Université Grenoble Alpes

# ANNALES DE L'INSTITUT FOURIER

Ian Agol & Franco Vargas Pallete

**Peripheral birationality for 3-dimensional convex co-compact $\mathrm{PSL}_2\,\mathbb{C}$ varieties**

Article à paraître, mis en ligne le 22 septembre 2025, 23 p.



Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
Article à paraître
*Mis en ligne le 22 septembre 2025.*

# PERIPHERAL BIRATIONALITY FOR 3-DIMENSIONAL CONVEX CO-COMPACT $\mathrm{PSL}_2\,\mathbb{C}$ VARIETIES

**by Ian AGOL & Franco VARGAS PALLETE (*)**

Abstract. — Let $M$ be a hyperbolizable 3-manifold with boundary, and let $\chi_0(M)$ be a component of the $\mathrm{PSL}_2\,\mathbb{C}$-character variety of $M$ that contains the convex co-compact characters. We show that the peripheral map $i_*\colon \chi_0(M) \to \chi(\partial M)$ to the character variety of $\partial M$ is a birational isomorphism with its image, and in particular is generically a one-to-one map. This generalizes work of Dunfield (one cusped hyperbolic 3-manifolds) and Klaff–Tillmann (finite volume hyperbolic 3-manifolds). We use the Bonahon–Schläfli formula and volume rigidity of discrete co-compact representations.

Résumé. — Soit $M$ une variété hyperbolisable de dimension 3 à bord, et soit $\chi_0(M)$ un composant de la variété de caractères $\mathrm{PSL}_2\,\mathbb{C}$ de $M$ qui contient les caractères co-compacts convexes. Nous montrons que l'application périphérique $i_*\colon \chi_0(M) \to \chi(\partial M)$ à la variété de caractères de $\partial M$ est un isomorphisme birationnel avec son image, et en particulier est génériquement injective. Cela généralise les travaux de Dunfield (variétés 3 hyperboliques cuspidées) et de Klaff–Tillmann (variétés 3 hyperboliques à volumes finis). Nous utilisons la formule de Bonahon–Schläfli et la rigidité volumique des représentations cocompactes discrètes.

## 1. Introduction

Given a connected 3-manifold $M$ with boundary and a representation $\rho\colon \pi_1(M) \to G$, $G$ a Lie group (which will usually be $\mathrm{PSL}_2(\mathbb{C})$), it is natural to ask to what extent $\rho$ is determined by $\rho\big|_{\pi_1(\partial M)}$? If the interior of $M$ admits a convex cocompact hyperbolic metric with holonomy $\rho$, then it is known that $\rho\big|_{\pi_1(\partial M)}$ determines $\rho$—in fact, the conformal structure of the boundary determines the hyperbolic metric by results of Ahlfors–Bers. For a manifold $M$ with one cusp, Dunfield [9] proved that a main

*Keywords:* character variety, hyperbolic volume and rigidity, Culler–Shalen theory.
2020 *Mathematics Subject Classification:* 57M50, 30F40.
(*) I. Agol's research was supported by a Simons Investigator Grant and the Mathematical Sciences Research Institute. F. Vargas Pallete's research was supported by NSF grant DMS-2001997.

component of the character variety (the Zariski component containing a discrete faithful representation) maps birationally to a factor of the $A$-polynomial. In particular, for a Zariski open subset of this component, the representation will be determined by its restriction to the boundary. This was extended by Klaff and Tillmann [14] to the multiple cusped case (see also Francaviglia [10], Francaviglia–Klaff [11]). However, in general there may be families of representations which are constant on $\pi_1 \partial M$.

We generalize the result of Dunfield to convex cocompact hyperbolic manifolds. In this case, the space of discrete faithful representations is determined by the Teichmüller space of the boundary, and its dimension is half the dimension of the character variety of the boundary. Because of the dimension agreeing, we know that the map is generically finite-to-one. We show that the induced map by inclusion from the main component of the character variety to its image in the character variety of the boundary is a birational map, so it is actually generically one-to-one.

The tools that Dunfield uses are the Schläfli formula and representation rigidity of Gromov which we adapt to the infinite volume case. An obvious issue with geometrically finite manifolds of negative Euler characteristic is that the volume is infinite. There is the notion of convex core volume, but this is only well-defined for discrete faithful representations. Rather than try to extend this to non-discrete representations, we choose pleated surfaces with the same bending locus to define a notion of volume which depends only on the restriction of the representation to the boundary. Although such volumes require some choices, and such choices are not defined everywhere, nevertheless we can show that there is a notion of volume determined by the restriction to the boundary, making use of a version of Schläfli's formula due to Bonahon [4]. The other tool, representation rigidity, then is proved at countably infinitely many representations for which there is an extension to a finite-volume hyperbolic orbifold representation. One could probably also extend the proof of volume rigidity to all representations whose restriction to the boundary is discrete and faithful, but rather than prove such a result, we decided to use what tools were already at hand. Once we have extended these two tools, the proof proceeds similarly to that of Dunfield. Suppose that a geometrically finite representation and another representation have the same peripheral holonomy. Then the volumes of the representations are the same. Hence by volume rigidity, they are both discrete, and hence conjugate.

### 1.1. Examples

Here are some examples for which the result can be proved more easily. Suppose one has a compression body $M$. The 3-manifold $M$ is constructed by attaching 2-handles to a surface $\Sigma$. Then the fundamental group of $M$ is the free product of the surface groups $\partial M \setminus \Sigma$. The compressible boundary component $\Sigma$ surjects the fundamental group of $M$, and hence the representation variety of $M$ embeds into the representation variety of the boundary component $\Sigma$.

A slightly less non-trivial example is that of a book of I-bundles. We say that $M = \mathcal{B} \cup_{\mathcal{A}} \mathcal{C}$ is a *book of I-bundles* if:

(1) $\mathcal{B}$ is a $I = [0,1]$ bundle over a (possibly disconnected) compact-with-boundary surface $\Sigma$. Each component of $\Sigma$ is referred as a *page*, inspired by the structure of the trivial case $\mathcal{B} = \Sigma \times [0,1]$.
(2) $\mathcal{C}$ is a disjoint collection of solid tori $D^2 \times S^1$. Each core curve of $\mathcal{C}$ is referred as a *binding*.
(3) $\mathcal{A} \subseteq \partial\mathcal{B}$ is a collection of annuli obtained by restricting the bundle to $\partial\Sigma$. We identify the components of $\mathcal{A}$ to homotopically non-trivial disjoint annuli in $\partial\mathcal{C}$.

In this case, when the representation of $\pi_1(M)$ is faithful, it determines the representations of each page. For each binding, there will be boundary components overlapping the pages meeting that binding. The representations of these boundary components determine how to "glue" together the representations of adjacent pages. Hence the boundary holonomy determines the full representation generically (since faithful representations are generic in the main component).

### 1.2. Outline

This paper is organized as follows. Section 2 explains the main tools we will use. Section 2.1 reviews $\mathrm{PSL}_2\,\mathbb{C}$ character varieties. Section 2.2 constructs the pleated surfaces we will use to compute the volume associate to a character. Sections 2.3, 2.4 deal with the definition of bending cocycle and its implementation in the Bonahon–Schläfli formula for change of volume. Sections 2.2, 2.3 and 2.4 start each with a brief outline of work of Bonahon [2, 4] followed by a formal description for our particular application. Section 3 uses all these tools and volume rigidity of co-compact characters to prove our main result (stated below). We finish with some remarks where we discuss the surjectivity and finite-to-one nature of the peripheral map, and also how to generalize the result for geometrically finite characters.

Main theorem. — *Let $M$ be a hyperbolizable compact 3-manifold with boundary. Let $\chi_0(M)$ be the connected component of the discrete and faithful representations. Then the map $i_*\colon \chi_0(M) \to \chi(\partial M)$ is a birational isomorphism onto its image.*


## Acknowledgments

Both authors are thankful to F. Bonahon for his interest and helpful comments.


## 2. Background

### 2.1. Character variety

For a comprehensive study of character varieties of 3-manifold groups, we refer the reader to [7] for $\mathrm{SL}_2\,\mathbb{C}$ characters and [5] for $\mathrm{PSL}_2\,\mathbb{C}$ characters. Here we present a factual recollection of the relevant definitions and results from [5].

Let $G$ be a finitely generated group. A $\mathrm{PSL}_2\,\mathbb{C}$-representation is a homomorphism $\rho\colon G \to \mathrm{PSL}_2\,\mathbb{C}$. The $\mathrm{PSL}_2\,\mathbb{C}$ *representation variety* $R(G)$ is defined by

$$R(G) = \left\{\rho \mid \rho\colon G \longrightarrow \mathrm{PSL}_2\,\mathbb{C} \text{ homomorphism}\right\}$$

In order to discuss the algebraic structure on $R(G)$, we recall some definitions and properties from algebraic geometry (see [12]). An *affine algebraic set* in $\mathbb{C}^m$ is the zero locus of a finite collection of polynomials with complex coefficients. Given $U, V$ affine algebraic sets, we say that $f\colon U \to V$ is a *regular map* if $f\colon U \to \mathbb{C}^m \supseteq V$ has polynomial coordinates and $f(U) \subseteq V$. Regular maps are in bijection (by taking the pull-back) with homomorphism between the coordinate rings of regular functions $A[V] \to A[U]$.

We say that two affine algebraic sets $U \subseteq \mathbb{C}^m$, $V \subseteq \mathbb{C}^n$ are isomorphic if there exists regular maps $f\colon U \to V$, $g\colon V \to U$ with polynomial coordinates so that $g \circ f = \mathrm{id}_U$, $f \circ g = \mathrm{id}_V$. An affine algebraic set is called *irreducible* if it is the zero locus of a finite collection of polynomials that generate a prime ideal. Every affine algebraic set $U$ is canonically decomposed (respecting isomorphisms) as the finite union of (irreducible) affine algebraic varieties, each of which is known as an *irreducible component* of $U$.

The adjoint representation $\mathrm{PSL}_2\,\mathbb{C} \to \mathrm{Aut}(\mathrm{sl}_2\,\mathbb{C})$ realizes $\mathrm{PSL}_2\,\mathbb{C}$ as an affine algebraic set. Hence if we take $\{g_1, \ldots, g_n\}$ a collection of generators of $G$, the map $R(G) \to (\mathrm{PSL}_2\,\mathbb{C})^n$, $\rho \mapsto \big(\rho(g_1), \ldots, \rho(g_n)\big)$ identifies $R(G)$ with an affine algebraic set. Since a different choice of generators will produce an isomorphic affine algebraic set, we identify $R(G)$ with an isomorphism class of affine algebraic sets.

We say two representations $\rho_1, \rho_2 \in R(G)$ are *conjugated* if there exists $g \in \mathrm{GL}_2\,\mathbb{C}$ so that $\rho_2 = g^{-1}\rho_1 g$. This defines an equivalence relation in $R(G)$, where any pair of equivalent representations belong to the same irreducible component of $R(G)$.

We say that a representation $\rho \in R(G)$ is *irreducible* if the only subspaces of $\mathbb{C}^2$ invariant by $\rho(G)$ are $\{0\}$ and $\mathbb{C}^2$, otherwise we say that $\rho$ is *reducible*. Irreducibility is preserved by conjugation. Moreover, the set of reducible representations is a subvariety of $R(G)$. Hence by an irreducible component of $R(G)$ of irreducible representations we refer to an irreducible component of $R(G)$ (in the algebro-geometric sense) so that the subvariety of reducible representations is proper.

We define the *character variety* $\chi(G)$ as the algebro-geometric quotient $R(G)/\,\mathrm{PSL}_2\,\mathbb{C}$ by considering the affine algebraic set matching the subring $A^{\mathrm{PSL}_2\,\mathbb{C}}\big[R(G)\big] \subseteq A\big[R(G)\big]$ of regular functions invariant by the natural $\mathrm{PSL}_2\,\mathbb{C}$ action. Hence we have a surjective regular map $\chi\colon R(G) \to \chi(G)$ that is constant in the $\mathrm{PSL}_2\,\mathbb{C}$ orbits, and for any $g \in G$ we have that the map $\tau_g\colon \chi(G) \to \mathbb{C}$, $\chi_\rho \mapsto \big(\mathrm{tr}\big(\rho(g)\big)\big)^2$ is a well-defined regular map.

We introduce some definitions for rational maps between affine algebraic varieties.

Definition 2.1. — *Let $X \subseteq \mathbb{C}^m$, $Y \subseteq \mathbb{C}^n$ be affine algebraic varieties. We say that $\phi = (\phi_1, \ldots, \phi_n)$ is a rational map from $X$ to $Y$ if each $\phi_1, \ldots, \phi_n$ is given by a rational function in $X$, and whenever defined, $\phi(x) \in Y$. We denote this by $\phi\colon X \dashrightarrow Y$. We say that $\phi\colon X \dashrightarrow Y$ is dominant if, given $U \subseteq X$ Zariski open set where $\phi$ is defined, then the Zariski closure of $\phi(U)$ is equal to $Y$. We say that $\phi$ is birational if there exists inverse $\psi\colon Y \dashrightarrow X$. In such case we say that $X$ and $Y$ are birationally isomorphic.*

Now we consider the convex co-compact $\mathrm{PSL}_2\,\mathbb{C}$ representations of a 3-manifold $M$ with boundary. These representations are irreducible and it is known (see for instance [8, Section 6]) that their Zariski closure in $R\big(\pi_1(M)\big)$ is an irreducible component $R_0$, so their characters have Zariski closure an irreducible component of $\chi(M)$. We denote this component by $\chi_0(M)$. By fixing paths from a basepoint to each component of the boundary $\partial M = \Sigma_1 \sqcup \cdots \sqcup \Sigma_k$ we have the regular maps between representation varieties induced by the inclusion $\partial M \hookrightarrow M$

$$i_{\ell *} \colon R\big(\pi_1(M)\big) \longrightarrow R\big(\pi_1(\Sigma_\ell)\big), \quad 1 \leqslant \ell \leqslant k.$$

Hence by taking the pullback we have ring homomorphisms

$$\varphi_\ell \colon A\big[R\big(\pi_1(\Sigma_\ell)\big)\big] \longrightarrow A\big[R\big(\pi_1(\partial M)\big)\big].$$

It is not hard to see that $\varphi_\ell$ is equivariant with respect to the natural $\mathrm{PSL}_2\,\mathbb{C}$ action on each ring, so in particular satisfies

$$\varphi_\ell\Big(A^{\mathrm{PSL}_2\,\mathbb{C}}\big[R\big(\pi_1(\Sigma_\ell)\big)\big]\Big) \subseteq A^{\mathrm{PSL}_2\,\mathbb{C}}\big[R\big(\pi_1(M)\big)\big].$$

This means that we have regular maps (which we also denote by $i_{\ell *}$) $i_{\ell *} \colon \chi\big(\pi_1(M)\big) \to \chi\big(\pi_1(\Sigma_\ell)\big)$ that make the following diagram commute.

$$\begin{array}{ccc} R\big(\pi_1(M)\big) & \xrightarrow{i_{\ell *}} & R\big(\pi_1(\Sigma_\ell)\big) \\ \chi\downarrow & & \downarrow\chi \\ \chi\big(\pi_1(M)\big) & \xrightarrow{i_{\ell *}} & \chi\big(\pi_1(\Sigma_\ell)\big) \end{array}$$

We should observe that while each $i_{\ell *}$ at the level of representation varieties depended on the choice of basepoint and paths in $M$, the maps $i_{\ell *}$ at the level of character varieties are well-defined, since a change of paths conjugates representations. We define then the *peripheral map* of $\chi(M)$ as the regular map

$$i_* = (i_{1*}, \ldots, i_{k*}) \colon \chi\big(\pi_1(M)\big) \longrightarrow \chi(\partial M) := \chi\big(\pi_1(\Sigma_1)\big) \times \cdots \times \chi\big(\pi_1(\Sigma_k)\big).$$

By Ahlfors–Bers [1] we have that $i_*$ evaluated at a convex co-compact $\mathrm{PSL}_2\,\mathbb{C}$ character of $\pi_1(M)$ gives a collection of convex co-compact $\mathrm{PSL}_2\,\mathbb{C}$ characters for each $\Sigma_\ell$. This collection of characters have closure included in the convex co-compact irreducible component of $\chi(\partial M)$, which we denote by $\chi_0(\partial M)$. Observe that the map $i_* \colon \chi_0(M) \to \overline{i_*\big(\chi_0(M)\big)} \subset \chi_0(\partial M)$ is a diffeomorphism between convex co-compact characters and their image. It is well-known (see for instance [13, Chapter I.3]) that $\dim_{\mathbb{C}}\big(i_*\big(\chi_0(M)\big)\big) \leqslant \dim_{\mathbb{C}}\big(\chi_0(M)\big)$, as dimension corresponds to the Krull dimension of local rings, and since $i_*$ has dense image then the induced map at the level of local rings is injective. Since dimension agrees with the dimension of smooth points (such as the convex co-compact characters), we know that the map

$$i_* \colon \chi_0(M) \longrightarrow \overline{i_*\big(\chi_0(M)\big)} \subset \chi_0(\partial M)$$

is a dominant (i.e. dense image) regular map between algebraic varieties of the same dimension.

### 2.2. Pleated surfaces

Let $S$ be a genus $g$ surface. Let us fix an auxiliary hyperbolic metric $m_0$ and $\Gamma = \{\gamma_i\}_{1\leqslant i\leqslant 3g-3}$ a maximal collection of oriented, disjoint, essential, pairwise non-isotopic, simple closed curves (i.e. an oriented pants decomposition). Take $\lambda_0$ to be the maximal lamination extension of $\Gamma$ so that any leaf of $\lambda_0$ that accumulates at $\gamma_i \in \Gamma$ does it in the direction of the orientation (see Figure 2.1). Equivalently, the orientation for each $\gamma_i \in \Gamma$ gives a preferred endpoint for any lift $\widetilde{\gamma_i}$ of $\gamma_i$ in the universal cover. Then any ideal triangle in the lift $(\widetilde{S}, \widetilde{\lambda_0})$ uses only preferred endpoints as ideal vertices.

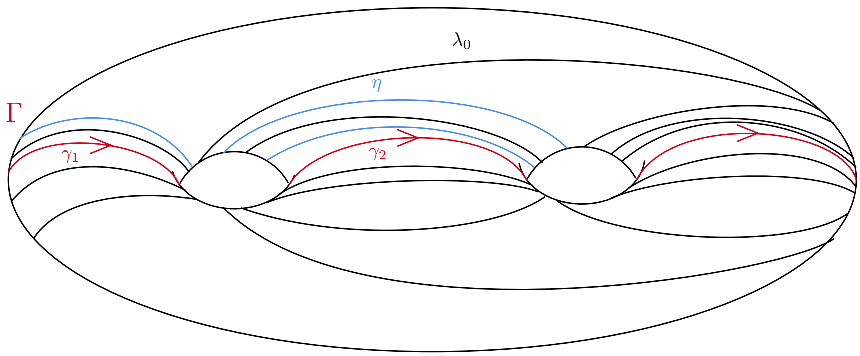


*Figure 2.1. Lamination $\lambda_0$ defined by the collection of oriented curves $\Gamma$. The component $\eta$ of $\lambda_0$ accumulates at $\gamma_1, \gamma_2$, according to their orientation.*

Take representations $\rho_t\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ so that $\rho(\gamma_i)$ is neither parabolic nor the identity (so there is a well-defined axis), and the endpoints of $\rho(\gamma_i), \rho(\gamma_j)$ are distinct for $i \neq j$. Assume as well that we have a given equivariant orientation/endpoint for axis of each lift of $\gamma_i$, which we denote by $\zeta$. If so we say that $\rho_t$ is $\Gamma$-adapted and we define the (abstract) pleated surface $(\widetilde{f}_\zeta = \widetilde{f}\colon \widetilde{S} \to \overline{\mathbb{H}^3}, \rho)$ as follows.

(1) For any lift $\widetilde{\gamma_i}$ of $\gamma_i$, map its preferred endpoint to the corresponding endpoint given by $\zeta$.

(2) For any lift of a component of $\lambda_0 \setminus \Gamma$, send it to the geodesic in $\mathbb{H}^3$ joining the corresponding preferred endpoints. This is possible since by assumption the preferred endpoints in $\partial_\infty \mathbb{H}^3$ are distinct.
(3) For any lift of an ideal triangle in $\widetilde{S} \setminus \widetilde{\lambda_0}$, send it to the ideal triangle spanned by the distinct corresponding endpoints.
(4) Finally, extend continuously to $\widetilde{\Gamma}$. This makes the map $\widetilde{f} \colon \widetilde{S} \to \overline{\mathbb{H}^3}$ equivariant by $\rho$.

Note that if every $\rho(\gamma_i)$ was loxodromic and the orientation $\zeta$ agrees with the orientation of $\Gamma$, this will be the classical notion of pleated surface. On the other hand if for some $\rho(\gamma_i)$ the orientations of $\zeta$ and $\Gamma$ disagree, then $\widetilde{f}$ is the classical notion of pleated surface for the lamination $\lambda_1$, where $\lambda_1$ is obtained from $\Gamma$ after changing the orientations to have full agreement with $\zeta$.

If $\rho(\gamma_i)$ is elliptic then the situation is a bit more delicate. The continuous extension of step (4) will send any lift $\widetilde{\gamma_i}$ entirely to the preferred endpoint given by $\zeta$. This means that under the pull-back metric on $S$ given by $\widetilde{f}$ we have that $\ell(\gamma_i) = 0$. This means that the pull-back metric of $S$ has a cusp at each elliptic $\gamma_i$ and hence is of finite type, which is consistent with the fact that the complex length of $\rho(\gamma_i)$ is purely imaginary.

## 2.3. Bending transverse cocycle

In [2, Section 6], Bonahon defines the bending transverse cocycle of a pleated surface. This means that for each arc $\alpha$ transverse to a lamination $\lambda$ we have a number $\beta(\alpha) \in \mathbb{R}/2\pi\mathbb{Z}$ called the *bending*, which represents the amount of turning made by the pleated surface between the geodesic faces containing the endpoints of $\alpha$. The cocycle $\beta$ is additive under finite subdivision of a transverse arcs and can be defined as follows. Consider all the geodesics of the lamination that $\alpha$ crosses, and the geodesic faces (or rather plaques, as denoted by Bonahon) of the pleated surface going between those geodesics. The boundary of the plaques define two curves $\eta_{1,2}$ in $\partial_\infty \mathbb{H}^3$. Then the bending of $\alpha$ is defined as the difference of angles between the end plaques minus the integral of the signed curvature of either $\eta_{1,2}$ (see [2, Lemma 23]).

For our fixed lamination $(\widetilde{S}, \widetilde{\lambda_0})$ we can finitely decompose any transverse arc into smaller transverse arcs so that each smaller arc intersects $\widetilde{\Gamma}$ at most once. This simplifies the description of the bending cocycle as follows.

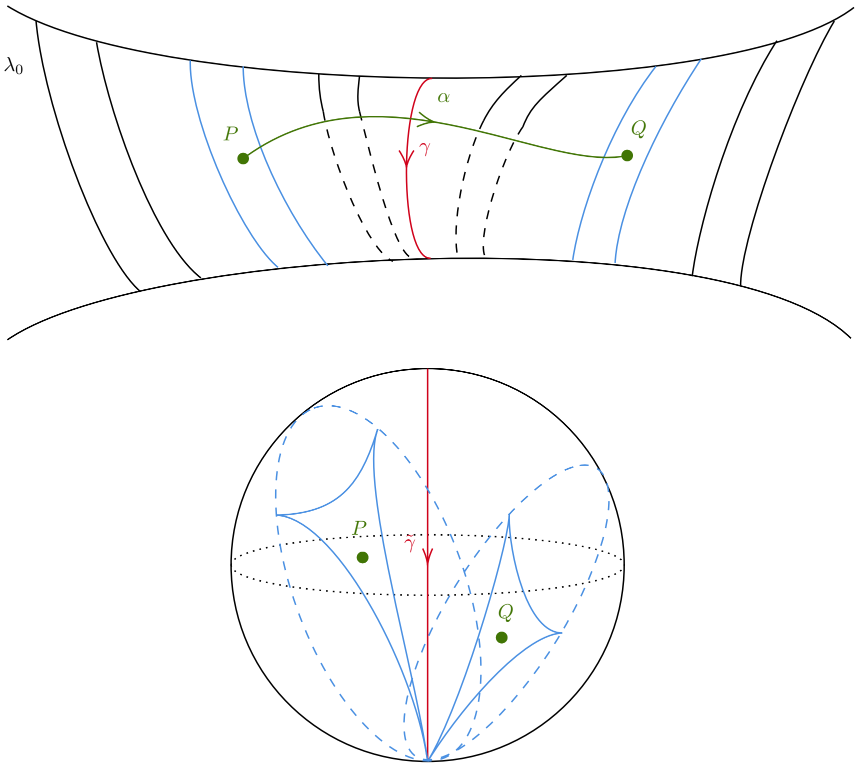


*Figure 2.2. Bending of a transverse curve $\alpha$ intersecting a unique component $\gamma$ of $\Gamma$.*

If an arc $\alpha$ does not intersect $\widetilde{\Gamma}$ then it only intersects finitely many geodesic lines of $\widetilde{\lambda_0}$. Then the bending is just given by the sum of the finitely many exterior dihedral angles involved.

Now say that $\alpha$ intersects exactly one component $\widetilde{\gamma_i}$ of $\widetilde{\Gamma}$. Furthermore, assume that $\alpha$ only intersects leaves of $\lambda_0$ that accumulate in $\gamma_i$. Then all plaques involved contain the preferred endpoint of $\widetilde{\gamma_i}$. This means that one of the two curves $\eta_{1,2}$ degenerates to a point, so we are only left with the angle of the two intersecting end plaques.

Now it is easy to see that for our definition of (abstract) pleated surface, the two above definitions are well-defined and are additive under finite subdivision. Then for a general transverse arc $\alpha$ we can take any finite subdivision so every arc is of one of the two cases analyzed above, and define the bending of $\alpha$ as the sum of bending. Since the cocycle was already additive between the two types, this shows that the bending cocycle is well-defined and additive under finite subdivision. Moreover, this implies that for a smooth 1-parameter family of $\Gamma$-adapted representations $\rho_t$ where the orientation/endpoint choice $\zeta$ varies continuously, the bending varies smoothly. This is because the family of preferred endpoints will vary smoothly, and the bending of any arc is decomposed as the finite sum of finitely many angles of planes defined by preferred endpoints. And while the bending cocycle is defined only up to an integer multiple of $2\pi$, its derivative is well-defined over $\mathbb{R}$.

For the end of this subsection, let us define the length of the derivative of the bending lamination, which we will denote by $\ell(b')$. In [2, Section 3] this quantity is computed by taking rectangles whose vertical sides are transverse to the lamination and whose horizontal sides are disjoint from it, integrating first along vertical segments a transverse Hölder distribution defined in terms of the derivative of the bending cocycle (see also [3]) and then integrating horizontally with respect the length measure. In our case for the lamination $\lambda_0$ we will simplify the definition as follows. Based at each endpoint chosen by $\zeta$ we pick a family of $\rho_0$-equivariant horoballs. Using such horoballs we calculate the (signed) length of a leave in $\lambda_0 \setminus \gamma$ as the signed length of the segment determined by the intersection with the fixed system of horoballs. Hence we define $\ell(b')$ as

$$\ell(b') := \sum_{l \in \lambda_0 \setminus \gamma} \ell(l).b'(l)$$

where $\ell(l)$ is the signed length we have defined for the leave $l$ and $b'(l)$ is the derivative of the bending angle along $l$. One can prove that $\ell(b')$ does not depend on the particular choice of equivariant horoballs, fact that we will see as a consequence of Proposition 2.2.

## 2.4. Volume variation

Let $\rho_t$ be a smooth 1-parameter family of $\Gamma$-adapted representations. Let $f_t$ be the $\lambda_0$ abstract pleated surface, $m_t$ the metric induced from $\mathbb{H}^3$ in $S$ by $f_t$, and let $V_t$ be the volume of a 3-chain bounded by $f_t$. Then we wish to establish that

$$V_t' = \frac{1}{2}\ell_t(b_t'),$$

where $b_t$ is the bending cocycle of $f_t$, and $\ell_t$ is the length of the (real-valued) transverse cocycle $b_t'$ with respect to the induced metric $m_t$. This is known for classical pleated surfaces as the Bonahon–Schläfli formula, as proved by Bonahon in [4, Theorem 1]. We will explain the outline of Bonahon's proof while providing a proof for our case under the definition for $\ell_t(b_t')$ we established at the end of the previous section. The main challenge is to carry out Bonahon's strategy in the case when elliptic transformations are involved.

Cover $\lambda_0$ by geodesic rectangles $R_1^0, \dots, R_m^0$ with disjoint interior, so that the components of $\lambda_0 \cap R_i^0$ are parallel to opposite horizontal sides of the rectangle. We refer to these rectangles as *horizontal rectangles*. Each horizontal rectangle $R_i^0$ can be collapsed vertically to an edge in order to obtain an embedded graph in $S$, and we can complete that graph to a triangulation $T_0$ of $S$, which lifts to a triangulation $\widetilde{T_0}$ of $\widetilde{S}$. Take then a $\rho_0$-equivariant map $g\colon \widetilde{S} \to \mathbb{H}^3$ that is polyhedral with respect to $\widetilde{T_0}$ and homotopic to $f_0$ through $\rho_0$-equivariant maps. For $t$ small, take $m_t$-geodesic rectangles $R_i^t$ so that the analogous statement holds for $\lambda_t$, $R_i^t$ is smooth on $t$, and up to isotopy on $S$ we have that $T_t = T_0$. Consider then $\rho_t$-equivariant maps $g_t\colon \widetilde{S} \to \mathbb{H}^3$ that are polyhedral with respect to $\widetilde{T_0}$ and homotopic to $f_t$ through $\rho_t$-equivariant maps. By the Schläfli formula for polyhedral maps, we can calculate the variation of volume for the quotient of a 3-chain bounded by $g_t$. Hence we can reduce the problem to calculate the variation of volume for the quotient of a $\rho_t$-equivariant homotopy between $f_t, g_t$. Since we can take maps $h_t\colon S \to S$ homotopic to the identity so that $h_t(R_i^t)$ is an arc of $T$, the volume of the quotient of the homotopy between $g_t$ and $g_t \circ \widetilde{h_t}$ is equal to 0. Then we can further reduce to calculate the variation of volume for the quotient of a $\rho_t$-equivariant homotopy $H_t$ between $f_t, g_t \circ \widetilde{h_t}$.

The next step is to divide the homotopy $H_t$ into a family of $\rho_t$ equivariant polyhedral pieces. Fixing a rectangle $R_i^t$, and given $g_t \circ \widetilde{h_t} = H_t(\,\cdot\,, 0)$ sends $R_i^t$ to a geodesic segment, we want extend $H$ to $R_i^t \times [0,1]$ by geodesic segments so that $f_t = H_t(\,\cdot\,, 1)$. In order to do so, for each component $R$ of $R_i^t \setminus \lambda_t$, we define $H_t\big(R \times [0,1]\big)$ so that decomposes into the union of a pyramid with square basis given by $R$, and a tetrahedra that shares a side with the pyramid (see Figure 2.3). Because $f_t, g_t \circ \widetilde{h_t}$ are $\rho_t$-equivariant, this family of polyhedra $P_t$ is $\rho_t$-equivariant. Since $(\lambda_t \cap R_i^t) \times [0,1]$ has 3-dimensional Lebesgue measure 0, we can focus solely in the family of polyhedra $P_t$ in order to calculate volumes.

Recall that the variation of volume of a polyhedra is given by the sum of half-lengths of edges times the variation of the dihedral angle. Then Bonahon [4] argues that for any interior edge and for any edge shared

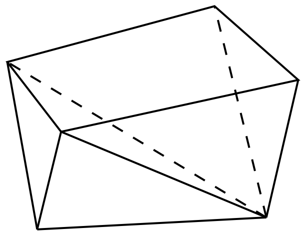

*Figure 2.3. 3-chain obtained as the union of a pyramid and a prism.*

with $g_t \circ \widetilde{h_t}$, the different contributions cancel out. As for edges appearing in $f_t$, their sum can be reinterpreted as the half-length of the variation of the bending cocycle. This is the delicate part of the argument, because as Bonahon points out, edges are not locally finite, so appropriate summability and convergence should be proved. Bonahon's statement covers the case when all $\gamma \in \Gamma$ are loxodromic, so we are left to justify when some $\gamma$ is elliptic given our choice of $\lambda_0$. Hence we concentrate on this case.

For our choice of $\lambda_0$ the polyhedral subdivision can be simplified in such a way that the questions of summability and convergence are easier to conclude. In the following proposition we prove the variational formula for volume while considering $S = \partial M$ and a path $\rho_t$ of $\mathrm{PSL}(2,\mathbb{C})$ representation of $\pi_1(M)$ that are adapted with respect an oriented pants decomposition. Under such constraints we define $V_t$ the volume enclosed by a abstract pleated surface on $S$ by extending equivariantly the pleated maps $f_t$ to $\widetilde{M}$ (the universal cover of $M$) and integrating along $M$ the pull-back of the volume form of $\mathbb{H}^3$ by $f$. As the boundary map is fixed equal to the pleated map, the volume $V_t$ is well-defined.

Proposition 2.2. — *Let $S$ be a closed hyperbolic surface (not necessarily connected), let $\Gamma$ be an oriented pants decomposition of $S$ and $\lambda_0$ the maximal lamination extension described in Section 2.2. Let $M$ be a closed 3-manifold whose boundary we identify with $S$, and assume that $\rho_t$ is a smooth 1-parameter family of representations of $\pi_1(M)$ to $\mathrm{PSL}(2,\mathbb{C})$ so that if they are restricted to $S$ they are $\Gamma$-adapted, and let $f_t$ be the $\rho_t$-equivariant (abstract) pleated surface along $\lambda_0$. If $V_t$, $m_t$ and $b_t$ are the volume bounded, induced metric in $S$, and the bending cocycle of $f_t$; then*

$$V_t' = \frac{1}{2}\ell_t(b_t'),$$

*where $\ell_t$ is the length of the (real-valued) transverse cocycle $b_t'$ with respect to the induced metric $m_t$.*

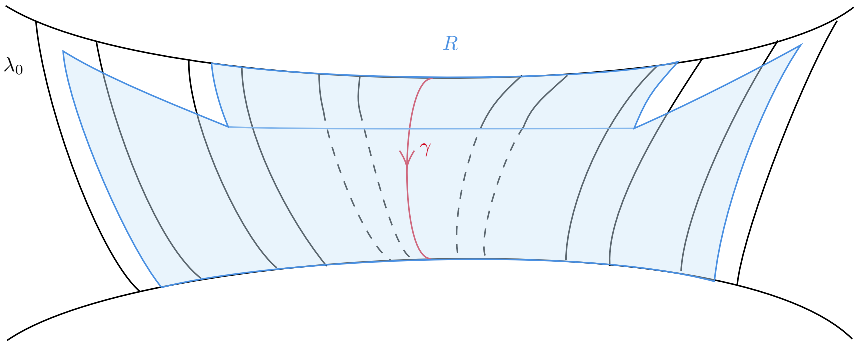


*Figure 2.4. Rectangle $R$ going around the closed geodesic $\gamma$.*

*Proof.* — Observe that it is sufficient to prove the statement for $t = 0$.

Cover each closed curve $\gamma$ in $\Gamma$ by the closure of a single horizontal rectangle $R^t_\gamma$ (see Figure 2.4). In the case that $\gamma$ is elliptic the vertical sides are not well-defined segments, but we have a well-defined pair of cusped cylinders which union we still denoted by $R^0_\gamma$ (similar for any other $t$ where $\rho_t(\gamma)$ is elliptic). Observe that in the complement of $\bigcup_{\gamma\in\Gamma} R^t_\gamma$ the lamination $\lambda_0$ is the finite union of geodesic segments, which we cover by finitely many horizontal rectangles following the pattern in Figure 2.5. We label these rectangles as $R^t_1, \ldots, R^t_{6g-6}$, so our total collection of horizontal rectangles is given by $\{R^t_\gamma\}_{\gamma\in\Gamma} \cup \{R^t_i\}_{1\leqslant i\leqslant 6g-6}$.

We take then the vertical collapsing of the horizontal rectangles and complete it to a graph $T$ given as the 1-skeleton of a triangulation, so that in each pair of pants in the complement of $\Gamma$ the graph $T$ is given as in Figure 2.6. In particular, $T$ contains a copy of each $\gamma \in \Gamma$. Our next goal is to define the $\rho_t$-equivariant polyhedral maps $g_t \colon \widetilde{S} \to \mathbb{H}^3$ bent along $T$, in order to differentiate the volume bounded by $g_t$ and the volume bounded between $f_t$ and $g_t \circ \widetilde{h}_t$, where $\widetilde{h}_t$ is the lift of the map that collapses the horizontal rectangle to $T$. In order to do so, we will further specify our choices.

Take $R^0_\gamma$ such that each horizontal side extends to a geodesic ray with the same endpoint as $\gamma$. In the collection of axis $\widetilde{\gamma} \subset \mathbb{H}^3$ under $\rho_0$ of $\gamma$ take a point $A_\gamma$ so that any $\rho_0(\gamma)$ translate of $A_\gamma$ does not belong to neither a $\rho_0$ lift of $\lambda_0 \setminus \gamma$ nor to the image of any lift of a vertex of $R^0_\gamma, R^0_i$ by $f_0$. This is possible because we are in the open set where $\widetilde{\gamma}$ is not included in the $\rho_0$ lift of $\lambda_0 \setminus \gamma$. Define $g_0$ on the 0-skeleton of $\widetilde{S}$ as a $\rho_0$-equivariant map that sends lifts of a vertex $v \in T$ in $\gamma$ to the $\rho\big(\pi_1(S)\big)$ orbit of $A_\gamma$. As $T$ is the

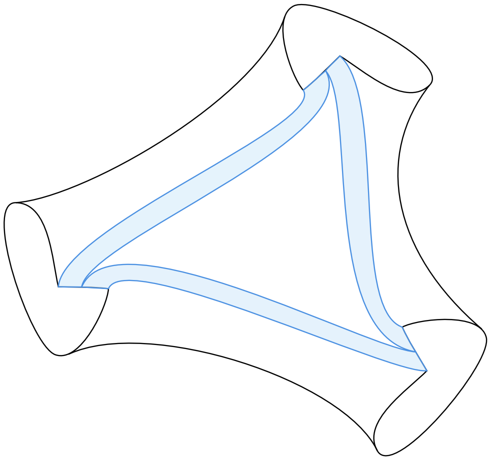

*Figure 2.5. Rectangles on the complement of the cuffs.*

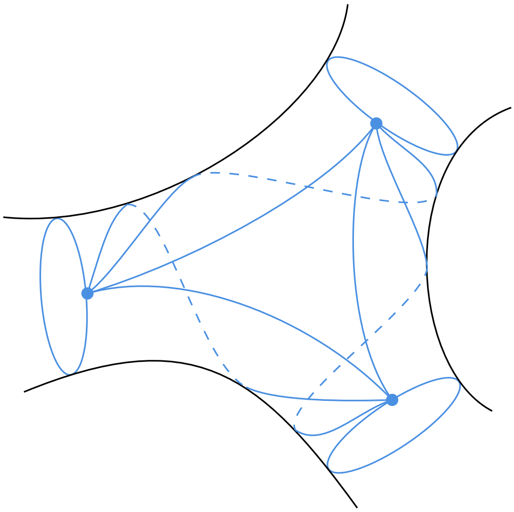

*Figure 2.6. Triangulation on a pair of pants.*

1-skeleton of a triangulation and the $\rho_0\big(\pi_1(S)\big)$ orbits of $A_\gamma$ are mutually disjoint, we can extend $g_0$ geodesically as a $\rho_0$-equivariant polyhedral map $g_0\colon \widetilde{S} \to \mathbb{H}^3$ bent along $T$. Moreover we extend this configuration for small values of $t$ so that $g_t$ varies smoothly as a polyhedral map. As stated in the outline, we take maps $h_t\colon S \to S$ homotopic to the identity so that $h_t(R_i^t)$ is an arc of $T$, and so that the volume of the quotient of the homotopy between $g_t$ and $g_t \circ \widetilde{h_t}$ is equal to 0. As the lamination itself generates a 0 volume set (which is easier to see for our case) we can focus on the variation of the volume of the polyhedra.

As $f_0$, $g_0 \circ \widetilde{h_0}$ send vertices of $\widetilde{S}$ to disjoint sets of orbits, we construct the $\rho_0$-equivariant polyhedral homotopy $H_0$ by taking 3-chains as in Figure 2.3 for each horizontal rectangle. Observe that some of these 3-chains could be collapsed or have the opposite orientation in $\mathbb{H}^3$, so we consider their signed volume. In order to apply the Schläfli formula for collapse polyhedra we need that even in cases when a face or an edge collapses it belongs to a well-defined plane or line that varies smoothly on $t$ around $t = 0$. Since for lifts of $\{R_i^0\}_{1\leqslant i\leqslant 6g-6}$ we have that $H_0$ is defined equivariantly on finitely many polyhedra, by moving slightly our choices of $A_\gamma$ and vertices of $\{R_i^0\}_{1\leqslant i\leqslant 6g-6}$ we can assume that even if a face or edge collapses it is contained in a well-defined plane or line that varies smoothly with $t$. Hence we allow $0, \pi$ in $\mathbb{R}/2\pi\mathbb{Z}$ as values for dihedral angles and 0 for lengths.

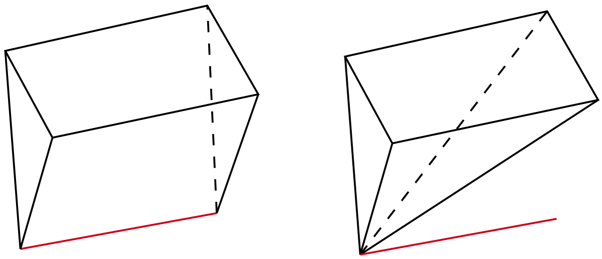

*Figure 2.7. Prisms obtained as the 3-chain in the cases when $\gamma$ is loxodromic (left) or elliptic (right).*

We look now with more attention to the polyhedra of $H_0$ involving $R_\gamma^0$, as we will use that description in the differentiation of volume. As lifts of the horizontal sides of $R_\gamma^0$, $\gamma$ and $\lambda_0 \cap R_\gamma^0$ have image all image by $f_0$ geodesic segments with a point at infinity in common, the associated 3-chains to $R_\gamma^0$ can be in fact considered as prisms. This is the combinatorial type we will take for these 3-chains. As in the case with the polyhedra in the previous paragraph, such prisms are not necessarily non-degenerate, as if $\gamma$ is elliptic then a side of the prism will collapse to a point. Regardless, since by choice the translates of $A_\gamma$ are disjoint from the boundary of the opposite rectangle, the planes containing a face of the prism are all well-defined. Similarly, each edge of the prism is contained in a well-defined geodesic line (in Figure 2.7 the collapsed segment belongs to $\widetilde{\gamma}$). Hence, even if the prism degenerates, we have a well-defined notion of angles between adjacent faces, where again we allow the values of $0, \pi$ in $\mathbb{R}/2\pi\mathbb{Z}$ for angles and 0 for lengths. Same follows for the 3-chains bounded by $g_0 \circ \widetilde{h}_0$, and we extend this configuration for $t$ small.

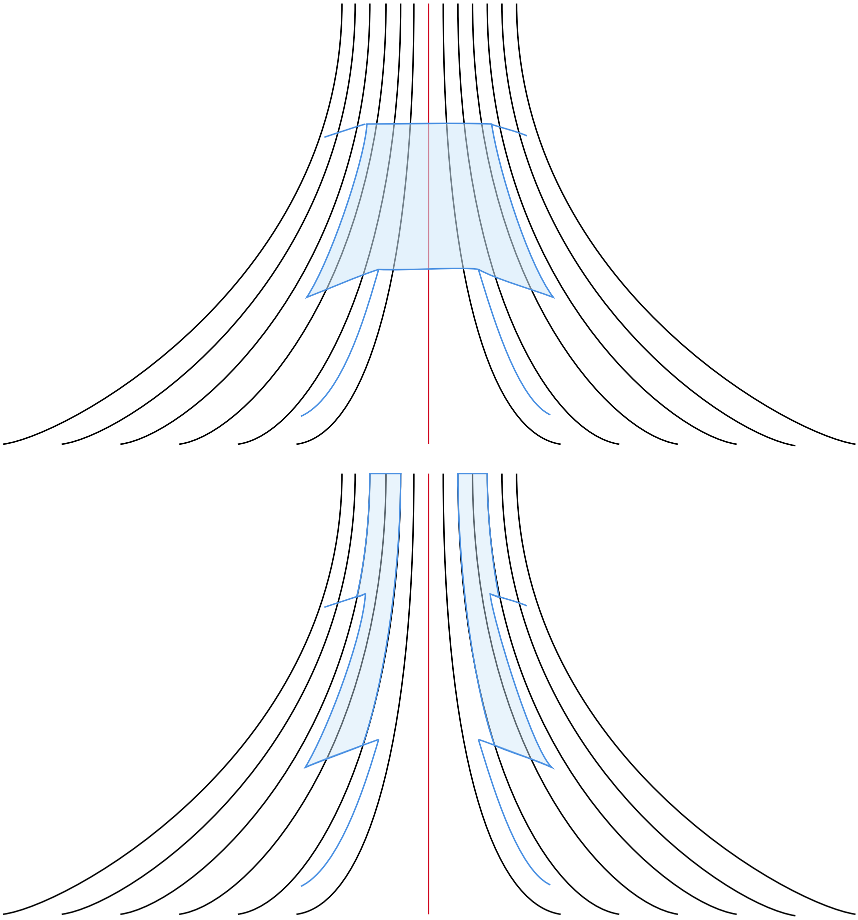

*Figure 2.8.*

For our particular construction we will rearrange the rectangles of $R^t_\gamma \setminus \lambda^t_0$ from the top to the bottom diagram of Figure 2.8. Doing so will stack all prisms to form finitely many tetrahedra with an ideal vertex such as in Figure 2.9. Polyhedra with ideal vertices have as well a Schläfli formula, where

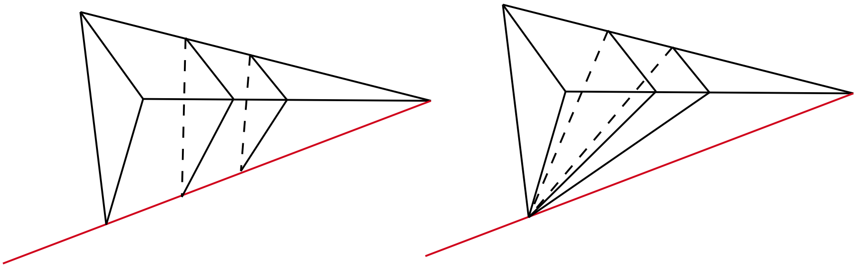

*Figure 2.9. Ideal prisms obtained in the cases when $\gamma$ is loxodromic (left) or elliptic (right).*

one takes a horoball at each ideal vertex and computes the (signed) length of an edge by taking the finite segment in the edge that goes between vertices/horoballs. Then the sum of (edge length)(derivative of dihedral angle) is well-defined, computes the variation of volume and it is independent of the choice of family of horoballs at ideal vertices. Observe then that for our purposes we are dealing with the volume of finitely many polyhedra, where some of them are ideal. Hence $V_0'$ can be computed as

$$V_0' = \frac{1}{2} \sum_{e \text{ edge}} \ell(e)\theta'(e),$$

where the sum is done along the finitely many edges in the aforementioned polyhedra, and $\ell(e), \theta(e)$ denote the (signed) length and dihedral angle of the edge, where ideal edges measure their length with respect to some prefixed horoball at the ideal vertices. Now we reorganize the summands $\ell(e)\theta'(e)$, taking advantage that edges are identified by the $\rho_t$ equivariance of the polyhedra. As edges are identified we will specify the particular dihedral angle of a polyhedra $P \ni e$ by writing $\theta(e, P)$. We divide the types of identified edges in four cases, depending if they are interior (i.e. not from the lamination) or not and if they are ideal or not.

(1) For an interior non-ideal edge $e$ we have that by $\rho_t$ equivariance all instances of $\theta'(e, P)$ add up to 0.
(2) For an interior ideal edge $e$, as its comes from each cuff associated to $\gamma \in \Gamma$, it appears in two ideal tetrahedra with the same angle (matching the rotation component of $\rho_t(\gamma)$) and opposite orientations, while on the ideal tetrahedra we compute the same (signed) length $\ell(e)$ by fixing the same horoball at the ideal point. Hence the summands $\ell(e)\theta'(e, P)$ cancel out.

(3) For non-ideal edges appearing on the lamination, the sum

$$\sum_{e\in P} \theta'(e,P)$$

is equal to the derivative of the bending of the lamination.

(4) For ideal edges appearing on the lamination the sum $\sum_{e\in P} \theta'(e,P)$ is equal to the derivative of the bending of the lamination, while $\ell(e)$ is the (signed) distance between its finite vertex and the horoball fixed at its ideal point. Observe that for a curve $\gamma \in \Gamma$ each of the cuffs uses the same horoball.

Hence after identifying edges and going through the cases, we are left with

$$V_0' = \frac{1}{2} \sum_{e \text{ lamination edge}} \ell(e) b'(e)$$

where $\ell$ denotes the length (or signed length between a finite vertex and a prefixed horoball) and $b'(e)$ is the derivative of the bending. As this right-hand side is our definition for $\ell_0(b_0')$, we have proved the proposition. $\square$

*Remark 2.3.* — In reality all we will need from Proposition 2.2 is that the derivative of volume depends only on information carried by the abstract pleated surface $f_t$, rather than the explicit formula. Nevertheless, it is of independent interest that the formula can be explained by a simple geometric description in our case.

*Remark 2.4.* — While Proposition 2.2 uses the 3-manifold $M$ to define the volume $V_t$, one can more generally define a relative volume by cropping by equivariant surfaces $S_t'$ homotopic to the abstract pleated surface that have 0 variation of volume (for instance, that the integral of the normal component of the variation is 0). While this relative volume $V_t$ depends on the choice of $S_t'$, its derivative $V_t'$ does not, and the conclusion of Proposition 2.2 holds.

## 3. Proof of main theorem

Now that we have all terms defined, let us restate our main result.

THEOREM 3.1. — *Let $M$ be a hyperbolizable compact 3-manifold with boundary. Let $\chi_0(M)$ be the connected irreducible component of the discrete and faithful representations. Then the map $i_*\colon \chi_0(M) \to \chi(\partial M)$ is a birational isomorphism onto its image.*

Since we want to use the Bonahon–Schläfli formula and volume rigidity, we first need a lemma saying that generically our pleated surface construction is well-defined.

LEMMA 3.2. — *Let $M$ be a hyperbolizable compact 3-manifold with boundary, and let $\lambda$ be a maximal geodesic lamination on $\partial M$ that contains a pants decomposition. Let $\chi_0(M)$ be the connected component of the discrete and faithful representations. Then the set*

$$\mathcal{P}(M,\lambda) = \left\{ [\rho] \in \chi_0(M) \;\middle|\; \begin{array}{l} (\partial M, \rho) \text{ has a pleated surface} \\ \text{with pleating locus } \lambda \end{array} \right\}$$

*is an open Zariski dense set in $\chi_0(M)$.*

*Proof.* — (Fixing a particular $\lambda$) We follow the description of Thurston [15]. Let $S \subseteq \partial M$ be a connected component with some fixed hyperbolic structure. Let $\mathcal{P} = \{\gamma_i\} \subset \lambda$ be the pants decomposition contained in the maximal geodesic lamination $\lambda$, so that $\lambda \setminus \mathcal{P}$ is a collection of geodesic lines accumulating at $\mathcal{P}$. Then a given $[\rho] \in \chi_0(M)$ has a pleating surface with pleating locus $\lambda$ if:

(1) $\rho(\gamma_i)$ is non-trivial and non-parabolic for all $\gamma_i \in \mathcal{P}$;
(2) the endpoints of $\rho(\gamma_i)$, $\rho(\gamma_j)$ are distinct for any $\gamma_i \neq \gamma_j \in \mathcal{P}$.

Indeed, if these conditions are satisfied, the lifts of $\mathcal{P}$ in $\widetilde{\lambda}$ can be mapped equivariantly to $\mathbb{H}^3$ by choosing the geodesic representatives $\rho(\gamma_i)$. And since any line $\ell \in \lambda \setminus \mathcal{P}$ accumulates to $\gamma_i \neq \gamma_j \in \mathcal{P}$ with different endpoints, then we can map a lift of $\ell$ to the unique geodesic joining distinct endpoints of the lifts of $\gamma_i, \gamma_j$. Hence for any ideal triangle in $S \setminus \lambda$ we have a map of its boundary to $M$, so there exists a corresponding ideal triangle in $M$. Such ideal triangles will make the realization of $\lambda$ in $M$.

What is left to see is that (1) and (2) define a Zariski dense set. The negative of (1) corresponds to $\mathrm{tr}^2\big(\rho(\gamma_i)\big) = 4$, which is a polynomial equation on the coefficients of $\rho(\gamma_i)$. Similarly, if $\rho(\gamma_i), \rho(\gamma_j)$ share an endpoint then $\mathrm{tr}^2\big(\rho(\gamma_i)\rho(\gamma_j)\rho(\gamma_i)^{-1}\rho(\gamma_j)^{-1}\big) = 4$, which is an algebraic equation on the coefficients of the commutator of $\rho(\gamma_i), \rho(\gamma_j)$. Finally, since these equations are not satisfied for convex co-compact representations, we have that the negatives of (1), (2) define a finite union of proper algebraic sets. Hence the intersection of the complements of these proper algebraic sets is a connected Zariski dense set in $\chi_0(M)$. □

Given that we have that the peripheral map $i_*\colon \chi_0(M) \to \overline{i_*\big(\chi_0(M)\big)} \subset \chi_0(\partial M)$ is dominant, we can use the following lemma to show that $i_*$ is essentially a finite-to-1 covering. Then the main theorem will follow from showing that $i_*$ is essentially injective.

Lemma 3.3 ([12, Proposition 7.16]). — *Let $X, Y$ be (complex) affine algebraic varieties, and let $f\colon X \to Y$ be a dominant rational map. If $X, Y$ have the same dimension, then there exist open Zariski dense subsets $X_0 \subseteq X$, $Y_0 \subseteq Y$, $f^{-1}(Y_0) = X_0$ and an integer $k$ so that the map $f\big|_{X_0}\colon X_0 \to Y_0$ is $k$-to-1.*

The idea to show that $k = 1$ for $i_*$ is to use volume rigidity for characters. In order to do so, we need to show first that volume only depends on the peripheral data.

Lemma 3.4. — *Let $M$ be a hyperbolizable compact 3-manifold with boundary, and let $\Gamma$ be an unoriented pants decomposition on $\partial M$. Let $\chi_0(M)$ be the connected component of the discrete and faithful representations. There exists an open Zariski dense subset $Z \subseteq \chi_0(M)$ so the following hold:*

(a) $\mathrm{vol}_\Gamma\colon Z \to \mathbb{R}$ *given by*

$$\mathrm{vol}_\Gamma(\rho) = \sum_{\Gamma' \text{ orientation on } \Gamma} \mathrm{vol}_{\lambda_0(\Gamma')}(\rho)$$

*is well-defined;*

(b) *on $W = i_*(Z)$ there is a well-defined map $V\colon W \to \mathbb{R}$ so the diagram commutes:*

$$\begin{array}{ccc} Z & \xrightarrow{i_*} & W \\ & \searrow^{\mathrm{vol}_\Gamma} & \downarrow V \\ & & \mathbb{R}. \end{array}$$

*Proof.* — On the set $\mathcal{P}\big(M, \lambda_0(\Gamma)\big)$ we can define $\mathrm{vol}_{\lambda_0(\Gamma)}$ as the volume interior to the pleated surface with pleating locus $\lambda_0(\Gamma)$. By the arguments explained in Section 2.4 this is a well-defined continuous function, although potentially non-differentiable. The reason for this is because we require a smooth equivariant family of endpoints for the lifts of $\Gamma$ in order to apply the Bonahon–Schläfli formula. As explained in Section 2.2, this is equivalent to choose an orientation $\Gamma'$ of $\Gamma$. Because we have to consider the case when $\rho(\gamma)$ is elliptic for some $\gamma \in \Gamma$, we cannot in principle choose a global smooth equivariant family of endpoints for the lifts of $\Gamma$. Instead, we choose all possible orientations at once and take the sum to obtain the smooth function $\mathrm{vol}_\Gamma$, whose derivative is given by the sum of Bonahon–Schläfli formulas for each orientation.

By intersecting with the Zariski dense sets of Lemma 3.3 we can assume that we have $Z \subseteq \chi_0(M)$, $W = i_*(Z) \subseteq \chi_0(\partial M)$ so that (a) is satisfied and $i_*\colon Z \to W$ is $k$-to-1. Our goal is to show that for any $\rho_1, \rho_2 \in Z$ with $i_*(\rho_1) = i_*(\rho_2)$ we have that $\mathrm{vol}_\Gamma(\rho_1) = \mathrm{vol}_\Gamma(\rho_2)$. Since $Z$ is connected, we can take a path $\rho_t$ in $Z$ with endpoints $\rho_1, \rho_2$. Observe that since the Bonahon–Schläfli formula depends exclusively on peripheral information, the change of volume $\mathrm{vol}_\Gamma$ on any lift of $i_*(\rho_t)$ through $i_*\colon Z \to W$ is always equal to $\mathrm{vol}_\Gamma(\rho_2) - \mathrm{vol}_\Gamma(\rho_1)$. Concatenate then consecutive lifts. Since we have a finite fiber, these consecutive lifts must contain a close loop. Then the change of volume on that close loop is equal to 0, but it is also a multiple of $\mathrm{vol}_\Gamma(\rho_2) - \mathrm{vol}_\Gamma(\rho_1)$. Then we must have that $\mathrm{vol}_\Gamma(\rho_2) = \mathrm{vol}_\Gamma(\rho_1)$, from where (b) follows. □

Now we are ready to prove the main theorem through volume rigidity.

*Proof of main theorem.* — By Lemmas 3.2, 3.3, 3.4 we have Zariski open subsets $Z \subseteq \chi_0(M)$, $W \subseteq \chi_0(\partial M)$ so that $i_*\colon Z \to W$ is a $k$-to-1 map and $\mathrm{vol}_\Gamma$ is constant over the fibers of $i_*$. By a result of Brooks [6, Theorem 1], there is a dense set $E \subset Z$ of convex co-compact characters that admit a co-compact extension by reflections. This extension, known also as the Thurston orbifold trick, is a co-finite extension made by considering system of orthogonal planes on each geometrically finite end and extend by their reflections.

Take then $\chi_\rho \in E$ and $\chi_{\rho'} \in Z$ so that $i_*(\chi_\rho) = i_*(\chi_{\rho'})$. Then there exists $G > \pi_1(M)$ with $\left[G : \pi_1(M)\right]$ finite and $\widetilde{\rho} \in R(G)$ so that $\widetilde{\rho}\big|_{\pi_1(M)} = \rho$ and $\mathbb{H}^3/\widetilde{\rho}(G)$ is a compact hyperbolic 3-manifold. Since $i_*(\rho') = i_*(\rho)$, we can find extension $\widetilde{\rho'} \in R(G)$ of $\rho'$. This is because $\rho'$ coincides (up to conjugation) as a representation with $\rho$ in each end, and the extension was made by reflecting on the system of orthogonal planes. And since $\mathrm{vol}_\Gamma(\rho) = \mathrm{vol}_\Gamma(\rho')$, then the volume of the complements of the reflecting planes in $\rho, \rho'$ are the same. This is because the defects between any two summands of $\mathrm{vol}_\Gamma$ or between the system of orthogonal planes and a summand of $\mathrm{vol}_\Gamma$ are determined by the representation of $\pi_1(\partial M)$, where $\rho$ and $\rho'$ coincide. But then this implies that the representations $\widetilde{\rho}, \widetilde{\rho'}$ have the same volume. As $\widetilde{\rho}$ corresponds to a compact hyperbolic 3-manifold, volume rigidity for compact hyperbolic groups (Gromov–Thurston–Goldman volume rigidity, see [9, Theorem 6.1]) implies that $\widetilde{\rho}, \widetilde{\rho'}$ are conjugated. Then it follows that $\chi_\rho = \chi_{\rho'}$.

Hence the map $i_*\colon Z \to W$ is 1-to-1 in $E \subset Z$, so it has to be that $k = 1$. This implies that $i_*$ is 1-to-1, and since we knew that $i_*$ was dominant, then $i_*$ is a birrational isomorphism (see the remark after the definition of birational map on p. 77 and Exercise 7.8 of [12]; this fact uses that we are working over characteristic 0). □

*Remarks 3.5.*

(1) In the case when $M$ is small we can say more about the map $i_*\colon \chi_0(M) \to \chi(\partial M)$. A 3-manifold $M$ is *small* if there does not exist incompressible, non-boundary parallel surface $\Sigma \subset M$. In this case we have that the map $i_*\colon \chi_0(M) \to \chi(\partial M)$ is a surjective, finite-to-one map. For if there is a character $\rho \in \chi(\partial M)$ with non-zero dimensional preimage or in the accumulation of the image of $i_*$, then there is an ideal point $p$ on $\chi_0(M)$ so that $i_*(p) = \rho$. By Culler–Shalen theory (done by Boyer–Zhang [5] for the $\mathrm{PSL}_2\,\mathbb{C}$ case) a meromorphic valuation at the ideal point $p$ produces a $\pi_1(M)$ action on a tree, from where an incompressible surface $\Sigma$ is produced. Since the vertices of the tree are taken by classes of valuation lattices on the field of meromorphic functions times itself, the fact $i_*(p) = \rho$ is well-defined implies that the boundary of $M$ acts trivially on such tree, so $\Sigma$ can't be boundary parallel.

(2) We can combine our approach and the work of [9, 14] to obtain a similar statement for $M$ geometrically finite. Let $(M, \mathcal{C})$ be a geometrically finite hyperbolic 3-manifold, where $\mathcal{C}$ is the collection of conjugacy classes in $\partial M$ corresponding to the rank-1 cusps. Denote by $\partial_{\mathcal{C}} M$ the boundary of $M$ after pinching the generators of $\mathcal{C}$. Then we can define the representation and character varieties $R(M,\mathcal{C}), \chi(M,\mathcal{C})$ as the subvarieties of $R(M), \chi(M)$ restricted to the condition that $\mathcal{C}$ are always mapped to parabolic elements in $\mathrm{PSL}_2\,\mathbb{C}$. Taking $\chi_0(M,\mathcal{C})$ as the irreducible component containing geometrically finite characters pinched at $\mathcal{C}$, then the map

$$i_*\colon \chi_0(M,\mathcal{C}) \longrightarrow \chi(\partial_{\mathcal{C}} M)$$

is a birational isomorphism with is image. Rank-2 cusps are dealt as in [9, 14], while for our pleated surface construction we extend generators of $\mathcal{C}$ to a pants decomposition of $\partial M$. Then the choice of endpoint of ideal triangles at a lift of an element of $\mathcal{C}$ is given by the unique parabolic fixed point.

## BIBLIOGRAPHY

[1] L. Ahlfors & L. Bers, “Riemann’s mapping theorem for variable metrics”, *Ann. Math. (2)* **72** (1960), p. 385-404.

[2] F. Bonahon, “Shearing hyperbolic surfaces, bending pleated surfaces and Thurston’s symplectic form”, *Ann. Fac. Sci. Toulouse, Math. (6)* **5** (1996), no. 2, p. 233-297.

[3] ———, “Transverse Hölder distributions for geodesic laminations”, *Topology* **36** (1997), no. 1, p. 103-122.

[4] ———, “A Schläfli-type formula for convex cores of hyperbolic 3-manifolds”, *J. Differ. Geom.* **50** (1998), no. 1, p. 25-58.

[5] S. Boyer & X. Zhang, “On Culler–Shalen seminorms and Dehn filling”, *Ann. Math. (2)* **148** (1998), no. 3, p. 737-801.

[6] R. Brooks, “Circle packings and co-compact extensions of Kleinian groups”, *Invent. Math.* **86** (1986), no. 3, p. 461-469.

[7] M. Culler & P. B. Shalen, “Varieties of group representations and splittings of 3-manifolds”, *Ann. Math. (2)* **117** (1983), no. 1, p. 109-146.

[8] D. Dumas & A. Kent, “Slicing, skinning, and grafting”, *Am. J. Math.* **131** (2009), no. 5, p. 1419-1429.

[9] N. M. Dunfield, “Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds”, *Invent. Math.* **136** (1999), no. 3, p. 623-657.

[10] S. Francaviglia, “Hyperbolic volume of representations of fundamental groups of cusped 3-manifolds”, *Int. Math. Res. Not.* (2004), no. 9, p. 425-459.

[11] S. Francaviglia & B. Klaff, “Maximal volume representations are Fuchsian”, *Geom. Dedicata* **117** (2006), p. 111-124.

[12] J. Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer, 1992.

[13] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer, 1977.

[14] B. Klaff & S. Tillmann, “A birationality result for character varieties”, *Math. Res. Lett.* **23** (2016), no. 4, p. 1099-1110.

[15] W. P. Thurston, “The geometry and topology of three-manifolds”, online at `https://library.slmath.org/nonmsri/gt3m/`, 2002.

Manuscrit reçu le 30 août 2023,
accepté le 21 mars 2024.

Ian AGOL
Department of Mathematics,
University of California, Berkeley,
970 Evans Hall #3840,
Berkeley, CA 94720 (USA)
ianagol@berkeley.edu

Franco VARGAS PALLETE
Department of Mathematics,
Yale University,
219 Prospect St, Floors 7-9,
New Haven, CT 06511 (USA)
franco.vargaspallete@yale.edu