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Bruno Colbois & Alexandre Girouard

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METRIC UPPER BOUNDS FOR STEKLOV AND LAPLACE EIGENVALUES

by Bruno COLBOIS & Alexandre GIROUARD (*)

Abstract. — We prove two upper bounds for the Steklov eigenvalues of a compact Riemannian manifold with boundary. The first involves the volume of the manifold and of its boundary, as well as packing and volume growth constants of the boundary and its distortion. Its proof is based on metric-measure space techniques. The second bound is in terms of the extrinsic diameter of the boundary and its injectivity radius. It is obtained from a concentration inequality, akin to Gromov–Milman concentration for closed manifolds. By applying these bounds to cylinders over closed manifolds, we obtain bounds for eigenvalues of the Laplace operator, in the spirit of Berger–Croke. For a family of manifolds that has uniformly bounded volume and boundary of fixed intrinsic geometry, we deduce that a large first nonzero Steklov eigenvalue implies that each boundary component is contained in a ball of small extrinsic radius.

Résumé. — Nous obtenons deux bornes supérieures pour les valeurs propres de Steklov d'une variété riemannienne compacte à bord. La première fait intervenir le volume de la variété et de son bord, des constantes d'empilement et de croissance du bord ainsi que sa distorsion. La preuve utilise une technique provenant de la théorie des espaces métriques mesurés. La deuxième borne dépend du diamètre extrinsèque du bord et de son rayon d'injectivité et découle d'une inégalité de concentration à la Gromov–Milman. En appliquant ces bornes à des cylindres, on obtient des bornes pour les valeurs propres du laplacien sur des variétés fermées, semblables à celles de Berger–Croke. Pour des variétés dont le volume est uniformément borné et dont le bord est de géométrie intrinsèque prescrite, nous déduisons qu'une grande valeur propre de Steklov implique que chaque composante du bord est contenue dans une boule extrinsèque de petit rayon.

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1. Introduction

Let M be a smooth connected compact Riemannian manifold of dimension $n+1 \geqslant 2$, with boundary $\Sigma = \partial M$. The Dirichlet-to-Neumann operator $\mathcal{D} : C^\infty(\Sigma) \to C^\infty(\Sigma)$ is defined by $\mathcal{D}f = \partial_\nu \widehat{f}$, where ν is the outward normal along the boundary Σ and where the function $\widehat{f} \in C^\infty(M)$ is the unique harmonic extension of f to the interior of M. The eigenvalues of $\mathcal{D}$ are known as *Steklov eigenvalues* of M. They form an unbounded sequence $0 = \sigma_0 \leqslant \sigma_1 \leqslant \sigma_2 \leqslant \cdots \to \infty$, where as usual each eigenvalue is repeated according to its multiplicity. The interplay of these eigenvalues with the geometry of M has been an active area of investigation in recent years. See [11, 21] for surveys and [2, 10, 13, 16, 19, 20, 26] for recent relevant results.

In this paper we study upper bounds for Steklov eigenvalues in terms of geometric quantities that are metric in nature: packing and growth constants, distortion between the intrinsic and extrinsic distances on the boundary, as well as diameters and injectivity radius of the boundary components. A recurring feature is that the bounds are linked to some comparison between intrinsic and extrinsic geometry of the boundary. They do not involve the curvature. See [14, 23, 27] for early use of similar techniques and [6, 7, 10] for some more recent results in the same spirit. See [12, 25, 28] for some bounds depending on curvature assumptions. Upper bounds for the eigenvalues λ_k of the Laplacian on a closed Riemannian manifold will also be obtained. They are in the spirit of [1, 17, 26], and of [14, 23].

Notation

We use two distances on the boundary Σ. The first one is the geodesic distance d_Σ. The second distance is induced on Σ from the geodesic distance d_M in M. In general, the letters M and Σ will be used to specify which distance is involved. For instance, for $x \in M$ we define the ball

$$B^M(x,r) := \big\{y \in M \,:\, d_M(x,y) < r\big\},$$

and for $x \in \Sigma$,

$$B^\Sigma(x,r) := \big\{y \in \Sigma \,:\, d_\Sigma(x,y) < r\big\}.$$

Similarly, we write $|\mathcal{O}|_M$ for the Riemannian measure of a Borel set $\mathcal{O} \subset M$, while $|\mathcal{O}|_\Sigma$ is the Riemannian measure of $\mathcal{O} \cap \Sigma$.

1.1. Upper bound in terms of metric invariants

For $x, y \in \Sigma$, we have $d_M(x,y) \leqslant d_\Sigma(x,y)$, where the convention is that for x and y in different connected components of Σ, we set $d_\Sigma(x,y) = +\infty$. Let $\Sigma_1, \dots, \Sigma_b$ be the connected components of the boundary Σ. The *distortion of* Σ_j *in* M is the number $\Lambda_j \in [1, \infty)$ defined by

$$\Lambda_j := \inf\big\{c \geqslant 1 \,:\, d_\Sigma(x,y) \leqslant c\, d_M(x,y),\ \forall\, x, y \in \Sigma_j\big\}. \tag{1.1}$$

The distortion of Σ in M is

$$\Lambda := \max\{\Lambda_1, \dots, \Lambda_b\}.$$

The distortion is a measure of how much the geodesic distance d_Σ differs from the induced distance $d_M\big|_\Sigma$. To state our first main result we also need two more geometric invariants. It follows from the compactness of Σ that there exists a *packing constant* $N_M \in \mathbb{N}$ for (Σ, d_M) and a *packing constant* N_Σ for (Σ, d_Σ), which satisfy the following properties:

- for each $r > 0$ and each $x \in \Sigma$, the extrinsic ball $B^M(x,r) \cap \Sigma$ can be covered by N_M extrinsic balls of radius $r/2$ centered at points $x_1, \dots, x_{N_M} \in \Sigma$:
$$B^M(x,r) \cap \Sigma \subset \bigcup_{i=1}^{N_M} B^M(x_i, r/2);$$
- for each $r > 0$ and each $x \in \Sigma$, the intrinsic ball $B^\Sigma(x,r)$ can be covered by N_Σ intrinsic balls of radius $r/2$ centered at points $x_1, \dots, x_{N_\Sigma} \in \Sigma$:
$$B^\Sigma(x,r) \subset \bigcup_{i=1}^{N_\Sigma} B^\Sigma(x_i, r/2).$$

There also exists a *growth constant* $\Gamma > 1$ for the metric-measure space $\big(\Sigma, d_\Sigma, |\cdot|_\Sigma\big)$, which satisfies the following property: for each $x \in \Sigma$ and each $r > 0$, $\big|B^\Sigma(x,r)\big|_\Sigma \leqslant \Gamma r^n$.

The first main result of this paper is a generalization of [10, Theorem 1.1], where M was a submanifold in some euclidean space.

Theorem 1.1. — *Let M be a smooth connected compact Riemannian manifold of dimension $n+1$ with boundary Σ. The following holds for each $k \geqslant 1$:*

$$\sigma_k \leqslant 2^{14} b^2 N_M^3 \Gamma \Lambda^2 \frac{|M|}{|\Sigma|^{\frac{n+2}{n}}} k^{2/n}. \tag{1.2}$$

The exponent $2/n$ on k is optimal: it cannot be replaced by any smaller number.

The proof of Theorem 1.1 will be presented in Section 3, where it will be deduced from a slightly more general statement (Theorem 3.1). The optimality of the exponent is discussed in Remark 1.5.

The volume $|M|$, the distortion Λ and the packing constant N_M depend on the geometry of M in its interior, while the constants $b, \Gamma, |\Sigma|$ only depend on the intrinsic geometry of the boundary Σ. In fact, the extrinsic packing constant N_M can be expressed in terms of the number of connected components of Σ, the distortion and of the intrinsic packing constant N_Σ of (Σ, d_Σ):

$$N_M = bN_\Sigma^{\log_2(2\Lambda)}.$$

This will be proved in Lemma 3.3. This leads to the following.

Corollary 1.2. — *Under the hypothesis of Theorem 1.1, the following holds for each $k \geqslant 1$:*

$$\sigma_k \leqslant 2^{14} b^5 N_\Sigma^{3\log_2(2\Lambda)} \Gamma \Lambda^2 \frac{|M|}{|\Sigma|^{\frac{n+2}{n}}} k^{2/n}. \tag{1.3}$$

In (1.3), apart from σ_k, only the distortion Λ and the volume of M depend on the geometry of M. All other geometric quantities are intrinsic to the boundary Σ. The importance of each geometric constant appearing in (1.3) will be discussed in Section 3.1.

While (1.3) is somewhat cumbersome, its strength is that its geometric dependance is completely explicit, with clear distinction between extrinsic and intrinsic features. The reader is invited to compare with [10, Theorem 1.1]. Note also that none of the geometric invariants appearing in (1.2) is superfluous. This will be discussed in Section 3.1.

Remark 1.3. — One can rewrite (1.2) in the following scale-invariant fashion:

$$\sigma_k |\Sigma|^{1/n} \leqslant \frac{2^{14}}{I(M)^{\frac{n+1}{n}}} b^2 N_M^3 \Gamma \Lambda^2 k^{2/n}, \tag{1.4}$$

where $I(M) = |\Sigma|/|M|^{\frac{n}{n+1}}$ is the isoperimetric ratio of M. One should compare this with [6, Theorem 1.3], which states that

$$\sigma_k |\Sigma|^{1/n} \leqslant \frac{\gamma(n)}{I(M)^{\frac{n-1}{n}}} k^{2/(n+1)}, \tag{1.5}$$

for domains M in a complete space that is conformally equivalent to a complete manifold with non-negative Ricci curvature. Our new inequality (1.4) applies to a much larger class of manifolds, since no curvature assumption is required.

There is a close link between the Steklov eigenvalues of a manifold M and the Laplace eigenvalues of its boundary ∂M. See [12]. In the situation where M is a cylinder this link is explicit and can be used to obtain new bounds on Laplace eigenvalues from known results on Steklov eigenvalues. Given a compact connected Riemannian manifold Σ, the eigenvalues of the Laplace operator $\Delta : C^\infty(\Sigma) \longrightarrow C^\infty(\Sigma)$ are written $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \to \infty$.

Corollary 1.4. — *Let Σ be a compact connected Riemannian manifold of dimension n. Then,*

$$\lambda_k |\Sigma|^{2/n} \leqslant 2^{16} \Gamma N_\Sigma^3 k^{2/n}, \tag{1.6}$$

where N_Σ is a packing constant of (Σ, d_Σ).

This result is similar in spirit to those presented in [23] and [14]. This is not surprising since the proof of Theorem 1.1 is based on a simplification of the main technical tool from [14], presented here as Lemma 3.2. One should compare this with [23, Remark 5.10].

Remark 1.5. — It follows from Weyl's law for λ_k that the power $2/n$ on k is optimal in (1.6), and therefore also in (1.2).

1.2. A Berger–Croke type inequality for Steklov and Laplace eigenvalues

Let M be a compact manifold with boundary Σ. Let $\Sigma_1, \ldots, \Sigma_b$ be the connected components of Σ. The extrinsic diameter of a connected component Σ_j is defined by

$$\mathrm{diam}_M(\Sigma_j) \coloneqq \sup\left\{ d_M(x, y) \,:\, x, y \in \Sigma_j \right\}.$$

The main result of this section is the following. Recall that the injectivity radius of a closed Riemannian manifold S is the largest number $r > 0$ such that the exponential map $\exp_p : B(0, r) \subset T_pS \to S$ is a diffeomorphism for all points $p \in S$.

Theorem 1.6. — *Let Σ_j be a connected component of the boundary Σ. Then,*

$$\sigma_k \leqslant K(n) \frac{|M|}{\mathrm{diam}_M(\Sigma_j)^{n+2}} \times \left(1 + \left(\frac{\mathrm{diam}_M(\Sigma_j)}{\mathrm{inj}(\Sigma_j)}\right)^n\right) k^{n+1}, \tag{1.7}$$

where $K(n)$ is a dimensional constant and $\mathrm{inj}(\Sigma_j)$ *is the injectivity radius of* Σ_j.

This theorem will be obtained as a corollary from the slightly more general Theorem 2.4. The proof is based on a simple concentration bound which is adapted from the work of Gromov and Milman [24]. See Lemma 2.1 below.

Remark 1.7. — The diameter of M itself does not appear in (1.7). This is not surprising, since one can modify M away from the boundary so as to obtain arbitrarily large diameter

$$\sup\{d_M(x,y)\ :\ x,y\in M\},$$

without significant change to σ_k, the extrinsic diameter $\mathrm{diam}_M(\Sigma)$ and the volume $|M|$. This can be performed for instance by replacing two small balls in M with a long thin tube joining them. See [18, Theorem 1.2].

Remark 1.8. — The presence of the injectivity radius in the denominator of (1.7) is essential. Indeed, let (M,g) be a compact Riemannian manifold of dimension $n\geqslant 3$ and let $C\subset M$ be a smooth embedded closed curve. For $\varepsilon>0$ small enough,

$$\Omega_\varepsilon := \{x\in M\ :\ d_M(x,C)>\varepsilon\}$$

is a connected domain with smooth boundary. In [2], Jade Brisson proved that $\sigma_1(\Omega_\varepsilon)\xrightarrow{\varepsilon\to 0}\infty$. Because $|\Omega_\varepsilon|$ and $\mathrm{diam}_M(\partial\Omega_\varepsilon)$ are uniformly bounded as $\varepsilon\to 0$, the injectivity radius could not be removed from (1.7).

For a family of manifolds that has uniformly bounded volume and boundary of fixed intrinsic geometry, we deduce that a large Steklov eigenvalue σ_k (for a fixed index k) implies that each boundary component is contained in a ball of small extrinsic radius.

Corollary 1.9. — *For $j\in\{1,\dots,b\}$, consider the intrinsic constant*

$$C(\Sigma_j) := K(n)^{\frac{1}{n+2}}\left(1+\left(\frac{\mathrm{diam}_\Sigma(\Sigma_j)}{\mathrm{inj}(\Sigma_j)}\right)^n\right)^{\frac{1}{n+2}}.$$

Let

$$\gamma_j = C(\Sigma_j)\left(\frac{|M|}{\sigma_k}\right)^{\frac{1}{n+2}}k^{\frac{n+1}{n+2}}.$$

Then there exists $x_j\in\Sigma_j$ such that $\Sigma_j\subset B^M(x_j,\gamma_j)$.

This behaviour should be compared with [15, Theorem 3], which essentially implies the following: if the first nonzero eigenvalue λ_1 of the

Laplacian on a closed Riemannian manifold M is large with respect to its packing constant, then the Riemannian measure of M is concentrated near one point. In contrast, if the first nonzero Steklov eigenvalue σ_1 is large, then Corollary 1.9 only claims that each connected component of the boundary is contained in a small ball, but there is nothing preventing these small balls from being far apart. This behaviour is meaningful and there are indeed examples where σ_1 is large while the boundary components remain far from each other. The easiest example is obtained by removing small balls from a closed manifold. This follows from [3, Theorem 1.5].

Berger [1] proved that on any closed Riemannian manifold Σ which admits an isometric involution without fixed points, the first nonzero eigenvalue of the Laplacian satisfies

$$\lambda_1 \leqslant K(n)\frac{|\Sigma|}{\operatorname{inj}(\Sigma)^{n+2}}. \tag{1.8}$$

This result was generalized by Croke, who proved in [17] that any closed Riemannian manifold satisfies the following for each $k \in \mathbb{N}$:

$$\lambda_k \leqslant K(n)\frac{|\Sigma|^2}{\operatorname{conv}(\Sigma)^{2n+2}}k^{2n}. \tag{1.9}$$

Here $\operatorname{conv}(\Sigma)$ is the *convexity radius* of Σ: the largest number $r > 0$ such that all geodesic balls $B(p,r) \subset \Sigma$ are geodesically convex. One should also see the recent paper [26] by Kokarev. Theorem 2.4 leads to the following improvement of the Berger and Croke inequalities.

COROLLARY 1.10. — *Let Σ be a closed Riemannian manifold. Then for each $k \in \mathbb{N}$,*

$$\lambda_k \operatorname{diam}(\Sigma)^2 \leqslant K(n)\frac{|\Sigma|}{\operatorname{inj}(\Sigma)^{n}}k^{n+1}. \tag{1.10}$$

Inequality (1.10) improves on Berger and Croke in several ways. For instance, the exponent on k is better. Perhaps more interestingly, because $\operatorname{conv}(\Sigma) \leqslant \operatorname{inj}(\Sigma) \leqslant \operatorname{diam}(\Sigma)$, inequality (1.10) allows a finer control for manifolds that have small injectivity radius and large diameter, as the following example shows.

Example 1.11. — The first nonzero Laplace eigenvalue of $\Sigma_L = \mathbb{S}^1_L \times \mathbb{S}^{n-1}$ behaves as $\lambda_1 \sim 1/L^2$ as $L \to +\infty$. Moreover, the volume of Σ_L is $|\Sigma_L| = L|\mathbb{S}^{n-1}|$, its injectivity and convexity radii are both equal to π. If $L > 0$ is large enough, the diameter of Σ is of order L. Whence, the upper bound of Berger and Croke reads

$$\lambda_1 \leqslant K(n)L|\mathbb{S}^{n-1}|/\pi^{n+2} \qquad \text{and} \qquad \lambda_1 \leqslant K(n)L^2|\mathbb{S}^{n-1}|^2/\pi^{2n+2}.$$

Both upper bounds diverge as $L \to +\infty$. On the other hand, our bound (1.10) gives the much more accurate

$$\lambda_1 \leqslant K(n)\frac{|\mathbb{S}^{n-1}|}{L\pi^n} \xrightarrow{L\to\infty} 0.$$

Plan of the paper

In Section 2 we present a concentration inequality akin to that of Gromov–Milman [24, Theorem 4.1], which we use to prove Theorem 1.6. To prove Theorem 1.1 some tools from metric geometry will then be used in Section 3. In particular, Lemma 3.3 links the packing constant of (Σ, d_M) and of (Σ, d_Σ).

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2. Upper bound and measure concentration for σ_k

The proof of Theorem 2.4 depends on the min-max characterization of Steklov eigenvalues:

$$\sigma_j = \min_{E\in\mathcal{H}_j}\max_{0\neq u\in E} R_M(u), \tag{2.1}$$

where $\mathcal{H}_j$ is the set of all $(j+1)$-dimensional subspaces in the Sobolev space $H^1(M)$, and where

$$R_M(u) = \frac{\int_M |\nabla u|^2 dV_M}{\int_\Sigma u^2\, dV_\Sigma}$$

is the Rayleigh–Steklov quotient of u. We start with a simple bound which is adapted from the work of Gromov and Milman [24].

Lemma 2.1. — *Let $A_i \subset \Sigma$ be disjoint measurable subsets, for $i = 1, \ldots, k+1$, with positive measures $\mu_i := |A_i|_\Sigma > 0$. Suppose these subsets are quantitatively separated:*

$$\rho := \frac{1}{2}\min_{i\neq j} d_M(A_i, A_j) > 0.$$

Then

$$\sigma_k \leqslant \frac{|M|}{\rho^2 \min \mu_i}.$$

Proof. — We use standard trial functions f_i that are supported in the pairwise-disjoint neighborhoods $A_i^\rho = \{x \in M : d_M(x, A_i) \leqslant \rho\}$ and have value 1 on A_i. These are defined by

$$f_i(x) = \begin{cases} 1 - \frac{1}{\rho} d_M(x, A_i) & \text{in } A_i^\rho, \\ 0 & \text{elsewhere.} \end{cases}$$

Observe that $\|\nabla f_i\|^2 = \frac{|A_i^\rho|_M}{\rho^2}$ and $\|f_i\|_\Sigma^2 \geqslant \mu_i$ since $f_i \equiv 1$ on A_i^ρ, so that

$$R(f_i) \leqslant \frac{|A_i^\rho|_M}{\rho^2 \mu_i} \leqslant \frac{|M|}{\rho^2 \min_i \mu_i}. \qquad \square$$

Remark 2.2. — If one uses A_i for $i = 1, 2, \ldots, 2k$ and supposes that $|A_1^\rho|_M \leqslant |A_2^\rho|_M \leqslant \cdots \leqslant |A_{2k+2}^\rho|_M$, then $|A_{k+1}^\rho|_M \leqslant |M|/k$ and one gets

$$\sigma_k \leqslant \frac{|M|}{k\rho^2 \min \mu_i}.$$

This trick is often useful in improving the exponent on k for bounds that are obtained using trial functions with disjoint supports. This will be used in the proof of Theorem 2.4 below.

Lemma 2.1 implies a concentration phenomenon when σ_1 is large in comparison to the other constants involved.

Proposition 2.3. — *Let M be a compact manifold with boundary Σ. Let $A \subset \Sigma$ be a subset of positive measure $\mu = |A|_\Sigma > 0$. Let $\rho > 0$. If $\sigma_1 \geqslant \frac{|M|}{\rho^2 \mu}$, then*

$$|A^{2\rho}|_\Sigma \geqslant |\Sigma| - \frac{|M|}{\sigma_1 \rho^2}.$$

This is particularly interesting for families of manifolds M_ε such that $\sigma_1 \to +\infty$, while $|\Sigma_\varepsilon|$ and $|M_\varepsilon|$ are independent of ε. In that case, the extrinsic neighborhood $A^{2\rho}$ contains all of the boundary in the limit, however small the number ρ is. This shows that the full boundary concentrates in the measure sense in the limit.

Proof of Proposition 2.3. — If $A^{2\rho} = \Sigma$, the statement is trivially true. Otherwise, define $B = \Sigma \setminus A^{2\rho}$, so that $d_M(A, B) = 2\rho$ as suggested by the notation. It follows from Lemma 2.1 that

$$\sigma_1 \leqslant \frac{|M|}{\rho^2 \min\{\mu, |\Sigma| - |A^{2\rho}|_\Sigma\}}.$$

Because σ_1 is large, that is $\sigma_1 > \frac{|M|}{\rho^2\mu}$, one has

$$\min\{\mu,\, |\Sigma| - |A^{2\rho}|_\Sigma\} = |\Sigma| - |A^{2\rho}|_\Sigma$$

so that

$$\sigma_1 \leqslant \frac{|M|}{\rho^2\big(|\Sigma| - |A^{2\rho}|_\Sigma\big)}.$$

The proof is completed by reorganizing this inequality. □

In order to prove Theorem 1.6, we will apply Lemma 2.1 to well-chosen balls in the boundary component Σ_j and obtain the following slightly more general result.

THEOREM 2.4. — *Let Σ_j be a connected component of the boundary Σ. Then,*

$$\sigma_k \leqslant K(n)\frac{|M|}{\operatorname{diam}_M(\Sigma_j)^2} \times \frac{1}{\min\{\operatorname{diam}_M(\Sigma_j)^n,\, \operatorname{inj}(\Sigma_j)^n\}} k^{n+1}, \tag{2.2}$$

where $K(n)$ is a dimensional constant and $\operatorname{inj}(\Sigma_j)$ *is the injectivity radius of Σ_j.*

Theorem 1.6 follows since

$$\begin{aligned}\frac{1}{\min\{\operatorname{diam}_M(\Sigma_j)^n,\, \operatorname{inj}(\Sigma_j)^n\}} &= \frac{1}{\operatorname{diam}_M(\Sigma_j)^n}\max\left\{1,\, \frac{\operatorname{diam}_M(\Sigma_j)^n}{\operatorname{inj}(\Sigma_j)^n}\right\}\\ &\leqslant \frac{1}{\operatorname{diam}_M(\Sigma_j)^n}\left(1 + \frac{\operatorname{diam}_M(\Sigma_j)^n}{\operatorname{inj}(\Sigma_j)^n}\right).\end{aligned}$$

Proof of Theorem 2.4. — Let $\delta = \operatorname{diam}_M(\Sigma_j)$. Consider $x_1, x_2, \dots, x_{2k} \in \Sigma_j$ such that

$$d_M(x_p, x_q) \geqslant \frac{\delta}{2k}, \quad \forall\, p \neq q.$$

To see that this is possible, consider points x_1, x_{2k} such that

$$d_M(x_1, x_{2k}) = \operatorname{diam}_M(\Sigma_j)$$

and use the concentric balls $B_M\big(x_1, \frac{p\delta}{2k}\big)$ with $p = 1, 2, \dots, 2k$. Because Σ_j is connected, it intersects each sphere $\partial B_M\big(x_1, \frac{i\delta}{2k}\big)$. Any sequence of points $x_p \in \partial B_M\big(x_1, \frac{p\delta}{2k}\big)$ will work.

Now, use Lemma 2.1 and its proof with $A_i = B_M\big(x_i, \frac{\delta}{8k}\big) \cap \Sigma_i$, and observe that the triangle inequality gives $\rho \geqslant \delta/4k$, where ρ is defined in Lemma 2.1. Hence, the Rayleigh quotients of the standard functions are controlled by

$$R(f_i) \leqslant \frac{|A_i^\rho|_M}{\big(\frac{\delta}{4k}\big)^2 \big|B_M\big(x_i, \frac{\delta}{8k}\big)\big|_\Sigma} \leqslant \frac{16|M|}{\delta^2 \big|B_M\big(x_i, \frac{\delta}{8k}\big)\big|_\Sigma} k^2.$$

Because the $2k$ sets A_i^ρ are disjoint, we can reorder them to ensure that $|A_i^\rho|_M \leqslant |M|/k$ for $i = 1, 2, \dots, k+1$. This leads to the improved bound

$$R(f_i) \leqslant \frac{16|M|}{\delta^2 \big|B_M\big(x_i, \frac{\delta}{8k}\big)\big|_\Sigma} k, \quad \text{for } i = 1, 2, \dots, k+1.$$

The main task is now to control the intrinsic volume $\big|B_M(x_i, \delta/8k)\big|_\Sigma$ from below. We split this in two cases.

Case 1. — If $\delta/8k \geqslant \operatorname{inj}(\Sigma_j)/4$ then, using that extrinsic balls are bigger than intrinsic balls of the same radius, Croke's inequality [17, Proposition 14] gives

$$\big|B_M(x_i, \delta/8k)\big|_\Sigma \geqslant \big|B_\Sigma(x_i, \delta/8k)\big|_\Sigma \geqslant \big|B_\Sigma(x_i, \operatorname{inj}(\Sigma_j)/4)\big|_\Sigma \geqslant c(n) \operatorname{inj}(\Sigma_j)^n$$

and

$$R(f_i) \leqslant c(n) \frac{|M|}{\delta^2 \operatorname{inj}(\Sigma_j)^n} k.$$

Case 2. — If $\delta/8k < \operatorname{inj}(\Sigma_j)$ then Croke's inequality gives

$$\big|B_M(x_i, \delta/8k)\big|_\Sigma \geqslant \big|B_\Sigma(x_i, \delta/8k)\big|_\Sigma \geqslant c(n)\delta^n/k^n$$

and

$$R(f_i) \leqslant c(n) \frac{|M|}{\delta^{n+2}} k^{n+1}.$$

Combining we get

$$R(f_i) \leqslant c(n) \frac{|M|}{\operatorname{diam}_M(\Sigma_j)^2} \times \frac{1}{\min\big\{\operatorname{diam}_M(\Sigma_j)^n, \operatorname{inj}(\Sigma_j)^n\big\}} k^{n+1}.$$

The result now follows from the min-max characterization of σ_k. □

Let us now prove the two corollaries that were stated in the introduction.

Proof of Corollary 1.9. — Because $\operatorname{diam}_M(\Sigma_j) \leqslant \operatorname{diam}_\Sigma(\Sigma_j)$, it follows from Theorem 1.6 that

$$\begin{aligned} \sigma_k &\leqslant K(n) \frac{|M|}{\operatorname{diam}_M(\Sigma_j)^{n+2}} \times \left(1 + \left(\frac{\operatorname{diam}_\Sigma(\Sigma_j)}{\operatorname{inj}(\Sigma_j)}\right)^n\right) k^{n+1} \\ &= C(\Sigma_j)^{n+2} \frac{|M|}{\operatorname{diam}_M(\Sigma_j)^{n+2}} k^{n+1}, \end{aligned}$$

where $C(\Sigma_j)$ depends only on the intrinsic geometry of Σ_j. This implies

$$\operatorname{diam}_M(\Sigma_j) \leqslant C(\Sigma_j) \left(\frac{|M|}{\sigma_k}\right)^{\frac{1}{n+2}} k^{\frac{n+1}{n+2}}.$$ □

Proof of Corollary 1.10. — The Steklov eigenvalues of the cylinder $M = [0, L] \times \Sigma$ have been computed in [6, Lemma 6.1]: for $L > 0$ small enough, $\sigma_k = \sqrt{\lambda_k} \tanh(\sqrt{\lambda_k} L)$. Notice that

$$\operatorname{diam}_M\big(\Sigma \times \{0\}\big) = \operatorname{diam}(\Sigma),$$

and $\operatorname{inj}(\Sigma) \leqslant \operatorname{diam}(\Sigma)$. It follows from (2.2) that

$$\sqrt{\lambda_k}\tanh(\sqrt{\lambda_k}L) \leqslant K(n)\frac{|M|}{\operatorname{diam}(\Sigma)^2} \times \frac{1}{\min\{\operatorname{diam}(\Sigma)^n,\, \operatorname{inj}(\Sigma)^n\}} k^{n+1}$$
$$= K(n)\frac{|\Sigma|L}{\operatorname{diam}(\Sigma)^2 \operatorname{inj}(\Sigma)^n} k^{n+1}.$$

Dividing by L on each side and taking the limit as $L \to 0$ completes the proof, since for each $c > 0$ the following holds:

$$\lim_{x\to 0} \frac{c\tanh(cx)}{x} = c^2. \qquad \square$$

3. Upper bounds in terms of distortion, packing and growth

The goal of this section is to prove Theorem 1.1. We will prove the following slightly more general result.

Theorem 3.1. — *Let M be a smooth connected compact Riemannian manifold of dimension $n+1$ with boundary Σ. Let Σ_0 be a connected component of the boundary. Let Γ, N_M be growth and extrinsic packing constants for Σ_0. Let Λ be the distortion of Σ_0 in M. Then, the following holds for each $k \geqslant 1$:*

$$\sigma_k \leqslant 2^{14} N_M^3 \Gamma \Lambda^2 \frac{|M|}{|\Sigma_0|^{\frac{n+2}{n}}} k^{2/n}. \tag{3.1}$$

Theorem 1.1 follows by taking Σ_0 to be the connected component of the boundary with the largest volume and observing that in this case $|\Sigma_0| \geqslant |\Sigma|/b$.

The strategy is similar to the one used to prove [10, Theorem 1.1]. It is based on the following result, which is a simplification of [7, Lemma 2.1].

Lemma 3.2. — *Let (X, d, μ) be a complete, locally compact metric measure space, where μ is a non-atomic finite measure. Assume that for all $r > 0$, there exists an integer N such that each ball of radius r can be covered by N balls of radius $r/2$. Let $K > 0$. If there exists a radius $r > 0$ such that, for each $x \in X$*

$$\mu\big(B(x,r)\big) \leqslant \frac{\mu(X)}{4N^2K},$$

then, there exist μ-measurable subsets $A_1, \ldots, A_K$ of X such that, $\forall\, i \leqslant K$, $\mu(A_i) \geqslant \frac{\mu(X)}{2NK}$ and, for $i \neq j$, $d(A_i, A_j) \geqslant 3r$.

See [10, Lemma 4.1] and the following paragraph for a discussion of Lemma 3.2.

Proof of Theorem 3.1. — We apply Lemma 3.2 to the metric measure space (X, d, μ) where $X = M$, and $d = d_M$ is the extrinsic distance. The measure μ is associated to the boundary component Σ_0: for a Borel subset $\mathcal{O}$ of M, we take

$$\mu(\mathcal{O}) = |\Sigma_0 \cap \mathcal{O}|_\Sigma.$$

In particular, $\mu(M)$ is the usual volume $|\Sigma_0|_\Sigma$ of this component. In order to estimate σ_k, we first construct $(2k+2)$ trial functions, so we will take $K = 2k+2$. The constant N in Lemma 3.2 is $N = N_M$ for Σ_0. Let Λ be the distortion of Σ_0 in M, so that $d_\Sigma(x, y) \leqslant \Lambda\, d_M(x, y)$, for each $x, y \in \Sigma_0$. For each $x \in \Sigma_0$, this implies

$$\big\{y \in \Sigma_0 : d_M(y, x) \leqslant r\big\} \subset \big\{y \in \Sigma_0 : d_\Sigma(y, x) \leqslant \Lambda r\big\}.$$

In other words, $B^M(x, r) \cap \Sigma_0 \subset B^\Sigma(x, \Lambda r)$. Recall that Γ is the growth constant of Σ_0. That is, for each $x \in \Sigma_0$ and each $r > 0$,

$$\mu\big(B^\Sigma(x, r)\big) \leqslant \Gamma r^n.$$

This implies that for all $r > 0$,

$$\mu\big(B^M(x, r)\big) \leqslant \Gamma \Lambda^n r^n.$$

Any $r < \big(\frac{|\Sigma_0|}{4N_M^2 \Lambda^n \Gamma K}\big)^{\frac{1}{n}}$ is such that

$$\mu\big(B^M(x, r)\big) \leqslant \frac{|\Sigma_0|}{4N_M^2 K}.$$

It follows from Lemma 3.2 that there are $2(k+1)$ measurable subsets $A_i \subset \Sigma_0$, $i = 1, \ldots, 2(k+1)$, that are $3r$-separated for d_M and satisfy

$$\mu(A_i) \geqslant \frac{|\Sigma_0|}{4N_M(2k+2)} \geqslant \frac{|\Sigma_0|}{16 N_M k}.$$

Taking

$$r = \left(\frac{|\Sigma_0|}{C_2 K}\right)^{\frac{1}{n}},$$

with $C_2 = 8\Gamma\Lambda^n N_M^2$ is enough to ensure that

$$r < \left(\frac{|\Sigma_0|}{4N_M^2 \Gamma \Lambda^n K}\right)^{\frac{1}{n}}.$$

As in the proof of Lemma 2.1, we use standard trial functions f_i that are supported in the pairwise-disjoint neighborhoods $A_i^r = \big\{x \in M :$

$d_M(x, A_i) \leqslant r\}$ and have value 1 on A_i. The Rayleigh–Steklov quotient of f_i satisfies

$$R(f_i) = \frac{\int_M |\nabla f_i|^2}{\int_\Sigma f_i^2} \leqslant \frac{1}{r^2} \frac{|A_i^r|_M}{|A_i|_\Sigma}.$$

As we dispose from $2(k+1)$ subsets A_i, with A_i^r and A_j^r disjoint if $i \neq j$, $k+1$ of them, say $A_1, \ldots, A_{k+1}$ satisfy

$$|A_i^r| \leqslant \frac{|M|}{k+1} \leqslant \frac{|M|}{k}.$$

So, for the function f_i associated to A_i, $i = 1, \ldots, k+1$, we have:

$$R(f_i) \leqslant \frac{|M|}{k} \left(\frac{C_2 K}{|\Sigma_0|} \right)^{\frac{2}{n}} \frac{16 N_M k}{|\Sigma_0|}.$$

Using that $K = 2(k+1) \leqslant 4k$ and

$$C_2 = 8\Gamma\Lambda^n N_M^2,$$

we obtain

$$\begin{aligned} R(f_i) &\leqslant 16 N_M (8\Gamma\Lambda^n N_M^2)^{\frac{2}{n}} \frac{|M|}{|\Sigma_0|^{\frac{n+2}{n}}} K^{2/n} \\ &= 4^{2+3/n+2/n} N_M^{\frac{n+4}{n}} \Gamma^{2/n} \Lambda^2 \frac{|M|}{|\Sigma_0|^{\frac{n+2}{n}}} k^{2/n}. \end{aligned}$$

Because $4^{2+2/n+3/n} \leqslant 4^7 = 2^{14}$, this completes the proof. □

Proof of Corollary 1.4. — As in the proof of Corollary 1.10, we consider the Steklov eigenvalues of the cylinder $M = [0, L] \times \Sigma$. For $L > 0$ small enough, $\sigma_k = \sqrt{\lambda_k} \tanh(\sqrt{\lambda_k} L)$, and it follows from (1.2) that

$$\sqrt{\lambda_k} \tanh(\sqrt{\lambda_k} L) \leqslant 2^{16} N^3 \Gamma \Lambda^2 \frac{L|\Sigma|}{|\Sigma|^{\frac{n+2}{n}}} k^{2/n}.$$

Dividing by L on each side and taking the limit as $L \to 0$ completes the proof. □

3.1. Importance of the geometric invariants

In this last section, we discuss the effectiveness of Theorem 1.1 and Corollary 1.2: our goal is to explain why the various geometric constants play a meaningful role. To do this, we will exhibit various families of manifolds M_ε that satisfy $\sigma_1(M_\varepsilon) \xrightarrow{\varepsilon \to 0} +\infty$, while all but one of the geometric constants appearing in (3.1) or (1.3) are independent of the parameter ε.

We start by stating and proving the Lemma which was used to obtain Corollary 1.2 from Theorem 1.1.

Lemma 3.3. — *Let $\Lambda \geqslant 1$ be the distortion of the boundary Σ in M. Let N_Σ be a packing constant for (Σ, d_Σ). Then $N_M = bN_\Sigma^{\log_2(2\Lambda)}$ is a packing constant for (Σ, d_M).*

Proof. — Let $\Sigma_1, \dots, \Sigma_b$ be the connected components of Σ. Let $p \in \Sigma$ and $r > 0$. Select one point $y_j \in B^M(p,r) \cap \Sigma_j$ whenever this intersection is nonempty. Then

$$B^M(p,r) \cap \Sigma \subset \bigcup_j B^M(y_j, 2r) \cap \Sigma_j \subset \bigcup_j B^\Sigma(y_j, 2\Lambda r).$$

For each j, there exist N_Σ balls $B^\Sigma(x_i, \Lambda r)$ with centers $x_i \in \Sigma_j$, that cover $B^\Sigma(y_j, 2\Lambda r)$. Each of these is covered by N_Σ balls of radius $\frac{\Lambda r}{2}$, and repeating this process $m+1 \in \mathbb{N}$ times leads to a cover of $B^\Sigma(y_j, 2\Lambda r)$ by N_Σ^{m+1} balls of radius $\frac{\Lambda r}{2^m}$. Now, for $m > 1 + \log_2(\Lambda)$ the radius of the covering balls is smaller than $r/2$. It follows that $B^M(p,r) \cap \Sigma$ is covered by at most $N := bN_\Sigma^{m+1}$ balls of radius $r/2$:

$$B^M(p,r) \cap \Sigma \subset \bigcup_{i=1}^N B^\Sigma(x_i, r/2) \subset \bigcup_{i=1}^N B^M(x_i, r/2). \qquad \square$$

3.1.1. Importance of the distortion Λ

In [4] the second author and D. Cianci constructed a family of Riemannian metrics g_ε on a manifold M of dimension at least 4, such $\sigma_1 \xrightarrow{\varepsilon \to 0} +\infty$, while $|M|$ is uniformly bounded and the restriction of g_ε to the boundary Σ does not depend on ε. If follows from (1.3) that the distortion Λ must also become arbitrarily large as $\varepsilon \to 0$. For manifolds of dimension 3, a similar example will follow from ongoing work by the second author and Polymerakis [22].

3.1.2. Importance of the volume of M

Given a compact manifold M of dimension at least three, with boundary Σ, in [8] we constructed a family of Riemannian metrics g_ε such that $\sigma_1 \xrightarrow{\varepsilon \to 0} +\infty$, while the distortion Λ is uniformly bounded above and the restriction of g_ε to the boundary Σ does not depend on ε. If follows from (1.3) that the volume $|M|$ must also be large.

3.1.3. Importance of the number of boundary components

In [9], we constructed a sequence of compact surfaces M_ℓ with boundary such that $\sigma_1 \xrightarrow{\ell\to\infty} +\infty$, and for each ℓ:

$$|\partial M_\ell| = |M_\ell| = \Lambda = 1.$$

Because the boundary is one-dimensional, the packing constant N_Σ and the growth constant Γ are also independent of ℓ. It follows from (1.3) that the number of connected components b of the boundary must satisfy $b \to \infty$. A similar construction works in arbitrary dimension. Indeed, the focus of [9] was on surfaces, and the examples were constructed by gluing a finite number of building blocks together following the pattern dictated by a graph. By using building blocks of arbitrary dimension $\geqslant 2$, we would obtain a similar construction and the proofs are exactly the same as in dimension 2.

3.1.4. Importance of the volume of the boundary

Let M be a closed Riemannian manifold, and consider the perforated domain $M_\varepsilon := M \setminus B(p,\varepsilon)$. Then $b = 1$ while N_Σ, Λ, Γ and $|M|$ are uniformly bounded. It is proved in [2] that $\sigma_1 \xrightarrow{\varepsilon\to 0} +\infty$.

3.1.5. Importance of the packing and growth constants

If Theorem 1.1 was true with Γ and N_M removed, then Corollary 1.4 would hold without these constants appearing. That is, a universal upper bound on $\lambda_1(\Sigma)|\Sigma|^{2/n}$ would be provided for each compact manifold Σ. However it was proved by the first author and J. Dodziuk [5] that any closed manifold Σ of dimension larger than 2 admits a Riemannian metric g with arbitrarily large Laplace spectral gap λ_1.

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Bruno COLBOIS
Université de Neuchâtel
Institut de Mathématiques
Rue Emile-Argand 11
CH-2000 Neuchâtel, Switzerland
bruno.colbois@unine.ch

Alexandre GIROUARD
Département de mathématiques et de statistique
Pavillon Alexandre-Vachon, Université Laval
Québec, QC, G1V 0A6, Canada
alexandre.girouard@mat.ulaval.ca