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Guillaume Aubrun & Alexander Müller-Hermes

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ASYMPTOTIC TENSOR POWERS OF BANACH SPACES

by Guillaume AUBRUN & Alexander MÜLLER-HERMES (*)

ABSTRACT. — We study the asymptotic behaviour of large tensor powers of normed spaces and of operators between them. We define the tensor radius of a finite-dimensional normed space X as the limit of the sequence $A_k^{1/k}$, where A_k is the equivalence constant between the projective and injective norms on $X^{\otimes k}$. We show that Euclidean spaces are characterized by the property that their tensor radius equals their dimension. Moreover, we compute the tensor radius for spaces with enough symmetries, such as the spaces ℓ_p^n. We also define the tensor radius of an operator T as the limit of the sequence $B_k^{1/k}$, where B_k is the injective-to-projective norm of $T^{\otimes k}$. We show that the tensor radius of an operator whose domain or range is Euclidean is equal to its nuclear norm, and give some evidence that this property might characterize Euclidean spaces.

RÉSUMÉ. — Nous étudions le comportement asymptotique des grandes puissances tensorielles des espaces normés et de leurs opérateurs. Nous définissons le rayon tensoriel d'un espace normé X de dimension finie comme la limite de la suite $A_k^{1/k}$, où A_k est la constante d'équivalence entre les normes injective et projective sur $X^{\otimes k}$. Nous montrons que les espaces euclidiens sont caractérisés par le fait que leur rayon tensoriel est égal à leur dimension. De plus, nous calculons le rayon tensoriel des espaces ayant sufisamment de symétries, comme les espaces ℓ_p^n. Nous définissons également le rayon tensoriel d'un opérateur T comme la limite de la suite $B_k^{1/k}$, où B_k est la norme injective-vers-projective de $T^{\otimes k}$. Nous montrons que le rayon tensoriel d'un opérateur défini sur un espace euclidien ou à valeurs dans un espace euclidien est égal à sa norme nucléaire, et suggérons que cette propriété pourrait caractériser les espaces euclidiens.

Keywords: tensor radius, projective and injective norms, Banach–Mazur distance, tensor powers.
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1. Introduction

For a real or complex Banach space X we denote its dual space by X^* and their respective unit balls by B_X and B_{X^*}. Two natural norms can be defined on the algebraic tensor product $X^{\otimes k}$: The *injective tensor norm* is given by

$$\|z\|_{\varepsilon_k(X)} = \sup\Big\{|(\lambda_1 \otimes \cdots \otimes \lambda_k)(z)| \,:\, \lambda_1, \dots, \lambda_k \in B_{X^*}\Big\},$$

for $z \in X^{\otimes k}$, and the *projective tensor norm* by

$$\|z\|_{\pi_k(X)} = \inf\Bigg\{\sum_{i=1}^{n} \Big\|x_i^{(1)}\Big\|_X \cdots \Big\|x_i^{(k)}\Big\|_X : n \in \mathbb{N}, \quad z = \sum_{i=1}^{n} x_i^{(1)} \otimes \cdots \otimes x_i^{(k)}\Bigg\}.$$

When X is finite-dimensional, the discrepancy between these norms is governed by the parameter

$$\rho_k(X) = \left(\sup_{z \in X^{\otimes k}} \frac{\|z\|_{\pi_k(X)}}{\|z\|_{\varepsilon_k(X)}} \right)^{\frac{1}{k}}.$$

More generally, given a linear operator $T : X \to Y$ between finite-dimensional normed spaces, we consider

$$\tau_k(T) = \left\|T^{\otimes k}\right\|^{1/k}_{\varepsilon_k(X) \to \pi_k(Y)} = \left(\sup_{z \in X^{\otimes k}} \frac{\left\|T^{\otimes k} z\right\|_{\pi_k(Y)}}{\|z\|_{\varepsilon_k(X)}} \right)^{\frac{1}{k}}, \tag{1.1}$$

such that $\rho_k(X) = \tau_k(\mathrm{id}_X)$. A standard subadditivity argument, detailed later as Lemma 3.1, shows the existence of the limits

$$\rho_\infty(X) = \lim_{k \to \infty} \rho_k(X), \quad \tau_\infty(T) = \lim_{k \to \infty} \tau_k(T).$$

We call these limits the *tensor radius of the space* X and the *tensor radius of the operator* T, respectively, by analogy with the spectral radius formula. The tensor radii are motivated by questions in quantum information theory where they have recently been applied by the authors [4]. In this article we aim to understand the properties of tensor radii and we begin by stating our main results. A follow-up to the present paper, including answers to some of the questions asked here, will appear in [5].

Main results on tensor radii of normed spaces

We show that the tensor radius is maximal precisely for Euclidean spaces:

Theorem 1.1. — *If X is an n-dimensional normed space, then $\sqrt{n} \leqslant \rho_\infty(X) \leqslant n$. Moreover, $\rho_\infty(X) = n$ if and only if X is Euclidean.*

We will prove Theorem 1.1 in Section 4.2. For a large class of normed spaces, including the spaces ℓ_p^n and their noncommutative analogues, we compute the tensor radius as a function of the Banach–Mazur distance (see Section 2.4) to the Euclidean space. Our argument applies to normed spaces with proportional John and Loewner ellipsoids (see Section 2.3), which includes all normed spaces with enough symmetries (see Section 2.5).

Theorem 1.2. — *If X is an n-dimensional normed space with proportional John and Loewner ellipsoids, then*

$$\rho_\infty(X) = \frac{n}{\mathrm{d}(X, \ell_2^n)}.$$

The proof of Theorem 1.2 is given in Section 4.3. In particular, we see that $\rho_\infty(\ell_\infty^n) = \sqrt{n}$, showing that the lower bound in Theorem 1.1 is sharp.

Main results on tensor radii of linear operators

Let $T : X \to Y$ be a linear operator between finite-dimensional normed spaces and let $\|T\|_{N(X\to Y)}$ denote its nuclear norm. It is elementary to show (see Section 3) that

$$\|T\|_{X\to Y} = \tau_1(T) \leqslant \tau_\infty(T) \leqslant \|T\|_{N(X\to Y)}. \tag{1.2}$$

In order to understand when the upper bound in (1.2) is sharp, we introduce the following property:

Definition 1.3. — *A pair of finite-dimensional normed spaces (X, Y) is said to have the* nuclear tensorization property *(NTP) if the relation $\tau_\infty(T) = \|T\|_{N(X\to Y)}$ holds for every operator $T : X \to Y$.*

It is an elementary fact that $\|\operatorname{id}_X\|_{N(X\to X)} = n$ for the identity operator $\operatorname{id}_X : X \to X$ on any n-dimensional normed space X. Since $\tau_\infty(\operatorname{id}_X) = \rho_\infty(X)$, Theorem 1.1 implies that the pair (X, X) fails to have the NTP whenever X is not Euclidean. The example of $(\ell_\infty^2, \ell_\infty^2)$ is elementary and quite instructive. We will state it here, before developing the general theory later:

Example 1.4. — In the following example, all spaces are over the reals. Using the isometric isomorphism between ℓ_1^2 and ℓ_∞^2, computing $\rho_\infty(\ell_\infty^2)$ is equivalent to computing $\tau_\infty(H)$ of the Hadamard matrix

$$H = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

seen as an operator from ℓ_∞^2 to ℓ_1^2. We have $\tau_1(H) = \|H\|_{\ell_\infty^2 \to \ell_1^2} = 1$ and it is elementary to compute $\|H\|_{N(\ell_\infty^2 \to \ell_1^2)} = 2$ (e.g., using [18, Proposition 8.7]). Consider $k \in \mathbb{N}$ and identify $\varepsilon_k(\ell_\infty^2)$ with $\ell_\infty^{2^k}$, as well as their dual spaces. We find

$$\tau_k(H)^k = \sup_{\alpha,\beta \in \{-1,1\}^{2^k}} \langle \alpha, H^{\otimes k}\beta \rangle \leqslant \left(\sqrt{2}\right)^k, \tag{1.3}$$

by the Cauchy–Schwarz inequality and the fact that $2^{k/2}H^{\otimes k}$ is an orthogonal matrix. Consider the vector $x = (1,1,1,-1)$, which is an eigenvector for $H^{\otimes 2}$ with eigenvalue $1/2$. When $k = 2p$ is even, the choice $\alpha = \beta = x^{\otimes p}$ shows that $\tau_k(H) = \sqrt{2}$. When k is odd, the inequality in (1.3) is strict (the left-hand side is a rational number and the right-hand side is irrational) and therefore $\tau_k(H) < \sqrt{2}$. We have $\tau_\infty(H) = \sqrt{2} < 2 = \|H\|_{N(\ell_\infty^2 \to \ell_1^2)}$ and $(\ell_\infty^2, \ell_1^2)$ does not have the NTP.

In Section 5.2 we find that many natural examples of pairs of normed spaces do not have the NTP. However, there are pairs of distinct normed spaces (X, Y) which have the NTP. For example, this is the case when either X or Y is Euclidean:

Theorem 1.5. — *If X, Y are finite-dimensional normed spaces and one of them is Euclidean, then (X, Y) has the NTP.*

We prove Theorem 1.5 in Section 5 and in Section 7 we generalize it to infinite-dimensional Banach spaces. It is a natural question, whether there exist pairs of non-Euclidean spaces with the NTP. We leave this question open.

Structure of the paper

Section 2 gathers background from Banach spaces theory. Section 3 discusses elementary properties of the tensor radii. A crucial ingredient to all our arguments is Lemma 3.4, which gives an upper bound on the tensor radius of an operator in terms of its factorization through Euclidean spaces.

This lemma is especially powerful when combined with the John and/or Loewner ellipsoids of normed spaces.

In Section 4, we focus on the tensor radius of a space and prove Theorems 1.1 and 1.2. The first step is Theorem 4.2, which states that the tensor radius of the space ℓ_2^n equals n. We obtain this result by showing that a uniformly chosen random vector is typically "very entangled".

In Section 5 we study the tensor radii of linear operators and focus on when they coincide with the nuclear norm. In Section 5.1 we show that the tensor radius for a pair (X, Y) of normed spaces coincides with the nuclear norm when either X or Y is Euclidean (thereby proving Theorem 1.5). In Section 5.2 we identify examples of spaces where the tensor radius does not coincide with the nuclear norm.

In Section 6 we study some natural questions about the tensor radius including whether it is a continuous or a norm (see Section 6.1), whether it is multiplicative (see Section 6.2), or for which spaces it attains its minimal possible value (see Section 6.3). While we answer some of these questions, we leave open many questions for future research. In Section 7 we discuss extensions of our work to infinite dimensional Banach spaces.

2. Notation and preliminaries

In most of the paper, we restrict to finite-dimensional normed spaces over a field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$. Extensions to infinite dimensions are briefly discussed in Section 7. Let X be a finite-dimensional real or complex normed space. We denote its unit ball by B_X and its dual space by X^*. If Y is another finite-dimensional normed space, we denote by $L(X, Y)$ the space of linear operators from X to Y.

2.1. Tensor norms

Let X, Y be finite-dimensional normed spaces. A *cross norm* is a norm $\|\cdot\|$ on $X \otimes Y$ satisfying the conditions

$$\|x \otimes y\| = \|x\|_X \|y\|_Y \quad \text{and} \quad \|x^* \otimes y^*\|_* = \|x^*\|_{X^*} \|y^*\|_{Y^*}$$

for every $x \in X$, $y \in Y$, $x^* \in X^*$, $y^* \in Y^*$, where $\|\cdot\|_*$ is the dual norm to $\|\cdot\|$. Important examples of cross norms are the *injective norm* given by

$$\|z\|_\varepsilon = \sup\left\{|(x^* \otimes y^*)(z)| : x^* \in B_{X^*},\ y^* \in B_{Y^*}\right\},$$

for $z \in X \otimes Y$, and the projective norm, defined as

$$\|z\|_\pi = \inf\left\{ \sum_{i=1}^n \|x_i\|_X \|y_i\|_Y : n \in \mathbb{N},\ z = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

These cross norms are extremal since any cross norm $\|\cdot\|$ on $X \otimes Y$ satisfies the inequalities $\|\cdot\|_\varepsilon \leqslant \|\cdot\| \leqslant \|\cdot\|_\pi$.

We denote by $X \otimes_\varepsilon Y$ (resp., $X \otimes_\pi Y$) the space $X \otimes Y$ equipped with the injective (resp., projective) norm. These norms are in duality: $(X \otimes_\pi Y)^*$ identifies with $X^* \otimes_\varepsilon Y^*$ and $(X \otimes_\varepsilon Y)^*$ identifies with $X^* \otimes_\pi Y^*$. All these definitions and properties have natural extensions to the tensor product of more than two spaces, leading to the definition of $\|\cdot\|_{\varepsilon_k(X)}$ and $\|\cdot\|_{\pi_k(X)}$ given in the introduction. We will denote by $\varepsilon_k(X)$ and $\pi_k(X)$ the space $X^{\otimes k}$ equipped with the injective and projective norm, respectively.

In the special case of Euclidean spaces, another cross norm plays a special role: if X, Y are Euclidean spaces, equipped with inner products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$, we may define uniquely an inner product on $X \otimes Y$ by the formula

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle_X \langle y_1, y_2 \rangle_Y$$

for every $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. The corresponding Euclidean norm, called the Hilbert–Schmidt norm and denoted $\|\cdot\|_{\mathrm{HS}}$, is a cross norm. This definition extends immediately to $k \geqslant 2$ factors and we denote by $\|\cdot\|_{\mathrm{HS}_k(X)}$ the Hilbert–Schmidt norm on $X^{\otimes k}$.

A *tensor norm* α is the data, for each pair (X, Y) of finite-dimensional normed spaces, of a cross norm $\|\cdot\|_{X \otimes_\alpha Y}$ on $X \otimes Y$, satisfying the following axiom called the *metric mapping property*: if X, X', Y, Y' are finite-dimensional normed space and $S \in L(X, X')$, $T \in L(Y, Y')$, then

$$\|S \otimes T\|_{X \otimes_\alpha Y \to X' \otimes_\alpha Y'} = \|S\| \cdot \|T\|.$$

Both the injective norm ε and the projective norm π are tensor norms.

2.2. Operators, nuclear norm and trace duality

Consider finite-dimensional normed spaces X, Y. The *operator norm* of an operator $T \in L(X, Y)$ is given by

$$\|T\|_{X \to Y} = \sup_{x \in B_X} \|Tx\|_Y,$$

and its *nuclear norm* by

$$\|T\|_{N(X \to Y)} = \inf \sum_i \|y_i\|_Y \|x_i^*\|_{X^*},$$

where the infimum ranges over so-called nuclear decompositions $T(\cdot) = \sum_i x_i^*(\cdot)y_i$ with $y_1,\dots,y_n \in Y$ and $x_1^*,\dots,x_n^* \in X^*$. If there is no ambiguity,we may write $\|T\|$ instead of $\|T\|_{X\to Y}$ and $\|T\|_N$ instead of $\|T\|_{N(X\to Y)}$. The operator and nuclear norms satisfy the *ideal property*: given operators $T \in L(X_0, X)$, $S \in L(X,Y)$, $R \in L(Y,Y_0)$, we have

$$\|RST\|_{X_0\to Y_0} \leqslant \|R\|_{Y\to Y_0}\, \|S\|_{X\to Y}\, \|T\|_{X_0\to X},$$
$$\|RST\|_{N(X_0\to Y_0)} \leqslant \|R\|_{Y\to Y_0}\, \|S\|_{N(X\to Y)}\, \|T\|_{X_0\to X}.$$

The space $L(X,Y)$ can be canonically identified with $X^* \otimes Y$; under this identification, the operator and nuclear norms on $L(X,Y)$ correspond respectively to the injective and projective norms on $X^* \otimes Y$. Then, the duality between injective and projective norms translates into the *trace duality*: for any $T \in L(X,Y)$, we have

$$\|T\|_{N(X\to Y)} = \sup_{\|Q\|_{Y\to X}\leqslant 1} |\mathrm{Tr}[QT]|. \tag{2.1}$$

2.3. John and Loewner ellipsoids

Here, we review standard facts about John and Loewner ellipsoids and refer to [18, Section 15] for details. We set $\mathbb{K}$ to be either $\mathbb{R}$ or $\mathbb{C}$ and let X be an n-dimensional normed space. If X is a complex vector space, we denote by $\overline{X}$ its conjugate vector space, i.e., the space X with the scalar multiplication defined by $(\lambda,x)\mapsto \bar{\lambda}x$ for all $(\lambda,x)\in \mathbb{C}\times X$. If X is a real vector space, we set $\overline{X} = X$. In the complex case, the identity map defines an isometric anti-isomorphism between X and $\overline{X}$. We may also identify ℓ_2^n and $(\ell_2^n)^*$ by the usual canonical anti-isomorphism. If $u : \ell_2^n \to X$ is a linear map, its adjoint $u^* : X^* \to (\ell_2^n)^*$ can then be considered as a linear map from $\overline{X}^*$ to ℓ_2^n by composing from both sides with the canonical anti-isomorphisms from above.

We fix a Haar measure on X which we call the *volume*. This measure is unique up to a multiplicative constant (see [9, p. 263, Theorem C]), and the precise choice is irrelevant for our purpose. More concretely, we may realize the volume as the pushforward of the Lebesgue measure on $\mathbb{K}^n$ under an arbitrary isomorphism between X and $\mathbb{K}^n$.

An ellipsoid in X is the image of the Euclidean unit ball $B_2^n \subset \ell_2^n$ under a linear bijective map. Among all ellipsoids containing B_X, there is a unique ellipsoid of minimal volume called the *Loewner ellipsoid* of X. If we denote by $\|\cdot\|_L$ the norm on X for which the Loewner ellipsoid is the unit ball, we have $\|\cdot\|_L \leqslant \|\cdot\|_X$ and a vector $x\in X$ such that $\|x\|_L = \|x\|_X = 1$ is called a

Loewner contact point. Dually, among all ellipsoids contained in B_X, there is a unique ellipsoid of maximal volume called the *John ellipsoid* of X. If we denote by $\|\cdot\|_J$ the norm on X for which the John ellipsoid is the unit ball, we have $\|\cdot\|_J \geqslant \|\cdot\|_X$ and a vector $x \in X$ such that $\|x\|_J = \|x\|_X = 1$ is called a *John contact point.*

These ellipsoids are usually characterized in terms of resolutions of identity involving contact points. A *resolution of identity* in the space ℓ_2^n is a finite family $(a_i, \lambda_i)_{i\in I}$ with a_i unit vectors in ℓ_2^n and $\lambda_i > 0$ such that

$$\mathrm{id}_{\ell_2^n} = \sum_{i\in I} \lambda_i \langle a_i, \cdot\rangle a_i.$$

We necessarily have $\sum \lambda_i = n$, as is seen by taking the trace on both sides.

Theorem 2.1 (see [18, Theorems 15.3 and 15.4]). — *Let X be an n-dimensional normed space and $u : \ell_2^n \to X$ be a linear bijection. Then*

(1) *The ellipsoid $u(B_2^n)$ is the John ellipsoid of X if and only if $\|u\| \leqslant 1$ and there exists a resolution of identity $(a_i, \lambda_i)_{i\in I}$ such that, for every $i \in I$,*

$$\|u(a_i)\|_X = 1 = \left\|(u^*)^{-1}(a_i)\right\|_{\overline{X}^*}.$$

(2) *The ellipsoid $u(B_2^n)$ is the Loewner ellipsoid of X if and only if $\|u^{-1}\| \leqslant 1$ and there exists a resolution of identity $(a_i, \lambda_i)_{i\in I}$ such that, for every $i \in I$,*

$$\|u(a_i)\|_X = 1 = \left\|(u^*)^{-1}(a_i)\right\|_{\overline{X}^*}.$$

We will need a corollary to Theorem 2.1. As explained above, if $v : X \to \ell_2^n$ is a linear map, its adjoint $v^* : (\ell_2^n)^* \to X^*$ can be considered as a linear map from ℓ_2^n to $\overline{X}^*$. With this convention we have the following identity

$$\overline{v^*(a)(x)} = \langle a, v(x)\rangle, \tag{2.2}$$

for all $x \in X$ and $a \in \ell_2^n$, where on the left-hand-side the element x is considered to be in the conjugate space $\overline{X}$. With this we can prove the following corollary to Theorem 2.1:

Corollary 2.2. — *Let X be an n-dimensional normed space.*

(1) *If $u : \ell_2^n \to X$ is such that $u(B_2^n)$ is the John ellipsoid of X, then*

$$\left\|(uu^*)^{-1}\right\|_{N\left(X\to\overline{X}^*\right)} \leqslant n.$$

(2) *If $u : \ell_2^n \to X$ is such that $u(B_2^n)$ is the Loewner ellipsoid of X, then*

$$\|uu^*\|_{N\left(\overline{X}^*\to X\right)} \leqslant n.$$

Proof. — Let $(a_i,\lambda_i)_{i\in I}$ a resolution of identity given by Theorem 2.1. For $i \in I$, set $x_i^* = (u^*)^{-1}(a_i)$, which is a unit vector in $\overline{X}^*$. For every $x \in X$, we have

$$\sum_{i\in I} \lambda_i \cdot \overline{x_i^*(x)} \cdot x_i^* = (u^*)^{-1}\left(\sum_{i\in I} \lambda_i \cdot \big\langle a_i, u^{-1}(x)\big\rangle \cdot a_i\right) = (u^*)^{-1}u^{-1}(x),$$

where we used (2.2) for the linear map $u^{-1} : X \to \ell_2^n$. Therefore, we have

$$\big\|(uu^*)^{-1}\big\|_{N(\overline{X}^*\to X)} \leqslant \sum_{i\in I} \lambda_i \|x_i^*\|_{X^*}^2 \leqslant n,$$

proving (1). The proof of (2) is similar. □

2.4. Banach–Mazur distance

Let X, Y be Banach spaces. The Banach–Mazur distance between X and Y is given by

$$\mathrm{d}(X,Y) = \inf\big\{\|U\|_{X\to Y}\,\big\|U^{-1}\big\|_{Y\to X} : U : X \longrightarrow Y \text{ linear bijection}\big\}.$$

For $1 \leqslant p \leqslant \infty$ and an integer $n \geqslant 1$, denote by ℓ_p^n the space $(\mathbb{K}^n, \|\cdot\|_p)$ where $\|\cdot\|_p$ is the usual p-norm. For an n-dimensional normed space X, we set

$$\mathrm{d}_X := \mathrm{d}(X, \ell_2^n).$$

A standard estimate, which can be deduced from Theorem 2.1, is that

$$\mathrm{d}_X \leqslant \sqrt{n}$$

for every n-dimensional normed space X.

2.5. Spaces with enough symmetries

Let X be a finite-dimensional normed space. We say that an invertible linear map $U : X \to X$ is a *symmetry* of X if it satisfies $\|Ux\|_X = \|x\|_X$ for every $x \in X$. The symmetries of X form a compact group which we denote by $G(X)$.

Following [18, Section 16], we say that X *has enough symmetries* if every linear map $T : X \to X$ satisfying $TU = UT$ for each symmetry $U \in G(X)$ is a multiple of id_X. The class of spaces with enough symmetries includes the spaces ℓ_p^n, and normed spaces obtained by equipping the space of matrices with a unitarily invariant norm such as the Schatten p-norms.

In a space with enough symmetries, there is (up to a positive scalar multiple) a unique inner product $\langle\cdot,\cdot\rangle$ which is invariant, i.e., such that $\langle U(x), U(y)\rangle = \langle x, y\rangle$ for every $x, y \in X$ and $U \in G(X)$. Since the inner product associated to either the John or the Loewner ellipsoid are invariant, it follows that for a space with enough symmetries, the John and Loewner ellipsoids are proportional.

Let X be a space with enough symmetries, and $\mathrm{d}U$ the Haar measure on $G(X)$. For any linear operator $T : X \to X$, we have

$$\int_{G(X)} U^{-1}TU\mathrm{d}U = \mathrm{Tr}[T]\frac{\mathrm{id}_X}{\dim(X)}, \tag{2.3}$$

where Tr denotes the trace on $L(X, X)$.

3. Basic properties of τ_k and τ_∞

3.1. Existence of τ_∞ and behaviour under transformations

We now investigate more systematically the properties of the quantities τ_k and τ_∞ (see (1.1) for the definition). We first show that

$$\|T\|_{X\to Y} \leqslant \tau_k(T) \leqslant \|T\|_{N(X\to Y)}. \tag{3.1}$$

The left inequality follows by restricting the supremum in (1.1) to tensors of the form $z = x^{\otimes k}$ for $x \in X$. To prove the right-hand side, consider a nuclear decomposition of the form $T(\cdot) = \sum_i x_i^*(\cdot)y_i$ with $x_1^*, \dots, x_n^* \in X^*$ and $y_1, \dots, y_n \in Y$. We have

$$\begin{aligned}\left\|T^{\otimes k}(z)\right\|_{\pi_k(Y)} &\leqslant \sum_{i_1,\dots,i_k} \left|\left(x_{i_1}^* \otimes \dots \otimes x_{i_k}^*\right)(z)\right| \cdot \|y_{i_1} \otimes \dots \otimes y_{i_k}\|_{\pi_k(Y)} \\ &\leqslant \sum_{i_1,\dots,i_k} \|x_{i_1}^*\|_{X^*} \dots \|x_{i_k}^*\|_{X^*} \|z\|_{\varepsilon_k(X)} \|y_{i_1}\|_Y \dots \|y_{i_k}\|_Y \\ &= \left(\sum_i \|x_i^*\|_{X^*}\|y_i\|_Y\right)^k \|z\|_{\varepsilon_k(X)},\end{aligned}$$

for every $z \in X^{\otimes k}$, and the result follows by optimizing over decompositions of T.

Lemma 3.1. — *Let X, Y be finite-dimensional normed spaces and $T : X \to Y$ a linear map. The limit of the sequence $(\tau_k(T))_{k\in\mathbb{N}}$ exists, and*

$$\tau_\infty(T) := \lim_{k\to\infty} \tau_k(T) = \sup_{k\geqslant 1} \tau_k(T).$$

Proof. — For any integers $k_1, k_2 \geqslant 1$, we have

$$(\tau_{k_1+k_2}(T))^{k_1+k_2} \geqslant (\tau_{k_1}(T))^{k_1}(\tau_{k_2}(T))^{k_2}, \tag{3.2}$$

as can be seen by restricting the supremum defining $\tau_{k_1+k_2}(T)$ to elements of the form $z_1 \otimes z_2$, for $z_1 \in X^{\otimes k_1}$, $z_2 \in X^{\otimes k_2}$. The conclusion follows by applying Fekete's lemma [7] to the sequence $(t_k)_k$ given by $t_k = k \log(\tau_k(T))$. □

By (3.2) the inequality $\tau_k(T) \leqslant \tau_l(T)$ holds whenever k divides l, but we have seen in Example 1.4 that the sequence $(\tau_k(T))_k$ is in general not monotonically increasing. An immediate consequence of (3.1) is the inequality

$$\tau_\infty(T) \leqslant \|T\|_{N(X\to Y)}. \tag{3.3}$$

The next lemma shows that the tensor radius behaves nicely under adjoints.

Lemma 3.2. — *Let X, Y be finite-dimensional normed spaces, and $T \in L(X,Y)$. Then $\tau_k(T) = \tau_k(T^*)$ for any $k \in \mathbb{N} \cup \{\infty\}$.*

Proof. — For any $k \geqslant 1$, we have

$$\left\|T^{\otimes k}\right\|_{\varepsilon_k(X)\to\pi_k(Y)} = \left\|(T^*)^{\otimes k}\right\|_{\varepsilon_k(Y^*)\to\pi_k(X^*)}$$

by the duality between the injective and projective norms. The result follows. □

By Lemma 3.2, a pair (X,Y) of finite-dimensional normed spaces has the NTP (see Definition 1.3) if and only if (Y^*, X^*) has the NTP. Tensor radii also satisfy the ideal property, in the following sense.

Lemma 3.3. — *Let X, X', Y, Y' be finite-dimensional normed spaces and $T \in L(X,Y)$. For $A \in L(X',X)$, $B \in L(Y,Y')$ and every $k \in \mathbb{N} \cup \{\infty\}$ we have*

$$\tau_k(BTA) \leqslant \|B\|\tau_k(T)\|A\|.$$

Proof. — It suffices to prove the result for finite k. We combine the ideal property of the operator norm and the metric mapping property of the injective and projective norms to obtain

$$\begin{aligned}\tau_k(BTA) &\leqslant \left\|A^{\otimes k}\right\|^{1/k}_{\varepsilon_k(X')\to\varepsilon_k(X)}\tau_k(T)\left\|B^{\otimes k}\right\|^{1/k}_{\pi_k(Y)\to\pi_k(Y')} \\ &= \|A\|\tau_k(T)\|B\|.\end{aligned}$$ □

3.2. Upper bound by factorization through ℓ_2

The following lemma is a useful tool to find upper bounds on $\tau_\infty(T)$ by considering factorizations of the operator T through Euclidean spaces.

LEMMA 3.4. — *Let X and Y denote finite-dimensional normed spaces. For any $d \in \mathbb{N}$ and any pair of linear operators*

$$Q_1 : \ell_2^d \longrightarrow Y \quad \text{and} \quad Q_2 : X \longrightarrow \ell_2^d,$$

we have

$$\tau_\infty(Q_1 Q_2) \leqslant \|Q_1 Q_1^*\|_{N(\overline{Y}^* \to Y)}^{\frac{1}{2}} \|Q_2^* Q_2\|_{N(X \to \overline{X}^*)}^{\frac{1}{2}}.$$

The operator Q_1^* appearing in Lemma 3.4 is seen as a linear operator from $\overline{Y}^*$ to ℓ_2^d as explained in Section 2.3. Similarly, the operator Q_2^* is seen as a linear operator from ℓ_2^d to $\overline{X}^*$.

Proof. — For any $k \in \mathbb{N}$, we have

$$\left\|(Q_1 Q_2)^{\otimes k}\right\|_{\varepsilon_k(X) \to \pi_k(Y)} \leqslant \left\|Q_2^{\otimes k}\right\|_{\varepsilon_k(X) \to \mathrm{HS}_k(\ell_2^d)} \left\|Q_1^{\otimes k}\right\|_{\mathrm{HS}_k(\ell_2^d) \to \pi_k(Y)}.$$

Fix an element $z \in X^{\otimes k}$ with $\|z\|_{\varepsilon_k(X)} \leqslant 1$, together with a nuclear decomposition $Q_2^* Q_2(\cdot) = \sum_i \overline{x_i'(\cdot)} x_i$ with $x_i, x_i' \in X^*$. Using (2.2), we find that

$$\begin{aligned}
\left\|Q_2^{\otimes k}(z)\right\|_{\mathrm{HS}_k(\ell_2^d)}^2 &= \left[(Q_2^* Q_2)^{\otimes k}(z)\right](z) \\
&\leqslant \sum_{i_1, \dots, i_k} |(x_{i_1} \otimes \cdots \otimes x_{i_k})(z)| \cdot \left|\overline{(x_{i_1}' \otimes \cdots \otimes x_{i_k}')(z)}\right| \\
&\leqslant \left(\sum_i \|x_i'\|_{X^*} \|x_i\|_{X^*}\right)^k.
\end{aligned}$$

Optimizing over z and over nuclear decompositions of $Q_2^* Q_2$, we conclude that

$$\left\|Q_2^{\otimes k}\right\|_{\varepsilon_k(X) \to \mathrm{HS}_k(\ell_2^d)} \leqslant \|Q_2^* Q_2\|_{N(X \to \overline{X}^*)}^{k/2}.$$

By taking adjoints and using the previous inequality, we obtain

$$\left\|Q_1^{\otimes k}\right\|_{\mathrm{HS}_k(\ell_2^d) \to \pi_k(Y)} = \left\|(Q_1^*)^{\otimes k}\right\|_{\varepsilon_k(Y^*) \to \mathrm{HS}_k(\ell_2^d)} \leqslant \|Q_1 Q_1^*\|_{N(\overline{Y}^* \to Y)}^{k/2}.$$

Combining the previous bounds and taking the limit $k \to \infty$ we have

$$\tau_\infty(Q_1 Q_2) \leqslant \|Q_1 Q_1^*\|_{N(\overline{Y}^* \to Y)}^{\frac{1}{2}} \|Q_2^* Q_2\|_{N(X \to \overline{X}^*)}^{\frac{1}{2}},$$

and the proof is finished. □

Remark 3.5. — Here is a reformulation of Lemma 3.4: given a map $T : X \to Y$, we have

$$\tau_\infty(T) \leqslant \inf\left\{ \left\| \sum \overline{x_i^*(\cdot)} x_i^* \right\|_{N(X \to \overline{X}^*)}^{1/2} \left\| \sum \overline{y_i(\cdot)} y_i \right\|_{N(\overline{Y}^* \to Y)}^{1/2} \right\}$$
$$\leqslant \|T\|_{N(X \to Y)},$$

where the infimum is taken over decompositions $T(\cdot) = \sum x_i^*(\cdot) y_i$, with $(x_i^*) \subset X^*$ and $(y_i) \in Y$.

We will often apply Lemma 3.4 together with Corollary 2.2 to obtain upper bounds that are often tight. We will use this approach in the following sections.

4. Tensor radii of normed spaces

In this section, we study the tensor radius $\rho_\infty(X) = \tau_\infty(\mathrm{id}_X)$ of a normed space X. We first gather some elementary properties.

Proposition 4.1. — *For n-dimensional normed spaces X, Y we have:*

(1) $\rho_k(X) = \rho_k(X^*)$ *for every* $k \in \mathbb{N} \cup \{\infty\}$.
(2) $\rho_k(X) \leqslant \mathrm{d}(X, Y)\rho_k(Y)$ *for every* $k \in \mathbb{N} \cup \{\infty\}$.
(3) $\rho_k(X) \leqslant n^{1-1/k}$ *for every* $k \in \mathbb{N}$*, and* $\rho_\infty(X) \leqslant n$.

Proof. — (1) is Lemma 3.2 applied to $T = \mathrm{id}_X$. The inequality (2) can be deduced from the ideal property (Lemma 3.3) by optimizing over linear bijections $U : Y \to X$. To obtain (3), consider an Auerbach basis (x_i) for X, i.e., such that $\|x_i\|_X = \|x_i^*\|_{X^*} = 1$, where (x_i^*) denotes the basis of X^* dual to (x_i). Any $z \in X^{\otimes k}$ can be expanded as

$$z = \sum_{i_1,\dots,i_{k-1}} x_{i_1} \otimes \cdots \otimes x_{i_{k-1}} \otimes y_{i_1 \dots i_{k-1}}$$

for some $y_{i_1,\dots,i_{k-1}} \in X$. We have

$$\begin{aligned} \|z\|_{\pi_k(X)} &\leqslant \sum_{i_1,\dots,i_{k-1}} \|y_{i_1 \dots i_{k-1}}\|_{X^*} \\ &\leqslant n^{k-1} \max_{i_1,\dots,i_{k-1}} \|y_{i_1 \dots i_{k-1}}\|_{X^*} \\ &\leqslant n^{k-1} \|z\|_{\varepsilon_k(X)} \end{aligned}$$

and the result follows. □

The quantity $\rho_2(X)$ (or rather, its square $\rho_2(X)^2$) was studied extensively in [3], where it has been shown that

$$n^{\frac{1}{4}-o(1)} \leqslant \rho_2(X) \leqslant \sqrt{n},$$

as $n = \dim(X)$ tends to infinity. Both of these estimates are sharp since we have, for example, $\rho_2(\ell_2^n) = \sqrt{n}$ and $\rho_2(\ell_1^n) \leqslant (2n)^{\frac{1}{4}}$.

4.1. Tensor radii of Euclidean spaces

Our first result determines the tensor radius of a Euclidean space.

THEOREM 4.2. — *For every $n \geqslant 1$, we have $\rho_\infty(\ell_2^n) = n$.*

Our proof of Theorem 4.2 uses the following lemma, which is an immediate extension of the $n = 2$ case which appears in [6, Proposition 8.28]. It is based on a standard random construction, whose proof we include for the reader's convenience.

LEMMA 4.3. — *Given integers $n, k \geqslant 2$, there exists a tensor $z \in (\ell_2^n)^{\otimes k}$ such that $\|z\|_{\mathrm{HS}_k(\ell_2^n)} = 1$ and*

$$\|z\|_{\varepsilon_k(\ell_2^n)} \leqslant \frac{C_n\sqrt{k\log k}}{n^{k/2}},$$

where C_n is a constant which does not depend on k.

Observe that Lemma 4.3 is not sharp for $k = 2$, since the minimal value of $\|z\|_{\varepsilon_2(\ell_2^n)}$ under the constraint that $\|z\|_{\mathrm{HS}_2(\ell_2^n)} = 1$ is equal to $1/\sqrt{n}$. We are going to apply the lemma for k tending to $+\infty$.

Proof. — The proof is a standard random construction. Consider a subset $\mathcal{N} \subset B_2^n$ which is a $\frac{1}{2k}$-net, i.e., such that for every $x \in B_2^n$ there is $x' \in \mathcal{N}$ such that $\|x - x'\| \leqslant \frac{1}{2k}$. There exists such an $\mathcal{N}$ with the property that $\operatorname{card} \mathcal{N} \leqslant (4k+1)^{2n}$ (see [19, Corollary 4.2.13]). Denote by $\mathcal{N}^{\otimes k}$ the set of vectors of the form $y_1 \otimes \cdots \otimes y_k$ for $y_1, \dots, y_k \in \mathcal{N}$.

Given $x_1, \dots, x_k \in B_2^n$, let $x_1', \dots, x_k' \in \mathcal{N}$ such that $\|x_i - x_i'\| \leqslant \frac{1}{2k}$ for every i. Let $\xi = x_1 \otimes \cdots \otimes x_k$ and $\xi' = x_1' \otimes \cdots \otimes x_k'$. For a tensor $z \in X^{\otimes k}$, we have by the triangle inequality

$$\begin{aligned}|\langle z,\xi\rangle| &\leqslant |\langle z,\xi'\rangle| + \sum_{i=1}^{k} \big|\big\langle z, x_1\otimes\cdots\otimes x_{i-1}\otimes(x_i-x_i')\otimes x_{i+1}'\otimes\cdots\otimes x_k'\big\rangle\big| \\ &\leqslant |\langle z,\xi'\rangle| + \sum_{i=1}^{k} \|z\|_{\varepsilon_k(\ell_2^n)}\|x_i-x_i'\| \\ &\leqslant \sup_{\eta\in\mathcal{N}^{\otimes k}} |\langle z,\eta\rangle| + \frac{1}{2}\|z\|_{\varepsilon_k(\ell_2^n)}.\end{aligned}$$

By taking the supremum over ξ, we obtain $\|z\|_{\varepsilon_k(\ell_2^n)} \leqslant 2\sup\{|\langle z,\eta\rangle| : \eta\in\mathcal{N}^{\otimes k}\}$. Assume now that z is chosen at random uniformly on the Hilbert–Schmidt unit sphere in $(\ell_2^n)^{\otimes k}$. We use the following lemma (see for example [6, Lemma 6.1 and eq. 4.32]).

Lemma 4.4. — *Let N be an integer. Choose z at random according to the uniform measure on the unit sphere in ℓ_2^N. For every finite subset $P\subset B_2^N$, we have*

$$\mathbb{E}\sup_{x\in P}|\langle z,x\rangle| \leqslant \frac{C\sqrt{\log\operatorname{card}(P)}}{\sqrt{N}},$$

where C is a constant.

Applying Lemma 4.4 with $N=n^k$ and $P=\mathcal{N}^{\otimes k}$ shows that

$$\mathbb{E}\|z\|_{\varepsilon_k(\ell_2^n)} \leqslant \frac{2C\sqrt{\log\operatorname{card}(\mathcal{N}^{\otimes k})}}{\sqrt{N}} \leqslant \frac{2C\sqrt{2nk\log(4k+1)}}{n^{k/2}}.$$

This implies the existence of a tensor z satisfying the desired condition with $C_n = O(\sqrt{n})$. □

An improved version of Lemma 4.3 can be proved in the real case: there is a tensor z in the real space $(\ell_2^n)^{\otimes k}$ such that $\|z\|_{\mathrm{HS}_k(\ell_2^n)}=1$ and $\|z\|_{\varepsilon_k(\ell_2^n)}\leqslant C_n/n^{k/2}$ (see [4, Lemma 4.4]). It seems to be unknown whether such an improvement is possible in the complex case, even for $n=2$ (see [6, Problem 8.27]).

Proof of Theorem 4.2. — By Proposition 4.1, we have $\rho_\infty(\ell_2^n)\leqslant n$ and we only need to prove that $\rho_\infty(\ell_2^n)\geqslant n$. Fix $k\geqslant 1$. The Hilbert–Schmidt norm on $(\ell_2^n)^{\otimes k}$ satisfies the inequality $\|z\|^2_{\mathrm{HS}_k(\ell_2^n)}\leqslant\|z\|_{\pi_k(\ell_2^n)}\|z\|_{\varepsilon_k(\ell_2^n)}$ for every $z\in(\ell_2^n)^{\otimes k}$. Therefore, we have

$$\rho_k(\ell_2^n) \geqslant \sup_{z\in(\ell_2^n)^{\otimes k}} \left(\frac{\|z\|_{\mathrm{HS}_k(\ell_2^n)}}{\|z\|_{\varepsilon_k(\ell_2^n)}}\right)^{2/k}.$$

Using Lemma 4.3 and taking the limit $k\to\infty$ shows that $\rho_\infty(\ell_2^n)\geqslant n$. □

4.2. Proof of Theorem 1.1

Let X be an n-dimensional normed space. Since $\rho_\infty(\ell_2^n) = n$, we obtain the lower bound

$$\rho_\infty(X) \geqslant \frac{\rho_\infty(\ell_2^n)}{\mathrm{d}_X} \geqslant \sqrt{n} \tag{4.1}$$

using Proposition 4.1 and Theorem 4.2.

It remains to show that $\rho_\infty(X) < n$ if we assume that X is not Euclidean. Let $Q_1 : \ell_2^n \to X$ such that $Q_1(B_2^n)$ is the John ellipsoid of X and set $Q_2 = Q_1^{-1}$. By Corollary 2.2, we have $\|Q_2^* Q_2\|_{N(X \to \overline{X}^*)} \leqslant n$. Since X is not Euclidean, we have $Q_1(B_2^n) \subsetneq B_X$ and therefore there is a unit vector $a_1 \in \ell_2^n$ such that $\|Q_1(a_1)\|_X < 1$. Next, we complete $\{a_1\}$ to an orthonormal basis $(a_1, \ldots, a_n)$ of ℓ_2^n and note that $\|Q_1(a_i)\|_X \leqslant 1$ for every i. For every $\varphi \in X^*$, we have

$$(Q_1 Q_1^*)(\varphi) = \sum_{i=1}^n \overline{(Q_1(a_i))(\varphi)} Q_1(a_i)$$

and therefore $\|Q_1 Q_1^*\|_{N(\overline{X}^* \to X)} \leqslant \sum_{i=1}^n \|Q_1(a_i)\|_X^2 < n$. Using Lemma 3.4, we find that $\rho_\infty(X) = \tau_\infty(\mathrm{id}_X) < n$.

4.3. Proof of Theorem 1.2

We start with an easy lemma: if the John and Loewner ellipsoids are proportional, they realize the infimum in the Banach–Mazur distance to the Euclidean space.

Lemma 4.5. — *Let X be an n-dimensional normed space with John ellipsoid $\mathcal{E}$ and with Loewner ellipsoid $\alpha\mathcal{E}$ for some $\alpha \geqslant 1$. Then $\alpha = \mathrm{d}_X$.*

Proof. — The inequality $\mathrm{d}_X \leqslant \alpha$ is immediate. Conversely, if an ellipsoid $\mathcal{F} \subset X$ and a number $\beta \geqslant 1$ satisfy $\mathcal{F} \subset B_X \subset \beta\mathcal{F}$, then by the definition of the John and Loewner ellipsoids,

$$\mathrm{vol}(\mathcal{F}) \leqslant \mathrm{vol}(\mathcal{E}), \quad \mathrm{vol}(\beta\mathcal{F}) \geqslant \mathrm{vol}(\alpha\mathcal{E})$$

from which we infer that $\beta \geqslant \alpha$. □

Proof of Theorem 1.2. — We already noticed in (4.1) that the lower bound $\rho_\infty(X) \geqslant n/\mathrm{d}_X$ holds for every n-dimensional space X. Set $\alpha = \mathrm{d}_X$, let $Q_1 : \ell_2^n \to X$ such that $Q_1(B_2^n)$ is the John ellipsoid of X and set $Q_2 = Q_1^{-1}$. By Lemma 4.5, $\alpha Q_1(B_2^n)$ is the Loewner ellipsoid of X. Applying both inequalities from Theorem 2.1 gives

$$\|Q_2^* Q_2\|_{N(X \to \overline{X}^*)} \leqslant n$$
$$\|(\alpha Q_1)(\alpha Q_1^*)\|_{N(\overline{X}^* \to X)} \leqslant n.$$

Using Lemma 3.4, this implies

$$\rho_\infty(X) \leqslant \|Q_2^* Q_2\|^{1/2}_{N(X \to \overline{X}^*)} \|Q_1 Q_1^*\|^{1/2}_{N(\overline{X}^* \to X)} \leqslant \frac{n}{\alpha},$$

as needed. □

As a corollary to Theorem 1.2, we compute the tensor radius of many natural examples of finite-dimensional normed spaces:

Corollary 4.6. — *For $n \geqslant 1$ and $1 \leqslant p \leqslant \infty$,*

- *the tensor radius of the space ℓ_p^n equals*

$$\rho_\infty\left(\ell_p^n\right) = n^{1-\left|\frac{1}{2}-\frac{1}{p}\right|}$$

- *the tensor radius of the space $\mathcal{S}_p^n$, defined as the space of $n \times n$ matrices equipped with Schatten p-norm $\|X\|_{\mathcal{S}_p} = (\mathrm{Tr}[|X|^p])^{1/p}$, equals*

$$\rho_\infty\left(\mathcal{S}_p^n\right) = n^{2-\left|\frac{1}{2}-\frac{1}{p}\right|}.$$

5. Tensor radii of operators

5.1. Proof of Theorem 1.5

We start by restating Theorem 1.5:

Theorem 5.1 (Theorem 1.5, restated). — *Consider $n \in \mathbb{N}$ and a finite-dimensional normed space Y. For every operator $T \in L(\ell_2^n, Y)$ or $T \in L(Y, \ell_2^n)$, we have*

$$\tau_\infty(T) = \|T\|_N. \tag{5.1}$$

Proof. — The inequality $\tau_\infty(T) \leqslant \|T\|_N$ holds by (3.3). Consider $T \in L(\ell_2^n, Y)$, and let $Q \in L(Y, \ell_2^n)$ such that $\|Q\| \leqslant 1$. Let $\mathrm{d}U$ be the Haar measure on the orthogonal group $O(n)$ (in the complex case, the same proof works by considering the unitary group $U(n)$). Define $R : \ell_2^n \to \ell_2^n$ by

$$R := \int_{O(n)} U^{-1} Q T U \,\mathrm{d}U. \tag{5.2}$$

By (2.3), we have $R = \alpha \, \mathrm{id}_{\ell_2^n}$ with $\alpha = \mathrm{Tr}[QT]/n$. On the other hand, for every $k \in \mathbb{N}$, we have

$$R^{\otimes k} = \int_{O(n)} \cdots \int_{O(n)} \big(U_1^{-1} \otimes \cdots \otimes U_k^{-1}\big)(QT)^{\otimes k}(U_1 \otimes \cdots \otimes U_k) \mathrm{d}U_1 \ldots \mathrm{d}U_k.$$

Using the triangle inequality, the ideal property of the operator norm, and taking the k^{th} root shows that $\tau_k(R) \leqslant \tau_k(QT)$. Taking the limit $k \to \infty$ gives

$$\tau_\infty(R) \leqslant \tau_\infty(QT) \leqslant \tau_\infty(T).$$

We have $\tau_\infty(R) = \alpha \tau_\infty(\mathrm{id}_{\ell_2^n}) = \mathrm{Tr}[QT]$, using Theorem 4.2. We proved that

$$\sup_{\|Q\| \leqslant 1} \mathrm{Tr}[QT] \leqslant \tau_\infty(T) \tag{5.3}$$

By trace duality (cf (2.1)), the left-hand side of (5.3) is exactly the nuclear norm $\|T\|_N$, completing the proof.

If $T \in L(Y, \ell_2^n)$, the proof is completely similar by considering the supremum of $\mathrm{Tr}[QT]$ over $Q : \ell_2^n \to Y$ such that $\|Q\| \leqslant 1$. □

The following corollary establishes a general lower bound on τ_∞ in terms of the nuclear norm. This also generalizes the bound from (4.1) on ρ_∞.

Corollary 5.2. — *Let X, Y be finite dimensional normed spaces. We have*

$$\tau_\infty(T) \geqslant \frac{\|T\|_N}{\min(\mathrm{d}_X, \mathrm{d}_Y)}, \tag{5.4}$$

for every operator $T \in L(X, Y)$.

Proof. — Assume that $\dim X = n$ and that $\mathrm{d}_X \leqslant \mathrm{d}_Y$ (if $\mathrm{d}_Y > \mathrm{d}_X$, then the proof works in the same way). Consider a bijection $U \in L(\ell_2^n, X)$. We have

$$\|T\|_N \leqslant \|TU\|_N \big\|U^{-1}\big\| = \tau_\infty(TU) \big\|U^{-1}\big\| \leqslant \tau_\infty(T) \|U\| \cdot \big\|U^{-1}\big\|$$

where the equality follows from Theorem 5.1, and the inequalities from the ideal property of τ_∞ and $\| \cdot \|_N$. The result follows by taking the infimum over U. □

5.2. Normed spaces without the nuclear tensorization property

Theorem 1.1 shows that for every non-Euclidean space X the pair (X, X) does not have the nuclear tensorization property (NTP, see Definition 1.3). Moreover, Theorem 1.5 shows that (X, Y) has the NTP when X or Y

is Euclidean, and one may ask whether the converse holds. In the case of spaces of the same dimension, the following criterion shows that a large class of examples fails the NTP.

Proposition 5.3. — *Let X, Y be finite-dimensional normed spaces of the same dimension n. Let $\mathcal{E}$ be the John ellipsoid of X and $\mathcal{F}$ be the Loewner ellipsoid of Y. Assume that $T : X \to Y$ is a linear map such that $T(\mathcal{E}) = \mathcal{F}$ and such that, whenever x is a John contact point of X, $T(x)$ is not a Loewner contact point of Y. Then $\tau_\infty(T) < \|T\|_{N(X\to Y)}$ and therefore (X, Y) fails the NTP.*

Proof. — Let $u : \ell_2^n \to X$ such that $u(B_2^n)$ is the John ellipsoid of X. By Corollary 2.2, we have the inequalities $\|(uu^*)^{-1}\|_{N(X\to\overline{X}^*)} \leqslant n$ and $\|(Tu)(Tu)^*\|_{N(\overline{Y}^*\to Y)} \leqslant n$. Applying Lemma 3.4 for $Q_1 = Tu$ and $Q_2 = u^{-1}$ gives

$$\tau_\infty(T) \leqslant \|(Tu)(Tu)^*\|^{1/2}_{N(\overline{Y}^*\to Y)} \left\|u^{-1}\left(u^{-1}\right)^*\right\|^{1/2}_{N\left(X\to\overline{X}^*\right)} \leqslant n.$$

Let $a \in \ell_2^n$ such that $\|a\|_2 = 1$. We have $\|u(a)\|_X \leqslant 1 \leqslant \|Tu(a)\|_Y$. The first inequality is an equality when $u(a)$ is a John contact point of X and the second inequality is an equality when $Tu(a)$ is a Loewner contact point of Y. By hypothesis, both cannot be equalities and therefore $\|u(a)\|_X < \|Tu(a)\|_Y$ and $\|T^{-1}\|_{Y\to X} < 1$ by compactness. By trace duality, we have

$$n = \mathrm{tr}(\mathrm{id}_X) \leqslant \|T\|_{N(X\to Y)} \left\|T^{-1}\right\|_{Y\to X} < \|T\|_{N(X\to Y)}$$

and therefore $\tau_\infty(T) < \|T\|_{N(X\to Y)}$. □

Proposition 5.3 can be applied when X (resp., Y) has few John (resp., Loewner) contact points. For example, real spaces with polyhedral unit balls have finitely many John and Loewner contact points; in this case the hypothesis of Proposition 5.3 is satisfied for a generic map T. We now consider the case of a pair (ℓ_∞^2, Y).

Proposition 5.4. — *Let Y be a finite-dimensional normed space. The pair (ℓ_∞^2, Y) has the NTP if and only if Y is Euclidean.*

Proof. — It suffices to show that (ℓ_∞^2, Y) fails the NTP is we assume that Y is not Euclidean. We use a classical characterization of Euclidean spaces [17]: a norm $\|\cdot\|$ is Euclidean if and only if the inequality $\|x + y\|^2 + \|x - y\|^2 \geqslant 4$ holds for every unit vectors x, y. (Note that while [17] considers only real spaces, the characterization extends easily to complex spaces, see [1, p. 3]). Since Y is not Euclidean, it contains unit vectors x, y such that $\|x + y\|^2 + \|x - y\|^2 < 4$.

Consider the operator $T : \ell_\infty^2 \to Y$ given by $T(a,b) = ax + by$. By [18, Proposition 8.7], we have $\|T\|_{N(\ell_\infty^2 \to Y)} = \|x\| + \|y\| = 2$. We now prove that

$$\tau_\infty(T) \leqslant \left(\|x+y\|^2 + \|x-y\|^2\right)^{1/2} < 2. \tag{5.5}$$

To prove (5.5), write $T = Q_1 Q_2$, where $Q_1 : \ell_2^2 \to Y$ is given by $Q_1(a,b) = a(x+y) + b(x-y)$ and $Q_2 : \ell_\infty^2 \to \ell_2^2$ is given by $Q_2(a,b) = \frac{1}{2}(a+b, a-b)$. By Lemma 3.4, we have

$$\tau_\infty(T) \leqslant \|Q_1 Q_1^*\|_{N\left(\overline{Y^*} \to Y\right)}^{1/2} \|Q_2^* Q_2\|_{N\left(\ell_\infty^2 \to \overline{\ell_1^2}\right)}^{1/2}$$

Since $Q_2^* Q_2$ is the map $(a,b) \mapsto \frac{1}{2}(\bar{a}, \bar{b})$, we have

$$\|Q_2^* Q_2\|_{N\left(\ell_\infty^2 \to \overline{\ell_1^2}\right)} = \frac{1}{2} \| \operatorname{id}_{\mathbb{K}^2} \|_{N\left(\ell_\infty^2 \to \ell_1^2\right)} = 1.$$

The map $Q_1 Q_1^*$ can be decomposed as $\overline{(x+y)(\cdot)}(x+y) + \overline{(x-y)(\cdot)}(x-y)$; this decomposition gives the bound $\|Q_1 Q_1^*\|_{N(Y^* \to Y)} \leqslant \|x+y\|^2 + \|x-y\|^2$. We obtained the inequality $\tau_\infty(T) < 2 = \|T\|_{N(\ell_\infty^2 \to Y)}$, showing that (ℓ_∞^2, Y) fails the NTP. ☐

Let Y be a normed space. We say that a subspace $Y' \subset Y$ is 1-*complemented* if there is a projection $P : Y \to Y'$ with $\|P\| = 1$. The next lemma shows that the NTP is inherited by complemented subspaces.

Lemma 5.5. — *Let X, Y be finite-dimensional normed spaces and $X' \subset X$, $Y' \subset Y$ be 1-complemented subspaces. If (X,Y) has the NTP, then (X', Y') has the NTP.*

Proof. — Let $\iota_1 : X' \to X$, $\iota_2 : Y' \to Y$ be the inclusion maps, and $P_1 : X \to X'$, $P_2 : Y \to Y'$ be projections of norm 1. Let $T \in L(X', Y')$ a linear map and set $S = \iota_2 T P_1 \in L(X,Y)$. The ideal property of the nuclear norm and the NTP for (X,Y) imply that

$$\|T\|_{N(X' \to Y')} = \|P_2 S \iota_1\|_{N(X' \to Y')} \leqslant \|S\|_{N(X \to Y)} = \tau_\infty(S).$$

Similarly, the ideal property of the tensor radius (see Lemma 3.3) implies that $\tau_\infty(S) \leqslant \tau_\infty(T)$. We conclude that $\tau_\infty(T) = \|T\|_{N(X' \to Y')}$ and the result follows. ☐

Let X be a finite-dimensional normed space containing a subspace isometric to ℓ_∞^2 (this includes in particular the spaces ℓ_∞^n and, in the real case, the spaces ℓ_1^n). Such a subspace is necessarily 1-complemented (this follows from the Hahn–Banach theorem). It follows from Proposition 5.4 and Lemma 5.5 that (X,Y) fails the NTP whenever Y is not Euclidean.

Our methods also imply that the ℓ_p-spaces over the reals, denoted by $\ell_p^n(\mathbb{R})$, only have the NTP in the Euclidean case.

PROPOSITION 5.6. — *Let $m, n \geqslant 2$ and $p, q \in [1, \infty]$. If $(\ell_p^m(\mathbb{R}), \ell_q^n(\mathbb{R}))$ has the NTP, then $p = 2$ or $q = 2$.*

Proof. — By Lemma 5.5, we may assume that $m = n = 2$ (ℓ_p^n contains a 1-complemented subspace isometric to ℓ_p^2). If $p \neq 2$, the John contact points of ℓ_p^2 (which coincide with the Loewner contact points of ℓ_{p*}^2 if $1/p + 1/p^* = 1$), are the following 4 points (up to normalization)

$$\begin{cases} (\pm 1, 0) \text{ or } (0, \pm 1) & \text{if } p > 2 \\ (\pm 1, \pm 1) & \text{if } p < 2. \end{cases}$$

If both $p \neq 2$ and $q \neq 2$, the fact that the pair (ℓ_p^2, ℓ_q^2) does not have the NTP follows from Proposition 5.3, by choosing $T : \mathbb{R}^2 \to \mathbb{R}^2$ to be a rotation of angle not multiple of $\pi/4$. The result follows. □

Some natural pairs of spaces are not covered by the criterion of Proposition 5.3. Examples of such pairs, for which we do not know whether they have the NTP, are

(1) the pair (X, X^*), where X is the space $\mathbb{R}^2$ equipped with the norm
$$\max(\|\cdot\|_2, (1+\varepsilon)\|\cdot\|_\infty)$$
for some small $\varepsilon > 0$,
(2) the pair (X, X^*), where X is the space $\ell_1^2 \otimes_\pi \ell_2^2$,
(3) in the complex case, the pair $(\ell_1^2, \ell_\infty^2)$ (the previous example is obtained by forgetting the complex structure). (Over $\mathbb{C}$, ℓ_1^2 and ℓ_∞^2 are not isometric, so this is not covered by Proposition 5.4).

Note that these examples have enough symmetries.

6. Further questions about tensor radii and some answers

6.1. Is the tensor radius continuous, or even a norm?

One may wonder whether τ_k is a norm on the space $L(X, Y)$. Here is an example showing that this is not the case in full generality.

Example 6.1. — Over the real field, consider the operators $S, T : \ell_1^2 \to \ell_1^2$ given by the following matrices

$$S = \begin{pmatrix} 1 & 1/3 \\ 1/3 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}.$$

One computes that $\tau_2(S) = \sqrt{2}$, $\tau_2(S) = \frac{3}{2\sqrt{2}}$ and $\tau_2(S+T) = \sqrt{\frac{155}{24}}$, so that

$$2.54\ldots \approx \tau_2(S+T) > \tau_2(S) + \tau_2(T) \approx 2.47\ldots$$

To obtain these values, we used the fact that the extreme points of the unit ball for the norm $\varepsilon_2(\ell_1^2)$ are given by $\pm e_i \otimes e_j$ and $\frac{1}{2}(\eta_1 e_1 \otimes e_1 + \eta_2 e_1 \otimes e_2 + \eta_3 e_2 \otimes e_1 + \eta_4 e_2 \otimes e_2)$, where (η_i) is a permutation of either $(1,1,1,-1)$ or $(1,-1,-1,-1)$.

Despite this counterexample, it turns out that τ_2 is a norm in the special case of $L(\ell_2^m, \ell_2^n)$. We have the following proposition, observed without proof in [10]

Proposition 6.2. — *Let m, n be integers. For $T \in L(\ell_2^m, \ell_2^n)$, the quantity $\tau_2(T)$ coincides with the Hilbert–Schmidt norm given by $\|T\|_{\mathrm{HS}} = (\mathrm{Tr}[TT^*])^{1/2}$.*

Proof. — Consider $T \in L(\ell_2^m, \ell_2^n)$. If $U \in L(\ell_2^m, \ell_2^m)$ and $V \in L(\ell_2^n, \ell_2^n)$ are isometries, we have $\tau_2(UTV) = \tau_2(T)$ by the ideal property of τ_2 (Lemma 3.3). Using the singular value decompositions, it suffices to prove the result when T is diagonal, i.e., of the form $T = \sum_{i=1}^{\min(m,n)} \lambda_i e_i e_i^*$ for $\lambda_i \geqslant 0$. We have then

$$\begin{aligned}\tau_2(T)^2 &= \sup\left\{ |\langle (T\otimes T)(x), y\rangle| : \|x\|_{\ell_2^m \otimes_\varepsilon \ell_2^m} \leqslant 1,\ \|y\|_{\ell_2^n \otimes_\varepsilon \ell_2^n} \leqslant 1 \right\} \\ &= \sup\left\{ \left|\mathrm{Tr}\left[TXTY^t\right]\right| : \|X\|_{\ell_2^m \to \ell_2^m} \leqslant 1,\ \|Y\|_{\ell_2^n \to \ell_2^n} \leqslant 1 \right\}\end{aligned}$$

after identifying tensors with operators. For X, Y as above, we have

$$\left|\mathrm{Tr}\left[TXTY^t\right]\right| \leqslant \|T\|_{\mathrm{HS}} \left\|XTY^t\right\|_{\mathrm{HS}} \leqslant \|T\|_{\mathrm{HS}}^2$$

with equality when $X = \mathrm{id}_{\ell_2^m}$, $Y = \mathrm{id}_{\ell_2^n}$, proving the result. □

We do not know whether τ_k is a norm on $L(\ell_2^m, \ell_2^n)$ for any other $k \in \mathbb{N}$. It would also be interesting to decide whether the tensor radius τ_∞ is a norm on $L(X,Y)$ (that question also appears in [11]). By Theorem 1.5 the answer to this question is positive when X or Y is Euclidean. However, in general we do not know the answer. We will now show a weaker version of the triangle inequality, which implies that the tensor radius is a continuous function.

Proposition 6.3. — *For finite-dimensional normed spaces X, Y and $S, T \in L(X,Y)$, we have*

$$\tau_\infty(S+T) \leqslant \tau_\infty(S) + \|T\|_N.$$

Using Corollary 5.2, the previous proposition implies:

Corollary 6.4 (Weak triangle inequality). — *Let X, Y be finite-dimensional normed spaces and $S, T \in L(X, Y)$. Then*

$$\tau_\infty(S+T) \leqslant \tau_\infty(S) + \min(\mathrm{d}_X, \mathrm{d}_Y)\tau_\infty(T).$$

We first prove a lemma:

Lemma 6.5. — *For $T \in L(X_1, Y_1)$, $y \in Y_2$, and $x^* \in X_2^*$, we have*

$$\|T \otimes yx^*\|_{X_1 \otimes_\varepsilon X_2 \to Y_1 \otimes_\pi Y_2} = \|T\|_{X_1 \to Y_1} \|x^*\|_{X_2^*} \|y\|_{Y_2}.$$

Proof. — For any $z \in X_1 \otimes X_2$ we have

$$\|(T \otimes yx^*)(z)\|_{Y_1 \otimes_\pi Y_2} = \|T(z_x)\|_{Y_1} \|y\|_{Y_2},$$

for $z_x = (\mathrm{id}_{X_1} \otimes x^*)(z) \in X_1$, by the metric mapping property of π. Since $\|z_x\|_{X_1} \leqslant \|x^*\|_{X_2^*} \|z\|_{X_1 \otimes_\varepsilon X_2}$, we conclude that

$$\|(T \otimes yx^*)(z)\|_{Y_1 \otimes_\pi Y_2} \leqslant \|T\|_{X_1 \to Y_1} \|x^*\|_{X_2^*} \|y\|_{Y_2} \|z\|_{X_1 \otimes_\varepsilon X_2},$$

showing one direction of the identity in the lemma. The other direction follows by inserting $z = z_1 \otimes z_2$ for a suitable choice of z_1 and z_2. ☐

Proof of Proposition 6.3. — Let $T = \sum_i y_i x_i^*$ denote a nuclear decomposition. For integers $j \leqslant k$, Lemma 6.5 implies

$$\begin{aligned}
&\left\| S^{\otimes(k-j)} \otimes T^{\otimes j} \right\|_{\varepsilon_{k-j}(X) \otimes_\varepsilon \varepsilon_j(X) \to \pi_{k-j}(Y) \otimes_\pi \pi_j(Y)} \\
&\leqslant \sum_{i_1, \ldots, i_j} \left\| S^{\otimes(k-j)} \otimes y_{i_1} x_{i_1}^* \otimes \cdots \otimes y_{i_j} x_{i_j}^* \right\|_{\varepsilon_{k-j}(X) \otimes_\varepsilon \varepsilon_j(X) \to \pi_{k-j}(Y) \otimes_\pi \pi_j(Y)} \\
&\leqslant \left\| S^{\otimes(k-j)} \right\|_{\varepsilon_{k-j}(X) \to \pi_{k-j}(Y)} \left(\sum_i \|y_i\|_Y \|x_i^*\|_{X^*} \right)^j.
\end{aligned}$$

Using the associativity and commutativity of the injective and projective tensor products, we may write

$$\begin{aligned}
\tau_k(S+T)^k &= \left\| (S+T)^{\otimes k} \right\|_{\varepsilon_k(X) \to \pi_k(Y)} \\
&\leqslant \sum_{j=0}^k \binom{k}{j} \left\| S^{\otimes(k-j)} \otimes T^{\otimes j} \right\|_{\varepsilon_{k-j}(X) \otimes_\varepsilon \varepsilon_j(X) \to \pi_{k-j}(Y) \otimes_\pi \pi_j(Y)} \\
&\leqslant \sum_{j=0}^k \binom{k}{j} \left\| S^{\otimes(k-j)} \right\|_{\varepsilon_{k-j}(X) \to \pi_{k-j}(Y)} \left(\sum_i \|y_i\|_Y \|x_i^*\|_{X^*} \right)^j \\
&\leqslant \left(\tau_\infty(S) + \sum_i \|y_i\|_Y \|x_i^*\|_{X^*} \right)^k.
\end{aligned}$$

We conclude by taking $k \to \infty$ and optimizing over nuclear decompositions of T. ☐

6.2. Is the tensor radius multiplicative?

Another natural question is whether the quantities τ_∞ and ρ_∞ are multiplicative under taking injective or projective tensor products. The examples for which we know ρ_∞ immediately give a counterexample:

$$\rho_\infty\big(\ell_2^d \otimes_\pi \ell_2^d\big) = \rho_\infty\big(S_1^d\big) = d^{3/2} < d^2 = \rho_\infty\big(\ell_2^d\big)^2.$$

However, it turns out that τ_∞ and ρ_∞ are submultiplicative in the following sense:

Proposition 6.6. — *Let X, Y be finite dimensional normed spaces and $T \in L(X,Y)$. Consider a cross norm $\otimes_\alpha$ on $X \otimes X$ and a cross norm $\otimes_\beta$ on $Y \otimes Y$. Then*

$$\tau_\infty([T \otimes T : X \otimes_\alpha X \longrightarrow Y \otimes_\beta Y]) \leqslant \tau_\infty(T)^2,$$

In particular, we have

$$\rho_\infty(X \otimes_\alpha X) \leqslant \rho_\infty(X)^2,$$

for any tensor norm $\otimes_\alpha$.

We start with the following lemma.

Lemma 6.7. — *For finite-dimensional normed spaces X and Y, we have*

$$\|\cdot\|_{\pi_k(X\otimes_\alpha Y)} \leqslant \|\cdot\|_{\pi_k(X\otimes_\pi Y)} \text{ and } \|\cdot\|_{\varepsilon_k(X\otimes_\alpha Y)} \geqslant \|\cdot\|_{\varepsilon_k(X\otimes_\varepsilon Y)},$$

for every cross norm $\otimes_\alpha$ on $X \otimes Y$.

Proof. — Combine the metric mapping property of π_k and ε_k and the inequality $\|\cdot\|_\varepsilon \leqslant \|\cdot\|_\alpha \leqslant \|\cdot\|_\pi$ on $X \otimes Y$. □

We can now present the proof of Proposition 6.6:

Proof of Proposition 6.6. — We consider the operator $T \otimes T : X \otimes_\alpha X \to Y \otimes_\beta Y$. For every $k \in \mathbb{N}$ we have

$$\begin{aligned}\tau_k(T\otimes T)^k &= \sup_{z\in(X\otimes X)^{\otimes k}} \frac{\big\|(T\otimes T)^{\otimes k}(z)\big\|_{\pi_k(Y\otimes_\alpha Y)}}{\|z\|_{\varepsilon_k(X\otimes_\beta X)}} \\ &\leqslant \sup_{z\in(X\otimes X)^{\otimes k}} \frac{\big\|(T\otimes T)^{\otimes k}(z)\big\|_{\pi_k(Y\otimes_\pi Y)}}{\|z\|_{\varepsilon_k(X\otimes_\varepsilon X)}} = \tau_{2k}(T)^{2k}.\end{aligned}$$

Taking the k^{th} root and the limit $k \to \infty$ finishes the proof. □

Note that $X \otimes_\alpha X$ has enough symmetries for any tensor norm α and any X with enough symmetries (see [14, p. 62]). Combining this fact with the previous proposition and with Theorem 1.2 leads to an estimate of independent interest:

Corollary 6.8. — *Let X be an n-dimensional normed space with enough symmetries and α a tensor norm. Then*

$$\mathrm{d}_{X\otimes_\alpha X} \geqslant \mathrm{d}_X^2.$$

A natural question is whether the assumption that X has enough symmetries can be removed. More generally, we could ask if for finite-dimensional normed spaces X and Y the inequality

$$\mathrm{d}_{X\otimes_\alpha Y} \geqslant \mathrm{d}_X \mathrm{d}_Y$$

holds for every tensor norm α.

6.3. When is the tensor radius minimal?

Another natural problem is to determine the n-dimensional spaces for which the tensor radius equals $\sqrt{n}$, the minimal possible value. This is achieved by both ℓ_1^n and ℓ_∞^n, but there are many more examples. The construction from [2, Example 1] produces a continuum of normed spaces $(\mathbb{R}^n, \|\cdot\|)$ for which B_2^n is the John ellipsoid and $\sqrt{n}B_2^n$ is the Loewner ellipsoid; by Theorem 1.2, each such space has a tensor radius equal to $\sqrt{n}$. Other examples can be produced by using Corollary 6.8: For any $n \in \mathbb{N}$ and any tensor norm α, consider the space $X = \ell_1^n \otimes_\alpha \ell_1^n$. The space X has enough symmetries and satisfies $\mathrm{d}_X = n = \sqrt{\dim(X)}$, and we conclude by Theorem 1.2 that $\rho_\infty(X) = \sqrt{\dim(X)}$. We should also note that Corollary 6.8 produces many examples of spaces with maximal distance to Euclidean in this way.

Another question concerns the lower bound

$$\rho_\infty(X) \geqslant \frac{n}{\mathrm{d}_X},$$

which holds for every n-dimensional normed space X: are there spaces for which this inequality is strict?

7. Infinite dimensions

While we focused exclusively on finite-dimensional normed spaces, the tensor radii also make sense for infinite-dimensional Banach spaces. Accordingly, given Banach spaces X, Y and $T : X \to Y$ a bounded linear operator, we may define $\tau_k(T)$ for $k \in \mathbb{N} \cup \{+\infty\}$ exactly as in the finite-dimensional case (note that in that, case, the supremum in (1.1) is taken over z in the algebraic product $X^{\otimes k}$). The elementary inequality $\tau_\infty(T) \leqslant \|T\|_N$ holds in generality (the nuclear norm $\|\cdot\|_N$ is defined in [16, p. 41]), and nuclear operators have a finite tensor radius . The class of operators for which τ_k is finite (at fixed k) has been discussed in a series of papers by John [10, 12, 13].

Whenever X is an infinite-dimensional Banach space, we have $\rho_3(X) = +\infty$ (i.e., the injective and projective norms are not equivalent on $X \otimes X \otimes X$) and therefore $\rho_\infty(X) = +\infty$. An argument for this is given in [11, Section 4.5]. This has to be compared with the famous example by Pisier [15] of an infinite-dimensional space X for which $\rho_2(X) < +\infty$, answering a question by Grothendieck [8].

We now explain how Theorem 5.1 translates to the infinite-dimensional setting. Consider two Banach spaces X and Y and assume that one of them is a Hilbert space. Let $T : X \to Y$ be a bounded operator. An alternative description of $\|T\|_N$ can be given by trace duality: it is equal to the *integral norm* of T, defined as

$$\|T\|_I = \sup\left\{\operatorname{Tr}[QT] : \|Q\|_{Y\to X} \leqslant 1,\ \operatorname{rank}(Q) < \infty\right\}. \tag{7.1}$$

We point out that, for general Banach spaces, the integral and nuclear norm are not equal. However, both quantities coincide when X or Y is a Hilbert space (or, more generally, when X^* or Y has the metric approximation property, see [16, Corollary 4.17]).

Theorem 7.1. — *Let H be a Hilbert space, X be a Banach space. Let $T \in L(X,H)$ and $S \in L(H,X)$ be bounded operators. Then $\tau_\infty(T) = \|T\|_N$ and $\tau_\infty(S) = \|S\|_N$.*

Proof. — For every finite rank operators $Q : H \to X$ and $R : X \to H$ such that $\|Q\| \leqslant 1$, $\|R\| \leqslant 1$, we have

$$\operatorname{Tr}[TQ] \leqslant \tau_\infty(T) \quad \text{and} \quad \operatorname{Tr}[RS] \leqslant \tau_\infty(S).$$

These inequalities are obtained by mimicking the proof of Theorem 5.1, identifying TQ or RS as operator on ℓ_2^n for some n. The result follows from (7.1). ☐

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Guillaume AUBRUN
Institut Camille Jordan,
Université Claude Bernard Lyon 1,
43 boulevard du 11 novembre 1918,
69622 Villeurbanne cedex (France)
aubrun@math.univ-lyon1.fr

Alexander MÜLLER-HERMES
Institut Camille Jordan,
Université Claude Bernard Lyon 1,
43 boulevard du 11 novembre 1918,
69622 Villeurbanne cedex (France)
Department of Mathematics,
University of Oslo,
P.O. Box 1053, Blindern,
0316 Oslo(Norway)
muellerh@math.uio.no