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ON L^p ESTIMATES FOR POSITIVITY-PRESERVING RIESZ TRANSFORMS RELATED TO SCHRÖDINGER OPERATORS

by Maciej KUCHARSKI & Błażej WRÓBEL (*)

Abstract. — We study the L^p, $1 \leqslant p \leqslant \infty$, boundedness for Riesz transforms of the form $V^a(-\frac{1}{2}\Delta + V)^{-a}$, where $a > 0$ and V is a non-negative potential. We prove that $V^a(-\frac{1}{2}\Delta + V)^{-a}$ is bounded on $L^p(\mathbb{R}^d)$ with $1 < p \leqslant 2$ whenever $a \leqslant 1/p$. We demonstrate that the $L^\infty(\mathbb{R}^d)$ boundedness holds if V satisfies an a-dependent integral condition that is resistant to small perturbations. Similar results with stronger assumptions on V are also obtained on $L^1(\mathbb{R}^d)$. In particular our L^∞ and L^1 results apply to non-negative locally bounded potentials V which globally have a power growth or an exponential growth.

We also discuss a counterexample showing that the $L^\infty(\mathbb{R}^d)$ boundedness may fail.

Résumé. — Nous étudions le caractère borné sur L^p, $1 \leqslant p \leqslant \infty$, pour les transformées de Riesz de la forme $V^a(-\frac{1}{2}\Delta + V)^{-a}$, où $a > 0$ et V est un potentiel non-négatif. Nous prouvons que $V^a(-\frac{1}{2}\Delta + V)^{-a}$ est bornée sur $L^p(\mathbb{R}^d)$ avec $1 < p \leqslant 2$ quand $a \leqslant 1/p$. Nous démontrons que le caractère borné sur $L^\infty(\mathbb{R}^d)$ est valable si V satisfait une condition intégrale dépendante de a et robuste aux petites perturbations. Des résultats similaires avec des hypothèses plus fortes sur V sont également obtenus sur $L^1(\mathbb{R}^d)$. En particulier, nos résultats L^∞ et L^1 s'appliquent aux potentiels non négatifs et localement bornés V qui ont globalement une croissance en puissance ou une croissance exponentielle.

Nous discutons également d'un contre-exemple montrant que le caractère borné sur $L^\infty(\mathbb{R}^d)$ peut échouer.

1. Introduction

In this paper we consider a class of Riesz transforms related to the Schrödinger operator

$$L = -\frac{1}{2}\Delta + V,$$

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with V being a non-negative potential in L^1_{loc}. The operator L is rigorously defined via quadratic forms, see Section 2. The Riesz transforms are formally given, for $a>0$, by

$$\begin{aligned} R_V^a f(x) &= V^a(x)\cdot\left(-\tfrac{1}{2}\Delta+V\right)^{-a} f(x) \\ &= \frac{V^a(x)}{\Gamma(a)}\cdot\int_0^\infty e^{-tL} f(x)\, t^{a-1}\mathrm{d}t, \end{aligned} \tag{1.1}$$

where e^{-tL} is the corresponding semigroup. We also set R_V^0 to be the identity operator. By the Trotter product formula the operators R_V^a are positivity preserving, unlike the Riesz transforms $\nabla L^{-1/2}$, which we do not study here. One can also see, cf. Proposition 2.4, that for $V\in L^1_{\mathrm{loc}}$ and $a=\frac{1}{2}$ the formal expression (1.1) gives rise to a contraction on $L^2(\mathbb{R}^d)$. For a general non-negative potential $V\in L^1_{\mathrm{loc}}$ we also know the $L^1(\mathbb{R}^d)$ boundedness of R_V^1, see for example [2, 14, 16]. Note that, apart from the case when V is constant, neither $R_V^{1/2}$ nor R_V^1 is a convolution operator.

Apart from the cases $a=\frac{1}{2}$ and $a=1$ there seem to be no L^p boundedness results for Riesz transforms R_V^a of general potentials $V\in L^1_{\mathrm{loc}}$. For V belonging to a reverse Hölder class L^p boundedness of R_V^a, $0<a<1$, is mentioned in [2, p. 1978]. We prove the following general result.

Theorem A (Theorem 2.6). — *Let $V\in L^1_{\mathrm{loc}}$ and take $p\in(1,2]$. Then for all $0\leqslant a\leqslant 1/p$ the Riesz transform R_V^a is bounded on L^p.*

Theorem A is derived as a consequence of the endpoint bounds for $R_V^{1/2}$ on $L^2(\mathbb{R}^d)$ (Proposition 2.4) and for R_V^1 on $L^1(\mathbb{R}^d)$ ([2, Theorem 4.3], see also [14, 16]) together with the interpolation result given below.

Theorem B (Theorem 2.5). — *Let $a_0>0$ and $a_1>0$. Assume that $V\in L^1_{\mathrm{loc}}$ is such that $R_V^{a_1}$ is bounded on L^{p_1} for some $p_1\in[1,\infty)$ and $R_V^{a_0}$ is bounded on L^1. Then, R_V^a is bounded on L^p for every p and a such that $1/p=\theta+(1-\theta)/p_1$ and $a=\theta a_0+(1-\theta)a_1$ with some $\theta\in(0,1)$.*

The above theorem is proved via Stein's complex interpolation theorem. It is worth emphasizing that when $p\in(1,2]$ the boundedness of R_V^a stated in Theorem A holds not only for $a=1/p$ but for all smaller a as well. This follows from Theorem B together with Corollary 2.3. However, this may be no longer true when $p=1$. The reason behind is eminent in the proof of Theorem B (Theorem 2.5); namely, the imaginary powers L^{iu}, $u\in\mathbb{R}$, are bounded on L^p, $p\in(1,2]$, but are unbounded on L^1.

The main purpose of our paper is to study the L^∞ and L^1 boundedness of R_V^a for specific classes of non-negative potentials V. We focus on obtaining results for which only large values of x matter and which are resistant to small perturbations of the potential V. Considering the L^∞ boundedness of R_V^a two particular cases of V serve as a good example of the possible situation. Firstly, if V is a non-negative constant function, say $V = c$, then $L = -\frac{\Delta}{2} + c$ and by (1.1) we have

$$R_c^a f = \frac{c^a}{\Gamma(a)} \int_0^\infty e^{-tc} t^{a-1} e^{t\Delta/2} f \, \mathrm{d}t.$$

Therefore, using the L^∞ contractivity of the heat semigroup $e^{t\Delta}$ we easily see that R_c^a is bounded on L^∞. Secondly, if $d \geqslant 3$ and $V \in L^q$, $q > d/2$, is a non-zero compactly supported function, then R_V^a is unbounded on L^∞ for all $a > 0$, see Proposition 3.3. Thus, the fact that V does not vanish outside a compact set is necessary for the boundedness of R_V^a on L^∞.

In what follows for two functions $A, B \colon \mathbb{R}^d \to [0, \infty)$ by $A(x) \approx B(x)$ we mean that for almost all $x \in \mathbb{R}^d$ we have $cA(x) \leqslant B(x) \leqslant CA(x)$ with two universal constants $0 < c < C$. We say that $A \approx B$ globally if $A(x) \approx B(x)$ for almost every x outside a compact set. The main classes of examples on $L^\infty(\mathbb{R}^d)$ which our theory admits are given below.

Theorem C. — *Let $V \colon \mathbb{R}^d \to [0, \infty)$ be a function in L^∞_{loc}. Then in all the three cases*

1. *$V(x) \approx 1$ globally*
2. *For some $\alpha > 0$ we have $V(x) \approx |x|^\alpha$ globally*
3. *For some $\beta > 1$ we have $V(x) \approx \beta^{|x|}$ globally*

each of the Riesz transforms R_V^a, $a > 0$, is bounded on $L^\infty(\mathbb{R}^d)$.

What lies at the heart of the proof of Theorem C is the Feynman–Kac formula. Theorem C is restated as Corollary 4.6 in Section 4, where it is deduced from Theorem 4.5. In order to apply Theorem 4.5 we need to verify two assumptions. Firstly V must be strictly positive far away along a line in $\mathbb{R}^d$. In this case Lemma 4.1 guarantees an exponential decay of the semigroup e^{-tL} on $L^\infty(\mathbb{R}^d)$. Secondly, we assume a specific interplay between the value $V(x)$ and the speed at which $V(y)$ decreases for y in a ball around x. The interplay is captured in condition (4.24) (the quantity $I^a(V)(x)$ being defined in (4.8)). It is easily verified that the assumptions of Theorem 4.5 are met in all the cases (1), (2), (3) of Theorem C.

We also prove an $L^1(\mathbb{R}^d)$ counterpart of Theorem C

Theorem D. — *Let $V \colon \mathbb{R}^d \to [0, \infty)$ be a function in L^∞_{loc}. Then in all the three cases*

1. *$V(x) \approx 1$ globally*
2. *For some $\alpha > 0$ we have $V(x) \approx |x|^\alpha$ globally*

(3) *For some* $\beta > 1$ *we have* $V(x) \approx \beta^{|x|}$ *globally*

each of the Riesz transforms R_V^a, $a > 0$, *is bounded on* $L^1(\mathbb{R}^d)$.

The proof of Theorem D also makes extensive use of the Feynman–Kac formula. However, such an approach seems better suited to $L^\infty(\mathbb{R}^d)$ estimates and thus the route to Theorem D is more complicated than in Theorem C. All the needed ingredients are justified in Section 5. Theorem D is restated there as Corollary 5.6 and the results needed to prove this corollary include Theorem 5.4 and Theorem 5.5. Note that in these results apart from condition (4.24) we need to control the speed at which $V(y)$ increases for y in a ball around x relative to the value of $V(x)$. This is similar to the conditions assumed in the case of L^∞ bounds.

Using Theorems C and D for $a = 1$, together with the argument from [27, Proof of Corollary 1.4, p. 174–175], we may also obtain $L^p(\mathbb{R}^d)$, $1 < p < \infty$, boundedness of the Riesz transforms $|\nabla L^{-1/2} f|$; here ∇ denotes the usual gradient on $\mathbb{R}^d$. As this is aside the main considerations of our paper we do not pursue it here.

The topic of Riesz transforms related to Schrödinger operators has been considered by a number of authors, both on $\mathbb{R}^d$ and on more general manifolds, see [1, 2, 3, 8, 9, 11, 12, 22, 27]. In the context of the Riesz transforms R_V^a the case $a = \frac{1}{2}$ has attracted most attention. For a general $V \in L^2_{\mathrm{loc}}$ it is known that $R_V^{1/2}$ is bounded on the $L^p(\mathbb{R}^d)$ spaces $1 < p \leqslant 2$, see Sikora [22, Theorem 11]. Our Theorem A extends the $L^p(\mathbb{R}^d)$ boundedness to R_V^a for $a \leqslant 1/p$. When the potential V is in the reverse Hölder class B_q for some $q \geqslant d/2$, then Shen proved that $R_V^{1/2}$ is bounded on $L^p(\mathbb{R}^d)$, $1 \leqslant p \leqslant 2q$, see [21, Theorem 5.10], and that R_V^1 is bounded on $L^p(\mathbb{R}^d)$, $1 \leqslant p \leqslant q$, see [21, Theorem 3.1]. Both results were later improved by Auscher and Ben Ali, see [2, Theorem 1.1 and Theorem 1.2] to $1 < q \leqslant \infty$. In particular this is true for V being a non-negative polynomial on $\mathbb{R}^d$. In fact, for such a V the Riesz transforms R_V^a, $a \geqslant 0$, are bounded both on $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$; this was proved by Dziubański [11, Theorem 4.5]. His proof uses nilpotent Lie group techniques for which it is important that V is a polynomial. Moreover, in the particular case of $V(x) = |x|^2$ Bongioanni and Torrea [4, Lemma 3] proved the $L^p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$, boundedness of R_V^a for all $a > 0$ by using explicitly the Mehler formula. Our proofs of Theorems C and D do not require explicit formulas and the examples listed there are resistant to small perturbations; for instance, we may take $V(x) = |x|^\alpha + E(x)$ with $\alpha > 0$, whenever the error term E is a locally bounded function of a lower order than $|x|^\alpha$ for large values of $|x|$.

The L^∞ boundedness of R^1_V was addressed by Urban and Zienkiewicz in [27]. In [27, Theorem 1.1] the authors proved the $L^\infty(\mathbb{R}^d)$ boundedness of R^1_V under the assumption that V is a non-negative polynomial satisfying a certain condition of C. Fefferman. This condition is of an algebraic nature. The estimates depend only on properties of the polynomial V and are independent of the dimension. Recently, the first author proved a dimension-free L^∞ bound for $R^{1/2}_V$ in the particular case of $V(x) = |x|^2$ and L being the harmonic oscillator, see [17, Theorem 8]. In fact it is proved there that the L^∞ norm of $R^{1/2}_{|x|^2}$ is less than 3. It is not clear whether one can prove dimension-free results on L^∞ as in [27] or [17] for R^1_V or $R^{1/2}_V$ for more general classes of potentials V. We hope to return to this topic in the near future.

It is perhaps noteworthy that in order to conclude the $L^p(\mathbb{R}^d)$, $p > 2$, boundedness of $R^{1/2}_V$, R^1_V or $|\nabla L^{-1/2}|$ the results available in the literature require that V satisfies at least a reverse Hölder condition. Such a V must then be a doubling weight. This is not required in our approach, for instance $V(x) = \beta^{|x|}$ is clearly non-doubling yet Theorems C and D apply.

We shall now describe the structure of our paper. Section 2 starts with definitions of the objects appearing throughout the paper. Then we prove several interpolation results for the Riesz transforms R^a_V, see Theorems 2.2 and 2.5 and Corollary 2.3. As an application, in Theorem 2.6 we obtain L^p boundedness of R^a_V for general non-negative potentials $V \in L^2_{\mathrm{loc}}$ within the range $1 < p \leqslant 2$, $0 \leqslant a \leqslant 1/p$. In Section 4 we prove Theorem 4.5 which gives sufficient conditions for the L^∞ boundedness of R^a_V and then we apply it to prove Theorem C. Section 5 is devoted to proving Theorems 5.4, 5.5 and 5.8 in which we present different conditions on V, a and p guaranteeing the L^1 boundedness of R^a_V and as a corollary Theorem D is proved.

Notation

Throughout the paper for $1 \leqslant p \leqslant \infty$ we denote by L^p the $L^p(\mathbb{R}^d)$ space with respect to the d-dimensional Lebesgue measure. For a function $f \in L^p$ we write $\|f\|_p := \|f\|_{L^p(\mathbb{R}^d)}$. Similar notation is also used for a bounded linear operator T on L^p; by $\|T\|_p$ we denote its norm. Although this is a slight collision of symbols it will cause no confusion later. For a Lebesgue-measurable subset $A \subseteq \mathbb{R}^d$ we denote by $|A|$ its Lebesgue measure. We say that f is a finitely simple function if it is a simple function supported in a compact subset of $\mathbb{R}^d$. Such functions are clearly dense in L^p, $1 \leqslant p < \infty$. For a set A we denote by $\mathbb{1}_A$ its characteristic function. The symbol $\mathbb{1}$

stands for the constant function 1. For $1 \leqslant p \leqslant \infty$ we denote by L^p_{loc} the space of functions which are locally in L^p. For $f \in L^1_{\mathrm{loc}}$ we denote by $\operatorname{supp} f$ its essential support. The space of smooth compactly supported functions on $\mathbb{R}^d$ is denoted by C_c^∞. For $x \in \mathbb{R}^d$ and $r > 0$ we denote by $B(x,r) := \{y \in \mathbb{R}^d \colon |x-y| \leqslant r\}$ the closed Euclidean ball of radius r.

The symbol $C_\square$ denotes a non-negative constant that depends only on the parameter $\square$. The exact value of $C_\square$ may change from one occurrence to another. We write C without subscript when the constant is universal in the sense that it may only depend on the dimension d or on the parameter of the Riesz transform $a > 0$.

It will be convenient to introduce an asymptotic notation. For two non-negative quantities A, B we write $A \lesssim B$ ($A \gtrsim B$) if there is an absolute constant $C > 0$ such that $A \leqslant CB$ ($A \geqslant CB$). We will write $A \approx B$ when $A \lesssim B$ and $A \gtrsim B$. In particular, if $A = A(x)$ and $B = B(x)$ are two non-negative functions on $\mathbb{R}^d$ then by $A \lesssim B$ we mean that $A(x) \leqslant CB(x)$ for almost all $x \in \mathbb{R}^d$; similar convention is applied to the symbols $\gtrsim$ and $\approx$. We say that a function $B \colon \mathbb{R}^d \to [0,\infty)$ controls a function $A \colon \mathbb{R}^d \to [0,\infty)$ globally if there exists a compact set F such that $A(x) \leqslant B(x)$ for almost all $x \notin F$. In this case we write $A \leqslant_g B$. Similarly, we say that any of the conditions $A \lesssim B$, $A \gtrsim B$ or $A \approx B$ holds globally if there exists a compact set F such that $A(x) \lesssim B(x)$, $A(x) \gtrsim B(x)$ or $A(x) \approx B(x)$, respectively, hold for almost every $x \notin F$. In this case we write, respectively, $A \lesssim_g B$, $A \gtrsim_g$, and $A \approx_g B$.

For a random variable X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $A \subseteq \mathbb{R}$ we denote $\mathbb{P}(X \in A) := \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\})$. We abbreviate *almost everywhere* and *almost every* to *a.e.*.

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2. Definitions and general results on L^p, $1 \leqslant p < \infty$

The main goal of this section is to define the Riesz transforms R_V^a, $a > 0$, on L^p and to prove L^p boundedness results for these operators valid for general classes of non-negative potentials V. Throughout this section we take $1 \leqslant p < \infty$. The case of $p = \infty$ is addressed in the next section.

Our general definition on L^p will be based on semigroups related to $-\frac{1}{2}\Delta + V$ that are given by the spectral theorem. Let $V \in L^1_{\mathrm{loc}}$ be an a.e. non-negative potential. This assumption is in force throughout the paper even if this is not stated explicitly. Whenever we write $V(x)$ we mean the value at x of a particular representative of the equivalence class of V in the space L^1_{loc}. The same is true for any expression in which similar ambiguity may arise. We follow closely the approach in [2, Section 3] (see also [7]) and define the Schrödinger operator L via quadratic forms. Consider the sesquilinear form

$$Q(u,v) = \int_{\mathbb{R}^d} \tfrac{1}{2}\langle \nabla u, \nabla v\rangle + V u \bar{v} \tag{2.1}$$

on the domain

$$\mathrm{Dom}(Q) = \{f \in L^2 : \nabla f \in L^2 \text{ and } V^{1/2} f \in L^2\},$$

where ∇f denotes the distributional gradient of f. We equip the domain with the norm

$$\|f\|_V = \left(\|f\|_2^2 + \tfrac{1}{2}\|\nabla f\|_2^2 + \left\|V^{1/2} f\right\|_2^2\right)^{1/2},$$

which turns it into a Hilbert space with $C_c^\infty(\mathbb{R}^d)$ as a dense subspace. Since Q is bounded below and non-negative, there is a unique positive self-adjoint operator L such that

$$\langle Lu, v\rangle = Q(u,v), \quad u \in \mathrm{Dom}(L),\, v \in \mathrm{Dom}(Q)$$

and its square root $L^{1/2}$, defined on $\mathrm{Dom}(L^{1/2}) = \mathrm{Dom}(Q)$, satisfies

$$\left\|L^{1/2} f\right\|_2^2 = \tfrac{1}{2}\|\nabla f\|_2^2 + \left\|V^{1/2} f\right\|_2^2, \quad f \in C_c^\infty(\mathbb{R}^d). \tag{2.2}$$

By [2, Section 3] the semigroup e^{-tL} is positivity-preserving and pointwise dominated by the heat semigroup, hence it is a contraction on L^p for $1 \leqslant p \leqslant \infty$.

Let $a > 0$. For $f \in L^p$, $1 \leqslant p < \infty$, and $\varepsilon > 0$ we define

$$(L + \varepsilon I)^{-a} f := \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL} f\, t^{a-1} e^{-\varepsilon t}\, \mathrm{d}t, \tag{2.3}$$

Since the semigroup e^{-tL} is a strongly continuous semigroup of contractions on L^p, the integral in (2.3) is well defined as a Bochner integral on L^p. It is also not hard to see that for $f \in L^2$ the operator defined by (2.3) coincides with $(L+\varepsilon I)^{-a}$ given by the spectral theorem. Moreover, if f is an a.e. non-negative function in L^p then

$$L^{-a}f(x) \coloneqq \lim_{\varepsilon\to 0^+} \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL} f(x)\, t^{a-1} e^{-\varepsilon t}\, \mathrm{d}t,$$

exists x-a.e. as a monotone pointwise limit being finite or infinite. In either case

$$L^{-a}f(x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL} f(x)\, t^{a-1}\, \mathrm{d}t,$$

by the monotone convergence theorem. For $a>0$ and a non-negative function $f \in L^p$ we let

$$R_V^a f(x) \coloneqq V^a(x) L^{-a} f(x), \qquad x \in \mathbb{R}^d. \tag{2.4}$$

This is well defined x-a.e. though possibly equal to infinity. Additionally, for $a=0$ we set R_V^0 to be $\mathbb{1}_{V\neq 0}$ times the identity operator.

Definition 2.1. — *Let $1 \leqslant p < \infty$ and $a>0$. We say that the Riesz transform R_V^a is bounded on L^p if there is a constant $C>0$ such that*

$$\|R_V^a f\|_p \leqslant C \|f\|_p,$$

for all non-negative finitely simple functions $f \in L^p$.

Note that if R_V^a is bounded on L^p, then for each finitely simple function f the quantity $R_V^a|f|$ given by (2.4) is finite for a.e. $x \in \mathbb{R}^d$. Since $|e^{-tL}f| \leqslant e^{-tL}|f|$ we see that in this case

$$V^a(x) \int_0^\infty e^{-tL} f(x)\, t^{a-1}\, \mathrm{d}t$$

is finite x-a.e.. Thus, whenever R_V^a is bounded on L^p the integral above is a natural definition of $R_V^a f$, first for finitely simple functions and then, by density, for arbitrary functions in L^p.

Using Stein's complex interpolation theorem and functional calculus for symmetric contraction semigroups [6] we now prove an interpolation result for the operators R_V^a. Similar method was applied in [2, Section 6]. There the authors proved the L^p boundedness of $R_V^{1/2}$ for $1<p<2(q+\varepsilon)$ by using Stein's complex interpolation theorem together with the L^p boundedness of R_V^1. They considered non-negative potentials belonging to a reverse Hölder class B_q.

Theorem 2.2. — *Let $0 \leqslant a_0 < a_1$. Assume that $V \in L^1_{\mathrm{loc}}$ is an a.e. non-negative potential such that $R_V^{a_0}$ is bounded on L^{p_0} and $R_V^{a_1}$ is bounded on L^{p_1} for some $p_0, p_1 \in (1,\infty)$. Then, R_V^a is bounded on L^p for every p and a such that $1/p = \theta/p_0 + (1-\theta)/p_1$ and $a = \theta a_0 + (1-\theta)a_1$ with some $\theta \in (0,1)$.*

Proof. — Let $\varepsilon > 0$ and denote $F(\varepsilon) := \{x \in \mathbb{R}^d \colon \varepsilon < V(x) < \varepsilon^{-1}\}$. It is enough to justify that

$$R^{a,\varepsilon} f(x) := (\mathbb{1}_{F(\varepsilon)} V^a)(x) \cdot \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL} f(x)\, t^{a-1} e^{-\varepsilon t}\, \mathrm{d}t,$$

satisfies for all simple functions f the bound

$$\|R^{a,\varepsilon} f\|_p \leqslant C \|f\|_p, \tag{2.5}$$

uniformly in $\varepsilon > 0$ and with $C > 0$ being a constant. Indeed, if (2.5) holds, then taking $\varepsilon \to 0^+$ we obtain the L^p boundedness of R_V^a, first (with the aid of monotone convergence theorem) for non-negative simple functions and then for all functions in L^p.

Thus, in the remainder of the proof we fix $\varepsilon > 0$ and focus on justifying (2.5). Denote $S = \{z \in \mathbb{C} \colon a_0 < \operatorname{Re} z < a_1\}$. Then, for $z \in \overline{S}$ and $\varepsilon > 0$ the function $m_z^\varepsilon(\lambda) = (\lambda + \varepsilon)^{-z}$ is a bounded function on $[0, \infty)$, hence, by the spectral theorem $(L + \varepsilon I)^{-z}$ is well defined as a bounded operator on L^2. We let

$$T_z f := (\mathbb{1}_{F(\varepsilon)} V^z) \cdot (L + \varepsilon I)^{-z} f, \qquad f \in L^2. \tag{2.6}$$

Since $(L + \varepsilon I)^{-b}$ given by the spectral theorem coincides with

$$\frac{1}{\Gamma(b)} \int_0^\infty e^{-tL} f\, t^{b-1} e^{-\varepsilon t}\, \mathrm{d}t,$$

for every $b > 0$, we have

$$R^{b,\varepsilon} f = T_b f, \qquad f \in L^2.$$

Thus, in order to justify (2.5) it suffices to prove a uniform in $\varepsilon > 0$ bound for the L^p norm of T_a.

This will be achieved by Stein's complex interpolation theorem. Note first that for f, g being finitely simple functions the pairing

$$h(z) = \langle T_z f, g \rangle, \qquad z \in \overline{S},$$

gives a function which is holomorphic in S. To see this observe that (2.3) still holds with complex $a \in S$. Combining this observation with the definition (2.6) of T_z it is easy to see that h is indeed holomorphic. Additionally, the spectral theorem implies the bound

$$|h(z)| \leqslant C(\varepsilon, f, g),$$

valid for $z \in \overline{S}$. Altogether $\{T_z\}_{z \in \overline{S}}$ is an analytic family of operators of admissible growth.

It remains to bound the operator T_z for $\operatorname{Re} z = a_0$ and $\operatorname{Re} z = a_1$; this is the place where we use the assumptions on $R_V^{a_j}$. Writing, for $z = a_j + i\tau$, $\tau \in \mathbb{R}$, $j = 0, 1$,

$$T_z = (\mathbb{1}_{F(\varepsilon)} V^z) \cdot (L + \varepsilon I)^{-z} = (\mathbb{1}_{F(\varepsilon)} V^{i\tau}) T_{a_j} (L + \varepsilon I)^{-i\tau}$$

we see that

$$\|T_z\|_{p_j} \leqslant \|T_{a_j}\|_{p_j} \|(L + \varepsilon I)^{-i\tau}\|_{p_j}. \tag{2.7}$$

Since $(L + \varepsilon I)$ generates a symmetric contraction semigroup and $p_j \in (1, \infty)$, by e.g. [6] the imaginary powers $(L + \varepsilon I)^{-i\tau}$ satisfy

$$\|(L + \varepsilon I)^{-i\tau}\|_{p_j} \lesssim e^{\pi|\tau|/2}, \tag{2.8}$$

uniformly in $\varepsilon > 0$. Moreover, we have

$$|T_{a_j}(f)(x)| = |R^{a_j,\varepsilon} f(x)| \leqslant R_V^{a_j} |f|(x), \qquad x \in \mathbb{R}^d.$$

Thus, coming back to (2.7) and using our assumptions on the L^{p_j} boundedness of $R_V^{a_j}$ we obtain, for $z = a_j + i\tau$, $j = 0, 1$,

$$\|T_z\|_{p_j} \lesssim e^{\pi|\tau|/2}, \qquad \tau \in \mathbb{R}.$$

In conclusion, applying Stein's complex interpolation theorem, see e.g. [15, Theorem 1.3.7], we obtain the L^p boundedness of R_V^a. □

Theorem 2.2 immediately leads to the following corollary.

Corollary 2.3. — *Let $a_0 \geqslant 0$, $a_1 \geqslant 0$, and assume that both $R_V^{a_1}$ and $R_V^{a_2}$ are bounded on L^p for some $1 < p < \infty$. Then R_V^a is bounded on L^p for every $a_0 \leqslant a \leqslant a_1$.*

Proof. — We apply Theorem 2.2 with $p_0 = p_1 = p$. □

It is straightforward to see that the Riesz transform $R_V^{1/2}$ is bounded on L^2. Using Corollary 2.3 we now extend the L^2 boundedness to the operators R_V^a with $0 \leqslant a \leqslant \frac{1}{2}$.

Proposition 2.4. — *Let $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ be an a.e. non-negative potential. If $0 \leqslant a \leqslant \frac{1}{2}$, then R_V^a extends to a contraction on $L^2(\mathbb{R}^d)$.*

Proof. — By formula (2.2) we have

$$\left\|V^{1/2} f\right\|_2 \leqslant \left\|L^{1/2} f\right\|_2, \qquad f \in C_c^\infty;$$

here $L^{1/2}$ is the self-adjoint operator with domain $\operatorname{Dom}(L^{1/2}) = \operatorname{Dom}(Q)$, while Q is the sesquilinear form given by (2.1). Using the fact that self-adjoint operators are closed we get $\operatorname{Dom}(L^{1/2}) \subseteq \operatorname{Dom}(V^{1/2})$ and

$$\left\|V^{1/2} f\right\|_2 \leqslant \left\|L^{1/2} f\right\|_2, \qquad f \in \operatorname{Dom}(L^{1/2}). \tag{2.9}$$

For each fixed $\varepsilon > 0$ the operator $(L + \varepsilon I)^{-1/2}$ is bounded on L^2 by the spectral theorem. Taking $f = (L + \varepsilon I)^{-1/2} g$ with $g \in L^2$ in (2.9) we get

$$\left\| V^{1/2}(L + \varepsilon I)^{-1/2} g \right\|_2 \leqslant \left\| L^{1/2}(L + \varepsilon I)^{-1/2} g \right\|_2, \qquad g \in L^2. \tag{2.10}$$

If g is a non-negative function on L^2 then by definitions (2.3), (2.4) and the monotone convergence theorem we have $\lim_{\varepsilon \to 0^+} \left\| V^{1/2}(L + \varepsilon I)^{-1/2} g \right\|_2 = \|R_V^{1/2} g\|_2$. The right-hand side of (2.10) converges to $\|g\|_2$ as $\varepsilon \to 0^+$ by the spectral theorem. Therefore we justified that $\|R_V^{1/2} g\|_2 \leqslant \|g\|_2$ for non-negative $g \in L^2$. This implies that $R_V^{1/2}$ is a contraction on L^2.

At this stage an application of Corollary 2.3 shows that R_V^a is bounded on L^2 whenever $0 \leqslant a \leqslant 1/2$. The contractivity of R_V^a is not a direct consequence of the corollary. However, it is easy to justify once we follow the proof of Theorem 2.2 and enhance inequality (2.8) to

$$\|(L + \varepsilon I)^{-i\tau}\|_2 \leqslant 1, \qquad \tau \in \mathbb{R}.$$

We omit details here. □

When $p_0 = 1$ we have a slightly weaker variant of Theorem 2.2 with the restriction $a_0, a_1 > 0$. This is due to the unboundedness of the imaginary powers $L^{i\tau}$, $\tau \in \mathbb{R}$, on L^1.

Theorem 2.5. — *Let $a_0 > 0$ and $a_1 > 0$. Assume that $V \in L^1_{\mathrm{loc}}$ is such that $R_V^{a_1}$ is bounded on L^{p_1} for some $p_1 \in [1, \infty)$ and $R_V^{a_0}$ is bounded on L^1. Then, R_V^a is bounded on L^p for every p and a such that $1/p = \theta + (1-\theta)/p_1$ and $a = \theta a_0 + (1-\theta) a_1$ with some $\theta \in (0,1)$.*

Proof. — The proof is similar to that of Theorem 2.2. For $\varepsilon > 0$ we define the sets $F(\varepsilon)$ and the operators $R^{a,\varepsilon}$ as in that proof. Once again it suffices to justify (2.5).

Assume without loss of generality that $a_0 < a_1$, let $S = \{z \in \mathbb{C} \colon a_0 < \operatorname{Re} z < a_1\}$ and define the family of operators $\{T_z\}_{z \in \overline{S}}$ as in (2.6). Since this time $a_0 > 0$ the formula

$$T_z f = (\mathbb{1}_{F(\varepsilon)} V^z) \cdot \frac{1}{\Gamma(z)} \int_0^\infty e^{-tL} f \, t^{z-1} e^{-\varepsilon t} \, \mathrm{d}t, \qquad f \in L^2, \tag{2.11}$$

holds for $z \in \overline{S}$. Moreover, $\{T_z\}_{z \in S}$ is a family of analytic operators of admissible growth; this can be justified as in the proof of Theorem 2.2. Hence, in order to apply Stein's complex interpolation theorem it remains to bound $\|T_z\|_{p_j}$ for $z = a_j + i\tau$, $j = 0, 1$. Using (2.11) and the asymptotics for the Gamma function $|\Gamma(a_j + i\tau)| \approx |\tau|^{a_j - 1/2} e^{-\pi|\tau|/2}$, see [19, 5.11.9], we obtain the pointwise bound

$$\begin{aligned}|T_z f(x)| &\lesssim e^{\pi|\tau|}(\mathbb{1}_{F(\varepsilon)}V^{a_j})(x)\cdot\int_0^\infty e^{-tL}|f|(x)\,t^{a_j-1}e^{-\varepsilon t}\,\mathrm{d}t\\ &\lesssim e^{\pi|\tau|}R_V^{a_j}|f|(x),\end{aligned}$$

valid for $z=a_j+i\tau$, $j=0,1$. Hence, the L^1 boundedness of $R_V^{a_0}$ together with the L^{p_1} boundedness of $R_V^{a_1}$ give

$$\|T_z\|_1 \lesssim e^{\pi|\tau|},\qquad z=a_0+i\tau,\quad \tau\in\mathbb{R},$$

and

$$\|T_z\|_{p_1} \lesssim e^{\pi|\tau|},\qquad z=a_1+i\tau,\quad \tau\in\mathbb{R}.$$

Thus, using Stein's complex interpolation theorem, we complete the proof. □

Analogously to the L^2 case one particular Riesz transform R_V^1 is always bounded on L^1, see [2, Theorem 4.3] and [14, 16]. Interpolating this result with Proposition 2.4 we obtain the following theorem.

Theorem 2.6. — *Let $V\in L^1_{\mathrm{loc}}$ and take $p\in(1,2]$. Then for all $0\leqslant a\leqslant 1/p$ the Riesz transform R_V^a is bounded on L^p.*

Proof. — The L^2 boundedness of $R_V^{1/2}$ is guaranteed by Proposition 2.4. The L^1 boundedness of R_V^1 is justified in [2, Theorem 4.3]. Hence, Theorem 2.5 gives the L^p boundedness of R_V^a whenever $a=\theta+(1-\theta)/2=1/p$. Finally, Corollary 2.3 extends the boundedness on L^p to $0\leqslant a\leqslant 1/p$. □

3. Definitions and a counterexample on L^∞

Here the approach from the previous section is invalid since e^{-tL} does not necessarily extend to a strongly continuous semigroup on L^∞. However, for certain classes of potentials the operator e^{-tL}, $t>0$, can be also expressed by the celebrated Feynman–Kac formula

$$e^{-tL}f(x)=\mathbb{E}_x\left[e^{-\int_0^t V(X_s)\mathrm{d}s}f(X_t)\right],\qquad f\in L^p, \tag{3.1}$$

where $1\leqslant p<\infty$. The expectation $\mathbb{E}_x$ is taken with regards to the Wiener measure of the standard d-dimensional Brownian motion $\{X_s\}_{s>0}$, starting at $x\in\mathbb{R}^d$; here $X_s=(X_s^1,\dots,X_s^d)$. Since the potential V is a.e. non-negative, identity (3.1) is true whenever $V\in L^2_{\mathrm{loc}}$ belongs to the local Kato class K_d^{loc}. This follows for example from [24, Remark 4.14] once we recall that for $V\in L^2_{\mathrm{loc}}$ the operator $-\Delta/2+V$ is essentially self-adjoint on C_c^∞, hence its Friedrichs extension is its unique self-adjoint extension. We will not need the definition of the local Kato class in our paper; for our purpose it is important to note that $L^q_{\mathrm{loc}} \subseteq K_d^{\mathrm{loc}}$ whenever $q \geqslant 1$ satisfies $q > d/2$, see [18, Lemma 4.105]. Therefore (3.1) is true for $V \in L^q_{\mathrm{loc}}$ whenever $q > d/2$ and $q \geqslant 2$, in particular for $V \in L^\infty_{\mathrm{loc}}$. The right-hand side of (3.1) makes sense also for $f \in L^\infty$, see [18, Section 4.2.4], which leads us to define for $t > 0$

$$e^{-tL} f(x) := \mathbb{E}_x\Big[e^{-\int_0^t V(X_s)\mathrm{d}s} f(X_t)\Big], \qquad f \in L^\infty. \tag{3.2}$$

To deal with measurability questions we need a technical lemma on the continuity of $e^{-tL}f$.

Lemma 3.1. — *Assume that $q > d/2$ and $q \geqslant 2$ and let $V \in L^q_{\mathrm{loc}}$ be an a.e. non-negative potential. Then for all $f \in L^\infty$ the function $e^{-tL}f(x)$ given by (3.2) is jointly continuous in $(t,x) \in (0,\infty) \times \mathbb{R}^d$. In particular $e^{-tL}(\mathbb{1})(x)$ is jointly continuous in t and x.*

Proof. — Since $L^q_{\mathrm{loc}} \subseteq K_d^{\mathrm{loc}}$ it follows from [24, Proposition 3.5] that e^{-tL} is an integral operator with its kernel $K_t(x,y)$ being a jointly continuous functions of (t,x,y). Since $V \geqslant 0$ we also have

$$K_t(x,y) \leqslant (2\pi t)^{-d/2} \exp\big(|x-y|^2/(2t)\big)$$

and therefore for each $N > 0$ it holds

$$\int_{|x-y|>N} K_t(x,y)|f(y)|\,\mathrm{d}y \leqslant \pi^{-d/2}\|f\|_\infty \int_{|w|\geqslant N/(\sqrt{2t})} e^{-|w|^2}\,\mathrm{d}w. \tag{3.3}$$

Consider now $(t,x) \to (t_0,x_0)$ and let $\varepsilon > 0$ be arbitrarily small. Splitting

$$e^{-tL}f(x) = \int_{|x-y|\leqslant N} K_t(x,y)f(y)\,\mathrm{d}y + \int_{|x-y|>N} K_t(x,y)f(y)\,\mathrm{d}y$$

and using (3.3) we see that for $N = N(\varepsilon)$ large enough holds

$$\left| e^{-tL}f(x) - \int_{|x-y|\leqslant N} K_t(x,y)f(y)\,\mathrm{d}y \right| \leqslant \varepsilon,$$

uniformly in $t_0/2 < t < 2t_0$ and $|x-x_0| < 1$. Moreover, for such (t,x) we see that $C\|f\|_{L^\infty}\mathbb{1}_{|y|\leqslant N+|x_0|+1}$ is an integrable majorant of $\mathbb{1}_{|x-y|\leqslant N}K_t(x,y)f(y)$. Thus, using Lebesgue's dominated convergence theorem we obtain

$$\limsup_{(t,x)\to(t_0,x_0)} \left|e^{-tL}f(x) - e^{-t_0L}f(x_0)\right| \leqslant 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary this completes the proof. □

Now, take $a > 0$ and let $V \in L^\infty_{\mathrm{loc}}$ be an a.e. non-negative potential. For a non-negative function $f \in L^\infty$ we define the Riesz transform R_V^a by

$$(3.4)\quad R_V^a f(x) = V^a(x) \cdot \frac{1}{\Gamma(a)} \int_0^\infty \mathbb{E}_x \left[e^{-\int_0^t V(X_s)\,\mathrm{d}s} f(X_t) \right] t^{a-1}\,\mathrm{d}t, \quad f \in L^\infty.$$

Note that by Lemma 3.1 the function $R_V^a f(x)$ is then a measurable function on $\mathbb{R}^d$ possibly being infinite for some x. Moreover, by (3.1) the L^∞ definition (3.4) coincides with the L^p definition (2.4) whenever f is a finitely simple function.

Since the semigroup is positivity preserving we have

$$(3.5)\quad |e^{-tL} f(x)| \leqslant e^{-tL}(\|f\|_\infty \mathbb{1})(x) = \|f\|_\infty e^{-tL}(\mathbb{1})(x), \qquad f \in L^\infty,$$

which leads to the following definition of the L^∞ boundedness of R_V^a.

Definition 3.2. — *We say that the Riesz transform R_V^a is bounded on L^∞ if*

$$(3.6)\quad \|R_V^a(\mathbb{1})\|_\infty < \infty.$$

Note that if (3.6) holds, then for every $f \in L^\infty$ by (3.5) we have $|R_V^a(f)(x)| \leqslant \|f\|_\infty R_V^a(\mathbb{1})(x)$ so that

$$\|R_V^a(f)\|_\infty \leqslant C\|f\|_\infty, \qquad f \in L^\infty$$

with $C = \|R_V^a(\mathbb{1})\|_\infty$.

Since

$$(3.7)\quad R_V^a(\mathbb{1})(x) = V^a(x) \cdot \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL}(\mathbb{1})(x)\, t^{a-1}\,\mathrm{d}t$$

it is apparent that in order for R_V^a to be finite a.e. on $\operatorname{supp} V$ the monotone function $t \mapsto e^{-tL}(\mathbb{1})(x)$ must converge to 0 as $t \to \infty$. This however is not always the case.

Proposition 3.3. — *Let $d \geqslant 3$ and let V be a non-negative potential on $\mathbb{R}^d$ which is compactly supported and not identically equal to zero. Assume that $V \in L^q(\mathbb{R}^d)$ with $q > d/2$ and $q \geqslant 2$. Then, for any $a > 0$ we have $R_V^a(\mathbb{1})(x) = \infty$ for x such that $V(x) \neq 0$. In particular R_V^a is unbounded on L^∞.*

Proof. — Fix $a > 0$. For $x \in \mathbb{R}^d$ we let $w(x) = \lim_{s\to\infty} e^{-sL}(\mathbb{1})(x)$. From [13, Lemma 2.4] there exist a constant $\delta > 0$ such that $\delta < w(x) \leqslant 1$ uniformly in $x \in \mathbb{R}^d$. Since by the semigroup property $w(x) = e^{-tL}(w)(x)$ for any $t > 0$, we see that $e^{-tL}(\mathbb{1}) \geqslant e^{-tL}(w)(x) \geqslant \delta$ uniformly in $x \in \mathbb{R}^d$.

Consequently, the integral $\int_0^\infty e^{-tL}(\mathbb{1})(x)\,t^{a-1}\,\mathrm{d}t$ is infinite for a.e. x and so is $R_V^a(\mathbb{1})(x)$ as long as $V(x)\neq 0$. □

The definition below is meant to guarantee the x-a.e. finiteness of $R_V^a f(x)$.

Definition 3.4. — *Let $V\in L^\infty_{\mathrm{loc}}$ be an a.e. non-negative potential and let $\delta>0$. We say that the semigroup e^{-tL} has an exponential decay of order δ (*$\mathrm{ED}(\delta)$ *for short) if there exists a constant $C>0$ such that*

$$\|e^{-tL}(\mathbb{1})\|_\infty \leqslant Ce^{-\delta t},\qquad t>0. \tag{ED(δ)}$$

The assumption $(ED(\delta))$ implies $|R_V^a f(x)|\leqslant C\delta^{-a}V^a(x)\|f\|_\infty$ x-a.e.. Note, however, that this may not be enough to conclude that $\|R_V^a(\mathbb{1})\|_\infty<\infty$.

4. L^∞ boundedness for classes of potentials

Throughout this section we assume that $V\in L^\infty_{\mathrm{loc}}$. Here our goal is to estimate the L^∞ norm of R_V^a for classes of potentials V. As mentioned in Definition 3.2 this is the same as estimating $\|R_V^a(\mathbb{1})\|_\infty$ with $R_V^a(\mathbb{1})$ defined by (3.7).

Before we dive into details, we prove a general result concerning the L^∞ decay of the semigroup e^{-tL} defined in (3.2). We will use Lemma 4.1 below to prove the L^∞ and L^1 boundedness of R_V^a for concrete examples of potentials V in Theorems C and D. Here π denotes a $(d-1)$-dimensional hyperplane in $\mathbb{R}^d$. For $N>0$ we let P be the strip

$$P=P_N:=\left\{x\in\mathbb{R}^d:\mathrm{dist}(x,\pi)\leqslant N\right\}$$

and set $\chi=\mathbb{1}_P$.

Lemma 4.1. — *Let $N>0$ and assume that the potential $V\in L^\infty_{\mathrm{loc}}$ is uniformly positive outside the strip P_N. More precisely we assume that V is non-negative a.e. and that there is $c>0$ such that $V(x)\geqslant c$ for a.e. x satisfying* $\mathrm{dist}(x,\pi)>N$. *Then the semigroup e^{-tL} has $ED(\delta)$ with $\delta=\frac12\min\left(c,\frac{1}{8N^2}\right)$. More precisely, there is a universal constant $C>0$ such that for $t>0$ and $x\in\mathbb{R}^d$ it holds*

$$e^{-tL}(\mathbb{1})(x)\leqslant C\,e^{-\delta t}.$$

To prove the above lemma we will need an auxiliary fact. Lemma 4.2 below can be deduced from [18, Lemma 4.105]. For the sake of completeness we give a more direct proof below.

Lemma 4.2. — *For all $k > 0$, $t > 0$, and $x \in \mathbb{R}^d$ we have*

$$\mathbb{E}_x\left[e^{2\int_0^t k\chi(X_s)\mathrm{d}s}\right] \leqslant C\, e^{8N^2k^2t}, \tag{4.1}$$

where $C > 0$ is a universal constant.

Proof. — We prove this fact in the case $\pi = \{0\} \times \mathbb{R}^{d-1}$ and $P = [-N, N] \times \mathbb{R}^{d-1}$. The general result follows from the invariance of Brownian motion under orthogonal transformations (see [20, p. 5]) and the fact that the bound is independent of x. Since in this case $\chi(X_s) = \mathbb{1}_{[-N,N]}(X_s^1)$ it suffices to prove the lemma in the dimension $d = 1$. In particular in the proof we take $x \in \mathbb{R}$.

The main tool of our proof is the local time of Brownian motion defined for $y \in \mathbb{R}$ in the one-dimensional case as

$$L_t(y) = \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{[y-\varepsilon,\, y+\varepsilon]}(Y_s)\,\mathrm{d}s,$$

where $\{Y_s\}_{s>0}$ is the standard one-dimensional Brownian motion starting at 0. It is well known that

$$\int_0^t f(Y_s)\,\mathrm{d}s = \int_{\mathbb{R}} f(y)L_t(y)\,\mathrm{d}y$$

for any locally integrable function f, see [5, (5.4)]. In particular, we have

$$\int_0^t \mathbb{1}_{[-N-x,\, N-x]}(Y_s)\,\mathrm{d}s = \int_{-N-x}^{N-x} L_t(y)\,\mathrm{d}y. \tag{4.2}$$

The law of $L_t(y)$ was computed by Takács [25]. From a paper of Doney and Yor [10], see the last identity in Section 3 on p. 277 (with $\mu = 0$ and $x = y$) and [10, eq. (1.4)], it follows that the distribution of $L_t(y)$ is given by

$$c_{y,t}\delta_0 + f_{y,t}(z)\,\mathrm{d}z$$

on $[0, +\infty)$, where δ_0 denotes the Dirac measure at 0,

$$f_{y,t}(z) = \frac{\sqrt{2}}{\sqrt{\pi t}}e^{-\frac{(|y|+z)^2}{2t}}, \qquad y \in \mathbb{R}, \quad z > 0, \tag{4.3}$$

and $c_{y,t} < 1$ is a normalizing constant which value is irrelevant for us.

Using (4.2) and Jensen's inequality for $x \in \mathbb{R}$ we obtain

$$\begin{aligned}\mathbb{E}_x\left[e^{2\int_0^t k\chi(X_s)\mathrm{d}s}\right] &= \mathbb{E}_0\left[e^{2\int_{-N-x}^{N-x} kL_t(y)\mathrm{d}y}\right]\\ &\leqslant \frac{1}{2N}\mathbb{E}_0\left[\int_{-N-x}^{N-x} e^{4NkL_t(y)}\,\mathrm{d}y\right]\\ &\leqslant \frac{1}{2N}\int_{-N-x}^{N-x}\left(1+\int_0^\infty e^{4Nkz} f_{y,t}(z)\,\mathrm{d}z\right)\mathrm{d}y\\ &= 1+\frac{1}{2N}\int_0^\infty e^{4Nkz}\int_{-N-x}^{N-x} f_{y,t}(z)\,\mathrm{d}y\,\mathrm{d}z\end{aligned}$$

The 1+ term in the second line comes from the atom of the distribution of $L_t(y)$ at $z=0$. Since the function $y \mapsto f_{y,t}(z)$ is radially decreasing, we can change the limits of the inner integral to $[-N,N]$, possibly increasing its value. Thus, using (4.3) gives

$$\begin{aligned}&1+\frac{1}{2N}\int_0^\infty e^{4Nkz}\int_{-N-x}^{N-x} f_{y,t}(z)\,\mathrm{d}y\,\mathrm{d}z\\ &\leqslant 1+\frac{1}{2N}\int_0^\infty e^{4Nkz}\int_{-N}^{N} f_{y,t}(z)\,\mathrm{d}y\,\mathrm{d}z\\ &= 1+\frac{\sqrt{2}}{N\sqrt{\pi t}}\int_0^\infty e^{4Nkz}\int_0^N e^{-\frac{(y+z)^2}{2t}}\,\mathrm{d}y\,\mathrm{d}z.\end{aligned} \tag{4.4}$$

First we deal with the inner integral. Calculating it yields

$$\int_0^N e^{-\frac{(y+z)^2}{2t}}\,\mathrm{d}y = \sqrt{\frac{\pi t}{2}}\left(\operatorname{erf}\left(\frac{z+N}{\sqrt{2t}}\right)-\operatorname{erf}\left(\frac{z}{\sqrt{2t}}\right)\right).$$

To estimate the expression above, note that $\operatorname{erf}'(y)=\frac{2e^{-y^2}}{\sqrt{\pi}}$, hence, by the mean value theorem

$$\operatorname{erf}\left(\frac{z+N}{\sqrt{2t}}\right)-\operatorname{erf}\left(\frac{z}{\sqrt{2t}}\right)=\frac{N}{\sqrt{2t}}\operatorname{erf}'(\theta),$$

for some $\theta > z/(\sqrt{2t})$ and thus

$$\int_0^N e^{-\frac{(y+z)^2}{2t}}\,\mathrm{d}y \lesssim Ne^{-\frac{z^2}{2t}}.$$

Plugging the above estimate into (4.4), we obtain

$$\begin{aligned}\mathbb{E}_x\left[e^{2\int_0^t k\chi(X_s)\mathrm{d}s}\right] &\lesssim 1+\sqrt{\frac{2}{\pi t}}\int_0^\infty e^{4Nkz-\frac{z^2}{2t}}\,\mathrm{d}z\\ &\lesssim e^{8N^2k^2t},\end{aligned}$$

which completes the proof of Lemma 4.2. □

Now we prove Lemma 4.1. In the proof the quadratic dependence on k on the right-hand side of (4.1) will be crucial.

Proof of Lemma 4.1. — We want to make use of the assumption that the potential V is uniformly positive outside the set P together with the previous lemma. We achieve this by an appropriate application of the Cauchy–Schwarz inequality.

Recall that $\chi = \mathbb{1}_P$ and take $k \in (0, c]$. Since the potential $2(V + k\chi)$ is bounded below by $2k$ using Cauchy–Schwarz inequality we estimate

$$\begin{aligned} e^{-tL}(\mathbb{1})(x) &= \mathbb{E}_x\left[e^{-\int_0^t V(X_s)\mathrm{d}s}\right] \\ &= \mathbb{E}_x\left[e^{-\int_0^t V(X_s)+k\chi(X_s)\mathrm{d}s}\, e^{\int_0^t k\chi(X_s)\mathrm{d}s}\right] \\ &\leqslant \left[\mathbb{E}_x e^{-2\int_0^t V(X_s)+k\chi(X_s)\mathrm{d}s}\right]^{1/2}\left[\mathbb{E}_x e^{2\int_0^t k\chi(X_s)\mathrm{d}s}\right]^{1/2} \\ &\leqslant e^{-kt}\mathbb{E}_x\left[e^{2\int_0^t k\chi(X_s)\mathrm{d}s}\right]^{1/2}. \end{aligned}$$

Applying Lemma 4.2 for each k satisfying $4N^2k^2 \leqslant \frac{k}{2}$ we get

$$e^{-tL}(\mathbb{1})(x) \lesssim e^{-kt+4N^2k^2t} \leqslant e^{-\frac{kt}{2}}, \qquad x \in \mathbb{R}^d.$$

In particular, the above estimate holds for $k = \min(c, (8N^2)^{-1})$ and the proof is completed. □

Now focus on our goal, which is estimating the quantity

$$\Gamma(a)\, R_V^a(\mathbb{1})(x) = V^a(x) \int_0^\infty e^{-tL}(\mathbb{1})(x)\, t^{a-1}\,\mathrm{d}t \tag{4.5}$$

independently of $x \in \mathbb{R}^d$. We will do this by splitting the integral in (4.5) into two parts and estimating them separately.

Before stating the result we need to introduce a quantity ρ which plays a crucial role in our assumptions. For $u \geqslant 1$ and $x \in \mathbb{R}^d$ we define

$$\rho = \rho_x(u) = \sup\left\{r \geqslant 0 : \tfrac{V(x)}{u} \leqslant V(y) \text{ for a.e. } y \in B(x,r)\right\}; \tag{4.6}$$

recall that $B(x,r)$ denotes the closed Euclidean ball of radius r in $\mathbb{R}^d$. Consequently, $\rho_x(u)$ is the radius of the largest closed ball around x in which the potential V is at least $V(x)/u$ a.e. We note that $\rho_x(u)$ is a non-decreasing function of u with values in $[0, \infty]$. We also set

$$r_k = r_k(x) = \rho_x(2^k) \quad \text{for } k = 0, 1, \ldots. \tag{4.7}$$

Our main assumption will be phrased in terms of

$$I^a(V)(x) := \int_1^{\max(1,V(x))} s^{a-1} e^{-\frac{\rho_x^2(s)}{4d}}\,\mathrm{d}s \qquad \text{for a.e. } x \in \mathbb{R}^d. \tag{4.8}$$

If $\rho_x(s) = \infty$, then we define $e^{-\frac{\rho_x^2(s)}{4d}} = 0$.

First we estimate the integral in (4.5) from 0 to 1. Recall that implicit constants in $\lesssim$ and $\approx$ do not depend on $x \in \mathbb{R}^d$ but may depend on $a > 0$.

Lemma 4.3. — *Let V be an a.e. non-negative potential and let $a > 0$. Then we have*

$$V(x)^a \int_0^1 e^{-tL}(\mathbb{1})(x)\, t^{a-1}\mathrm{d}t \lesssim I^a(V)(x) + 1 \qquad \textit{for a.e. } x \in \mathbb{R}^d.$$

Proof. — First if $V(x) \leqslant 2$, then

$$V(x)^a \int_0^1 e^{-tL}(\mathbb{1})(x)\, t^{a-1}\mathrm{d}t \lesssim 1.$$

From now on we focus on the other case $V(x) > 2$. Define $K = K(x) = \lfloor \log_2 V(x) \rfloor$. For fixed $x \in \mathbb{R}^d$ and $k = 0, 1, 2, \dots$ we introduce the sets

$$A_k = \left\{ y \in \mathbb{R}^d : \frac{V(x)}{2^k} \leqslant V(y) \right\} \tag{4.9}$$

and

$$\Omega_k = \{\omega \in \Omega : X_s(\omega) \in A_k \text{ for almost all } s \in [0,t]\},$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying probability space for the d-dimensional Brownian motion $\{X_s\}_{s>0}$ started at 0.

Note that both the families $\{A_k\}$ and $\{\Omega_k\}$ are increasing in k. Using the Feynman–Kac formula (3.2) we write

$$\begin{aligned} e^{-tL}(\mathbb{1})(x) &= \mathbb{E}_x\left[e^{-\int_0^t V(X_s)\mathrm{d}s}\, \mathbb{1}_{\Omega_0}\right] + \sum_{k=1}^{K} \mathbb{E}_x\left[e^{-\int_0^t V(X_s)\mathrm{d}s}\, \mathbb{1}_{\Omega_k \cap \Omega_{k-1}^c}\right] \\ &\quad + \mathbb{E}_x\left[e^{-\int_0^t V(X_s)\mathrm{d}s}\, \mathbb{1}_{\Omega_K^c}\right] \\ &\leqslant e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} \mathbb{P}\big(\Omega_k \cap \Omega_{k-1}^c\big) + \mathbb{P}(\Omega_K^c). \end{aligned} \tag{4.10}$$

We need to estimate the probabilities in the above formula. This will be achieved with the aid of

$$\mathbb{P}(\Omega_k^c) \leqslant \mathbb{P}\bigg(\sup_{0\leqslant s\leqslant t} |X_s - x| \geqslant r_k\bigg). \tag{4.11}$$

Before moving further we focus on justifying (4.11). To prove this inequality we will show that

$$\left\{\omega \in \Omega : \sup_{0\leqslant s\leqslant t} |X_s(\omega) - x| < r_k\right\} \subseteq \Omega_k$$

up to a set of $\mathbb{P}$ measure 0. More precisely, we will demonstrate that for $\mathbb{P}$ almost all $\omega \in \Omega$ we have the implication

$$\text{if } \sup_{0\leqslant s\leqslant t} |X_s(\omega) - x| < r_k \text{ then } X_s(\omega) \in A_k \text{ for a.e. } s \in [0,t]. \tag{4.12}$$

To this end take $\omega \in \Omega$ such that $\sup_{0\leqslant s\leqslant t} |X_s(\omega) - x| < r_k$. Using the definitions (4.6) and (4.7) of ρ and r_k we see that there is a set $N \subseteq \mathbb{R}^d$ of measure 0 such that

$$\text{if } X_s(\omega) \notin N \text{ then } \frac{V(x)}{2^k} \leqslant V(X_s(\omega)),$$

By the definition (4.9) of A_k this statement is the same as the implication

$$\text{if } X_s(\omega) \notin N \text{ then } X_s(\omega) \in A_k.$$

Define $f_\omega(s) := X_s(\omega)$, $s \in [0,t]$, and let $\widetilde{N}(\omega) = f_\omega^{-1}[N] \subseteq [0,t]$. Then $s \notin \widetilde{N}(\omega)$ if and only if $X_s(\omega) \notin N$. We shall now demonstrate that $\left|\widetilde{N}(\omega)\right| = 0$ for $\mathbb{P}$ almost all $\omega \in \Omega$. Observe that

$$\left|\widetilde{N}(\omega)\right| = |\{s \in [0,t] : X_s(\omega) \in N\}| = \int_0^t \mathbb{1}_{\{X_s(\omega)\in N\}}(s,\omega)\,\mathrm{d}s.$$

Calculating the expected value of the above expression and using Fubini's theorem give

$$\begin{aligned}\mathbb{E}\left[\left|\widetilde{N}\right|\right] = \mathbb{E}\left[\int_0^t \mathbb{1}_{\{X_s(\omega)\in N\}}(s,\omega)\,\mathrm{d}s\right] &= \int_0^t \mathbb{E}\left[\mathbb{1}_{\{X_s(\omega)\in N\}}(s,\omega)\right]\mathrm{d}s \\ = \int_0^t \mathbb{P}(X_s(\omega) \in N)\,\mathrm{d}s = 0.\end{aligned}$$

The last equality follows from the fact that $|N| = 0$ and that each of the variables X_s has a continuous distribution. Since $\left|\widetilde{N}(\omega)\right|$ is non-negative, it has to be 0 for $\mathbb{P}$ almost all $\omega \in \Omega$.

Hence we have proved that for $\mathbb{P}$ almost all $\omega \in \Omega$ there is a set $\widetilde{N}(\omega) \subseteq [0,t]$ of Lebesgue measure 0 and such that

$$\text{if } s \notin \widetilde{N}(\omega) \text{ then } X_s(\omega) \in A_k.$$

This proves (4.12) and in consequence (4.11).

Now we come back to calculating the probabilities in (4.10). The right-hand side of inequality (4.11) is the probability that X_s exits the ball of radius r_k centered at x. We can estimate it from above by the probability that X_s exits an inscribed cube whose sides are parallel to the coordinate axes. The length of its diagonal equals $a\sqrt{d} = 2r_k$, where a is the cube's side length, so we get

$$\begin{aligned}
(4.13)\quad & \mathbb{P}\left(\sup_{0\leqslant s\leqslant t} |X_s - x| \geqslant r_k\right) \\
&\leqslant \mathbb{P}\left(\sup_{0\leqslant s\leqslant t} \max_i \left|X_s^i - x_i\right| \geqslant \frac{a}{2}\right) \\
&= \mathbb{P}\left(\max_i \sup_{0\leqslant s\leqslant t} \left|X_s^i - x_i\right| \geqslant \frac{a}{2}\right) \\
&\leqslant d\cdot\mathbb{P}\left(\sup_{0\leqslant s\leqslant t} \left|X_s^1 - x_1\right| \geqslant \frac{a}{2}\right) \\
&\leqslant d\cdot\mathbb{P}\left(\sup_{0\leqslant s\leqslant t} (X_s^1 - x_1) \geqslant \frac{a}{2}\right) + d\cdot\mathbb{P}\left(\inf_{0\leqslant s\leqslant t} (X_s^1 - x_1) \leqslant -\frac{a}{2}\right) \\
&= 2d\cdot\mathbb{P}\left(\sup_{0\leqslant s\leqslant t} (X_s^1 - x_1) \geqslant \frac{a}{2}\right) \\
&= 4d\cdot\mathbb{P}\left((X_t^1 - x_1) \geqslant \frac{a}{2}\right) \\
&\leqslant 4d\,\mathrm{erfc}\left(\frac{r_k}{\sqrt{2td}}\right) \\
&\leqslant 4de^{-\frac{r_k^2}{2td}}.
\end{aligned}$$

The last equality in (4.13) follows from the reflection principle for Brownian motion, while the last inequality is a well-known bound for the complementary error function erfc, see e.g. [19, eq. (7.8.3)].

Consequently,

$$(4.14)\qquad \mathbb{P}(\Omega_k^c) \leqslant 4de^{-\frac{r_k^2}{2td}}$$

and coming back to (4.10) for $0 < t < 1$ we get

$$(4.15)\qquad \begin{aligned}
e^{-tL}(\mathbb{1})(x) &\lesssim e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} e^{-\frac{r_{k-1}^2}{2td}} + e^{-\frac{r_K^2}{2td}} \\
&\leqslant e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} e^{-\frac{r_{k-1}^2}{2d}} + e^{-\frac{r_K^2}{2d}}
\end{aligned}$$

Integrating and multiplying this inequality by $V(x)^a$ gives

$$(4.16)\qquad V(x)^a \int_0^1 e^{-tL}(\mathbb{1})(x)\, t^{a-1}\mathrm{d}t \lesssim 1 + \sum_{k=1}^{K} 2^{ka} e^{-\frac{r_{k-1}^2}{2d}} + V(x)^a e^{-\frac{r_K^2}{2d}}.$$

Then, for $k \geqslant 2$ we estimate each of the terms in the sum by an integral recalling that $r_k(x) = \rho_x(2^k)$ and using the fact that $\rho_x(u)$ is a non-decreasing function of u

$$2^{ka} e^{-\frac{r_{k-1}^2}{2d}} \leqslant \int_{k-2}^{k-1} 2^{(u+2)a} e^{-\frac{\rho_x^2(2^u)}{2d}} \,\mathrm{d}u. \tag{4.17}$$

The last term in (4.16) is estimated in a similar manner using additionally the fact that $V(x)^a \leqslant \int_{K-1}^{K} 2^{(u+2)a} \,\mathrm{d}u$. This yields

$$V(x)^a e^{-\frac{r_K^2}{2d}} \leqslant \int_{K-1}^{K} 2^{(u+2)a} e^{-\frac{\rho_x^2(2^u)}{2d}} \,\mathrm{d}u. \tag{4.18}$$

We estimate the first term of the sum in (4.16) by a constant and plug this, (4.17) and (4.18) into (4.16), which results in

$$\begin{aligned} 1 + \sum_{k=1}^{K} 2^{ka} e^{-\frac{r_{k-1}^2}{2d}} + V(x)^a e^{-\frac{r_K^2}{2d}} &\lesssim 1 + \int_0^K 2^{ua} e^{-\frac{\rho_x^2(2^u)}{2d}} \mathrm{d}u \\ &\leqslant 1 + \int_0^{\log_2 V(x)} 2^{ua} e^{-\frac{\rho_x^2(2^u)}{2d}} \mathrm{d}u. \end{aligned} \tag{4.19}$$

Finally we substitute $s = 2^u$ to get

$$\begin{aligned} 1 + \int_0^{\log_2 V(x)} 2^{ua} e^{-\frac{\rho_x^2(2^u)}{2d}} \mathrm{d}u &\approx 1 + \int_1^{V(x)} s^{a-1} e^{-\frac{\rho_x^2(s)}{2d}} \mathrm{d}s \\ &\leqslant 1 + I^a(V)(x). \end{aligned} \tag{4.20}$$

□

In the next lemma we estimate the second part of the integral from (4.5).

Lemma 4.4. — *Let V be an a.e. non-negative potential and suppose that, for some $\delta > 0$, the semigroup e^{-tL} satisfies* (ED(δ))*. Take $a > 0$. Then we have*

$$V(x)^a \int_1^\infty e^{-tL}(\mathbb{1})(x)\, t^{a-1} \mathrm{d}t \lesssim I^a(V)(x) + 1, \qquad x \in \mathbb{R}^d. \tag{4.21}$$

Proof. — Using the semigroup property and the positivity-preserving property of $\{e^{-tL}\}_{t>0}$ for $t \geqslant 1$ we obtain

$$\begin{aligned} e^{-tL}(\mathbb{1})(x) = e^{-(t/2)L}[e^{-(t/2)L}(\mathbb{1})](x) &\leqslant \left\| e^{-(t/2)L}(\mathbb{1}) \right\|_\infty e^{-(t/2)L}(\mathbb{1})(x) \\ &\leqslant C e^{-\delta t/2} e^{-(1/2)L}(\mathbb{1})(x), \end{aligned}$$

where the last two inequalities follow from (ED(δ)) and (3.1). Plugging this into (4.21) we get

$$V(x)^a \int_1^\infty e^{-tL}(\mathbb{1})(x)\, t^{a-1} \mathrm{d}t \lesssim V(x)^a e^{-L/2}(\mathbb{1})(x). \tag{4.22}$$

Now we are left with proving that $V^a(x)e^{-L/2}(\mathbb{1})(x) \lesssim I^a(V)(x) + 1$. If $V(x) \leqslant 2$, then this is true. Assume that $V(x) > 2$ and let $K(x) = \lfloor \log_2 V(x) \rfloor$. Recall that by (4.15) we have

$$e^{-L/2}(\mathbb{1})(x) \lesssim e^{-\frac{V(x)}{2}} + \sum_{k=1}^{K} e^{-\frac{V(x)}{2^{k+1}}} e^{-\frac{r_{k-1}^2}{2d}} + e^{-\frac{r_K^2}{2d}}.$$

Since $V(x)^a e^{-\frac{V(x)}{2^{k+1}}} \leqslant \left(\frac{2^{k+1}a}{e}\right)^a$, repeating calculations as in (4.16)–(4.20) we get

$$V(x)^a e^{-L/2}(\mathbb{1})(x) \lesssim 1 + \sum_{k=1}^{K} 2^{ka} e^{-\frac{r_{k-1}^2}{2d}} + V(x)^a e^{-\frac{r_K^2}{2d}} \lesssim 1 + I^a(V)(x). \tag{4.23}$$

In view of (4.22) this completes the proof of the lemma. □

Together, Lemma 4.3 and Lemma 4.4 lead to the following conclusion.

THEOREM 4.5. — *Let $V \in L^\infty_{\mathrm{loc}}$ be an a.e. non-negative potential. Suppose that the semigroup e^{-tL} has exponential decay of order $\delta > 0$ (see* (ED(δ))*). If*

$$I^a(V) \lesssim_g 1 \tag{4.24}$$

for some $a > 0$, then the operator R_V^a is bounded on L^∞.

Proof. — We need to estimate the quantity

$$V^a(x) \int_0^\infty e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, \mathrm{d}t \tag{4.25}$$

independently of x. Take $N > 0$ such that $I^a(V)(x) \lesssim 1$ for almost all $|x| > N$. Then by Lemma 4.3 and Lemma 4.4 the expression (4.25) is uniformly bounded for a.e. $|x| > N$. If on the other hand $|x| \leqslant N$, then, since $V \in L^\infty_{\mathrm{loc}}$ and the semigroup satisfies (ED(δ)), the expression (4.25) is uniformly bounded x-a.e. □

As an application of this theorem, we prove that R_V^a is bounded on $L^\infty(\mathbb{R}^d)$ if V is of the order of a power function or an exponential function. The corollary below is a restatement of one of our main results — Theorem C.

COROLLARY 4.6. — *Let $V\colon \mathbb{R}^d \to [0,\infty)$ be a function in L^∞_{loc}. Then in all the three cases*

(1) *$V(x) \approx 1$ globally*
(2) *For some $\alpha > 0$ we have $V(x) \approx |x|^\alpha$ globally*
(3) *For some $\beta > 1$ we have $V(x) \approx \beta^{|x|}$ globally*

each of the Riesz transforms R_V^a, $a > 0$, is bounded on $L^\infty(\mathbb{R}^d)$.

Remark. — More generally, the theorem also holds if in (2) and (3) we take an arbitrary norm on $\mathbb{R}^d$ instead of the Euclidean norm $|\cdot|$. The proof is the same mutatis mutandis.

Proof. — In the proof implicit constants in $\lesssim$, $\gtrsim$, and $\approx$ do not depend on $x \in \mathbb{R}^d$ but may depend on $a > 0$, $\alpha > 0$ or $\beta > 1$.

Clearly in all three cases the assumptions of Lemma 4.1 are satisfied, so the semigroup satisfies (ED(δ)) and we only need to check that (4.24) holds.

In the first case $V(x)$ is bounded for almost all sufficiently large values of $|x|$ and so is $I^a(V)(x)$ for all $a > 0$.

In the second case we need to estimate from below $\rho_x(s)$ appearing in $I^a(V)$. We shall prove that $\rho_x(s) \geqslant |x|/2$ provided s and $|x|$ are large enough. Let N be such that for some $0 < m < M$ it holds

$$m|x|^\alpha < V(x) < M|x|^\alpha \qquad \text{for a.e.} \qquad |x| \geqslant N. \tag{4.26}$$

Take $|x| \geqslant 2N$ and assume that $|x-y| \leqslant |x|/2$. Then $2|x| \geqslant |y| \geqslant |x|/2 \geqslant N$ so that (4.26) holds with y in place of x. Consequently, $V(x) \approx V(y)$ for such x and y so that for s larger than some threshold depending only on N, m and M it holds $V(y) \geqslant V(x)/s$. This means that for a.e. $|x| \geqslant 2N$ and uniformly large enough $s \geqslant 1$ we have $\rho_x(s) \geqslant |x|/2$. Thus, for any $a > 0$ we obtain

$$I^a(V)(x) \lesssim_g 1 + |x|^{a\alpha} e^{-\frac{|x|^2}{16d}} \lesssim_g 1. \tag{4.27}$$

as desired.

Finally we handle the third case. We shall prove that $\rho_x(s) \geqslant \frac{1}{2}\min\big(|x|, \log_\beta s\big)$ provided s and $|x|$ are large enough. Let $N > 0$ be such that for some $0 < m \leqslant 1 \leqslant M$ we have

$$m\beta^{|x|} < V(x) < M\beta^{|x|} \qquad \text{for a.e.} \qquad |x| \geqslant N. \tag{4.28}$$

Take $|x| \geqslant 2N$, $s > 4$, and assume that $|x-y| \leqslant \frac{1}{2}\min\big(|x|, \log_\beta s\big)$. Then, similarly to the previous paragraph, $|x| \approx |y| \geqslant N$ and (4.28) also holds with y in place of x. Therefore, for such x and y we have $\beta^{|y|-|x|} \approx V(y)/V(x)$. In particular $|y| - |x| - \gamma \leqslant \log_\beta V(y) - \log_\beta V(x)$, for some $\gamma > 0$ independent of x and y. Hence, we reach

$$-\frac{1}{2}\min\big(|x|, \log_\beta s\big) - \gamma \leqslant \log_\beta V(y) - \log_\beta V(x). \tag{4.29}$$

Taking s large enough we see that $-\frac{1}{2}\log_\beta s - \gamma \geqslant -\log_\beta s$ and coming back to (4.29) we obtain $V(x)/s \leqslant V(y)$. In conclusion, we proved that $\rho_x(s) \geqslant$

$\frac{1}{2}\min\big(|x|, \log_\beta s\big)$ for a.e. $|x| \geqslant 2N$ when s is large enough (independently of x). Now, using (4.28) we obtain the uniform in $|x| \geqslant 2N$ bound

$$I^a(V)(x) \lesssim_g 1 + \int_1^{\beta^{|x|}} s^{a-1} e^{-\frac{(\log_\beta s)^2}{16d}} \,\mathrm{d}s + \int_{\beta^{|x|}}^{M\beta^{|x|}} s^{a-1} e^{-\frac{|x|^2}{16d}} \,\mathrm{d}s \lesssim_g 1, \tag{4.30}$$

This completes the treatment of the third case and also the proof of Corollary 4.6. □

5. L^1 boundedness for classes of potentials

In this section we estimate the L^1 norm of the operator R_V^a for $a > 0$ and various non-negative potentials $V \in L^\infty_{\mathrm{loc}}$. Recall that the assumption $V \in L^\infty_{\mathrm{loc}}$ guarantees the validity of the Feynman–Kac formula (3.1).

The idea is to estimate the L^∞ norm of the adjoint operator which formally is

$$(L^{-a}V^a)f = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL}(V^a f)\, t^{a-1} \,\mathrm{d}t.$$

Using the positivity-preserving property of e^{-tL} the task naturally reduces to estimating the L^∞ norm of the function

$$\Gamma(a)L^{-a}(V^a)(x) \coloneqq \int_0^\infty e^{-tL}(V^a)(x)\, t^{a-1} \,\mathrm{d}t. \tag{5.1}$$

Since V may be unbounded, the expression $e^{-tL}(V^a)(x)$ may be infinite for some x in which case the x-measurability of the integral (5.1) is not clear. To remedy the situation we formally define

$$\Gamma(a)L^{-a}(V^a)(x) \coloneqq \lim_{N\to\infty} \int_0^\infty e^{-tL}(V^a \mathbb{1}_{|V|<N})(x)\, t^{a-1} e^{-t/N} \,\mathrm{d}t. \tag{5.2}$$

Note that each of the integrals in (5.2) is finite and measurable by Lemma 3.1, hence the limit gives a measurable function by the monotone convergence theorem. A short duality argument shows that if $L^{-a}(V^a) \in L^\infty$, then indeed R_V^a is bounded on L^1 with $\|R_V^a\|_1 \leqslant \|L^{-a}(V^a)\|_\infty$.

Throughout this section we estimate the L^∞ norm of $L^{-a}(V^a)$ in the form (5.1). This is allowed since by the assumptions which we will impose on V both $e^{-tL}(V^a)(x)$ and the integral (5.1) will turn out to be finite x-a.e.. This permits us to take $N = \infty$ in (5.2).

In what follows for $x \in \mathbb{R}^d$ and $u \geqslant 1$ we let

$$\sigma = \sigma_x(u) = \sup\{r \geqslant 0 : V(y) \leqslant uV(x) \ \text{ for a.e. } y \in B(x,r)\}.$$

Consequently, $\sigma_x(u)$ is the radius of the largest closed ball around x in which the potential V is at most $uV(x)$ a.e. We remark that $\sigma_x(u)$ is a non-decreasing function of u with values in $[0,\infty]$. Using the quantity $\sigma_x(u)$ we define

$$J^a(V)(x) \coloneqq \min(1, V(x)^a) \int_1^\infty s^{a-1} e^{-\sigma_x^2(s)/8}\, \mathrm{d}s, \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{5.3}$$

If $V \in L^\infty$ and $uV(x) \geqslant \|V\|_\infty$, then $V(y) \leqslant uV(x)$ for a.e. $y \in B(x,r)$ with arbitrarily large $r > 0$. In this case $\sigma_x(u) = \infty$ and by convention $e^{-\sigma_x^2(u)/8} = 0$. This is the case for instance if $V \in L^\infty$ is of constant order for large x.

We begin with estimating the integral (5.1) from 0 to 1. Recall that implicit constants in $\lesssim$ and $\approx$ are allowed to depend on d and $a > 0$.

Proposition 5.1. — *Let $V \in L^\infty_{\mathrm{loc}}$ be an a.e. non-negative potential and take $a > 0$. Then the inequality*

$$\int_0^1 e^{-tL}(V^a)(x)\, t^{a-1} \mathrm{d}t \lesssim (J^a(V)(x) + 1)(I^a(V)(x) + 1) \tag{5.4}$$

holds for a.e. $x \in \mathbb{R}^d$ that satisfies $V(x) \neq 0$. Moreover, if V is an a.e. non-negative potential which satisfies the growth estimate $V(x) \lesssim \exp\big(|x|^2/(4a)\big)$ for a.e. $x \in \mathbb{R}^d$, then

$$\int_0^1 e^{-tL}(V^a)(x)\, t^{a-1} \mathrm{d}t \lesssim \exp\big(|x|^2\big), \qquad x \in \mathbb{R}^d. \tag{5.5}$$

Proof.

Proof of (5.4). — Here we consider $x \in \mathbb{R}^d$ such that $V(x) \neq 0$. Recall that

$$A_k = \left\{ y \in \mathbb{R}^d : \frac{V(x)}{2^k} \leqslant V(y) \right\}$$

and

$$\Omega_k = \{\omega \in \Omega : X_s(\omega) \in A_k \text{ for almost all } s \in [0,t]\}.$$

Here we shall also need

$$B_j = \big\{ y \in \mathbb{R}^d :\ 2^j V(x) < V(y) \leqslant 2^{j+1} V(x) \big\}$$

and

$$\Psi_j = \Psi_j^t \coloneqq \{\omega \in \Omega : X_t(\omega) \in B_j\}.$$

Note that if $V(x) \neq 0$ then the sets $\{B_j\}_{j\in\mathbb{Z}}$ are pairwise disjoint and

$$(5.6)\quad \begin{aligned} e^{-tL}(V^a)(x) &= e^{-tL}\Bigg(\sum_{j\leqslant 0} \mathbb{1}_{B_j} V^a\Bigg)(x) + e^{-tL}\Bigg(\sum_{j>0} \mathbb{1}_{B_j} V^a\Bigg)(x) \\ &\lesssim V(x)^a e^{-tL}(\mathbb{1})(x) + \sum_{j>0} V(x)^a 2^{ja} e^{-tL}(\mathbb{1}_{B_j})(x). \end{aligned}$$

We shall prove that the estimates

$$(5.7)\quad \int_0^1 e^{-tL}(V^a)(x) t^{a-1} \mathrm{d}t \lesssim (I^a(V)(x)+1)\left(\int_1^\infty s^{a-1} e^{-\sigma_x^2(s)/8}\, \mathrm{d}s + 1\right)$$

and

$$(5.8)\quad \begin{aligned} &\int_0^1 e^{-tL}(V^a)(x) t^{a-1} \mathrm{d}t \\ &\qquad \lesssim I^a(V)(x) + 1 + V(x)^a \left(\int_1^\infty s^{a-1} e^{-\sigma_x^2(s)/8}\, \mathrm{d}s\right) \end{aligned}$$

hold for x such that $V(x) \neq 0$. The inequalities (5.7) and (5.8) imply (5.4).

We prove (5.7) first. Let $K = \max(1, \lfloor \log_2 V(x) \rfloor)$ and for $k = 1, \dots, K$ and $j \in \mathbb{Z}$ denote

$$r_k = \rho_x(2^k), \qquad s_j = \sigma_x(2^j).$$

Estimating the second term in (5.6) we use the Feynman–Kac formula (3.1) with $f = V^a \mathbb{1}_{B_j}$ to write

$$\sum_{j>0} e^{-tL}(V^a \mathbb{1}_{B_j})(x) \lesssim V^a(x) \sum_{j>0} 2^{ja} e^{-tL}(\mathbb{1}_{B_j})(x).$$

Using again (3.1), proceeding as in the proof of Lemma 4.3 and applying (4.14) we obtain

$$\begin{aligned} &e^{-tL}(\mathbb{1}_{B_j})(x) \\ &\quad \leqslant e^{-tV(x)} \mathbb{P}(\Psi_j) + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} \mathbb{P}\big(\Omega_{k-1}^c \cap \Psi_j\big) + \mathbb{P}(\Omega_K^c \cap \Psi_j) \\ &\quad \leqslant \mathbb{P}(\Psi_j)^{1/2} \left(e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} \left[\mathbb{P}\big(\Omega_{k-1}^c\big)\right]^{1/2} + [\mathbb{P}(\Omega_K^c)]^{1/2} \right) \\ &\quad \lesssim \mathbb{P}(\Psi_j)^{1/2} \left(e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} e^{-\frac{r_{k-1}^2}{4td}} + e^{-\frac{r_K^2}{4td}} \right) \end{aligned}$$

Further, we have $\Psi_j \subseteq \{\omega \in \Omega \colon X_t(\omega) \notin B(x, s_j)\}$ up to a set of $\mathbb{P}$ measure 0. Indeed, a.e. $y \in B(x, s_j)$ satisfies $V(y) \leqslant 2^j V(x)$, hence it lies outside

B_j. Here we also use the fact that X_t has a continuous distribution. Thus we reach

$$\tag{5.9} \begin{aligned} \mathbb{P}(\Psi_j) \leqslant \mathbb{P}(|X_t - x| \geqslant s_j) &= \frac{1}{(2\pi t)^{d/2}} \int_{|y|\geqslant s_j} e^{-\frac{|y|^2}{2t}} \,\mathrm{d}y \\ &\leqslant \frac{e^{-s_j^2/(4t)}}{(2\pi t)^{d/2}} \int_{|y|\geqslant s_j} e^{-\frac{|y|^2}{4t}} \,\mathrm{d}y \lesssim e^{-s_j^2/(4t)} \end{aligned}$$

so that

$$e^{-tL}(\mathbb{1}_{B_j})(x) \lesssim e^{-s_j^2/(8t)} \left(e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} e^{-\frac{r_{k-1}^2}{4td}} + e^{-\frac{r_K^2}{4td}} \right).$$

Putting the above bound in (5.6) and replacing the sum over j with an integral as in (4.18) and (4.19) we reach

$$\begin{aligned} &\sum_{j>0} V(x)^a 2^{ja} e^{-tL}(\mathbb{1}_{B_j})(x) \\ &\lesssim V(x)^a \left(e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} e^{-\frac{r_{k-1}^2}{4td}} + e^{-\frac{r_K^2}{4td}} \right) \sum_{j>0} 2^{ja} e^{-s_j^2/(8t)} \\ &\lesssim V(x)^a \left(e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} e^{-\frac{r_{k-1}^2}{4td}} + e^{-\frac{r_K^2}{4td}} \right) \int_1^\infty s^{a-1} e^{-\sigma_x^2(s)/(8t)} \,\mathrm{d}s. \end{aligned}$$

The first term on the right-hand side of (5.6) was already estimated in the proof of Lemma 4.3 by

$$V(x)^a e^{-tL}(\mathbb{1})(x) \leqslant V(x)^a \left(e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} e^{-\frac{r_{k-1}^2}{2td}} + e^{-\frac{r_K^2}{2td}} \right),$$

see (4.15). Hence, coming back to (5.6) we reach

$$\begin{aligned} e^{-tL}(V^a)(x) \lesssim V(x)^a &\left(\int_1^\infty s^{a-1} e^{-\sigma_x^2(s)/8} \,\mathrm{d}s + 1 \right) \\ &\times \left(e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} e^{-\frac{r_{k-1}^2}{4td}} + e^{-\frac{r_K^2}{4td}} \right) \end{aligned}$$

We use the above inequality to estimate $\int_0^1 e^{-tL}(V^a)(x)\, t^{a-1} \,\mathrm{d}t$. From this point on the proof is a repetition of the argument in (4.16)–(4.20) that leads to (5.7).

Now we pass to the proof of (5.8). This time we merely estimate $e^{-tL}(\mathbb{1}_{B_j})(x)$ by $\mathbb{P}(\Psi_j)$. In view of (5.6) and (5.9) proceeding as in the proof of (5.7) we thus obtain

$$\begin{aligned} e^{-tL}(V^a)(x) &\lesssim V(x)^a\left(e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} e^{-\frac{r_{k-1}^2}{2td}} + e^{-\frac{r_K^2}{2td}}\right) \\ &\qquad + V(x)^a \sum_{j>0} 2^{ja} e^{-s_j^2/(4t)} \\ &\lesssim V(x)^a\left(e^{-tV(x)} + \sum_{k=1}^{K} e^{-\frac{tV(x)}{2^k}} e^{-\frac{r_{k-1}^2}{2td}} + e^{-\frac{r_K^2}{2td}}\right) \\ &\qquad + V(x)^a \int_1^\infty s^{a-1} e^{-\sigma_x^2(s)/8}\,\mathrm{d}s. \end{aligned}$$

Once again we integrate the above expression by repeating the argument in (4.16)–(4.20) and obtain (5.8).

Proof of (5.5). — The growth assumption on V implies that

$$\mathbb{E}_x[V(X_t)^a] \lesssim (2\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-|y-x|^2/(2t)} e^{|y|^2/4}\,\mathrm{d}y.$$

Then, a short calculation leads to

$$\mathbb{E}_x[V(X_t)^a] \lesssim \exp\big(|x|^2\big), \qquad t<1.$$

Thus, using the Feynman–Kac formula (3.1) we estimate

$$e^{-tL}(V^a)(x) \leqslant \mathbb{E}_x[V(X_t)^a] \lesssim \exp\big(|x|^2\big),$$

so that

$$\int_0^1 e^{-tL}(V^a)(x)\, t^{a-1}\mathrm{d}t \lesssim \exp\big(|x|^2\big).$$

This completes the proof of Proposition 5.1. □

Now we pass to the integral (5.1) restricted to the range $[1,\infty)$. We shall prove several results with varying assumptions on the potential V. For this reason the treatment here is significantly more complicated than in Section 4.

We start with a counterpart of Proposition 5.1. To this end we need yet another quantity

$$K_c^a(V)(x) \coloneqq \min(1, V(x)^a) \int_1^\infty e^{-c\sigma_x(s)} s^{a-1}\,\mathrm{d}s, \qquad \text{for a.e. } x\in\mathbb{R}^d,$$

where $a, c > 0$. Note that this is essentially larger than $J^a(V)(x)$ defined by (5.3) and used in Proposition 5.1. Indeed, observe that for each $c > 0$ there is a constant M independent of x and s such that $\frac{\sigma_x^2(s)}{8} \geqslant c\sigma_x(s) - M$

for all $s \geqslant 1$ and $x \in \mathbb{R}^d$, which means that $e^{-\sigma_x^2(s)/8} \leqslant e^M e^{-c\sigma_x(s)}$ and in turn

$$J^a(V)(x) \lesssim K_c^a(V)(x). \tag{5.10}$$

Proposition 5.2. — *Let V be an a.e. non-negative potential. Assume that the semigroup e^{-tL} satisfies $(\mathrm{ED}(\delta))$ with some $\delta > 0$. Let $a > 0$, take $b > a$ and define*

$$c = \min\left(\frac{b-a}{8b}, \frac{\delta a}{4b}\right). \tag{5.11}$$

Then

$$\int_1^\infty e^{-tL}(V^a)(x)\, t^{a-1}\mathrm{d}t \lesssim (K_c^a(V)(x)+1)(I^b(V)(x)+1) \tag{5.12}$$

uniformly in every x such that $V(x) \neq 0$.

Moreover, if V is of exponential growth η, i.e.

$$V(x) \lesssim e^{\eta|x|}, \tag{5.13}$$

with $\eta < \sqrt{\delta}/(\sqrt{2d}a)$, then

$$\int_1^\infty e^{-tL}(V^a)(x)\, t^{a-1}\mathrm{d}t \lesssim \exp\left(\sqrt{d}a\eta|x|\right), \qquad x \in \mathbb{R}^d. \tag{5.14}$$

Remark. — The implicit constants in (5.12), (5.14) possibly depend on a, b, δ, η.

Proof.

Proof of (5.12). — Using the splitting into the sets B_j as in (5.6) and the Feynman–Kac formula (3.1) we obtain

$$\begin{aligned} e^{-tL}(V^a)(x) &\lesssim V(x)^a e^{-tL}(\mathbb{1})(x) + \sum_{j>0} V(x)^a 2^{ja} e^{-tL}(\mathbb{1}_{B_j})(x) \\ &\lesssim V(x)^a e^{-tL}(\mathbb{1})(x) + \sum_{j>0} V(x)^a 2^{ja} \mathbb{E}_x\left[e^{-\int_0^t V(X_s)\mathrm{d}s}\mathbb{1}_{\Psi_j}\right] \end{aligned}$$

By Lemma 4.4 we have

$$\int_1^\infty V(x)^a e^{-tL}(\mathbb{1})(x)\, t^{a-1}\,\mathrm{d}t \lesssim I^a(V)(x)+1 \lesssim I^b(V)(x)+1.$$

Hence, we only focus on the integral over the second term, namely $\int_1^\infty S_x(t)\, t^{a-1}\,\mathrm{d}t$ with

$$S_x(t) := \sum_{j>0} V(x)^a 2^{ja}\mathbb{E}_x\left[e^{-\int_0^t V(X_s)\mathrm{d}s}\mathbb{1}_{\Psi_j}\right].$$

Let $p = b/a$ and let q be its conjugate exponent. Then Hölder's inequality gives

$$\begin{aligned} S_x(t) &\leqslant \sum_{j>0} V(x)^a 2^{ja} \left(\mathbb{E}_x \left[e^{-p \int_0^t V(X_s)\mathrm{d}s} \right] \right)^{1/p} \left(\mathbb{E}_x [\mathbb{1}_{\Psi_j}] \right)^{1/q} \\ &\lesssim \sum_{j>0} V(x)^a 2^{ja} \left(e^{-tL}(\mathbb{1})(x) \right)^{1/p} \mathbb{P}(\Psi_j)^{1/q}. \end{aligned} \tag{5.15}$$

Using (5.15) we shall prove that

$$\int_1^\infty S_x(t)\, t^{a-1}\, \mathrm{d}t \lesssim (I^b(V)(x) + 1) \left(\int_1^\infty e^{-c\sigma_x(s)} s^{a-1}\, \mathrm{d}s + 1 \right). \tag{5.16}$$

and

$$\int_1^\infty S_x(t)\, t^{a-1}\, \mathrm{d}t \lesssim V(x)^a \left(\int_1^\infty e^{-c\sigma_x(s)} s^{a-1}\, \mathrm{d}s \right). \tag{5.17}$$

These two inequalities imply that

$$\int_1^\infty S_x(t)\, t^{a-1}\, \mathrm{d}t \lesssim (K_c^a(V)(x) + 1)(I^b(V)(x) + 1),$$

and thus are enough to complete the proof of (5.12).

We start with (5.16). Using monotonicity, the semigroup property, and (ED(δ)) we obtain that

$$e^{-tL}(\mathbb{1})(x) = e^{-tL/2}(e^{-tL/2}(\mathbb{1}))(x) \lesssim e^{-\delta t/2} e^{-L/2}(\mathbb{1})(x).$$

Hence, (5.15) gives

$$S_x(t) \leqslant e^{-\delta t/(2p)} \left(V(x)^{ap} e^{-L/2}(\mathbb{1})(x) \right)^{1/p} \cdot \sum_{j>0} 2^{ja} \mathbb{P}(\Psi_j)^{1/q}.$$

Since $ap = b$ a repetition of the computation in (4.23) shows that

$$S_x(t) \lesssim (I^b(V)(x) + 1) \cdot e^{-\delta t/(2p)} \cdot \sum_{j>0} 2^{ja} \mathbb{P}(\Psi_j)^{1/q}. \tag{5.18}$$

Now, using the estimate (5.9) for $\mathbb{P}(\Psi_j)$ we obtain

$$\sum_{j>0} 2^{ja} \mathbb{P}(\Psi_j)^{1/q} \lesssim \sum_{j>0} 2^{ja} e^{-s_j^2/(4tq)}. \tag{5.19}$$

Consider the integral

$$\int_1^\infty e^{-\delta t/(2p)} e^{-s_j^2/(4tq)} t^{a-1}\, \mathrm{d}t.$$

We split it at $t = s_j$ and estimate each part separately:

$$\int_1^\infty e^{-\delta t/(2p)} e^{-s_j^2/(4tq)} t^{a-1}\,\mathrm{d}t \leqslant \int_1^{s_j} e^{-s_j^2/(4tq)} t^{a-1}\,\mathrm{d}t + \int_{s_j}^\infty e^{-\delta t/(2p)} t^{a-1}\,\mathrm{d}t \lesssim e^{-s_j/(8q)} + e^{-\delta s_j/(4p)} \lesssim e^{-cs_j}.$$

Recall that $c = \min((b-a)/(8b), \delta a/(4b))$. Formally, the splitting above only works when $s_j \geqslant 1$, however, the estimate

$$\int_1^\infty e^{-\delta t/(2p)} e^{-s_j^2/(4tq)} t^{a-1}\,\mathrm{d}t \lesssim e^{-cs_j}$$

remains true for any $s_j \geqslant 0$. Consequently, integrating (5.19) we get

$$\int_1^\infty e^{-\delta t/(2p)} \cdot \sum_{j>0} 2^{ja} \mathbb{P}(\Psi_j)^{1/q} t^{a-1}\,\mathrm{d}t \leqslant \sum_{j>0} 2^{ja} e^{-cs_j} \lesssim \int_1^\infty e^{-c\sigma_x(s)} s^{a-1}\,\mathrm{d}s, \tag{5.20}$$

where in the last inequality above we used the fact that $s_j = \sigma_x(2^j)$. Combining (5.20) with (5.18) gives (5.16).

We pass to the proof of (5.17). Note that (5.15) and the assumption (ED(δ)) imply

$$S_x(t) \lesssim e^{-\delta t/p} \sum_{j>0} V(x)^a 2^{ja} \mathbb{P}(\Psi_j)^{1/q},$$

thus, an application of (5.20) produces

$$\int_1^\infty S_x(t)\, t^{a-1}\,\mathrm{d}t \lesssim V(x)^a \int_1^\infty e^{-c\sigma_x(s)} s^{a-1}\,\mathrm{d}s,$$

and (5.17) is justified.

Proof of (5.14). — Using the Feynman–Kac formula (3.2) and Cauchy–Schwarz inequality we obtain

$$e^{-tL}(V^a)(x) \leqslant \mathbb{E}_x\left[V^{2a}(X_t)\right]^{1/2} \mathbb{E}_x\left[e^{-2\int_0^t V(X_s)\mathrm{d}s}\right]^{1/2} \leqslant \mathbb{E}_x\left[V^{2a}(X_t)\right]^{1/2} \left(e^{-tL}(\mathbb{1})(x)\right)^{1/2}.$$

Hence, the assumptions (ED(δ)) and (5.13) give

$$e^{-tL}(V^a)(x) \lesssim e^{-\delta t/2} \Big(\mathbb{E}_x e^{2\eta a|X_t|}\Big)^{1/2}.$$

We claim that the proof of (5.14) will be completed if we show that

$$\mathbb{E}_x e^{2\eta a|X_t|} \lesssim \exp\Big(2d\eta^2 a^2 t + 2\sqrt{d}\eta a|x|\Big). \tag{5.21}$$

Indeed, the above estimate leads to

$$\begin{aligned}\int_1^\infty e^{-tL}(V^a)(x)\, t^{a-1}\,\mathrm{d}t &\lesssim e^{\sqrt{d}\eta a|x|}\int_1^\infty \exp\big(-\delta t/2 + d\eta^2 a^2 t\big)\, t^{a-1}\,\mathrm{d}t\\ &\lesssim e^{\sqrt{d}\eta a|x|},\end{aligned}$$

where in the last inequality we used the assumption $\eta < \sqrt{\delta}/(\sqrt{2d}a)$.

It remains to justify (5.21). Since

$$\begin{aligned}\mathbb{E}_x\Big[e^{2\eta a|X_t|}\Big] &= \frac{1}{(2\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{2\eta a|z|}e^{-\frac{|x-z|^2}{2t}}\,\mathrm{d}z\\ &\leqslant \frac{1}{(2\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{2\eta a\sum_{i=1}^d |z_i|}e^{-\frac{|x-z|^2}{2t}}\,\mathrm{d}z\\ &= \prod_{i=1}^d \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{2\eta a|z_i|}e^{-\frac{|x_i-z_i|^2}{2t}}\,\mathrm{d}z_i\end{aligned} \tag{5.22}$$

it suffices to focus on each of the factors in the above product separately. A simple computation shows that

$$\begin{aligned}\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{2\eta a|z_i|}e^{-\frac{|x_i-z_i|^2}{2t}}\,\mathrm{d}z_i &\leqslant e^{2\eta a|x_i|}\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{2\eta a|z_i-x_i|}e^{-\frac{|x_i-z_i|^2}{2t}}\,\mathrm{d}z_i\\ &= e^{2\eta a|x_i|}\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{2\eta a|y|}e^{-\frac{|y|^2}{2t}}\,\mathrm{d}y\\ &\leqslant 2e^{2\eta a|x_i|}\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{2\eta a y}e^{-\frac{|y|^2}{2t}}\,\mathrm{d}y\\ &= 2e^{2\eta a|x_i|}e^{(2\eta a)^2 t/2} = 2e^{2\eta a|x_i|}e^{2\eta^2 a^2 t}.\end{aligned}$$

Hence, coming back to (5.22) and using the inequality $\sum_{i=1}^d |x_i| \leqslant \sqrt{d}|x|$ we obtain

$$\mathbb{E}_x\Big[e^{2\eta a|X_t|}\Big] \leqslant 2^d e^{2d\eta^2 a^2 t}\prod_{i=1}^d e^{2\eta a|x_i|} \lesssim \exp\Big(2d\eta^2 a^2 t + 2\sqrt{d}\eta a|x|\Big),$$

thus proving the claim (5.21).

The proof of Proposition 5.2 is thus completed. □

By a comparison with the Hermite semigroup we can improve Proposition 5.2 in the full range $a > 0$ for potentials V which grow at infinity faster than $|x|^2$.

PROPOSITION 5.3. — *Let c, b, N be positive constants. Assume that $V \in L^\infty_{\mathrm{loc}}$ is an a.e. non-negative potential that satisfies $c|x|^2 \leqslant V(x)$ for a.e. $|x| \geqslant N$ and $V(x) \lesssim e^{b|x|^2}$. Denote $\mu = \frac{d^{1/3}}{5N^2}$. Then, for each $0 < a \leqslant \frac{\mu \tanh \frac{\mu}{2}}{4b}$ we have*

$$\int_1^\infty e^{-tL}(V^a)(x)\, t^{a-1} \mathrm{d}t \lesssim 1, \qquad x \in \mathbb{R}^d.$$

Proof. — Denote by ω a C_c^∞ function which is equal to $c|x|^2$ for $|x| \leqslant N$, is bounded by $c|x|^2$, and vanishes for $|x| \geqslant 2N$. Then, for all $k \in (0,1]$, we have

$$V(x) + k\omega(x) \geqslant ck|x|^2, \qquad \text{for a.e. } x \in \mathbb{R}^d.$$

Hence, using (3.2) and Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
(5.23)\quad & e^{-tL}(V^a)(x) \\
&= \mathbb{E}_x\left[e^{-\int_0^t V(X_s)\mathrm{d}s} V^a(X_t)\right] \\
&= \mathbb{E}_x\left[e^{-\int_0^t (V+k\omega)(X_s)\mathrm{d}s} V^a(X_t) \cdot e^{k\int_0^t \omega(X_s)\mathrm{d}s}\right] \\
&\leqslant \left(\mathbb{E}_x\left[e^{-2\int_0^t (V+k\omega)(X_s)\mathrm{d}s} V^{2a}(X_t)\right]\right)^{1/2} \cdot \left(\mathbb{E}_x e^{2k\int_0^t \omega(X_s)\mathrm{d}s}\right)^{1/2} \\
&\leqslant \left(\mathbb{E}_x\left[e^{-2ck\int_0^t |X_s|^2\mathrm{d}s} V^{2a}(X_t)\right]\right)^{1/2} \cdot \left(\mathbb{E}_x e^{2k\int_0^t \omega(X_s)\mathrm{d}s}\right)^{1/2} \\
&= \left(e^{-t(-\frac{\Delta}{2}+2ck|x|^2)}(V^{2a})(x)\right)^{1/2} \cdot \left(\mathbb{E}_x e^{2k\int_0^t \omega(X_s)\mathrm{d}s}\right)^{1/2}.
\end{aligned}$$

In what follows we denote

$$\gamma = \gamma(c,k) = 2\sqrt{ck}.$$

Throughout the proof the implicit constants in $\lesssim$ depend on $k \in (0,1]$, thus also on γ. Appropriate k and γ will be fixed at a later stage. From [26, 4.1.2] or [23, 1.4] we deduce that

$$e^{-t(-\frac{\Delta}{2}+2ck|x|^2)} f(x) = e^{-t(-\frac{\Delta}{2}+\frac{\gamma^2}{2}|x|^2)} f(x) = \left(\frac{\gamma}{2\pi}\right)^{d/2} \int_{\mathbb{R}^d} K_t^\gamma(x,y) f(y)\,\mathrm{d}y,$$

with

$$\begin{aligned}
K_t^\gamma(x,y) &= \frac{1}{(\sinh \gamma t)^{d/2}} \exp\left(-\frac{\gamma}{2}\left(|x|^2 + |y|^2\right)\coth \gamma t + \frac{\gamma\langle x, y\rangle}{\sinh \gamma t}\right) \\
&= \frac{1}{(\sinh \gamma t)^{d/2}} \exp\left(-\frac{\gamma|x-y|^2}{4\tanh\frac{\gamma t}{2}} - \frac{\gamma \tanh\frac{\gamma t}{2}}{4}|x+y|^2\right).
\end{aligned}$$

Using the upper bound on V we estimate $e^{-t(-\frac{\Delta}{2}+\frac{\gamma^2}{2}|x|^2)}(V^{2a})$ as follows

$$
\begin{aligned}
(5.24)\quad & e^{-t(-\frac{\Delta}{2}+\frac{\gamma^2}{2}|x|^2)}(V^{2a})(x) \\
&\lesssim \frac{1}{(\sinh\gamma t)^{d/2}}\int_{\mathbb{R}^d} V(y)^{2a}\exp\left(-\frac{\gamma|x-y|^2}{4\tanh\frac{\gamma t}{2}}-\frac{\gamma\tanh\frac{\gamma t}{2}}{4}|x+y|^2\right)\mathrm{d}y \\
&\lesssim e^{-\frac{d\gamma t}{2}}\int_{\mathbb{R}^d}\exp\left(2ab|y|^2-\frac{\gamma|x-y|^2}{4\tanh\frac{\gamma t}{2}}-\frac{\gamma\tanh\frac{\gamma t}{2}}{4}|x+y|^2\right)\mathrm{d}y
\end{aligned}
$$

Rewriting the exponents we obtain

$$
\begin{aligned}
2ab|y|^2 &- \frac{\gamma|x-y|^2}{4\tanh\frac{\gamma t}{2}}-\frac{\gamma\tanh\frac{\gamma t}{2}}{4}|x+y|^2 \\
&= \left(2ab-\frac{\gamma\coth\gamma t}{2}\right)\left|y+\frac{\gamma\operatorname{csch}\gamma t}{4ab-\gamma\coth\gamma t}x\right|^2 \\
&\qquad -\left(\frac{\gamma\coth\gamma t}{2}+\frac{(\gamma\operatorname{csch}\gamma t)^2}{8ab-2\gamma\coth\gamma t}\right)|x|^2.
\end{aligned}
$$

We see that for the integral in (5.24) to be finite the quantity $\varphi(t):=2ab-\frac{\gamma\coth\gamma t}{2}$ has to be negative for all $t\geqslant 1$, which is satisfied for $a\leqslant\frac{\gamma\tanh\frac{\gamma}{2}}{4b}$ since $\frac{\gamma\tanh\frac{\gamma}{2}}{4b}<\frac{\gamma\coth\gamma t}{4b}$. For such a we have $\varphi(t)\leqslant\frac{\gamma}{2}(\tanh\frac{\gamma}{2}-\coth\gamma t)$ and

$$
\begin{aligned}
\int_{\mathbb{R}^d} & \exp\left(2ab|y|^2-\frac{\gamma|x-y|^2}{4\tanh\frac{\gamma t}{2}}-\frac{\gamma\tanh\frac{\gamma t}{2}}{4}|x+y|^2\right)\mathrm{d}y \\
&= \exp\left(-\left(\frac{\gamma\coth\gamma t}{2}+\frac{(\gamma\operatorname{csch}\gamma t)^2}{4\varphi(t)}\right)|x|^2\right)\int_{\mathbb{R}^d} e^{\varphi(t)|y|^2}\mathrm{d}y \\
&\leqslant \exp\left(-\frac{\gamma}{2}\left(\coth\gamma t+\frac{\operatorname{csch}^2\gamma t}{\tanh\frac{\gamma}{2}-\coth\gamma t}\right)|x|^2\right)\left(-\frac{\pi}{\varphi(t)}\right)^{d/2}.
\end{aligned}
$$

Denoting $\psi(t):=\coth\gamma t+\frac{\operatorname{csch}^2\gamma t}{\tanh\frac{\gamma}{2}-\coth\gamma t}$ a calculation gives

$$
\psi'(t)=-\frac{\gamma\operatorname{csch}^2\gamma t\;\cdot\;\left(-1+\tanh^2\frac{\gamma}{2}\right)}{\left(\tanh\frac{\gamma}{2}-\coth\gamma t\right)^2}.
$$

Since ψ' is positive the function ψ is strictly increasing. Moreover it has a zero at $t=\frac{1}{2}$ so that for $t\geqslant 1$ we have $\psi(t)\geqslant\psi(1)=\delta>0$ and thus we can continue the previous calculation as follows

$$\exp\left(-\frac{\gamma}{2}\left(\coth\gamma t+\frac{\operatorname{csch}^2\gamma t}{\tanh\frac{\gamma}{2}-\coth\gamma t}\right)|x|^2\right)\left(-\frac{\pi}{\varphi(t)}\right)^{d/2} \lesssim e^{-\frac{\gamma\delta|x|^2}{2}}(-\varphi(t))^{-d/2}$$

Next we need to handle the term $(-\varphi(t))^{-d/2}$. Since $a \leqslant \frac{\gamma\tanh\frac{\gamma}{2}}{4b}$ we see that

$$(-\varphi(t))^{-d/2} \lesssim \left(\gamma\left(\coth\gamma t-\tanh\frac{\gamma}{2}\right)\right)^{-d/2} \lesssim 1, \qquad t \geqslant 1.$$

Finally plugging the above estimates in (5.24) we get

$$e^{-t(-\frac{\Delta}{2}+\frac{\gamma^2}{2}|x|^2)}(V^{2a})(x) \lesssim e^{-\frac{d\gamma t}{2}}e^{-\frac{\gamma\delta|x|^2}{2}}, \tag{5.25}$$

uniformly in $x\in\mathbb{R}^d$ and $t \geqslant 1$.

Next we estimate $\left(\mathbb{E}_x e^{2k\int_0^t \omega(X_s)\mathrm{d}s}\right)^{1/2}$. Since $\omega \leqslant 4cN^2\mathbb{1}_P$ for $P = [-2N,2N]\times\mathbb{R}^{d-1}$, we can apply Lemma 4.2 with $k' = 4ckN^2$, which gives

$$\mathbb{E}_x e^{2k\int_0^t \omega(X_s)\mathrm{d}s} \lesssim e^{512c^2k^2N^6t} = e^{32\gamma^4N^6t} \tag{5.26}$$

Combining (5.25) and (5.26) and coming back to (5.23) we reach

$$\int_1^\infty e^{-tL}(V^a)(x)\, t^{a-1}\mathrm{d}t \lesssim e^{-\frac{\gamma\delta|x|^2}{4}}\int_1^\infty e^{-\frac{d\gamma t}{4}}e^{16\gamma^4N^6t}t^{a-1}\mathrm{d}t \lesssim 1, \quad x\in\mathbb{R}^d,$$

provided that $\gamma < \frac{d^{1/3}}{4N^2}$. This can be achieved by taking $k = \min(1,\mu^2/(4c))$, since for such k we have

$$\gamma = 2\sqrt{ck} \leqslant \mu < \frac{d^{1/3}}{4N^2}.$$

The proof of Proposition 5.3 is thus completed. □

We shall now derive L^1 boundedness of R_V^a using Proposition 5.1 together with one of the Propositions 5.2, 5.3 and 5.7.

Combining Proposition 5.1 and Proposition 5.2 we get a theorem on the L^1 boundedness of R_V^a. Note that this theorem inherits the stronger assumptions on V from Proposition 5.2. Its advantage is the allowance of large a when $V(x) \lesssim e^{\eta|x|}$ with small η. This is useful for instance when $V(x) \approx_g |x|^\alpha$.

Theorem 5.4. — *Let V be an a.e. non-negative potential having an exponential growth* (5.13) *for some $\eta > 0$ and such that e^{-tL} has an exponential decay* (ED(δ)) *of an order $\delta > 0$. Let $0 < a < \delta^{1/2}(2d)^{-1/2}\eta^{-1}$,* *take $b > a$ and let c be the constant defined in* (5.11)*. If*

$$K_c^a(V)(x) \lesssim_g 1 \qquad \text{and} \qquad I^b(V)(x) \lesssim_g 1,$$

then R_V^a is bounded on L^1.

Proof. — By duality it suffices to estimate the L^∞ norm of

$$\begin{aligned}(5.27)\quad \frac{1}{\Gamma(a)}\int_0^\infty e^{-tL}(V^a)t^{a-1}\,\mathrm{d}t &= \frac{1}{\Gamma(a)}\int_0^1 e^{-tL}(V^a)t^{a-1}\,\mathrm{d}t + \frac{1}{\Gamma(a)}\int_1^\infty e^{-tL}(V^a)t^{a-1}\,\mathrm{d}t \\ &=: L + G.\end{aligned}$$

Using the bound $e^{\eta|x|} \lesssim e^{|x|^2/(4a)}$ and (5.5) from Proposition 5.1 we see that

$$L(x) \lesssim C(N),$$

whenever $|x| \leqslant N$. Then (5.10) together with (5.4) from Proposition 5.1 gives

$$\|L\|_\infty \lesssim 1.$$

The estimate

$$\|G\|_\infty \lesssim 1$$

is a straightforward consequence of our assumptions and Proposition 5.2. □

Proposition 5.1 and Proposition 5.3 allow us to improve Theorem 5.4 for potentials that grow at least as a constant times $|x|^2$. The improvement comes from the replacement of the condition $K_c^a(V)(x) \lesssim_g 1$ by $J^a(V)(x) \lesssim 1$. This is useful e.g. for potentials $V(x) = \beta^{|x|}$, $\beta > 1$, for which $K_c^a(V)$ may be unbounded.

THEOREM 5.5. — *Let $0 < a < \infty$ and let V be an a.e. non-negative potential which satisfies, for some $c > 0$ the estimate $c|x|^2 \lesssim_g V(x)$. Assume that for all $\varepsilon > 0$ we have $V(x) \lesssim_\varepsilon e^{\varepsilon|x|^2}$. If*

$$J^a(V)(x) \lesssim_g \qquad \text{and} \qquad I^a(V)(x) \lesssim_g 1,$$

then R_V^a is bounded on L^1.

Proof. — We use the splitting (5.27) again. The estimate $\|G\|_\infty \lesssim 1$ is a consequence of Proposition 5.3. Indeed, the assumption $V(x) \lesssim e^{\varepsilon|x|^2}$ with arbitrarily small $\varepsilon > 0$ implies that we can apply Proposition 5.3 with arbitrarily large $a > 0$. The bound $\|L\|_\infty \lesssim 1$ follows from the assumptions and Proposition 5.1 as in the proof of Theorem 5.4. □

As a corollary of Theorems 5.4 and 5.5 we obtain the $L^1(\mathbb{R}^d)$ boundedness of R_V^a for various classes of potentials. The corollary below is a restatement of Theorem D from the introduction.

Corollary 5.6. — *Let $V\colon \mathbb{R}^d \to [0,\infty)$ be a function in L^∞_{loc}. Then in all the three cases*

(1) *$V(x) \approx 1$ globally*
(2) *For some $\alpha > 0$ we have $V(x) \approx |x|^\alpha$ globally*
(3) *For some $\beta > 1$ we have $V(x) \approx \beta^{|x|}$ globally*

each of the Riesz transforms R_V^a, $a > 0$, is bounded on $L^1(\mathbb{R}^d)$.

Remark. — Similarly to Corollary 4.6 the Euclidean norm $|\cdot|$ in (2) and (3) can be replaced by an arbitrary norm on $\mathbb{R}^d$.

Proof. — In the proof implicit constants in $\lesssim$, $\gtrsim$, and $\approx$ do not depend on $x \in \mathbb{R}^d$ but may depend on $a > 0$, $\alpha > 0$ or $\beta > 1$.

Note that in all three cases the assumptions of Lemma 4.1 are satisfied so that the semigroup e^{-tL} satisfies (ED(δ)).

In case (1) we merely use (ED(δ)) and obtain

$$\frac{1}{\Gamma(a)}\int_0^\infty e^{-tL}(V^a)(x)t^{a-1}\,\mathrm{d}t \lesssim \frac{1}{\Gamma(a)}\int_0^\infty \|e^{-tL}(\mathbb{1})\|_\infty\, t^{a-1}\,\mathrm{d}t \lesssim 1,$$

uniformly in $x \in \mathbb{R}^d$.

In the treatment of the remaining cases we will apply Theorem 5.4 in case (2) and Theorem 5.5 in case (3).

We start with case (2); the task is to check that the assumptions of Theorem 5.4 hold. Clearly (5.13) is true for any $\eta > 0$. In the proof of Corollary 4.6 we justified in (4.27) that $I^b(V)(x) \lesssim_g 1$ for any $b > 0$. Finally we need to control $K_c^a(V)(x)$. To this end we shall estimate $\sigma_x(s)$ from below. Let C, N, m and M be non-negative constants such that

$$m|x|^\alpha < V(x) < M|x|^\alpha \quad \text{for a.e.} \quad |x| > N$$

and

$$V(x) \leqslant C \quad \text{for a.e.} \quad |x| \leqslant N.$$

Take $|x| \geqslant N$ and assume that $|x-y| < \varepsilon|x|s^{1/\alpha}$, where $\varepsilon > 0$ is a constant to be determined in a moment. Then

$$|y| \leqslant |x| + |x-y| \leqslant |x|(1+\varepsilon s^{1/\alpha})$$

so that for $|y| > N$ we have

$$V(y) \leqslant M|y|^\alpha \leqslant M|x|^\alpha\left(1+\varepsilon s^{1/\alpha}\right)^\alpha \leqslant MA|x|^\alpha(1+\varepsilon^\alpha s)$$

for some constant $A \geqslant 1$ depending only on α. On the other hand

$$V(x) \geqslant m|x|^{\alpha}$$

so taking ε such that $MA\varepsilon^{\alpha} = m/2$ we see that the inequality $|x-y| < \varepsilon|x|s^{1/\alpha}$ implies

$$V(y) \leqslant MA|x|^{\alpha}(1+\varepsilon^{\alpha}s) \leqslant MA|x|^{\alpha} + sV(x)/2 \leqslant \left(\frac{MA}{m} + \frac{s}{2}\right)V(x) \leqslant sV(x),$$

whenever s is large enough (independently of x). Thus we proved that $\sigma_x(s) \geqslant \varepsilon|x|s^{1/\alpha}$ for such s and a.e. $|x| \geqslant N$. Consequently,

$$K_c^a(V)(x) \lesssim_g 1 + \int_1^{\infty} e^{-c\varepsilon|x|s^{1/\alpha}} s^{a-1}\,\mathrm{d}s \lesssim_g 1$$

for any $a, c > 0$ and an application of Theorem 5.4 completes the proof in case (2).

Finally we justify case (3). It is clear that $c|x|^2 \lesssim_g V(x) \lesssim e^{\varepsilon|x|^2}$ for some $c > 0$ and all $\varepsilon > 0$. Moreover, in the proof of Corollary 4.6 in (4.30) we justified that $I^a(V)(x) \lesssim_g 1$. Thus, in order to use Theorem 5.5 it remains to estimate $J^a(V)(x)$. Similarly, to case (2) we shall estimate $\sigma_x(s)$ from below. Let $M > 0$ be a constant such that $V(y) \leqslant M\beta^{|y|}$, for a.e. $y \in \mathbb{R}^d$ and let N, m be non-negative constants such that $m\beta^{|x|} < V(x)$ for a.e. $|x| \geqslant N$. Take $|x| \geqslant N$, $s \geqslant 1$ and assume that $|x-y| < \frac{1}{2}\log_{\beta} s$. Then we have $|y| \leqslant |x| + \frac{1}{2}\log_{\beta} s$, so that

$$V(y) \leqslant Ms^{1/2}\beta^{|x|} \leqslant \frac{M}{m}s^{1/2}V(x) \leqslant sV(x),$$

for s large enough (independently of y and x). In other words we proved that $\sigma_x(s) \geqslant \frac{1}{2}\log_{\beta} s$ whenever $|x| \geqslant N$ and s is uniformly large enough. Consequently,

$$J^a(V)(x) \lesssim_g 1 + \int_1^{\infty} e^{-(\log_{\beta} s)^2/32} s^{a-1}\,\mathrm{d}s \lesssim_g 1$$

for any $a > 0$ and an application of Theorem 5.5 completes the proof in case (3). ☐

We finish this section with improved results for Riesz transforms R_V^a in the range $0 < a < 1$. These results are not needed in the proof of Corollary 5.6, however they might by useful in other cases.

Using the L^1 boundedness of R_V^1 one may improve Proposition 5.2 in the range $0 \leqslant a \leqslant 1$.

Proposition 5.7. — *Let $a \leqslant 1$ and assume that e^{-tL} satisfies* (ED(δ)) *with some $\delta > 0$. Then the estimate*

$$\int_1^\infty e^{-tL}(V^a)(x)\, t^{a-1} \mathrm{d}t \lesssim 1$$

holds uniformly in $x \in \mathbb{R}^d$.

Proof. — Observe that for $a \leqslant 1$ we have

$$e^{-tL}(V^a)(x) \leqslant e^{-tL}(V)(x) + e^{-tL}(\mathbb{1})(x),$$

so that

$$\int_1^\infty e^{-tL}(V^a)(x)\, t^{a-1} \mathrm{d}t \leqslant \int_1^\infty e^{-tL}(V)(x)\, t^{a-1} \mathrm{d}t + \int_1^\infty e^{-tL}(\mathbb{1})(x)\, t^{a-1} \mathrm{d}t. \tag{5.28}$$

From e.g. [2, Theorem 4.3] we see that the operator R_V^1 is bounded on L^1 which, by duality, means that the first integral in (5.28) is bounded independently of x. Boundedness of the second integral follows from (ED(δ)). □

Finally, combining Proposition 5.7 and Proposition 5.1 we obtain an improved version of Theorem 5.4 in the range $0 < a \leqslant 1$.

Theorem 5.8. — *Let $0 < a \leqslant 1$ and let V be an a.e. non-negative potential which satisfies the growth estimate $V(x) \lesssim \exp\big(|x|^2/(4a)\big)$ and such that e^{-tL} has an exponential decay* (ED(δ)) *for some $\delta > 0$. If*

$$J^a(V)(x) \lesssim_g 1 \qquad \textit{and} \qquad I^a(V)(x) \lesssim_g 1,$$

then R_V^a is bounded on L^1.

Proof. — We use the splitting (5.27). The estimate $\|G\|_\infty \lesssim 1$ is an immediate consequence of Proposition 5.7. The bound $\|L\|_\infty \lesssim 1$ follows from the assumptions and Proposition 5.1 as in the proof of Theorem 5.4. □

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