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GENERALIZED SUITA CONJECTURES WITH JETS AND WEIGHTS

by Wang XU & Xiangyu ZHOU (*)

Abstract. — We survey different approaches to Suita's conjecture and its various generalizations. We present a new and unified proof for generalized Suita conjectures with jets and weights, which is based on the concavity of certain minimal L^2 integrals and the necessary condition for linearity. Additionally, we provide some examples and counterexamples for the equalities in generalized Suita conjectures.

Résumé. — Nous passons en revue différentes approches de la conjecture de Suita et de ses diverses généralisations. Nous présentons une nouvelle preuve unifiée des conjectures généralisées de Suita avec jets et poids, basée sur la concavité de certaines L^2-intégrales minimales et la condition nécessaire de linéarité. De plus, nous donnons quelques exemples et contre-exemples pour les égalités dans les conjectures généralisées de Suita.

1. Introduction

In [28], Suita conjectured an inequality between the Bergman kernel and the logarithmic capacity of a hyperbolic Riemann surface. Later, Ohsawa [26] noticed a connection between the L^2 extension problem and Suita's conjecture, and he was able to prove a weaker inequality. By proving L^2 extension theorems with optimal estimates, Błocki [5] (for planar domains) and Guan–Zhou [17] (for Riemann surfaces) solved Suita's conjecture. By carefully using the optimal L^2 extension theorem with 'gain' established in [19], Guan–Zhou [19] also settled the equality part of the conjecture (i.e. to characterize when the equality holds). Since then, various approaches (see [4, 6, 10]) and generalizations (see [7, 8, 9, 18]) to Suita's conjecture have emerged.

Keywords: Suita conjecture, Bergman kernel, logarithmic capacity, Azukawa indicatrix, Hartogs domain.
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The first purpose of this article is to survey the progress made in Suita's conjecture and its generalizations. We also present a new approach to one dimensional generalizations with jets (see [8]) or weights (see [18, 19]), which is based on the concavity of certain minimal L^2 integrals (see [13]) and the necessary condition for linearity (see [30]). Actually, we prove a result unifying [8] and [18, 19] (see Theorem 4.6). We also construct a family of counterexamples for the equality in higher order Suita conjecture (see Theorem 5.1), which contrasts with the phenomenon observed in simply/doubly connected planar domains.

Our approach is also applicable to higher dimensional generalizations (see [7, 9]), and we obtain a necessary condition for the equality case (see Proposition 6.1). To the authors' knowledge, the only known example for the equality case is the biholomorphic image of a balanced domain (with a possible closed pluripolar set removed). In this article, we provide a new family of examples (see Theorem 7.1 and 7.2) for the equality in higher dimensional Suita conjecture.

2. Capacities and kernels on Riemann surfaces

In this section, Ω is a potential-theoretical hyperbolic Riemann surface, which means that Ω admits a negative non-constant subharmonic function. Then Ω has non-trivial Green's functions (see [11]). Recall that the *Bergman kernel* of Ω is

$$\kappa_\Omega(z) := \sup\left\{\sqrt{-1}F(z)\wedge\overline{F(z)} : F\in\Gamma(\Omega,K_\Omega),\ \int_\Omega \tfrac{\sqrt{-1}}{2}F\wedge\overline{F}\leqslant 1\right\},$$

and the *exact Bergman kernel* of Ω is

$$\widetilde{\kappa}_\Omega(z) := \sup\left\{\sqrt{-1}\partial f(z)\wedge\overline{\partial f(z)} : f\in\mathcal{O}(\Omega),\ \int_\Omega \tfrac{\sqrt{-1}}{2}\partial f\wedge\overline{\partial f}\leqslant 1\right\}.$$

Let (V,w) be a coordinate chart of Ω. We write $\kappa_\Omega\big|_V = B_\Omega|\mathrm{d}w|^2$, $\widetilde{\kappa}_\Omega\big|_V = \widetilde{B}_\Omega|\mathrm{d}w|^2$ and $c_D(z) := \sqrt{\pi\widetilde{B}_\Omega(z)}$. By definition, $\widetilde{B}_\Omega\leqslant B_\Omega$. Recall that the *logarithmic capacity* of Ω is locally defined by

$$c_\beta(z_0) := \lim_{z\to z_0}\exp\big(G_\Omega(z,z_0)-\log\big|w(z)-w(z_0)\big|\big),$$

and the *analytic capacity* of Ω is locally defined by

$$c_B(z_0) := \sup\left\{\big|\tfrac{\partial f}{\partial w}(z_0)\big| : f\in\mathcal{O}(\Omega),\ f(z_0)=0,\ \sup_\Omega|f|\leqslant 1\right\}.$$

Clearly, $c_\beta|\mathrm{d}w|$ and $c_B|\mathrm{d}w|$ are globally defined conformal invariants.

In the following, we collect some results on the comparison between these conformal invariants.

Theorem 2.1 (see [19]). — $c_B \leqslant c_\beta$. *Moreover,* $c_B(z_0) = c_\beta(z_0)$ *for some* $z_0 \in \Omega$ *if and only if there exists a holomorphic function* $g \in \mathcal{O}(\Omega)$ *such that* $\log|g| = G_\Omega(\,\cdot\,, z_0)$.

Proof. — Let $\mathcal{F}_{z_0} := \left\{f \in \mathcal{O}(\Omega) : f(z_0) = 0,\ \sup_\Omega|f| \leqslant 1\right\}$. Since $\mathcal{F}_{z_0}$ is a normal family, there exists $h \in \mathcal{F}_{z_0}$ with $\left|\frac{\partial h}{\partial w}(z_0)\right| = c_B(z_0)$. If $c_B(z_0) = 0$, there is nothing to prove. In the following, we assume that $c_B(z_0) > 0$. By the maximum principle, $|h| < 1$ everywhere. Since $\log|h| < 0$ is subharmonic on Ω and $\log\big|h(z)\big| - \log\big|w(z) - w(z_0)\big|$ is bounded near z_0, we know $\log|h| \leqslant G_\Omega(\,\cdot\,, z_0)$, and then

$$\begin{aligned} c_B(z_0) &= \left|\frac{\partial h}{\partial w}(z_0)\right| \\ &= \lim_{z \to z_0} \exp\big(\log\big|h(z)\big| - \log\big|w(z) - w(z_0)\big|\big) \\ &\leqslant \lim_{z \to z_0} \exp\big(G_\Omega(z, z_0) - \log\big|w(z) - w(z_0)\big|\big) \\ &= c_\beta(z_0). \end{aligned}$$

Therefore, $c_B(z_0) \leqslant c_\beta(z_0)$ in general.

If $c_B(z_0) > 0$, then $\varphi := \log|h| - G_\Omega(\,\cdot\,, z_0) \leqslant 0$ is a subharmonic function on Ω and $\varphi(z_0) = \log \frac{c_B(z_0)}{c_\beta(z_0)}$. If $c_B(z_0) = c_\beta(z_0)$, then $\varphi(z_0) = 0$. By the maximum principle, $\varphi \equiv 0$, i.e. $\log|h| = G_\Omega(\,\cdot\,, z_0)$. Conversely, if there exists $g \in \mathcal{O}(\Omega)$ such that $\log|g| = G_\Omega(\,\cdot\,, z_0)$, then $g \in \mathcal{F}_{z_0}$ and $c_\beta(z_0) = \left|\frac{\partial g}{\partial w}(z_0)\right| \leqslant c_B(z_0)$. This implies $c_B(z_0) = c_\beta(z_0)$. $\square$

Theorem 2.2 (Sakai [27]). — $c_D \leqslant c_B$. *Moreover,* $c_D(z_0) = c_B(z_0) > 0$ *for some* $z_0 \in \Omega$ *if and only if* Ω *is conformally equivalent to the unit disc less a possible closed set which is expressed as the union of at most a countable number of compact sets of class* $\mathcal{N}_B$.

Theorem 2.3 (Suita [28]). — $\pi B_\Omega \geqslant c_B^2$. *Moreover,* $\pi B_\Omega(z_0) = c_B(z_0)^2$ *for some* $z_0 \in \Omega$ *if and only if* Ω *is conformally equivalent to the unit disc less a possible closed set of inner capacity zero.*

Recall that a compact set E in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is of class $\mathcal{N}_B$ if all bounded holomorphic functions on $\widehat{\mathbb{C}} \backslash E$ are constant, and a closed set E in $\mathbb{D}$ has inner capacity zero if and only if E is polar.

In 1972, Suita [28] conjectured that the curvature of $c_\beta|\mathrm{d}w|$ is not greater than -4, i.e.

$$-\frac{4}{c_\beta^2}\frac{\partial^2 \log c_\beta}{\partial w \partial \overline{w}} \leqslant -4,$$

and the equality holds at some point if and only if Ω is conformally equivalent to the unit disc less a possible closed set of inner capacity zero.

According to [28], $\frac{\partial^2}{\partial w \partial \overline{w}}(\log c_\beta) = \pi B_\Omega$, then the inequality in Suita's conjecture is equivalent to

$$\pi B_\Omega(z) \geqslant c_\beta(z)^2.$$

For a doubly connected planar domain Ω with no degenerate boundary component, Suita [28] proved that $\pi B_\Omega > c_\beta^2$.

In [26], Ohsawa observed a connection between the L^2 extension problem and the inequality in Suita's conjecture, and he proved that $750\pi B_\Omega(z) \geqslant c_\beta(z)^2$. Since then, there are many attempts to sharpen the estimate. In 2012, by proving L^2 extension theorems with optimal estimates, Błocki [5] (for planar domains) and Guan–Zhou [17] (for Riemann surfaces) solved the inequality part of the conjecture. Later, Guan–Zhou [19] also settled the equality part of the conjecture through a careful use of the optimal L^2 extension theorem with 'gain'.

Theorem 2.4 (Błocki [5]; Guan–Zhou [17, 19]). — *$\pi B_\Omega \geqslant c_\beta^2$. Moreover, $\pi B_\Omega(z_0) = c_\beta(z_0)^2$ for some $z_0 \in \Omega$ if and only if Ω is conformally equivalent to the unit disc less a possible closed polar set.*

In summary, one has

$$\pi \widetilde{B}_\Omega \leqslant c_B^2 \leqslant c_\beta^2 \leqslant \pi B_\Omega,$$

and Theorems 2.1 to 2.4 also give the necessary and sufficient conditions for these inequalities to become equalities.

3. Various approaches to the Suita conjecture

After [5, 17, 19], there are several new approaches to the Suita conjecture. For the inequality part, Błocki [6] gave a new proof based on the tensor power trick, and Berndtsson–Lempert [4] presented another proof based on the log-psh variation of fibrewise Bergman kernels. Recently, Dong [10] proposed a simplified proof for the equality part by using Maitani–Yamaguchi's [24] variation formula for fibrewise Bergman kernels. In [30, Section 5.2], we presented a slightly different proof for the equality part of Suita's conjecture, which is based on the concavity of certain minimal L^2 integrals and the necessary condition for linearity.

In the following, we compare these different approaches to the Suita conjecture. We shall adjust the original notations to ensure consistency. As before, Ω is a hyperbolic Riemann surface, (V, w) is a connected coordinate chart around $z_0 \in \Omega$, $\kappa_\Omega = B_\Omega |\mathrm{d}w|^2$ is the Bergman kernel, $c_\beta |\mathrm{d}w|$ is the logarithmic capacity, and $G \coloneqq G_\Omega(\,\cdot\,, z_0)$ is the Green function. Suita [28] conjectured that $\pi B_\Omega(z_0) \geqslant c_\beta(z_0)^2$, and the equality holds if and only if Ω is conformally equivalent to $\mathbb{D}$ less a possible closed polar set.

The inequality part of Suita's conjecture

The approach of Błocki [5] *and Guan–Zhou* [19]. — As noticed by Ohsawa [26], proving the inequality is equivalent to proving an L^2 extension theorem with optimal estimate, i.e. to find a holomorphic 1-form $F \in \Gamma(\Omega, K_\Omega)$ with $F(z_0) = \mathrm{d}w$ and

$$\int_\Omega \frac{\sqrt{-1}}{2} F \wedge \overline{F} \leqslant \frac{\pi}{c_\beta(z_0)^2}.$$

The existence of such F would imply $B_\Omega(z_0) \geqslant \left(\int_\Omega \frac{\sqrt{-1}}{2} F \wedge \overline{F}\right)^{-1} \geqslant \pi^{-1} c_\beta(z_0)^2$. By proving certain optimal L^2 extension theorems, Błocki and Guan–Zhou solved the inequality part of the conjecture. □

The approach of Błocki [6]. — Using Donnelly–Fefferman's L^2 estimates of $\bar\partial$, together with a tensor power trick, Błocki showed that, for a pseudoconvex domain $D \Subset \mathbb{C}^n$,

$$B_D(z) \geqslant \frac{1}{e^{2na}\,\mathrm{Vol}\big(\{G_D(\,\cdot\,, z) < -a\}\big)}, \quad z \in D, a \in \mathbb{R}_+. \tag{3.1}$$

Here, G_D is the pluricomplex Green function of D. In dimension 1, for $a \gg 1$, the sublevel set $\{G_D(\,\cdot\,, z) < -a\}$ is almost a disc with radius $e^{-a} c_\beta(z; D)^{-1}$. The right-hand side converges to $\pi^{-1} c_\beta(z; D)^2$ as $a \to +\infty$, and then $B_D(z) \geqslant \pi^{-1} c_\beta(z; D)^2$. □

The approach of Berndtsson–Lempert [4]. — For each $t \geqslant 0$, define $\Omega_t \coloneqq \{2G < -t\}$ and $B(t) \coloneqq B_{\Omega_t}(z_0)$. Consider the following Stein manifold:

$$\mathcal{X} = \big\{(\tau, z) \in \mathbb{C} \times \Omega : 2G(z) + \operatorname{Re}\tau < 0\big\}.$$

By the log-psh variation of fibrewise Bergman kernels (see [3, 24]), $\tau \mapsto \log B(\operatorname{Re}\tau)$ is a psh function, then $t \mapsto \log B(t)$ is a convex function. By the local behavior of $G_\Omega(\,\cdot\,, z_0)$ near z_0, one has $B(t) \sim \pi^{-1} c_\beta(z_0)^2 e^t$ as $t \to +\infty$. Since the convex function $k(t) \coloneqq \log B(t) - t$ is bounded from above as $t \to +\infty$, $k(t)$ must be decreasing. Therefore, $k(0) \geqslant \lim_{t\to+\infty} k(t)$, which implies $B_\Omega(z_0) \geqslant \pi^{-1} c_\beta(z_0)^2$.

There is a slightly different proof due to Guan [12]. Since $B(t)$ is the reciprocal of certain minimal L^2 integral on Ω_t, by a general concavity property (see [13, Proposition 4.1]), $r \mapsto \frac{1}{B(-\log r)}$ is a concave increasing function on $(0,1]$. Therefore, $\frac{1}{rB(-\log r)}$ is decreasing in r and $e^{k(t)} = e^{-t}B(t)$ is decreasing in t. As a consequence, $B_\Omega(z_0) \geqslant \pi^{-1}c_\beta(z_0)^2$. Notice that, in this particular case, $B(t) \leqslant Ce^t$ for some $C > 0$, then the convexity of $\log B(t)$ implies the concavity of $\frac{1}{B(-\log r)}$. □

The equality part of Suita's conjecture

The approach of Guan–Zhou [19]. — After a suitable change of coordinate, we may assume that $G(w) \equiv \log\left|c_\beta(z_0)w\right|$ on V. Since $\pi B_\Omega(z_0) = c_\beta(z_0)^2$, there exists a unique holomorphic 1-form $F \in \Gamma(\Omega, K_\Omega)$ such that $F(z_0) = \mathrm{d}w$ and $\int_\Omega \frac{\sqrt{-1}}{2} F \wedge \overline{F} = \pi c_\beta(z_0)^{-2}$. Given $r_1 < r_2 < r_3 < 0$ such that $\left\{2G < r_3\right\} \Subset V$, let $d_1(t) \equiv 1$ and let $d_2(t)$ be a smooth function on $(-\infty, 0)$ so that $d_1 \equiv d_2$ on $(-\infty, r_1) \cup (r_3, 0)$, $d_1 > d_2$ on (r_1, r_2), $d_1 < d_2$ on (r_2, r_3), $d_2(t)e^t$ is increasing on $(-\infty, 0)$, and $\int_{-\infty}^0 d_2(t)e^t \,\mathrm{d}t = \int_{-\infty}^0 d_1(t)e^t \,\mathrm{d}t = 1$.

According to [19, Theorem 2.2], there exists a holomorphic 1-form $F' \in \Gamma(\Omega, K_\Omega)$ with $F'(z_0) = \mathrm{d}w$ and $\int_\Omega \frac{\sqrt{-1}}{2} d_2(2G) F' \wedge \overline{F'} \leqslant \pi c_\beta(z_0)^{-2}$. By careful computations,

$$\int_\Omega \frac{\sqrt{-1}}{2} F' \wedge \overline{F'} \leqslant \int_\Omega \frac{\sqrt{-1}}{2} d_2(2G) F' \wedge \overline{F'} \leqslant \pi c_\beta(z_0)^{-2}.$$

On the other hand, $\int_\Omega \frac{\sqrt{-1}}{2} F' \wedge \overline{F'} \geqslant B_\Omega(z_0)^{-1} = \pi c_\beta(z_0)^{-2}$. Since the minimal element is unique, one has $F \equiv F'$, and then

$$\int_\Omega \frac{\sqrt{-1}}{2} F \wedge \overline{F} = \int_\Omega \frac{\sqrt{-1}}{2} d_2(2G) F \wedge \overline{F}.$$

By careful computations, this equality implies $F\big|_V \equiv \mathrm{d}w$ (see [19, Lemma 4.21]). In summary:

> if $\pi B_\Omega(z_0) = c_\beta(z_0)^2$ and (V, w) is a connected coordinate chart around z_0 such that $G\big|_V \equiv \log\left|c_\beta(z_0)w\right|$, then there exists a global holomorphic 1-form $F \in \Gamma(\Omega, K_\Omega)$ with $F\big|_V \equiv \mathrm{d}w$.

Using this fact and the theory of Riemann surfaces, Guan–Zhou constructed a holomorphic function $g \in \mathcal{O}(\Omega)$ such that $G = \log|g|$. By Theorems 2.1 and 2.3, $c_B(z_0)^2 = c_\beta(z_0)^2 = \pi B_\Omega(z_0)$, and Ω is conformally equivalent to $\mathbb{D}$ less a possible closed polar set. □

The approach of Dong [10]. — For each $t \geqslant 0$, put $\Omega_t = \{2G < -t\}$ and $B(t) = B_{\Omega_t}(z_0)$. Let $\kappa_t(\cdot,\cdot)$ be the Bergman kernel of Ω_t, i.e. $\kappa_t(x,y) = \sqrt{-1}\sum_\alpha \phi_t^\alpha(x) \wedge \overline{\phi_t^\alpha(y)}$, where $\{\phi_t^\alpha\}_\alpha$ is a complete orthonormal basis of $A^2(\Omega_t, K_\Omega)$. If we write $\kappa_t(\cdot, z_0) = K_t(\cdot) \wedge \overline{\mathrm{d}w}$, then $K_t \in \Gamma(\Omega_t, K_\Omega)$ is the unique holomorphic 1-form with minimal L^2-norm such that $K_t(z_0) = B(t)\,\mathrm{d}w$. Recall from [4] that $k(t) := \log B(t) - t$ is a decreasing function. If $\pi B_\Omega(z_0) = c_\beta(z_0)^2$, then $k(t)$ is constant and $B(t) \equiv B_\Omega(z_0)e^t$.

Using the variation formula of Maitani–Yamaguchi [24], Dong proved that $K_0\big|_{\Omega_t} \equiv K_t e^{-t}$ for all $t \geqslant 0$ such that K_t is zero-free. In this proof, he needed to approximate Ω by smoothly bordered Riemann surfaces while keeping the equality $\pi B_\Omega(z_0) = c_\beta(z_0)^2$.

Assume that (V, w) is a connected coordinate chart around z_0 such that $G\big|_V \equiv \log\big|c_\beta(z_0)w\big|$, then $\Omega_t \Subset V$ and $K_t(w) \equiv B(t)\,\mathrm{d}w$ for $t \gg 1$. Consequently, $K_0\big|_V \equiv B_\Omega(z_0)\,\mathrm{d}w$. Set $F := K_0/B_\Omega(z_0)$, then $F\big|_V \equiv \mathrm{d}w$. By careful analysis, Dong showed that $g := F/(2\partial G)$ is a holomorphic function on Ω and $G = \log|g|$. By [25, Theorem 1], Ω is conformally equivalent to $\mathbb{D}$ less a possible closed polar set. □

The approach of Xu–Zhou [30]. — For each $t \geqslant 0$, set $\Omega_t = \{2G < -t\}$ and $B(t) = B_{\Omega_t}(z_0)$. Let $F_t \in \Gamma(\Omega_t, K_\Omega)$ be the unique holomorphic 1-form with minimal L^2-norm such that $F_t(z_0) = \mathrm{d}w$, then $B(t) = \|F_t\|^{-2}$. By the concavity of minimal L^2 integrals (see [13]), $r \mapsto \|F_{-\log r}\|^2$ is a concave function. If $\pi B_\Omega(z_0) = c_\beta(z_0)^2$, then $B(t) \equiv B_\Omega(z_0)e^t$ and $\|F_{-\log r}\|^2 \equiv r/B_\Omega(z_0)$ is linear in r. By the necessary condition for linearity (see [30, Remark 5.3]), $F_0\big|_{\Omega_t} \equiv F_t$ for any $t \geqslant 0$. If (V, w) is a connected coordinate chart around z_0 such that $G\big|_V \equiv \log\big|c_\beta(z_0)w\big|$, then $F_s \equiv \mathrm{d}w$ for $s \gg 1$ and then $F_0\big|_V = \mathrm{d}w$. The other part of the proof is the same as in [19]. □

Remark 3.1. — The first part of all three proofs is to find a holomorphic 1-form $F \in \Gamma(\Omega, K_\Omega)$ such that $F\big|_V \equiv \mathrm{d}w$, in which (V, w) is a connected coordinate chart around z_0 with $G\big|_V \equiv \log\big|c_\beta(z_0)w\big|$. But the approaches are different. Having such an F, one can construct $g \in \mathcal{O}(\Omega)$ such that $\log|g| = G_\Omega(\cdot, z_0)$.

We remark that, without requiring $\pi B_\Omega(z_0) = c_\beta(z_0)^2$ in advance, the existence of $g \in \mathcal{O}(\Omega)$ satisfying $\log|g| = G_\Omega(\cdot, z_0)$ guarantees the rigidity. This fact is implicitly contained in Suita's article [28, p. 213]. It also follows from a theorem of Minda [25]: if $f\colon X \to Y$ is a holomorphic map between hyperbolic Riemann surfaces, and $G_Y\big(f(a), f(b)\big) = G_X(a, b)$ for some $a \neq b$, then f is injective and $Y \backslash f(X)$ is a closed set of capacity zero.

4. One dimensional generalizations

In this section, Ω is a hyperbolic Riemann surface, z_0 is a distinguished point of Ω and (V, w) is a coordinate chart around z_0. Let $p\colon \mathbb{D} \to \Omega$ be a universal covering of Ω. Recall that the group of deck transformations $\mathrm{Deck}(\mathbb{D}/\Omega)$ is isomorphic to the fundamental group $\pi_1(\Omega)$. Therefore, any $\sigma \in \pi_1(\Omega)$ can be identified with an element in $\mathrm{Aut}(\mathbb{D})$ which we shall also denote by σ. Moreover, any such automorphism satisfies $p \circ \sigma = p$.

Lemma 4.1. — *If f_1 and f_2 are holomorphic functions on a connected complex manifold M such that $|f_1| \equiv |f_2|$, then $f_1 \equiv \alpha f_2$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.*

Proof. — Apply the Riemann extension theorem and the maximum principle to f_1/f_2. □

Lemma 4.2. — *There exists a $g \in \mathcal{O}(\mathbb{D})$ such that $\log|g| = p^* G_\Omega(\,\cdot\,, z_0)$.*

Proof. — By the Weierstrass theorem for open Riemann surfaces, there is an $h \in \mathcal{O}(\Omega)$ so that $h(z_0) = 0$, $\mathrm{d}h(z_0) \neq 0$ and $h\big|_{\Omega\setminus\{z_0\}} \neq 0$. Since $p^*\big(G_\Omega(\,\cdot\,, z_0) - \log|h|\big)$ is harmonic on $\mathbb{D}$, there exists $f \in \mathcal{O}(\mathbb{D})$ such that $\operatorname{Re} f = p^*\big(G_\Omega(\,\cdot\,, z_0) - \log|h|\big)$. Let $g := p^*(h)\exp(f)$, then $g \in \mathcal{O}(\mathbb{D})$ and $\log|g| = p^* G_\Omega(\,\cdot\,, z_0)$. □

Let $g \in \mathcal{O}(\mathbb{D})$ be a holomorphic function such that $\log|g| = p^* G_\Omega(\,\cdot\,, z_0)$. For any $\sigma \in \pi_1(\Omega)$, we have $|g| = \exp\big(p^* G_\Omega(\,\cdot\,, z_0)\big) = \exp\big(\sigma^* p^* G_\Omega(\,\cdot\,, z_0)\big) = |\sigma^* g|$, which implies $\sigma^* g/g$ is a constant of modulus one. Clearly,

$$\chi_{z_0}\colon \sigma \in \pi_1(\Omega) \longmapsto \sigma^* g/g \in \mathbb{S}^1$$

is a group homomorphism, which is independent of the choice of g.

Let η be a harmonic function on Ω, then there exists a holomorphic function $\xi \in \mathcal{O}(\mathbb{D})$ so that $|\xi| = \exp(p^*\eta)$. For any $\sigma \in \pi_1(\Omega)$, we have $|\xi| = \exp(p^*\eta) = \exp(\sigma^* p^* \eta) = |\sigma^*\xi|$, and then $\sigma^*\xi/\xi$ is a constant of modulus one. Clearly,

$$\chi_\eta\colon \sigma \in \pi_1(\Omega) \longmapsto \sigma^*\xi/\xi \in \mathbb{S}^1$$

is also a group homomorphism, which is independent of the choice of ξ.

Given a group homomorphism $\chi \in \mathrm{Hom}\big(\pi_1(\Omega), \mathbb{S}^1\big)$, we define

$$\mathcal{O}^\chi(\Omega) := \big\{f \in \mathcal{O}(\mathbb{D}) : \sigma^* f = \chi(\sigma) f \text{ for all } \sigma \in \pi_1(\Omega)\big\},$$
$$\Gamma^\chi(\Omega) := \big\{F \in \Gamma(\mathbb{D}, K_\mathbb{D}) : \sigma^* F = \chi(\sigma) F \text{ for all } \sigma \in \pi_1(\Omega)\big\}.$$

A typical element $f \in \mathcal{O}^\chi(\Omega)$ (resp. $F \in \Gamma^\chi(\Omega)$) is called a multiplicative function (resp. Prym differential). Recall that the *multiplicative Bergman kernel* (or χ-Bergman kernel) of Ω is defined by

$$\kappa_\Omega^\chi(z) := \sup\left\{\sqrt{-1}F(z) \wedge \overline{F(z)} : F \in \Gamma^\chi(\Omega), \int_\Omega \frac{\sqrt{-1}}{2} F \wedge \overline{F} \leqslant 1\right\}.$$

Since $p_*(F \wedge \overline{F})$ is well-defined on Ω, we can simply write $F \wedge \overline{F}$ on Ω.

The extended Suita conjecture (see Yamada [31]) is the following.

Extended Suita conjecture. — *$\pi\kappa_\Omega^\chi \geqslant c_\beta^2 |\mathrm{d}w|^2$ and the equality holds at $z_0 \in \Omega$ if and only if $\chi = \chi_{z_0}$.*

Notice that, if $\chi = \chi_{z_0}$, then $\pi\kappa_\Omega^\chi(z_0) = c_\beta(z_0)^2|\mathrm{d}w|^2$ (see [31, Theorem 7]).

There is an equivalent formulation in terms of *weighted Bergman kernels*. Given a harmonic function η on Ω, we define

$$\kappa_{\Omega,\eta}(z) = \sup\left\{\sqrt{-1}F(z) \wedge \overline{F(z)} : F \in \Gamma(\Omega, K_\Omega), \int_\Omega \frac{\sqrt{-1}}{2} F \wedge \overline{F} e^{-2\eta} \leqslant 1\right\}.$$

Then the extended Suita conjecture is equivalent to the following.

Conjecture. — *$\pi\kappa_{\Omega,\eta} \geqslant c_\beta^2 e^{2\eta}|\mathrm{d}w|^2$ and the equality holds at $z_0 \in \Omega$ if and only if $\chi_\eta\chi_{z_0} = 1$.*

The inequality part of the conjecture was proved in Guan–Zhou [18] and the equality part was proved in Guan–Zhou [19].

Theorem 4.3 (Guan–Zhou [18, 19]). — *$\pi\kappa_{\Omega,\eta}(z_0) \geqslant c_\beta(z_0)^2 e^{2\eta(z_0)}|\mathrm{d}w|^2$ and the equality holds if and only if $\chi_\eta\chi_{z_0} = 1$.*

Lemma 4.4. — *Let η be a harmonic function on Ω, then $\chi_\eta\chi_{z_0}^k = 1$ for some $k \in \mathbb{N}$ if and only if there exists a holomorphic function $\widehat{g} \in \mathcal{O}(\Omega)$ such that $\log|\widehat{g}| = kG_\Omega(\,\cdot\,, z_0) + \eta$.*

Proof. — We choose $g, \xi \in \mathcal{O}(\mathbb{D})$ such that $\log|g| = p^*G_\Omega(\,\cdot\,, z_0)$ and $|\xi| = \exp(p^*\eta)$. If $\chi_\eta\chi_{z_0}^k = 1$, then

$$\sigma^*(\xi g^k) = \chi_\eta(\sigma)\xi \cdot \big(\chi_{z_0}(\sigma)g\big)^k = \xi g^k$$

for any $\sigma \in \pi_1(\Omega)$. As a consequence, $\widehat{g} := p_*(\xi g^k)$ is a well-defined holomorphic function on Ω. Since

$$\log|\xi g^k| = kp^*G_\Omega(\,\cdot\,, z_0) + p^*\eta,$$

it is clear that $\log|\widehat{g}| = kG_\Omega(\,\cdot\,, z_0) + \eta$.

Conversely, if there is a $\widehat{g} \in \mathcal{O}(\Omega)$ such that $\log|\widehat{g}| = kG_\Omega(\,\cdot\,, z_0) + \eta$, then $|p^*\widehat{g}| = |\xi g^k|$. Hence, $\xi g^k = c \cdot p^*\widehat{g}$ for some $c \in \mathbb{S}^1$. For any $\sigma \in \pi_1(\Omega)$, we have $p^*\widehat{g} = \sigma^*(p^*\widehat{g})$, this implies

$$\xi g^k = \sigma^*(\xi g^k) = \chi_\eta(\sigma)\xi \cdot \big(\chi_{z_0}(\sigma) g\big)^k.$$

Therefore, $\chi_\eta(\sigma)\chi_{z_0}(\sigma)^k = 1$ for all $\sigma \in \pi_1(\Omega)$. □

Recall that Ω is a hyperbolic Riemann surface and (V, w) is a coordinate chart around $z_0 \in \Omega$. Denote by $\mathfrak{m}_{z_0}$ the unique maximal ideal of $\mathcal{O}_{z_0}$. Given $m \in \mathbb{N}$, consider the generalized Bergman kernel

$$B_\Omega^{(m)}(z_0) := \sup\left\{ \left|\frac{\partial^m f}{\partial w^m}(z_0)\right|^2 \,\middle|\, \begin{array}{l} F \in \Gamma(\Omega, K_\Omega) \text{ with } F\big|_V = f\,\mathrm{d}w, \\ \int_\Omega \frac{\sqrt{-1}}{2} F \wedge \overline{F} \leqslant 1,\ [f]_{z_0} \in \mathfrak{m}_{z_0}^m \end{array} \right\}.$$

Clearly, $B_\Omega^{(m)}(z_0)|\mathrm{d}w|^{2m+2}$ is independent of the choice of (V, w). For a planar domain $\Omega \subset \mathbb{C}_w$, following the method of [5], Błocki–Zwonek [8] proved that

$$\pi B_\Omega^{(m)}(z_0) \geqslant m!\,(m+1)!\,c_\beta(z_0)^{2m+2}. \tag{4.1}$$

By modifying Guan–Zhou's proof for the equality part of Suita's conjecture, Li [21] obtained an equivalent condition for (4.1) to become an equality (also see [22]).

Theorem 4.5 (see [8] and [21]). — $\pi B_\Omega^{(m)}(z_0) \geqslant m!\,(m+1)!\,c_\beta(z_0)^{2m+2}$. *Moreover, the equality holds if and only if there exists a holomorphic function* $\widehat{g} \in \mathcal{O}(\Omega)$ *such that* $\log|\widehat{g}| = (m+1)G_\Omega(\,\cdot\,, z_0)$.

In the following, we illustrate that the method of [30, Section 5.2] is also applicable to Theorems 4.3 and 4.5. For simplicity, it is better to consider a unified version.

Theorem 4.6. — *Let Ω be a hyperbolic Riemann surface, η be a harmonic function on Ω and (V, w) be a coordinate chart around $z_0 \in \Omega$. For $m \in \mathbb{N}$, we define*

$$B_{\Omega,\eta}^{(m)}(z_0) := \sup\left\{ \left|\frac{\partial^m f}{\partial w^m}(z_0)\right|^2 \,\middle|\, \begin{array}{l} F \in \Gamma(\Omega, K_\Omega) \text{ with } F\big|_V = f\,\mathrm{d}w, \\ \int_\Omega \frac{\sqrt{-1}}{2} F \wedge \overline{F} e^{-2\eta} \leqslant 1,\ [f]_{z_0} \in \mathfrak{m}_{z_0}^m \end{array} \right\}.$$

Then

$$\pi B_{\Omega,\eta}^{(m)}(z_0) \geqslant m!\,(m+1)!\,c_\beta(z_0)^{2m+2} e^{2\eta(z_0)}. \tag{4.2}$$

Moreover, the equality holds if and only if $\chi_\eta \chi_{z_0}^{m+1} = 1$, *if and only if there is a holomorphic function* $\widehat{g} \in \mathcal{O}(\Omega)$ *such that* $\log|\widehat{g}| = (m+1)G_\Omega(\,\cdot\,, z_0) + \eta$. *In this case,*

$$F := \partial \widehat{g} - 2\widehat{g}\partial\eta \in \Gamma(\Omega, K_\Omega)$$

is extremal with respect to $B_{\Omega,\eta}^{(m)}(z_0)$.

Here, a holomorphic 1-form $F \in \Gamma(\Omega, K_\Omega)$ (with $F\big|_V = f\,\mathrm{d}w$) is said to be *extremal* with respect to $B_{\Omega,\eta}^{(m)}(z_0)$, if the following conditions are satisfied:

$$[f]_{z_0} \in \mathfrak{m}_{z_0}^m, \quad B_{\Omega,\eta}^{(m)}(z_0) = \frac{1}{\int_\Omega \frac{\sqrt{-1}}{2} F \wedge \overline{F} e^{-2\eta}} \left| \frac{\partial^m f}{\partial w^m}(z_0) \right|^2.$$

Clearly, an extremal 1-form always exists and it is unique up to non-zero multiplicative constants.

Remark 4.7. — Obviously, Theorem 2.4 corresponds to the case of $m = 0$ and $\eta \equiv 0$, Theorem 4.3 is the case of $m = 0$ and Theorem 4.5 is the case of $\eta \equiv 0$. Theorem 4.6 was announced in [29]. We notice that Guan–Mi–Yuan [15] obtained a result generalizing this theorem: they characterized the linearity of certain minimal L^2 integrals on hyperbolic Riemann surfaces by using the solution of the extended Suita conjecture (i.e. Theorem 4.3). But our purpose is different, we give a new and unified proof to the inequality part and the necessity part of Theorems 2.4, 4.3 and 4.5. For completeness, we also include a proof for the sufficiency part.

Let us recall the concavity of minimal L^2 integrals and the necessary condition for linearity (see [30, Remark 5.3] and [14, Theorem 1.3]). For simplicity, we only focus on a special case, which is enough to prove Theorem 4.6.

Proposition 4.8. — *Let* Ω *be a hyperbolic Riemann surface,* φ *be a harmonic function on* Ω *and* $\psi = 2(m+1)G_\Omega(\,\cdot\,, z_0)$. *For each* $t \geqslant 0$, *let* $\Omega_t := \{\psi < -t\}$ *and*

$$\mathcal{A}_t := \left\{ F \in \Gamma(\Omega_t, K_\Omega) : \|F\|_{\mathcal{A}_t}^2 = \int_{\Omega_t} \frac{\sqrt{-1}}{2} F \wedge \overline{F} e^{-\varphi} < +\infty \right\}.$$

Let F *be a holomorphic 1-form defined in a neighbourhood of* z_0. *For each* $t \geqslant 0$, *let* $F_t \in \mathcal{A}_t$ *be the unique element with minimal norm that coincides with* F *up to order* m *at* z_0.

Set $I(t) := \|F_t\|_{\mathcal{A}_t}^2$. *Then* $r \mapsto I(-\log r)$ *is a concave increasing function on* $(0, 1]$ *and* $I(0) \leqslant I(t)e^t \leqslant I(s)e^s$ *for any* $0 \leqslant t \leqslant s$. *Moreover, if* $r \mapsto I(-\log r)$ *is linear on* $(0, 1]$, *then* $F_t \equiv F_0\big|_{\Omega_t}$ *for any* $t > 0$.

Proof of Theorem 4.6. — Let $p\colon \mathbb{D} \to \Omega$ be a universal covering, let $\xi \in \mathcal{O}(\mathbb{D})$ and $g \in \mathcal{O}(\mathbb{D})$ be holomorphic functions so that $|\xi| = \exp(p^*\eta)$ and $\log|g| = p^*G_\Omega(\,\cdot\,, z_0)$. Shrinking V if necessary, we may assume that V is connected and p is biholomorphic on any connected component of $p^{-1}(V)$. Let U be a component of $p^{-1}(V)$. We define $h \coloneqq p_*\big(g\big|_U\big)$ and $\zeta \coloneqq p_*\big(\xi\big|_U\big)$, then $G_\Omega(\,\cdot\,, z_0) = \log|h|$ and $|\zeta| = e^\eta$ on V. After a suitable change of coordinate, we further assume that $w \equiv c_\beta(z_0)^{-1}h$ on V. We will keep these notations throughout the proof.

Let $\psi = 2(m+1)G_\Omega(\,\cdot\,, z_0)$ and $\varphi = 2\eta$. For each $t \geqslant 0$, we define $\Omega_t \coloneqq \big\{\psi < -t\big\}$ and $\mathcal{A}_t$ as in Proposition 4.8. Let $F_t \in \mathcal{A}_t$ be the unique element with minimal norm that coincides with $w^m\,\mathrm{d}w$ up to order m at z_0. We write $B(t) \coloneqq B^{(m)}_{\Omega_t,\eta}(z_0)$, then

$$\int_{\Omega_t} \tfrac{\sqrt{-1}}{2} F_t \wedge \overline{F_t} e^{-2\eta} = \frac{(m!)^2}{B(t)}.$$

By Proposition 4.8, $r \mapsto \frac{1}{B(-\log r)}$ is a concave function on $(0,1]$ and

$$B(s)e^{-s} \leqslant B(t)e^{-t} \leqslant B(0) = B^{(m)}_{\Omega,\eta}(z_0), \quad 0 \leqslant t \leqslant s.$$

Inequality part. — We may choose $s \gg 1$ so that $\Omega_s \Subset V$. Recall that $G_\Omega(\,\cdot\,, z_0) = \log\big|c_\beta(z_0)w\big|$ and $|\zeta| = e^\eta$ on V. Therefore,

$$\Omega_s = \Big\{|w| < c_\beta(z_0)^{-1}\exp\big(\tfrac{-s}{2m+2}\big)\Big\}$$

is an open disc in (V, w). Assume that $F' \coloneqq u\,\mathrm{d}w \in \mathcal{A}_s$ coincides with $w^m\,\mathrm{d}w$ up to order m at z_0. Then $v \coloneqq u/(\zeta w^m)$ is a holomorphic function on $\mathbb{D}(0;r)$ satisfying $v(0) = 1/\zeta(z_0)$, where $r \coloneqq c_\beta(z_0)^{-1}\exp\big(\frac{-s}{2m+2}\big)$. We may expand v into a power series with normal convergence on $\mathbb{D}(0;r)$:

$$v(w) = \sum_{k=0}^{\infty} a_k w^k \quad \text{with } a_0 = \frac{1}{\zeta(z_0)}.$$

By direct computations,

$$\begin{aligned}\int_{\Omega_s} \tfrac{\sqrt{-1}}{2} F' \wedge \overline{F'} e^{-2\eta} &= \int_{\mathbb{D}(0;r)} |v(w)|^2 |w|^{2m}\,\mathrm{d}\lambda_w \\ &= \sum_{k=0}^{\infty} |a_k|^2 \int_{\mathbb{D}(0;r)} |w|^{2m+2k}\,\mathrm{d}\lambda_w \\ &= \sum_{k=0}^{\infty} \frac{\pi}{m+k+1} |a_k|^2 r^{2m+2k+2}.\end{aligned}$$

Clearly, the above expression is minimized when v is a constant function. As a consequence, for such $s \gg 1$, $F_s(w) \equiv \zeta(z_0)^{-1}\zeta(w)w^m\,\mathrm{d}w$ and

$$\begin{aligned}\frac{(m!)^2}{B(s)} &= \int_{\Omega_s} \frac{\sqrt{-1}}{2} F_s \wedge \overline{F_s} e^{-2\eta}\\ &= \frac{\pi}{m+1}|a_0|^2 r^{2m+2}\\ &= \frac{\pi}{m+1} c_\beta(z_0)^{-2m-2} e^{-2\eta(z_0)} e^{-s}.\end{aligned}$$

For any $0 \leqslant t \ll s$, we have

$$\begin{aligned}B^{(m)}_{\Omega,\eta}(z_0) &\geqslant B(t)e^{-t}\\ &\geqslant B(s)e^{-s}\\ &= \pi^{-1} m!\,(m+1)!\, c_\beta(z_0)^{2m+2} e^{2\eta(z_0)}.\end{aligned} \tag{4.3}$$

this proves the inequality part of the theorem.

Equality part: necessity. — In the following, we assume that the equality in (4.2) holds. According to inequality (4.3),

$$B(t)e^{-t} \equiv \pi^{-1} m!\,(m+1)!\, c_\beta(z_0)^{2m+2} e^{2\eta(z_0)} \quad (\forall\, t \geqslant 0),$$

and then $\frac{1}{B(-\log r)}$ is a linear function of r. By Proposition 4.8, $F_t \equiv F_0\big|_{\Omega_t}$ for any $t \geqslant 0$. Since $F_s(w) \equiv \zeta(z_0)^{-1}\zeta(w)w^m\,\mathrm{d}w$ for $s \gg 1$, we conclude that

$$F_0\big|_V = \zeta(z_0)^{-1}\zeta(w)w^m\,\mathrm{d}w.$$

Multiplying F_0 by a constant, we obtain a holomorphic 1-form $F \in \Gamma(\Omega, K_\Omega)$ such that $F\big|_V = \zeta\,\mathrm{d}(h^{m+1})$ and F is extremal with respect to $B^{(m)}_{\Omega,\eta}(z_0)$.

According to the definitions of ζ and h, we know $p^*F = \xi\,\mathrm{d}(g^{m+1})$ on U. By the uniqueness of analytic continuation, $p^*F \equiv \xi\,\mathrm{d}(g^{m+1})$ on $\mathbb{D}$. For any $\sigma \in \pi_1(\Omega)$, we have $p^*F = \sigma^*(p^*F)$; this implies

$$\xi\,\mathrm{d}(g^{m+1}) = \sigma^*\big(\xi\,\mathrm{d}(g^{m+1})\big) = \chi_\eta(\sigma)\chi_{z_0}(\sigma)^{m+1}\cdot \xi\,\mathrm{d}(g^{m+1}).$$

Therefore, $\chi_\eta(\sigma)\chi_{z_0}(\sigma)^{m+1} = 1$ for all $\sigma \in \pi_1(\Omega)$, which means that $\chi_\eta\chi_{z_0}^{m+1} = 1$.

In this case, $\widehat{g} := p_*(\xi g^{m+1}) \in \mathcal{O}(\Omega)$ and $F := p_*\big(\xi\,\mathrm{d}(g^{m+1})\big) \in \Gamma(\Omega, K_\Omega)$ are well-defined, $\log|\widehat{g}| = (m+1)G_\Omega(\,\cdot\,, z_0) + \eta$, and F is extremal with respect to $B^{(m)}_{\Omega,\eta}(z_0)$. We want a neat formula for the extremal 1-form. Notice that

$$\xi\,\mathrm{d}(g^{m+1}) = \xi\partial(\xi^{-1}\cdot \xi g^{m+1}) = \partial(\xi g^{m+1}) - (\xi g^{m+1})\cdot \xi^{-1}\partial\xi.$$

Differentiating $|\xi|^2 = \exp(2p^*\eta)$, we get

$$\bar{\xi}\partial\xi = 2\partial(p^*\eta)\exp(2p^*\eta) = 2p^*(\partial\eta)|\xi|^2,$$

which implies $\xi^{-1}\partial\xi = 2p^*(\partial\eta)$. As a consequence,

$$\begin{aligned} F &= p_*\big(\xi\,\mathrm{d}(g^{m+1})\big) \\ &= p_*\big(\partial(\xi g^{m+1}) - (\xi g^{m+1})\cdot 2p^*(\partial\eta)\big) \\ &= \partial\widehat{g} - 2\widehat{g}\partial\eta \\ &= e^{2\eta}\partial(e^{-2\eta}\widehat{g}). \end{aligned}$$

Similarly, since $|g|^2 = \exp(2p^*G)$, where $G := G_\Omega(\,\cdot\,, z_0)$, we have

$$\xi\,\mathrm{d}(g^{m+1}) = (m+1)(\xi g^{m+1})\cdot g^{-1}\,\mathrm{d}g = (m+1)(\xi g^{m+1})\cdot 2p^*(\partial G),$$

which implies $F = p_*\big(\xi\,\mathrm{d}(g^{m+1})\big) = 2(m+1)\widehat{g}\partial G$ on $\Omega\backslash\{z_0\}$.

Equality part: sufficiency. — Finally, we assume that $\chi_\eta\chi_{z_0}^{m+1} = 1$, then

$$\widehat{g} := p_*(\xi g^{m+1}) \in \mathcal{O}(\Omega) \qquad \text{and} \qquad F := p_*\big(\xi\,\mathrm{d}(g^{m+1})\big) \in \Gamma(\Omega, K_\Omega)$$

are well-defined objects on Ω. The above proof suggests that F is extremal with respect to $B^{(m)}_{\Omega,\eta}(z_0)$. Clearly, to verify this guess, we only need to prove $\int_\Omega F\wedge\overline{F'}e^{-2\eta} = 0$ for any holomorphic 1-form $F' \in \mathcal{A}_0$ (with $F'\big|_V = f\,\mathrm{d}w$) satisfying $[f]_{z_0} \in \mathfrak{m}_{z_0}^{m+1}$.

Since $[\widehat{g}]_{z_0} \in \mathfrak{m}_{z_0}^{m+1}$ and $\widehat{g} \neq 0$ elsewhere, we know that $F'' := F'/\widehat{g}$ is a holomorphic 1-form on Ω. Since $F = e^{2\eta}\partial(e^{-2\eta}\widehat{g})$ and $|\widehat{g}|^2 = e^{2(m+1)G+2\eta}$, we have

$$\begin{aligned} \int_\Omega F\wedge\overline{F'}e^{-2\eta} &= \int_\Omega \partial(e^{-2\eta}\widehat{g})\wedge\overline{\widehat{g}F''} \\ &= \int_\Omega \partial\big(e^{-2\eta}|\widehat{g}|^2\big)\wedge\overline{F''} \\ &= \int_\Omega \partial\big(e^{2(m+1)G}\big)\wedge\overline{F''}. \end{aligned}$$

We take a sequence of subdomains $D_j \ni z_0$ such that $\overline{D_j} \subset D_{j+1}$, $\Omega = \bigcup_j D_j$ and each D_j is bounded by analytic curves (see [2, p. 144]). Then $G_j := G_{D_j}(\,\cdot\,, z_0)$ is continuous up to $\overline{D_j}$ and $G_j \equiv 0$ on ∂D_j. By the reflection principle, G_j has a harmonic extension in some neighbourhood of ∂D_j. By Stokes' formula,

$$\int_{D_j}\partial\big(e^{2(m+1)G_j}\big)\wedge\overline{F''} = \int_{\partial D_j} e^{2(m+1)G_j}\overline{F''} = \int_{\partial D_j}\overline{F''} = \int_{D_j}\mathrm{d}\overline{F''} = 0.$$

Notice that $e^{2(m+1)G_j}$ and $e^{2(m+1)G}$ have no singularity at z_0, and $G_j - G$ is a harmonic function on D_j. Since $D_j \nearrow \Omega$, it is clear that $G_j \searrow G$. By Harnack's theorem, $G_j - G$ decreases to 0 uniformly on any compact subset of Ω. Using the estimates on derivatives, $\partial(G_j - G) \to 0$ uniformly on any compact subset of Ω. Let $\alpha_j := \partial\big(e^{2(m+1)G_j}\big)$ and $\alpha := \partial\big(e^{2(m+1)G}\big)$. Since

$$\begin{aligned}
&\partial\big(e^{2(m+1)G_j}\big) - \partial\big(e^{2(m+1)G}\big)\\
&\quad = \partial\Big[\big(e^{2(m+1)(G_j-G)} - 1\big)e^{2(m+1)G}\Big]\\
&\quad = \big(e^{2(m+1)(G_j-G)} - 1\big)\partial\big(e^{2(m+1)G}\big) + 2(m+1)e^{2(m+1)G_j}\partial(G_j - G),
\end{aligned}$$

we conclude that $\alpha_j \to \alpha$ uniformly on any compact subset of Ω.

We recall a useful formula for Green's function $G = G_\Omega(\,\cdot\,, z_0)$:

$$\int_\Omega \sqrt{-1}\partial G \wedge \bar{\partial} G e^{aG} = \frac{\pi}{a} \quad (\forall\ a > 0).$$

In particular,

$$\int_\Omega \sqrt{-1}\alpha \wedge \bar{\alpha} = 4(m+1)^2 \int_\Omega \sqrt{-1}\partial G \wedge \bar{\partial} G e^{4(m+1)G} = (m+1)\pi.$$

Similarly, $\int_{D_j} \sqrt{-1}\alpha_j \wedge \overline{\alpha_j} = (m+1)\pi$. Recall that $F' = \widehat{g}F'' \in \mathcal{A}_0$, then

$$\|F'\|^2_{\mathcal{A}_0} := \int_\Omega \sqrt{-1}F' \wedge \overline{F'}e^{-2\eta} = \int_\Omega \sqrt{-1}F'' \wedge \overline{F''}e^{2(m+1)G} < +\infty.$$

We may assume that $\big\{2(m+1)G < -t_0\big\} \Subset D_1$ for some $t_0 \gg 1$, then

$$\int_{\Omega\setminus D_1} \sqrt{-1}F'' \wedge \overline{F''} \leqslant e^{t_0}\|F'\|^2_{\mathcal{A}_0} < +\infty.$$

Recall that $\int_{D_j} \alpha_j \wedge \overline{F''} = 0$. For any integers $j \geqslant k \geqslant 1$, we have

$$\begin{aligned}
&\left|\int_\Omega \alpha \wedge \overline{F''}\right|\\
&\qquad = \left|\int_\Omega \alpha \wedge \overline{F''} - \int_{D_j} \alpha_j \wedge \overline{F''}\right|\\
&\qquad \leqslant \left|\int_{\Omega\setminus D_k} \alpha \wedge \overline{F''}\right| + \left|\int_{D_j\setminus D_k} \alpha_j \wedge \overline{F''}\right| + \left|\int_{D_k} (\alpha_j - \alpha) \wedge \overline{F''}\right|\\
&\qquad \leqslant \big(\|\alpha\|_{L^2(\Omega)} + \|\alpha_j\|_{L^2(D_j)}\big)\|F''\|_{L^2(\Omega\setminus D_k)} + \left|\int_{D_k} (\alpha_j - \alpha) \wedge \overline{F''}\right|.
\end{aligned}$$

Since $\|F''\|_{L^2(\Omega\setminus D_1)} < +\infty$ and $D_k \nearrow \Omega$, the first term converges to 0 as $k \to +\infty$. For fixed k, since $\alpha_j \to \alpha$ uniformly on $\overline{D_k}$, the second term converges to 0 as $j \to +\infty$. Let $j \to +\infty$ and then $k \to +\infty$, we conclude that $\int_\Omega \alpha \wedge \overline{F''} = 0$.

In summary, provided $\chi_\eta \chi_{z_0}^{m+1} = 1$, we prove that $F := p_*\big(\xi\, \mathrm{d}(g^{m+1})\big)$ *is extremal with respect to* $B_{\Omega,\eta}^{(m)}(z_0)$. Recall that $F\big|_V = \zeta\, \mathrm{d}(h^{m+1})$ and $h \equiv c_\beta(z_0) w$ on V. Therefore,

$$F\big|_V = (m+1) c_\beta(z_0)^{m+1} \zeta(w) w^m \, \mathrm{d}w.$$

If we write $F\big|_V = f(w)\, \mathrm{d}w$, then $[f]_{z_0} \in \mathfrak{m}_{z_0}^m$ and

$$\left| \frac{\partial^m f}{\partial w^m}(z_0) \right|^2 = \big((m+1)!\big)^2 c_\beta(z_0)^{2m+2} e^{2\eta(z_0)}.$$

On the other hand, since $F = 2(m+1)\widehat{g}\partial G$ on $\Omega \setminus \{z_0\}$, we have

$$\int_\Omega \tfrac{\sqrt{-1}}{2} F \wedge \overline{F} e^{-2\eta} = 4(m+1)^2 \int_\Omega \tfrac{\sqrt{-1}}{2} \partial G \wedge \bar{\partial} G e^{2(m+1)G} = (m+1)\pi.$$

Since F is extremal with respect to $B_{\Omega,\eta}^{(m)}(z_0)$, we conclude that

$$B_{\Omega,\eta}^{(m)}(z_0) = \frac{\big((m+1)!\big)^2 c_\beta(z_0)^{2m+2} e^{2\eta(z_0)}}{(m+1)\pi}.$$

This completes the proof. □

5. The equality in higher order Suita conjecture

In this section, we restrict ourselves to the case of planar domains.

By the Riemann mapping theorem, any simply connected domain $\Omega \subsetneq \mathbb{C}$ is conformally equivalent to the unit disk, then it is clear that $\pi B_\Omega^{(m)}(z_0) = m!\,(m+1)!\, c_\beta(z_0)^{2m+2}$ for all $z_0 \in \Omega$ and $m \in \mathbb{N}$.

Next, we consider a domain $\Omega \subset \mathbb{C}$ of finite connectivity $n \geqslant 2$. Since isolated points are removable singularities for L^2 holomorphic functions and upper bounded subharmonic functions, we may assume that no connected component of $\widehat{\mathbb{C}} \setminus \Omega$ reduces to a point. After a conformal transformation, *we assume that* Ω *is bounded by* n *analytic curves* $\Gamma_1, \ldots, \Gamma_n$. Let ω_j be the harmonic measure of Γ_j with respect to Ω. By the reflection principle, ω_j and $G_\Omega(\,\cdot\,, z_0)$ ($\forall\, z_0 \in \Omega$) have harmonic extensions in some neighbourhood of $\partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_n$. It is known that the period $\int_{\Gamma_j} {}^*\mathrm{d}G_\Omega(\,\cdot\,, z_0)$ equals to $2\pi\omega_j(z_0)$, where ${}^*\mathrm{d}u = -\sqrt{-1}(\partial u - \bar{\partial} u)$ denotes the conjugate differential of $\mathrm{d}u$. (See [1] for details.)

If $n = 2$, then such Ω is conformally equivalent to some annulus $A_R = \big\{z \in \mathbb{C} : 1 < |z| < R\big\}$. In a joint work [22] with Li, by studying the multi-valued harmonic conjugate of $G_{A_R}(\,\cdot\,, z_0)$, we showed that, if $|z_0| = \exp\big(\frac{k}{m+1} \log R\big)$ for some integer $k \in [1, m]$, then there exists a holomorphic function $g \in \mathcal{O}(A_R)$ such that $\log|g| = (m+1)G_{A_R}(\cdot, z_0)$. According to Theorem 4.5, for any integers $1 \leqslant k \leqslant m$, we have

$$\pi B^{(m)}_{A_R}(z_0) = m!\,(m+1)!\,c_\beta(z_0; A_R)^{2m+2}, \quad |z_0| = R^{\frac{k}{m+1}}. \tag{5.1}$$

This also follows from an explicit formula for the Green's function of A_R (see [20]):

$$G_{A_R}(z, a) = \log \frac{\left|(1 - a^{-1}z)\Pi(a, z)\right|}{|z|^{s(a)}}, \quad 1 < a < R,$$

where

$$\Pi(a, z) := \frac{\prod_{\nu=1}^{\infty}\left(1 - \frac{z}{a}R^{-2\nu}\right)\left(1 - \frac{a}{z}R^{-2\nu}\right)}{\prod_{\nu=1}^{\infty}(1 - azR^{-2\nu})\left(1 - \frac{1}{az}R^{-2\nu+2}\right)}$$

and

$$s(a) := 1 - \frac{\log a}{\log R}.$$

If $a = \exp\left(\frac{k}{m+1}\log R\right)$ for some integer $k \in [1, m]$, then $s(a) = 1 - \frac{k}{m+1}$ and

$$g_a(z) := \frac{\left((1 - a^{-1}z)\Pi(a, z)\right)^{m+1}}{z^{m+1-k}}$$

is a holomorphic function on A_R such that $\log|g_a| = (m+1)G_{A_R}(\cdot, a)$.

As pointed out by Guan–Sun–Yuan [16], given $z_0 \in \Omega$ and $m \in \mathbb{N}$, there exists a holomorphic function $g \in \mathcal{O}(\Omega)$ satisfying $\log|g| = (m+1)G_\Omega(\cdot, z_0)$ if and only if $(m+1)\omega_j(z_0) \in \mathbb{Z}$ for all $1 \leqslant j \leqslant n$. By Theorem 4.5, these conditions are equivalent to $\pi B^{(m)}_\Omega(z_0) = m!\,(m+1)!\,c_\beta(z_0)^{2m+2}$. Since $0 < \omega_j < 1$ on Ω and $\omega_1 + \cdots + \omega_n \equiv 1$, in this case, one has

$$1 = \omega_1(z_0) + \cdots + \omega_n(z_0) \geqslant \tfrac{n}{m+1}.$$

Consequently, if the equality in m-order Suita conjecture holds somewhere in an n-connected domain, then it is necessary that $m \geqslant n - 1$. (Every domain considered here is bounded by analytic curves!) Moreover, Guan–Sun–Yuan [16] showed that, in any 3-connected domain Ω, there exist some $z_0 \in \Omega$ and large $m \in \mathbb{N}$ such that $\pi B^{(m)}_\Omega(z_0) = m!\,(m+1)!\,c_\beta(z_0)^{2m+2}$.

In summary, if $\Omega \subsetneq \mathbb{C}$ is *simply connected*, then for any $m \geqslant 0$, the equality in m-order Suita conjecture holds for every point of Ω; if Ω is *doubly connected*, then for any $m \geqslant 1$, the equality in m-order Suita conjecture holds for all points on m analytic curves.

It is natural to ask, if Ω is 3-*connected*, can we find a point $z_0 \in \Omega$ such that the equality in 2-order Suita conjecture holds? However, the following counterexample shows that this is impossible in general.

Theorem 5.1. — *Given any integers $n \geqslant 3$ and $M \gg 1$, there exists a family of smoothly bounded n-connected domain $\Omega \subset \mathbb{C}$ such that no point of Ω can satisfy the equality in m-order Suita conjecture, where $m = 0, 1, \ldots, M$.*

Proof. — Let $a, \varepsilon \in (0,1)$ be positive constants to be specified later. Define

$$\varphi_1(z) = \frac{z+a}{1+az} \qquad \text{and} \qquad \varphi_2(z) = \frac{z-a}{1-az},$$

they are automorphisms of the unit disk $\mathbb{D}$. Let

$$D_1 = \left\{z \in \mathbb{D} : \left|\varphi_1(z)\right| \leqslant \varepsilon\right\} \qquad \text{and} \qquad D_2 = \left\{z \in \mathbb{D} : \left|\varphi_2(z)\right| \leqslant \varepsilon\right\}.$$

By the property of linear fractional transformations, D_1 and D_2 are closed disks in $\mathbb{D}$:

$$D_1, D_2 = \left\{z \in \mathbb{C} : \left|z \pm \frac{a(1-\varepsilon^2)}{1-a^2\varepsilon^2}\right| \leqslant \frac{\varepsilon(1-a^2)}{1-a^2\varepsilon^2}\right\}.$$

If $\varepsilon < a$, then D_1 and D_2 are disjoint. Let $D_3, \ldots, D_{n-1}$ be arbitrary disjoint closed disks in $\mathbb{D} \setminus (D_1 \cup D_2)$. Denote $\Gamma_j = \partial D_j$ and $\Gamma_n = \partial\mathbb{D}$. Then

$$\Omega := \mathbb{D} \setminus \left(\textstyle\bigcup_{j=1}^{n-1} D_j\right)$$

is an n-connected domain bounded by circles $\Gamma_1, \ldots, \Gamma_n$.

Let ω_j be the harmonic measure of Γ_j with respect to Ω, i.e. ω_j is a harmonic function on Ω, taking boundary values 1 on Γ_j and 0 on the other contours. Notice that $\log\left|\varphi_1(z)\right| / \log\varepsilon$ is a harmonic function on Ω, taking boundary values 1 on Γ_1 and is nonnegative on the other contours. By the maximum principle,

$$\omega_1(z) \leqslant \frac{\log\left|\varphi_1(z)\right|}{\log\varepsilon}, \quad z \in \Omega.$$

Let $c := \inf\left\{\left|\varphi_1(z)\right| : z \in \mathbb{D}, \operatorname{Re} z \geqslant 0\right\}$. Since $\varphi_1 \in \operatorname{Aut}(\mathbb{D})$ and $\varphi_1(-a) = 0$, it clear that $0 < c < 1$. Therefore,

$$\omega_1(z) \leqslant \frac{\log c}{\log\varepsilon}, \quad z \in \Omega \cap \left\{z : \operatorname{Re} z \geqslant 0\right\}.$$

Similarly, $\omega_2 \leqslant \frac{\log c}{\log\varepsilon}$ on $\Omega \cap \left\{z : \operatorname{Re} z \leqslant 0\right\}$. Notice that the constant c depends only on a. Indeed, we can show that $c = \left|\varphi_1(0)\right| = a$.

Let $a \in (0,1)$ be arbitrarily given, we choose $\varepsilon \ll 1$ so that $\frac{\log c}{\log\varepsilon} < \frac{1}{M+1}$, i.e. $0 < \varepsilon < c^{M+1}$. Since $\omega_1 < \frac{1}{M+1}$ on $\Omega \cap \left\{z : \operatorname{Re} z \geqslant 0\right\}$, all the curves $\omega_1 = q$, with $q \in \left(\frac{1}{2}\mathbb{Z} \cup \cdots \cup \frac{1}{M+1}\mathbb{Z}\right) \cap \mathbb{Q}_+$, completely lie in the left half-plane. Similarly, all the curves $\omega_2 = q'$, with $q' \in \left(\frac{1}{2}\mathbb{Z} \cup \cdots \cup \frac{1}{M+1}\mathbb{Z}\right) \cap \mathbb{Q}_+$, completely lie in the right half-plane. Since these curves are disjoint, for any integer $m = 0, 1, \ldots, M$, we cannot find a point $z_0 \in \Omega$ such that $\omega_1(z_0), \omega_2(z_0) \in \frac{1}{m+1}\mathbb{Z}$. Therefore, no point of Ω can satisfy the equality in m-order Suita conjecture, where $m = 0, 1, \ldots, M$. $\square$

6. Higher dimensional generalizations

In this section, D is a bounded domain in $\mathbb{C}^n$. Given a measurable function φ on D which is locally bounded from above, the weighted Bergman space is defined by

$$A^2(D; e^{-\varphi}) := \left\{ f \in \mathcal{O}(D) : \|f\|^2 = \int_D |f|^2 e^{-\varphi} \,\mathrm{d}\lambda < +\infty \right\},$$

and the weighted Bergman kernel is

$$B_D(z; e^{-\varphi}) := \sup\big\{ |f(z)|^2 : f \in A^2(D; e^{-\varphi}),\ \|f\|^2 \leqslant 1 \big\}.$$

If $\varphi \equiv 0$, we shall simplify these notations as $A^2(D)$ and $B_D(z)$.

Denote by $G_D(\,\cdot\,, z)$ the pluricomplex Green function of D with a pole at $z \in D$, then the *Azukawa pseudometric* of D is defined by

$$A_D(z; X) := \varlimsup_{\lambda \to 0} \big(G_D(z + \lambda X, z) - \log|\lambda| \big), \quad z \in D,\ X \in \mathbb{C}^n.$$

Clearly, $A_D(z; \cdot\,) \in \mathrm{psh}(\mathbb{C}^n)$ and $A_D(z; \tau X) = A_D(z; X) + \log|\tau|$ for any $\tau \in \mathbb{C}$. Therefore, the *Azukawa indicatrix*

$$I_D(z) := \big\{ X \in \mathbb{C}^n : A_D(z; X) < 0 \big\}$$

is a *balanced* pseudoconvex domain in $\mathbb{C}^n$. (Recall that a set $U \subset \mathbb{C}^n$ is said to be balanced, if $\tau z \in U$ for every $z \in U$ and $\tau \in \mathbb{C}$ with $|\tau| \leqslant 1$.)

For simplicity, we assume that $z_0 = 0$ and $\mathbb{B}^n(0; r) \subset D \subset \mathbb{B}^n(0; R)$. For each $a \geqslant 0$, let $D_a := \big\{ G_D(\,\cdot\,, 0) < -a \big\}$. Since $\log\big(|z|/R\big) \leqslant G_D(z, 0) \leqslant \log\big(|z|/r\big)$, it is easy to see that $\mathbb{B}^n(0; r) \subset e^a D_a \subset \mathbb{B}^n(0; R)$ and $\mathbb{B}^n(0; r) \subset I_D(0) \subset \mathbb{B}^n(0; R)$. Here we use the standard convention: given $U \subset \mathbb{C}^n$ and $c > 0$, then cU is a set defined by $\big\{ cz : z \in U \big\}$.

In the following, we assume that $D \subset \mathbb{C}^n$ is *hyperconvex*, which means that there exists a negative continuous psh exhaustion function on D. In this case, Zwonek [32] proved that A_D is continuous on $D \times \mathbb{C}^n$ and

$$A_D(z; X) = \lim_{\substack{w \to z,\, w \neq z \\ (w-z)/|w-z| \to X/|X|}} \left(G_D(w, z) - \log \tfrac{|w-z|}{|X|} \right) \quad (X \neq 0).$$

For any $0 < \varepsilon \ll 1$, we can find an $a_\varepsilon > 0$ such that $e^a D_a \subset (1 + \varepsilon) I_D(0)$ for all $a > a_\varepsilon$. Otherwise, there exist $\varepsilon > 0$, $a_j \to +\infty$ and $X_j \in \mathbb{C}^n$ such that $X_j \in e^{a_j} D_{a_j}$ ($\Leftrightarrow G(e^{-a_j} X_j, 0) < -a_j$) and $X_j \notin (1 + \varepsilon) I_D(0)$ ($\Leftrightarrow$

$A(0;X_j) \geqslant \log(1+\varepsilon))$. Since $(1+\varepsilon)r \leqslant |X_j| \leqslant R$, we may assume that $X_j \to X^*$ as $j \to +\infty$. Using the regularity of A_D, we reach a contradiction:

$$A_D(0;X^*) = \lim_{j\to+\infty} A_D(0;X_j) \geqslant \log(1+\varepsilon),$$
$$A_D(0;X^*) = \lim_{j\to+\infty}\left(G_D(e^{-a_j}X_j,0) - \log\tfrac{|e^{-a_j}X_j|}{|X^*|}\right) \leqslant 0.$$

Similarly, for any $0<\varepsilon\ll 1$, we can find an $a'_\varepsilon > 0$ such that $e^a D_a \supset (1-\varepsilon)I_D(0)$ for all $a > a'_\varepsilon$. In summary, for any $0<\varepsilon \ll 1$, we have

$$(1-\varepsilon)I_D(0) \subset e^a D_a \subset (1+\varepsilon)I_D(0) \quad \text{for all } a \gg 1. \tag{6.1}$$

As a consequence, $\lim_{a\to+\infty} e^{2na}\operatorname{Vol}(D_a) = \operatorname{Vol}\big(I_D(0)\big)$. Using (3.1), Błocki–Zwonek [7] obtained the following generalization to Suita's conjecture:

$$B_D(z_0) \geqslant \frac{1}{\operatorname{Vol}\big(I_D(z_0)\big)}. \tag{6.2}$$

Via approximation, (6.2) holds for general pseudoconvex domains in $\mathbb{C}^n$.

Let $H(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha$ be a homogeneous polynomial of degree m on $\mathbb{C}^n$, for any holomorphic function $f(z)$, we define

$$\partial_z^H f \coloneqq \sum_{|\alpha|=m} c_\alpha \partial_z^\alpha f = \sum_{|\alpha|=m} c_\alpha \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}}.$$

Błocki–Zwonek [9] introduced the following generalized Bergman kernel:

$$B_D^H(w) \coloneqq \sup\left\{ \left|\partial_z^H f(w)\right|^2 : f\in A^2(D),\ \int_D |f|^2\,\mathrm{d}\lambda \leqslant 1,\ [f]_w \in \mathfrak{m}_w^m \right\}.$$

At first, we assume that $D \ni z_0$ is a bounded hyperconvex domain. Assume that $z_0 = 0$ and let $D_a \coloneqq \big\{G_D(\,\cdot\,,0) < -a\big\}$. By the monotonicity property of Bergman kernels and (6.1),

$$\lim_{a\to+\infty} e^{-2(n+m)a} B_{D_a}^H(0) = \lim_{a\to+\infty} B_{e^a D_a}^H(0) = B_{I_D(0)}^H(0).$$

Using a tensor power trick, Błocki–Zwonek [9] proved that

$$a \longmapsto e^{-2(n+m)a} B_{D_a}^H(0)$$

is a decreasing function on $[0,+\infty)$, and then

$$B_D^H(z_0) \geqslant B_{I_D(z_0)}^H(0). \tag{6.3}$$

Via approximation, (6.3) is true for general pseudoconvex domains. Clearly, if $H \equiv 1$, then $B_\bullet^H$ are the usual Bergman kernels and (6.3) reduces to (6.2).

If $D \subset \mathbb{C}$ is a planar domain, then $I_D(z_0) = \mathbb{D}\big(0; c_\beta(z_0)^{-1}\big)$, where $c_\beta(z_0)$ is the logarithmic capacity of D at z_0. In this case, (6.2) reduces to Suita's conjecture. Let $H(z) = z^m$, then $B_D^H = B_D^{(m)}$. By direct computations, $B^H_{I_D(z_0)}(0) = \pi^{-1} m!\,(m+1)!\, c_\beta(z_0)^{2m+2}$ and then (6.3) reduces to (4.1).

In the following, we apply the approach of Section 4 to prove (6.3).

Recall that $D \ni z_0$ is a bounded pseudoconvex domain in $\mathbb{C}^n$ and $H(z)$ is a homogeneous polynomial of degree m. There exists some $f \in A^2(D)$ such that $[f]_{z_0} \in \mathfrak{m}_{z_0}^m$ and $\partial_z^H f(z_0) = 1$. We obtain such an f by solving certain $\bar\partial$-equation with L^2 estimate. Alternatively, we can apply [30, Corollary 1.4] directly. Let $\psi := 2(n+m)G_D(\,\cdot\,, z_0)$. For each $t \geqslant 0$, let $\mathcal{D}_t := \big\{\psi < -t\big\}$ and let $f_t \in A^2(\mathcal{D}_t)$ be the unique holomorphic function with minimal L^2 norm such that

$$[f_t]_{z_0} \in \mathfrak{m}_{z_0}^m \qquad \text{and} \qquad \partial_z^H f_t(z_0) = 1.$$

Let $I(t) := \|f_t\|^2_{A^2(\mathcal{D}_t)}$, then it is clear that $B^H_{\mathcal{D}_t}(z_0) = I(t)^{-1}$.

Notice that, if $\widetilde{f}$ is a holomorphic function such that $|\widetilde{f} - f|^2 e^{-\psi}$ is locally integrable near z_0, then $[\widetilde{f}]_{z_0} \in \mathfrak{m}_{z_0}^m$ and $\partial_z^H \widetilde{f}(z_0) = 1$. Therefore, the arguments of [30, Section 5.1] can be applied without any change, and we conclude that (see also [14]):

(i) $r \mapsto I(-\log r)$ is a concave increasing function on $(0,1]$;
(ii) if $r \mapsto I(-\log r)$ is linear, then $f_0\big|_{\mathcal{D}_t} \equiv f_t$ for any $t \geqslant 0$.

By concavity, $r \mapsto I(-\log r)/r$ is decreasing on $(0,1]$, then $t \mapsto e^{-t} B^H_{\mathcal{D}_t}(z_0)$ is also decreasing on $[0,+\infty)$. This monotonicity was proved in [9] by using a tensor power trick. (Recall that $D_a := \big\{G_D(\,\cdot\,,z_0) < -a\big\} = \mathcal{D}_{2(n+m)a}$.) The remaining part of the proof is the same as [9]: if D is hyperconvex, then

$$\lim_{t\to+\infty} e^{-t} B^H_{\mathcal{D}_t}(z_0) = \lim_{a\to+\infty} e^{-2(n+m)a} B^H_{D_a}(z_0) = B^H_{I_D(z_0)}(0),$$

and the monotonicity implies the inequality (6.3); via approximation, (6.3) is true for general pseudoconvex domains. Actually, we have something more.

Proposition 6.1. — *Let $D \ni z_0$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $H(z)$ be a holomorphic homogeneous polynomial of degree m on $\mathbb{C}^n$. For each $t \geqslant 0$, let $\mathcal{D}_t := \big\{2(n+m)G_D(\,\cdot\,,z_0) < -t\big\}$ and $B(t) := B^H_{\mathcal{D}_t}(z_0)$.*

(1) *For any $N \in \mathbb{N}_+$, $r \mapsto B\big(-\frac{1}{N}\log r\big)^{-N}$ is a concave function on $(0,1]$.*
(2) *If $B_D^H(z_0) = B^H_{I_D(z_0)}(0)$, then $f_0\big|_{\mathcal{D}_t} \equiv f_t$ for any $t \geqslant 0$, where $f_t \in A^2(\mathcal{D}_t)$ is the unique holomorphic function with minimal L^2 norm such that $[f_t]_{z_0} \in \mathfrak{m}_{z_0}^m$ and $\partial_z^H f_t(z_0) = 1$.*

Proof. — We assume the same notations as above. By conclusion (i), $r \mapsto B(-\log r)^{-1}$ is a concave function on $(0,1]$. Given $N \in \mathbb{N}_+$, we consider the product domain $\widetilde{D} := D \times \cdots \times D \subset \mathbb{C}^{nN}$ and $\widetilde{z_0} := (z_0, \ldots, z_0) \in \widetilde{D}$. By the same reason,

$$r \in (0,1] \longmapsto \left(B^{H\times\cdots\times H}_{\{\Psi<\log r\}}(\widetilde{z_0})\right)^{-1}$$

is a concave function, where $\Psi := 2(n+m)NG_{\widetilde{D}}(\,\cdot\,, \widetilde{z_0})$. By the product properties of pluricomplex Green functions (see [20]) and generalized Bergman kernels (see [9]),

$$\left\{\Psi < \log r\right\} = \left\{\psi < \tfrac{\log r}{N}\right\} \times \cdots \times \left\{\psi < \tfrac{\log r}{N}\right\},$$

$$B^{H\times\cdots\times H}_{\{\Psi<\log r\}}(\widetilde{z_0}) = \left(B^{H}_{\{\psi<\log r/N\}}(z_0)\right)^{N} = B\left(-\tfrac{1}{N}\log r\right)^{N}.$$

Therefore, $r \mapsto B\left(-\frac{1}{N}\log r\right)^{-N}$ is a concave function on $(0,1]$.

If D is hyperconvex, we know

$$B^H_D(z_0) \geqslant e^{-t} B^H_{\mathcal{D}_t}(z_0) \geqslant B^H_{I_D(z_0)}(0), \quad t > 0.$$

Via approximation, this is true for general pseudoconvex domains. Hence, if $B^H_D(z_0) = B^H_{I_D(z_0)}(0)$, then $B^H_{\mathcal{D}_t}(z_0) = e^t B^H_D(z_0)$ for all t. In this case, $r \mapsto I(-\log r) = r/B^H_D(z_0)$ is linear, then it follows from conclusion (ii) that $f_0\big|_{\mathcal{D}_t} \equiv f_t$ for any $t \geqslant 0$. □

If $D \subset \mathbb{C}$ is a planar domain, then (6.3) reduces to (4.1), and Theorem 4.5 gives a full characterization for the equality case of (4.1). However, in higher dimensions, such a characterization is unknown yet. Nevertheless, the above proposition gives a necessary condition for the equality of (6.3).

7. The equality in higher dimension Suita conjecture

In this section, we study the equality case of higher order Suita conjecture. The first example is well-known; the second example is one dimensional, it was included for completeness. As an application of Theorem 4.6, we also give a new family of examples for which (6.3) becomes an equality.

(1) Let $D = \left\{z \in \mathbb{C}^n : h(z) < 1\right\}$ be a *bounded balanced pseudoconvex domain*, where $h\colon \mathbb{C}^n \to [0,\infty)$ is homogeneous (which means that $h(\tau z) = |\tau| h(z)$ for any $\tau \in \mathbb{C}$) and $\log h$ is psh. As $G_D(\,\cdot\,,0) \equiv \log h$, we know $D_a = \left\{G_D(\,\cdot\,,0) < -a\right\} = e^{-a}D$ and $I_D(0) = D$. Let $H(z)$ be a homogeneous polynomial of degree m on $\mathbb{C}^n$, then

$$B^H_D(0) = e^{-2(n+m)a} B^H_{D_a}(0) = B^H_{I_D(0)}(0) \quad (\forall\, a \geqslant 0).$$

(2) Let $\Omega = \{z \in \mathbb{C} : 1 < |z| < R\}$ be an *annulus* and $H(z) = z^m$. We choose a point $z_0 \in \Omega$ with $|z_0| = \exp\big(\frac{k}{m+1}\log R\big)$, where $k \in [1, m]$ is an integer. According to equation (5.1),

$$B^H_\Omega(z_0) = \pi^{-1} m!\,(m+1)!\, c_\beta(z_0;\Omega)^{2m+2} = B^H_{I_\Omega(z_0)}(0).$$

(3) Let Ω be a bounded domain in $\mathbb{C}$ and η be a harmonic function on Ω. Let $D = \{w \in \mathbb{C}^n : h(w) < 1\}$ be a bounded balanced pseudoconvex domain in $\mathbb{C}^n$, where $h\colon \mathbb{C}^n \to [0,\infty)$ is homogeneous and $\log h$ is psh. We consider the following *generalized Hartogs domain*:

$$\widetilde{\Omega} = \{(z,w) \in \Omega \times \mathbb{C}^n : h(w) < e^{-\eta(z)}\}.$$

Let ϕ be a subharmonic exhaustion function of Ω, then

$$\max\left\{\phi(z), -\frac{1}{\log h(w) + \eta(z)}\right\}$$

is a psh exhaustion function of $\widetilde{\Omega}$. Hence, $\widetilde{\Omega} \subset \mathbb{C}^{n+1}$ is pseudoconvex.

For any $z_0 \in \Omega$, $\psi(z,w) := \max\{G_\Omega(z,z_0), \log h(w) + \eta(z)\}$ is a negative psh function on $\widetilde{\Omega}$. Clearly, $\psi(z,w)$ has a logarithmic pole at $(z_0, 0)$. By the definition of pluricomplex Green functions,

$$G_{\widetilde{\Omega}}\big((z,w),(z_0,0)\big) \geqslant \max\{G_\Omega(z,z_0), \log h(w) + \eta(z)\}.$$

Now we estimate the Azukawa pseudometric of $\widetilde{\Omega}$ at $(z_0, 0)$. For any non-zero $(X,Y) \in \mathbb{C} \times \mathbb{C}^n$,

$$\begin{aligned}
&A_{\widetilde{\Omega}}\big((z_0,0);(X,Y)\big)\\
&\quad := \varlimsup_{\lambda\to 0}\Big(G_{\widetilde{\Omega}}\big((z_0+\lambda X, \lambda Y),(z_0,0)\big) - \log|\lambda|\Big)\\
&\quad \geqslant \varlimsup_{\lambda\to 0}\Big(\max\{G_\Omega(z_0+\lambda X, z_0), \log h(\lambda Y) + \eta(z_0+\lambda X)\} - \log|\lambda|\Big)\\
&\quad = \varlimsup_{\lambda\to 0} \max\{G_\Omega(z_0+\lambda X, z_0) - \log|\lambda|, \log h(Y) + \eta(z_0+\lambda X)\}.
\end{aligned}$$

Notice that, if $X \neq 0$, then $\lim_{\lambda\to 0}\exp\big(G_\Omega(z_0+\lambda X, z_0) - \log|\lambda X|\big) = c_\beta(z_0)$, where $c_\beta(z_0)$ is the logarithmic capacity of Ω at z_0. Therefore,

$$\begin{aligned}
&A_{\widetilde{\Omega}}\big((z_0,0);(X,Y)\big)\\
&\geqslant \max\Big\{\lim_{\lambda\to 0}\big(G_\Omega(z_0+\lambda X, z_0) - \log|\lambda|\big), \log h(Y) + \lim_{\lambda\to 0}\eta(z_0+\lambda X)\Big\}\\
&= \max\{\log\big|c_\beta(z_0)X\big|, \log h(Y) + \eta(z_0)\}.
\end{aligned}$$

Consequently,

$$I_{\widetilde{\Omega}}\big((z_0,0)\big) \subset \left\{(X,Y) \in \mathbb{C}\times\mathbb{C}^n : |X| < c_\beta(z_0)^{-1},\ h(Y) < e^{-\eta(z_0)}\right\} \tag{7.1}$$

and

$$\operatorname{Vol}\big(I_{\widetilde{\Omega}}\big((z_0,0)\big)\big) \leqslant \pi c_\beta(z_0)^{-2} \times \operatorname{Vol}(D) e^{-2n\eta(z_0)}. \tag{7.2}$$

Then we compute the Bergman kernel of $\widetilde{\Omega}$ at $(z_0,0)$. For any $r>0$, since $rD=\big\{w\in\mathbb{C}^n : h(w)<r\big\}$ is a balanced domain, it is clear that

$$\int_{rD} \big|f(w)\big|^2 \,\mathrm{d}\lambda_w \geqslant \operatorname{Vol}(D)\big|f(0)\big|^2 r^{2n}, \quad f\in\mathcal{O}(rD).$$

Notice that the integral on the left-hand side may be infinite. For any $\widetilde{f}\in A^2(\widetilde{\Omega})$, we have

$$\begin{aligned}\int_{\widetilde{\Omega}} |\widetilde{f}|^2 \,\mathrm{d}\lambda &= \int_{\Omega}\bigg(\int_{e^{-\eta(z)}D} \big|\widetilde{f}(z,w)\big|^2 \,\mathrm{d}\lambda_w\bigg)\mathrm{d}\lambda_z \\ &\geqslant \operatorname{Vol}(D)\int_{\Omega}\big|\widetilde{f}(z,0)\big|^2 e^{-2n\eta(z)} \,\mathrm{d}\lambda_z.\end{aligned}$$

It follows that $\widetilde{f}(\,\cdot\,,0)\in A^2(\Omega;e^{-2n\eta})$ and

$$\begin{aligned}B_{\widetilde{\Omega}}\big((z_0,0)\big) &= \sup\left\{\frac{\big|\widetilde{f}(z_0,0)\big|^2}{\int_{\widetilde{\Omega}}|\widetilde{f}|^2\,\mathrm{d}\lambda} : \widetilde{f}\in A^2(\widetilde{\Omega})\right\} \\ &\leqslant \sup\left\{\frac{\big|g(z_0)\big|^2}{\operatorname{Vol}(D)\int_{\Omega}|g|^2 e^{-2n\eta}\,\mathrm{d}\lambda} : g\in A^2(\Omega;e^{-2n\eta})\right\} \\ &= \frac{B_{\Omega,n\eta}(z_0)}{\operatorname{Vol}(D)}.\end{aligned}$$

On the other hand, for any $g\in A^2(\Omega;e^{-2n\eta})$, we define $\widetilde{g}(z,w) := g(z)$; then

$$\widetilde{g}\in A^2(\widetilde{\Omega}) \qquad \text{and} \qquad \int_{\widetilde{\Omega}}|\widetilde{g}|^2\,\mathrm{d}\lambda = \operatorname{Vol}(D)\int_{\Omega}|g|^2 e^{-2n\eta}\,\mathrm{d}\lambda.$$

Then it is clear that

$$B_{\widetilde{\Omega}}\big((z_0,0)\big) = \frac{B_{\Omega,n\eta}(z_0)}{\operatorname{Vol}(D)}. \tag{7.3}$$

If D is the unit ball in $\mathbb{C}^n$, (7.3) is also known as Ligocka's formula [23].

Combining (7.3), (6.2) and (7.2), we get

$$\frac{B_{\Omega,n\eta}(z_0)}{\operatorname{Vol}(D)} = B_{\widetilde{\Omega}}\big((z_0,0)\big) \geqslant \frac{1}{\operatorname{Vol}\big(I_{\widetilde{\Omega}}\big((z_0,0)\big)\big)} \geqslant \frac{c_\beta(z_0)^2 e^{2n\eta(z_0)}}{\pi\operatorname{Vol}(D)}.$$

By Theorem 4.3, if $\chi_{z_0}\chi_\eta^n=1$ (equivalently, there exists some $\widehat{g}\in\mathcal{O}(\Omega)$ such that $\log|\widehat{g}| = G_\Omega(\,\cdot\,,z_0)+n\eta$), then the above inequalities become equalities.

THEOREM 7.1. — *Let Ω be a bounded domain in $\mathbb{C}$ and η be a harmonic function on Ω. Let D be a bounded balanced pseudoconvex domain in $\mathbb{C}^n$ and $\widetilde{\Omega} = \{(z,w) \in \Omega \times \mathbb{C}^n : w \in e^{-\eta(z)}D\}$. If $\chi_{z_0}\chi_\eta^n = 1$ for some $z_0 \in \Omega$, then*

$$B_{\widetilde{\Omega}}\big((z_0,0)\big) = \frac{1}{\mathrm{Vol}\big(I_{\widetilde{\Omega}}\big((z_0,0)\big)\big)}.$$

It is not hard to generalize the above example to the case of (6.3).

THEOREM 7.2. — *Let Ω be a bounded domain in $\mathbb{C}$ and η be a harmonic function on Ω. Let D be a bounded balanced pseudoconvex domain in $\mathbb{C}^n$ and*

$$\widetilde{\Omega} = \{(z,w) \in \Omega \times \mathbb{C}^n : w \in e^{-\eta(z)}D\}.$$

Let $H_2(w)$ be a homogeneous polynomial of degree k on $\mathbb{C}^n$ and $H(z,w) := z^m H_2(w)$. If $\chi_{z_0}^{m+1}\chi_\eta^{n+k} = 1$ for some $z_0 \in \Omega$, then

$$B^H_{\widetilde{\Omega}}\big((z_0,0)\big) = B^H_W\big((0,0)\big), \quad \text{where } W := I_{\widetilde{\Omega}}\big((z_0,0)\big).$$

Proof. — Since rD is a balanced domain, any $f \in \mathcal{O}(rD)$ can be written as a compactly convergent series $f = \sum_{l=0}^{\infty} f_l$, where each f_l is a homogeneous polynomial of degree l. It is clear that $[f_k]_0 \in \mathfrak{m}_0^k$, $\partial_w^{H_2} f_k(0) = \partial_w^{H_2} f(0)$ and

$$\int_{rD} |f|^2 \,\mathrm{d}\lambda = \sum_l \int_{rD} |f_l|^2 \,\mathrm{d}\lambda \geqslant \int_{rD} |f_k|^2 \,\mathrm{d}\lambda.$$

Notice that these integrals may be infinite. By the definition of generalized Bergman kernels,

$$\int_{rD} |f|^2 \,\mathrm{d}\lambda \geqslant \int_{rD} |f_k|^2 \,\mathrm{d}\lambda \geqslant \frac{\big|\partial_w^{H_2} f_k(0)\big|^2}{B_{rD}^{H_2}(0)} = \frac{\big|\partial_w^{H_2} f(0)\big|^2}{B_D^{H_2}(0) r^{-2(n+k)}}.$$

For any $\widetilde{f} \in A^2(\widetilde{\Omega})$ with $[\widetilde{f}]_{(z_0,0)} \in \mathfrak{m}_{(z_0,0)}^{m+k}$, we have

$$\begin{aligned}\int_{\widetilde{\Omega}} |\widetilde{f}|^2 \,\mathrm{d}\lambda &= \int_\Omega \left(\int_{e^{-\eta(z)}D} \big|\widetilde{f}(z,w)\big|^2 \,\mathrm{d}\lambda_w \right) \mathrm{d}\lambda_z \\ &\geqslant \frac{1}{B_D^{H_2}(0)} \int_\Omega \big|\partial_w^{H_2} \widetilde{f}(z,0)\big|^2 e^{-2(n+k)\eta(z)} \,\mathrm{d}\lambda_z.\end{aligned}$$

Therefore, $\partial_w^{H_2}\widetilde{f}(\,\cdot\,,0) \in A^2\big(\Omega; e^{-2(n+k)\eta}\big)$. Since $\big[\partial_w^{H_2}\widetilde{f}(\,\cdot\,,0)\big]_{z_0} \in \mathfrak{m}_{z_0}^m$, we know

$$\begin{aligned}B^H_{\widetilde{\Omega}}\big((z_0,0)\big) &\leqslant \sup_{\tilde{f}} \frac{\big|\partial_z^m \partial_w^{H_2} \widetilde{f}(z_0,0)\big|^2}{\int_\Omega \big|\partial_w^{H_2}\widetilde{f}(z,0)\big|^2 e^{-2(n+k)\eta(z)} \,\mathrm{d}\lambda_z / B_D^{H_2}(0)} \\ &\leqslant B^{(m)}_{\Omega,(n+k)\eta}(z_0) \times B_D^{H_2}(0).\end{aligned}$$

Here, the supremum is taken over all $\widetilde{f} \in A^2(\widetilde{\Omega})$ with $[\widetilde{f}]_{(z_0,0)} \in \mathfrak{m}_{(z_0,0)}^{m+k}$.

It is easy to find a homogeneous polynomial u of degree k on $\mathbb{C}^n$ such that $\int_D |u|^2 \,\mathrm{d}\lambda = 1$ and $B_D^{H_2}(0) = |\partial_w^{H_2} u(0)|^2$. For any $g \in A^2\big(\Omega; e^{-2(n+k)\eta}\big)$ with $[g]_{z_0} \in \mathfrak{m}_{z_0}^m$, let $\widetilde{g}(z,w) := g(z)u(w)$, then $\widetilde{g} \in A^2(\widetilde{\Omega})$ and $[\widetilde{g}]_{(z_0,0)} \in \mathfrak{m}_{(z_0,0)}^{m+k}$. Therefore,

$$\begin{aligned} B^{(m)}_{\Omega,(n+k)\eta}(z_0) &= \sup_g \frac{\big|\partial_z^m g(z_0)\big|^2}{\int_\Omega |g|^2 e^{-2(n+k)\eta}\,\mathrm{d}\lambda} \\ &= \sup_g \frac{\big|\partial^H \widetilde{g}(z_0,0)\big|^2 / \big|\partial_w^{H_2} u(0)\big|^2}{\int_{\widetilde{\Omega}} |\widetilde{g}|^2 \,\mathrm{d}\lambda} \\ &\leqslant \frac{B^H_{\widetilde{\Omega}}\big((z_0,0)\big)}{B_D^{H_2}(0)}. \end{aligned}$$

Here, the supremum is taken over all $g \in A^2\big(\Omega; e^{-2(n+k)\eta}\big)$ with $[g]_{z_0} \in \mathfrak{m}_{z_0}^m$.

Denote by W the Azukawa indicatrix of $\widetilde{\Omega}$ at $(z_0,0)$. According to (7.1), $W \subset U \times V$, where $U := \mathbb{D}\big(0; c_\beta(z_0)^{-1}\big)$ and $V := e^{-\eta(z_0)} D$. By the monotonicity property and the product property of generalized Bergman kernels,

$$\begin{aligned} B_W^H\big((0,0)\big) &\geqslant B^H_{U\times V}\big((0,0)\big) \\ &= B_U^{(m)}(0) \times B_V^{H_2}(0) \\ &= \tfrac{m!\,(m+1)!}{\pi} c_\beta(z_0)^{2m+2} \times e^{2(n+k)\eta(z_0)} B_D^{H_2}(0). \end{aligned}$$

Combining these results with (6.3), we get

$$\begin{aligned} B^{(m)}_{\Omega,(n+k)\eta}(z_0) \times B_D^{H_2}(0) &= B^H_{\widetilde{\Omega}}\big((z_0,0)\big) \\ &\geqslant B_W^H\big((0,0)\big) \\ &\geqslant \tfrac{m!\,(m+1)!}{\pi} c_\beta(z_0)^{2m+2} e^{2(n+k)\eta(z_0)} \times B_D^{H_2}(0). \end{aligned}$$

According to Theorem 4.6, if $\chi_{z_0}^{m+1}\chi_\eta^{n+k} = 1$ (equivalently, there exists $\widehat{g} \in \mathcal{O}(\Omega)$ such that $\log|\widehat{g}| = (m+1)G_\Omega(\,\cdot\,, z_0) + (n+k)\eta)$, then the above inequalities become equalities. □

Remark 7.3. — It is easy to find (Ω,η) satisfying the requirements of Theorem 7.2. Let Ω' be an arbitrary bounded domain in $\mathbb{C}$, we choose a holomorphic function $0 \not\equiv f \in \mathcal{O}(\Omega')$ having at least a zero $z_0 \in \Omega'$. Define $\Omega := \big(\Omega' \setminus f^{-1}(0)\big) \cup \{z_0\}$; then z_0 is the only zero of $f \in \mathcal{O}(\Omega)$. Let $m := \mathrm{ord}_{z_0}(f) - 1$ and

$$\eta := \tfrac{1}{n+k}\big(\log|f| - (m+1)G_\Omega(\,\cdot\,, z_0)\big),$$

then η is a harmonic function on Ω and $\chi_{z_0}^{m+1}\chi_\eta^{n+k} = 1$.

Clearly, all balanced domains are contractible. In contrast, the $\widetilde{\Omega}$ defined in Theorem 7.2 has the same homotopy type as Ω, which could be very complicated.

Using the transformation rule under biholomorphism and the product property of generalized Bergman kernels (see [9]), one can construct more and more examples from (1), (2) and (3). It would be interesting to find a full characterization for the equality of (6.3).

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