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ON THE OPTIMAL CONTROLLABILITY TIME FOR LINEAR HYPERBOLIC SYSTEMS WITH TIME-DEPENDENT COEFFICIENTS

by Jean-Michel CORON & Hoai-Minh NGUYEN (*)

Abstract. — The optimal time for the controllability of linear hyperbolic systems in one-dimensional space with one-side controls has been obtained recently for time-independent coefficients in our previous works. In this paper, we consider linear hyperbolic systems with time-varying zero-order terms. We show the possibility that the optimal time for the null-controllability becomes significantly larger than the one of the time-invariant setting even when the zero-order term is indefinitely differentiable. When the analyticity with respect to time is imposed for the zero-order term, we also establish that the optimal time is the same as in the time-independent setting.

Résumé. — Le temps optimal pour la contrôlabilité des systèmes hyperboliques linéaires sur un espace unidimensionnel avec des contrôles unilatéraux a été obtenu récemment pour des coefficients indépendants du temps dans nos travaux antérieurs. Dans cet article, nous considérons des systèmes hyperboliques linéaires avec des termes d'ordre zéro variables dans le temps. Nous montrons la possibilité que le temps optimal pour la contrôlabilité nulle devienne significativement plus grand que celui du cadre invariant dans le temps, même lorsque le terme d'ordre zéro est indéfiniment différentiable. Lorsque l'analyticité par rapport au temps est imposée pour le terme d'ordre zéro, nous établissons également que le temps optimal est le même que dans le cadre indépendant du temps.

1. Introduction and statement of the main results

Hyperbolic systems in one-dimensional space are frequently used in the modeling of many systems such as traffic flow [1], heat exchangers [50], fluids in open channels [22, 26, 27, 28], and phase transition [23]. Many other

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interesting examples can be found in [5] and the references therein. The optimal time for the controllability of hyperbolic systems in one-dimensional space with one-side controls has been derived recently for time-independent coefficients [16, 18]. In this paper, we consider hyperbolic systems with time-varying zero-order terms, which are known to be controllable in some positive time. In this paper, we show the possibility that the optimal time for the null-controllability becomes significantly larger than the one of the time-invariant setting even when the zero-order term is indefinitely differentiable. When the analyticity with respect to time is imposed for the zero-order term, we also establish that the optimal time is the same as in the time-independent setting. The first result is quite surprising since the zero-order term does not interfere with the characteristic flows of the system. The latter result complementary to the first one can then be viewed as an extension of a well-known controllability property of linear differential equations: if a linear control system is controllable in some positive time and is *analytic*, then it is controllable in any time greater than the optimal time, which is 0.

Let us first briefly discuss known results for the time-independent coefficients to underline the phenomena. Consider the system

$$\partial_t u(t,x) = \Sigma(x)\partial_x u(t,x) + C(x)u(t,x) \quad \text{for } (t,x) \in \mathbb{R}_+ \times (0,1). \tag{1.1}$$

Here $u = (u_1, \dots, u_n)^\mathsf{T} \colon \mathbb{R}_+ \times (0,1) \to \mathbb{R}^n$ $(n \geqslant 2)$, Σ and C are $(n \times n)$ real, matrix-valued functions defined in $[0,1]$. We assume that, for every $x \in [0,1]$, the matrix $\Sigma(x)$ is diagonalizable with $m \geqslant 1$ distinct positive eigenvalues and $k = n - m \geqslant 1$ distinct negative eigenvalues. We also assume that $C \in \big(L^\infty([0,1])\big)^{n\times n}$. Using Riemann coordinates, one might assume that $\Sigma(x)$ is of the form

$$\Sigma(x) = \operatorname{diag}\big(-\lambda_1(x), \dots, -\lambda_k(x), \lambda_{k+1}(x), \dots, \lambda_n(x)\big), \tag{1.2}$$

where[(1)]

$$-\lambda_1(x) < \dots < -\lambda_k(x) < 0 < \lambda_{k+1}(x) < \dots < \lambda_{k+m}(x). \tag{1.3}$$

In what follows, we assume that

$$\lambda_i \text{ is of class } C^2 \text{ on } [0,1] \text{ for } 1 \leqslant i \leqslant n\,(= k+m), \tag{1.4}$$

and denote

$$u_- = (u_1, \dots, u_k)^\mathsf{T} \qquad \text{and} \qquad u_+ = (u_{k+1}, \dots, u_{k+m})^\mathsf{T}.$$

(1) Thus $\Sigma_{ii} = \lambda_i$ for $k+1 \leqslant i \leqslant k+m$ and $\Sigma_{ii} = -\lambda_i$ for $1 \leqslant i \leqslant k$.

We are interested in the following type of boundary conditions and boundary controls. The boundary conditions at $x = 0$ are given by

$$u_-(t,0) = Bu_+(t,0) \quad \text{for } t \geqslant 0, \tag{1.5}$$

for some $(k \times m)$ real *constant* matrix B, and at $x = 1$

$$u_+(t,1) \text{ is controlled for } t \geqslant 0. \tag{1.6}$$

Let us recall that the control system (1.1), (1.5), and (1.6) is null-controllable (resp. exactly controllable) at time $T > 0$ if, for every initial datum $u_0\colon (0,1) \to \mathbb{R}^n$ in $\left[L^2(0,1)\right]^n$ (resp. for every initial datum $u_0\colon (0,1) \to \mathbb{R}^n$ in $\left[L^2(0,1)\right]^n$ and for every (final) state $u_T\colon (0,1) \to \mathbb{R}^n$ in $\left[L^2(0,1)\right]^n$), there is a control $U\colon (0,T) \to \mathbb{R}^m$ in $\left[L^2(0,T)\right]^m$ such that the solution of (1.1), (1.5), and (1.6) (with $u_+ = U$) satisfying $u(t=0,x) = u_0(x)$ vanishes (resp. reaches u_T) at the time T: $u(t=T,\cdot\,) = 0$ (resp. $u(t=T,\cdot\,) = u_T$). Moreover, the control system (1.1), (1.5), and (1.6) is approximately controllable at time $T > 0$ if, for every $\delta > 0$, for every initial datum $u_0\colon (0,1) \to \mathbb{R}^n$ in $\left[L^2(0,1)\right]^n$ and for every state $u_T\colon (0,1) \to \mathbb{R}^n$ in $\left[L^2(0,1)\right]^n$, there is a control $U\colon (0,T) \to \mathbb{R}^m$ in $\left[L^2(0,T)\right]^m$ such that the solution of (1.1), (1.5), and (1.6) (with $u_+ = U$) satisfying $u(t=0,x) = u_0(x)$ is such that $\left\|u(T,\cdot\,) - u_T\right\|_{L^2(0,1)} < \delta$.

Throughout this paper, we consider broad solutions in L^2 with respect to t and x for an initial datum in $\left[L^2(0,1)\right]^n$ and a control in $\left[L^2(0,T)\right]^m$ (see, for example, [37, Section 3]). In particular, the solutions belong to $C\big([0,T];\left[L^2(0,1)\right]^n\big)$ and $C\big([0,1];\left[L^2(0,T)\right]^n\big)$. The well-posedness for broad solutions for system (1.1), (1.5), and (1.6) even when Σ and C depend also on t is standard.

Set

$$\tau_i := \int_0^1 \frac{1}{\lambda_i(\xi)}\,\mathrm{d}\xi \quad \text{for } 1 \leqslant i \leqslant n. \tag{1.7}$$

The exact controllability, the null-controllability, and the boundary stabilization problem of hyperbolic systems in one-dimensional space have been widely investigated in the literature for almost half a century, see, e.g., [5] and the references therein. Concerning the exact controllability and the null-controllability related to (1.1), (1.5) and (1.6), the pioneer works date back to the ones of Rauch and Taylor [39] and Russell [41]. In particular, it was shown, see [41, Theorem 3.2], that system (1.1), (1.5), and (1.6) is null-controllable for time $\tau_k + \tau_{k+1}$, and is exactly controllable at the same time if $k = m$ and B is invertible. The extension of this result for quasilinear systems was initiated by Greenberg and Li [25] and Slemrod [42].

A recent efficient way in the study of the stabilization and the controllability of system (1.1), (1.5), and (1.6) is via a backstepping approach. The backstepping approach for the control of partial differential equations was pioneered by Krstic and his coauthors (see [34] for a concise introduction). The backstepping method is now frequently used for various control problems, modeling by partial differential equations in one-dimensional space. For example, it has been used to stabilize the wave equations [33, 43, 46], the parabolic equations in [44, 45], nonlinear parabolic equations [49], and to obtain the null-controllability of the heat equation [15]. The standard backstepping approach relies on the Volterra transform of the second kind. It is worth noting that, in some situations, more general transformations have to be considered as for Korteweg–de Vries equations [7], Kuramoto–Sivashinsky equations [14], Schrödinger's equation [11], and hyperbolic equations with internal controls [52].

The use of the backstepping approach for the hyperbolic system in one-dimensional space was first proposed by Coron et al. [20] for 2×2 system ($m = k = 1$). Later, this approach has been extended and now can be applied for general pairs (m, k), see [3, 12, 16, 18, 21, 30, 31].

Set

$$T_{\rm opt} := \begin{cases} \max\big\{\tau_1 + \tau_{m+1}, \dots, \tau_k + \tau_{m+k}, \tau_{k+1}\big\} & \text{if } m \geqslant k, \\ \max\big\{\tau_{k+1-m} + \tau_{k+1}, \tau_{k+2-m} + \tau_{k+2}, \dots, \\ \qquad\qquad \tau_k + \tau_{k+m}\big\} & \text{if } m < k. \end{cases} \tag{1.8}$$

Involving the backstepping technique, we established [16, 18] that the null-controllability holds at $T_{\rm opt}$ for generic B and C, and the null-controllability holds for any $T > T_{\rm opt}$ under the condition $B \in \mathcal{B}$. Here

$$\mathcal{B} := \left\{ B \in \mathbb{R}^{k \times m} \;\middle|\; \begin{array}{l} \text{such that (1.10) holds} \\ \text{for } 1 \leqslant i \leqslant \min\{k, m-1\} \end{array} \right\}, \tag{1.9}$$

where

$$\begin{array}{c} \text{the } i \times i \text{ matrix formed from the last } i \text{ columns} \\ \text{and the last } i \text{ rows of } B \text{ is invertible.} \end{array} \tag{1.10}$$

Roughly speaking, the condition $B \in \mathcal{B}$ allows us to implement l controls corresponding to the fastest positive speeds to control l components corresponding to the lowest negative speeds.(2) It is clear that $B \in \mathcal{B}$ for almost every $k \times m$ matrix B. It is worth noting that the condition $T > T_{\rm opt}$ is

(2) The i direction ($1 \leqslant i \leqslant n$) is called positive (resp. negative) if Σ_{ii} is positive (resp. negative).

necessary in the sense that there are counterexamples for $T = T_{\text{opt}}$ see [16, Assertion 2) of Theorem 1.1]. The optimality of T_{opt} was established under the additional condition (1.10) being valid with $i = m$ when $k \geqslant m$, see [16, Proposition 1.6]. Our results improved the time to reach the null-controllability obtained previously. Similar conclusions hold for the exact controllability under the natural conditions $m \geqslant k$ and (1.10) for $1 \leqslant i \leqslant k$ (see [16, 18, 31]). When the system is homogeneous, i.e., $C \equiv 0$, we established that the null-controllability can be achieved via a time-independent feedback even for the quasilinear setting [17]. We also constructed Lyapunov functions which yield the null-controllability for such a system at the optimal time T_{opt} [19].

In this paper, we are interested in hyperbolic systems with time-dependent coefficients in one-dimensional space. More precisely, instead of (1.1), (1.5), and (1.6), we deal with

$$\partial_t u(t,x) = \Sigma(x)\partial_x u(t,x) + C(t,x)u(t,x) \quad \text{for } (t,x) \in \mathbb{R}_+ \times (0,1), \tag{1.11}$$

and (1.5), and (1.6).

The first result of the paper reveals that the optimal time for the null-controllability of system (1.11), (1.5), and (1.6) might be significantly larger than the one for the time-independent setting even when Σ is constant and C is indefinitely differentiable. More precisely, we have

Theorem 1.1. — *Let $k \geqslant 1$, $m \geqslant 2$, and Σ be constant such that* (1.3) *holds. Assume that*

$$B_{k,1} \neq 0, \quad B_{k,\ell} \neq 0, \quad B_{k,j} = 0 \quad \text{for } 2 \leqslant j \leqslant m \text{ with } j \neq \ell, \tag{1.12}$$

for some $2 \leqslant \ell \leqslant m$. There exists $C \in C^\infty\big([0,+\infty) \times [0,1]\big)$ such that for all $\varepsilon > 0$, system (1.11), (1.5), *and* (1.6) *is* neither *approximately controllable* nor *null-controllable at time*

$$T = \tau_k + \tau_{k+1} - \varepsilon. \tag{1.13}$$

Remark 1.2. — The definition of the approximate controllability and of the null-controllability for system (1.11), (1.5), and (1.6) is similar to the ones corresponding to (1.1), (1.5), and (1.6).

Remark 1.3. — There are infinitely many matrices $B \in \mathcal{B}$ satisfying condition (1.12). Note that (1.12) is only on the k-row of B.

Remark 1.4. — In a recent work, Coron et al. [13] establish the null-controllability of (1.11), (1.5), and (1.6) for time $\tau_k + \tau_{k+1}$ for all $k \times m$ matrices B. They also obtain stabilizing feedbacks and derive similar results when Σ depends on t. Combining Theorem 1.1 and their results, one obtains the optimality for the time $\tau_k + \tau_{k+1}$ when $m \geqslant 2$ and $k \geqslant 1$, and for a large class of B.

The proof of Theorem 1.1 is based on constructing counter-examples for the unique continuation property of the adjoint system (see also Section 2). The construction is inspired by the one given in the proof of [16, Assertion 2) of Theorem 1.1] but much more involved.

When the analyticity of C with respect to time is imposed, the situation changes dramatically. To state our results in this direction, we first introduce some notations. For a non-empty interval (a, b) of $\mathbb{R}$ and a Banach space $\mathcal{X}$, we denote

$$\mathcal{H}\big((a,b);\mathcal{X}\big) = \big\{\Phi\colon (a,b) \to \mathcal{X};\ \Phi \text{ is analytic}\big\}. \tag{1.14}$$

When the space $\mathcal{X}$ is clear, we simply call a $\Phi \in \mathcal{H}\big((a,b);\mathcal{X}\big)$ that Φ is analytic in (a, b). For $m \geqslant k$, set

$$\mathcal{B}_e := \big\{B \in \mathbb{R}^{k\times m};\ \text{such that (1.10) holds for } 1 \leqslant i \leqslant k\big\}. \tag{1.15}$$

Denote

$$T_1 = \tau_k + \tau_{k+1}. \tag{1.16}$$

Our main results for the analytic setting are the following two theorems. The first one on the null-controllability is the following.

Theorem 1.5. — *Let $k \geqslant m \geqslant 1$, and let $B \in \mathcal{B}$ be such that* (1.10) *holds for $i = m$. Assume that $C \in \mathcal{H}\big(I; \big[L^\infty(0,1)\big]^{n\times n}\big)$ for some open interval I containing $[0, T_1]$. System* (1.11), (1.5), *and* (1.6) *starting from time* 0 *is null-controllable at any time $T > T_{\rm opt}$.*

The second one on the exact controllability is the following.

Theorem 1.6. — *Let $m \geqslant k \geqslant 1$, and let $B \in \mathcal{B}_e$. Assume, for some open interval I containing $[T_{\rm opt} - T_1, T_{\rm opt}]$, that $C \in \mathcal{H}\big(I; \big[L^\infty(0,1)\big]^{n\times n}\big)$ System* (1.11), (1.5), *and* (1.6) *starting from time* 0 *is exactly controllable at any time $T > T_{\rm opt}$.*

Remark 1.7. — The analyticity of C is imposed for some negative time in Theorem 1.6.

Except for the case where $m = 1$ for which $T_1 = T_{\rm opt}$, Theorems 1.5 and 1.6 are new to our knowledge. Theorems 1.1, 1.5, and 1.6 reveal the crucial role of the analytic assumption of the coefficients on the optimal controllability time. It is well-known that if a linear control system modeled by differential equations is controllable in some positive time T and is *analytic*, then it is controllable in any time greater than the optimal time, which is 0, see, e.g., [9, Chapter 1] or [47, Chapter 3]. Theorems 1.5 and 1.6, which are complementary to Theorem 1.1, can be thus viewed as an extension of this well-known result for linear hyperbolic systems in one-dimensional space.

A related context to Theorem 1.6 is the one of the wave equation. For the wave equation with time-independent coefficients, the controllability is known under the sharp geometric control condition due to Bardos, Lebeau, and Rauch in [4] (see also [39]). What is missing from the celebrated Bardos, Lebeau, and Rauch result/analysis to deal with time-varying, first, and zero-order terms is the unique continuation property in this context. When the coefficients are analytic in time, the unique continuation property is derived via Carleman's estimates due to Tataru–Hörmander–Robbiano–Zuily [29, 40, 48] (see also [35] for a discussion) and therefore, the controllability of the wave equations follows in this case. Related results concerning the Schrödinger equation are due to Anantharaman, Léautaud, and Macià [2]. Theorem 1.1 in this paper shows that one cannot replace the analyticity assumption by the smoothness assumption at least in a very related context of the wave equation, the context of hyperbolic systems in one-dimensional space.

We now say a few words on the proofs of Theorems 1.5 and 1.6. Theorem 1.6 is derived from Theorem 1.5 using our arguments in the proof of [18, Theorem 3]. The proof of Theorem 1.5 is inspired by the analysis in [18], in which we established similar results for the time-independent setting. The crucial part of the analysis is then to locate the essential, analytic nature of the system, the smoothness is not sufficient as shown previously in Theorem 1.1. This is done by exploring both the original system and its dual one. The proof also involves the theory of perturbations of analytic compact operators, see, e.g., [32]. As a consequence of our analysis, we also obtain the unique continuation principle for hyperbolic systems for the optimal time in the analytic setting (see Proposition 3.18), which has its own interest. Our approach is thus quite different from the one used previously for the wave system. The strategy of the proof is described in more details at the beginning of Section 3.

The paper is organized as follows. The proofs of Theorems 1.1, 1.5, and 1.6 are given in Sections 2, 3, and 4, respectively. In the appendix, we establish properties of hyperbolic systems used in Section 3. The situation is non-standard in the sense that the domains considered are not rectangles and the boundary conditions are involved. The analysis is delicate and has its own interest.

2. Analysis in the smooth setting — Proof of Theorem 1.1

In order to establish Theorem 1.1, we show that the unique continuation property does not hold for the adjoint system if the time T is given by (1.13) for some choices of C. To this end, we first introduce some notations and derive the adjoint system (Lemma 2.1). We then recall the relation between the approximate controllable and the null-controllable properties of the system considered and the unique continuation property of its adjoint system (Lemma 2.2). Finally, we give the proof of Theorem 1.1 by constructing examples that violate the unique continuation property.

Fix $T > 0$ and define

$$\mathcal{F}_T\colon \left[L^2(0,T)\right]^m \longrightarrow \left[L^2(0,1)\right]^n$$
$$\mathcal{F}_T(U) \longmapsto u(T,\cdot),$$

where u is the unique solution of system (1.11), (1.5), and (1.6) with $u_+(\cdot,1) = U$ and with $u(0,\cdot) = 0$. Denote

$$\Sigma_- = \operatorname{diag}(-\lambda_1,\dots,-\lambda_k) \quad \text{and} \quad \Sigma_+ = \operatorname{diag}(\lambda_{k+1},\dots,\lambda_{k+m}).$$

As usual, we obtain the following result on the adjoint system.

Lemma 2.1. — *Let $T > 0$. We have, for $\varphi \in \left[L^2(0,1)\right]^n$,*

$$\mathcal{F}_T^*(\varphi) = \Sigma_+(1) v_+(\cdot,1) \quad \textit{in } (0,T),$$

where v is the unique broad solution of the system

$$\partial_t v(t,x) = \Sigma(x)\partial_x v(t,x) + \big(\Sigma'(x) - C^{\mathsf{T}}(t,x)\big)v(t,x) \quad \textit{for } (t,x) \in (0,T)\times(0,1), \tag{2.1}$$

with, for $0 < t < T$,

$$v_-(t,1) = 0, \tag{2.2}$$

$$\Sigma_+(0)v_+(t,0) = -B^{\mathsf{T}}\Sigma_-(0)v_-(t,0), \tag{2.3}$$

and

$$v(t = T,\cdot) = \varphi \quad \textit{in } (0,1). \tag{2.4}$$

The proof of Lemma 2.1 is standard and omitted, see, e.g., [18, proof of Lemma 1] for a closely related context.

From Lemma 2.1, one derives the following necessary condition for the approximate controllability and the null-controllability of system (1.11), (1.5), and (1.6) in time T, whose proof is standard and omitted (see, e.g., [9, Theorems 2.43 and 2.44]; these theorems are dealing with stationary linear control systems, but these theorems and their proofs also hold for time-varying linear systems).

Lemma 2.2. — *Let $T > 0$. System* (1.11), (1.5), *and* (1.6) *starting at time 0 is* neither *approximately controllable in time* T nor *null-controllable in time T if there exists $\varphi \in \left[L^2(0,1)\right]^n$ such that the broad solution v of the adjoint problem* (2.1), (2.2), *and* (2.3) *is such that*

$$v_+(\,\cdot\,,1) = 0 \quad \text{in } (0,T) \qquad \text{and} \qquad v(0,\cdot\,) \in \left[L^2(0,1)\right]^n \setminus \{0\}.$$

The rest of this section containing two subsections is devoted to the proof of Theorem 1.1. In the first subsection, we prove Theorem 1.1 in the case $m = 2$ and $k = 1$ to highlight the structure of the matrix C and to make the ideas of the proof clear. The proof in the general case is given in the second subsection.

2.1. Proof of Theorem 1.1 in the case $k = 1$ and $m = 2$

It suffices to consider the case where ε is small. This will be assumed later on.

Since Σ is constant, it follows that, for all $1 \leqslant i \leqslant n = m + k = 3$,

$$\tau_i = 1/\lambda_i.$$

Assume the (1×2) matrix B is given by

$$B = (a, b).$$

The condition (1.12) ($\ell = 2$ in this case) then becomes

$$a \neq 0 \qquad \text{and} \qquad b \neq 0. \tag{2.5}$$

The adjoint system (see Lemma 2.1) is

$$\partial_t v = \Sigma \partial_x v - C^{\mathsf{T}} v \quad \text{in } (0,T) \times (0,1), \tag{2.6}$$

and

$$v_1(t,1) = 0,$$

$$\begin{pmatrix} v_2 \\ v_3 \end{pmatrix}(t,0) = \begin{pmatrix} a\lambda_1\lambda_2^{-1}v_1 \\ b\lambda_1\lambda_3^{-1}v_1 \end{pmatrix} = \begin{pmatrix} \gamma_2 v_1 \\ \gamma_3 v_1 \end{pmatrix}(t,0) \tag{2.7}$$

for $t \in (0,T)$, where $\gamma_2 = a\lambda_1\lambda_2^{-1}$ and $\gamma_3 = b\lambda_1\lambda_3^{-1}$. We will consider the matrix C of the form:

$$C(t,x) = \begin{pmatrix} 0 & 0 & -\alpha(t,x) \\ 0 & 0 & -\beta(t,x) \\ 0 & 0 & 0 \end{pmatrix}.$$

The goal is to find smooth functions α and β such that there exists a smooth solution v of the adjoint system (2.6) and (2.7) which also satisfies

$$v_2(\,\cdot\,,1) = 0 = v_3(\,\cdot\,,1) \quad \text{in } (0,T) \qquad \text{and} \qquad v(0,\cdot\,) \not\equiv 0 \quad \text{in } [0,1]. \tag{2.8}$$

As a consequence of this fact, the unique continuation property of the adjoint system is violated. The conclusion then follows from Lemma 2.2.

Note that, from (2.6) and the choice of C, one has, for $(t,x) \in (0,T) \times (0,1)$,

$$\begin{cases} \partial_t v_1 = -\lambda_1 \partial_x v_1, \\ \partial_t v_2 = \lambda_2 \partial_x v_2, \\ \partial_t v_3 = \lambda_3 \partial_x v_3 + \alpha v_1 + \beta v_2. \end{cases} \tag{2.9}$$

We now construct α and β by first deriving their constraints under some special requirements on their structure (see (2.11) and (2.12) below). Considering the equation of v_3 in (2.9), by the characteristic method, for $\tau_3 \leqslant t + \tau_3 \leqslant T$, we have, *if* $v_3(t,1) = 0$ for $\tau_3 \leqslant t + \tau_3 \leqslant T$, then

$$\begin{aligned} v_3(t+\tau_3, 0) = \int_0^1 \tau_3 \alpha(t + \tau_3 s, 1-s) v_1(t+\tau_3 s, 1-s)\,\mathrm{d}s \\ + \int_0^1 \tau_3 \beta(t+\tau_3 s, 1-s) v_2(t+\tau_3 s, 1-s)\,\mathrm{d}s \end{aligned} \tag{2.10}$$

for $\tau_3 \leqslant t + \tau_3 \leqslant T$. We will make the following useful/important assumption concerning the structure of α and β from now on:

$$\tau_3 \alpha(t+\tau_3 s, 1-s) = \widetilde{\alpha}(t+\tau_3) \quad \text{for } s \in [0,1],\ t \in (0,T), \tag{2.11}$$

$$\tau_3 \beta(t+\tau_3 s, 1-s) = \widetilde{\beta}(t+\tau_3) \quad \text{for } s \in [0,1],\ t \in (0,T), \tag{2.12}$$

for some functions $\widetilde{\alpha}$ and $\widetilde{\beta}$ constructed later. With this assumption, the LHS of (2.11) and (2.12) are thus constant with respect to $s \in [0,1]$. Given two functions $\widetilde{\alpha}$ and $\widetilde{\beta}$ defined in $\mathbb{R}$, one can verify that (2.11) and (2.12) hold if

$$\alpha(t,x) = \tau_3^{-1} \widetilde{\alpha}(t+\tau_3 x) \qquad \text{and} \qquad \beta(t,x) = \tau_3^{-1} \widetilde{\beta}(t+\tau_3 x) \quad \text{for } t \in \mathbb{R},\ x \in [0,1]. \tag{2.13}$$

Using (2.11) and (2.12), one can rewrite (2.10) under the form

$$\begin{aligned} v_3(t+\tau_3, 0) = \int_0^1 \widetilde{\alpha}(t+\tau_3) v_1(t+\tau_3 s, 1-s)\,\mathrm{d}s \\ + \int_0^1 \widetilde{\beta}(t+\tau_3) v_2(t+\tau_3 s, 1-s)\,\mathrm{d}s \end{aligned} \tag{2.14}$$

for $\tau_3 \leqslant t + \tau_3 \leqslant T$. Replacing s by $1 - s$ and $t + \tau_3$ by t, using (2.14), we can rewrite (2.13) under the form

$$v_3(t,0) = \widetilde{\alpha}(t) \int_0^1 v_1(-\tau_3 s + t, s)\,\mathrm{d}s + \widetilde{\beta}(t) \int_0^1 v_2(-\tau_3 s + t, s)\,\mathrm{d}s \tag{2.15}$$

for $t \in (\tau_3, T)$.

We now construct v, α, and β. In what follows, we only consider the solution v of the adjoint system (backward system) satisfying the following condition at time T (final condition):

$$\begin{cases} v_2(T,\cdot\,) = v_3(T,\cdot\,) = 0 & \\ v_1(T,x) = 0 & \text{for } 0 \leqslant x \leqslant \frac{T-\tau_2}{\tau_1}. \end{cases} \tag{2.16}$$

Concerning the value $\frac{T-\tau_2}{\tau_1}$ given above, it is worth noting that the point $\big(\frac{T-\tau_2}{\tau_1}, T\big)$ is on the characteristic flow of v_1 passing the point $(0, \tau_2)$ (see the left figure of Figure 2.1).

It follows from (2.9) and the remark above on the value of $\frac{T-\tau_2}{\tau_1}$ that $v_1(t,0) = 0$ in (τ_2, T) (see the left figure of Figure 2.1). This implies that $v_2(t,0) = 0$ in (τ_2, T) since $v_2(t,0) = \gamma_2 v_1(t,0)$ in $(0,T)$ by (2.7). Combining this with the condition $v_2(T,\cdot\,) = 0$ in $(0,1)$, we derive from the fact $\partial_t v_2 = \lambda_2 \partial_x v_2$ (see (2.9)) that

$$v_2(1,\cdot\,) = 0 \quad \text{in } (0,T) \tag{2.17}$$

(see the left figure of Figure 2.1).

The goal now is to appropriately choose non-zero $v_1(T,\cdot\,) \in C_c^\infty\big(\frac{T-\tau_2}{\tau_1}, 1\big)$ and smooth $\widetilde{\alpha}$ and $\widetilde{\beta}$ such that $v_3(\,\cdot\,,1) = 0$ in $(0,T)$; therefore (2.8) holds by (2.17). It is worth noting that

$$\text{such a solution of the adjoint problem is in } C^\infty\big([0,T] \times [0,1]\big) \tag{2.18}$$

since the compatibility conditions at $(T,0)$ and $(T,1)$ are satisfied for all orders.

Since $\partial_t v_1 = -\lambda_1 \partial_x v_1$ by (2.9) and $v_1(t,0) = \gamma_2^{-1} v_3(t,0)$ by (2.7), it suffices to determine non-zero $v_3(\,\cdot\,,0) \neq 0$ in (τ_3, τ_2) such that $v_3(\,\cdot\,,1) = 0$ in $(0,T)$.

For $t \in (\tau_3, \tau_2)$, let $\gamma_1(t)$ be the abscissa of the intersection of the line passing $(0,t)$ and $(1, t-\tau_3)$ (see the right figure of Figure 2.1). Using the fact that $v_1(t,1) = 0$ for $t \in (0, \tau_1 + \tau_3)$, one has, for $t \in (\tau_3, \tau_2)$ and with $\theta_1 = \frac{1}{\tau_1 + \tau_3}$,

$$\int_0^1 v_1(-\tau_3 s + t, s)\,\mathrm{d}s = \int_0^{\gamma_1(t)} v_1(-\tau_3 s + t, s)\,\mathrm{d}s = \theta_1 \int_{\tau_3}^t v_1(s,0)\,\mathrm{d}s. \tag{2.19}$$

In the last identity, we also used the fact $\partial_t v_1 = -\lambda_1 \partial_x v_1$ by (2.9).

Similarly, for $t \in (\tau_3, \tau_2)$, let $\gamma_2(t)$ be the abscissa of the intersection of the line passing $(0, t)$ and $(1, t - \tau_3)$, and the line passing $(0, \tau_2)$ and $(1, 0)$ (see the right figure of Figure 2.1). We have, with $\theta_2 = \frac{1}{\tau_2 - \tau_3}$,

$$\begin{aligned} \int_0^1 v_2(-\tau_3 s + t, s) \, \mathrm{d}s &= \int_0^{\gamma_2(t)} v_2(-\tau_3 s + t, s) \, \mathrm{d}s \\ &= \theta_2 \int_t^{\tau_2} v_2(s, 0) \, \mathrm{d}s. \end{aligned} \tag{2.20}$$

Using the boundary condition in (2.7), we derive from (2.15) that

$$v_3(t, 0) = \widehat{\alpha}(t) \int_{\tau_3}^t v_3(s, 0) \, \mathrm{d}s + \widehat{\beta}(t) \int_t^{\tau_2} v_3(s, 0) \, \mathrm{d}s \quad \text{for } t \in (\tau_3, \tau_2), \tag{2.21}$$

where

$$\widehat{\alpha} = \gamma_3^{-1} \theta_1 \widetilde{\alpha} \qquad \text{and} \qquad \widehat{\beta} = \gamma_2 \gamma_3^{-1} \theta_2 \widetilde{\beta}. \tag{2.22}$$

Since

$$\tau_1 + \tau_2 - \varepsilon = T,$$

it follows that, for $\varepsilon > 0$ being small enough so that $T > \tau_1 + \tau_3$,

$$\widetilde{I} := (\tau_3, \tau_2) \cap (T - \tau_1, T) = (T - \tau_1, \tau_2) \neq \emptyset.$$

We are ready to choose $v(T, \cdot)$ in $\big(\frac{T - \tau_2}{\tau_1}, 1\big)$, α, and β. Fix $\varphi \in C_c^\infty(\mathbb{R})$ such that

$$\operatorname{supp} \varphi \subset \widetilde{I} \qquad \text{and} \qquad \int_{tI} \varphi = 1. \tag{2.23}$$

Determine $v_1(T, \cdot)$ in $\big(\frac{T - \tau_2}{\tau_1}, 1\big)$ such that

$$v_3(t, 0) = \varphi(t) \quad \text{for } t \in (T - \tau_1, \tau_2) \tag{2.24}$$

via the relations $v_3(t, 0) = \gamma_3 v_1(t, 0)$ and $\partial_t v_1 = -\lambda_1 \partial_x v_1$, and define α and β by (2.13), (2.22), and

$$\widehat{\alpha}(t) = \widehat{\beta}(t) = \varphi(t) \quad \text{for } t \in \mathbb{R}. \tag{2.25}$$

As mentioned in (2.18), the solution v of the adjoint problem imposing (2.16) is in $C^\infty\big([0, T] \times [0, 1]\big)$. By (2.17), it remains to verify that

$$v_3(\cdot, 1) = 0 \quad \text{in } (0, T).$$

Since $v_1(\cdot, 1) = 0$ in $(0, T)$, $v_1(T, x) = 0$ for $x \in \big(0, \frac{T - \tau_2}{\tau_1}\big)$, and $\partial_t v_1 = -\lambda_1 \partial_x v_1$, it follows that $v_1(t, 0) = 0$ for $0 \leqslant t \leqslant T - \tau_1$ and $\tau_2 \leqslant t \leqslant T$ (see the left figure of Figure 2.1). Since $v_3(t, 0) = \gamma_3 v_1(t, 0)$ by (2.7), it follows that

$$v_3(t, 0) = 0 \quad \text{for } 0 \leqslant t \leqslant T - \tau_1 \text{ and } \tau_2 \leqslant t \leqslant T.$$

Combining this with (2.24), one has

$$v_3(t,0) = \varphi(t) \quad \text{in } (0,T) \tag{2.26}$$

since $\operatorname{supp}\varphi \subset \widetilde{I}$ by (2.23).

One can check that (2.15) holds in $(T-\tau_1, \tau_2)$ by (2.21), (2.23), (2.24), and (2.25). Since $\operatorname{supp}\varphi \subset (T-\tau_1,\tau_2)$, both sides of (2.15) are 0 for $t \in (\tau_3, T-\tau_1) \cup (\tau_2, T)$ thanks to (2.22), (2.25) and (2.26). We have thus just proved that (2.15) holds in (τ_3, T). It follows from (2.10) that $v_3(\,\cdot\,,1) = 0$ in $(0, T-\tau_3)$.

Since $\tau_3\alpha(t+\tau_3 s, 1-s) = \widetilde{\alpha}(t+\tau_3)$ and $\tau_3\beta(t+\tau_3 s, 1-s) = \widetilde{\beta}(t+\tau_3)$ for $s \in [0,1]$ by (2.11), it follows from the support condition in (2.23), and (2.25) that

$$\alpha(t+\tau_3 s, 1-s) = 0 = \beta(t+\tau_3 s, 1-s) \quad \text{for } t \geqslant T-\tau_3 \text{ and for } s \in [0,1].$$

Using the characteristic method, one can check from the equation of v_3 in (2.9) (see also (2.10)) that $v_3(\,\cdot\,,1) = 0$ in $(T-\tau_3, T)$.

The proof is complete. □

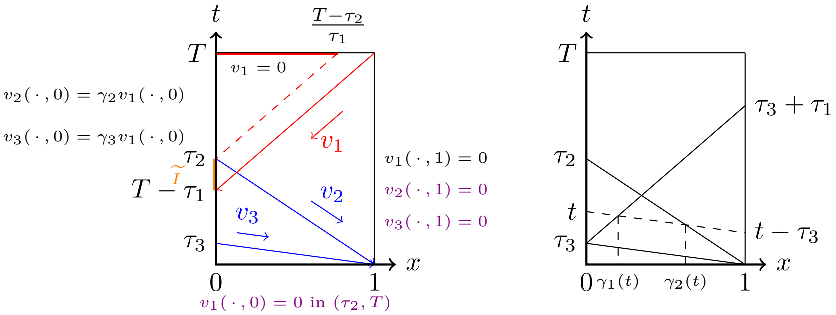

Figure 2.1. Left: geometry of Theorem 1.1 in the case $k = 1$ and $m = 2$. Right: on the definition of γ_1 and γ_2 for $t \in (\tau_3, \tau_2)$: $\gamma_1(t)$ is the abscissa of the intersection of the line passing $(0,t)$ and $(1, t-\tau_3)$, and the line passing $(0,\tau_3)$ and $(1, \tau_3+\tau_1)$; $\gamma_2(t)$ is the abscissa of the intersection of the line passing $(0,t)$ and $(1,t-\tau_3)$, and the line passing $(0,\tau_2)$ and $(1,0)$.

2.2. Proof of Theorem 1.1

The proof of Theorem 1.1 in the general case is in the spirit of the one in the case $k = 1$ and $m = 2$ but more involved. The goal is to construct examples that violate the unique continuation property for the adjoint system (see Lemma 2.1 and Lemma 2.2).

In what follows, we will assume that

$$T \geqslant \max\{T_{\text{opt}}, \tau_k + \tau_{k+\ell}\},$$

where T_{opt} is defined by (1.8); hence ε is assumed to be sufficiently small (note that $\tau_k + \tau_{k+1} > \max\{T_{\text{opt}}, \tau_k + \tau_{k+\ell}\}$ since $2 \leqslant \ell \leqslant m$). We will consider the coefficient $C(t,x)$ satisfying the following structure:

$$C_{i,j}(t,x) = \begin{cases} -\alpha(t,x) & \text{if } (i,j) = (k, k+\ell), \\ -\beta(t,x) & \text{if } (i,j) = (k+1, k+\ell), \\ 0 & \text{otherwise}, \end{cases} \tag{2.27}$$

where α and β are two smooth functions defined later.

Since Σ is constant and C satisfies (2.27), concerning the adjoint system, system (2.1) is equivalent to, for $(t,x) \in (0,T) \times (0,1)$,

$$\partial_t v_j(t,x) = \Sigma_{j,j} \partial_x v_j(t,x) \quad \text{if } 1 \leqslant j \leqslant n \text{ with } j \neq k+\ell, \tag{2.28}$$

and

$$\partial_t v_{k+\ell}(t,x) = \lambda_{k+\ell} \partial_x v_{k+\ell}(t,x) + \alpha(t,x) v_k(t,x) + \beta(t,x) v_{k+1}(t,x) \tag{2.29}$$

($\Sigma_{j,j} = -\lambda_j$ if $1 \leqslant j \leqslant k$ and $\Sigma_{j,j} = \lambda_j$ otherwise).

Under appropriate choices of α and β determined later, we will construct a smooth solution v of the adjoint system (2.28), (2.29), (2.30), and (2.31), where

$$v_-(t,1) = 0 \quad \text{for } t \in [0,T], \tag{2.30}$$

$$\Sigma_+(0) v_+(t,0) = -B^{\mathsf{T}} \Sigma_-(0) v_-(t,0) \quad \text{for } t \in [0,T]. \tag{2.31}$$

Moreover, we require that v satisfies the following *additional* conditions:

$$v_+(\,\cdot\,,1) = 0 \quad \text{in } (0,T) \qquad \text{and} \qquad v(0,\cdot\,) \not\equiv 0 \quad \text{in } (0,1). \tag{2.32}$$

As a consequence, the unique continuation property does not hold for the adjoint system. By Lemma 2.2, the conclusion of Theorem 1.1 follows.

We now concentrate on this construction. To this end, we first derive the constraints on v, α, and β. From (2.29), we have, for $\tau_{k+\ell} \leqslant t + \tau_{k+\ell} \leqslant T$ and $0 \leqslant s \leqslant 1$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\big(v_{k+\ell}(t + \tau_{k+\ell}s, 1-s)\big) &= \tau_{k+\ell}\alpha(t + \tau_{k+\ell}s, 1-s) v_k(t + \tau_{k+\ell}s, 1-s) \\ &\quad + \tau_{k+\ell}\beta(t + \tau_{k+\ell}s, 1-s) v_{k+1}(t + \tau_{k+\ell}s, 1-s). \end{aligned}$$

This implies, for $\tau_{k+\ell} \leqslant t + \tau_{k+\ell} \leqslant T$,

$$\begin{aligned} &v_{k+\ell}(t+\tau_{k+\ell}, 0) \\ &\quad = \int_0^1 \tau_{k+\ell}\alpha(t+\tau_{k+\ell}s, 1-s) v_k(t+\tau_{k+\ell}s, 1-s)\,\mathrm{d}s \\ &\quad\quad + \int_0^1 \tau_{k+\ell}\beta(t+\tau_{k+\ell}s, 1-s) v_{k+1}(t+\tau_{k+\ell}s, 1-s)\,\mathrm{d}s \\ &\quad\quad + v_{k+\ell}(t, 1). \end{aligned} \tag{2.33}$$

It follows that *if* $v_{k+\ell}(t, 1) = 0$ for $\tau_{k+\ell} \leqslant t + \tau_{k+\ell} \leqslant T$, then

$$\begin{aligned} &v_{k+\ell}(t+\tau_{k+\ell}, 0) \\ &\quad = \int_0^1 \tau_{k+\ell}\alpha(t+\tau_{k+\ell}s, 1-s) v_k(t+\tau_{k+\ell}s, 1-s)\,\mathrm{d}s \\ &\quad\quad + \int_0^1 \tau_{k+\ell}\beta(t+\tau_{k+\ell}s, 1-s) v_{k+1}(t+\tau_{k+\ell}s, 1-s)\,\mathrm{d}s \end{aligned} \tag{2.34}$$

for $\tau_{k+\ell} \leqslant t + \tau_{k+\ell} \leqslant T$.

We will make the following assumption on the structure of α and β:

$$\tau_{k+\ell}\alpha(t+\tau_{k+\ell}s, 1-s) = \widetilde{\alpha}(t+\tau_{k+\ell}) \quad \text{for } s \in [0,1],\ t \in [0,T], \tag{2.35}$$

and

$$\tau_{k+\ell}\beta(t+\tau_{k+\ell}s, 1-s) = \widetilde{\beta}(t+\tau_{k+\ell}) \quad \text{for } s \in [0,1],\ t \in [0,T] \tag{2.36}$$

for some functions $\widetilde{\alpha}$ and $\widetilde{\beta}$ constructed later. Given $\widetilde{\alpha}$ and $\widetilde{\beta}$ defined in $\mathbb{R}$, one can verify that (2.35) and (2.36) hold if

$$\alpha(t,x) = \tau_{k+\ell}^{-1}\widetilde{\alpha}(t+\tau_{k+\ell}x) \qquad \text{and} \qquad \beta(t,x) = \tau_{k+\ell}^{-1}\widetilde{\beta}(t+\tau_{k+\ell}x). \tag{2.37}$$

Under conditions (2.35) and (2.36), by replacing first s by $1-s$ and then $t+\tau_{k+\ell}$ by t, identity (2.34) can be then written as, for $t \in (\tau_{k+\ell}, T)$,

$$v_{k+\ell}(t,0) = \widetilde{\alpha}(t)\int_0^1 v_k(-\tau_{k+\ell}s+t, s)\,\mathrm{d}s + \widetilde{\beta}(t)\int_0^1 v_{k+1}(-\tau_{k+\ell}s+t, s)\,\mathrm{d}s. \tag{2.38}$$

We write (2.31) as

$$v_+(t,0) = -\Sigma_+^{-1}B^{\mathsf{T}}\Sigma_- v_-(t,0) \quad \text{for } t \in [0,T]. \tag{2.39}$$

In what follows, we consider the solution v of the adjoint system (2.28), (2.29), (2.30), and (2.31) (backward system) which satisfies the following condition at time T (final condition):

$$v_1(T,\cdot) = \ldots = v_{k-1}(T,\cdot) = v_{k+1}(T,\cdot) = \ldots = v_{k+m}(T,\cdot) = 0, \tag{2.40}$$

and

$$v_k(T, x) = 0 \quad \text{for } 0 \leqslant x \leqslant \frac{T - \tau_{k+1}}{\tau_k} < 1 \text{ since } T < \tau_k + \tau_{k+1}. \tag{2.41}$$

Concerning the value $\frac{T-\tau_{k+1}}{\tau_k}$, it is worth noting that the characteristic flow of v_k passing the point $(0, \tau_{k+1})$ will pass the point $\big(\frac{T-\tau_{k+1}}{\tau_k}, T\big)$ (see Figure 2.2).

From the system of v (2.28), (2.29), (2.30), and (2.31), the solution v is then uniquely determined by the value of $v_k(T, x)$ for $\frac{T-\tau_{k+1}}{\tau_k} < x \leqslant 1$.

As in the case $m = 2$ and $k = 1$, we have

$$v_1(t, 0) = \ldots = v_{k-1}(t, 0) = 0 \quad \text{for } t \in (0, T). \tag{2.42}$$

We then derive from (2.39) and (2.42) that

$$v_{k+1}(t, 0) = \gamma_{k+1} v_k(t, 0) \qquad \text{and} \qquad v_{k+\ell}(t, 0) = \gamma_{k+\ell} v_k(t, 0) \quad \text{for } t \in (0, T), \tag{2.43}$$

where

$$\gamma_{k+1} := \lambda_{k+1}^{-1} \lambda_k B_{k,1} \overset{(1.12)}{\neq} 0 \qquad \text{and} \qquad \gamma_{k+\ell} := \lambda_{k+\ell}^{-1} \lambda_k B_{k,\ell} \overset{(1.12)}{\neq} 0. \tag{2.44}$$

Using (2.42), (2.39), and the last condition in (1.12), we have

$$v_{k+j}(\,\cdot\,, 0) = 0 \quad \text{in } (0, T) \text{ for } 2 \leqslant j \leqslant m \text{ with } j \neq \ell. \tag{2.45}$$

Combining (2.45) with the fact that $v_{k+j}(T, \cdot\,) = 0$ for $2 \leqslant j \leqslant m$ with $j \neq \ell$ by (2.40), and using the equation of v_{k+j} for $2 \leqslant j \leqslant m$ with $j \neq \ell$ in (2.28), we reach

$$v_{k+j}(t, 1) = 0 \quad \text{in } (0, T) \text{ for } 2 \leqslant j \leqslant m \text{ with } j \neq \ell. \tag{2.46}$$

We next construct v_k in $\big(\frac{T-\tau_{k+1}}{\tau_k}, 1\big)$ such that $v_k \in C^\infty\big([0, 1]\big)$ with compact support in $\big(\frac{T-\tau_{k+1}}{\tau_k}, 1\big)$ and (2.32) holds. From (2.46), it suffices to check

$$v_{k+1}(t, 1) = v_{k+\ell}(t, 1) = 0 \quad \text{in } (0, T). \tag{2.47}$$

As in the case $m = 2$ and $k = 1$, one has, for $t \in (\tau_{k+\ell}, \tau_{k+1})$,

$$\int_0^1 v_k(-\tau_{k+\ell} s + t, s) \, \mathrm{d}s = \theta_k \int_{\tau_{k+\ell}}^t v_k(s, 0) \, \mathrm{d}s, \tag{2.48}$$

and

$$\int_0^1 v_{k+1}(-\tau_{k+\ell} s + t, s) \, \mathrm{d}s = \theta_{k+1} \int_t^{\tau_{k+1}} v_{k+1}(s, 0) \, \mathrm{d}s, \tag{2.49}$$

where

$$\theta_k = \frac{1}{\tau_k + \tau_{k+\ell}} \qquad \text{and} \qquad \theta_{k+1} = \frac{1}{\tau_{k+1} - \tau_{k+\ell}}. \tag{2.50}$$

Using (2.48) and (2.49), we derive from (2.38), after taking into account (2.43) that, for $t \in (\tau_{k+\ell}, \tau_{k+1})$

$$v_{k+\ell}(t,0) = \widehat{\alpha}(t) \int_{\tau_{k+\ell}}^{t} v_{k+\ell}(s,0)\,\mathrm{d}s + \widehat{\beta}(t) \int_{t}^{\tau_{k+1}} v_{k+\ell}(s,0)\,\mathrm{d}s, \tag{2.51}$$

where

$$\widehat{\alpha} = \gamma_{k+\ell}^{-1}\theta_k \widetilde{\alpha} \qquad \text{and} \qquad \widehat{\beta} = \gamma_{k+1}\gamma_{k+\ell}^{-1}\theta_{k+1}\widetilde{\beta}. \tag{2.52}$$

Since

$$\tau_k + \tau_{k+1} - \varepsilon \overset{(1.13)}{=} T,$$

it follows that, at least if $\varepsilon > 0$ is small enough so that $T > \tau_{k+\ell}$,

$$\widetilde{I} \coloneqq (\tau_{k+\ell}, \tau_{k+1}) \cap (T - \tau_k, T) = (T - \tau_k, \tau_{k+1}) \neq \emptyset.$$

We are ready to construct the example. Fix $\varphi \in C_c^\infty(\mathbb{R})$ such that

$$\operatorname{supp} \varphi \subset \widetilde{I} \qquad \text{and} \qquad \int_{\widetilde{I}} \varphi = 1. \tag{2.53}$$

Consider v at the time T given by (2.40) and (2.41), and $v_k(T,\cdot)$ in $\left(\frac{T-\tau_{k+1}}{\tau_k}, 1\right)$ being determined via the relations $v_{k+\ell}(t,0) = \gamma_{k+\ell} v_k(t,0)$ and $\partial_t v_1 = -\lambda_1 \partial_x v_1$ such that

$$v_{k+\ell}(t,0) = \varphi(t) \quad \text{for } t \in (T - \tau_k, \tau_{k+1}). \tag{2.54}$$

Set

$$\widehat{\alpha}(t) = \widehat{\beta}(t) = \varphi(t) \quad \text{for } t \in \mathbb{R}. \tag{2.55}$$

The function α and β are then defined by (2.37) and (2.52).

As mentioned previously, it suffices to check (2.47).

It is clear that (2.51) holds for $t \in (T - \tau_k, \tau_{k+1})$. Since $\partial_t v_k = -\lambda_k v_k$ by (2.28), $v_k(T,\cdot) = 0$ for $0 \leqslant x \leqslant \frac{T-\tau_{k+1}}{\tau_k}$ and $v_k(t,1) = 0$ in $(0,T)$ by (2.30), we derive that $v_k(t,0) = 0$ for $0 \leqslant t \leqslant T - \tau_k$ and for $\tau_{k+1} \leqslant t \leqslant T$ (see Figure 2.2). It follows then from (2.43) that $v_{k+l}(t,0) = 0$ for $0 \leqslant t \leqslant T - \tau_k$ and for $\tau_{k+1} \leqslant t \leqslant T$. Combining this with the support condition of φ and (2.54), one obtains

$$v_{k+l}(t,0) = \varphi(t) \quad \text{in } (0,T). \tag{2.56}$$

One can also check that (2.51) holds for $t \in (\tau_{k+\ell}, \tau_{k+1})$ by (2.53) and (2.56). This implies (2.38) holds for $t \in (\tau_{k+\ell}, \tau_{k+1})$. On the other hand, for $\tau_{k+1} \leqslant t \leqslant T$, both sides of (2.38) are 0 by (2.56) and (2.55). We have just proved that

$$v_{k+l}(t,1) = 0 \quad \text{for } 0 \leqslant t \leqslant T - \tau_{k+\ell}. \tag{2.57}$$

Using (2.55), the equation of $v_{k+\ell}$ in (2.29), as in the proof in the case $k=1$ and $m=2$, one has

$$v_{k+l}(t,1) = 0 \quad \text{for } T - \tau_{k+\ell} \leqslant t \leqslant T. \tag{2.58}$$

Combining (2.57) and (2.58), we obtain

$$v_{k+l}(t,1) = 0 \quad \text{for } t \in (0,T). \tag{2.59}$$

It remains to check $v_{k+1}(t,1)=0$ in $(0,T)$. Indeed, from (2.43) and (2.56), one derives that $v_{k+1}(t,0) = 0$ for $\tau_{k+1} \leqslant t \leqslant T$ (see Figure 2.2). Since $v_{k+1}(T,\cdot) = 0$ in $(0,1)$ by (2.40), it follows since $\partial_t v_{k+1} = \lambda_{k+1} \partial_x v_{k+1}$ by (2.28) that $v_{k+1}(t,1) = 0$ for $t \in (0,T)$. □

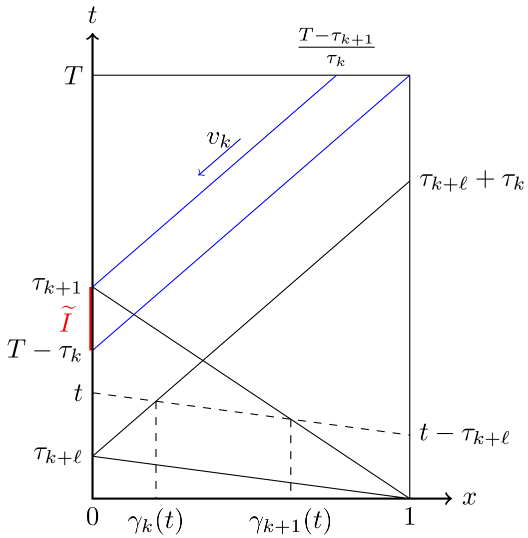

Figure 2.2. On the definition of γ_k and γ_{k+1} for $t \in (\tau_{k+\ell}, \tau_{k+1})$: $\gamma_k(t)$ is the abscissa of the intersection of the line passing $(0,t)$ and $(1, t - \tau_{k+\ell})$, and the line passing $(0, \tau_{k+\ell})$ and $(1, \tau_{k+\ell} + \tau_k)$; $\gamma_{k+1}(t)$ is the abscissa of the intersection of the line passing $(0,t)$ and $(1, t - \tau_{k+\ell})$, and the line passing $(0, \tau_{k+1})$ and $(1, 0)$.

3. Null-controllability in the analytic setting — Proof of Theorem 1.5

This section is devoted to the proof of Theorem 1.5. Recall that T_{opt} is defined by (1.8) and T_1 is defined by (1.16). The proof is divided into three steps described below.

Step 1: For each τ such that $[\tau, \tau + T_{\text{opt}}] \subset I$, we introduce/characterize the space $H(\tau)$ ($\subset \left[L^2(0,1)\right]^n$), which is of finite dimension, for which (i) one can steer[(3)] any element in $H(\tau)^\perp$ at time τ to 0 in time T_{opt} and (ii) one cannot steer any element in $H(\tau) \setminus \{0\}$ at time τ to 0 in time T_{opt} (see Proposition 3.6 and Proposition 3.7). Moreover, we show that $H(\cdot)$ is analytic in a neighborhood I_1 of $[0, T_1 - T_{\text{opt}}]$ except for a discrete subset, which is removable (see Lemma 3.8).[(4)]

Step 2: For each $\tau \in I_1$ a neighborhood of $[0, T_1 - T_{\text{opt}}]$, we introduce/characterize the subspace $J(\tau)$ of $H(\tau)$ (see Lemma 3.12) for which (i) one can steer every element φ in $J(\tau)$ from time τ to 0 in time $T_{\text{opt},+}$, i.e., in time $T_{\text{opt}} + \delta$ for all $\delta > 0$ and (ii) with $M(\tau)$ being the orthogonal complement of $J(\tau)$ in $H(\tau)$, one cannot steer every element φ in $M(\tau) \setminus \{0\}$ from time τ to 0 in time $T_{\text{opt},+}$. We also show that there exists a constant ε_0 such that, roughly speaking, the following property holds: if $\tau \in I_1$ and $\varphi \in M(\tau)\setminus\{0\}$, then one *cannot* steer φ from time τ to 0 in time $T_{\text{opt}} + \varepsilon_0$ (see Proposition 3.17).

Step 3: We give the proof of Theorem 1.5 using Steps 1 and 2.

Let us make some comments on these three steps before proceeding them. Concerning Step 1, the fact that $H(\tau)$ is of finite dimension already appeared in our previous analysis [18]. Some necessary conditions on $H(\tau)$ are derived in [18] and the starting point of the analysis there is the backstepping technique. In this paper, the (complete) characterization of $H(\tau)$ is given and it plays a crucial role in our proof of Theorem 1.5. This characterization can be obtained by first applying the backstepping technique (and then by using similar ideas given here). However, this way requires a quite strong assumption on the analyticity of C in the step of using the backstepping technique (see Remark 3.19). To avoid it, we implement a new

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(3) Here and in what follows, for a closed subspace E of $[L^2(0,1)]^n$, we denote Prof_E the projection to E, and $E^\perp$ its orthogonal complement, both with respect to the standard $L^2(0,1)$-scalar product.

(4) The analyticity of $H(\tau)$ is understood via the analyticity of the mapping $\text{Prof}_{H(\tau)}$. This convention is used throughout the paper.

approach applied directly to the original system. The analysis is though strongly inspired/guided by our understanding in the form obtained via the backstepping. A part of the technical points in this step is to establish the well-posedness of hyperbolic equations with *unusual* boundary conditions (the boundary condition of a component can be given both on the left at $x = 0$ for some interval of time and on the right at $x = 1$ for some other interval of time), and in a domain which is not necessary to be a rectangle in xt plane. The analysis is interesting but delicate and is presented in the appendix. After characterizing $H(\cdot)$, the analyticity of $H(\cdot)$ is established by suitably applying the theory of perturbations of analytic compact operators, see, e.g., [32]. These results are given in Section 3.1. Concerning Steps 1 and 2, the characterizations of all states for which one can steer from time τ to 0 in time $T_{\rm opt}$ or in time $T_{{\rm opt},+}$ can be done for $C \in \big[L^\infty\big(I \times (0,1)\big)\big]^{n\times n}$. The analyticity of C is not required for this purpose. It is in the proof of the existence of ε_0, given in Step 2, that the analyticity of C plays a crucial role. The approach proposed in this paper is quite robust and might be applied to other contexts.

The rest of this section containing four subsections is organized as follows. In the first subsection, we introduce notations and present preliminary results related to observability inequalities, which are the starting point of our analysis. Steps 1, 2, and 3 are then given in the second, third, and fourth subsections, respectively.

In what follows in this section, I denotes an open interval containing $[0, T_1]$ where $T_1 = \tau_k + \tau_{k+1}$ is given in (1.16). Throughout this section, we assume that Σ verifies (1.2), (1.3), (1.4), and $C \in \big(L^\infty(I \times [0,1])\big)^{n\times n}$ is real.

3.1. Preliminaries

Fix $\tau \in I$ and $T > 0$ such that $[\tau, \tau + T] \subset I$. Define

$$\mathcal{F}_{\tau,T} \colon \big[L^2(\tau, \tau+T)\big]^m \longrightarrow \big[L^2(0,1)\big]^n$$
$$U \longmapsto u(\tau + T, \cdot),$$

where u is the unique solution of the system

$$\partial_t u(t,x) = \Sigma(x)\partial_x u(t,x) + C(t,x)u(t,x) \quad \text{for } (t,x) \in (\tau, \tau+T) \times (0,1), \tag{3.1}$$

$$u_-(t,0) = Bu_+(t,0) \quad \text{for } t \in (\tau, \tau+T), \tag{3.2}$$

$$u_+(t,1) = U(t) \quad \text{for } t \in (\tau, \tau+T), \tag{3.3}$$

$$u(t=\tau,\cdot\,)=0 \quad \text{in } (0,1). \tag{3.4}$$

Set, for $(t,x)\in I\times(0,1)$,

$$\mathbf{C}(t,x)=\Sigma'(x)-C^{\mathsf{T}}(t,x). \tag{3.5}$$

The following result, which is Lemma 2.1 translated in time, provides the formula for the adjoint $\mathcal{F}^*_{\tau,T}$ of $\mathcal{F}_{\tau,T}$.

LEMMA 3.1. — *Let $\tau\in I$ and $T>0$ such that $[\tau,\tau+T]\subset I$. We have, for $\varphi\in\left[L^2(0,1)\right]^n$,*

$$\mathcal{F}^*_{\tau,T}(\varphi)(\,\cdot\,)=\Sigma_+(1)v_+(\,\cdot\,,1) \quad \textit{in } (\tau,\tau+T),$$

where v is the unique broad solution of the system

$$\partial_t v(t,x)=\Sigma(x)\partial_x v(t,x)+\mathbf{C}(t,x)v(t,x) \quad \textit{for } (t,x)\in(\tau,\tau+T)\times(0,1), \tag{3.6}$$

with, for $0<t<T$,

$$v_-(t,1)=0, \tag{3.7}$$

$$\Sigma_+(0)v_+(t,0)=-B^{\mathsf{T}}\Sigma_-(0)v_-(t,0), \tag{3.8}$$

and

$$v(\tau+T,\cdot\,)=\varphi \quad \textit{in } (0,1). \tag{3.9}$$

Using the same method, we also obtain the following two results, see, e.g., the proof of [18, Lemma 2] for the analysis.

LEMMA 3.2. — *Let $\tau\in I$ and $T>0$ such that $[\tau,\tau+T]\subset I$. Assume that u is a broad solution of (3.1)–(3.3) such that $u_+(\,\cdot\,,1)=0$ in $(\tau,\tau+T)$. Then, for $\varphi\in\left[L^2(0,1)\right]^n$, we have*(5)

$$\int_0^1\big\langle u(\tau+T,x),v(\tau+T,x)\big\rangle\,\mathrm{d}x=\int_0^1\big\langle u(\tau,x),v(\tau,x)\big\rangle\,\mathrm{d}x,$$

where v is a solution of (3.6)–(3.9).

LEMMA 3.3. — *Let $\tau\in I$ and $T>0$ such that $[\tau,\tau+T]\subset I$. Assume that u is a broad solution of (3.1)–(3.3). Then*

$$\int_0^1\big\langle u(\tau+T,x),v(\tau+T,x)\big\rangle\,\mathrm{d}x=\int_0^1\big\langle u(\tau,x),v(\tau,x)\big\rangle\,\mathrm{d}x,$$

where v is a solution of (3.6)–(3.8) satisfying $v_+(\,\cdot\,,1)=0$.

---

(5) The notation $\langle\,\cdot\,,\cdot\,\rangle$ stands for the Euclidean scalar product in $\mathbb{R}^\ell$ for $\ell\geqslant 1$.

Applying the Hilbert uniqueness method, see e.g. [9, Chapter 2] and [38], we have the following result.

Lemma 3.4. — *Let E be a closed subspace of $\big[L^2(0,1)\big]^n$. System* (3.1)–(3.3) *is null-controllable at the time $\tau + T$ for initial datum at time τ in E if and only if, for some positive constant $C_{\tau,T}$,*

$$\int_{\tau}^{\tau+T} \big|v_+(t,1)\big|^2 \,\mathrm{d}t \geqslant C_{\tau,T} \int_0^1 \big|\mathrm{Proj}_E\, v(\tau,x)\big|^2 \,\mathrm{d}x \quad \forall\, \varphi \in \big[L^2(0,1)\big]^n, \tag{3.10}$$

where v is the broad solution of the adjoint system (3.6)–(3.9).

3.2. Characterization of states at time τ steered to 0 in time T_{opt}

In what follows in Section 3,

$$\text{denote } I = (\alpha,\beta) \text{ and set } I_1 = (\alpha, \beta - T_{\mathrm{opt}}). \tag{3.11}$$

Recall that I is an open interval containing $[0, T_1]$ where $T_1 = \tau_k + \tau_{k+1}$ given in (1.16). It is clear that if $\tau \in I_1$ then $\tau + T_{\mathrm{opt}} \in I$.

We first characterize states which can be steered at time τ to 0 in time $T \geqslant T_{\mathrm{opt}}$, i.e., the corresponding solution is 0 at time $\tau + T$. To this end, we first introduce the space $H(\tau, T)$.

Definition 3.5. — *Let $k \geqslant m \geqslant 1$ and let $B \in \mathcal{B}$ be such that* (1.10) *holds for $i = m$. Let $\tau \in I$ and $T > 0$ be such that $\tau + T \in I$. Let $H(\tau,T)$ be the set of all $\varphi \in \big[L^2(0,1)\big]^n$ such that there exists a broad solution v of the system*

$$\partial_t v(t,x) = \Sigma(x)\partial_x v(t,x) + \mathbf{C}(t,x)v(t,x) \quad \text{for } (t,x) \in (\tau, \tau+T)\times(0,1), \tag{3.12}$$

with, for $\tau < t < \tau + T$,

$$v_-(t,1) = 0, \tag{3.13}$$

$$\Sigma_+(0)v_+(t,0) = -B^{\mathsf{T}}\Sigma_-(0)v_-(t,0), \tag{3.14}$$

$$v_+(t,1) = 0, \tag{3.15}$$

and

$$v(\tau, \cdot\,) = \varphi. \tag{3.16}$$

Recall that $\mathbf{C}$ is defined by $\mathbf{C}(t,x) = \Sigma'(x) - C^{\mathsf{T}}(t,x)$, see (3.5).

We will show later that in the case $T \geqslant T_{\mathrm{opt}}$, $H(\tau,T)$ characterizes the space which cannot steer to 0 in time T (see (i) and (ii) of Proposition 3.7). In what follows, we denote, for notational ease,

$$H(\tau) := H(\tau, T_{\mathrm{opt}}). \tag{3.17}$$

We have the following result concerning $H(\tau)$.

Proposition 3.6. — *Let $k \geqslant m \geqslant 1$ and let $B \in \mathcal{B}$ be such that* (1.10) *holds for $i = m$. Let I be an open interval containing $[0,T_1]$ and assume that $C \in \left[L^\infty\big(I \times (0,1)\big)\right]^{n\times n}$. There exist:*

- *a compact operator $\mathcal{K}(\tau)\colon \left[L^2(0,1)\right]^n \to \left[L^2(0,1)\right]^n$, and*
- *a continuous linear operator*

$$\mathcal{L}(\tau)\colon \left[L^2(0,1)\right]^n \longrightarrow \left[L^2(0,T_{\mathrm{opt}} - \tau_{k-m+1})\right]^m,$$

defined for $\tau \in I_1$ such that they are uniformly bounded in I_1 and

$$H(\tau) = \left\{\varphi \in \left[L^2(0,1)\right]^n;\ \varphi + \mathcal{K}(\tau)\varphi = 0 \text{ and } \mathcal{L}(\tau)\varphi = 0\right\}. \tag{3.18}$$

Assume in addition that $C \in \mathcal{H}\big(I, \left[L^\infty(0,1)\right]^{n\times n}\big)$. Then $\mathcal{K}$ and $\mathcal{L}$ are analytic in I_1.

Recall that I_1 is defined in (3.11).

Using essentially the compactness of $\mathcal{K}(\tau)$, we can derive the null-controllability property of $H(\tau)$ for the system (3.1)–(3.3). For later use, we state this in a slightly more general version.

Proposition 3.7. — *Let $k \geqslant m \geqslant 1$ and let $B \in \mathcal{B}$ be such that* (1.10) *holds for $i = m$. Let I be an open interval containing $[0,T_1]$, and let $\tau \in I$ and $T \geqslant T_{\mathrm{opt}}$ be such that $\tau + T \in I$. Then the following two facts, concerning system* (3.1)–(3.3)*, hold:*

(i) *one can steer $\varphi \in H(\tau,T)^\perp$ at time τ to 0 at time $\tau + T$;*
(ii) *one cannot steer any element φ in $H(\tau,T) \setminus \{0\}$ at time τ to 0 at time $\tau + T$.*

Assume that the assumptions in Theorem 1.6 hold. Let $\tau \in I$ and $T > T_{\mathrm{opt}}$. Assume that $\tau + T_1 \in I$. We later prove that $H(\tau,T) = \{0\}$ (see Proposition 3.18) which is the unique continuation principle corresponding to (3.12)–(3.15).

As a consequence of Proposition 3.6 and the theory of analytic compact operators, see, e.g., [32], we can prove the following result.

Lemma 3.8. — *Let I be an open interval containing $[0, T_1]$ and assume that $C \in \mathcal{H}\big(I, \big[L^\infty(0,1)\big]^{n\times n}\big)$. Then $H(\tau)$ is analytic in I_1, where I_1 is defined by* (3.11)*, except for a discrete set, which is removable.*[6]

The proofs of Propositions 3.6 and 3.7, and Lemma 3.8 are given in the next three subsections, respectively.

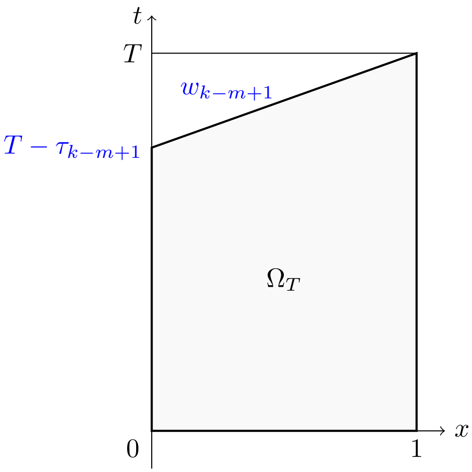

Figure 3.1. Geometry of the set Ω_T when Σ is constant.

Before entering the details of the proofs, we introduce some notations which are used several times later on. We first deal with the characteristic flows. Extend λ_i in $\mathbb{R}$ with $1 \leqslant i \leqslant k+m$ by $\lambda_i(0)$ for $x < 0$ and $\lambda_i(1)$ for $x > 1$. For $(s, \xi) \in [0, +\infty) \times [0,1]$, define $x_i(t, s, \xi)$ for $t \in \mathbb{R}$ by

$$\frac{\mathrm{d}}{\mathrm{d}t} x_i(t,s,\xi) = \lambda_i\big(x_i(t,s,\xi)\big) \quad \text{and} \quad x_i(s,s,\xi) = \xi \qquad \text{if } 1 \leqslant i \leqslant k, \tag{3.19}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} x_i(t,s,\xi) = -\lambda_i\big(x_i(t,s,\xi)\big) \quad \text{and} \quad x_i(s,s,\xi) = \xi \qquad \text{if } k+1 \leqslant i \leqslant k+m. \tag{3.20}$$

[6] Recall that the analyticity of $H(\,\cdot\,)$ means the analyticity of $\mathrm{Proj}_{H(\,\cdot\,)}$.

Let us define Ω_T by

$$\begin{aligned}&\Omega_T \text{ is the region of points } (t,x) \in (0,+\infty) \times (0,1) \text{ such that}\\ &\text{in the } xt\text{-plane they are below the characteristic flow}\\ &\text{of } v_{k-m+1} \text{ passing the point } (1,T)\end{aligned} \tag{3.21}$$

(see Figure 3.1). For simplicity of the notation, we also denote

$$\Omega = \Omega_{T_{\mathrm{opt}}}. \tag{3.22}$$

3.2.1. Proof of Proposition 3.6

Fix $\tau \in I_1$. Let v be a broad solution of the adjoint system

$$\partial_t v(t,x) = \Sigma(x)\partial_x v(t,x) + \mathbf{C}(t+\tau,x)v(t,x) \quad \text{for } (t,x) \in (0,T_{\mathrm{opt}}) \times (0,1), \tag{3.23}$$

with, for $0 < t < T_{\mathrm{opt}}$,

$$v_-(t,1) = 0, \tag{3.24}$$

$$\Sigma_+(0)v_+(t,0) = -B^{\mathsf{T}}\Sigma_-(0)v_-(t,0), \tag{3.25}$$

such that v also satisfies

$$v_+(t,1) = 0 \quad \text{for } t \in (0,T_{\mathrm{opt}}). \tag{3.26}$$

Recall that $\mathbf{C}$ is defined in (3.5).

For $1 \leqslant i \leqslant k \leqslant j \leqslant k+m$, we denote, for a vector $v \in \mathbb{R}^{k+m}$,

$$v_{-,\geqslant i} = (v_i, \dots, v_k)$$

and

$$v_{<i,\geqslant j} = (v_1, \dots, v_{i-1}, v_j, \dots, v_{k+m}).$$

Using condition (1.10) with $i = 1$, one can write the last equation of (3.25) in an equivalent form:

$$v_{-,\geqslant k}(t,0) = Q_k v_{<k,\geqslant k+m}(t,0), \tag{3.27}$$

for some $1 \times k$ matrix Q_k.

Using condition (1.10) with $i = 2$, one can write the last two equations of (3.25) in an equivalent form:

$$v_{-,\geqslant k-1}(t,0) = Q_{k-1} v_{<k-1,\geqslant k+m-1}(t,0), \tag{3.28}$$

for some $2 \times k$ matrix Q_{k-1}.

...

Using condition (1.10) with $i = m-1$, one can write the last $(m-1)$ equations of (3.25) in an equivalent form:

$$v_{-,\geqslant k-m+2}(t,0) = Q_{k-m+2} v_{<k-m+2,\geqslant k+2}(t,0), \tag{3.29}$$

for some $(m-1)\times k$ matrix Q_{k-m+2}.

Using condition (1.10) with $i = m$, one can write the last m equations of (3.25) in an equivalent form:

$$v_{-,\geqslant k-m+1}(t,0) = Q_{k-m+1} v_{<k-m+1,\geqslant k+1}(t,0), \tag{3.30}$$

for some $m \times k$ matrix Q_{k-m+1}.

Given $f \in \left[L^2(0,T_{\rm opt})\right]^n$ and $g \in \left[L^2(0,1)\right]^m$, we consider the system

$$w_t(t,x) = \Sigma(x)\partial_x w(t,x) + \mathbf{C}(t+\tau,x)w(t,x) \quad \text{for } (t,x)\in\Omega, \tag{3.31}$$

$$w(\,\cdot\,,1) = f \quad \text{in } (0,T_{\rm opt}), \tag{3.32}$$

$$w_+(0,\cdot\,) = g \quad \text{in } (0,1), \tag{3.33}$$

$$w_{-,\geqslant k}(t,0) = Q_k w_{<k,\geqslant k+m}(t,0) \quad \text{for } t\in(T_{\rm opt}-\tau_k, T_{\rm opt}-\tau_{k-1}), \tag{3.34}$$

$$w_{-,\geqslant k-1}(t,0) = Q_{k-1} w_{<k-1,\geqslant k+m-1}(t,0) \quad \text{for } t\in(T_{\rm opt}-\tau_{k-1}, T_{\rm opt}-\tau_{k-2}), \tag{3.35}$$

$$\dots$$

$$w_{-,\geqslant k-m+2}(t,0) = Q_{k-m+2} w_{<k-m+2,\geqslant k+2}(t,0) \quad \text{for } t\in(T_{\rm opt}-\tau_{k-m+2}, T_{\rm opt}-\tau_{k-m+1}), \tag{3.36}$$

(see Figure 3.2). Recall that $\Omega = \Omega_{T_{\rm opt}}$ where Ω_T is defined in (3.21).

For $\tau \in I_1$, define

$$\begin{aligned} \mathcal{T}(\tau)\colon \left[L^2(0,T)\right]^n \times \left[L^2(0,1)\right]^m &\longrightarrow \left[L^2(\Omega)\right]^n \\ (f,g) &\longmapsto w, \end{aligned} \tag{3.37}$$

where w is the broad solution of (3.31)–(3.36) (see Definition A.1 for the definition of broad solutions and Theorem A.2 for their existence and uniqueness, both in the appendix).

We now introduce the operators $\mathcal{K}$ and $\mathcal{L}$. Set

$$w = \mathcal{T}(\tau)\big(0, v_+(0,\cdot\,)\big),$$

where v is a broad solution of (3.23)–(3.26). The definitions of $\mathcal{K}(\tau)$ and $\mathcal{L}(\tau)$ are based on the fact that $v = w$ in Ω and the fact

$$v_+(t,0) = -\Sigma_+(0)B^{\mathsf{T}}\Sigma_-(0)v_-(t,0) \quad \text{in } (0,T_{\rm opt}-\tau_{k-m+1}) \tag{3.38}$$

by (3.25). Moreover, in the definition of $\mathcal{T}(\tau)$, we only use the last equation of (3.38) in (3.34), the last two equations of (3.38) in (3.35), ..., and the last m equations of (3.38) in (3.36).

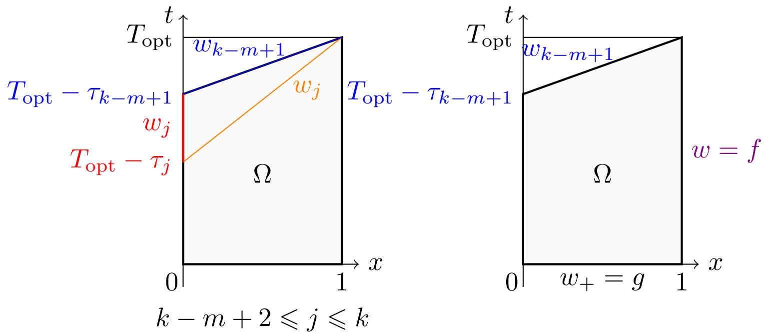

Figure 3.2. Geometry of the setting considered in the proof of Proposition 3.6 when Σ is constant. The boundary conditions imposed at $x = 0$ for w_j with $k - m + 2 \leqslant j \leqslant k$ are given on the left, and the boundary conditions imposed at $x = 1$ and $t = 0$ are given on the right.

We now introduce some notations which are used in the definition of $\mathcal{K}$ and $\mathcal{L}$. For $x \in [0, 1]$ and $1 \leqslant j \leqslant k + m$, let $\tau(j, x) \in [0, +\infty)$ be such that

$$x_j\big(\tau(j, x), 0, x\big) = 0 \quad \text{for } k + 1 \leqslant j \leqslant k + m,$$

and

$$x_j\big(\tau(j, x), 0, x\big) = 1 \quad \text{for } 1 \leqslant j \leqslant k$$

(see Figure 3.3). Recall that $x_j(t, s, \xi)$ is defined in (3.19) and (3.20).

We begin with the definition of $\mathcal{K}(\tau)\colon \big[L^2(0,1)\big]^n \to \big[L^2(0,1)\big]^n$. The goal is to provide the supplementary requirements on $w_+(0, \cdot\,)$ which are complementary to the conditions (3.38) missing from the definition of $\mathcal{T}(\tau)$ and the fact $v_-(0, \cdot\,) = w_-(0, \cdot\,)$ which is not required in the definition of $\mathcal{T}(\tau)$ (see Lemma 3.10 given at the end of this section). We define $\mathcal{K}(\tau)$ as follows, with $w = \mathcal{T}(\tau)(0, \varphi_+)$ and $\varphi_+ = (\varphi_{k+1}, \dots, \varphi_{k+m})$:

- for $1 \leqslant j \leqslant m$ and $x \in (0, 1)$,

$$\text{(3.39)} \quad \big(\mathcal{K}(\tau)(\varphi)(x)\big)_{k+j} = \Big(\Sigma_+(0)^{-1} B^{\mathsf{T}} \Sigma_-(0) w_-\big(\tau(k + j, x), 0\big)\Big)_j + \int_0^{\tau(k+j,x)} \Big(\mathbf{C}\big(t + \tau, x_{k+j}(t, 0, x)\big) w\big(t, x_{k+j}(t, 0, x)\big)\Big)_{k+j} \,\mathrm{d}t;$$

- for $1 \leqslant j \leqslant k$ and $x \in (0, 1)$,

$$\text{(3.40)} \quad \big(\mathcal{K}(\tau)(\varphi)(x)\big)_j = \int_0^{\tau(j,x)} \Big(\mathbf{C}\big(t + \tau, x_j(t, 0, x)\big) w\big(t, x_j(t, 0, x)\big)\Big)_j \,\mathrm{d}t.$$

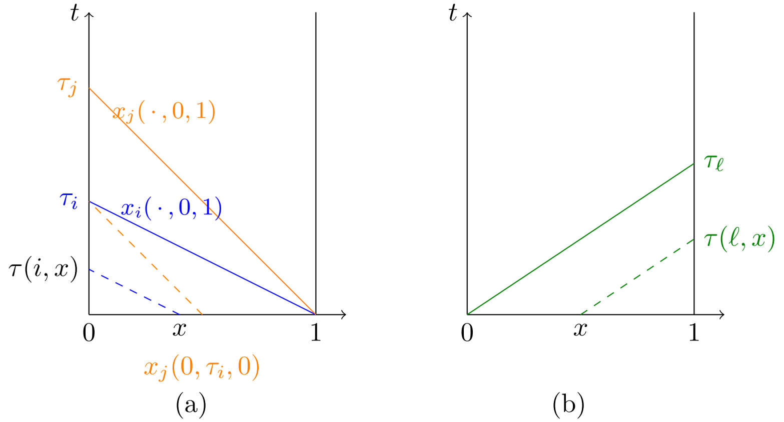

Figure 3.3. Σ is constant. (a) The definition of $x_i(\cdot, 0, 1)$, $x_j(\cdot, 0, 1)$, $x_j(0, \tau_i, 0)$, and $\tau_j(x)$ for $k+1 \leqslant j < i \leqslant k+m$. (b) The definition of $x_\ell(\cdot, 0, 0)$ and $\tau(\ell, x)$ for $1 \leqslant \ell \leqslant k$.

We now check the properties of $\mathcal{K}$ stated in the theorem. It is clear that $\mathcal{K}(\tau)$ is linear. Using Proposition A.10 in the appendix, one can derive that $\mathcal{K}(\tau)$ is uniformly bounded in I_1 and is analytic in I_1 if $C \in \mathcal{H}\big(I, \big[L^\infty(0,1)\big]^{n \times n}\big)$. From the definition of $\mathcal{K}(\tau)$ in (3.39) and (3.40), we claim that

$$\mathcal{K}(\tau) \text{ is compact.} \tag{3.41}$$

Indeed, let $(\varphi^{(l)})$ be a bounded sequence in $\big[L^2(0,1)\big]^n$. Define

$$w^{(l)} = \mathcal{T}\big(\tau, 0, \varphi_+^{(l)}\big) \quad \text{in } \Omega,$$

and denote

$$\|w^{(l)}\|_{\mathcal{Y}} = \max\Big\{ \sup_{x \in [0,1]} \|w_i^{(l)}\|_{L^2(\Omega_x)}, \sup_{t \in [0, T_{\mathrm{opt}}]} \|w_i^{(l)}\|_{L^2(\Omega_t)}; 1 \leqslant i \leqslant n \Big\},$$

where

$$\Omega_t = \big\{ y \in \mathbb{R}; (t,y) \in \Omega \big\} \qquad \text{and} \qquad \Omega_x = \big\{ s \in \mathbb{R}; (s,x) \in \Omega \big\}.$$

Applying Theorem A.2 in the appendix, we derive that $\big(\|w^{(l)}\|_{\mathcal{Y}}\big)$ is a bounded sequence. Without loss of generality, one might assume that

$$\begin{gathered} w^{(l)} \text{ is weakly convergent in } L^2(\Omega) \\ \text{and } \varphi_+^l \text{ s weakly convergent in } \big[L^2(0,1)\big]^m. \end{gathered} \tag{3.42}$$

Claim 3.9. — *We claim that, for a.e.* $x \in [0,1]$, *the quantity on the RHS of* (3.40) *with* w *being replaced by* $w^{(l)}$ *can be written under the form (for* $1 \leqslant j \leqslant k$*)*

$$\int_{\Omega} F_j(y,t,x) w^{(l)}(y,t)\,\mathrm{d}y\,\mathrm{d}t + \int_0^1 G_j(y,x)\varphi_+^{(l)}(y)\,\mathrm{d}y, \tag{3.43}$$

and similarly the quantity on the RHS of (3.39) *with* w *being replaced by* $w^{(l)}$ *can be written under the form (for* $1 \leqslant j \leqslant m$*)*

$$\int_{\Omega} F_{k+j}(y,t,x) w^{(l)}(y,t)\,\mathrm{d}y\,\mathrm{d}t + \int_0^1 G_{k+j}(y,x)\varphi_+^{(l)}(y)\,\mathrm{d}y, \tag{3.44}$$

with x *being a parameter, for some function* $F_j(\,\cdot\,,x), F_{k+j}(\,\cdot\,,x) \in \mathbb{R}^{n\times n}$ *defined in* Ω *such that* $F_j(y,t,x)$, $F_{j+k}(y,t,x)$ *are measurable functions for two variables* $(t,y) \in \Omega$, *and for some function* $G_j(\,\cdot\,,x)$, $G_{k+j}(\,\cdot\,,x) \in \mathbb{R}^n$ *defined in* $(0,1)$ *such that* $G_j(y,x)$, $G_{k+j}(y,x)$ *are measurable function with respect to* $y \in (0,1)$. *Moreover,* $\big|F_j(t,x,y)\big|$, $\big|F_{k+j}(t,x,y)\big|$ *and* $\big|G_j(y,x)\big|$, $\big|G_{k+j}(y,x)\big|$ *are bounded by a positive constant independent of* (t,y) *and* x *as well.*

It is clear from Claim 3.9 and the dominated convergence theorem that $\big(\mathcal{K}(\tau)(\varphi)\big)_j$ converges in $L^2(0,1)$ for $1 \leqslant j \leqslant k+m$ by (3.39) and (3.40). Therefore, the compactness of $\mathcal{K}(\tau)$ follows.

Proof. — We now establish Claim 3.9 in three steps.

Step 1. — Consider $1 \leqslant j \leqslant k$. We prove (3.43).

Note that, for a.e. $x \in [0,1]$, the requirements in the definition of broad solutions given in (A.8)–(A.12) in Definition A.1 for $w^{(l)}$ hold for a.e. $t \in \big[0, \tau(j,x)\big]$. Consider such an x. Since

$$\partial_t w_j^{(l)}(t,x) = \Sigma_{jj}\partial_x w_j^{(l)}(t,x) + \big(\mathbf{C}(\tau+t,x)w^{(l)}(t,x)\big)_j,$$

in the integral sense for a.e. $t \in \big[0,\tau(j,x)\big]$ as given in Definition A.1, and

$$\int_0^{\tau(j,x)} \Big(\mathbf{C}\big(t+\tau, x_j(t,0,x)\big) w^{(l)}\big(t, x_j(t,0,x)\big)\Big)_j\,\mathrm{d}t = w_j^{(l)}\big(\tau(j,x),1\big), \tag{3.45}$$

using the separation of constant method, one can rewrite the LHS of (3.45) as

$$\int_0^{\tau(j,x)} \Big\langle D_j\big(t+\tau, x_j(t,0,x)\big), w^{(l,j)}\big(t, x_j(t,0,x)\big)\Big\rangle\,\mathrm{d}t. \tag{3.46}$$

Here $D_j \in \mathbb{R}^{n-1}$ is computed by the separation of constant method, and $w^{(l,j)}$ is the vector obtained from $w^{(l)}$ without its j-component. We now replace the value of $w^{(l,j)}\big(t, x_j(t,0,x)\big)$ by the corresponding expressions in (A.8)–(A.12). Note that in this case, the other endpoints belong to the subset of the boundary of Ω with $x = 1$ or the subset of the boundary of Ω with $t = 0$ for the components of $w_+^{(l)}$. We then obtain

$$\int_0^{\tau(j,x)} \Big(\mathbf{C}\big(t+\tau, x_j(t,0,x)\big) w^{(l)}\big(t, x_j(t,0,x)\big)\Big)_j \,\mathrm{d}t = \int_\Omega F_j(y,t,x) w^{(l)}(y,t)\,\mathrm{d}y\,\mathrm{d}t + \int_0^1 G_j(y,x) w_+^l(y)\,\mathrm{d}y, \tag{3.47}$$

with x being a parameter, for some function $F_j(\,\cdot\,, x) \in \mathbb{R}^{n\times n}$ defined in Ω such that $F_j(y,t,x)$ is a measurable function for two variables $(t,y) \in \Omega$, and for some function $G_j(\,\cdot\,, x) \in \mathbb{R}^n$ defined in $(0,1)$ such that $F_j(y,x)$ is a measurable function with respect to $y \in (0,1)$. Moreover, $\big|F_j(t,x,y)\big|$ and $\big|G_j(y,x)\big|$ are bounded by a positive constant independent of (t,y) and x as well since $\mathbf{C}$ is bounded. The proof of Step 1 is complete.

In the proof of Steps 2 and 3 below, we always use the variation of constant method as in Step 1 to eliminate the $w_p^{(l)}$ component in the integral along the characteristic flows of the p-th direction for $1 \leqslant p \leqslant n$ to bring a variant of (3.45) into a variant of (3.46).

Step 2. — Consider $1 \leqslant j \leqslant m$. We prove that the second term of the RHS of (3.39) can be written as in (3.44).

The proof of Step 2 is in the spirit of Step 1. We now replace the values of $w_p^{(l)}\big(t, x_{k+j}(t,0,x)\big)$ $(p \neq k+j)$ as in Step 1 by the integral on the characteristic flow for the p-component and the values at the endpoint where the boundary of the p-th component is given. One has two cases.

Case 1. — If such an endpoint belongs to the part of the boundary of Ω for which the boundary conditions are prescribed (by 0 for $x = 1$ or $\varphi_+^{(l)}$ for $t = 0$), then as in Step 1, we obtain the form (3.44) for the contribution from such a component.

Case 2. — If such an endpoint belongs to the part of the boundary of Ω for which the boundary conditions given in (3.34)–(3.36) are used, then these conditions are taken into account so that one can write the value of $w_p^{(l)}$ as a linear combination of the components of

$$w_{(p)}^{(l)} \text{ at this end point}$$

where $w_{(p)}^{(l)}$ containing only the components of $w^{(l)}$ at that endpoint for which the other endpoints (in $\partial\Omega$) of the corresponding characteristic flows in $\overline{\Omega}$ passing this endpoint are on the boundary of Ω with $x = 1$ or $t = 0$

for which their value are prescribed (by $0 \in \mathbb{R}^n$ for $x = 1$ or $\varphi_+^{(l)} \in \mathbb{R}^m$ for $t = 0$). We now can replace the value of $w_{(p)}^{(l)}$ at this endpoint by the integrals along the corresponding characteristic flows passing that endpoint and the values at the other endpoints. We thus obtain the form (3.44) for the contribution for such p-th component of $w^{(l)}$ as in Step 1.

Combining Case 1 and Case 2, we obtain (3.44).

Step 3. — Consider $1 \leqslant j \leqslant m$. We prove that the first term of the RHS of (3.39) can be written as in (3.44).

We have, for $x \in (0,1)$,

$$\tau(k+j,x) \leqslant \tau(k+j,1) = \tau_{k+j} \overset{(1.3)}{\leqslant} \tau_{k+1}. \tag{3.48}$$

It follows from the definition of T_{opt} (1.8) that

$$\tau(k+j,x) \leqslant T_{\text{opt}} - \tau_{k-m+1}. \tag{3.49}$$

This is the essential fact for the proof of Step 3 and thus for the compactness of $H(\tau)$.

Taking this into account and using (3.34)–(3.36), one can replace the first term in the RHS of (3.39) for a.e. $x \in (0,1)$ as a linear combination of the components of

$$w^{(l)}_{<k, \geqslant k+m}\big(\tau(k+j,x),0\big) \quad \text{if } \tau(k+j,x) \in (T_{\text{opt}} - \tau_k, T_{\text{opt}} - \tau_{k-1}) \text{ by (3.34)}, \tag{3.50}$$

$$w^{(l)}_{<k-1, \geqslant k+m-1}\big(\tau(k+j,x),0\big) \quad \text{if } \tau(k+j,x) \in (T_{\text{opt}} - \tau_{k-1}, T_{\text{opt}} - \tau_{k-2}) \text{ by (3.35)} \tag{3.51}$$

$$\dots$$

$$w^{(l)}_{<k-m+2, \geqslant k+2}\big(\tau(k+j,x),0\big) \quad \text{if } \tau(k+j,x) \in (T_{\text{opt}} - \tau_{k-m+2}, T_{\text{opt}} - \tau_{k-m+1}) \text{ by (3.36)}. \tag{3.52}$$

It is clear that the first term of the RHS of (3.39) is a linear combination of the components of

$$w_-^{(l)}\big(\tau(k+j,x),0\big) \quad \text{if } \tau(k+j,x) \in (0, T_{\text{opt}} - \tau_k). \tag{3.53}$$

We now replace the values of the components of $w^{(l)}\big(\tau(k+j,x),0\big)$ in (3.50)–(3.53) by the integrals on the corresponding characteristic flows and the values of the endpoints. Note that the values of the endpoints are prescribed (by 0 for $x = 1$ or $\varphi_+^{(l)}$ for $t = 0$). We now can proceed as in Step 2 to derive the conclusion.

The proof of Claim 3.9 is complete. ☐

We now introduce the operator $\mathcal{L}(\tau)$. The goal is to complement the missing conditions in (3.38) in $(0, T_{\rm opt} - \tau_{k-m+1})$. In order to make the proof easier/shorter, we put all the information in (3.38) in the definition of $\mathcal{L}(\tau)$. We then define $\mathcal{L}(\tau)$ as follows, with $\varphi \in \big[L^2(0,1)\big]^n$ and $w = \mathcal{T}\big(\tau)(0, \varphi_+\big)$,

$$\mathcal{L}(\tau)(\varphi)(t) = \Sigma_+(0) w_+(t,0) + B^{\mathsf{T}} \Sigma_-(0) w_-(t,0) \quad \text{for } t \in (0, T_{\rm opt} - \tau_{k-m+1}). \tag{3.54}$$

From the definitions of $\mathcal{K}(\tau)$ and $\mathcal{L}(\tau)$, we derive from Lemma 3.10 below that $H(\tau) \subset \big\{\varphi \in \big[L^2(0,1)\big]^n;\ \varphi + \mathcal{K}(\tau)\varphi = 0 \text{ and } \mathcal{L}(\tau)(\varphi) = 0\big\}$. It remains to prove that

$$\Big\{\varphi \in \big[L^2(0,1)\big]^n;\ \varphi + \mathcal{K}(\tau)\varphi = 0 \text{ and } \mathcal{L}(\tau)\varphi = 0\Big\} \subset H(\tau). \tag{3.55}$$

To this end, we introduce another operator $\widehat{\mathcal{T}}(\tau)$ related to $\mathcal{T}(\tau)$. Consider the system, for $(f,g) \in \big[L^2(0,1)\big]^n \times \big[L^2(0,1)\big]^m$,

$$\partial_t \widehat{w}(t,x) = \Sigma(x) \partial_x \widehat{w}(t,x) + \mathbf{C}(t+\tau, x) \widehat{w}(t,x) \quad \text{for } (t,x) \in (0, T_{\rm opt}) \times (0,1), \tag{3.56}$$

$$\widehat{w}(\,\cdot\,, 1) = f \quad \text{in } (0, T_{\rm opt}), \tag{3.57}$$

$$\widehat{w}_+(0, \cdot\,) = g \quad \text{in } (0,1), \tag{3.58}$$

$$\widehat{w}_i(T_{\rm opt}, \cdot\,) = 0 \quad \text{in } (0,1), \text{ for } 1 \leqslant i \leqslant k-m, \tag{3.59}$$

$$\widehat{w}_{-,\geqslant k}(t,0) = Q_k \widehat{w}_{<k,\geqslant k+m}(t,0) \quad \text{for } t \in (T_{\rm opt} - \tau_k, T_{\rm opt} - \tau_{k-1}), \tag{3.60}$$

$$\widehat{w}_{-,\geqslant k-1}(t,0) = Q_{k-1} \widehat{w}_{<k-1,\geqslant k+m-1}(t,0) \quad \text{for } t \in (T_{\rm opt} - \tau_{k-1}, T_{\rm opt} - \tau_{k-2}), \tag{3.61}$$

$$\dots$$

$$\widehat{w}_{-,\geqslant k-m+2}(t,0) = Q_{k-m+2} \widehat{w}_{<k-m+2,\geqslant k+2}(t,0) \quad \text{for } t \in (T_{\rm opt} - \tau_{k-m+2}, T_{\rm opt} - \tau_{k-m+1}), \tag{3.62}$$

$$\widehat{w}_{-,\geqslant k-m+1}(t,0) = Q_{k-m+1} \widehat{w}_{<k-m+1,\geqslant k+1}(t,0) \quad \text{for } t \in (T_{\rm opt} - \tau_{k-m+1}, T_{\rm opt}) \tag{3.63}$$

(it is at this stage that the condition (1.10) with $i = m$ is required!).

For $\tau \in I_1$ given in (3.11), define

$$\begin{aligned} \widehat{\mathcal{T}}(\tau) \colon \big[L^2(0,1)\big]^n \times \big[L^2(0,1)\big]^m &\longrightarrow \big[L^2\big((0,T_{\rm opt}) \times (0,1)\big)\big]^n \\ (f,g) &\longmapsto \widehat{w}, \end{aligned} \tag{3.64}$$

where $\widehat{w}$ is the unique broad solution of (3.56)–(3.63) (see Theorem A.12 in the appendix with $T = T_{\rm opt}$ for the existence and uniqueness of broad solutions; the definition of broad solutions is similar to Definition A.1).

It is clear that

(3.65) $\mathcal{T}(\tau)(0,g)$ is the restriction of $\widehat{\mathcal{T}}(\tau)(0,g)$ in Ω for $g \in \big[L^2(0,1)\big]^m$

since they have the same definition in Ω.

Fix

$$\varphi_0 \in \Big\{\varphi \in \big[L^2(0,1)\big]^n;\ \varphi + \mathcal{K}(\tau)\varphi = 0 \text{ and } \mathcal{L}(\tau)\varphi = 0\Big\}. \tag{3.66}$$

Denote

$$w = \mathcal{T}(\tau)(0,\varphi_{0,+}) \qquad \text{and} \qquad \widehat{w} = \widehat{\mathcal{T}}(\tau)(0,\varphi_{0,+}).$$

Then, by (3.65),

$$\widehat{w} = w \quad \text{in } \Omega. \tag{3.67}$$

Since $\varphi + \mathcal{K}(\tau)(\varphi) = 0$, we have (see Lemma 3.10 below)

$$w(0,\cdot\,) = \varphi_0 \quad \text{in } (0,1).$$

Since $\mathcal{L}(\tau)(\varphi_0) = 0$, we obtain (see (3.54))

$$\Sigma_+(0)\widehat{w}_+(t,0) = -B^{\mathsf{T}}\Sigma_-(0)\widehat{w}_-(t,0) \quad \text{for } t \in (0, T_{\text{opt}} - \tau_{k-m+1}). \tag{3.68}$$

On the other hand, by the definition of $\widehat{\mathcal{T}}(\tau)$ (in particular, condition (3.63)), one has,

$$\Sigma_+(0)\widehat{w}_+(t,0) = -B^{\mathsf{T}}\Sigma_-(0)\widehat{w}_-(t,0) \quad \text{for } t \in (T_{\text{opt}} - \tau_{k-m+1}, T_{\text{opt}}). \tag{3.69}$$

Combining (3.68) and (3.69) yields

$$\Sigma_+(0)\widehat{w}_+(t,0) = -B^{\mathsf{T}}\Sigma_-(0)\widehat{w}_-(t,0) \quad \text{for } t \in (0, T_{\text{opt}}). \tag{3.70}$$

Thus $\widehat{w}$ is a solution of (3.23)–(3.25) satisfying (3.26) with $\widehat{w}(0,\cdot\,) = w(0,\cdot\,) = \varphi_0$.

The proof of Proposition 3.6 is complete. □

In the proof of Proposition 3.6, we used the following lemma. Recall that $\mathcal{K}(\tau)$ is defined by (3.39) and (3.40).

Lemma 3.10. — *Let $\varphi \in \big[L^2(0,1)\big]^n$ and set $w = \mathcal{T}(\tau)(0,\varphi_+)$ where $\varphi_+ = (\varphi_{k+1},\dots,\varphi_{k+m})$. We have:*

(i) *the following boundary condition:*

$$w_+(t,0) = -\Sigma_+(0)B^{\mathsf{T}}\Sigma_-(0)w_-(t,0) \quad \textit{in } (0, T_{\text{opt}} - \tau_{k-m+1})$$

holds if and only if $\big(\mathcal{K}(\tau)(\varphi)\big)_{k+j} + \varphi_{k+j} = 0$ in $(0,1)$ for $1 \leqslant j \leqslant m$, and $\mathcal{L}(\tau)(\varphi) = 0$;

(ii) *assertion $w_-(0,\cdot\,) = \varphi_-$ in $(0,1)$, where $\varphi_- = (\varphi_1,\dots,\varphi_k)$, holds if and only if $\big(\mathcal{K}(\tau)(\varphi)\big)_j + \varphi_j = 0$ in $(0,1)$ for $1 \leqslant j \leqslant k$.*

Proof. — We begin with (i). We have, for $1 \leqslant j \leqslant m$,

$$\frac{\mathrm{d}}{\mathrm{d}t} w_{k+j}\big(t, x_{k+j}(t,0,x)\big) = \Big(\mathbf{C}\big(t+\tau, x_{k+j}(t,0,x)\big) w\big(t, x_{k+j}(t,0,x)\big)\Big)_{k+j}.$$

Integrating from 0 to $\tau(k+j,x)$ yields, for $1 \leqslant j \leqslant m$ and for $x \in \big(0, x_{k+j}(0, \tau_{k+m}, 0)\big)$,

$$w_{k+j}(0,x) = w_{k+j}\big(\tau(k+j,x), 0\big) - \int_0^{\tau(k+j,x)} \Big(\mathbf{C}\big(t+\tau, x_{k+j}(t,0,x)\big) w\big(t, x_{k+j}(t,0,x)\big)\Big)_{k+j} \mathrm{d}t.$$

Assertion (i) follows by (3.39) and the definition of $\mathcal{L}(\tau)$.

We next deal with (ii). For $1 \leqslant j \leqslant k$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} w_j\big(t, x_j(t,0,x)\big) = \Big(\mathbf{C}\big(t+\tau, x_j(t,0,x)\big) w_j\big(t, x_j(t,0,x)\big)\Big)_j.$$

Integrating from 0 to $\tau(j,x)$ yields, for $1 \leqslant j \leqslant k$ and for $x \in (0,1)$,

$$w_j(0,x) = w_j\big(\tau(j,x), 1\big) - \int_0^{\tau(j,x)} \Big(\mathbf{C}\big(t+\tau, x_j(t,0,x)\big) w\big(t, x_j(t,0,x)\big)\Big)_j \mathrm{d}t.$$

Since $w_-\big(\,\cdot\,, 1\big) = 0$ by the definition of $\mathcal{T}(\tau)$, it follows that, for $1 \leqslant j \leqslant k$ and $x \in (0,1)$,

$$w_j(0,x) = -\int_0^{\tau(j,x)} \Big(\mathbf{C}\big(t+\tau, x_j(t,0,x)\big) w\big(t, x_j(t,0,x)\big)\Big)_j \mathrm{d}t.$$

Assertion (ii) follows by (3.40). □

3.2.2. Proof of Proposition 3.7

We begin with assertion (ii). Let $\varphi \in H(\tau, T) \setminus \{0\}$ be arbitrary. From the definition of $H(\tau,T)$ in Definition 3.5, there exists a broad solution v of (3.12)–(3.16).

Set

$$v^{(\tau)}(t,x) = v(t-\tau, x) \quad \text{for } (t,x) \in (\tau, \tau+T)$$

and let w be a solution of (3.1)–(3.3) in which u is replaced by w, with $w(\tau,\cdot\,) = v^{(\tau)}(\tau,\cdot\,) = \varphi$.[(7)] By Lemma 3.3, we have

$$\begin{aligned}\int_0^1 \big\langle w(T+\tau,x), v^{(\tau)}(T+\tau,x)\big\rangle \,\mathrm{d}x &= \int_0^1 \big\langle w(\tau,x), v^{(\tau)}(\tau,x)\big\rangle \,\mathrm{d}x \\ &= \int_0^1 |\varphi|^2 \\ &\neq 0.\end{aligned}$$

[(7)] Condition (3.3) means that $w_+(t,1) \in [L^2(\tau,\tau+T)]^m$.

Therefore, one cannot steer φ from time τ to 0 at time $\tau + T$.

We next establish assertion (i) by a contradiction argument. Assume that this is not true. Since $T \geqslant T_{\rm opt}$, it follows that $H(\tau, T) \subset H(\tau, T_{\rm opt})$, which is a subspace of $\big[L^2(0,1)\big]^n$ of finite dimension thanks to the compactness of $\mathcal{K}(\tau)$ by Proposition 3.6. By Lemma 3.4 applied to $E = H(\tau, T)^{\perp}$, there exists a sequence of solutions (v_N) of (3.23)–(3.25) with $T_{\rm opt}$ being replaced by T such that

$$\lim_{N \to +\infty} \big\| v_{N,+}(\,\cdot\,, 1) \big\|_{L^2(0,T)} = 0 \quad \text{and} \quad \big\| \operatorname{Proj}_{H(\tau,T)^{\perp}} v_N(0, \cdot\,) \big\|_{L^2(0,1)} = 1. \tag{3.71}$$

Set

$$\varphi_N = \operatorname{Prof}_{H(\tau,T)} v_N(0, \cdot\,) \in H(\tau, T) \subset \big[L^2(0,1)\big]^n$$

and let V_N be the corresponding solution with respect to φ_N given in Definition 3.5. Replacing v_N by $v_N - V_N$ if necessary, without loss of generality, one can assume in addition that $v_N(0, \cdot\,) \in H(\tau, T)^{\perp}$, which yields in particular that $\big\| v_N(0, \cdot\,) \big\|_{L^2(0,1)} = \big\| \operatorname{Proj}_{H(\tau,T)^{\perp}} v_N(0, \cdot\,) \big\|_{L^2(0,1)} = 1$. This will be assumed from now on.

Consider $f_N \in \big[L^2(0,T)\big]^n$ defined by

$$f_N = v_N(\,\cdot\,, 1) \quad \text{in } (0, T). \tag{3.72}$$

Since $v_{N,-}(\,\cdot\,, 1) = 0$ in $(0, T)$ and $\lim_{N \to +\infty} \big\| v_{N,+}(\,\cdot\,, 1) \big\|_{L^2(0,T)} = 0$, it follows that

$$\lim_{N \to +\infty} f_N = 0 \quad \text{in } \big[L^2(0,T)\big]^n. \tag{3.73}$$

As in the proof of Lemma 3.10, we derive from (3.73) that

$$\mathcal{K}(\tau) v_N(0, \cdot\,) + v_N(0, \cdot\,) = g_N \quad \text{in } (0,1)$$

for some $(g_N) \to 0$ in $\big[L^2(0,1)\big]^n$. Since $\mathcal{K}(\tau)$ is compact by Proposition 3.6), without loss of generality, one might assume that

$$v_N(0, \cdot\,) \longrightarrow \varphi \quad \text{in } \big[L^2(0,1)\big]^n,$$

and hence $\varphi \in H(\tau, T)^{\perp}$ by (3.73).

We define $\mathcal{T}(\tau, T)$ and $\widehat{\mathcal{T}}(\tau, T)$ as defining $\mathcal{T}(\tau)$ and $\widehat{\mathcal{T}}(\tau)$ with $T_{\rm opt}$ being replaced by T (and thus Ω is replaced by Ω_T). Recall that Ω_T is defined in (3.21). As in (3.65), we also have

$$\widehat{\mathcal{T}}(\tau, T)(f, g) = \mathcal{T}(\tau, T)(f, g) \quad \text{in } \Omega_T \qquad \text{for } f \in \big[L^2(0,T)\big]^n,\ g \in \big[L^2(0,1)\big]^m, \tag{3.74}$$

since $\widehat{\mathcal{T}}(\tau, T)$ and $\mathcal{T}(\tau, T)$ have the same requirements in Ω_T.

Set

$$u = \widehat{\mathcal{T}}(\tau, T)(0, \varphi_+) \qquad \text{and} \qquad u_N = \widehat{\mathcal{T}}(\tau, T)\big(f_N, v_{N,+}(0, \cdot)\big).$$

Then, by (3.74),

$$u_N = v_N \quad \text{in } \Omega_T \tag{3.75}$$

since $\mathcal{T}(\tau, T)\big(f_N, v_{N,+}(0, \cdot)\big) = v_N$ in Ω_T by (3.72).

Since:

- $\Sigma_+(0) u_{N,+}(t, 0) = -B^{\mathsf{T}} \Sigma_-(0) u_{N,+}(t, 0)$ for $t \in (T - \tau_{k-m+1}, T)$ by (3.63) with $T_{\rm opt}$ being replaced by T;
- $\Sigma_+(0) v_{N,+}(t, 0) = -B^{\mathsf{T}} \Sigma_-(0) v_{N,+}(t, 0)$ in $(0, T - \tau_{k-m+1})$;

it follows from (3.75) that

$$\Sigma_+(0) u_{N,+}(t, 0) = -B^{\mathsf{T}} \Sigma_-(0) u_{N,+}(t, 0) \quad \text{for } t \in (0, T). \tag{3.76}$$

Using the continuity of $\widehat{\mathcal{T}}(\tau, T)$ (see Theorem A.12 in the appendix), we derive from (3.76) that

$$\Sigma_+(0) u_+(t, 0) = -B^{\mathsf{T}} \Sigma_-(0) u_-(t, 0) \quad \text{for } t \in (0, T), \tag{3.77}$$

and since $v_N = \mathcal{T}(\tau, T)\big(f_N, v_{N,+}(0, \cdot)\big)$ in Ω_T we obtain

$$\varphi(x) = \widehat{\mathcal{T}}(\tau, T)(0, \varphi_+)(0, x) = u(0, x) \quad \text{for } x \in (0, 1). \tag{3.78}$$

Since $u(\cdot, 1) = 0$ by the definition of $\widehat{\mathcal{T}}(\tau, T)$, we derive from (3.77) and (3.78) that

$$\varphi \in H(\tau, T).$$

Thus $\varphi = 0$ since $\varphi \in H(\tau, T)^\perp$. We deduce that

$$0 = \big\|\mathrm{Proj}_{H(\tau,T)^\perp} \varphi\big\| = \lim_{N \to +\infty} \big\|\mathrm{Proj}_{H(\tau,T)^\perp} v_N(0, \cdot)\big\| = 1.$$

We have a contradiction. Assertion (i) is proved.

The proof is complete. □

Remark 3.11. — Note that in the proof of (ii) of Proposition 3.7, one does not require that condition $T \geqslant T_{\rm opt}$. In fact, assertion (ii) holds for $T > 0$ arbitrary.

3.2.3. Proof of Lemma 3.8

Set, for $\tau \in I_1$,

$$E(\tau) = \Big\{\varphi \in \big[L^2(0, 1)\big]^n;\ \varphi + \mathcal{K}(\tau)\varphi = 0\Big\}, \tag{3.79}$$

From (3.18) in Proposition 3.6, we have, for $\tau \in I_1$,

$$H(\tau) = E(\tau) \cap \Big\{\varphi \in \big[L^2(0,1)\big]^n;\ \mathcal{L}(\tau)\varphi = 0\Big\}. \tag{3.80}$$

Since $K(\tau)$ is compact, it follows that the eigenvalue -1 of $K(\tau)$ is isolated for each $\tau \in I_1$. From [32, Section 3 of Chapter 7] (see also [32, Section 3 of Chapter 2]), for each $\tau_0 \in I_1$ there is a $\gamma = \gamma(\tau_0) > 0$ (γ depends on τ_0) such that the sum of the eigenprojections $\mathcal{P}(\tau)$ for all the (generalized) eigenvalues of $\mathcal{K}(\tau)$ lying inside $\big\{z \in \mathbb{C} : |z+1| < \gamma\big\}$ is analytic when τ is in a small neighborhood O_{τ_0} of τ_0. Set $P(\tau) = \mathcal{P}(\tau)\big(\big[L^2(0,1)\big]^n\big)$. We thus have, for $\tau \in O_{\tau_0}$,

$$\begin{aligned} E(\tau) &\overset{(3.79)}{=} \Big\{\varphi \in \big[L^2(0,1)\big]^n;\ \varphi + \mathcal{K}(\tau)\varphi = 0\Big\} \\ &= \big\{\varphi \in P(\tau);\ \varphi + \mathcal{K}(\tau)\varphi = 0\big\}. \end{aligned}$$

It follows that, for $\tau \in O_{\tau_0}$,

$$H(\tau) = \big\{\varphi \in P(\tau);\ \varphi + \mathcal{K}(\tau)\varphi = 0 \text{ and } \mathcal{L}(\tau)\varphi = 0\big\}. \tag{3.81}$$

We now can use the theory of the perturbation of the null-space of analytic matrices. Applying [24, Theorem S6.1, pp. 388–389] and using (3.81), we derive that[(8)]

$$\begin{gathered} H(\tau) \text{ is analytic in } O_{\tau_0} \text{ except for a discrete subset,} \\ \text{which is removable.} \end{gathered} \tag{3.82}$$

The conclusion follows since τ_0 is arbitrary in I_1. The proof is complete. □

3.3. Characterization of states at time τ steered to 0 in time $T_{\mathrm{opt},+}$

Fix $\gamma_0 > 0$ such that $[0, T_1] \subset (\alpha + \gamma_0, \beta - \gamma_0)$. Recall that $I = (\alpha, \beta)$ contains $[0, T_1]$, see (3.11). Set

$$I_2 = (\alpha + \gamma_0, \beta - \gamma_0 - T_{\mathrm{opt}}). \tag{3.83}$$

(8) One way to apply the theory of the perturbation of the null-space of analytic matrices can be done as follows. One can first locally choose an analytic orthogonal basis $\{\varphi_1(\tau), \dots, \varphi_\ell(\tau)\}$ of $P(\tau)$. We then represent the operator $\mathrm{Id} + \mathcal{K}(\tau)$ (where Id denotes the identity map) in this basis after noting that it is an application from $P(\tau)$ into $P(\tau)$. We also represent $\mathcal{L}(\tau)$ using the set $\{\mathcal{L}(\tau)(\varphi_1(\tau)), \dots, \mathcal{L}(\tau)(\varphi_\ell(\tau))\}$.

Given $0 < \varepsilon < \gamma_0$ and $\tau \in I_2$, consider the system, for $V \in \big[L^2(0,\varepsilon)\big]^m$,

$$(3.84)\qquad \begin{cases} \partial_t v(t,x) = \Sigma(x)\partial_x v(t,x) + C(t+\tau,x)v(t,x) & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } (t,x) \in (0,\varepsilon)\times(0,1), & \\ v_-(t,0) = Bv_+(t,0) & \text{for } t \in (0,\varepsilon), \\ v_+(t,1) = V(t) & \text{for } t \in (0,\varepsilon), \\ v(0,\cdot) = 0 & \text{in } (0,1). \end{cases}$$

Define[9]

$$\begin{aligned} \mathcal{T}^c_{\tau,\varepsilon} \colon \big[L^2(0,\varepsilon)\big]^m &\longrightarrow \big[L^2(0,1)\big]^n \\ V &\longmapsto v(\varepsilon,\cdot), \end{aligned}$$

where v is the solution of (3.84). Consider two subsets $Y_{\tau,\varepsilon}$ and $A_{\tau,\varepsilon}$ of $\big[L^2(0,1)\big]^n$ defined by[10]

$$(3.85)\qquad Y_{\tau,\varepsilon} = \mathcal{T}^c_{\tau,\varepsilon}\big\{\big[L^2(0,\varepsilon)\big]^m\big\} \quad\text{and}\quad A_{\tau,\varepsilon} = \operatorname{Proj}_{H(\tau+\varepsilon)}\{Y_{\tau,\varepsilon}\}.$$

Given $0 < \varepsilon < \gamma_0$ and $\tau \in I_2$, we also define[11]

$$\begin{aligned} \mathcal{T}^I_{\tau,\varepsilon} \colon \big[L^2(0,1)\big]^n &\longrightarrow \big[L^2(0,1)\big]^n \\ \varphi &\longmapsto w(\varepsilon,\cdot), \end{aligned}$$

where w is the solution of

$$(3.86)\qquad \begin{cases} \partial_t w(t,x) = \Sigma(x)\partial_x w(t,x) + C(t+\tau,x)w(t,x) & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } (t,x) \in (0,\varepsilon)\times(0,1), & \\ w_-(t,0) = Bw_+(t,0) & \text{for } t \in (0,\varepsilon), \\ w_+(t,1) = 0 & \text{for } t \in (0,\varepsilon), \\ w(0,\cdot) = \varphi & \text{in } (0,1). \end{cases}$$

Set, for $0 < \varepsilon < \gamma_0$ and for $\tau \in I_2$,

$$(3.87)\qquad J(\tau,\varepsilon) := \big\{\varphi \in H(\tau);\ \operatorname{Proj}_{H(\tau+\varepsilon)} \mathcal{T}^I_{\tau,\varepsilon}(\varphi) \in A_{\tau,\varepsilon}\big\}.$$

The motivation for the definition of $\mathcal{T}^c_{\tau,\varepsilon}$ and $\mathcal{T}^I_{\tau,\varepsilon}$ is given in the following result.

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(9) The sub-index c means that controls are used.
(10) The letter A means the attainability.
(11) The sub-index I means that initial data are considered.

LEMMA 3.12. — *Let $0 < \varepsilon < \gamma_0$ and $\tau \in I_2$. Then $J(\tau, \varepsilon)$ is the space of (functions) states in $H(\tau)$ such that one can steer them from time τ to 0 at time $\tau + T_{\text{opt}} + \varepsilon$. As a consequence, for $\tau \in I_2$,*

$$J(\tau, \varepsilon') \subset J(\tau, \varepsilon) \quad \text{for } 0 < \varepsilon' < \varepsilon < \gamma_0, \tag{3.88}$$

and the limit $J(\tau)$ of $J(\tau, \varepsilon)$ as $\varepsilon \to 0_+$ exists.

Remark 3.13. — The monotone property of $J(\tau, \varepsilon)$ with respect to ε given in (3.88) will play a role in our analysis.

Remark 3.14. — The analyticity of C in I is not required in Lemma 3.12.

Proof of Lemma 3.12. — Given $\varphi \in J(\tau, \varepsilon)$, by the definition of $J(\tau, \varepsilon)$, there exists $\widehat{V} \in \left[L^2(0, \varepsilon)\right]^m$ such that

$$\operatorname{Proj}_{H(\tau+\varepsilon)} \widehat{w}(\varepsilon, \cdot\,) = 0,$$

where $\widehat{w}$ defined in $(0, \varepsilon) \times (0,1)$ is the solution of the system

$$\begin{cases} \partial_t \widehat{w}(t,x) = \Sigma(x)\partial_x \widehat{w}(t,x) + C(t+\tau, x)\widehat{w}(t,x) & \text{for } (t,x) \in (0,\varepsilon) \times (0,1), \\ \widehat{w}_-(t,0) = B\widehat{w}_+(t,0) & \text{for } t \in (0,\varepsilon), \\ \widehat{w}_+(t,1) = \widehat{V} & \text{for } t \in (0,\varepsilon), \\ \widehat{w}(0,\cdot\,) = \varphi & \text{in } (0,1). \end{cases}$$

It follows that, by the properties of $H(\tau+\varepsilon) = H(\tau+\varepsilon, T_{\text{opt}})$ in Proposition 3.7, there exists $\widetilde{V} \in \left[L^2(\varepsilon, T_{\text{opt}} + \varepsilon)\right]^m$ such that

$$\widetilde{w}(T_{\text{opt}} + \varepsilon, \cdot\,) = 0 \quad \text{in } (0,1),$$

where $\widetilde{w}$ defined in $(\varepsilon, T_{\text{opt}} + \varepsilon) \times (0,1)$ is the solution of the system

$$\begin{cases} \partial_t \widetilde{w}(t,x) = \Sigma(x)\partial_x \widetilde{w}(t,x) + C(t+\tau, x)\widetilde{w}(t,x) \\ \qquad\qquad\qquad\qquad\qquad \text{for } (t,x) \in (\varepsilon, T_{\text{opt}}+\varepsilon) \times (0,1), \\ \widetilde{w}_-(t,0) = B\widetilde{w}_+(t,0) \qquad \text{for } t \in (\varepsilon, T_{\text{opt}}+\varepsilon), \\ \widetilde{w}_+(t,1) = \widetilde{V} \qquad \text{for } t \in (\varepsilon, T_{\text{opt}}+\varepsilon), \\ \widetilde{w}(\varepsilon,\cdot\,) = \widehat{w}(\varepsilon,\cdot\,) \qquad \text{in } (0,1). \end{cases}$$

Let w be defined in $(0, T_{\text{opt}} + \varepsilon) \times (0,1)$ by $\widehat{w}$ in $(0,\varepsilon) \times (0,1)$ and by $\widetilde{w}$ in $(\varepsilon, T_{\text{opt}} + \varepsilon) \times (0,1)$. Set

$$\mathbf{w}(t,x) = w(t-\tau, x) \quad \text{in } (\tau, \tau + T_{\text{opt}} + \varepsilon) \times (0,1).$$

Then $\mathbf{w}$ is a solution starting from φ at time τ and arriving at 0 at time $\tau + T_{\text{opt}} + \varepsilon$, i.e.,

$$\begin{cases} \partial_t \mathbf{w}(t,x) = \Sigma(x)\partial_x \mathbf{w}(t,x) + C(t,x)\mathbf{w}(t,x) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } (t,x) \in (\tau, \tau + T_{\text{opt}} + \varepsilon) \times (0,1), \\ \mathbf{w}_-(t,0) = B\mathbf{w}_+(t,0) \qquad\qquad \text{for } t \in (\tau, \tau + T_{\text{opt}} + \varepsilon), \\ \mathbf{w}(\tau, \cdot\,) = \varphi \quad \text{and} \quad \mathbf{w}(\tau + T_{\text{opt}} + \varepsilon, \cdot\,) = 0 \quad \text{in } (0,1). \end{cases}$$

We have thus proved that one can steer $\varphi \in J(\tau, \varepsilon)$ at time τ to 0 at time $\tau + T_{\text{opt}} + \varepsilon$.

Conversely, let $\varphi \in H(\tau)$ be such that one can steer φ at time τ to 0 at time $\tau + T_{\text{opt}} + \varepsilon$ using a control $W \in \left[L^2(\tau, \tau + T_{\text{opt}} + \varepsilon)\right]^m$. Let $\mathbf{w}$ be the corresponding solution, and set $w(t,x) = \mathbf{w}(t+\tau, x)$ in $(0, T_{\text{opt}} + \varepsilon) \times (0,1)$. Since $\mathbf{w}(\tau + \varepsilon, \cdot\,)$ is steered from time $\tau + \varepsilon$ to 0 at time $\tau + T_{\text{opt}} + \varepsilon$, it follows from the properties of $H(\tau + \varepsilon) = H(\tau + \varepsilon, T_{\text{opt}})$ in Proposition 3.7 that

$$\operatorname{Proj}_{H(\tau+\varepsilon)} \mathbf{w}(\tau + \varepsilon, \cdot\,) = 0.$$

In other words,

$$\operatorname{Proj}_{H(\tau+\varepsilon)} w(\varepsilon, \cdot\,) = 0.$$

This yields that $\varphi \in J(\tau, \varepsilon)$.

We thus proved that $J(\tau, \varepsilon)$ is the space of (functions) states in $H(\tau)$ such that one can steer them from time τ to 0 at time $\tau + T_{\text{opt}} + \varepsilon$. The other conclusions of Lemma 3.12 are direct consequences of this fact and the details of the proof are omitted. □

Concerning $A_{\tau,\varepsilon}$, we have the following result.

Lemma 3.15. — *Let $0 < \varepsilon < \gamma_0$. Assume that $C \in \mathcal{H}\big(I; \left[L^\infty(0,1)\right]^{n\times n}\big)$. We have:*

$A_{\tau,\varepsilon}$ is analytic in I_2 except for a discrete set, which is removable.

Recall that $A_{\tau,\varepsilon}$ is defined in (3.85) and I_2 is defined in (3.83).

Proof. — Denote

$$l = \max_{\substack{\tau \in I_2 \\ H \text{ is continuous at } \tau+\varepsilon}} \dim A_{\tau,\varepsilon} < +\infty,$$

since $H(\tau)$ is analytic in I_1 except for a discrete subset, which is removable. Fix $\tau_0 \in I_2$ such that $\dim A_{\tau_0,\varepsilon} = l$ and fix $\xi_1, \dots, \xi_l \in \left[L^2(0,\varepsilon)\right]^m$ such that

$$\left\{\operatorname{Proj}_{H(\tau_0+\varepsilon)} \mathcal{T}^c_{\tau_0,\varepsilon}(\xi_j);\ 1 \leqslant j \leqslant l\right\} \text{ is an orthogonal basis of } A_{\tau_0,\varepsilon}.$$

Since, for fixed ε, $\mathcal{T}^c_{\cdot,\varepsilon}$ is analytic in I_2 and $H(\cdot+\varepsilon)$ is analytic in I_2 except for a discrete subset which is removable, it follows that

$$\dim \operatorname{span}\big\{\operatorname{Proj}_{H(\tau+\varepsilon)} \mathcal{T}^c_{\tau,\varepsilon}(\xi_j);\ 1 \leqslant j \leqslant l\big\} = l \quad \text{in } I_2 \text{ except for a discrete subset.} \tag{3.89}$$

This in turn implies, by the property of l,

$$A(\tau,\varepsilon) = \operatorname{span}\big\{\operatorname{Proj}_{H(\tau+\varepsilon)} \mathcal{T}^c_{\tau,\varepsilon}(\xi_j);\ 1 \leqslant j \leqslant l\big\} \quad \text{in } I_2 \text{ except for a discrete subset.} \tag{3.90}$$

Combining (3.89) and (3.90) yields the conclusion. □

Let

(3.91) $M(\tau)$ be the orthogonal complement of $J(\tau)$ in $H(\tau)$.

Recall that $J(\tau)$ is the limit of $J(\tau,\varepsilon)$ as $\varepsilon \to 0_+$, see Lemma 3.12. It is clear that for each $\tau \in I_1$, there exists some $\varepsilon_\tau > 0$ such that one cannot steer any $\varphi \in M(\tau)\setminus\{0\}$ at time τ to 0 at time $\tau + T_{\mathrm{opt}} + \varepsilon_\tau$. The constant ε_τ can be chosen independently of $\varphi \in M(\tau)\setminus\{0\}$, for example, one can take ε_τ so that $J(\tau,\varepsilon) = J(\tau)$ for $0 \leqslant \varepsilon \leqslant \varepsilon_\tau/2$. The analyticity of C is not required for this purpose. Nevertheless, when the analyticity of C in I is imposed, one can obtain a *uniform* lower bound for ε_τ for $\tau \in I_2$ in a sense which will be precise now. To establish this property, for $0 < \varepsilon < \gamma_0$ and $\tau \in I_2$, we first write $J(\tau,\varepsilon)$ under the form

$$J(\tau,\varepsilon) = \big\{\varphi \in H(\tau);\ \operatorname{Proj}_{A_{\tau,\varepsilon}} \operatorname{Proj}_{H(\tau+\varepsilon)} \mathcal{T}^I_{\tau,\varepsilon}(\varphi) - \operatorname{Proj}_{H(\tau+\varepsilon)} \mathcal{T}^I_{\tau,\varepsilon}(\varphi) = 0\big\}.$$

Since the operator

$$\operatorname{Proj}_{A_{\cdot,\varepsilon}} \operatorname{Proj}_{H(\cdot+\varepsilon)} \mathcal{T}^I_{\cdot,\varepsilon} - \operatorname{Proj}_{H(\cdot+\varepsilon)} \mathcal{T}^I_{\cdot,\varepsilon} \text{ is analytic in } I_2$$
$$\text{except for a discrete subset, which is removable,}$$

one has, as in the proof of Lemma 3.8 (in particular the derivation of (3.82) from (3.81)),

$$J(\cdot,\varepsilon) \text{ is analytic in } I_2$$
$$\text{except for a discret set, which is removable.}$$

We derive that for each $n \in \mathbb{N}$ with $1/n < \gamma_0$, there exists a discrete subset D_n of I_2 such that[12]

$$J(\tau, 1/n) \text{ is analytic in } I_2$$
$$\text{except for a discrete set } D_n, \text{ which is removable.}$$

[12] Replacing γ_0 by $\gamma_0/2$ if necessary, one can even assume that D_n is finite.

As a consequence, one has

$$\dim J(\,\cdot\,, 1/n) \text{ is constant in } I_2 \setminus D_n. \tag{3.92}$$

Set

$$D = \bigcup_{\substack{n\in\mathbb{N}\\ 1/n<\gamma_0}} D_n \tag{3.93}$$

and fix $\tau_0 \in I_2 \setminus D$. There exists $0 < \varepsilon_0 < \gamma_0$ such that

$$J(\varepsilon, \tau_0) = J(\tau_0) \quad \text{for } 0 < \varepsilon < \varepsilon_0.$$

It follows from Lemma 3.12 and (3.92) that, for $0 < \varepsilon < \varepsilon_0$ and $\tau, \tau' \in I_2\setminus D$, one has

$$J(\tau, \varepsilon) = J(\tau) \qquad \text{and} \qquad \dim J(\tau) = \dim J(\tau'). \tag{3.94}$$

We thus proved the following fact.

Lemma 3.16. — *There exists a discrete set D and $0 < \varepsilon_0 < \gamma_0$ such that*[13]

$$\dim M(\tau) = \dim M(\tau') \quad \text{for } \tau, \tau' \in I_2 \setminus D,$$

and one cannot steer any $v \in M(\tau)\setminus\{0\}$ from time τ to 0 at time $\tau+T_{\text{opt}}+\varepsilon_0$ for $\tau \in I_2 \setminus D$.

We now summarize the results which have been derived in this section.

Proposition 3.17. — *Assume that the assumptions of Theorem 1.5 hold. There exist an orthogonal decomposition of $H(\tau)$ via $H(\tau) = J(\tau) \otimes M(\tau)$ for $\tau \in I_1$, a discrete subset D of I_2, and a constant $\varepsilon_0 > 0$ such that the following four properties hold:*

(i) *for $\varphi \in J(\tau)$, one can steer v at time τ to 0 at time $\tau + T_{\text{opt}} + \delta$ for all $\delta > 0$;*

(ii) *for $\varphi \in M(\tau) \setminus \{0\}$, there exists $\varepsilon_\tau > 0$ such that one cannot steer φ at time τ to 0 at time $\tau + T_{\text{opt}} + \delta$ for $0 < \delta < \varepsilon_\tau$;*

(iii) $\dim M(\tau) = \dim M(\tau')$ *for $\tau, \tau' \in I_2 \setminus D$;*

(iv) *for $\tau \in I_2 \setminus D$, and $\varphi \in M(\tau) \setminus \{0\}$, one cannot steer φ at time τ to 0 at time $\tau + T_{\text{opt}} + \varepsilon_0$.*

Proposition 3.17 also gives the characterization of states which can be steered at time τ to 0 at time $\tau + T_{\text{opt}} + \delta$ for all $\delta > 0$. Indeed, one has, for $\tau \in I_1$:

---

(13) The set mentioned here is the union of the set D given in (3.93) and the set of $\tau \in I_2$ such that $\dim H(\tau)$ is constant, which is discrete. For notational ease, we still use the same notation D.

- for $v \in H(\tau)^{\perp} \cup J(\tau)$, one can steer v at time τ to 0 at time $\tau + T_{\text{opt}} + \delta$ for all $\delta > 0$;
- for $v \in M(\tau) \setminus \{0\}$, there exists $\varepsilon_\tau > 0$ such that one cannot steer v at time τ to 0 at time $\tau + T_{\text{opt}} + \delta$ for $0 < \delta < \varepsilon_\tau$.

3.4. Null-controllability in time $T_{\text{opt},+}$ — Proof of Theorem 1.5

We first assume that $0 \notin D$ (recall that D is defined by (3.93)). We will prove that $M(0) = \{0\}$ by contradiction, and the conclusion follows from Proposition 3.17. Assume that there exists $\varphi \in M(0) \setminus \{0\}$. Since $M(0) \subset H(0, T_{\text{opt}} + \varepsilon_0)$ by assertion (iv) of Proposition 3.17, it follows from Proposition 3.7 that there exists a solution $v^{(0)}$ of the system

$$\partial_t v^{(0)}(t,x) = \Sigma(x)\partial_x v^{(0)}(t,x) + \mathbf{C}(t,x)v^{(0)}(t,x) \quad \text{for } (t,x) \in (0, T_{\text{opt}} + \varepsilon_0) \times (0,1), \tag{3.95}$$

with, for $t \in (0, T_{\text{opt}} + \varepsilon_0)$,

$$v^{(0)}(t,1) = 0, \tag{3.96}$$

$$\Sigma_+(0)v_+^{(0)}(t,0) = -B^{\mathsf{T}}\Sigma_-(0)v_-^{(0)}(t,0), \tag{3.97}$$

$$v^{(0)}(t=0,\cdot\,) = \varphi \quad \text{in } (0,1). \tag{3.98}$$

Fix $t_1 \in (\varepsilon_0/3, \varepsilon_0/2) \setminus D$ (recall that D is discrete). By Definition 3.5, one has

$$v^{(0)}(t_1,\cdot\,) \in H(t_1, T_{\text{opt}} + \varepsilon_0 - t_1).$$

This in turn implies that, since $H(t_1, T_{\text{opt}} + \varepsilon_0 - t_1) = M(t_1) = H(t_1, T_{\text{opt}} + \varepsilon_0)$ by assertion (iv) of Proposition 3.17,

$$v^{(0)}(t_1,\cdot\,) \in H(t_1, T_{\text{opt}} + \varepsilon_0).$$

By Proposition 3.7 again, there exists a solution $v^{(1)}$ of the system

$$\partial_t v^{(1)}(t,x) = \Sigma(x)\partial_x v^{(1)}(t,x) + \mathbf{C}(t,x)v^{(1)}(t,x) \quad \text{for } (t,x) \in (t_1, t_1 + T_{\text{opt}} + \varepsilon_0) \times (0,1), \tag{3.99}$$

with, for $t \in (t_1, t_1 + T_{\text{opt}} + \varepsilon_0)$,

$$v^{(1)}(t,1) = 0, \tag{3.100}$$

$$\Sigma_+(0)v_+^{(1)}(t,0) = -B^{\mathsf{T}}\Sigma_-(0)v_-^{(1)}(t,0), \tag{3.101}$$

$$v^{(1)}(t=t_1,\cdot\,) = v^{(0)}(t_1,\cdot\,) \quad \text{in } (0,1). \tag{3.102}$$

Consider the solution v of the adjoint system (3.6)–(3.8) for the time interval $(0, t_1 + T_{\text{opt}} + \varepsilon_0)$ with $v(t_1 + T_{\text{opt}} + \varepsilon_0, \cdot\,) = v^{(1)}(t_1 + T_{\text{opt}} + \varepsilon_0, \cdot\,)$ (backward system). One can check that

$$v(t, \cdot\,) = v^{(1)}(t, \cdot\,) \quad \text{for } t \in (t_1, t_1 + T_{\text{opt}} + \varepsilon_0)$$

and, since $v^{(1)}(t_1, \cdot\,) = v^{(0)}(t_1, \cdot\,)$,

$$v(t, \cdot\,) = v^{(0)}(t, \cdot\,) \quad \text{for } t \in (0, t_1).$$

For notational ease, we will denote this v by $v^{(1)}$. We thus proved that there exists a solution $v^{(1)}$ of (3.6)–(3.8) such that

$$v^{(1)}(\,\cdot\,, 1) = 0 \quad \text{in } (0, t_1 + T_{\text{opt}} + \varepsilon_0),$$

and

$$v^{(1)}(0, \cdot\,) = \varphi \quad \text{in } (0, 1).$$

Continuing this process, there exist $0 = t_0 < t_1 < \cdots < t_{N-1} \leqslant T_1 - T_{\text{opt}} < t_N < \beta - T_{\text{opt}}$ and a family of $v^{(\ell)}$ with $1 \leqslant \ell \leqslant N$ such that $t_\ell \in I \setminus D$,

$$\partial_t v^{(\ell)}(t, x) = \Sigma(x)\partial_x v^{(\ell)}(t, x) + \mathbf{C}(t, x)v^{(\ell)}(t, x) \quad \text{for } (t, x) \in (0, t_\ell + T_{\text{opt}} + \varepsilon_0) \times (0, 1), \tag{3.103}$$

with, for $t \in (0, t_\ell + T_{\text{opt}} + \varepsilon_0)$,

$$v^{(\ell)}(t, 1) = 0, \tag{3.104}$$

$$\Sigma_+(0)v_+^{(\ell)}(t, 0) = -B^{\mathsf{T}}\Sigma_-(0)v_-^{(\ell)}(t, 0), \tag{3.105}$$

$$v^{(\ell)}(t = 0, \cdot\,) = \varphi(\,\cdot\,) \quad \text{in } (0, 1), \tag{3.106}$$

and

$$\varepsilon_0/3 \leqslant t_\ell - t_{\ell-1} \leqslant \varepsilon_0/2.$$

This implies, by Proposition 3.7, that one cannot steer φ from time 0 to 0 at time T_1. We have a contradiction since the system is null-controllable at the time T_1. The conclusion follows in the case $0 \in I_2 \setminus D$.

The proof in the general case can be derived from the previous case by noting that, using the same arguments, one has

$$M(\tau_0) = \{0\} \text{ for } \tau_0 \in I_2 \setminus D \text{ and } \tau_0 \text{ is close to } 0.$$

The details are omitted.

The proof is complete. □

The proof of Theorem 1.5 also yields the following unique continuation principle.

PROPOSITION 3.18. — *Let $k \geqslant m \geqslant 1$ and let $B \in \mathcal{B}$ be such that* (1.10) *holds for $i = m$. Assume that $C_1 \in \mathcal{H}\big(I, \left[L^\infty(0,1)\right]^{n\times n}\big)$ for some open interval I containing $[0, T_1]$. Let $\tau \in I$ and $T > T_{\mathrm{opt}}$. Assume that $\tau + T_1 \in I$. Let v be a solution of the system*

$$\partial_t v(t,x) = \Sigma(x)\partial_x v(t,x) + C_1(t,x)v(t,x) \quad \text{for } (t,x) \in (\tau, \tau+T)\times(0,1), \tag{3.107}$$

with, for $\tau < t < \tau + T$,

$$v_-(t,1) = 0, \tag{3.108}$$

$$\Sigma_+(0)v_+(t,0) = -B^{\mathsf{T}}\Sigma_-(0)v_-(t,0), \tag{3.109}$$

$$v_+(t,1) = 0. \tag{3.110}$$

Then $v = 0$.

Recall that $T_1 = \tau_k + \tau_{k+1}$ by (1.16).

Proof. — The conclusion of (3.18) follows from the proof of Theorem 1.5 applied to $C(t,x)$ defined by $\Sigma'(x) - C(t,x)^{\mathsf{T}} = C_1(t,x)$. □

The unique continuation result stated in Proposition 3.18 can be seen as a variant of the unique continuation principle for the wave equations whose first and zero-order terms are analytic in time due to Tataru–Hörmander–Robbiano–Zuily. Our strategy was mentioned at the beginning of Section 3. We do not know if such a unique continuation principle can be proved using Carleman's estimate as in the wave setting. It is worth noting that if this is possible then the analyticity of C_1 in time must be taken into account by Theorem 1.1. More importantly the conditions $B \in \mathcal{B}$ and (1.10) holding for $i = m$ have to be essentially used in the proof process since it is known that the unique continuation does not hold without this assumption even in the case $C_1 \equiv 0$. The advantage of Carleman's estimate might be that the analyticity of C_1 is only required for a neighborhood of $[0, T_{\mathrm{opt}}]$ instead of $[0, T_1]$.

Remark 3.19. — It is natural to compare the direct approach here with the one involving the backstepping technique. In the time-invariant setting, both approaches yield the same result since (1.10) with $i = m$ is not imposed to establish the compactness of $\mathcal{K}(\tau)$ (see the first step of the proof of Proposition 3.6). Nevertheless, (equivalent) control-forms obtained from the backstepping approach are easier to handle/understand. The analysis in this paper is strongly inspired/guided by such control-forms. In the time-varying setting, one might derive the same conclusion under the assumption that C is analytic in $\mathbb{R}$ and its holomorphic extension in $\{z \in \mathbb{C};\ |\Im(z)| < \gamma\}$ is bounded for some $\gamma > 0$. This quite strong assumption on the analyticity of C comes from the construction of the kernel in the step of using backstepping.

4. Exact controllability in the analytic setting — Proof of Theorem 1.6

Theorem 1.6 can be derived from Theorem 1.5, as in the proof of [18, Theorem 3]. For the convenience of the reader, we reproduce the proof.

We first consider the case $m = k$. Let $T > T_{\rm opt}$ be such that $T \in I$. Set

$$\widetilde{w}(t,x) = w(T-t,x) \quad \text{for } t \in (0,T),\ x \in (0,1).$$

Then

$$\widetilde{w}_-(t,0) = \widetilde{B}^{-1}\widetilde{w}_+(t,0),$$

with $\widetilde{w}_-(t,\cdot) = (w_{2k},\dots,w_{k+1})^{\mathsf{T}}(T-t,\cdot)$, and $\widetilde{w}_+(t,\cdot) = (w_k,\dots,w_1)^{\mathsf{T}}(T-t,\cdot)$, and $\widetilde{B}_{ij} = B_{pq}$ with $p = k-i+1$ and $q = k-j+1$. Note that the $i \times i$ matrix formed from the first i columns and rows of $\widetilde{B}$ is invertible. Using the Gaussian elimination method, one can find $(k \times k)$ matrices $T_1,\dots,T_N$ such that

$$T_N \cdots T_1 \widetilde{B} = \widetilde{U},$$

where $\widetilde{U}$ is a $(k \times k)$ upper triangular matrix, and T_i $(1 \leqslant i \leqslant N)$ is the matrix given by the operation which replaces a row p by itself plus a multiple of a row q for some $1 \leqslant q < p \leqslant N$. It follows that

$$\widetilde{B}^{-1} = \widetilde{U}^{-1} T_N \cdots T_1.$$

One can check that $\widetilde{U}^{-1}$ is an invertible, upper triangular matrix, and $T_N \cdots T_1$ is an invertible, lower triangular matrix. It follows that the $i \times i$ matrix formed from the last i columns and rows of $\widetilde{B}^{-1}$ is the product of the matrix formed from the last i columns and rows of $\widetilde{U}^{-1}$ and the matrix formed from the last i columns and rows of $T_N \cdots T_1$. Therefore, $\widetilde{B}^{-1} \in \mathcal{B}$ and (1.10) with B being replaced by $\widetilde{B}^{-1}$ holds for $i = k = m$. One can also check that the exact controllability of the system for $w(\,\cdot\,,\cdot)$ at the time T from time 0 is equivalent to the null-controllability of the system for $\widetilde{w}(\,\cdot\,,\cdot)$ at the same time from time 0. The conclusion of Theorem 1.6 now follows from Theorem 1.5 by noting that $C(\,\cdot - T,\cdot)$ is analytic in a neighborhood of $[0,T_1]$.

The case $m > k$ can be obtained from the case $m = k$ as follows. Consider $\widehat{w}(\cdot,\cdot)$ the solution of the system

$$\partial_t \widehat{w}(t,x) = \widehat{\Sigma}(x)\partial_x \widehat{w}(t,x) + \widehat{C}(t,x)\widehat{w}(t,x),$$
$$\widehat{w}_-(t,0) = \widehat{B}\widehat{w}_+(t,0), \text{ and } \widehat{w}_+(t,1) \text{ are controls.}$$

Here

$$\widehat{\Sigma} = \operatorname{diag}(-\widehat{\lambda}_1, \dots, -\widehat{\lambda}_m, \widehat{\lambda}_{m+1}, \dots, \widehat{\lambda}_{2m}),$$

with $\widehat{\lambda}_j = -(1+m-k-j)\varepsilon^{-1}$ for $1 \leqslant j \leqslant m-k$ with positive small ε, $\widehat{\lambda}_j = \lambda_{j-(m-k)}$ if $m-k+1 \leqslant j \leqslant m$, and $\widehat{\lambda}_{j+m} = \lambda_{j+k}$ for $1 \leqslant j \leqslant m$,

$$\widehat{C}(t,x) = \begin{pmatrix} 0_{m-k,m-k} & 0_{m-k,n} \\ 0_{n,m-k} & C(t,x) \end{pmatrix},$$

and

$$\widehat{B} = \begin{pmatrix} I_{m-k} & 0_{m-k,m} \\ 0_{m-k,m} & B \end{pmatrix},$$

where I_ℓ denotes the identity matrix of size $\ell \times \ell$ for $\ell \geqslant 1$. Here $0_{i,j}$ denotes the zero matrix of size $i \times j$ for $i, j, \ell \geqslant 1$. Then the exact controllability of w at the time T from time 0 can be derived from the exact controllability of $\widehat{w}$ at the same time from time 0. One then can deduce the conclusion of Theorem 1.6 from the case $m = k$ using Theorem 1.5 by noting that the optimal time for the system of $\widehat{w}$ converges to the optimal time for the system of w as $\varepsilon \to 0_+$. □

Appendix A. Hyperbolic systems in non-rectangle domains

In this section, we give the meaning of broad solutions used to define $\mathcal{T}(\tau, T)$ and $\widehat{\mathcal{T}}(\tau, T)$ for $T \geqslant T_{\rm opt}$ and thus to define $\mathcal{T}(\tau)$ and $\widehat{\mathcal{T}}(\tau)$, which are introduced in the proof of Proposition 3.6 and Proposition 3.7. We also study their well-posedness and establish the boundedness and the analyticity of $\mathcal{T}(\tau)$ under appropriate assumptions. The key point of the analysis is to find suitably weighted norms in order to apply the fixed point arguments. This matter is non-standard and subtle (see Remark A.9). In this section, we assume that $k \geqslant m \geqslant 1$ although the arguments are quite robust and also work for the case $m > k \geqslant 1$ under appropriate modifications.

Let $T \geqslant T_{\rm opt}$, $F \in \left[L^\infty(\Omega_T)\right]^{n\times n}$, $(f,g) \in \left[L^2(0,T)\right]^n \times \left[L^2(0,1)\right]^m$, and $\gamma \in \left[L^2(\Omega_{T,k,t})\right]^n$. Recall that Ω_T is defined in (3.21) and $\Omega = \Omega_{T_{\rm opt}}$. We first deal with the following system, which is slightly more general than the system (3.31)–(3.36) with T_{opt} being replaced by T:

$$\partial_t w(t,x) = \Sigma(x)\partial_x w(t,x) + F(t,x)w(t,x) + \gamma(t,x) \quad \text{for } (t,x) \in \Omega_T, \tag{A.1}$$

$$w(\cdot,1) = f \quad \text{in } (0,T), \tag{A.2}$$

$$w_+(0,\cdot) = g \quad \text{in } (0,1), \tag{A.3}$$

$$w_{-,\geqslant k}(t,0) = Q_k w_{<k,\geqslant k+m}(t,0) \quad \text{for } t \in (T-\tau_k, T-\tau_{k-1}), \tag{A.4}$$

$$w_{-,\geqslant k-1}(t,0) = Q_{k-1} w_{<k-1,\geqslant k+m-1}(t,0) \quad \text{for } t \in (T-\tau_{k-1}, T-\tau_{k-2}), \tag{A.5}$$

$$\dots$$

$$w_{-,\geqslant k-m+2}(t,0) = Q_{k-m+2} w_{<k-m+2,\geqslant k+2}(t,0) \quad \text{for } t \in (T-\tau_{k-m+2}, T-\tau_{k-m+1}). \tag{A.6}$$

Given a subset O of $\mathbb{R}^2$ and a point $(t,x) \in \mathbb{R}^2$, we denote

$$O_t = \big\{y \in \mathbb{R}; (t,y) \in O\big\} \qquad \text{and} \qquad O_x = \big\{s \in \mathbb{R}; (s,x) \in O\big\}. \tag{A.7}$$

We next give the definition of the broad solutions of system (A.1)–(A.6).

Definition A.1. — *Let:*

- $T \geqslant T_{\text{opt}}$, $F \in \big[L^\infty(\Omega_T)\big]^{n\times n}$;
- $(f,g) \in \big[L^2(0,T)\big]^n \times \big[L^2(0,1)\big]^m$;
- $\gamma \in \big[L^2(\Omega_{T,k,t})\big]^n$.

A vector-valued function $w \in \mathcal{Y}_T := \big[L^2(\Omega_T)\big]^n \cap C\big([0,T]; \big[L^2(\Omega_{T,t})\big]^n\big) \cap C\big([0,1]; \big[L^2(\Omega_{T,x})\big]^n\big)$ *is called a broad solution of* (A.1)–(A.6) *if for almost every* $(t_1,\xi_1) \in \Omega_T$, *the following conditions hold:*[14]

(1) *for* $1 \leqslant j \leqslant k-m+1$,

$$w_j(t_1,\xi_1) = \int_t^{t_1} \Big(F\big(s, x_j(s,t_1,\xi_1)\big) w\big(s, x_j(s,t_1,\xi_1)\big)\Big)_j \,\mathrm{d}s + \int_t^{t_1} \gamma_j\big(s, x_j(s,t_1,\xi_1)\big)\,\mathrm{d}s + f_j(t), \tag{A.8}$$

where t is such that $x_j(t,t_1,\xi_1) = 1$;

[14] A function $\varphi \in L^2(\Omega)$ is said to be in $C([0,T_{\text{opt}}]; L^2(\Omega_t))$ if $(t_n) \subset [0,T_{\text{opt}}]$ converging to t then

$$\lim_{n\to+\infty}\Big(\|f(t_n,\cdot) - f(t,\cdot)\|_{L^2(\Omega_{t_n}\cap\Omega_t)} + \|f(t_n,\cdot)\|_{L^2(\Omega_{t_n}\setminus\Omega_t)} + \|f(t,\cdot)\|_{L^2(\Omega_t\setminus\Omega_{t_n})}\Big) = 0.$$

Similar meaning is used for $C([0,1]; L^2(\Omega_x))$.

(2) *for* $k-m+2 \leqslant j \leqslant k$,

$$w_j(t_1,\xi_1) = \int_t^{t_1} \Big(F\big(s,x_j(s,t_1,\xi_1)\big)w\big(s,x_j(s,t_1,\xi_1)\big)\Big)_j \,\mathrm{d}s + \int_t^{t_1} \gamma_j\big(s,x_j(s,t_1,\xi_1)\big)\,\mathrm{d}s + f_j(t), \tag{A.9}$$

if $t \in (0,T)$ *where* t *is such that* $x_j(t,t_1,\xi_1)=1$, *otherwise*

$$w_j(t_1,\xi_1) = \int_{\hat t}^{t_1} \Big(F\big(s,x_j(s,t_1,\xi_1)\big)w\big(s,x_j(s,t_1,\xi_1)\big)\Big)_j \,\mathrm{d}s + \int_{\hat t}^{t_1} \gamma_j\big(s,x_j(s,t_1,\xi_1)\big)\,\mathrm{d}s + \big(Q_l w_{<l,\geqslant l+m}(\widehat{t},0)\big)_{j-l+1}, \tag{A.10}$$

if $\widehat{t} \in (T-\tau_l, T-\tau_{l-1})$ *where* $\widehat{t}$ *is such that* $x_j(\widehat{t},t_1,\xi_1)=0$;

(3) *for* $k+1 \leqslant j \leqslant k+m$,

$$w_j(t_1,\xi_1) = \int_t^{t_1} \Big(F\big(s,x_j(s,t_1,\xi_1)\big)w\big(s,x_j(s,t_1,\xi_1)\big)\Big)_j \,\mathrm{d}s + \int_t^{t_1} \gamma_j\big(s,x_j(s,t_1,\xi_1)\big)\,\mathrm{d}s + f_j(t), \tag{A.11}$$

if $t \in (0,T)$ *where* t *is such that* $x_j(t,t_1,\xi_1)=1$, *otherwise*

$$w_j(t_1,\xi_1) = \int_0^{t_1} \Big(F\big(s,x_j(s,t_1,\xi_1)\big)w\big(s,x_j(s,t_1,\xi_1)\big)\Big)_j \,\mathrm{d}s + \int_0^{t_1} \gamma_j\big(s,x_j(s,t_1,\xi_1)\big)\,\mathrm{d}s + g_{j-k}(\eta), \tag{A.12}$$

where $\eta \in (0,1)$ *is such that* $x_j(0,t_1,\xi_1)=\eta$.

Recall that the characteristic flow x_j with $1 \leqslant j \leqslant k+m$ is defined in (3.19) and (3.20).

In this definition, the term $Q_l w_{<l,\geqslant l+m}(\widehat{t},0)$ in (A.10) is required to be replaced by the corresponding expression in the RHS of (A.8), or (A.9), or (A.11), or (A.12) with $(\widehat{t},0)$ standing for (t_1,ξ_1).

The well-posedness of broad solutions of (A.1)–(A.6) is given in the following theorem.

Theorem A.2. — *Let* $T \geqslant T_{\mathrm{opt}}$, $F \in \big[L^\infty(\Omega_T)\big]^{n\times n}$, $(f,g) \in \big[L^2(0,T)\big]^n \times \big[L^2(0,1)\big]^m$, *and* $\gamma \in \big[L^2(\Omega_{T,k,t})\big]^n$. *There exists a unique broad solution* $w \in \mathcal{Y}_T$ *of* (A.1)–(A.6). *Moreover,*

$$\|w\|_{\mathcal{Y}_T} \leqslant C\big(\|f\|_{L^2(0,T)} + \|g\|_{L^2(0,1)} + \|\gamma\|_{L^2(\Omega_T)}\big), \tag{A.13}$$

for some positive constant C depending on an upper bound of $\|F\|_{L^\infty(\Omega)}$, T, and Σ.

Here and in what follows, we denote

$$\|w\|_{\mathcal{Y}_T} = \max\left\{\sup_{x\in[0,1]} \|w_i\|_{L^2(\Omega_{T,x})}, \sup_{t\in[0,T]} \|w_i\|_{L^2(\Omega_{T,t})};\ 1 \leqslant i \leqslant n\right\}.$$

Remark A.3. — The analysis of Theorem A.2 can be easily extended to cover the case where source terms in L^2 are added in (A.4)–(A.6).

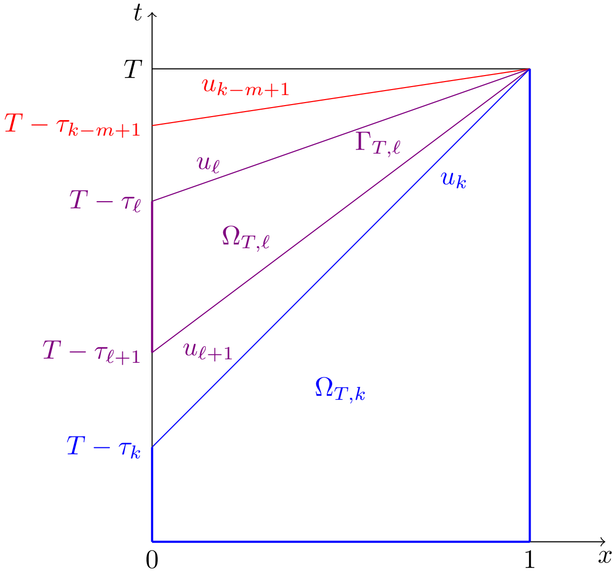

Figure A.1. Geometry of $\Omega_{T,\ell}$ and $\Gamma_{T,\ell}$ with $k-m+1 \leqslant \ell \leqslant k$ for a constant Σ.

Before giving the proof of Theorem A.2, let us introduce some notations. For $k-m+1 \leqslant \ell \leqslant k-1$, let $\Omega_{T,\ell}$ be the region of Ω_T between the characteristic curves of x_ℓ and $x_{\ell+1}$ both passing the point $(T,1)$ in the xt-plane. We also denote $\Omega_{T,k}$ the region of Ω_T below the characteristic curve of x_k passing the point $(T,1)$ in the xt-plane. Let $\Gamma_{T,\ell}$ with $k-m+1 \leqslant \ell \leqslant k$ be the boundary part of $\Omega_{T,\ell}$ formed by the characteristic curve of x_ℓ passing the point $(T,1)$. See Figure A.1.

The proof of Theorem A.2 is based on two lemmas below. The first one is the following.

Lemma A.4. — *Let* $T \geqslant T_{\mathrm{opt}}$, $F \in \left[L^\infty(\Omega_{T,k})\right]^{n\times n}$, $(f,g) \in \left[L^2(0,T)\right]^n \times \left[L^2(0,1)\right]^m$, *and* $\gamma \in \left[L^2(\Omega_{T,k})\right]^n$. *There exists a unique broad solution* $w \in \mathcal{Y}_{T,k} \coloneqq \left[L^2(\Omega_{T,k})\right]^n \cap C\big([0,T]; \left[L^2(\Omega_{T,k,t})\right]^n\big) \cap C\big([0,1]; \left[L^2(\Omega_{T,k,x})\right]^n\big)$ *of the system*

$$\partial_t w(t,x) = \Sigma(x)\partial_x w(t,x) + F(t,x)w(t,x) + \gamma(t,x) \quad \text{for } (t,x) \in \Omega_{T,k}, \tag{A.14}$$

$$w(\,\cdot\,,1) = f \quad \text{in } (0,T), \tag{A.15}$$

$$w_+(0,\cdot\,) = g \quad \text{in } (0,1). \tag{A.16}$$

Moreover,

$$\|w\|_{\mathcal{Y}_{T,k}} \leqslant C\big(\|f\|_{L^2(0,T)} + \|g\|_{L^2(0,1)} + \|\gamma\|_{L^2(\Omega_{T,k})}\big) \tag{A.17}$$

for some positive constant C *depending only on an upper bound of* $\|F\|_{L^\infty(\Omega_{T,k})}$ *and* T, *and* Σ.

Here and in what follows, we denote, for $k-m+1 \leqslant \ell \leqslant k$,

$$\|w\|_{\mathcal{Y}_{T,\ell}} = \max\left\{ \sup_{x\in[0,1]} \|w_i\|_{L^2(\Omega_{T,\ell,x})}, \sup_{t\in[0,T]} \|w_i\|_{L^2(\Omega_{T,\ell,t})};\ 1 \leqslant i \leqslant n \right\}.$$

The broad solutions considered in Lemma A.4 are defined similarly as the ones of (A.1)–(A.6) given in Definition A.1 as follows.

Definition A.5. — *Let* $T \geqslant T_{\mathrm{opt}}$, $F \in \left[L^\infty(\Omega_{T,k})\right]^{n\times n}$, *and* $(f,g) \in \left[L^2(0,T)\right]^n \times \left[L^2(0,1)\right]^m$, *and* $\gamma \in \left[L^2(\Omega_{T,k})\right]^n$. *A vector-valued function* $w \in \mathcal{Y}_{T,k}$ *is called a broad solution of* (A.14)–(A.16) *if for almost every* $(t_1,\xi_1) \in \Omega_{T,k}$, *the following conditions hold:*

(1) *for* $1 \leqslant j \leqslant k$,

$$w_j(t_1,\xi_1) = \int_t^{t_1} \Big(F\big(s,x_j(s,t_1,\xi_1)\big)w\big(s,x_j(s,t_1,\xi_1)\big)\Big)_j \,\mathrm{d}s + \int_t^{t_1} \gamma_j\big(s,x_j(s,t_1,\xi_1)\big)\,\mathrm{d}s + f_j(t), \tag{A.18}$$

where t *is such that* $x_j(t,t_1,\xi_1) = 1$;

(2) *for* $k+1 \leqslant j \leqslant k+m$,

$$w_j(t_1,\xi_1) = \int_t^{t_1} \Big(F\big(s,x_j(s,t_1,\xi_1)\big)w\big(s,x_j(s,t_1,\xi_1)\big)\Big)_j \,\mathrm{d}s + \int_t^{t_1} \gamma_j\big(s,x_j(s,t_1,\xi_1)\big)\,\mathrm{d}s + f_j(t), \tag{A.19}$$

if $t \in (0,T)$ where t is such that $x_j(t,t_1,\xi_1) = 1$, otherwise

$$w_j(t_1,\xi_1) = \int_0^{t_1} \Big(F\big(s, x_j(s,t_1,\xi_1)\big) w\big(s, x_j(s,t_1,\xi_1)\big)\Big)_j \, \mathrm{d}s + \int_0^{t_1} \gamma_j\big(s, x_j(s,t_1,\xi_1)\big) \, \mathrm{d}s + g_{j-k}(\eta), \tag{A.20}$$

where $\eta \in (0,1)$ is such that $x_j(0,t_1,\xi_1) = \eta$.

Proof of Lemma A.4. — For $v \in \big[L^2(\Omega_{T,k})\big]^n$, set

$$T_k(v)(t,x) = e^{Lx} v(t,x) \quad \text{for } (t,x) \in \Omega_{T,k},$$

where L is a large positive constant determined later.

We now introduce

$$\|v\|_{\Omega_{T,k}} := \max\Big\{ \sup_{x\in[0,1]} \big\|(T_k v)_i\big\|_{L^2(\Omega_{T,k,x})}, \sup_{t\in[0,T]} \big\|(Tv)_i\big\|_{L^2(\Omega_{T,k,t})};\ 1 \leqslant i \leqslant n\Big\}.$$

One can check that $\mathcal{Y}_{T,k}$ equipped with the norm $\|\cdot\|_{\Omega_{T,k}}$ is a Banach space. It is also clear that $\|\cdot\|_{\Omega_{T,k}}$ is equivalent to $\|\cdot\|_{\mathcal{Y}_{T,k}}$.

The proof is now based on a fixed-point argument. To this end, define $\mathcal{F}_k$ from $\mathcal{Y}_{T,k}$ into itself as follows: for $v \in \mathcal{Y}_{T,k}$, and for $(t_1,\xi_1) \in \Omega_{T,k}$ and $1 \leqslant j \leqslant k+m$,

$$\big(\mathcal{F}_k(v)\big)_j(t_1,\xi_1) \text{ is the RHS of (A.18), or (A.19), or (A.20)} \tag{A.21}$$

under the corresponding conditions.

We claim that, for L large enough, $\mathcal{F}_k$ is a contraction mapping from $\mathcal{Y}_{T,k}$ equipped with the norm $\|\cdot\|_{\Omega_{T,k}}$ into itself; and the conclusion follows then.

For $v \in \mathcal{Y}_{T,k}$, one can check that $\mathcal{F}(v) \in \mathcal{Y}_{T,k}$.

Let $v, w \in \mathcal{Y}_{T,k}$ be arbitrary. Fix $\xi_1 \in [0,1]$. Let $1 \leqslant j \leqslant k$. We have for $(t_1,\xi_1) \in \Omega_{T,k}$, by (A.18),

$$\mathcal{F}(v)_j(t_1,\xi_1) - \mathcal{F}(w)_j(t_1,\xi_1) = \int_t^{t_1} \Big(F\big(s, x_j(s,t_1,\xi_1)\big)(v-w)\big(s, x_j(s,t_1,\xi_1)\big)\Big)_j \, \mathrm{d}s, \tag{A.22}$$

where $t = t(t_1, \xi_1)$ is such that $x_j(t, t_1, \xi_1) = 1$. This implies

$$\int_{\Omega_{T,k,\xi_1}} e^{2L\xi_1} \big|\mathcal{F}(v)_j(t_1,\xi_1) - \mathcal{F}(w)_j(t_1,\xi_1)\big|^2 \, \mathrm{d}t_1 \\ \leqslant C \int_{\Omega_{T,k,\xi_1}} \operatorname{sign}(t - t_1) \int_{t_1}^{t} e^{2L\xi_1} |v - w|^2\big(s, x_j(s, t_1, \xi_1)\big) \, \mathrm{d}s \, \mathrm{d}t_1,$$

where $\operatorname{sign}(\theta) = 1$ if $\theta > 0$ and -1 if $\theta < 0$. Here and in what follows in this proof, C denotes a positive constant which depends only on an upper bound of $\|F\|_{L^\infty(\Omega_{T,k})}$ and T, and Σ, and can change from one place to another.

Since

$$e^{2L\xi_1} |v - w|^2\big(s, x_j(s, t_1, \xi_1)\big) \\ = e^{2L(\xi_1 - x_j(s,t_1,\xi_1))} e^{2L x_j(s,t_1,\xi_1)} |v - w|^2\big(s, x_j(s, t_1, \xi_1)\big),$$

and, for s between t_1 and t,

$$\xi_1 - x_j(s, t_1, \xi_1) \leqslant 0,$$

by a change of variables $x = x_j(s, t_1, \xi_1)$,[15] one obtains, for $1 \leqslant j \leqslant k$,

$$\begin{aligned} &\int_{\Omega_{T,k,\xi_1}} e^{2L\xi_1} \big|\mathcal{F}(v)_j(t_1,\xi_1) - \mathcal{F}(w)_j(t_1,\xi_1)\big|^2 \, \mathrm{d}t_1 \\ &\qquad \leqslant C \int_{\Omega_{T,k;x \geqslant \xi_1}} e^{2L(\xi_1 - x)} e^{2Lx} |v - w|^2(s, x) \, \mathrm{d}s \, \mathrm{d}x \\ &\qquad \leqslant \frac{C}{L} \|v - w\|^2_{\Omega_{T,k}}. \end{aligned} \tag{A.23}$$

We next consider $k + 1 \leqslant j \leqslant k + m$. Using (A.19) and (A.20), similar to (A.23) for $1 \leqslant j \leqslant k$, we also reach (A.23) for $k + 1 \leqslant j \leqslant k + m$. Combining this with (A.23) for $1 \leqslant j \leqslant k$ yields

$$\int_{\Omega_{T,k,\xi_1}} e^{2L\xi_1} \big|\mathcal{F}(v)(t_1,\xi_1) - \mathcal{F}(w)(t_1,\xi_1)\big|^2 \, \mathrm{d}t_1 \leqslant \frac{C}{L} \|v - w\|^2_{\Omega_{T,k}}. \tag{A.24}$$

Fix $t_1 \in [0, T]$. Let $1 \leqslant j \leqslant k$. From (A.22), we obtain, for $(t_1, \xi_1) \in \Omega_{T,k}$,

$$\int_{\Omega_{T,k,t_1}} e^{2L\xi_1} \big|\mathcal{F}(v)_j(t_1,\xi_1) - \mathcal{F}(w)_j(t_1,\xi_1)\big|^2 \, \mathrm{d}\xi_1 \\ \leqslant C \int_{\Omega_{T,k,t_1}} \operatorname{sign}(t - t_1) \int_{t_1}^{t} e^{2L\xi_1} |v - w|^2\big(s, x_j(s, t_1, \xi_1)\big) \, \mathrm{d}s \, \mathrm{d}t_1.$$

[15] x_j is continuously differentiable with respect to s, t_1, ξ_1 when $x_j(s, t_1, \xi_1)$ is in $\overline{\Omega}$ since Σ is of class C^2.

Similar to (A.23), we obtain, for $1 \leqslant j \leqslant k$,

$$\begin{aligned}\int_{\Omega_{T,k,t_1}} e^{2L\xi_1} \big|\mathcal{F}(v)_j(t_1,\xi_1) - \mathcal{F}(w)_j(t_1,\xi_1)\big|^2 \,\mathrm{d}\xi_1 \\ \leqslant C \int_{\Omega_{T,k};x \geqslant \xi_1} e^{2L(\xi_1 - x)} e^{2Lx} |v-w|^2(s,x)\,\mathrm{d}s\,\mathrm{d}t_1 \\ \leqslant \frac{C}{L} \|v-w\|^2_{\Omega_{T,k}}.\end{aligned} \tag{A.25}$$

Using (A.19) and (A.20), similar to (A.25) for $1 \leqslant j \leqslant k$, we also reach (A.25) for $k+1 \leqslant j \leqslant k+m$. Combining this with (A.25) for $1 \leqslant j \leqslant k$ yields

$$\int_{\Omega_{T,t_1}} e^{-2L\xi_1} \big|\mathcal{F}(v)(t_1,\xi_1) - \mathcal{F}(w)(t_1,\xi_1)\big|^2 \,\mathrm{d}\xi_1 \leqslant \frac{C}{L} \|v-w\|^2_{\Omega_{T,k}}. \tag{A.26}$$

The claim now follows from (A.24) and (A.26). The proof is complete. □

The second lemma used in the proof of Theorem A.2 is the following.

Lemma A.6. — *Let $k-m+1 \leqslant \ell \leqslant k-1$, $T \geqslant T_{\mathrm{opt}}$, $F \in \big[L^\infty(\Omega_{T,\ell})\big]^{n\times n}$, $\gamma \in \big[L^2(\Omega_\ell)\big]^n$, and $h_j \in L^2(\Gamma_{T,\ell+1})$ for $1 \leqslant j \leqslant k+m$ and $j \neq \ell+1$. There exists a unique broad solution*

$$w \in \mathcal{Y}_{T,\ell} := \big[L^2(\Omega_{T,\ell})\big]^n \cap C\big([0,T];\big[L^2(\Omega_{T,\ell,t})\big]^n\big) \cap C\big([0,1];\big[L^2(\Omega_{T,\ell,x})\big]^n\big)$$

of the system

$$\partial_t w(t,x) = \Sigma(x)\partial_x w(t,x) + F(t,x)w(t,x) + \gamma(t,x) \quad \text{for } (t,x) \in \Omega_{T,\ell}, \tag{A.27}$$

$$w_j = h_j \quad \text{on } \Gamma_{T,\ell+1}, \ \text{ for } 1 \leqslant j \leqslant k+m \text{ and } j \neq \ell+1, \tag{A.28}$$

$$w_{-,\geqslant \ell+1}(0,\cdot) = Q_{\ell+1} w_{<\ell+1,\geqslant k+\ell+1} \quad \text{for } t \in (T-\tau_{\ell+1}, T-\tau_\ell). \tag{A.29}$$

Moreover,

$$\|w\|_{\mathcal{Y}_{T,\ell}} \leqslant C \Bigg(\sum_{1 \leqslant j \leqslant k+m; j \neq \ell+1} \|h_j\|_{L^2(\Gamma_{T,\ell})} + \|\gamma\|_{L^2(\Omega_{T,\ell})} \Bigg)$$

for some positive constant C depending only on Σ, the upper bound of T and an upper bound of $\|F\|_{L^\infty(\Omega_{T,\ell})}$.

Remark A.7. — The analysis of Lemma A.6 can be easily extended to cover the case where source terms in L^2 are added in (A.29).

The broad solutions considered in Lemma A.6, which are in the same spirit of the ones in Lemma A.4, are defined as follows.

Definition A.8. — *Let $k-m+1 \leqslant \ell \leqslant k-1$, $T \geqslant T_{\text{opt}}$, $F \in \big[L^\infty(\Omega_{T,\ell})\big]^{n\times n}$, $\gamma \in \big[L^2(\Omega_{T,\ell})\big]^n$, and $h_j \in L^2(\Gamma_{T,\ell+1})$ for $1 \leqslant j \leqslant k+m$ and $j \neq \ell+1$. A vector-valued function $w \in \mathcal{Y}_{T,\ell}$ is called a broad solution of* (A.27)–(A.29) *if for almost every $(t_1,\xi_1) \in \Omega_{T,\ell}$, the following conditions hold:*

(1) *for $1 \leqslant j \leqslant \ell$ and for $k+1 \leqslant j \leqslant k+m$,*

$$w_j(t_1,\xi_1) = \int_t^{t_1} \Big(F\big(s,x_j(s,t_1,\xi_1)\big)w\big(s,x_j(s,t_1,\xi_1)\big)\Big)_j \,\mathrm{d}s + \int_t^{t_1} \gamma_j\big(s,x_j(s,t_1,\xi_1)\big)\,\mathrm{d}s + h_j(t), \tag{A.30}$$

where t is such that $x_j(t,t_1,\xi_1) \in \Gamma_{T,\ell+1}$;

(2) *for $\ell+1 \leqslant j \leqslant k$,*

$$w_j(t_1,\xi_1) = \int_{\hat t}^{t_1} \Big(F\big(s,x_j(s,t_1,\xi_1)\big)w\big(s,x_j(s,t_1,\xi_1)\big)\Big)_j \,\mathrm{d}s + \int_{\hat t}^{t_1} \gamma_j\big(s,x_j(s,t_1,\xi_1)\big)\,\mathrm{d}s + \big(Q_{\ell+1}w_{<\ell+1,\geqslant \ell+m+1}\big)_{j-\ell}(\widehat{t},0) \tag{A.31}$$

if $\widehat{t} \in (T-\tau_{\ell+1}, T-\tau_\ell)$ where $\widehat{t}$ is such that $x_j(\widehat{t},t_1,\xi_1) = 0$, otherwise

$$w_j(t_1,\xi_1) = \int_t^{t_1} \Big(F\big(s,x_j(s,t_1,\xi_1)\big)w\big(s,x_j(s,t_1,\xi_1)\big)\Big)_j \,\mathrm{d}s + \int_t^{t_1} \gamma_j\big(s,x_j(s,t_1,\xi_1)\big)\,\mathrm{d}s + h_j(t), \tag{A.32}$$

where t is such that $x_j(t,t_1,\xi_1) \in \Gamma_{T,\ell+1}$.

As in Definition A.1, the term $Q_{\ell+1}w_{<\ell+1,\geqslant\ell+m+1}(\widehat{t},0)$ in (A.31) is required to be replaced by the corresponding expression in the RHS of (A.30) with $(\widehat{t},0)$ standing for (t_1,ξ_1).

Proof of Lemma A.6. — The key part of the proof is to introduce an appropriate weighted norm, which is adapted to the geometry and the boundary conditions considered, for which the fixed point argument works (see Remark A.9 for comments on this point).

We begin with the case where Σ is constant. For $1 \leqslant j \leqslant k+m$, let $\vec{v}_j$ be the unit vector parallel to the characteristic curve of x_j directed to the boundary for which the boundary condition for v_j is given ($\vec{v}_j$ is parallel to $(1,\Sigma_{jj})^\mathsf{T}$ in the xt-plane). Set

$$G_1 = \big\{\vec{v}_j;\ 1 \leqslant j \leqslant \ell,\ k+1 \leqslant j \leqslant k+m\big\} \quad\text{and}\quad G_2 = \big\{\vec{v}_j;\ \ell+1 \leqslant j \leqslant k\big\}.$$

Here are some useful observations. There exist two non-zero vectors $\vec{u}_1$ and $\vec{u}_2$ such that:

(a1) $G_1 \cup G_2 \cup \{\vec{u}_1\}$ lies strictly on one side of the line containing $\vec{u}_2$;

(a2) G_1 is a subset of the open, solid, cone centered at the origin and formed by $\vec{u}_1$ and $\vec{u}_2$, i.e., in the set $\{s_1\vec{u}_1 + s_2\vec{u}_2;\ s_1, s_2 > 0\}$;

(a3) G_2 is a subset of the open, solid, cone centered at the origin and formed by $\vec{u}_1$ and $-\vec{u}_2$, i.e., in the set $\{s_1\vec{u}_1 - s_2\vec{u}_2;\ s_1, s_2 > 0\}$.

(For example, one can choose $\vec{u}_1 = (0, -1)^\mathsf{T}$ and $\vec{u_2}$ is close to $\vec{v}_\ell$ but with a larger slope in the xt-plane, see Figure A.2.)

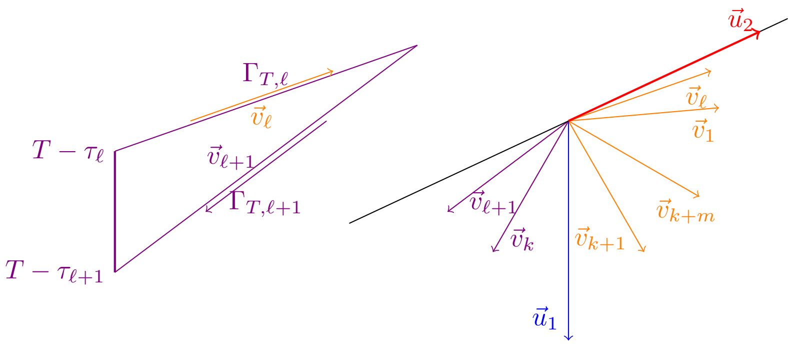

Figure A.2. Geometry of $\vec{v}_j$ for $1 \leqslant j \leqslant n$, and $\vec{u}_1$ and $\vec{u}_2$ for $\Omega_{T,\ell}$ when Σ is constant.

We are ready to introduce the weighted norm used. For $v \in \left[L^2(\Omega_{T,\ell})\right]^n$, set

$$T_\ell(v)(t,x) = e^{Ly_1(t,x)}v(t,x) \quad \text{for } (t,x) \in \Omega_{T,\ell}, \tag{A.33}$$

where $y_1(t,x)$ is the first component of $(y_1, y_2)(t,x)$ which is the coordinate of (t,x) corresponding to the basis $\vec{u}_1$ and $\vec{u}_2$ (in the xt-plane).

We now introduce

$$\|v\|_{\Omega_{T,\ell}} := \max\Big\{ \sup_{x\in[0,1]} \big\|(T_\ell v)_i\big\|_{L^2(\Omega_{T,\ell,x})}, \sup_{t\in[0,T]} \big\|(T_\ell v)_i\big\|_{L^2(\Omega_{T,\ell,t})};\ 1 \leqslant i \leqslant n \Big\}. \tag{A.34}$$

One can check that $\mathcal{Y}_{T,\ell}$ equipped with the norm $\|\cdot\|_{\Omega_{T,\ell}}$ is a Banach space. It is also clear that $\|\cdot\|_{\Omega_{T,\ell}}$ is equivalent to $\|\cdot\|_{\mathcal{Y}_{T,\ell}}$.

The proof is now based on a fixed point argument as in the one of Lemma A.4. To this end, define $\mathcal{F}_\ell$ from $\mathcal{Y}_{T,\ell}$ equipped with the norm

$\|\cdot\|_{\Omega_{T,\ell}}$ into itself as follows: for $v \in \mathcal{Y}_{T,\ell}$ and for $(t_1, \xi_1) \in \Omega_{T,\ell}$,

$$\big(\mathcal{F}_\ell(v)\big)_i(t_1, \xi_1) \text{ is the RHS of (A.30), or (A.31), or (A.32)} \tag{A.35}$$

under the corresponding conditions.

Fix $\xi_1 \in [0,1]$. Let $1 \leqslant j \leqslant \ell$ or $k+1 \leqslant j \leqslant k+m$. We have, for $(t_1, \xi_1) \in \Omega_{T,\ell}$, by (A.30),

$$\begin{aligned} &\mathcal{F}(v)_j(t_1,\xi_1) - \mathcal{F}(w)_j(t_1,\xi_1) \\ &\qquad = \int_t^{t_1} \Big(F\big(s, x_j(s,t_1,\xi_1)\big)(v-w)\big(s, x_j(s,t_1,\xi_1)\big)\Big)_j \,\mathrm{d}s, \end{aligned} \tag{A.36}$$

where t is such that $x_j(t, t_1, \xi_1) \in \Gamma_{T,\ell+1}$. This implies

$$\begin{aligned} &\int_{\Omega_{T,\ell,\xi_1}} e^{2Ly_1(t_1,\xi_1)} \big|\mathcal{F}(v)_j(t_1,\xi_1) - \mathcal{F}(w)_j(t_1,\xi_1)\big|^2 \,\mathrm{d}t_1 \\ &\leqslant C \int_{\Omega_{T,\ell,\xi_1}} \operatorname{sign}(t-t_1) \int_{t_1}^{t} e^{2Ly_1(t_1,\xi_1)} |v-w|^2\big(s, x_j(s,t_1,\xi_1)\big) \,\mathrm{d}s \,\mathrm{d}t_1. \end{aligned} \tag{A.37}$$

Here and in what follows in this proof, C (resp. c) denotes a positive constant which depends only on an upper bound of $\|F\|_{L^\infty(\Omega_{T,k})}$ and T, and Σ, and can change from one place to another.

We have

$$\begin{aligned} &e^{2Ly_1(t_1,\xi_1)} |v-w|^2\big(s, x_j(s,t_1,\xi_1)\big) \\ &= e^{2L(y_1(t_1,\xi_1) - y_1(s, x_j(s,t_1,\xi_1)))} e^{2Ly_1(s, x_j(s,t_1,\xi_1))} |v-w|^2\big(s, x_j(s,t_1,\xi_1)\big), \end{aligned} \tag{A.38}$$

and, for s between t_1 and t,

$$y_1(t_1,\xi_1) - y_1\big(s, x_j(s,t_1,\xi_1)\big) \leqslant -c\big|\xi_1 - x_j(s,t_1,\xi_1)\big| \tag{A.39}$$

by (a2) and the definition of G_1.

Making a change of variables $x = x_j(s,t_1,\xi_1)$, we derive from (A.37) that, for $1 \leqslant j \leqslant \ell$ or $k+1 \leqslant j \leqslant k+m$,

$$\begin{aligned} &\int_{\Omega_{T,\ell,\xi_1}} e^{2Ly_1(t_1,\xi_1)} \big|\mathcal{F}(v)_j(t_1,\xi_1) - \mathcal{F}(w)_j(t_1,\xi_1)\big|^2 \,\mathrm{d}t_1 \\ &\qquad \leqslant C \int_{\Omega_{T,\ell}} e^{-cL|\xi_1 - x|} e^{2Ly_1(s,x)} |v-w|^2(s,x) \,\mathrm{d}s \,\mathrm{d}x \leqslant \frac{C}{L} \|v-w\|^2_{\Omega_{T,\ell}}. \end{aligned} \tag{A.40}$$

We next deal with $\ell+1 \leqslant j \leqslant k$. Set

$$\Omega_{T,\ell,\xi_1,1} = \big\{t_1 \in [0,T];\ \text{(A.31) holds}\big\}$$

and

$$\Omega_{T,\ell,\xi_1,2} = \big\{t_1 \in [0,T];\ \text{(A.32) holds}\big\}.$$

We have, by (A.31), for $t_1 \in \Omega_{T,\ell,\xi_1,1}$,

$$\begin{aligned}\mathcal{F}(v)_j(t_1,\xi_1) - \mathcal{F}(w)_j(t_1,x_1) \\ = \int_{\hat{t}}^{t_1} \Big(F\big(s, x_j(s,t_1,\xi_1)\big)(v-w)\big(s, x_j(s,t_1,\xi_1)\big)\Big)_j \,\mathrm{d}s \\ + \big(Q_{\ell+1}(v-w)_{<\ell+1,\geqslant \ell+m+1}\big)_{j-\ell}(\widehat{t},0)\end{aligned} \tag{A.41}$$

where $\widehat{t} = \widehat{t}(t_1,\xi_1)$ is such that $x_j(\widehat{t},t_1,\xi_1) = 0$.

We next estimate

$$\int_{\Omega_{T,\ell,\xi_1,1}} \operatorname{sign}(\widehat{t} - t_1) \int_{t_1}^{\hat{t}} e^{2Ly_1(t_1,\xi_1)} |v-w|^2\big(s, x_j(s,t_1,\xi_1)\big) \,\mathrm{d}s \,\mathrm{d}t_1.$$

We have, for s between t_1 and $\widehat{t}$,

$$y_1(t_1,\xi_1) - y_1\big(s, x_j(s,t_1,\xi_1)\big) \leqslant -c\big|\xi_1 - x_j(s,t_1,\xi_1)\big| \tag{A.42}$$

by (a3) and the definition of G_2.

Making a change of variables $x = x_j(s,t_1,\xi_1)$, we derive from (A.38) that

$$\begin{aligned}\int_{\Omega_{T,\ell,\xi_1,1}} \operatorname{sign}(\widehat{t} - t_1) \int_{t_1}^{\hat{t}} e^{2Ly_1(t_1,\xi_1)} |v-w|^2\big(s, x_j(s,t_1,\xi_1)\big) \,\mathrm{d}s \,\mathrm{d}t_1 \\ \leqslant C \int_{\Omega_{T,\ell}} e^{-cL|\xi_1 - x|} e^{-2Ly_1(s,x)} |v-w|^2(s,x) \,\mathrm{d}s \,\mathrm{d}x.\end{aligned}$$

This implies

$$\begin{aligned}\int_{\Omega_{T,\ell,\xi_1}} \operatorname{sign}(\widehat{t} - t_1) \int_{t_1}^{\hat{t}} e^{2Ly_1(t_1,\xi_1)} |v-w|^2\big(s, x_j(s,t_1,\xi_1)\big) \,\mathrm{d}s \,\mathrm{d}t_1 \\ \leqslant \frac{C}{L} \|v-w\|^2_{\Omega_{T,\ell}}.\end{aligned} \tag{A.43}$$

By (A.40), we also have

$$\begin{aligned}\int_{\Omega_{T,\ell,0}} e^{2Ly_1(\hat{t},0)} \big|Q_{\ell+1}(v-w)_{<\ell+1,\geqslant \ell+m+1}(\widehat{t},0)\big|^2 \,\mathrm{d}\widehat{t} \\ \leqslant \frac{C}{L} \|v-w\|^2_{\Omega_{T,\ell}}.\end{aligned} \tag{A.44}$$

Using (A.42), and making a change of variable $\widehat{t} = \widehat{t}(t_1,\xi_1)$, we derive that

$$\begin{aligned}\int_{\Omega_{T,\ell,\xi_1}} e^{Ly_1(t_1,\xi_1)} \big|Q_{\ell+1}(v-w)_{<\ell+1,\geqslant \ell+m+1}\big(\widehat{t}(t_1,\xi_1),0\big)\big|^2 \,\mathrm{d}t_1 \\ \leqslant C \int_{\Omega_{T,\ell,0}} e^{2Ly_1(\hat{t},0)} \big|Q_{\ell+1}(v-w)_{<\ell+1,\geqslant \ell+m+1}(\widehat{t},0)\big|^2 \,\mathrm{d}\widehat{t}.\end{aligned} \tag{A.45}$$

Combining (A.31), (A.43), (A.44), and (A.45) yields, for $\ell+1 \leqslant j \leqslant k$,

$$\int_{\Omega_{T,\ell,\xi_1,1}} e^{2Ly_1(t_1,\xi_1)}\left|\mathcal{F}(v)_j(t_1,\xi_1)-\mathcal{F}(w)_j(t_1,x_1)\right|^2 \mathrm{d}t_1 \leqslant \frac{C}{L}\|v-w\|^2_{\Omega_{T,\ell}}. \tag{A.46}$$

Using similar arguments, we also obtain, for $\ell+1 \leqslant j \leqslant k$,

$$\int_{\Omega_{T,\ell,\xi_1,2}} e^{2Ly_1(t_1,\xi_1)}\left|\mathcal{F}(v)_j(t_1,\xi_1)-\mathcal{F}(w)_j(t_1,x_1)\right|^2 \mathrm{d}t_1 \leqslant \frac{C}{L}\|v-w\|^2_{\Omega_{T,\ell}}. \tag{A.47}$$

We derive from (A.46) and (A.47) that

$$\int_{\Omega_{T,\ell,\xi_1}} e^{2Ly_1(t_1,\xi_1)}\left|\mathcal{F}(v)_j(t_1,\xi_1)-\mathcal{F}(w)_j(t_1,x_1)\right|^2 \mathrm{d}t_1 \leqslant \frac{C}{L}\|v-w\|^2_{\Omega_{T,\ell}}. \tag{A.48}$$

From (A.40) and (A.48), we obtain

$$\int_{\Omega_{T,\ell,\xi_1}} e^{2Ly_1(t_1,\xi_1)}\left|\mathcal{F}(v)(t_1,\xi_1)-\mathcal{F}(w)(t_1,x_1)\right|^2 \mathrm{d}t_1 \leqslant \frac{C}{L}\|v-w\|^2_{\Omega_{T,\ell}}. \tag{A.49}$$

For $t_1 \in (0, T_{\mathrm{opt}})$, by the same approach used to derive (A.49), we also have

$$\int_{\Omega_{T,\ell t_1}} e^{2Ly_1(t_1,\xi_1)}\left|\mathcal{F}(v)(t_1,\xi_1)-\mathcal{F}(w)(t_1,x_1)\right|^2 \mathrm{d}\xi_1 \leqslant \frac{C}{L}\|v-w\|^2_{\Omega_{T,\ell}}. \tag{A.50}$$

The conclusion in the case where Σ is constant now follows from (A.49) and (A.50).

We next make necessary modifications to derive the conclusion in the general case. The idea is to find a replacement for $y_1(t,x)$ which is *increasing* when one follows the characteristic flows directed to the boundary for which the boundary conditions are imposed. To this end, for $1 \leqslant j \leqslant k+m$, let $\vec{v}_j = \vec{v}_j(t,x)$ be the unit vector tangent to the characteristic curve of x_j at the point (t,x) directed to the boundary where the boundary condition for v_j is given. The vector $\vec{v}_j(t,x)$ is parallel to $\big(1, \Sigma_{jj}(x)\big)^{\mathsf{T}}$ in the xt-plane so that one can choose it independent of t and in fact, we will do. We will denote it by $\vec{v}_j(x)$ from now on. Set

$$G_1(x) = \big\{\vec{v}_j(x);\ 1 \leqslant j \leqslant \ell,\ k+1 \leqslant j \leqslant k+m\big\}$$

and

$$G_2(x) = \big\{\vec{v}_j(x);\ \ell+1 \leqslant j \leqslant k\big\}.$$

Let $\varphi(x)$ be such that $\vec{v}_\ell(x)$ is parallel to and has the same direction with $\big(\varphi(x), 1\big)^{\mathsf{T}}$. Set, in the xt-plane,

$$\vec{u}_1(x) = (0, -1)^{\mathsf{T}},$$

and

$$\vec{u}_2(x) = \big(\varphi(x) - \varepsilon, 1\big)^{\mathsf{T}},$$

where ε is a constant that is positive and sufficiently small, the smallness of ε is independent of x, such that, $\varphi(x) > 2\varepsilon$, and:

(a1) $G_1(x) \cup G_2(x) \cup \big\{\vec{u}_1(x)\big\}$ lies on one side of the line containing $\vec{u}_2(x)$;

(a2) $G_1(x)$ is a subset of the open solid cone centered at the origin and formed by $\vec{u}_1(x)$ and $\vec{u}_2(x)$, i.e., in the set $\big\{s_1\vec{u}_1(x) + s_2\vec{u}_2(x);\ s_1, s_2 > 0\big\}$;

(a3) $G_2(x)$ is a subset of the open solid cone centered at the origin and formed by $\vec{u}_1(x)$ and $-\vec{u}_2(x)$, i.e., in the set $\big\{s_1\vec{u}_1(x) - s_2\vec{u}_2(x);\ s_1, s_2 > 0\big\}$.

Fix such a positive constant ε. For a point $(x_0, t_0) \in \Omega_{T,\ell}$, let $\big(x(s), t(s)\big)$ for $s \in [\alpha, \beta] \subset \mathbb{R}$ be a (piecewise) C^1 regular curve in $\overline{\Omega}_{T,\ell}$ (in the xt-plane) starting from $(T, 1)$ and arriving at (x_0, t_0).[16] We first claim that

$$\text{(A.51)}\quad \int_\alpha^\beta y_1\big(x'(s), t'(s), x(s), t(s)\big)\big|\big(x'(s), t'(s)\big)\big|\,\mathrm{d}s \text{ depends on } (t_0, x_0)$$
$$\text{but is independent of the curve and the parametrization.}$$

Here $y_1\big(t'(s), x'(s), t(s), x(s)\big)$ is the first coordinate of the vector

$$\big(t'(s), x'(s)\big)/\big|\big(t'(s), x'(s)\big)\big|$$

in the bases $\vec{u}_1\big(t(s), x(s)\big)$ and $\vec{u}_2\big(t(s), x(s)\big)$.

We now establish the claim. For notational ease, we assume that

$$\big|\big(t'(s), x'(s)\big)\big| = 1.$$

We first compute $y_1\big(t'(s), x'(s), t(s), x(s)\big)$. Let a and b in $\mathbb{R}$ be such that

$$\big(x'(s), t'(s)\big) = a(0, -1) + b\big(\varphi\big(x(s)\big) - \varepsilon, 1\big).$$

[16] Regularity means that $(x'(s), t'(s)) \neq (0, 0)$ for $s \in [\alpha, \beta]$ such that $(x'(s), t'(s))$ is well-defined.

We have

$$a = -t'(s) + \frac{x'(s)}{\varphi\big(x(s)\big) - \varepsilon} \qquad \text{and} \qquad b = \frac{x'(s)}{\varphi\big(x(s)\big) - \varepsilon}.$$

Thus

$$y_1\big(t'(s), x'(s), t(s), x(s)\big) = -t'(s) + \frac{x'(s)}{\varphi\big(x(s)\big) - \varepsilon}.$$

It follows that

$$\int_\alpha^\beta y_1\big(x'(s), t'(s), x(s), t(s)\big)\,\mathrm{d}s = -t_0 + \Phi(x_0), \tag{A.52}$$

where

$$\Phi(\xi) = \int_0^\xi \frac{1}{\varphi(s) - \varepsilon}\,\mathrm{d}s \quad \text{for } \xi \in [0,1].$$

The claim is proved.

Define

$$Y_1 \colon \Omega_{T,\ell} \longrightarrow \mathbb{R}, \qquad (t,x) \longmapsto -t + \Phi(x).$$

The proof in the general case follows as in the constant case with T_ℓ now defined by

$$T_\ell(v)(t,x) = e^{LY_1(t,x)} v(t,x). \tag{A.53}$$

One just notes that (A.39) and (A.42) hold with y_1 replaced by Y_1. Indeed, one has

$$\begin{aligned} &Y_1\big(s, x_j(s,t_1,\xi_1)\big) - Y_1(t_1,\xi_1) \\ &\quad = \int_{t_1}^s y_1\big(\partial_\theta x_j(\theta,t_1,\xi_1), 1, x_j(\theta,t_1,\xi_1), \theta\big) \Big|\big(\partial_\theta x_j(\theta,t_1,\xi_1), 1\big)\Big|\,\mathrm{d}\theta \\ &\quad \geqslant C\,\mathrm{sign}(s-t_1) \int_{t_1}^s y_1\Big(\vec{v}_j\big(x_j(\theta,t_1,\xi_1)\big), 1, x_j(\theta,t_1,\xi_1), \theta\Big)\,\mathrm{d}\theta \\ &\quad \geqslant C|t_1 - s| \\ &\quad \geqslant C\big|x_j(s,t_1,\xi_1) - \xi_1\big|. \end{aligned}$$

The details are omitted. □

We are ready to give the following.

Proof of Theorem A.2. — We first prove the uniqueness. Assume that $f = 0$, $g = 0$, and $\gamma = 0$. Then the restriction of w into $\Omega_{T,k}$ is 0 by Lemma A.4. It follows that the restriction of w into $\Omega_{T,k-1} = 0$ by Lemma A.6, …, the restriction of w into $\Omega_{T,k-m+1} = 0$ by Lemma A.6. Therefore, $w = 0$ in Ω_T.

To establish the existence, we proceed as follows. Let $w^{(k)}$ be the unique broad solution in $\Omega_{T,k}$ corresponding to (f,g), let $w^{(k-1)}$ be the unique broad solution in $\Omega_{T,k-1}$ where the data on $\Gamma_{T,k}$ come from $w^{(k)}$, ..., let $w^{(k-m+1)}$ be the unique broad solution in $\Omega_{T,k-m+1}$ where the data on $\Gamma_{T,k-m+2}$ come from $w^{(k-m+2)}$. (17) The corresponding solution is obtained by gluing these solutions together. The proof is complete. □

Remark A.9. — The introduction of appropriately weighted norms plays a crucial role in the proof of the well-posedness of broad solutions considered so far in this section, in particular in the proof of Lemma A.6. The introduction of weighted norms in order to be able to apply the fixed point argument used in establishing the well-posedness of the hyperbolic system is not new. The standard one is e^{-Lt} where L is a large positive number, see e.g. [36, (1.18), p. 78] or [6, (3.36), p. 50], while the weight e^{-Lx} is used in [10, 51] to prove exponential stability; see also [8, V defined in Section 3.2] for the Euler equations of incompressible fluids. In [16], we used the weight $e^{-L_1 x - L_2 t}$ where L_1 and L_2 are two large positive numbers with L_2 being much larger than L_1. The introduction of $e^{-L_1 x}$ in the weight is to handle the non-local term from the boundary condition imposed on the right (at $x = 1$) considered there. In these settings, t-direction has a privileged role. In the settings considered in this section, the domain is not a rectangle with respect to t and x, and the boundary conditions are quite complicated. Therefore, the time direction and the space direction play almost the same role here. In the setting of Lemma A.4, the privileged direction is x-direction so the weighted norm is chosen of the form e^{Lx}. In Lemma A.6, the *new* weighted norm introduced in (A.34) with T_ℓ given by (A.33) or (A.53) adapts the geometry and the boundary conditions, imposed in a nontrivial way. It is interesting to note that Y_1 is a *non-linear* function of t and x. The analysis here is inspired by [16] (see also [17]).

As a consequence of Theorem A.2, we can prove the following result.

Proposition A.10. — *Let* $C \in \big[L^\infty\big(I \times (0,1)\big)\big]^{n\times n}$ *for some open interval* I *containing* $[0, T_1]$. *Define, for* $\tau \in I_1$, *where* I_1 *is defined by* (3.11),

$$\mathcal{T}(\tau)\colon \big[L^2(0,T_{\mathrm{opt}})\big]^n \times \big[L^2(0,1)\big]^m \longrightarrow \mathcal{Y} \qquad (f,g) \longmapsto w, \tag{A.54}$$

where w *is the solution of* (3.31)–(3.36). *Then* $\mathcal{T}(\tau)$ *is uniformly bounded in* I_1. *Assume in addition that* $C \in \mathcal{H}\big(I, \big[L^\infty(0,1)\big]^{n\times n}\big)$. *Then* $\mathcal{T}(\cdot)$ *is analytic in* I_1.

(17) The data coming from $w^{(k)}$ on $\Gamma_{T,k}$, ..., $w^{(k-m+2)}$ on $\Gamma_{T,k-m+2}$ are given by the RHS of (A.18)–(A.20) in Definition A.5 for $(t_1, \xi_1) \in \Gamma_{T,k}$, and (A.30)–(A.32) in Definition A.8 for $(t_1, \xi_1) \in \Gamma_{T,\ell}$ with $\ell = k-1, \dots, k-m+1$, respectively.

Proof. — By Theorem A.2, for each $(f,g) \in \big[L^2(0,1)\big]^n \times \big[L^2(0,1)\big]^m$, there exists a unique broad solution $w \in \mathcal{Y}$ of (3.31)–(3.36). Hence $\mathcal{T}(\tau)$ is well-defined. The uniform boundedness of $\mathcal{T}$ is also a direct consequence of Theorem A.2, in particular of (A.13).

We next deal with the analyticity of $\mathcal{T}$ and thus assume that $C \in \mathcal{H}\big(I, \big[L^\infty(0,1)\big]^{n\times n}\big)$. Fix τ_0 in a sufficiently small neighborhood of I_1 (in the complex plane). We will prove that $\mathcal{T}$ is differentiable at τ_0 in the complex sense. For notational ease, we will assume that $\tau_0 = 0$.

Fix $(f,g) \in \big[L^2(0,T_{\mathrm{opt}})\big]^n \times \big[L^2(0,1)\big]^n$. Set $w^{(\tau)} = \mathcal{T}(\tau)(f,g)$ in Ω for τ in a small neighborhood (in the complex plane) of 0 and let $v \in \mathcal{Y}$ be the unique broad solution of the system

$$v_t(t,x) = \Sigma(x)\partial_x v(t,x) + \mathbf{C}(t,x)v(t,x) + \mathbf{C}_\tau(t,x)w^{(0)}(t,x) \quad \text{for } (t,x) \in \Omega, \tag{A.55}$$

$$v(\,\cdot\,,1) = 0 \quad \text{in } (0,T_{\mathrm{opt}}), \tag{A.56}$$

$$v_+(0,\cdot\,) = 0 \quad \text{in } (0,1), \tag{A.57}$$

$$v_{-,\geqslant k}(t,0) = Q_k v_{<k,\geqslant k+m}(t,0) \quad \text{for } t \in (T_{\mathrm{opt}} - \tau_k, T_{\mathrm{opt}} - \tau_{k-1}), \tag{A.58}$$

$$v_{-,\geqslant k-1}(t,0) = Q_{k-1} v_{<k-1,\geqslant k+m-1}(t,0) \quad \text{for } t \in (T_{\mathrm{opt}} - \tau_{k-1}, T_{\mathrm{opt}} - \tau_{k-2}), \tag{A.59}$$

$$\dots$$

$$v_{-,\geqslant k-m+2}(t,0) = Q_{k-m+2} v_{<k-m+2,\geqslant k+2}(t,0) \quad \text{for } t \in (T_{\mathrm{opt}} - \tau_{k-m+2}, T_{\mathrm{opt}} - \tau_{k-m+1}). \tag{A.60}$$

Here $\mathbf{C}_\tau(\tau,x)$ denotes the derivative of $\mathbf{C}(\tau,x)$ with respect to τ in the complex sense. The existence and uniqueness of v follow from Theorem A.2.

We claim that

(A.61) the derivative of $\mathcal{T}$ at 0 is given by $\mathcal{T}_1$ where $\mathcal{T}_1(f,g) = v$ in Ω

(the derivative of $\mathcal{T}$ is considered in the complex sense). To this end, for τ in a small neighborhood (in the complex plane) of 0 but not 0, we consider $dw \in \mathcal{Y}$ defined by

$$dw := \frac{1}{\tau}\big(w^{(\tau)} - w^{(0)} - \tau v\big) \quad \text{in } \Omega.$$

Then $dw \in \mathcal{Y}$ is a broad solution of the system

$$\begin{aligned} \partial_t dw(t,x) &= \Sigma(x)\partial_x dw(t,x) + \mathbf{C}(t,x)dw(t,x) \\ &+ \frac{1}{\tau}\big(\mathbf{C}(t+\tau,x) - \mathbf{C}(t,x)\big)w^{(\tau)}(t,x) - \mathbf{C}_\tau(t,x)w^{(0)}(t,x) \quad \text{in } \Omega, \end{aligned} \tag{A.62}$$

and (A.56)–(A.60) hold with v replaced by dw. We derive from Theorem A.2 that

$$\begin{aligned}\|dw\|_{\mathcal{Y}} &\leqslant C\big(\|w^{(\tau)}\|_{L^2(\Omega)} + \|w^{(0)}\|_{L^2(\Omega)}\big)\\ &\leqslant C\big(\|f\|_{L^2(0,T_{\rm opt})} + \|g\|_{L^2(0,1)}\big).\end{aligned} \tag{A.63}$$

Using the definition of dw, we can write the last two terms in (A.62) under the form

$$\begin{aligned}&\frac{1}{\tau}\big(\mathbf{C}(t+\tau,x) - \mathbf{C}(t,x)\big)\big(w^{(0)} + \tau dw + \tau v\big) - \mathbf{C}_\tau(t,x)w^{(0)}(t,x)\\ &\quad = \frac{1}{\tau}\big(\mathbf{C}(t+\tau,x) - \mathbf{C}(t,x) - \tau\mathbf{C}_\tau(t,x)\big)w^{(0)}(t,x)\\ &\quad\quad + \frac{1}{\tau}\big(\mathbf{C}(t+\tau,x) - \mathbf{C}(t,x)\big)(\tau dw + \tau v).\end{aligned} \tag{A.64}$$

Note that the $L^2(\Omega)$-norm of the RHS of (A.64) is bounded by

$$C|\tau|\big(\|w^{(0)}\|_{L^2(\Omega)} + \|dw\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\big).$$

Applying Theorem A.2 again, we derive from (A.63) that

$$\|dw\|_{\mathcal{Y}} \leqslant C|\tau|\big(\|w^{(0)}\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} + \|f\|_{L^2(0,T_{\rm opt})} + \|g\|_{L^2(0,1)}\big). \tag{A.65}$$

By noting that

$$\|w^{(0)}\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \leqslant C\big(\|f\|_{L^2(0,T_{\rm opt})} + \|g\|_{L^2(0,1)}\big),$$

claim (A.61) follows from (A.65). The proof is complete. □

Remark A.11. — Let $C \in \big[L^\infty\big(I \times (0,1)\big)\big]^{n\times n}$ for some open interval I containing $[0, T_1]$. One can prove that $\mathcal{T}(\tau)$ is strongly continuous, i.e., $\mathcal{T}(\tau)(f,g) \to \mathcal{T}(\tau_0)(f,g)$ in $\mathcal{Y}$ as $\tau \to \tau_0$ in I_1 for all $(f,g) \in \big[L^2(0,T_{\rm opt})\big]^n \times \big[L^2(0,1)\big]^m$. Indeed, let us assume that $\tau_0 = 0$ for notational ease. Set $w^{(\tau)} = \mathcal{T}(\tau)(f,g)$ in Ω for $\tau \in I_1$ and for $(f,g) \in \big[L^2(0,T_{\rm opt})\big]^n \times \big[L^2(0,1)\big]^m$. Denote $\delta w = w^{(\tau)} - w^{(0)}$ in Ω. We have, in Ω

$$\begin{aligned}\partial_t \delta w(t,x) = \Sigma(x)\partial_x\delta w(t,x) + \mathbf{C}(t+\tau,x)\delta w(t,x)&\\ + \big(\mathbf{C}(t+\tau,x) - \mathbf{C}(t,x)\big)w^{(0)}(t,x)&,\end{aligned}$$

and δw satisfies the same boundary conditions as dw. Applying Theorem A.2, one has

$$\|\delta w\|_{\mathcal{Y}} \leqslant C\|g\|_{L^2(\Omega)},$$

where $g(t,x) = \big(\mathbf{C}(t+\tau,x) - \mathbf{C}(t,x)\big)w^{(0)}(t,x)$. Since $\|g\|_{L^2(\Omega)} \to 0$ as $\tau \to 0$, the conclusion follows.

We next discuss the broad solutions used in the definition of $\widehat{\mathcal{T}}(\tau, T)$ with $T \geqslant T_{\text{opt}}$, and their well-posedness. Let $F \in \left[L^\infty\big((0,T)\times(0,1)\big)\right]^{n\times n}$, $(f,g) \in \left[L^2(0,T)\right]^n \times \left[L^2(0,1)\right]^m$, and let $q \in \left[L^2(0,1)\right]^{k-m}$.(18) Consider the system

$$\text{(A.66)} \quad \partial_t \widehat{w}(t,x) = \Sigma(x)\partial_x \widehat{w}(t,x) + F(t,x)\widehat{w}(t,x) + \gamma(t,x) \quad \text{for } (t,x) \in (0,T)\times(0,1),$$

$$\text{(A.67)} \quad \widehat{w}(\,\cdot\,,1) = f \quad \text{in } (0,T),$$

$$\text{(A.68)} \quad \widehat{w}_+(0,\cdot\,) = g \quad \text{in } (0,1),$$

$$\text{(A.69)} \quad \widehat{w}_j(T,\cdot\,) = q_j \quad \text{in } (0,1), \text{for } 1 \leqslant j \leqslant k-m,$$

$$\text{(A.70)} \quad \widehat{w}_{-,\geqslant k}(t,0) = Q_k \widehat{w}_{<k,\geqslant k+m}(t,0) \quad \text{for } t \in (T-\tau_k, T-\tau_{k-1}),$$

$$\text{(A.71)} \quad \widehat{w}_{-,\geqslant k-1}(t,0) = Q_{k-1} \widehat{w}_{<k-1,\geqslant k+m-1}(t,0) \quad \text{for } t \in (T-\tau_{k-1}, T-\tau_{k-2}),$$

$$\dots$$

$$\text{(A.72)} \quad \widehat{w}_{-,\geqslant k-m+2}(t,0) = Q_{k-m+2} \widehat{w}_{<k-m+2,\geqslant k+2}(t,0) \quad \text{for } t \in (T-\tau_{k-m+2}, T-\tau_{k-m+1}),$$

$$\text{(A.73)} \quad \widehat{w}_{-,\geqslant k-m+1}(t,0) = Q_{k-m+1} \widehat{w}_{<k-m+1,\geqslant k+1}(t,0) \quad \text{for } t \in (T-\tau_{k-m+1}, T).$$

We have the following result, which implies the well-posedness of $\widehat{\mathcal{T}}(\tau, T)$.

Theorem A.12. — *Let:*

- $T \geqslant T_{\text{opt}}$;
- $F \in \left[L^\infty\big((0,T)\times(0,1)\big)\right]^{n\times n}$;
- $(f,g) \in \left[L^2(0,T)\right]^n \times \left[L^2(0,1)\right]^m$;
- $q \in \left[L^2(0,1)\right]^{k-m}$;(19)
- $\gamma \in \left[L^2\big((0,T)\times(0,1)\big)\right]^n$.

There exists a unique broad solution

$$\text{(A.74)} \quad \widehat{w} \in \widehat{\mathcal{Y}}_T := \left[L^2\big((0,T)\times(0,1)\big)\right]^n \cap C\big([0,T]; \left[L^2(0,1)\right]^n\big) \cap C\big([0,1]; \left[L^2(0,T)\right]^n\big)$$

of (A.66)–(A.73). *Moreover,*

$$\|\widehat{w}\|_{\widehat{\mathcal{Y}}_T} \leqslant C\big(\|f\|_{L^2(0,T)} + \|g\|_{L^2(0,1)} + \|q\|_{L^2(0,1)} + \|\gamma\|_{L^2((0,T)\times(0,1))}\big),$$

(18) q is irrelevant when $k = m$.
(19) q is irrelevant when $k = m$.

for some positive constant C depending only on an upper bound of $\|F\|_{L^\infty(\Omega_\ell)}$ and T, and Σ.

Here we denote

$$\|\widehat{w}\|_{\hat{\mathcal{Y}}_T} = \max\left\{\sup_{x\in[0,1]} \|\widehat{w}_i\|_{L^2(0,T)},\ \sup_{t\in[0,T]} \|\widehat{w}_i\|_{L^2(0,1)};\ 1 \leqslant i \leqslant n\right\}.$$

Remark A.13. — The analysis of Theorem A.12 can be extended to cover the case where source terms in L^2 are added in (A.70)–(A.73).

The definition of broad solutions $\widehat{w} \in \widehat{\mathcal{Y}}_T$ of (A.66)–(A.73) is similar to the one given in Definition A.1 and left to the reader. The proof of (A.12) is similar to the one of Theorem A.2. Nevertheless, in addition to Lemmas A.4 and A.6, we also use the following.

Lemma A.14. — *Let $T \geqslant T_{\text{opt}}$. Set $\Omega_{T,k-m} = [0,T]\times(0,1)\setminus\Omega_T$. Let $F \in \left[L^\infty(\Omega_{T,k-m})\right]^{n\times n}$, $\gamma \in \left[L^2(\Omega_{T,k-m})\right]^n$, $h_j \in L^2(\Gamma_{T,k-m+1})$ for $1 \leqslant j \leqslant k+m$ and $j \neq k-m+1$, and let $q_j \in L^2(\Gamma_{T,k-m})$ for $1 \leqslant j \leqslant k-m$ where $\Gamma_{T,k-m} = \{T\}\times(0,1)$. There exists a unique broad solution $w \in \mathcal{Y}_{T,k-m} := \left[L^2(\Omega_{T,k-m})\right]^n \cap C\big([0,T]; \left[L^2(\Omega_{T,k-m,t})\right]^n\big) \cap C\big([0,1]; \left[L^2(\Omega_{T,k-m,x})\right]^n\big)$ of the system*

$$\partial_t w(t,x) = \Sigma(x)\partial_x w(t,x) + F(t,x)w(t,x) + \gamma(t,x) \quad \text{for } (t,x) \in \Omega_{T,k-m}, \tag{A.75}$$

$$w_j = h_j \quad \text{on } \Gamma_{T,k-m+1}, \text{ for } 1 \leqslant j \leqslant k+m \text{ and } j \neq k-m+1, \tag{A.76}$$

$$w_j = q_j \quad \text{on } \Gamma_{T,k-m}, \text{ for } 1 \leqslant j \leqslant k-m, \tag{A.77}$$

$$w_{-,\geqslant k-m+1}(0,\cdot) = Q_{k-m+1} w_{<k-m+1,\geqslant k+1} \tag{A.78}$$

$$\text{for } t \in (T-\tau_{k-m+1}, T). \tag{A.79}$$

Moreover,

$$\|w\|_{\mathcal{Y}_{T,k-m}} \leqslant C\Bigg(\sum_{1\leqslant j\leqslant k+m; j\neq k-m+1} \|h_j\|_{L^2(\Gamma_{T,k-m+1})} + \sum_{1\leqslant j\leqslant k-m} \|q_j\|_{L^2(\Gamma_{T,k-m})} + \|\gamma\|_{L^2(\Omega_{T,k-m})}\Bigg),$$

for some positive constant C depending only T, Σ, and on an upper bound of $\|F\|_{L^\infty(\Omega_{T,k-m})}$.

Recall that $k \geqslant m \geqslant 1$ in this section.

Remark A.15. — The analysis of Lemma A.14 can be extended to cover the case where source terms in L^2 are added in (A.79).

Proof. — The proof of Lemma A.14 is similar to the one of Lemma A.6. We just mention here how to define G_1, G_2 and determine $\vec{u_1}$ and $\vec{u}_2$ in the general case (Σ is not required to be constant). For $1 \leqslant j \leqslant k+m$, let $\vec{v}_j = \vec{v}_j(t,x)$ be the unit vector tangent to the characteristic curve of x_j at the point (t,x) directed to the boundary where the boundary condition for v_j is given. The vector $\vec{v}_j(t,x)$ is parallel to $\big(1, \Sigma_{jj}(x)\big)^{\mathsf{T}}$ in the xt-plane so that we can choose it independent of t and in fact, we will do. We denote it by $\vec{v}_j(x)$ from now on. Set

$$G_1(x) = \big\{\vec{v}_j(x);\ 1 \leqslant j \leqslant k-m,\ k+1 \leqslant j \leqslant k+m\big\}$$

and

$$G_2(x) = \big\{\vec{v}_j(x);\ k-m+1 \leqslant j \leqslant k\big\}.$$

Let $\varphi(x)$ be such that $\vec{v}_1(x)$ is parallel to and has the same direction with $\big(\varphi(x), 1\big)^{\mathsf{T}}$. Set, in the xt-plane,

$$\vec{u}_1(x) = (0,-1)^{\mathsf{T}},$$

and

$$\vec{u}_2(x) = \big(\varphi(x) - \varepsilon, 1\big)^{\mathsf{T}} \text{ if } k > m, \text{ otherwise } \vec{u_2} = (1,0)^{\mathsf{T}},$$

where ε is positive and sufficiently small, the smallness of ε is independent of x, such that, $\varphi(x) > 2\varepsilon$ (the choice of ε is irrelevant when $k=m$), and:

- (a1) $G_1(x) \cup G_2(x) \cup \big\{\vec{u}_1(x)\big\}$ lies on one side of the line containing $\vec{u}_2(x)$;
- (a2) $G_1(x)$ is a subset of the open solid cone centered at the origin and formed by $\vec{u}_1(x)$ and $\vec{u}_2(x)$;
- (a3) $G_2(x)$ is a subset of the open solid cone centered at the origin and formed by $\vec{u}_1(x)$ and $-\vec{u}_2(x)$.

The rest of the proof is then almost unchanged and left to the reader. □

BIBLIOGRAPHY

[1] S. Amin, F. M. Hante & A. M. Bayen, "On stability of switched linear hyperbolic conservation laws with reflecting boundaries", in *Hybrid systems: computation and control*, Lecture Notes in Computer Science, vol. 4981, Springer, 2008, p. 602-605.

[2] N. Anantharaman, M. Léautaud & F. Macià, "Wigner measures and observability for the Schrödinger equation on the disk", *Invent. Math.* **206** (2016), no. 2, p. 485-599.

[3] J. Auriol & F. Di Meglio, "Minimum time control of heterodirectional linear coupled hyperbolic PDEs", *Automatica* **71** (2016), p. 300-307.

[4] C. Bardos, G. Lebeau & J. Rauch, "Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary", *SIAM J. Control Optim.* **30** (1992), no. 5, p. 1024-1065.

[5] G. Bastin & J.-M. Coron, *Stability and boundary stabilization of 1-D hyperbolic systems*, Progress in Nonlinear Differential Equations and their Applications, vol. 88, Birkhäuser, 2016, xiv+307 pages.

[6] A. Bressan, *Hyperbolic systems of conservation laws*, Oxford Lecture Series in Mathematics and its Applications, vol. 20, Oxford University Press, 2000, xii+250 pages.

[7] E. Cerpa & J.-M. Coron, "Rapid stabilization for a Korteweg–de Vries equation from the left Dirichlet boundary condition", *IEEE Trans. Autom. Control* **58** (2013), no. 7, p. 1688-1695.

[8] J.-M. Coron, "On the null asymptotic stabilization of the two-dimensional incompressible Euler equations in a simply connected domain", *SIAM J. Control Optim.* **37** (1999), no. 6, p. 1874-1896.

[9] ———, *Control and nonlinearity*, Mathematical Surveys and Monographs, vol. 136, American Mathematical Society, 2007, xiv+426 pages.

[10] J.-M. Coron, G. Bastin & B. d'Andréa Novel, "Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems", *SIAM J. Control Optim.* **47** (2008), no. 3, p. 1460-1498.

[11] J.-M. Coron, L. Gagnon & M. Morancey, "Rapid stabilization of a linearized bilinear 1-D Schrödinger equation", *J. Math. Pures Appl. (9)* **115** (2018), p. 24-73.

[12] J.-M. Coron, L. Hu & G. Olive, "Finite-time boundary stabilization of general linear hyperbolic balance laws via Fredholm backstepping transformation", *Automatica* **84** (2017), p. 95-100.

[13] J.-M. Coron, L. Hu, G. Olive & P. Shang, "Boundary stabilization in finite time of one-dimensional linear hyperbolic balance laws with coefficients depending on time and space", *J. Differ. Equations* **271** (2021), p. 1109-1170.

[14] J.-M. Coron & Q. Lü, "Local rapid stabilization for a Korteweg–de Vries equation with a Neumann boundary control on the right", *J. Math. Pures Appl. (9)* **102** (2014), no. 6, p. 1080-1120.

[15] J.-M. Coron & H.-M. Nguyen, "Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach", *Arch. Ration. Mech. Anal.* **225** (2017), p. 993-1023.

[16] ———, "Optimal time for the controllability of linear hyperbolic systems in one-dimensional space", *SIAM J. Control Optim.* **57** (2019), no. 2, p. 1127-1156.

[17] ———, "Finite-time stabilization in optimal time of homogeneous quasilinear hyperbolic systems in one dimensional space", *ESAIM, Control Optim. Calc. Var.* **26** (2020), article no. 119 (24 pages).

[18] ———, "Null-controllability of linear hyperbolic systems in one dimensional space", *Syst. Control Lett.* **148** (2021), article no. 104851 (8 pages).

[19] ———, "Lyapunov functions and finite-time stabilization in optimal time for homogeneous linear and quasilinear hyperbolic systems", *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **39** (2022), no. 5, p. 1235-1260.

[20] J.-M. Coron, R. Vazquez, M. Krstic & G. Bastin, "Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping", *SIAM J. Control Optim.* **51** (2013), no. 3, p. 2005-2035.

[21] F. Di Meglio, R. Vazquez & M. Krstic, "Stabilization of a system of $n + 1$ coupled first-order hyperbolic linear PDEs with a single boundary input", *IEEE Trans. Autom. Control* **58** (2013), no. 12, p. 3097-3111.

[22] V. Dos Santos & C. Prieur, "Boundary control of open channels with numerical and experimental validations", *IEEE Trans. Control Sys. Technol.* **16** (2008), no. 6, p. 1252-1264.

[23] P. Goatin, "The Aw–Rascle vehicular traffic flow model with phase transitions", *Math. Comput. Modelling* **44** (2006), no. 3-4, p. 287-303.

[24] I. Gohberg, P. Lancaster & L. Rodman, *Matrix polynomials*, Classics in Applied Mathematics, vol. 58, Society for Industrial and Applied Mathematics, 2009, xxiv+409 pages.

[25] J. M. Greenberg & T. Li, "The effect of boundary damping for the quasilinear wave equation", *J. Differ. Equations* **52** (1984), no. 1, p. 66-75.

[26] M. Gugat & G. Leugering, "Global boundary controllability of the de St. Venant equations between steady states", *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **20** (2003), no. 1, p. 1-11.

[27] M. Gugat, G. Leugering & E. J. P. Georg Schmidt, "Global controllability between steady supercritical flows in channel networks", *Math. Methods Appl. Sci.* **27** (2004), no. 7, p. 781-802.

[28] J. de Halleux, C. Prieur, J.-M. Coron, B. d'Andréa Novel & G. Bastin, "Boundary feedback control in networks of open channels", *Automatica* **39** (2003), no. 8, p. 1365-1376.

[29] L. Hörmander, "On the uniqueness of the Cauchy problem under partial analyticity assumptions", in *Geometrical optics and related topics (Cortona, 1996)*, Progress in Nonlinear Differential Equations and their Applications, vol. 32, Birkhäuser, 1997, p. 179-219.

[30] L. Hu, F. Di Meglio, R. Vazquez & M. Krstic, "Control of homodirectional and general heterodirectional linear coupled hyperbolic PDEs", *IEEE Trans. Autom. Control* **61** (2016), no. 11, p. 3301-3314.

[31] L. Hu & G. Olive, "Minimal time for the exact controllability of one-dimensional first-order linear hyperbolic systems by one-sided boundary controls", *J. Math. Pures Appl. (9)* **148** (2021), p. 24-74.

[32] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer, 1995, xxii+619 pages.

[33] M. Krstic, B.-Z. Guo, A. Balogh & A. Smyshlyaev, "Output-feedback stabilization of an unstable wave equation", *Automatica* **44** (2008), no. 1, p. 63-74.

[34] M. Krstic & A. Smyshlyaev, *Boundary control of PDEs*, Advances in Design and Control, vol. 16, Society for Industrial and Applied Mathematics, 2008, x+192 pages.

[35] C. Laurent & M. Léautaud, "Quantitative unique continuation for operators with partially analytic coefficients. Application to approximate control for waves", *J. Eur. Math. Soc.* **21** (2019), no. 4, p. 957-1069.

[36] T. Li & W. C. Yu, *Boundary value problems for quasilinear hyperbolic systems*, Duke University Mathematics Series, vol. 5, Duke University Press, 1985, viii+325 pages.

[37] T. Li, W. C. Yu & W. X. Shen, "Second initial-boundary value problems for quasilinear hyperbolic-parabolic coupled systems", *Chin. Ann. Math.* **2** (1981), no. 1, p. 65-90.

[38] J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*, Recherches en Mathématiques Appliquées, vol. 8, Masson, 1988, x+541 pages.

[39] J. Rauch & M. Taylor, "Exponential decay of solutions to hyperbolic equations in bounded domains", *Indiana Univ. Math. J.* **24** (1974), p. 79-86.

[40] L. Robbiano & C. Zuily, "Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients", *Invent. Math.* **131** (1998), no. 3, p. 493-539.

[41] D. L. Russell, "Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions", *SIAM Rev.* **20** (1978), no. 4, p. 639-739.

[42] M. Slemrod, "Boundary feedback stabilization for a quasilinear wave equation", in *Control theory for distributed parameter systems and applications (Vorau, 1982)*, Lecture Notes in Control and Information Sciences, vol. 54, Springer, 1983, p. 221-237.

[43] A. Smyshlyaev, E. Cerpa & M. Krstic, "Boundary stabilization of a 1-D wave equation with in-domain antidamping", *SIAM J. Control Optim.* **48** (2010), no. 6, p. 4014-4031.

[44] A. Smyshlyaev & M. Krstic, "Closed-form boundary state feedbacks for a class of 1-D partial integro-differential equations", *IEEE Trans. Autom. Control* **49** (2004), no. 12, p. 2185-2202.

[45] ———, "On control design for PDEs with space-dependent diffusivity or time-dependent reactivity", *Automatica* **41** (2005), no. 9, p. 1601-1608.

[46] ———, "Boundary control of an anti-stable wave equation with anti-damping on the uncontrolled boundary", *Syst. Control Lett.* **58** (2009), no. 8, p. 617-623.

[47] E. D. Sontag, *Mathematical control theory*, second ed., Texts in Applied Mathematics, vol. 6, Springer, 1998, xvi+531 pages.

[48] D. Tataru, "Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem", *Commun. Partial Differ. Equations* **20** (1995), no. 5-6, p. 855-884.

[49] R. Vazquez & M. Krstic, "Control of 1-D parabolic PDEs with Volterra nonlinearities. I. Design", *Automatica* **44** (2008), no. 11, p. 2778-2790.

[50] C.-Z. Xu & G. Sallet, "Exponential stability and transfer functions of processes governed by symmetric hyperbolic systems", *ESAIM, Control Optim. Calc. Var.* **7** (2002), p. 421-442.

[51] ———, "Exponential stability and transfer functions of processes governed by symmetric hyperbolic systems", *ESAIM, Control Optim. Calc. Var.* **7** (2002), p. 421-442.

[52] C. Zhang, "Finite-time internal stabilization of a linear 1-D transport equation", *Syst. Control Lett.* **133** (2019), article no. 104529 (8 pages).

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