique family of elements of $K[\![Z_{1,0}, Z_{2,0}, Z_{3,0}]\!]$ such that:

(1) for every $s \geqslant 1$, one has $\mathcal{Z}_{3,s} = z_{3,s} \pmod{\langle Z_{1,0}, Z_{2,0}, Z_{3,0} \rangle}$;
(2) for every $s \geqslant 2$, one has

$$Z_{1,s} Z_{3,0} - Z_{2,0} Z_{2,s} + \sum_{r=1}^{s} \bigl(Z_{1,s-r} \cdot \mathcal{Z}_{3,r} - Z_{2,s-r} Z_{2,r} \bigr) = 0.$$

Explicit truncations of the series $\mathscr{Z}_{3,s}$ may be obtained by applying effectively Hensel's lemma, in other words by successive approximations, though explicit computations quickly become cumbersome. For example one has

$$\mathscr{Z}_{3,1} = \frac{Z_{2,1}^2}{Z_{1,1}} + \frac{Z_{1,0}Z_{1,2}Z_{2,1}^2}{Z_{1,1}^3} - \frac{2Z_{1,0}Z_{2,1}Z_{2,2}}{Z_{1,1}^2} - \frac{Z_{1,2}Z_{3,0}}{Z_{1,1}} + \frac{2Z_{2,0}Z_{2,2}}{Z_{1,1}} \pmod{\langle Z_{1,0}, Z_{2,0}, Z_{3,0}\rangle^2}$$

and

$$\begin{aligned}\mathscr{Z}_{3,2} = &-\frac{Z_{1,2}Z_{2,1}^2}{Z_{1,1}^2} + \frac{2Z_{2,1}Z_{2,2}}{Z_{1,1}} - \frac{2Z_{1,0}Z_{1,2}^2Z_{2,1}^2}{Z_{1,1}^4} + \frac{4Z_{1,0}Z_{1,2}Z_{2,1}Z_{2,2}}{Z_{1,1}^3}\\ &+ \frac{Z_{1,0}Z_{1,3}Z_{2,1}^2}{Z_{1,1}^3} - \frac{Z_{1,0}Z_{2,2}^2}{Z_{1,1}^2} - \frac{2Z_{1,0}Z_{2,1}Z_{2,3}}{Z_{1,1}^2} + \frac{Z_{1,2}^2Z_{3,0}}{Z_{1,1}^2}\\ &- \frac{2Z_{1,2}Z_{2,0}Z_{2,2}}{Z_{1,1}^2} - \frac{Z_{1,3}Z_{3,0}}{Z_{1,1}} + \frac{2Z_{2,0}Z_{2,3}}{Z_{1,1}} \pmod{\langle Z_{1,0}, Z_{2,0}, Z_{3,0}\rangle^2}.\end{aligned}$$

Then the formal neighborhood of $\eta_{\boldsymbol{n}}$ in $\mathscr{L}_\infty(V_\sigma)$ is isomorphic to the formal spectrum of

$$K[\![Z_{1,0}, Z_{2,0}, Z_{3,0}]\!]/\langle Z_{1,0}Z_{3,0} - Z_{2,0}^2, Z_{1,1}Z_{3,0} + Z_{1,0}\mathscr{Z}_{3,1} - 2Z_{2,0}Z_{2,1}\rangle.$$

Note that it is not clear that the latter is the completion of an essentially of finite type local K-algebra.

6.11.

Using our comparison theorem, the computation of the formal neighborhood of $\eta_{\boldsymbol{n}}$ in $\mathscr{L}_\infty(V_\sigma)$ may also be done in the following much more straightforward way. First we compute the formal scheme $\mathcal{W}(\boldsymbol{n})$ defined in 6.6. We have the following equality in $k[Z_{1,0}, Z_{2,0}, Z_{3,0}, t]$:

$$\begin{aligned}F\big|_{Z_j = t + Z_{j,0}} &= (t + Z_{1,0})(t + Z_{3,0}) - (t + Z_{2,0})^2\\ &= (Z_{1,0} + Z_{3,0} - 2Z_{2,0})t + Z_{1,0}Z_{3,0} - Z_{2,0}^2.\end{aligned}$$

We deduce that

$$\mathcal{W}(\boldsymbol{n}) = \mathsf{Spf}\left(\frac{k[\![Z_{1,0}, Z_{2,0}, Z_{3,0}]\!]}{\langle Z_{1,0}Z_{3,0} - Z_{2,0}^2, Z_{1,0} + Z_{3,0} - 2Z_{2,0}\rangle}\right)$$

and that the formal neighborhood of $\eta_{\boldsymbol{n}}$ in $\mathscr{L}_\infty(V_\sigma)$ is isomorphic to

$$\mathsf{Spf}\left(\frac{K[\![Z_{1,0}, Z_{2,0}, Z_{3,0}]\!]}{\langle Z_{1,0}Z_{3,0} - Z_{2,0}^2, Z_{1,0} + Z_{3,0} - 2Z_{2,0}\rangle}\right).$$

In addition, it is not difficult to see that $\mathcal{W}(\boldsymbol{n})$ is isomorphic to

$$\mathsf{Spf}\big(k[\![Z_{1,0}, Z_{2,0}]\!]/\langle Z_{1,0}^2\rangle\big).$$

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