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SOME OBSERVATIONS ON DEFORMED DONALDSON–THOMAS CONNECTIONS

by Kotaro KAWAI (*)

ABSTRACT. — A deformed Donaldson–Thomas (dDT) connection is a Hermitian connection of a Hermitian line bundle over a G_2-manifold X satisfying a certain nonlinear PDE. This is considered to be the mirror of a (co)associative cycle in the context of mirror symmetry. It can also be considered as an analogue of a G_2-instanton. In this paper, we see that some important observations that appear in other geometric problems are also found in the dDT case as follows.

(1) A dDT connection exists if a 7-manifold has full holonomy G_2 and the G_2-structure is "sufficiently large". (2) The dDT equation is described as the zero of a certain multi-moment map. (3) The gradient flow equation of a Chern–Simons type functional of Karigiannis and Leung, whose critical points are dDT connections, agrees with the Spin(7) version of the dDT equation on a cylinder with respect to a certain metric on a certain space. This can be considered as an analogue of the observation in instanton Floer homology for 3-manifolds.

RÉSUMÉ. — Une connexion déformée de Donaldson–Thomas (dDT) est une connexion hermitienne d'un fibré en ligne hermitien sur une variété G_2 X satisfaisant une certaine EDP non linéaire. Ceci est considéré comme le miroir d'un cycle (co)associatif dans le contexte de la symétrie miroir. On peut également le considérer comme un analogue d'un G_2-instanton. Dans cet article, nous voyons que certaines observations importantes qui apparaissent dans d'autres problèmes géométriques se retrouvent également dans le cas dDT comme suit.

(1) Une connexion dDT existe si une variété 7 possède une holonomie complète G_2 et que la structure G_2 est « suffisamment grande ». (2) L'équation dDT est décrite comme le zéro d'une certaine application multi-moments. (3) L'équation de flux de gradient d'une fonctionnelle de type Chern–Simons de Karigiannis et Leung, dont les points critiques sont des connexions dDT, concorde avec la version Spin(7) de l'équation dDT sur un cylindre par rapport à une certaine métrique sur un certain espace. Ceci peut être considéré comme un analogue de l'observation en homologie de Floer instanton pour les variétés 3.

Keywords: mirror symmetry, gauge theory, G_2-manifold, deformed Donaldson–Thomas connections.
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1. Introduction

Let X^7 be a 7-manifold with a G_2-structure $\varphi \in \Omega^3(X)$. For the definition of G_2-structures, see for example [10, Section 2.2]. Denote by g, vol and $*$ the induced Riemannian metric, volume form and Hodge star operator, respectively. We use the same sign convention as the paper above. In particular, we have the decomposition $\Lambda^2 T^*X = \Lambda^2_7 \oplus \Lambda^2_{14}$, where

$$\begin{aligned} \Lambda^2_7 &= \left\{ \beta \in \Lambda^2 T^*X \,\middle|\, *(\varphi \wedge \beta) = 2\beta \right\}, \\ \Lambda^2_{14} &= \left\{ \beta \in \Lambda^2 T^*X \,\middle|\, *(\varphi \wedge \beta) = -\beta \right\}. \end{aligned} \tag{1.1}$$

Let $(L, h) \to X$ be a smooth complex Hermitian line bundle over X. We denote by $\mathcal{A}_0$ the affine space of Hermitian connections on (L, h). Given $\nabla \in \mathcal{A}_0$, we regard its curvature F_∇ as a $\sqrt{-1}\mathbb{R}$-valued closed 2-form on X.

Definition 1.1. — *A Hermitian connection* $\nabla \in \mathcal{A}_0$ *satisfying*

$$\frac{1}{6} F_\nabla^3 + F_\nabla \wedge *\varphi = 0 \tag{1.2}$$

is called a deformed Donaldson–Thomas (dDT) connection*.*

DDT connections appeared in the context of mirror symmetry. They were introduced in [11] as "mirrors" of calibrated (associative) submanifolds. Historically, deformed Hermitian Yang–Mills (dHYM) connections were introduced first in [12] as "mirrors" of special Lagrangian submanifolds. There is also a similar notion of dDT connections for a manifold with a Spin(7)-structure ([8, 11]). As the names indicate, dDT connections can also be considered as analogues of Donaldson–Thomas connections (G_2-instantons).

Thus it is natural to expect that dDT connections would have similar properties to associative submanifolds and G_2-instantons. We show that it is indeed the case in [9, 10]. For example, the moduli space of dDT connections is b^1-dimensional and canonically orientable if we perturb the G_2-structure. Any dDT connection on a compact G_2-manifold is a global minimizer of the "mirror volume" and its value is topological by the "mirror" of associator equality. We could also prove similar statements in the Spin(7) case in [7, 10]. Moreover, dDT connections are given by critical points of the Chern–Simons type functional in [6, Theorem 5.13]. The variational characterization is known only for the G_2 case, and no such characterization is known for the Spin(7) case.

This paper is organized as follows. In Section 2, we study the existence of a dDT connection. Known examples of dDT connections are either trivial or constructed in [4, 13], and are very few in number. So it would be important to consider the existence problem. We first see that the formal "large radius limit" of the defining equation of dDT connections is that of G_2-instantons. Thus it is natural to expect that dDT connections for a "sufficiently large" G_2-structure will behave like G_2-instantons. Moreover, it is known that any complex Hermitian line bundle admits a G_2-instanton on a compact holonomy G_2-manifold. Then we show the following from these facts.

Theorem 1.2 (Theorem 2.2). — *Suppose that (X, φ) is a compact holonomy G_2-manifold. Let $(L, h) \to X$ be a smooth complex Hermitian line bundle over X. If φ is rescaled by a sufficiently large factor, there exists a dDT connection with respect to the rescaled φ.*

In Section 3, we formulate the dDT equation in terms of a multi-moment map. The multi-moment map is a generalization of the moment map introduced in [14, 15]. The dHYM equation is described as the zero of a certain moment map on an infinite dimensional symplectic manifold ([2, Section 2], [1, Section 2.1]). Analogously, we show that the dDT equation is described as the zero of a certain multi-moment map.

Theorem 1.3 (Theorem 3.4). — *The dDT equation is described as the zero of the multi-moment map defined in Theorem 3.4.*

In Section 4, we study the gradient flow of the Karigiannis–Leung functional introduced in [6] whose critical points are dDT connections. It is known that the gradient flow equation of the Chern–Simons functional on an oriented 3-manifold X^3 agrees with the ASD equation on $\mathbb{R} \times X^3$. This is an important observation in instanton Floer homology for 3-manifolds. We show that there is an analogous relation between dDT equations for G_2- and Spin(7)-manifolds using the Karigiannis–Leung functional. This will establish a new link between 3, 4-manifold theory and G_2-, Spin(7)-geometry, and we might define analogues of instanton Floer homology using dDT connections.

Theorem 1.4 (Theorem 4.3). — *The gradient flow equation of a Chern–Simons type functional of Karigiannis and Leung, whose critical points are dDT connections, agrees with the* $\mathrm{Spin}(7)$ *version of the dDT equation on a cylinder with respect to a metric $\mathcal{G}$ on a space $\mathcal{A}_{ac}$ defined at the beginning of Section 4.2.*

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2. Large radius limit

In this section, we show the existence of a dDT connection if a 7-manifold has full holonomy G_2 and the G_2-structure is "sufficiently large".

Suppose that (X, φ) is a compact holonomy G_2-manifold. Let $(L, h) \to X$ be a smooth complex Hermitian line bundle over X. Set

$$\mathcal{A}_0 = \{\text{Hermitian connections of } (L, h)\} = \nabla_0 + \sqrt{-1}\Omega^1 \cdot \mathrm{id}_L,$$

where $\nabla_0 \in \mathcal{A}_0$ is any fixed connection and Ω^1 is the space of 1-forms on X. Denote by $\mathcal{G}_U$ the group of unitary gauge transformations of (L, h), which acts on $\mathcal{A}_0$. Explicitly,

$$\mathcal{G}_U = \left\{ f \cdot \mathrm{id}_L \,\middle|\, f \in \Omega^0_{\mathbb{C}},\ |f| = 1 \right\} \cong C^\infty\left(X, S^1\right),$$

where $\Omega^0_{\mathbb{C}}$ is the space of $\mathbb{C}$-valued smooth functions, and the action $\mathcal{G}_U \times \mathcal{A}_0 \to \mathcal{A}_0$ is defined by $(\lambda, \nabla) \mapsto \lambda^* \nabla := \lambda^{-1} \circ \nabla \circ \lambda$. When $\lambda = f \cdot \mathrm{id}_L$ for $f \in C^\infty(X, S^1)$, we have

$$\lambda^* \nabla = \lambda^{-1} \circ \nabla \circ \lambda = \nabla + f^{-1} \mathrm{d}f \cdot \mathrm{id}_L\,. \tag{2.1}$$

Thus the $\mathcal{G}_U$-orbit through $\nabla \in \mathcal{A}_0$ is given by $\nabla + \mathcal{K}_U \cdot \mathrm{id}_L$, where

$$\mathcal{K}_U := \left\{ f^{-1} \mathrm{d}f \in \sqrt{-1}\Omega^1 \,\middle|\, f \in \Omega^0_{\mathbb{C}},\ |f| = 1 \right\}. \tag{2.2}$$

Note that the curvature 2-form F_∇ is invariant under the action of $\mathcal{G}_U$.

Consider the family of G_2-structures

$$\left\{ \varphi_r := r^3 \varphi \right\}_{r > 0},$$

all of which induce holonomy G_2 metrics. The defining equation of dDT connections with respect to φ_r is given by

$$0 = \mathcal{F}_r(\nabla) := \frac{1}{6} F_\nabla^3 + r^4 F_\nabla \wedge * \varphi.$$

Thus, formally taking the "large radius limit", which means the leading behaviour of $\mathcal{F}_r(\nabla)$ as $r \to \infty$, we obtain

$$F_\nabla \wedge * \varphi = 0.$$

This is exactly the defining equation of G_2-instantons. Thus it is natural to expect that dDT connections for a sufficiently large G_2-structure will behave like G_2-instantons. The following is well-known for G_2-instantons on a smooth complex Hermitian line bundle, but we give the proof for completeness.

Lemma 2.1. — *On a compact holonomy G_2-manifold (X^7, φ), there is a unique G_2-instanton on a smooth complex Hermitian line bundle $L \to X$ up to the action of $\mathcal{G}_U$.*

Proof. — For any $\nabla \in \mathcal{A}_0$, we have $\mathrm{d}F_\nabla = 0$. So it defines a cohomology class $[F_\nabla] \in \sqrt{-1}H^2(X, \mathbb{R})$, which is known to be equal to $-2\pi\sqrt{-1}c_1(L)$. Then there exists a 1-form $\alpha \in \sqrt{-1}\Omega^1$ such that $F_\nabla + \mathrm{d}\alpha$ is harmonic by Hodge theory.

Denote by $\Omega^k_\ell \subset \Omega^k$ the subspace of the space of k-forms corresponding to the ℓ-dimensional irreducible representation of G_2. For more details, see for example [10, Section 2.2]. Denote by $\mathcal{H}^k$ the space of harmonic k-forms on X and set $\mathcal{H}^k_\ell = \mathcal{H}^k \cap \Omega^k_\ell$. Then by [5, Theorem 10.2.4], we have $\mathcal{H}^2_7 \cong \mathcal{H}^1_7 = \mathcal{H}^1 = \{0\}$. Thus we have

$$F_{\nabla + \alpha \cdot \mathrm{id}_L} = F_\nabla + \mathrm{d}\alpha \in \sqrt{-1}\mathcal{H}^2 = \sqrt{-1}\mathcal{H}^2_7 \oplus \mathcal{H}^2_{14} = \sqrt{-1}\mathcal{H}^2_{14},$$

which implies that $F_{\nabla + \alpha \cdot \mathrm{id}_L} \wedge *\varphi = 0$.

If $\nabla' = \nabla + (\alpha + \alpha') \cdot \mathrm{id}_L$ for $\alpha' \in \sqrt{-1}\Omega^1$ is also a G_2-instanton, we have $0 = F_{\nabla'} \wedge *\varphi = \mathrm{d}\alpha' \wedge *\varphi$, which is equivalent to

$$-\mathrm{d}\alpha' = *(\mathrm{d}\alpha' \wedge \varphi) = *\mathrm{d}(\alpha' \wedge \varphi). \tag{2.3}$$

Since $\mathrm{d}\Omega^1 \cap \mathrm{d}^*\Omega^3 = \{0\}$, we have $\mathrm{d}\alpha' = 0$. Since $H^1(X, \mathbb{R}) = \{0\}$ by [5, Theorem 10.2.4] again and $\sqrt{-1}\mathbb{R}$-valued exact 1-forms are contained in $\mathcal{K}_U$, the G_2-instanton is unique up to the action of $\mathcal{G}_U$. □

Using this, we can show the following.

Theorem 2.2. — *Suppose that (X, φ) is a compact holonomy G_2-manifold. Let $(L, h) \to X$ be a smooth complex Hermitian line bundle over X. Then for sufficiently large $r > 0$, there exists a dDT connection with respect to φ_r.*

Proof. — Define a map $\mathcal{F} : [0,1] \times \mathcal{A}_0 \to \sqrt{-1}\mathrm{d}\Omega^5$ by

$$\mathcal{F}(s, \nabla) = \frac{s^4}{6} F_\nabla^3 + F_\nabla \wedge *\varphi.$$

Then $\mathcal{F}(0, \cdot)^{-1}(0)/\mathcal{G}_U$, which is a point by Lemma 2.1, is the moduli space of G_2-instantons with respect to φ and $\mathcal{F}(s, \cdot)^{-1}(0)/\mathcal{G}_U$ for $s \neq 0$ is the moduli space of dDT connections with respect to $\varphi_{1/s}$.

We want to apply the implicit function theorem to show the statement. Fix a G_2-instanton $\nabla_0 \in \mathcal{F}(0, \cdot)^{-1}(0)$, whose existence is guaranteed by Lemma 2.1. Denote by the linearization $(\mathrm{d}\mathcal{F})_{(0,\nabla_0)} : \mathbb{R} \oplus \sqrt{-1}\Omega^1 \to \sqrt{-1}\mathrm{d}\Omega^5$ of $\mathcal{F}$ at $(0, \nabla_0)$. Then we have

$$(\mathrm{d}\mathcal{F})_{(0,\nabla_0)}\left(0, \sqrt{-1}b\right) = \sqrt{-1}\mathrm{d}b \wedge *\varphi.$$

Lemma 2.3. — *We have*

$$\ker(\mathrm{d}\mathcal{F})_{(0,\nabla_0)} = \mathbb{R} \oplus \sqrt{-1}\mathrm{d}\Omega^0, \qquad \mathrm{Im}(\mathrm{d}\mathcal{F})_{(0,\nabla_0)} = \sqrt{-1}\mathrm{d}\Omega^5.$$

Proof. — The first equation is proved as in (2.3). For the second equation, the Hodge decomposition implies that $\mathrm{d}^*\Omega^2 = \mathrm{d}^*\mathrm{d}\Omega^1$. For any $b \in \Omega^1$, we have

$$\mathrm{d}^*\mathrm{d}b = d^*(db + *(\varphi \wedge \mathrm{d}b)) \in \mathrm{d}^*\Omega^2_7,$$

where we use the fact that φ is closed. Recall also the sign convention (1.1). This implies that $\mathrm{d}^*\Omega^2 = \mathrm{d}^*\Omega^2_7$. Then

$$\mathrm{d}\Omega^5 = *\mathrm{d}^*\Omega^2 = *\mathrm{d}^*\Omega^2_7 = \mathrm{d}\Omega^5_7.$$

Since Ω^5_7 is spanned by $b \wedge *\varphi$ for $b \in \Omega^1$, the proof is completed. □

By the Hodge decomposition and $H^1(X, \mathbb{R}) = \{0\}$, we have $\Omega^1 = \mathrm{d}\Omega^0 \oplus \mathrm{d}^*\Omega^2$. By this and Lemma 2.3, we see that $(\mathrm{d}\mathcal{F})_{(0,\nabla_0)}|_{\sqrt{-1}\mathrm{d}^*\Omega^2} : \sqrt{-1}\mathrm{d}^*\Omega^2 \to \sqrt{-1}\mathrm{d}\Omega^5$ is an isomorphism. Hence, we can apply the implicit function theorem (after the Banach completion) and we see that $\mathcal{F}(s, \cdot)^{-1}(0) \neq \emptyset$ for sufficiently small s.

Finally, we explain how to recover the regularity of elements in $\mathcal{F}(s, \cdot)^{-1}(0)$ after the Banach completion. Since the curvature is invariant under the addition of closed 1-forms, there exists $a_s \in \Omega^1$ such that

$$(*_s) \qquad \mathcal{F}\left(s, \nabla_0 + \sqrt{-1}a_s \cdot \mathrm{id}_L\right) = 0, \qquad \mathrm{d}^*a_s = 0$$

for sufficiently small s. In particular, $(*_0)$ is given by $\mathrm{d}a_0 \wedge *\varphi = \mathrm{d}^*a_0 = 0$, which is an overdetermined elliptic equation. Overdetermined ellipticity is an open condition, so we see that $(*_s)$ is also overdetermined elliptic for sufficiently small s. Hence we can find a smooth element in $\mathcal{F}(s, \cdot)^{-1}(0)$ around $(0, \nabla_0)$ and the proof is completed. □

3. The multi-moment map

It is known that there is a moment map picture in the dHYM case. In particular, the dHYM equation is described as the zero of a certain moment map on an infinite dimensional symplectic manifold. See for example [2, Section 2] or the survey article [1, Section 2.1]. Analogously, we show that the dDT equation is described as the zero of a certain multi-moment map. First, recall the definition of the multi-moment map in [14, 15].

Definition 3.1. — *Let X be a smooth manifold and $c \in \Omega^3$ be a closed 3-form on X. Suppose that a Lie group G acts on X preserving c. Denote by $\mathfrak{g}$ the Lie algebra of G and set*

$$\mathcal{P}_{\mathfrak{g}} = \ker\left(L : \Lambda^2\mathfrak{g} \longrightarrow \mathfrak{g}\right) \subset \Lambda^2\mathfrak{g},$$

where L is the linear map induced by the Lie bracket. (Note that $\mathcal{P}_{\mathfrak{g}} = \Lambda^2\mathfrak{g}$ if G is abelian.) Denote by u^* the vector field on X generated by $u \in \mathfrak{g}$. For a two vector $p = \sum_j u_j \wedge v_j \in \Lambda^2\mathfrak{g}$, set

$$p^* = \sum_j u_j^* \wedge v_j^*, \qquad i(p^*)c = \sum_j c\left(u_j^*, v_j^*, \cdot\right).$$

Denote by $\langle \cdot, \cdot \rangle : \Lambda^2\mathfrak{g}^* \times \Lambda^2\mathfrak{g} \to \mathbb{R}$ the canonical pairing.

Then a map $\nu : X \to \mathcal{P}_{\mathfrak{g}}^*$ is called a multi-moment map *if it is G-equivariant and satisfies*

$$\mathrm{d}\langle \nu, p\rangle = i(p^*)c$$

for any $p \in \mathcal{P}_{\mathfrak{g}}$.

Let X be a compact 7-manifold with a coclosed G_2-structure φ ($\mathrm{d}{*\varphi} = 0$) and $(L, h) \to X$ be a smooth complex Hermitian line bundle over X. Let $\mathcal{A}_0$ be the space of Hermitian connections of (L, h). Define a map $\mathcal{F}_{G_2} : \mathcal{A}_0 \to \sqrt{-1}\Omega^6$ by

$$\mathcal{F}_{G_2}(\nabla) = \frac{1}{6}F_\nabla^3 + F_\nabla \wedge *\varphi.$$

Then the space of dDT connections is given by $\mathcal{F}_{G_2}^{-1}(0)$. Denote by $\mathcal{G}_U$ the group of unitary gauge transformations of (L, h) acting $\mathcal{A}_0$ canonically as in (2.1). Since $\mathcal{G}_U = C^\infty(X, S^1)$, the Lie algebra $\mathfrak{g}_U$ of $\mathcal{G}_U$ is identified with the space $\sqrt{-1}\Omega^0$ of $\sqrt{-1}\mathbb{R}$-valued functions on X. Note that $\mathcal{P}_{\mathfrak{g}} = \Lambda^2\mathfrak{g}$ since $\mathcal{G}_U$ is abelian. Define a 3-form $\Theta \in \Omega^3(\mathcal{A}_0)$ on $\mathcal{A}_0$ by

$$\Theta_\nabla(\alpha_1, \alpha_2, \alpha_3) = \sqrt{-1}\int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \left(\frac{1}{2}F_\nabla^2 + *\varphi\right),$$

where $\nabla \in \mathcal{A}_0$ and $\alpha_1, \alpha_2, \alpha_3 \in \sqrt{-1}\Omega^1 = T_\nabla\mathcal{A}_0$. We first show the following Lemma 3.2 as required in Definition 3.1. Then we show that there exists a multi-moment map ν for $(\mathcal{A}_0, \Theta, \mathcal{G}_U)$ and the dDT equation is regarded as the zero of ν, where we need the coclosed assumption on φ.

Lemma 3.2. — *The 3-form Θ is $\mathcal{G}_U$-invariant and closed.*

Proof. — Take any $\nabla \in \mathcal{A}_0$, $\lambda = f \cdot \mathrm{id}_L \in \mathcal{G}_U$, where $f \in C^\infty(X, S^1)$, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \sqrt{-1}\Omega^1 \cong T_\nabla\mathcal{A}_0$. Identify α_j with a vector field on $\mathcal{A}_0$ by

$$(\alpha_j)_{\widetilde{\nabla}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\widetilde{\nabla} + t\alpha_j \cdot \mathrm{id}_L\right)\Big|_{t=0}$$

for $\widetilde{\nabla} \in \mathcal{A}_0$. We first show the $\mathcal{G}_U$-invariance of Θ. That is,

$$\Theta_{\lambda^* \nabla}\big(\lambda_*(\alpha_1), \lambda_*(\alpha_2), \lambda_*(\alpha_3)\big) = \Theta_\nabla(\alpha_1, \alpha_2, \alpha_3). \tag{3.1}$$

By (2.1), we compute

$$\lambda_*(\alpha_j)_\nabla = \lambda_* \frac{\mathrm{d}}{\mathrm{d}t} (\nabla + t\alpha_j \cdot \mathrm{id}_L)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\nabla + \left(t\alpha_j + f^{-1}df\right) \cdot \mathrm{id}_L\right)\Big|_{t=0} = (\alpha_j)_{\lambda^*\nabla}.$$

Since $F_{\lambda^*\nabla} = F_\nabla$, we obtain (3.1).

Next, we show the closedness of Θ. Note that $[\alpha_i, \alpha_j] = 0$. Then it follows that

$$\begin{aligned} \mathrm{d}\Theta(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_1 (\Theta(\alpha_2, \alpha_3, \alpha_4)) - \alpha_2 (\Theta(\alpha_1, \alpha_3, \alpha_4)) \\ + \alpha_3 (\Theta(\alpha_1, \alpha_2, \alpha_4)) - \alpha_4 (\Theta(\alpha_1, \alpha_2, \alpha_3)). \end{aligned}$$

Since

$$\begin{aligned} \alpha_i (\Theta(\alpha_j, \alpha_k, \alpha_\ell))_\nabla &= \sqrt{-1} \frac{\mathrm{d}}{\mathrm{d}t} \int_X \alpha_j \wedge \alpha_k \wedge \alpha_\ell \wedge \left(\frac{1}{2} F^2_{\nabla + t\alpha_i \cdot \mathrm{id}_L} + *\varphi\right)\Bigg|_{t=0} \\ &= \sqrt{-1} \int_X \alpha_j \wedge \alpha_k \wedge \alpha_\ell \wedge \mathrm{d}\alpha_i \wedge F_\nabla, \end{aligned}$$

we have

$$(\mathrm{d}\Theta)_\nabla(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sqrt{-1} \int_X \mathrm{d}(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge F_\nabla) = 0,$$

which implies that $\mathrm{d}\Theta = 0$. □

We also need the following lemma.

Lemma 3.3. — *We have*

$$\Omega^1 = \left\{ \sum_{j=1}^N f_1^j \mathrm{d} f_2^j \;\middle|\; N \in \mathbb{N}, f_1^j, f_2^j \in \Omega^0 \right\}.$$

Proof. — Take any 1-form $\alpha \in \Omega^1$. We first show that for any $x \in X$, there exists an open neighborhood U_x of x and smooth functions $\{\widetilde{f}^1_{x,j}, \widetilde{f}^2_{x,j}\}_{j=1}^7$ on X such that

$$\alpha|_{U_x} = \sum_{j=1}^7 \widetilde{f}^1_{x,j} \; \mathrm{d} \; \widetilde{f}^2_{x,j}\Big|_{U_x}. \tag{3.2}$$

Indeed, take any local coordinates $(V, (x^1, \dots, x^7))$ of x and set

$$\alpha|_V = \sum_{j=1}^7 \alpha_j \mathrm{d}x^j.$$

We can take a cutoff function h such that h has compact support in V and $h = 1$ on an open neighborhood U_x of x. Then setting $\widetilde{f}^1_{x,j} = h\alpha_j$ and $\widetilde{f}^2_{x,j} = hx_j$, which are smooth functions on X, we obtain (3.2).

Since $\{U_x\}_{x\in X}$ is an open cover of X and X is compact, there exists $x_1, \dots, x_N \in X$ such that $\{U_{x_p}\}_{p=1}^N$ covers X. Denote by $\{h_p\}_{p=1}^N$ the partition of unity subordinate to $\{U_{x_p}\}_{p=1}^N$. Set

$$f^1_{p,j} = h_p \widetilde{f}^1_{x_p,j} \qquad f^2_{p,j} = \widetilde{f}^2_{x_p,j}.$$

Then we have $\alpha = \sum_{p=1}^N \sum_{j=1}^7 f^1_{p,j} \mathrm{d} f^2_{p,j}$. Indeed, take any $x \in X$. We may assume that $x \in U_{x_1} \cap \ldots \cap U_{x_k}$ and $x \notin U_p$ for $p = k+1, \dots, N$. Then $\sum_{j=1}^7 (\widetilde{f}^1_{x_p,j}\ \mathrm{d}\widetilde{f}^2_{x_p,j})_x = \alpha_x$ for $p = 1, \dots, k$ by (3.2) and $h_p(x) = 0$ for $p = k+1, \dots, N$. Hence

$$\begin{aligned}\sum_{p=1}^N \sum_{j=1}^7 \left(f^1_{p,j} \mathrm{d} f^2_{p,j}\right)_x &= \sum_{p=1}^k h_p(x) \sum_{j=1}^7 \left(\widetilde{f}^1_{x_p,j} \mathrm{d}\widetilde{f}^2_{x_p,j}\right)_x \\ &= \sum_{p=1}^k h_p(x)\alpha_x \\ &= \sum_{p=1}^N h_p(x)\alpha_x = \alpha_x. \end{aligned}$$ □

Denote by Z^6 the space of closed 6-forms on X. Define a map $\iota_{Z^6} : Z^6 \to \Lambda^2 \mathfrak{g}_U^*$ by

$$\iota_{Z^6}(\xi)(f_1, f_2) = \int_X \xi \wedge \frac{1}{2}(f_1 \mathrm{d} f_2 - f_2 \mathrm{d} f_1) = \int_X \xi \wedge f_1 \mathrm{d} f_2$$

for $\xi \in Z^6$ and $f_1, f_2 \in \sqrt{-1}\Omega^0 = \mathfrak{g}_U$.

Theorem 3.4. — *Define a $\mathcal{G}_U$-invariant map $\nu : \mathcal{A}_0 \to \Lambda^2 \mathfrak{g}_U^*$ by*

$$\nu(\nabla) = \iota_{Z^6}\left(\sqrt{-1}\mathcal{F}_{G_2}(\nabla)\right).$$

Then we have $d\langle \nu, p\rangle = i(p^*)\Theta$ for any $p \in \Lambda^2 \mathfrak{g}_U$.

Since we assume that $\mathrm{d} * \varphi = 0$, we see that $\sqrt{-1}\mathcal{F}_{G_2}(\nabla) \in Z^6$ for any $\nabla \in \mathcal{A}_0$. By Lemma 3.3, ι_{Z^6} is injective. Hence we have $\nu^{-1}(0) = \mathcal{F}_{G_2}^{-1}(0)$. In this sense, we can regard the dDT equation as the zero of a multi-moment map.

Proof. — First note that the vector field f^* generated by $f \in \sqrt{-1}\Omega^0 = \mathfrak{g}_U$ is given by

$$f^*_\nabla = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tf}\right)^* \nabla\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla + e^{-tf}\mathrm{d}e^{tf} \cdot \mathrm{id}_L\right)\bigg|_{t=0} = \mathrm{d}f$$

at $\nabla \in \mathcal{A}_0$. Hence for any $f_1, f_2 \in \sqrt{-1}\Omega^0 = \mathfrak{g}_U$ and $\alpha \in \sqrt{-1}\Omega^1 = T_\nabla \mathcal{A}_0$, we have

$$
\begin{aligned}
&\Theta_\nabla\big((f_1^*)_\nabla, (f_2^*)_\nabla, \alpha\big) \\
&= \sqrt{-1}\int_X \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \alpha \wedge \left(\frac{1}{2}F_\nabla^2 + *\varphi\right) \\
&= \sqrt{-1}\int_X f_1 \mathrm{d}f_2 \wedge \mathrm{d}\alpha \wedge \left(\frac{1}{2}F_\nabla^2 + *\varphi\right) = \sqrt{-1}\int_X f_1 \mathrm{d}f_2 \wedge (\mathrm{d}\mathcal{F}_{G_2})_\nabla(\alpha),
\end{aligned}
$$

where $(\mathrm{d}\mathcal{F}_{G_2})_\nabla : \sqrt{-1}\Omega^1 \to \sqrt{-1}\Omega^6$ is the linearization of $\mathcal{F}_{G_2}$ at $\nabla \in \mathcal{A}_0$. Hence we obtain

$$
\begin{aligned}
\Theta_\nabla\left((f_1^*)_\nabla, (f_2^*)_\nabla, \alpha\right) &= \iota_{Z^6}\left(\sqrt{-1}(\mathrm{d}\mathcal{F}_{G_2})_\nabla(\alpha)\right)(f_1, f_2) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\iota_{Z^6}\left(\sqrt{-1}\mathcal{F}_{G_2}(\nabla + t\alpha \cdot \mathrm{id}_L)\right)\Big|_{t=0}(f_1, f_2) \\
&= \left(\mathrm{d}\langle \nu, f_1 \wedge f_2\rangle\right)_\nabla(\alpha). \qquad \square
\end{aligned}
$$

In the dHYM case, the "$\mathcal{J}$ functional" defined in [2, Remark 2.15] or [1, Lemma 2.6 (ii)] is convex along geodesics and the critical points are solutions of the dHYM equation. Hence it plays an important role in the existence problem.

In the dDT case, there is a functional whose critical points are dDT connections. See Section 4.2. However, no metric has yet been found that makes the functional convex along geodesics. Since no such results have been found for associative submanifolds, it might be difficult to relate the functional to the existence problem.

However, as we see in the next section, we have an observation as in the case of instanton Floer homology for 3-manifolds by using the functional in Section 4.2. We might develop the theory like instanton Floer homology using dDT connections.

4. Gradient flow of the Karigiannis–Leung functional

It is known that the gradient flow equation of the Chern–Simons functional on an oriented 3-manifold X^3 agrees with the ASD equation on $\mathbb{R} \times X^3$. See for example [3, Section 2.5.3]. This is an important observation in instanton Floer homology for 3-manifolds. We show that there is an analogous relation between dDT equations for G_2- and Spin(7)-manifolds.

Let X^7 be a 7-manifold with a G_2-structure φ and $(L, h) \to X^7$ be a smooth complex Hermitian line bundle over X^7. Let $\{\nabla_t\}_{t\in\mathbb{R}}$ be a family of

Hermitian connections of $(L,h) \to X^7$. We identify this with a connection $\widetilde{\nabla}$ of $\pi^*L \to \mathbb{R} \times X^7$, where $\pi : \mathbb{R} \times X^7 \to X^7$ is the projection. If we set

$$\nabla_t = \nabla_0 + \sqrt{-1}a_t \cdot \mathrm{id}_L,$$

where $a_t \in \Omega^1(X^7)$, we have $\widetilde{\nabla} = \pi^*\nabla_0 + \sqrt{-1}\pi^* a_t \cdot \mathrm{id}_{\pi^*L}$ and the curvature $F_{\widetilde{\nabla}}$ of $\widetilde{\nabla}$ is given by

$$F_{\widetilde{\nabla}} = \sqrt{-1}\mathrm{d}t \wedge \frac{\partial \pi^* a_t}{\partial t} + \pi^* F_{\nabla_t}.$$

4.1. The Spin(7)-dDT condition on $\mathbb{R} \times X^7$

The product $\mathbb{R} \times X^7$ admits a canonical Spin(7)-structure. We write down the condition that $F_{\widetilde{\nabla}}$ is a Spin(7)-dDT connection, a dDT connection for a manifold with a Spin(7)-structure. For simplicity, set

$$\dot{a}_t := \frac{\partial \pi^* a_t}{\partial t}, \qquad E_t := -\sqrt{-1}\pi^* F_{\nabla_t}.$$

Lemma 4.1. — *The connection $\widetilde{\nabla}$ is a* Spin(7)*-dDT connection if and only if*

$$-*\varphi \wedge E_t + \frac{1}{6}E_t^3 - \left(1 - \frac{1}{2} * \left(\varphi \wedge E_t^2\right)\right) * \dot{a}_t + *(\dot{a}_t \wedge E_t \wedge \varphi) \wedge *E_t = 0 \tag{4.1}$$

$$\frac{1}{2}\varphi \wedge *E_t^2 - \dot{a}_t \wedge E_t \wedge \varphi = 0. \tag{4.2}$$

Proof. — Denote by $*_8$ and $* = *_7$ the Hodge star operators on $\mathbb{R} \times X^7$ and X^7, respectively. Then, $\widetilde{\nabla}$ is a Spin(7)-dDT connection (in the sense of [8, Definition 1.3]) if and only if

$$\begin{aligned} \left\langle F_{\widetilde{\nabla}} + \frac{1}{6} *_8 F_{\widetilde{\nabla}}^3,\ \mathrm{d}t \wedge b + i(b^\sharp)\varphi \right\rangle &= 0, \\ \left\langle F_{\widetilde{\nabla}}^2, \mathrm{d}t \wedge i(b^\sharp) * \varphi - b \wedge \varphi \right\rangle &= 0 \end{aligned} \tag{4.3}$$

for any $b \in \Omega^1(X^7)$ by [8, Lemma 3.4]. Since

$$\begin{aligned} \frac{1}{6} *_8 F_{\widetilde{\nabla}}^3 &= -\frac{\sqrt{-1}}{6} *_8 \left(3\mathrm{d}t \wedge \dot{a}_t \wedge E_t^2 + E_t^3\right) \\ &= \sqrt{-1}\left(-\frac{1}{2} * \left(\dot{a}_t \wedge E_t^2\right) - \frac{1}{6}\mathrm{d}t \wedge *E_t^3\right), \end{aligned}$$

(4.3) is equivalent to

$$\left\langle \dot{a}_t - \frac{1}{6} * E_t^3, b \right\rangle + \left\langle E_t - \frac{1}{2} * \left(\dot{a}_t \wedge E_t^2 \right), i(b^\sharp)\varphi \right\rangle = 0, \tag{4.4}$$

$$\left\langle 2\dot{a}_t \wedge E_t, i(b^\sharp) * \varphi \right\rangle - \left\langle E_t^2, b \wedge \varphi \right\rangle = 0. \tag{4.5}$$

We compute

$$\left\langle E_t, i(b^\sharp)\varphi \right\rangle = *(E_t \wedge *(i(b^\sharp)\varphi)) = *(E_t \wedge b \wedge *\varphi) = \langle *\varphi \wedge E_t, *b \rangle$$

and

$$\begin{aligned} &\left\langle -\frac{1}{2} * \left(\dot{a}_t \wedge E_t^2 \right), i(b^\sharp)\varphi \right\rangle \\ &\qquad = -\frac{1}{2} * \left(\dot{a}_t \wedge E_t^2 \wedge i(b^\sharp)\varphi \right) \\ &\qquad = -\frac{1}{2} * \left(i(b^\sharp) \left(\dot{a}_t \wedge E_t^2 \right) \wedge \varphi \right) \\ &\qquad = -\frac{1}{2} * \left(E_t^2 \wedge \varphi \right) \cdot \langle \dot{a}_t, b \rangle + * \left(\dot{a}_t \wedge E_t \wedge (i(b^\sharp)E_t) \wedge \varphi \right). \end{aligned}$$

Since $i(b^\sharp)E_t = - * (b \wedge *E_t)$, we have

$$\begin{aligned} * \left(\dot{a}_t \wedge E_t \wedge (i(b^\sharp)E_t) \wedge \varphi \right) &= \langle \dot{a}_t \wedge E_t \wedge \varphi, b \wedge *E_t \rangle \\ &= -\langle *(\dot{a}_t \wedge E_t \wedge \varphi) \wedge *E_t, *b \rangle. \end{aligned}$$

Then, we see that (4.4) is equivalent to (4.1). Similarly, since

$$\begin{aligned} \left\langle 2\dot{a}_t \wedge E_t, i(b^\sharp) * \varphi \right\rangle &= -2 * (\dot{a}_t \wedge E_t \wedge b \wedge \varphi) = 2\langle \dot{a}_t \wedge E_t \wedge \varphi, *b \rangle, \\ -\left\langle E_t^2, b \wedge \varphi \right\rangle &= - * \left(b \wedge \varphi \wedge *E_t^2 \right) \qquad = -\left\langle \varphi \wedge *E_t^2, *b \right\rangle, \end{aligned}$$

we see that (4.5) is equivalent to (4.2). □

Hence, eliminating $*(\dot{a}_t \wedge E_t \wedge \varphi)$ from (4.1) by (4.2), we obtain

$$\begin{aligned} - *\varphi \wedge E_t + \frac{1}{6} E_t^3 + \frac{1}{2} * \left(\varphi \wedge *E_t^2 \right) \wedge *E_t \\ = \left(1 - \frac{1}{2} * \left(\varphi \wedge E_t^2 \right) \right) * \dot{a}_t. \end{aligned} \tag{4.6}$$

Remark 4.2. — If $1 - *(\varphi \wedge E_t^2)/2 \neq 0$, (4.1) and (4.2) are equivalent to (4.6) by Proposition A.3.

4.2. The Karigiannis–Leung functional

Karigiannis and Leung [6] introduced the functional whose critical points are dDT connections. We first review it.

Let X^7 be a compact 7-manifold with a coclosed G_2-structure φ ($d{*}\varphi=0$) and let $(L,h)\to X^7$ be a smooth complex Hermitian line bundle. Denote by $\mathcal{A}_0$ the space of Hermitian connections of (L,h). Define a 1-form Θ on $\mathcal{A}_0$ by

$$\Theta_\nabla\left(\sqrt{-1}b\right)=\int_X \sqrt{-1}b\wedge\left(\frac{1}{6}F_\nabla^3+F_\nabla\wedge*\varphi\right)$$

for $\nabla\in\mathcal{A}_0$ and $\sqrt{-1}b\in\sqrt{-1}\Omega^1=T_\nabla\mathcal{A}_0$. Then we see that $\Theta_\nabla=0$ if and only if ∇ is a dDT connection. We can show that Θ is closed as in the proof of Lemma 3.2. Since $\mathcal{A}_0$ is contractible, there exists $\mathcal{F}:\mathcal{A}_0\to\mathbb{R}$ such that $d\mathcal{F}=\Theta$. Hence we see that dDT connections are critical points of $\mathcal{F}$.

Now, we study the relation between Spin(7)-dDT connections on $\mathbb{R}\times X^7$ and the Karigiannis–Leung functional $\mathcal{F}$. Set

$$\mathcal{A}_{ac}:=\left\{\nabla\in\mathcal{A}_0\;\middle|\;1+\frac{1}{2}*\left(\varphi\wedge F_\nabla^2\right)>0\right\}.$$

This type of the subset is also considered in the dHYM case. For example, see the survey article [1, Definition 2.1]. By the mirror of the associator equality in [10, Theorem 5.1], it will be natural to call a Hermitian connection ∇ satisfying $1+*(\varphi\wedge F_\nabla^2)/2>0$ *almost calibrated* as in the dHYM case.

Define a metric $\mathcal{G}$ on $\mathcal{A}_{\mathrm{ac}}$ by

$$\mathcal{G}_\nabla\left(\sqrt{-1}a,\sqrt{-1}b\right)=\int_X\langle a,b\rangle_\nabla\left(1+\frac{1}{2}*\left(\varphi\wedge F_\nabla^2\right)\right)\mathrm{vol}$$

where $\nabla\in\mathcal{A}_{\mathrm{ac}}, \sqrt{-1}a,\sqrt{-1}b\in\sqrt{-1}\Omega^1=T_\nabla\mathcal{A}_{\mathrm{ac}}$, vol is the induced volume form from φ, and $\langle\cdot,\cdot\rangle_\nabla$ is the induced metric on the space of differential forms from $(\mathrm{id}_{TX}+(-\sqrt{-1}F_\nabla)^\sharp)^*\varphi$. Here, $(-\sqrt{-1}F_\nabla)^\sharp$ is an endomorphism of TX defined by $g((-\sqrt{-1}F_\nabla)^\sharp(u),v)=-\sqrt{-1}F_\nabla(u,v)$ for $u,v\in TX$, where g is the induced metric (on TX) from φ. Note that $(-\sqrt{-1}F_\nabla)^\sharp$ is skew-symmetric with respect to g. Explicitly, if we denote by g_∇ the induced metric (on TX) from $(\mathrm{id}_{TX}+(-\sqrt{-1}F_\nabla)^\sharp)^*\varphi$, we have $g_\nabla=(\mathrm{id}_{TX}+(-\sqrt{-1}F_\nabla)^\sharp)^*g$ and $\langle\cdot,\cdot\rangle_\nabla$ is the induced metric from g_∇.

The following is the main theorem of this paper.

THEOREM 4.3. — *The gradient flow equation of $\mathcal{F}$ with respect to $\mathcal{G}$ on $\mathcal{A}_{ac}$ agrees with the* Spin(7)*-dDT equation on $\mathbb{R}\times X^7$.*

Proof. — We first deduce the gradient flow equation and compare it with the computation in Section 4.1. Take any $\nabla\in\mathcal{A}_{ac}$ and $b\in\Omega^1$. Set

$$E_\nabla=-\sqrt{-1}F_\nabla\in\Omega^2.$$

Denote by $\langle \cdot, \cdot \rangle$ the induced metric on the space of differential forms from φ. Then we compute

$$(4.7)\quad \begin{aligned} (d\mathcal{F})_\nabla \left(\sqrt{-1}b\right) &= \int_X \sqrt{-1}b \wedge \left(\frac{1}{6}F_\nabla^3 + F_\nabla \wedge *\varphi\right) \\ &= \int_X \left\langle b, * \left(\frac{1}{6}E_\nabla^3 - E_\nabla \wedge *\varphi\right)\right\rangle \mathrm{vol}\,. \end{aligned}$$

By Proposition A.1, we have

$$* \left(\frac{1}{6}E_\nabla^3 - E_\nabla \wedge *\varphi\right) = \left(\left(\mathrm{id}_{TX} - \left(E_\nabla^\sharp\right)^2\right)^{-1}\right)^* \eta_\nabla,$$

where

$$\eta_\nabla = * \left(- * \varphi \wedge E_\nabla + \frac{1}{6}E_\nabla^3 + \frac{1}{2} * \left(\varphi \wedge *E_\nabla^2\right) \wedge *E_\nabla\right) \in \Omega^1.$$

Since

$$\begin{aligned} \mathrm{id}_{TX} - \left(E_\nabla^\sharp\right)^2 &= \left(\mathrm{id}_{TX} - E_\nabla^\sharp\right)\left(\mathrm{id}_{TX} + E_\nabla^\sharp\right) \\ &= {}^t\left(\mathrm{id}_{TX} + E_\nabla^\sharp\right)\left(\mathrm{id}_{TX} + E_\nabla^\sharp\right), \end{aligned}$$

where ${}^t(\mathrm{id}_{TX} + E_\nabla^\sharp)$ is the transpose of $\mathrm{id}_{TX} + E_\nabla^\sharp$ with respect to g, we have

$$(4.8)\quad \begin{aligned} &\left\langle b, * \left(\frac{1}{6}E_\nabla^3 - E_\nabla \wedge *\varphi\right)\right\rangle \\ &= \left\langle b, \left(\left(\mathrm{id}_{TX} - \left(E_\nabla^\sharp\right)^2\right)^{-1}\right)^* \eta_\nabla \right\rangle \\ &= \left\langle \left(\left(\mathrm{id}_{TX} + E_\nabla^\sharp\right)^{-1}\right)^* b, \left(\left(\mathrm{id}_{TX} + E_\nabla^\sharp\right)^{-1}\right)^* \eta_\nabla \right\rangle = \langle b, \eta_\nabla \rangle_\nabla\,. \end{aligned}$$

Then by (4.7) and (4.8), the gradient vector field of $\mathcal{F}$ with respect to $\mathcal{G}$ is given by

$$\mathcal{A}_{ac} \ni \nabla \longmapsto \frac{\sqrt{-1}\eta_\nabla}{1 - \frac{1}{2} * \left(\varphi \wedge E_\nabla^2\right)} \in \sqrt{-1}\Omega^1.$$

Thus a family $\{\nabla_t\}_{t \in \mathbb{R}} \subset \mathcal{A}_{ac}$ satisfies the gradient flow of $\mathcal{F}$ with respect to $\mathcal{G}$ if and only if

$$(4.9)\quad \begin{aligned} \dot{a}_t &= \frac{\eta_{\nabla_t}}{1 - \frac{1}{2} * \left(\varphi \wedge E_{\nabla_t}^2\right)} \\ &= \frac{* \left(- * \varphi \wedge E_t + \frac{1}{6}E_t^3 + \frac{1}{2} * \left(\varphi \wedge *E_t^2\right) \wedge *E_t\right)}{1 - \frac{1}{2} * \left(\varphi \wedge E_t^2\right)}, \end{aligned}$$

where $\nabla_t = \nabla_0 + \sqrt{-1}a_t \cdot \mathrm{id}_L$, $a_t \in \Omega^1$, $\dot{a}_t = \partial a_t / \partial t$ and $E_t = E_{\nabla_t} = -\sqrt{-1}F_{\nabla_t}$. Then we see that (4.9) is equivalent to (4.6). By Remark 4.2, this is equivalent to the Spin(7)-dDT equation on $\mathbb{R} \times X^7$. □

By Theorem 4.3, we will have to consider the deformation theory of the Spin(7)-dDT connections on $\mathbb{R}\times X^7$ next for the analogue of instanton Floer homology for 3-manifolds. Deformations of Spin(7)-dDT connections on a compact manifold with a Spin(7)-structure are studied in [7, Theorem 1.2], but there are some technical assumptions. We will have to deal with more technical issues, including these, to develop the deformation theory on a cylinder.

Appendix A. Algebraic Computations

In this appendix, we give some algebraic computations needed in the proof of Theorem 4.3.

Set $V = \mathbb{R}^7$ and let g be the standard inner product on V. Denote by $*$ the standard Hodge star operator on V. For a 2-form $F \in \Lambda^2 V^*$, define $F^\sharp \in \mathrm{End}(V)$ by

$$g(F^\sharp(u), v) = F(u, v)$$

for $u, v \in V$. Then, $F^\sharp$ is skew-symmetric, and hence, $\det(I + F^\sharp) > 0$, where I is the identity matrix. We also have

$$\begin{aligned} \det\bigl(I - (F^\sharp)^2\bigr) &= \det\left(I + F^\sharp\right)\det\left(I - F^\sharp\right) \\ &= \det\left(I + F^\sharp\right)\det\left(I + {}^tF^\sharp\right) \\ &= \left(\det\left(I + F^\sharp\right)\right)^2 > 0, \end{aligned}$$

where ${}^tF^\sharp$ is the transpose of $F^\sharp$ with respect to g. Define a 3-form $\varphi \in \Lambda^3 V^*$ by

$$\varphi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

where $\{e_i\}_{i=1}^7$ is a standard oriented basis of V with the dual basis $\{e^i\}_{i=1}^7$ of V^* and $e^{i_1\cdots i_k}$ is short for $e^{i_1} \wedge \cdots \wedge e^{i_k}$. The stabilizer of φ is known to be the exceptional 14-dimensional simple Lie group G_2. The elements of G_2 preserve the standard inner product g and volume form vol. The group G_2 acts canonically on $\Lambda^k V^*$, and $\Lambda^2 V^*$ is decomposed as $\Lambda^2 V^* = \Lambda^2_7 V^* \oplus \Lambda^2_{14} V^*$, where $\Lambda^2_\ell V^*$ is a ℓ-dimensional irreducible subrepresentation of G_2 in $\Lambda^2 V^*$. For more details, see for example [10, Section 2.2]. Set

$$F = F_7 + F_{14} = i(u)\varphi + F_{14} \in \Lambda^2_7 V^* \oplus \Lambda^2_{14} V^*$$

for $u \in V$.

PROPOSITION A.1. — *For a 2-form $F \in \Lambda^2 V^*$, set $\xi = -*\varphi \wedge F + F^3/6 \in \Lambda^6 V^*$. Then we have*

$$\left(I - (F^\sharp)^2\right)^* * \xi = * \left(\xi + \frac{1}{2} * \left(\varphi \wedge *F^2\right) \wedge *F \right).$$

Proof. — Since $(I - (F^\sharp)^2)^* * \xi = *\xi - ((F^\sharp)^2)^* * \xi$, we only have to compute $((F^\sharp)^2)^* * \xi$. Set

$$F_{ij} = F(e_i, e_j).$$

We have $F^\sharp = \sum_{i,j} F_{ij} e^i \otimes e_j$, which implies that $(F^\sharp)^2 = \sum_{i,j,k} F_{ij} F_{jk} e^i \otimes e_k$. Then we compute

$$\left((F^\sharp)^2\right)^* * \xi = \sum_{i,j,k} F_{ij} F_{jk} * \xi(e_k) \cdot e^i = -\sum_j \langle i(e_j)F, *\xi \rangle \cdot i(e_j)F.$$

Since

$$\begin{aligned} \langle i(e_j)F, *\xi \rangle &= * \left(*\xi \wedge *(i(e_j)F)\right) = - * \left(*\xi \wedge e^j \wedge *F\right) \\ &= \left\langle e^j, *(*\xi \wedge *F) \right\rangle, i(e_j)F = - * \left(e^j \wedge *F\right), \end{aligned}$$

we have

$$\begin{aligned} \left((F^\sharp)^2\right)^* * \xi &= * \left(* \left(*\xi \wedge *F\right) \wedge *F\right) \\ &= * \left(* \left(* \left(- * \varphi \wedge F + \frac{F^3}{6} \right) \wedge *F \right) \wedge *F \right). \end{aligned} \tag{A.1}$$

LEMMA A.2. — *We have*

$$(*F^3) \wedge *F = 0, \qquad * \left(\varphi \wedge *F^2\right) = -6 i(u) F.$$

Proof. — We can prove the first equation as in [9, Lemma C.2]. For any $v \in V$, set

$$v^\flat = g(v, \cdot) \in V^*.$$

We compute

$$v^\flat \wedge (*F^3) \wedge *F = (*F^3) \wedge *(i(v)F) = F^3 \wedge i(v)F = i(v)(F^4/4) = 0,$$

which implies the first equation. Similarly, for any $v \in V$, we have

$$v^\flat \wedge \varphi \wedge *F^2 = *(v^\flat \wedge \varphi) \wedge F^2 = -i(v) * \varphi \wedge F^2 = *\varphi \wedge i(v) F^2 = 2 i(v) F \wedge F \wedge *\varphi.$$

Since $F \wedge *\varphi = i(u)\varphi \wedge *\varphi = 3 * u^\flat$ by for example [9, Lemma B.1], we obtain

$$v^\flat \wedge \varphi \wedge *F^2 = 6 \left\langle u^\flat, i(v)F \right\rangle \mathrm{vol} = 6 \left\langle v^\flat \wedge u^\flat, F \right\rangle \mathrm{vol} = -6 \left\langle v^\flat, i(u)F \right\rangle \mathrm{vol},$$

which implies the second equation. □

Then by (A.1), Lemma A.2 and the equation $F\wedge *\varphi = i(u)\varphi\wedge *\varphi = 3*u^\flat$, we obtain

$$\begin{aligned}\left((F^\sharp)^2\right)^* * \xi &= *\big(*\left(*\left(- * \varphi \wedge F\right) \wedge *F\right) \wedge *F\big)\\ &= *\big(*\big(-3u^\flat \wedge *F\big) \wedge *F\big)\\ &= 3*\left((i(u)F) \wedge *F\right)\\ &= -\frac{1}{2} * \big(*\big(\varphi \wedge *F^2\big) \wedge *F\big)\end{aligned}$$

and the proof is completed. □

Proposition A.3. — *For a 1-form $a \in V^*$ and a 2-form $F \in \Lambda^2 V^*$ such that $1 - *(\varphi \wedge F^2)/2 \neq 0$,*

$$\text{(A.2)}\quad \begin{aligned} &- *\varphi \wedge F + \frac{1}{6}F^3\\ &\qquad - \left(1 - \frac{1}{2} * \big(\varphi \wedge F^2\big)\right) * a + *(a \wedge F \wedge \varphi) \wedge *F = 0,\end{aligned}$$

$$\text{(A.3)}\qquad \frac{1}{2}\varphi \wedge *F^2 - a \wedge F \wedge \varphi = 0$$

if and only if

$$\text{(A.4)}\quad - * \varphi \wedge F + \frac{1}{6}F^3 + \frac{1}{2} * \big(\varphi \wedge *F^2\big) \wedge *F = \left(1 - \frac{1}{2} * \big(\varphi \wedge F^2\big)\right) * a.$$

Proof. — Eliminating $a \wedge F \wedge \varphi$ from (A.2) by (A.3), we obtain (A.4). Conversely, (A.4) implies (A.3) by the following Lemma A.4. By (A.4), the left hand side of (A.2) is computed as

$$\begin{aligned}&-\frac{1}{2} * \big(\varphi \wedge *F^2\big) \wedge *F + *(a \wedge F \wedge \varphi) \wedge *F\\ &\qquad\qquad = * \left(-\frac{1}{2}\varphi \wedge *F^2 + a \wedge F \wedge \varphi\right) \wedge *F,\end{aligned}$$

which vanishes by (A.3). □

Lemma A.4. — *For any 2-form $F \in \Lambda^2 V^*$, we have*

$$\begin{aligned}&* \left(- * \varphi \wedge F + \frac{1}{6}F^3 + \frac{1}{2} * \big(\varphi \wedge *F^2\big) \wedge *F\right) \wedge F \wedge \varphi\\ &\qquad\qquad = \frac{1}{2}\left(1 - \frac{1}{2} * \big(\varphi \wedge F^2\big)\right) \varphi \wedge *F^2.\end{aligned}$$

Proof. — Fix any $v \in V$ and set

$$J_1 = v^\flat \wedge * \left(- * \varphi \wedge F + \frac{1}{6} F^3 \right) \wedge F \wedge \varphi,$$

$$J_2 = v^\flat \wedge * \left(\frac{1}{2} * \left(\varphi \wedge * F^2 \right) \wedge * F \right) \wedge F \wedge \varphi.$$

We compute J_1 and J_2. We have

$$\begin{aligned} J_1 &= * \left(i(v) \left(* \varphi \wedge F - \frac{1}{6} F^3 \right) \right) \wedge * (2F_7 - F_{14}) \\ &= i(v) \left(* \varphi \wedge F - \frac{1}{6} F^3 \right) \wedge (2F_7 - F_{14}) \\ &= \left(-3 * (v^\flat \wedge u^\flat) - \frac{1}{2} i(v) F \wedge F^2 \right) \wedge (2F_7 - F_{14}), \end{aligned}$$

where we use $* \varphi \wedge F = 3 * u^\flat$. We also have

$$\begin{aligned} -3 * \left(v^\flat \wedge u^\flat \right) \wedge (2F_7 - F_{14}) &= -3 \left\langle v^\flat \wedge u^\flat, 2F_7 - F_{14} \right\rangle \mathrm{vol} \\ &= -3 \left\langle v^\flat, i(u) F \right\rangle \mathrm{vol} \end{aligned}$$

as $i(u) F_7 = i(u) i(u) \varphi = 0$, and

$$\begin{aligned} &\left(-\frac{1}{2} i(v) F \wedge F^2 \right) \wedge (2F_7 - F_{14}) \\ &= -\frac{1}{2} i(v) F \wedge \left(F_7^2 + 2F_7 \wedge F_{14} + F_{14}^2 \right) \wedge (2F_7 - F_{14}) \\ &= -\frac{1}{2} \left(i(v) F_7 + i(v) F_{14} \right) \wedge \left(2F_7^3 + 3F_7^2 \wedge F_{14} - F_{14}^3 \right) \\ &= -\frac{1}{2} \left\{ i(v) F_7 \wedge \left(3F_7^2 \wedge F_{14} - F_{14}^3 \right) + i(v) F_{14} \wedge \left(2F_7^3 + 3F_7^2 \wedge F_{14} \right) \right\}, \end{aligned}$$

where we use

$$i(v) F_7 \wedge F_7^3 = i(v) \left(F_7^4 / 4 \right) = 0 \quad \text{and} \quad i(v) F_{14} \wedge F_{14}^3 = i(v) \left(F_{14}^4 / 4 \right) = 0.$$

By [9, (B.7)], we have

$$F_7^3 = 6 |u|^2 * u^\flat. \tag{A.5}$$

Then

$$\begin{aligned} 3 i(v) F_7 \wedge F_7^2 \wedge F_{14} = i(v) F_7^3 \wedge F_{14} &= -6 |u|^2 * \left(v^\flat \wedge u^\flat \right) \wedge F_{14} \\ &= 6 |u|^2 \left\langle v^\flat, i(u) F \right\rangle \mathrm{vol}, \\ 2 i(v) F_{14} \wedge F_7^3 = 12 |u|^2 i(v) F_{14} \wedge * u^\flat &= 12 |u|^2 \left\langle F_{14}, v^\flat \wedge u^\flat \right\rangle \mathrm{vol} \\ &= -12 |u|^2 \left\langle v^\flat, i(u) F \right\rangle \mathrm{vol}. \end{aligned}$$

Hence we obtain

$$J_1 = \left(-3+3|u|^2\right)\left\langle v^\flat, i(u)F\right\rangle \mathrm{vol} + \frac{1}{2}\left(i(v)F_7\wedge F_{14}^3 - 3i(v)F_{14}\wedge F_7^2\wedge F_{14}\right). \tag{A.6}$$

Next, we compute J_2. By Lemma A.2, we have

$$\begin{aligned} J_2 &= v^\flat \wedge *\left(-3i(u)F\wedge *F\right)\wedge *(2F_7 - F_{14}) \\ &= 3*\left(i(u)F\wedge *F\right)\wedge *\big(i(v)(-2F_7+F_{14})\big) \\ &= 3i(u)F\wedge *F\wedge i(v)(-2F_7+F_{14}). \end{aligned}$$

Since

$$\begin{aligned} i(u)F\wedge *F &= i(u)F_{14}\wedge\left(\frac{1}{2}F_7\wedge\varphi - F_{14}\wedge\varphi\right) \\ &= \frac{1}{2}\left(i(u)(F_{14}\wedge F_7\wedge\varphi) - F_{14}\wedge F_7\wedge i(u)\varphi\right) - \frac{1}{2}i(u)F_{14}^2\wedge\varphi \\ &= -\frac{1}{2}F_7^2\wedge F_{14} - \frac{1}{2}\left(i(u)\left(F_{14}^2\wedge\varphi\right) - F_{14}^2\wedge i(u)\varphi\right) \\ &= \frac{1}{2}\left(|F_{14}|^2 * u^\flat - F_7^2\wedge F_{14} + F_7\wedge F_{14}^2\right), \end{aligned}$$

we have

$$J_2 = \frac{3}{2}\left(-F_7^2\wedge F_{14} + F_7\wedge F_{14}^2\right)\wedge i(v)\left(-2F_7+F_{14}\right) + \frac{3}{2}|F_{14}|^2 * u^\flat\wedge i(v)(-2F_7+F_{14}).$$

We compute

$$\begin{aligned} &\left(-F_7^2\wedge F_{14} + F_7\wedge F_{14}^2\right)\wedge i(v)(-2F_7+F_{14}) \\ &\qquad = 2i(v)F_7\wedge F_7^2\wedge F_{14} - 2i(v)F_7\wedge F_7\wedge F_{14}^2 \\ &\qquad\qquad - i(v)F_{14}\wedge F_7^2\wedge F_{14} + i(v)F_{14}\wedge F_7\wedge F_{14}^2. \end{aligned}$$

By (A.5), it follows that

$$2i(v)F_7\wedge F_7^2\wedge F_{14} = \frac{2}{3}i(v)F_7^3\wedge F_{14} = 4|u|^2\left\langle v^\flat, i(u)F\right\rangle \mathrm{vol}.$$

Since $-2i(v)F_7\wedge F_7\wedge F_{14}^2 = -i(v)F_7^2\wedge F_{14}^2 = F_7^2\wedge i(v)F_{14}^2 = 2i(v)F_{14}\wedge F_7^2\wedge F_{14}$, we have

$$-2i(v)F_7\wedge F_7\wedge F_{14}^2 - i(v)F_{14}\wedge F_7^2\wedge F_{14} = i(v)F_{14}\wedge F_7^2\wedge F_{14}.$$

We also have

$$i(v)F_{14}\wedge F_7\wedge F_{14}^2 = \frac{1}{3}i(v)F_{14}^3\wedge F_7 = -\frac{1}{3}F_{14}^3\wedge i(v)F_7$$

and

$$\begin{aligned}\frac{3}{2}|F_{14}|^2 * u^\flat \wedge i(v)(-2F_7 + F_{14}) &= \frac{3}{2}|F_{14}|^2\left\langle -2F_7 + F_{14}, v^\flat \wedge u^\flat \right\rangle \mathrm{vol} \\ &= -\frac{3}{2}|F_{14}|^2\left\langle v^\flat, i(u)F \right\rangle \mathrm{vol}\,.\end{aligned}$$

Hence we obtain

$$\begin{aligned}J_2 = &\left(6|u|^2 - \frac{3}{2}|F_{14}|^2\right)\left\langle v^\flat, i(u)F\right\rangle \mathrm{vol} \\ &+ \frac{3}{2} i(v)F_{14} \wedge F_7^2 \wedge F_{14} - \frac{1}{2} i(v) F_7 \wedge F_{14}^3.\end{aligned} \tag{A.7}$$

Then by (A.6) and (A.7), we obtain

$$\begin{aligned}J_1 + J_2 &= 3\left(-1 + 3|u|^2 - \frac{1}{2}|F_{14}|^2\right)\left\langle v^\flat, i(u)F\right\rangle \mathrm{vol} \\ &= 3\left(-1 + \frac{1}{2} * \left(\varphi \wedge F^2\right)\right)\left\langle v^\flat, i(u)F\right\rangle \mathrm{vol},\end{aligned}$$

where we use $*(\varphi \wedge F^2) = *\left(F \wedge *(2F_7 - F_{14})\right) = 2|F_7|^2 - |F_{14}|^2 = 6|u|^2 - |F_{14}|^2$ by [9, Lemma B.1]. Then it follows that

$$\begin{aligned}&* \left(- * \varphi \wedge F + \frac{1}{6}F^3 + \frac{1}{2} * \left(\varphi \wedge *F^2\right) \wedge *F\right) \wedge F \wedge \varphi \\ &\qquad = 3\left(-1 + \frac{1}{2} * \left(\varphi \wedge F^2\right)\right) * (i(u)F).\end{aligned}$$

Since $\varphi \wedge *F^2 = -6 * (i(u)F)$ by Lemma A.2, the proof of Lemma A.4 is completed. □

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