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HECKE'S THEOREM ON THE DIFFERENT FOR 3-MANIFOLDS

by Will SAWIN & Mark SHUSTERMAN (*)

ABSTRACT. — Hecke has shown that the different of an extension of number fields is a square in the ideal class group. We prove an analog for branched covers of closed 3-manifolds saying that the branch divisor is a square in the first homology group.

RÉSUMÉ. — Hecke a montré que la différente d'une extension de corps de nombres est un carré dans le groupe des classes d'idéaux. Nous prouvons un analogue pour les revêtements ramifiés de 3-variétés fermées en disant que le diviseur de ramification est un carré dans le premier groupe d'homologie.

1. Introduction

Let E/F be an extension of number fields, let $\mathcal{O}_E$ be the ring of integers of E, and let $\mathrm{Cl}(\mathcal{O}_E)$ be the class group of $\mathcal{O}_E$. One associates to the extension E/F the different $\mathcal{D}_{E/F}$, an ideal in $\mathcal{O}_E$, see [5, Chapter 3]. Hecke has shown that as an element of $\mathrm{Cl}(\mathcal{O}_E)$, the different $\mathcal{D}_{E/F}$ is a square, namely there exists an ideal class $J \in \mathrm{Cl}(\mathcal{O}_E)$ such that $J^2 = \mathcal{D}_{E/F}$ in $\mathrm{Cl}(\mathcal{O}_E)$. Hecke's proof uses a reciprocity formula for Gauss sums, see [1, 2] for a proof and a discussion of related results.

An analog of Hecke's theorem for finite separable extensions of fields of fractions of Dedekind domains fails in general, see [3]. However, there exists an analog in case E/F is a finite separable extension of function fields of curves over finite fields of odd characteristic, see [1]. Another geometric analog of Hecke's theorem, based on similarities between the inverse of the different and the canonical bundle on a curve, is the theory of theta characteristics.

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In this work we consider an analog of Hecke's theorem for 3-manifolds, as suggested by arithmetic topology. We refer to [4] for the analogy between rings of integers and primes on the one hand, and 3-manifolds and knots on the other hand. The analog of $\mathrm{Spec}(\mathcal{O}_F)$ is a closed (not necessarily oriented) 3-manifold M. The map $\mathrm{Spec}(\mathcal{O}_E) \to \mathrm{Spec}(\mathcal{O}_F)$ is replaced by a cover $\pi\colon \widetilde{M} \to M$ branched over a link $L \subset M$, so $\widetilde{M}$ is a closed 3-manifold and $\pi^{-1}(M \setminus L)$ is a covering space of $M \setminus L$. The inverse image of L under π is a link $\widetilde{L}$ in $\widetilde{M}$.

For a prime ideal $\mathfrak{p}$ of $\mathcal{O}_E$ we denote by $e_\mathfrak{p}$ its ramification index, namely the largest positive integer e for which $\mathfrak{p}^e$ contains $\mathfrak{p} \cap \mathcal{O}_F$. We view $\mathrm{Spec}(\mathcal{O}_E) \to \mathrm{Spec}(\mathcal{O}_F)$ as branched over the primes of $\mathcal{O}_E$ that ramify, so $\widetilde{L}$ is our analog for $\mathcal{R}_{E/F} = \{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_E) : e_\mathfrak{p} > 1\}$. The analogy is perhaps closest in case $\mathrm{Spec}(\mathcal{O}_E) \to \mathrm{Spec}(\mathcal{O}_F)$ is tamely ramified, namely $e_\mathfrak{p}$ is coprime to $|\mathcal{O}_E/\mathfrak{p}|$ for every $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_E)$. In this case the different of E/F is given by

$$\mathcal{D}_{E/F} = \prod_{\mathfrak{p} \in \mathcal{R}_{E/F}} \mathfrak{p}^{e_\mathfrak{p} - 1}.$$

The prime ideals in $\mathcal{R}_{E/F}$ are analogous to the components of the link $\widetilde{L}$. For each component $\widetilde{K}$ of this link, let the ramification index $e_{\widetilde{K}}$ be the number of times the image under π of a small loop around $\widetilde{K}$ wraps around $\pi(\widetilde{K})$. An analog of $\mathrm{Cl}(\mathcal{O}_E)$ is $H_1(\widetilde{M}, \mathbb{Z})$, and a homology class is a square if and only if its image in

$$H_1\big(\widetilde{M}, \mathbb{Z}\big) \otimes_\mathbb{Z} \mathbb{Z}/2\mathbb{Z} \cong H_1\big(\widetilde{M}, \mathbb{Z}/2\mathbb{Z}\big)$$

vanishes. Our analogy of $\mathcal{D}_{E/F}$, or rather of its class in $\mathrm{Cl}(\mathcal{O}_E)/\,\mathrm{Cl}(\mathcal{O}_E)^2$, is the branch divisor

$$\mathcal{D}_\pi = \sum_{\widetilde{K} \text{ a component of } \widetilde{L}} \big(e_{\widetilde{K}} - 1\big)[\widetilde{K}] \in H_1\big(\widetilde{M}, \mathbb{Z}/2\mathbb{Z}\big)$$

of π. Since we are working with $\mathbb{Z}/2\mathbb{Z}$-coefficients, it is not necessary to fix an orientation of $\widetilde{K}$, nor is the sign of $e_{\widetilde{K}}$ significant.

Theorem 1.1. — *Let $\widetilde{M}$ and M be closed 3-manifolds, and let $\pi\colon \widetilde{M} \to M$ be a cover branched over a link in M. Then the branch divisor $\mathcal{D}_\pi$ represents the trivial class in $H_1(\widetilde{M}, \mathbb{Z}/2\mathbb{Z})$.*

2. A central extension of the hyperoctahedral group

Let n be a positive integer, and let S_n be the symmetric group. Recall the hyperoctahedral group

$$B_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n \tag{2.1}$$

where S_n acts on $(\mathbb{Z}/2\mathbb{Z})^n$ by permuting the coordinates.

Let H_n be the group consisting of pairs $(a, b) \in (\mathbb{Z}/2\mathbb{Z})^n \times \mathbb{Z}/2\mathbb{Z}$ with group law

$$(a_1, b_1)(a_2, b_2) = \left(a_1 + a_2, b_1 + b_2 + \sum_{1 \leqslant i < j \leqslant n} a_{1,i}\, a_{2,j} \right).$$

A straightforward computation shows that this law is associative, and that the inverse of (a, b) is

$$\left(a, b + \sum_{1 \leqslant i < j \leqslant n} a_i\, a_j \right).$$

Projection onto the first factor exhibits H_n as a central extension of $(\mathbb{Z}/2\mathbb{Z})^n$ by $\mathbb{Z}/2\mathbb{Z}$.

For $1 \leqslant i \leqslant n$ we denote by e_i the i^{th} unit vector in $(\mathbb{Z}/2\mathbb{Z})^n$, set $x_i = (e_i, 0) \in H_n$, and $\epsilon = (0, 1) \in H_n$. We denote the unit element $(0, 0) \in H_n$ by 1. We can check that

$$x_i^2 = \epsilon^2 = 1, \quad 1 \leqslant i \leqslant n, \tag{2.2}$$

that

$$x_i x_j = \epsilon x_j x_i, \quad 1 \leqslant i, j \leqslant n,\ i \neq j, \tag{2.3}$$

and that

$$\epsilon x_i = x_i \epsilon, \quad 1 \leqslant i \leqslant n. \tag{2.4}$$

Furthermore, the relations in (2.2), (2.3), and (2.4) among the generators $x_1, \ldots, x_n, \epsilon$ define the group H_n since using these relations every word in $x_1, \ldots, x_n, \epsilon$ can be brought to the form $x_{i_1}, \ldots, x_{i_k} \epsilon^\delta$ with $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ and $\delta \in \{0, 1\}$.

We therefore have an action of S_n on H_n by automorphisms via

$$\sigma(x_i) = x_{\sigma(i)},\ \sigma(\epsilon) = \epsilon, \quad \sigma \in S_n,\ 1 \leqslant i \leqslant n.$$

Let $G_n = H_n \rtimes S_n$ be the semidirect product defined using this action. Since $\epsilon \in H_n$ is central and S_n-invariant, it lies in the center of G_n, so

$$G_n/\langle \epsilon \rangle = G_n/\{1, \epsilon\} \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n = B_n$$

where the last equality is by definition of B_n in Eq. (2.1). We see that G_n is a central extension of B_n by $\mathbb{Z}/2\mathbb{Z}$. We denote by β_n the class in $H^2(B_n, \mathbb{Z}/2\mathbb{Z})$ corresponding to this extension.

Let $\sigma, \tau \in B_n$ be two elements that commute, let $\widetilde{\sigma}, \widetilde{\tau}$ be lifts to G_n, and define

$$\phi(\sigma,\tau) = [\widetilde{\sigma}, \widetilde{\tau}] = \widetilde{\sigma}\widetilde{\tau}\widetilde{\sigma}^{-1}\widetilde{\tau}^{-1} \in \langle \epsilon \rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

Since G_n is a central extension of B_n, the above is indeed independent of the choice of lifts. As every element in $\mathbb{Z}/2\mathbb{Z}$ is its own inverse, we see that

$$\phi(\sigma,\tau) = [\widetilde{\sigma}, \widetilde{\tau}] = [\widetilde{\tau}, \widetilde{\sigma}]^{-1} = [\widetilde{\tau}, \widetilde{\sigma}] = \phi(\tau,\sigma). \tag{2.5}$$

We denote by

$$C_{B_n}(\sigma) = \{\tau \in B_n : \sigma\tau = \tau\sigma\}$$

the centralizer of σ in B_n.

Proposition 2.1. — *For every $\sigma \in B_n$ the map that sends $\tau \in C_{B_n}(\sigma)$ to $\phi(\sigma,\tau)$ is a homomorphism.*

Proof. — For every $\tau_1, \tau_2 \in C_{B_n}(\sigma)$ we have

$$\phi(\sigma,\tau_1\tau_2) = \widetilde{\sigma}\widetilde{\tau_1}\widetilde{\tau_2}\widetilde{\sigma}^{-1}\widetilde{\tau_2}^{-1}\widetilde{\tau_1}^{-1}, \quad \phi(\sigma,\tau_1)\phi(\sigma,\tau_2) = \widetilde{\sigma}\widetilde{\tau_1}\widetilde{\sigma}^{-1}\widetilde{\tau_1}^{-1}[\widetilde{\sigma}, \widetilde{\tau_2}]$$

so after cancelling $\widetilde{\sigma}\widetilde{\tau_1}$, it remains to check that

$$\widetilde{\tau_2}\widetilde{\sigma}^{-1}\widetilde{\tau_2}^{-1}\widetilde{\tau_1}^{-1} = \widetilde{\sigma}^{-1}\widetilde{\tau_1}^{-1}[\widetilde{\sigma}, \widetilde{\tau_2}].$$

After multiplying by $\widetilde{\sigma}$ from the left, we just need to check that $[\widetilde{\sigma}, \widetilde{\tau_2}]$ commutes with $\widetilde{\tau_1}$. This is indeed the case because $[\widetilde{\sigma}, \widetilde{\tau_2}]$ lies in the central subgroup $\{1, \epsilon\}$ of G_n. □

Corollary 2.2. — *For every $\tau \in B_n$ the map that sends $\sigma \in C_{B_n}(\tau)$ to $\phi(\sigma,\tau)$ is a homomorphism.*

Proof. — For $\sigma_1, \sigma_2 \in C_{B_n}(\tau)$ we get from Eq. (2.5) and Proposition 2.1 that

$$\phi(\sigma_1\sigma_2,\tau) = \phi(\tau,\sigma_1\sigma_2) = \phi(\tau,\sigma_1)\phi(\tau,\sigma_2) = \phi(\sigma_1,\tau)\phi(\sigma_2,\tau)$$

as required. □

Proposition 2.3. — *For a k-cycle $\sigma = (i_1 \dots i_k) \in S_n \leqslant B_n$, and*

$$\tau = e_{i_1} + \dots + e_{i_k} \in (\mathbb{Z}/2\mathbb{Z})^n \leqslant B_n$$

we have $\phi(\sigma,\tau) = \epsilon^{k-1}$. For every $\alpha \in S_n \leqslant B_n$ with $\alpha(i_1) = i_1, \dots, \alpha(i_k) = i_k$ we have $\phi(\alpha,\tau) = 1$.

Proof. — We take $\widetilde{\sigma} = (i_1, \dots, i_k)$, $\widetilde{\tau} = x_{i_1}, \dots, x_{i_k}$ and get that

$$\begin{aligned}
\phi(\sigma, \tau) &= \widetilde{\sigma}\widetilde{\tau}\widetilde{\sigma}^{-1} \cdot \widetilde{\tau}^{-1} \\
&= \sigma(x_{i_1}, \dots, x_{i_k}) \cdot (x_{i_1}, \dots, x_{i_k})^{-1} \\
&= x_{\sigma(i_1)}, \dots, x_{\sigma(i_k)} \cdot (x_{i_1}, \dots, x_{i_k})^{-1} \\
&= x_{i_2}, \dots, x_{i_k} x_{i_1} \cdot (x_{i_1}, \dots, x_{i_k})^{-1} \\
&= \epsilon^{k-1} x_{i_1}, \dots, x_{i_k} \cdot (x_{i_1}, \dots, x_{i_k})^{-1} \\
&= \epsilon^{k-1}.
\end{aligned}$$

Taking $\widetilde{\alpha} = \alpha$ we see that

$$\begin{aligned}
\phi(\alpha, \tau) &= \widetilde{\alpha}\widetilde{\tau}\widetilde{\alpha}^{-1} \cdot \widetilde{\tau}^{-1} \\
&= \alpha(x_{i_1}, \dots, x_{i_k}) \cdot (x_{i_1}, \dots, x_{i_k})^{-1} \\
&= x_{\alpha(i_1)}, \dots, x_{\alpha(i_k)} \cdot (x_{i_1}, \dots, x_{i_k})^{-1} \\
&= 1
\end{aligned}$$

as claimed. □

Corollary 2.4. — *Let $\sigma \in S_n \leqslant B_n$ whose disjoint cycles are*

$$C_1 = (i_{1,1}, \dots, i_{1,d_1}), \dots, C_j = \big(i_{j,1}, \dots, i_{j,d_j}\big), \quad \sum_{r=1}^{j} d_r = n,$$

and let $\tau \in C_{B_n}(\sigma)$. Then there exists a (unique) choice of $\tau' \in C_{S_n}(\sigma)$ and $\lambda_1, \dots, \lambda_j \in \mathbb{Z}/2\mathbb{Z}$ such that

$$\tau = \tau' v, \quad v = \sum_{r=1}^{j} \lambda_r \big(e_{i_{r,1}} + \cdots + e_{i_{r,d_r}}\big) \tag{2.6}$$

and

$$\phi(\sigma, \tau) = \epsilon^{\sum_{r=1}^{j} \lambda_r (d_r - 1)}.$$

Proof. — The ability to express τ as in Eq. (2.6) is immediate from the definition of the group law in B_n. From Proposition 2.1, Corollary 2.2, and Proposition 2.3 we therefore get that

$$\begin{aligned}
\phi(\sigma, \tau) &= \phi\left(\sigma, \tau' \cdot \sum_{r=1}^{j} \lambda_r (e_{i_{r,1}} + \cdots + e_{i_{r,d_r}})\right) \\
&= \phi(\sigma, \tau') \cdot \prod_{r=1}^{j} \phi\big(\sigma, e_{i_{r,1}} + \cdots + e_{i_{r,d_r}}\big)^{\lambda_r}
\end{aligned}$$

$$= [\sigma, \tau'] \cdot \prod_{r=1}^{j} \prod_{s=1}^{j} \phi\big(C_s, e_{i_{r,1}} + \cdots + e_{i_{r,d_r}}\big)^{\lambda_r}$$
$$= 1 \cdot \prod_{r=1}^{j} \epsilon^{\lambda_r(d_r-1)}$$
$$= \epsilon^{\sum_{r=1}^{j} \lambda_r(d_r-1)}$$

as required. □

We keep the notation of Corollary 2.4 and denote by $O_1, \dots, O_z$ the orbits of the action by conjugation of the subgroup of S_n generated by τ' on $\{C_1, \dots, C_j\}$. For $1 \leqslant y \leqslant z$ we let $I_y \subseteq \{1, \dots, n\}$ be the set of all indices that appear in one of the cycles in O_y, and define the permutation $\tau'_y \in S_n$ by

$$\tau'_y(i) = \begin{cases} \tau'(i) & i \in I_y \\ i & i \notin I_y. \end{cases}$$

We have a disjoint union

$$\bigcup_{y=1}^{z} I_y = \{1, \dots, n\}$$

hence $\tau' = \tau'_1 \cdots \tau'_z$ and the permutations $\tau'_1, \dots, \tau'_z$ commute. We put

$$\tau_y = \tau'_y v_y, \quad v_y = \sum_{\substack{1 \leqslant r \leqslant j \\ C_r \in O_y}} \lambda_r\big(e_{i_{r,1}} + \cdots + e_{i_{r,d_r}}\big)$$

and get that

$$\tau = \tau'_1 v_1, \dots, \tau'_z v_z \tag{2.7}$$

where the factors $\tau'_1 v_1, \dots, \tau'_z v_z$ commute.

3. Proof of Theorem 1.1

It suffices to show, for each $\alpha \in H^1(\widetilde{M}, \mathbb{Z}/2\mathbb{Z})$, that the pairing of the branch divisor $\mathcal{D}_\pi$ with α vanishes, namely

$$\sum_{\widetilde{K} \text{ a component of } \widetilde{L}} (e_{\widetilde{K}} - 1)\Big\langle [\widetilde{K}], \alpha \Big\rangle = 0$$

or equivalently

$$\sum_{K \text{ a component of } L} \quad \sum_{\widetilde{K} \text{ a component of } \pi^{-1}(K)} (e_{\widetilde{K}} - 1)\Big\langle [\widetilde{K}], \alpha \Big\rangle = 0.$$

Associated to α is a degree two covering space $N \to \widetilde{M}$. Let n be the degree of $\pi\colon \widetilde{M} \to M$ which is locally constant away from L, thus constant. Away from L, we get that N is a degree 2 covering space of a degree n covering space, hence has monodromy group contained in the wreath product

$$S_2 \wr S_n = (\mathbb{Z}/2\mathbb{Z}) \wr S_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n = B_n.$$

We thus have a map $H^2(B_n, \mathbb{Z}/2\mathbb{Z}) \to H^2(M \setminus L, \mathbb{Z}/2\mathbb{Z})$, and we denote by $\gamma \in H^2(M \setminus L, \mathbb{Z}/2\mathbb{Z})$ the image of β_n. Here β_n is the class in $H^2(B_n, \mathbb{Z}/2\mathbb{Z})$ corresponding to the central extension

$$\{1\} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow G_n \longrightarrow B_n \longrightarrow \{1\}$$

of B_n by $\mathbb{Z}/2\mathbb{Z}$.

Consider a tubular neighborhood Q of L and let $S = \partial Q$ be its boundary, a union of tori. Each such torus T corresponds to a unique component K of L - the boundary of a tubular neighborhood of K is T. Since S bounds a 3-manifold in $M \setminus L$, i.e. the complement of the tubular neighborhood Q, our cohomology class γ integrates to 0 on S. It follows that

$$\sum_{T \text{ a component of } S} \int_T \gamma = 0.$$

It is therefore sufficient to prove that

$$\int_T \gamma = \sum_{\widetilde{K} \text{ a component of } \pi^{-1}(K)} (e_{\widetilde{K}} - 1)\Big\langle [\widetilde{K}], \alpha \Big\rangle. \tag{3.1}$$

Since T is a torus, a covering of T with monodromy B_n, i.e. a homomorphism from $\pi_1(T)$ to B_n, is given by a pair of elements $m, \ell \in B_n$ that commute, where m represents a meridian and ℓ represents a longitude. From the standard cell decomposition of the torus, we can see that

$$\int_T \gamma = \phi(m, \ell).$$

Since the $\mathbb{Z}/2\mathbb{Z}$-covering $N \to \widetilde{M}$ is unbranched over every component $\widetilde{K}$ of $\pi^{-1}(K)$, the monodromy of the meridian m does not swap the two components of the covering, and therefore m is (up to conjugation) contained in $S_n \leqslant B_n$.

We shall use here the notation of Corollary 2.4 and the paragraph following it for $\sigma = m$ and $\tau = \ell$, in particular we write $\ell = \ell' v$ as in (2.6). The components of $\pi^{-1}(K)$ are naturally in bijection with the orbits of the action by conjugation of the subgroup of S_n generated by ℓ' on the set of disjoint cycles $\{C_1, \dots, C_j\}$ of m. We denote by $O_{\tilde{K}}$ the orbit corresponding to a component $\widetilde{K}$ of $\pi^{-1}(K)$. As in (2.7), we can write

$$\ell = \prod_{\widetilde{K}} \ell_{\tilde{K}}, \quad \ell_{\tilde{K}} = \ell'_{\tilde{K}} v_{\tilde{K}}, \quad v_{\tilde{K}} = \sum_{\substack{1 \leqslant r \leqslant j \\ C_r \in O_{\tilde{K}}}} \lambda_r \big(e_{i_{r,1}} + \dots + e_{i_{r,d_r}}\big).$$

We denote the number of cycles in $O_{\tilde{K}}$ by $t(\widetilde{K})$, note that each such cycle is of length $e_{\tilde{K}}$, and set

$$d_{\tilde{K}} = \#\{1 \leqslant r \leqslant j : C_r \in O_{\tilde{K}},\ \lambda_r = 1\}.$$

It follows from Corollary 2.4 that $\phi(m, \ell_{\tilde{K}}) \equiv (e_{\tilde{K}} - 1) d_{\tilde{K}} \mod 2$, so from Corollary 2.2 we get that

$$\begin{aligned} \phi(m, \ell) &= \sum_{\widetilde{K} \text{ a component of } \pi^{-1}(K)} \phi(m, \ell_{\tilde{K}}) \\ &\equiv \sum_{\widetilde{K} \text{ a component of } \pi^{-1}(K)} (e_{\tilde{K}} - 1)\, d_{\tilde{K}} \mod 2. \end{aligned}$$

It is therefore enough to show that $d_{\tilde{K}} \equiv \langle [\widetilde{K}], \alpha \rangle \mod 2$.

Let C be a longitude curve in a tubular neighborhood of $\widetilde{K}$. Then $[C] = [\widetilde{K}]$ as homology classes in $H_1(\widetilde{M}, \mathbb{Z}/2\mathbb{Z})$, so it suffices to show that $d_{\tilde{K}} \equiv \langle [C], \alpha \rangle \mod 2$. The projection of $[C]$ to T is

$$a[m] + t(\widetilde{K})[\ell]$$

for some $a \in \mathbb{Z}$. Thus, the action of C on the covering space $N \to M$ is given by $m^a \ell^{t(\tilde{K})}$. We have

$$m^a \ell^{t(\tilde{K})} = m^a (\ell' v)^{t(\tilde{K})} = m^a \ell'^{t(\tilde{K})} \cdot \Big(v + \ell'(v) + \dots + \ell'^{t(\tilde{K})-1}(v)\Big).$$

The pairing $\langle [C], \alpha \rangle$ is nonzero if and only if the monodromy along C of the covering $N \to \widetilde{M}$ is nontrivial, which happens if and only if the action of $m^a \ell^{t(\tilde{K})}$ sends one branch of this covering to the other, and that occurs if and only if the k^{th} entry of $v + \ell'(v) + \dots + \ell'^{t(\tilde{K})-1}(v)$ is nonzero for some (equivalently, every) index $1 \leqslant k \leqslant n$ that belongs to one of the cycles in $O_{\tilde{K}}$. It is therefore sufficient to show that

$$d_{\tilde{K}} \equiv \Big(v + \ell'(v) + \dots + \ell'^{t(\tilde{K})-1}(v)\Big)_k \mod 2.$$

We have

$$\begin{aligned} \Big(v + \ell'(v) + \dots + \ell'^{t(\tilde{K})-1}(v)\Big)_k &= v_k + \ell'(v)_k + \dots + \ell'^{t(\tilde{K})-1}(v)_k \\ &= v_k + v_{\ell'^{-1}(k)} + \dots + v_{\ell'^{-t(\tilde{K})+1}(k)}. \end{aligned}$$

By the orbit-stabilizer theorem, each of the $t(\widetilde{K})$ cycles in $O_{\tilde{K}}$ contains exactly one of the $t(\widetilde{K})$ elements $k, \ell'^{-1}(k), \dots, \ell'^{-t(\tilde{K})+1}(k)$. Thus, from (2.6) we get that

$$v_k + v_{\ell'^{-1}(k)} + \cdots + v_{\ell'^{-t(\tilde{K})+1}(k)} = \sum_{\substack{1 \leqslant r \leqslant j \\ C_r \in O_{\tilde{K}}}} \lambda_r \equiv d_{\tilde{K}} \mod 2,$$

as desired.

4. An alternative argument in the oriented case

We sketch here an alternative proof of Theorem 1.1 in case the manifolds M and $\widetilde{M}$ are orientable. Fix triangulations of M and of $\widetilde{M}$. The skeleta of the barycentric subdivision represent the dual of the terms in the total Stiefel–Whitney class of M and of $\widetilde{M}$. Since M and $\widetilde{M}$ are closed orientable 3-manifolds, they are parallelizable, so the aforementioned total Stiefel–Whitney classes are zero. Viewed as classes in $H_1(\widetilde{M}, \mathbb{Z}/2\mathbb{Z})$, the pullback under π of the 1-skeleton of our triangulation of M, and the 1-skeleton of our triangulation of $\widetilde{M}$ differ by $\mathcal{D}_\pi$. It follows that $\mathcal{D}_\pi$ represents the trivial class in $H_1(\widetilde{M}, \mathbb{Z}/2\mathbb{Z})$ as required.

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