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A UNIQUENESS RESULT FOR A TWO-DIMENSIONAL VARIATIONAL PROBLEM

by Benjamin LLEDOS

Abstract. — We investigate the uniqueness of the solutions for a non-strictly convex problem in the Calculus of Variations of the form $\int \varphi(\nabla v) - \lambda v$. Here, φ is a convex function define on $\mathbb{R}^2$ and λ is Lipschitz continuous. We establish the uniqueness of the solutions when the gradient of λ is small and give some counterexamples when that is not the case. The proof is based on the global Lipschitz regularity of the minimizers and on the study of their level sets.

Résumé. — Nous étudions l'unicité des solutions d'un problème non strictement convexe en calcul des variations de la forme $\int \varphi(\nabla v) - \lambda v$. Ici, φ est une fonction convexe définie sur $\mathbb{R}^2$ et λ est une fonction lipschitzienne. Nous établissons l'unicité des solutions lorsque le gradient de λ est petit et donnons des contre-exemples lorsque ce n'est pas le cas. La preuve est basée sur la régularité lipschitzienne globale des minimiseurs et sur l'étude de leurs ensembles de niveau.

1. Introduction

1.1. A model case

The motivation of this article is to study non-strictly convex problems in the Calculus of Variations in dimension two, as in the following model case:

$$\widetilde{\mathcal{I}}_\lambda : u \longmapsto \int_\Omega F(\nabla u(x)) - \lambda(x)u(x)\mathrm{d}x \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^2$, $\lambda \in L^\infty(\Omega)$ and $F(y) = f(|y|)$ with

$$f(t) = \begin{cases} \frac{1}{2}|t|^2 & \text{if } |t| \leqslant 1, \\ |t| - \frac{1}{2} & \text{if } 1 < |t| < 2, \\ \frac{1}{4}|t|^2 + \frac{1}{2} & \text{if } 2 \leqslant |t|. \end{cases} \tag{1.2}$$

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For this functional, the admissible functions u belong to $W_{\psi}^{1,2}(\Omega)$, which is the subset of the Sobolev space $W^{1,2}(\Omega)$ of functions that have a prescribed trace $\psi : \mathbb{R}^2 \to \mathbb{R}$ on the Lipschitz boundary $\partial\Omega$ of Ω. The objective is to demonstrate the uniqueness of solutions for the subsequent minimization problem:

$$\widetilde{\mathcal{P}}_\lambda : \min_{u \in W_{\psi}^{1,2}(\Omega)} \widetilde{\mathcal{I}}_\lambda(u).$$

When $\lambda \equiv \lambda_0 \in \mathbb{R}_+$, this problem studied by Kawohl, Stara and Wittum in [15] arises as the convexification of a non-convex problem of shape optimization in the theory of elasticity. In this example, f is the convexification of the minimum of two parabolas: $t \mapsto \frac{1}{2}|t|^2$ and $t \mapsto \frac{1}{4}|t|^2 + \frac{1}{2}$. Observe in particular that f is affine on the interval $(1, 2)$. Since f is convex but not strictly convex, there is no obvious reason for $\widetilde{\mathcal{P}}_\lambda$ to have only one solution.

In fact, the authors of [15] rely on the assumption that the level sets of one minimizer u are star-shaped. Furthermore, they suppose that the boundary of the set in Ω where $f'(|\nabla u|) = 1$ is piecewise $\mathcal{C}^1$. In this paper, we do not require such additional assumptions.

There is no general answer to the question of uniqueness for non strictly convex problems in the Calculus of Variations, especially when λ is not constant. For this reason, we restrict our attention to the framework (1.1) where f can be replaced by more general convex functions provided that they are strictly convex around the origin and at infinity.

1.2. Main results

More precisely, let $g : \mathbb{R} \to \mathbb{R}$ be an even $\mathcal{C}^1$ convex function with $g(t) > g(0) = 0$ for all $t \neq 0$. Additionally, we assume that $g \in \mathcal{C}^{1,1}_{\mathrm{loc}}(\mathbb{R}\backslash\{0\})$.

We suppose that g has p-growth for $p > 1$, namely, there exist $C_1 > 0$ and $C_2 > 0$ such that:

$$C_1|t|^p \leqslant g(t) \leqslant C_2(1 + |t|^p) \text{ for all } t \in \mathbb{R}. \tag{1.3}$$

We introduce the following set of strict convexity of g:

$$\mathcal{SC} = \left\{ \begin{matrix} x \in \mathbb{R}, \quad \forall\, y \in \mathbb{R}\backslash\{x\}, \quad \forall\, t \in\,]0,1[, \\ g(tx + (1-t)y) < tg(x) + (1-t)g(y) \end{matrix} \right\}. \tag{1.4}$$

For instance, $\mathcal{SC} = (-\infty, -2) \cup (-1, 1) \cup (2, +\infty)$ when g is equal to the function f in (1.2).

We make some *structural assumptions* on $\mathcal{SC}$:

- The set $\mathcal{SC}$ has finitely many connected components, in particular $\mathcal{SC}$ is open and

$$\mathcal{SC} \cap \mathbb{R}_+ = \bigcup_{n=0}^{N} \mathcal{SC}_n \tag{1.5}$$

  with $\mathcal{SC}_0 := [0, b_0)$, $\mathcal{SC}_n := (a_n, b_n)$ for every $n \in \mathbb{N}^*$, $n < N$ and $\mathcal{SC}_N := (a_N, +\infty)$. For every $n \in \mathbb{N}$, $n < N$ we introduce $d_n := g'(b_n) = g'(a_{n+1})$.
- We assume that g is $\mathcal{C}^2$ and $g'' > 0$ on $\mathcal{SC}\backslash\{0\}$ and that g is strongly convex at $+\infty$ in the following sense:

$$\liminf_{t\to+\infty} \frac{tg''(t)}{g'(t)} > 0.$$

We define $\varphi(\cdot) := g(|\cdot|)$ and for $\lambda \in L^\infty(\Omega)$ we introduce the following functional:

$$\mathcal{I}_\lambda : u \longmapsto \int_\Omega \varphi(\nabla u(x)) - \lambda(x)u(x)\mathrm{d}x$$

on $W^{1,p}_\psi(\Omega)$, where Ω is an open simply connected bounded set of $\mathbb{R}^2$ with a Lipschitz continuous boundary $\partial\Omega$ and ψ is a Lipschitz-continuous function on $\partial\Omega$. Here, $W^{1,p}_\psi(\Omega)$ is the subset of those functions in $W^{1,p}(\Omega)$ that are equal to ψ on $\partial\Omega$.

We introduce the minimization problem:

$$\mathcal{P}_\lambda : \min_{u\in W^{1,p}_\psi(\Omega)} \mathcal{I}_\lambda(u).$$

The main result of the paper is the following:

Theorem 1.1. — *Let Ω be a simply connected bounded open set of $\mathbb{R}^2$. We assume that Ω has a $\mathcal{C}^{1,1}$ boundary, $\psi \in \mathcal{C}^{1,1}(\mathbb{R}^2)$, λ is Lipschitz continuous on $\overline{\Omega}$, $\min_{x\in\overline{\Omega}} \lambda(x) > 0$. There exists a positive constant*

$$C := C\left(N,\ |\Omega|,\ \max_{\overline{\Omega}} \lambda,\ \min_{\overline{\Omega}} \lambda,\ \|\psi\|_{\mathcal{C}^{1,1}(\Omega)},\ \kappa\right)$$

where κ is the essential infimum of the curvature of $\partial\Omega$, such that if $\|\nabla\lambda\|_{L^\infty(\Omega)} \leqslant C$ then $\mathcal{P}_\lambda$ admits a unique solution on $W^{1,p}_\psi(\Omega)$.

Remark 1.2. — When λ is constant, a more general result, true in any dimension, can be found in [17].

Moreover, the boundedness condition on $\nabla\lambda$ is natural since:

Proposition 1.3. — *There exists $\lambda \in \mathcal{C}^\infty(\overline{B_1(0)})$ with $\min_{\overline{B_1(0)}} \lambda > 0$ such that $\mathcal{P}_\lambda$ has more than one solution on $W^{1,p}_0(B_1(0))$.*

1.3. Ideas of the proof

We want to prove the uniqueness of the solution for the variational problem $\mathcal{P}_\lambda$. According to classical theory (see [10, 14]), there exists at least one minimizer, u, to the problem $\mathcal{P}_\lambda$. This function u is bounded, globally Hölder continuous by [14, Theorem 7.8] and locally Lipschitz continuous by [7, Theorem 1.1].

When $\lambda = 0$, the proof is substantially simplified. In this case, the strategy has been developed by Marcellini in [19] under additional assumptions, and the proof itself in a general framework is due to Lussardi and Mascolo in [18]. The proof in those papers is divided into two parts:

(Part 1) If u and v are two solutions of the same problem, then v is constant on the level sets of u.

(Part 2) The level sets of u intersect the boundary $\partial\Omega$ of Ω. Since u and v are equal on $\partial\Omega$, they are equal on Ω.

As noted in Remark 1.2, a shorter proof can be found in [17] when $\lambda \equiv \lambda_0 \in \mathbb{R}_+$. However, when $\lambda \in W^{1,\infty}(\Omega)$ the proof requires new ideas and turns out to be fairly intricate. Although Part 1 remains true, Part 2 is not. The term $u \mapsto \int_\Omega \lambda u$ changes the geometry of the level lines, which may not intersect the boundary $\partial\Omega$ of Ω. It is even possible that only one level set intersects the boundary, see Proposition 2.10.

A very important subset of Ω is the following:

Proposition 1.4. — *There exists an open set U such that for every minimizer u of the same problem $\mathcal{P}_\lambda$, one has $u \in \mathcal{C}^1(U)$ and for every $x \in U$, $|\nabla u(x)| \in \mathcal{SC}\backslash\{0\}$ while for a.e. $x \notin U, |\nabla u(x)| \notin \mathcal{SC}\backslash\{0\}$.*

When $\lambda \neq 0$, the set $U \cup \partial\Omega$ has the same role as $\partial\Omega$ in Part 2 when there is no lower order term. However, the fact that $u = v$ on U is far from being obvious. Nevertheless, if u and v are two minimizers of the same problem, we can easily see that ∇u and ∇v are equal on U. Additionally, we can even prove that this is also true on the level sets that intersect U. The aim of the proof is to demonstrate that $u = v$ or $\nabla u = \nabla v$ on the level sets of u and v. Consequently, for a.e. $x \in \Omega$, the Lipschitz map $w(x) := u(x) - v(x)$ is equal to 0 or $\nabla w(x) = 0$, thus $u - v = w = 0$.

This idea of using $U \cup \partial\Omega$ comes from a paper by Bouchitté and Bousquet [5]. However, in their study, that fact that $\mathcal{SC}$ is of the form $(1, +\infty)$ implies that the boundary of every connected component of U intersects $\partial\Omega$. Since $\nabla u = \nabla v$ on U and $u = v$ on $\partial\Omega$, they readily deduce that $u = v$ on U and this part of the proof is easier. We would like to clarify to the reader that this paper is not a generalization of [5] because g has no singularity at the origin, unlike in [5]. This singularity of g in [5] and in its generalization [16] creates some regularity issues that are not present in this study.

For instance, in our situation, we have that $\max(\alpha, g'(|\nabla u|)) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ for any $\alpha > 0$ thanks to [20]. Then, we prove that $\max(d_0, g'(|\nabla u|))$ has a representative that is absolutely continuous on almost every level sets.

The other major difference between this paper and [5, 16] lies in the continuity of $\max(1, g'(|\nabla u|)$ over Ω, which permits the demonstration of their results for higher dimensions. Conversely, we heavily rely on two results that exclusively hold in dimension two. Firstly, a general regularity result for Lipschitz continuous functions, see Theorem 2.8 below and secondly, the Jordan curve theorem. The latter is the reason why we assume that Ω simply connected: this prevents the existence of holes inside the connected components of the upper level set $E_s := \{u > s\}$ for $s \in \mathbb{R}$.

When λ is sufficiently small, it can be proven that almost every level set intersects a connected component of U in which $|\nabla u| < b_0$. As a result, we are able to establish the following theorem:

Theorem 1.5. — *We assume that φ is as in Section 1.2 and $0 \leqslant \lambda(x) \leqslant d_0 h_\Omega$ for a.e. $x \in \Omega$. Then, the problem $\mathcal{P}_\lambda$ admits a unique minimizer.*

Here, $d_0 = g'(b_0)$ and h_Ω is the *Cheeger* constant of Ω:

Definition 1.6. — *The Cheeger constant of Ω is defined as:*

$$h_\Omega = \inf_{D \subset \overline{\Omega}} \frac{\operatorname{Per}\big(D, \mathbb{R}^2\big)}{|D|}$$

where

$$\operatorname{Per}\big(D, \mathbb{R}^2\big) = \sup\left\{ \int_D \operatorname{div} \theta \,\middle|\, \theta \in C^1_c\big(\mathbb{R}^2; \mathbb{R}^2\big),\ |\theta(x)| \leqslant 1,\ \forall\, x \in \mathbb{R}^2 \right\}$$

is called the Perimeter *of the set D. A set $D \subset \overline{\Omega}$ of finite perimeter with $|D| > 0$ is said to be a* Cheeger set *if* $\operatorname{Per}(D, \mathbb{R}^2) = h_\Omega |D|$.

The proof of the main theorem is based on an induction argument related to the family $\{d_n,\ n \in \mathbb{N},\ 0 \leqslant n < N\}$ with the previous theorem as the initialization step.

We study the connected components $l_s(u)$ of $L_s(u) := u^{-1}(s) \subset \mathbb{R}^2$ such that $l_s(u)$ is a closed simple curve. If $l_s(u) \cap \partial\Omega \neq \emptyset$, then $u - v$ is constant on $l_s(u)$ and $u - v = 0$ on $\partial\Omega$, implying $u - v$ on $l_s(u)$. Hence, we can assume that $l_s(u) \Subset \Omega$. Utilizing the Jordan curve theorem, we can define

F_s as the bounded connected component of $\mathbb{R}^2\backslash l_s(u)$. When $l_s(u)\cap U=\emptyset$ we use the following proposition:

Proposition 1.7. — *There exists a representative f_0 of the function $\max(d_0, g'(|\nabla u|))$ such that for a.e. $s\in\mathbb{R}$, if $l_s(u)\cap U=\emptyset$ then f_0 is equal to a constant $C(l_s(u))\in\{d_i, 0\leqslant i<N\}$ on $l_s(u)$.*

Another important result is a maximum principle proved in Section 5 for smooth approximations of our problem $\mathcal{P}_\lambda$. To begin with, we regularize the problem to obtain a sequence $(u_n)_{n\in\mathbb{N}}$ of smooth minimizers of smooth problems $\mathcal{P}_{\lambda_n}$, where $(g_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ are smooth approximations of g and λ. In Section 4, we use the fact that the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ generates Young measures $(\nu_x)_{x\in\Omega}$ to prove that $g_n'(\nabla u_n)\to g'(\nabla u)$ a.e. in Ω when $n\to+\infty$. For such approximations, we have:

Proposition 1.8. — *For a.e. $s\in\mathbb{R}$, if $l_s(u)$ is a connected component of $L_s(u)$ which is a closed simple curve and such that $l_s(u)\Subset\Omega$ then*

$$\sup_{l_s(u)}\max\big(d_0, g_n'(|\nabla u_n|)\big)=\sup_{F_s} g_n'(|\nabla u_n|).$$

Plan of the paper

In the next section, we recall some classical results, and we introduce the notations and notions useful throughout the article. In Section 3, we study the regularity properties of the level sets of the minimizers. In the subsequent Section 4, we prove that $\max(\alpha, g'(|\nabla u|)\in W^{1,2}_{\mathrm{loc}}(\Omega)$. We prove the maximum principle for $\max(d_0,|\sigma_n|)$ in Section 5. Section 6 is dedicated to the proof of Theorem 1.1 and Theorem 1.5. In the last section, we present a possible extension to the main theorem.

2. Preliminary results

In this section, we introduce some known results related to this problem.

2.1. Direct methods

We know from the direct method in the calculus of variations (see [10, 14]) that the problem $\mathcal{P}_\lambda$ has at least one minimizer. We recall that every minimizer u is bounded, globally Hölder continuous according to [14, Theorem 7.8] and locally Lipschitz continuous by [7, Theorem 1.1].

We begin this subsection by observing that the minimum of a solution is attained on the boundary of Ω.

Proposition 2.1. — *Let u be a minimizer of $\mathcal{P}_\lambda$ on $W^{1,p}_\psi(\Omega)$ with $\lambda \in L^\infty(\Omega)$ and $\lambda(x) \geqslant 0$ for a.e. $x \in \Omega$. Then $\min_{\overline{\Omega}} u = \min_{\partial\Omega} \psi$.*

Proof. — Since $\min_{\overline{\Omega}} u \leqslant c := \min_{\partial\Omega} \psi$, we have to prove that $\min_{\overline{\Omega}} u \geqslant \min_{\partial\Omega} \psi$. We introduce $w := \max(u, c)$. If there exists a point $x \in \Omega$ such that $u(x) < c$, then by continuity of u, the set $\{u < c\}$ has a positive measure. We have $w = u$ and $\nabla w = \nabla u$ on $\{u > c\}$. Moreover, since $\{u < c\}$ has a positive measure we have that:

$$0 = \int_{\{u \leqslant c\}} g(|\nabla w|) < \int_{\{u \leqslant c\}} g(|\nabla u|)$$

and

$$-\int_{\{u \leqslant c\}} \lambda w = -\int_{\{u \leqslant c\}} \lambda c \leqslant -\int_{\{u \leqslant c\}} \lambda u.$$

Hence, $I_\lambda(w) < I_\lambda(u)$ on Ω, which contradicts the fact that u is a minimizer. Thus, $u \geqslant c$ on Ω. ☐

We prove that the gradients of two minimizers of the same problem are collinear. This feature is useful in numerous following proofs.

Lemma 2.2. — *Let u and v be two minimizers of $\mathcal{P}_\lambda$ with $\lambda \in L^\infty(\Omega)$. Then for a.e. $x \in \Omega$, $\nabla u(x)$ and $\nabla v(x)$ are collinear and g is affine on the interval $[|\nabla u(x)|, |\nabla v(x)|]$.*

Proof. — Since u is a solution of P_λ,

$$\mathcal{I}_\lambda(u) \leqslant \mathcal{I}_\lambda\left(\frac{u+v}{2}\right).$$

By the fact that g is non-decreasing and the convexity of g and of the Euclidean norm,

$$\begin{aligned}\mathcal{I}_\lambda\left(\frac{u+v}{2}\right) &= \int_\Omega g\left(\left|\frac{\nabla u + \nabla v}{2}\right|\right) - \lambda \frac{u+v}{2} \\ &\leqslant \frac{1}{2}\int_\Omega \big(g(|\nabla u|) - \lambda u\big) + \frac{1}{2}\int_\Omega \big(g(|\nabla v|) - \lambda v\big) \\ &= \frac{1}{2}\mathcal{I}_\lambda(u) + \frac{1}{2}\mathcal{I}_\lambda(v).\end{aligned}$$

Since v is another solution,

$$\mathcal{I}_\lambda(u) = \frac{1}{2}\mathcal{I}_\lambda(u) + \frac{1}{2}\mathcal{I}_\lambda(v).$$

This implies that

$$\int_\Omega g\left(\left|\frac{\nabla u+\nabla v}{2}\right|\right) = \int_\Omega \frac{1}{2}\big(g(|\nabla u|)+g(|\nabla v|)\big).$$

Hence, for a.e. $x \in \Omega$,

$$g\left(\left|\frac{\nabla u(x)+\nabla v(x)}{2}\right|\right) = \frac{1}{2}\big(g(|\nabla u(x)|)+g(|\nabla v(x)|)\big). \tag{2.1}$$

Thus, for a.e. $x \in \Omega$, $g \circ |\cdot|$ is affine on the segment $[\nabla u(x), \nabla v(x)]$. In view of the definition of g and the strict convexity of the lower level sets of $|\cdot|$, it follows that $\nabla u(x)$ and $\nabla v(x)$ are collinear for a.e. $x \in \Omega$. Additionally, g is affine on $[|\nabla u(x)|, |\nabla v(x)|]$ for a.e. $x \in \Omega$. □

We use a result from [3] to introduce a set where ∇u is continuous and $|\nabla u|$ only takes values in $\mathcal{SC}\backslash\{0\}$, as defined in Section 1.4.

Proposition 2.3. — *When $\lambda \in \mathcal{C}^0(\overline{\Omega})$, there exists an open set U such that $u \in \mathcal{C}^1(U)$ and for every $x \in U$, $|\nabla u(x)| \in \mathcal{SC}\backslash\{0\}$, while for a.e. $x \notin U$, $|\nabla u(x)| \notin \mathcal{SC}\backslash\{0\}$.*

Proof. — By [3, Theorem 6.1], for a.e. $x \in \Omega$ such that $|\nabla u(x)| \in \mathcal{SC}\backslash\{0\}$ there exists a neighborhood $\mathcal{V}$ of x such that $u \in \mathcal{C}^{1,\alpha}(\mathcal{V})$. Since $\mathcal{SC}\backslash\{0\}$ is open, there exists $\epsilon > 0$ such that for every $x' \in B_\epsilon(x)$, $|\nabla u(x')| \in \mathcal{SC}\backslash\{0\}$. Let U be the set of such x, then U is open and for a.e. $x \notin U$, $|\nabla u(x)| \notin \mathcal{SC}\backslash\{0\}$. □

One of the interests of this set is the following:

Proposition 2.4. — *The set U does not depend on the choice of a minimizer. Moreover, let u and v be two minimizers of $\mathcal{P}_\lambda$, then $\nabla u = \nabla v$ on U.*

Proof. — Let us consider two minimizers u and v of the same problem. We define, respectively, U_u and U_v as the open sets of the previous proposition for u and v. By Lemma 2.2 and strict convexity of g on $\mathcal{SC}$, we have that $\nabla u = \nabla v$ a.e. on U_u. Hence, $v \in \mathcal{C}^1(U_u)$ and for every $x \in U_u$, $|\nabla v(x)| \in \mathcal{SC}\backslash\{0\}$. Thus, by definition of U_v, we have that $U_u \subset U_v$. To prove the other inclusion, we just have to exchange u and v. Hence, $U_u = U_v = U$ and $\nabla u = \nabla v$ on U. □

A direct consequence of this result is that:

Remark 2.5. — For every connected component U_i of U, $u-v$ is constant on U_i.

To conclude this section, we introduce the weak Euler–Lagrange equation associated to $\mathcal{P}_\lambda$:

$$\operatorname{div}\big(\nabla\varphi(\nabla u)\big) = -\lambda \text{ on } \Omega. \tag{2.2}$$

Remark 2.6. — By Lemma 2.2 the function $\nabla\varphi(\nabla u) = g'(|\nabla u|)\frac{\nabla u}{|\nabla u|}$ is independent of the choice of the minimizer of $\mathcal{P}_\lambda$ and will be denoted by σ.

2.2. Lipschitz regularity of a minimizer u and its level lines

In this subsection, we recall some Lipschitz regularity results for u and its level lines.

We use the following result from [16, Theorem 1.4] that states that our minimizers are globally Lipschitz-continuous on Ω:

Proposition 2.7. — *We assume that Ω has a $\mathcal{C}^{1,1}$ boundary and $\psi \in \mathcal{C}^{1,1}(\mathbb{R}^2)$. Then any minimizer u of $\mathcal{P}_\lambda$ is globally Lipschitz-continuous on Ω. Moreover, there exists $L > 0$ such that*

$$\|\nabla u\|_{L^\infty(\Omega)} \leqslant L\big(p,\ C_1,\ |\Omega|,\ \|\lambda\|_{L^\infty(\Omega)},\ \|\psi\|_{\mathcal{C}^{1,1}(\mathbb{R}^2)},\ \kappa\big)$$

where κ is the essential infimum of the curvature of $\partial\Omega$ and C_1 is introduced in (1.3).

For a function $f : \mathbb{R}^2 \to \mathbb{R}$ and for every $s \in \mathbb{R}$, we define $L_s^*(f)$ as the union of all connected components $l_s(f)$ of $L_s(f) = f^{-1}(s) \subset \mathbb{R}^2$ such that $\mathcal{H}^1(l_s(f)) > 0$. Here, $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure.

We state the following theorem from [1, Theorem 2.5] on the level sets of Lipschitz continuous functions.

Theorem 2.8. — *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz continuous function with compact support. For a.e. $s \in \mathbb{R}$, we have:*

- $\mathcal{H}^1(L_s(f)\backslash L_s^*(f)) = 0$.
- *Every connected component $l_s(f)$ of $L_s(f)$ that is not a point is a closed simple curve with a Lipschitz parametrization γ_s.*
- *$L_s^*(f)$ has a countable number of connected components.*

It follows that the level lines of a minimizer u have a Lipschitz-continuous parametrization:

Remark 2.9. — Let u be a globally Lipschitz-continuous minimizer of $\mathcal{P}_\lambda$ with $\lambda \in L^\infty(\Omega)$. We extend it outside Ω by ψ, which can be assumed to have compact support. For a.e. $s \in \mathbb{R}$, every connected component of $L_s^*(u) \subset \mathbb{R}^2$ has a Lipschitz-continuous parametrization.

2.3. Explicit solution on the ball and counter-example

The unique solution on $W_0^{1,p}(B_r(x_0))$ for our problem $\mathcal{P}_\lambda$, in dimension two, given that λ is a positive constant, can be expressed explicitly by utilizing [8, Theorem 1].

Proposition 2.10. — *When $\Omega = B_r(x_0)$ and λ is constant, the problem $\mathcal{P}_\lambda$ admits a unique minimizer on $W_0^{1,p}(B_r(x_0))$. We can compute it explicitly:*

$$u(x) := C - \frac{2}{\lambda} g^*\left(\frac{\lambda}{2}|x - x_0|\right)$$

with $g^*(x) := \sup_{y\in\mathbb{R}} \langle x, y\rangle - g(y)$ being the Legendre transform of g, and C the constant such that $u \in W_0^{1,p}(B_r(x_0))$.

The following proposition uses the Euler–Lagrange equation (2.2) to prove that a function is a minimizer.

Proposition 2.11. — *Let u be in $W^{1,p}(\Omega)$ and φ a convex function. If there exists $\lambda \in L^\infty(\Omega)$ such that $\operatorname{div} \nabla\varphi(\nabla u) = -\lambda$, then u is a minimizer of $\mathcal{P}_\lambda$ on $W_u^{1,p}(\Omega)$.*

Proof. — By convexity of φ, for every $w \in W_u^{1,p}(\Omega)$ we have:

$$\int_\Omega \varphi(\nabla w) \geqslant \int_\Omega \varphi(\nabla u) + \big\langle \nabla\varphi(\nabla u), \nabla w - \nabla u\big\rangle.$$

Since $\operatorname{div} \nabla\varphi(\nabla u) = -\lambda$ we get:

$$\int_\Omega \big\langle \nabla\varphi(\nabla u), \nabla w - \nabla u\big\rangle = \int_\Omega \lambda(w - u).$$

Hence, $\mathcal{I}_\lambda(w) \geqslant \mathcal{I}_\lambda(u)$ for every $w \in W_u^{1,p}(\Omega)$. Thus, u is a minimizer of $\mathcal{P}_\lambda$ on $W_u^{1,p}(\Omega)$. □

We apply this result to show that when λ is not constant, we can have more than one solution.

Proposition 2.12. — *Let $g : \mathbb{R}_+ \to \mathbb{R}$ be a non-strictly convex function such that $g(0) < g(t)$ for every $t > 0$. We assume that $g \in \mathcal{C}^1([0, +\infty))$ with $g'(0) = 0$. Then, there exists $\lambda_\infty \in \mathcal{C}^\infty(\overline{B_1(0)})$, $\lambda_\infty > 0$ such that $\mathcal{P}_{\lambda_\infty}$ has an infinite number of solutions on $W_0^{1,p}(B_1)$ with $\varphi(\cdot) = g(|\cdot|)$ and*

$$\mathcal{I}_{\lambda_\infty}(u) := \int_{B_1} \varphi(\nabla u(x)) - \lambda_\infty(x)u(x)\mathrm{d}x.$$

Proof. — We construct two different radial solutions u and v of the same problem. For every $x \in B_1(0)$, we set $u(x) := \widetilde{u}(|x|)$ and $v(x) := \widetilde{v}(|x|)$. Our goal is to define $\widetilde{u}'$ and $\widetilde{v}'$ on $(0,1)$.

Since g is not strictly convex on $\mathbb{R}_+$, there exist $a, b \in \mathbb{R}_+$ such that g' is constant on (a,b) and $g'(t) \neq g'(a)$ for every $t \notin [a,b]$. Since $g(0) < g(t)$ for every $t > 0$ and $g'(0) = 0$, we have that $a > 0$.

Let us introduce a smooth increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(t) = t$ for every $t \geqslant 0$ small and $f(t) = g'(a)$ for every $t \geqslant \frac{1}{2}$. We use the fact that for every $x > 0$ if $x \in \partial g^*(y)$ then $g'(x) = y$. Hence, for every $t > 0$, $g'(\partial g^*(f(t))) = \{f(t)\}$. For every $t \in (0,1)$, we set $\widetilde{u}'(t) = -x_t$ with $x_t \in \partial g^*(f(t))$ such that $\widetilde{u}'$ is measurable. Such a choice is possible by [9, Theorem 5.3, p. 151]. In order to define $\widetilde{v}'$, we set $\widetilde{v}'(t) = \widetilde{u}'(t)$ on $(0, \frac{1}{2})$ and $\widetilde{v}'(t) = -b$ for every $t > \frac{1}{2}$.

Hence, $g'(|\widetilde{v}'|) = g'(|\widetilde{u}'|) = f$ is a smooth function. Then, we can set $u(x) := \int_{|x|}^{1} -\widetilde{u}'(t)\mathrm{d}t$ and $v(x) := \int_{|x|}^{1} -\widetilde{v}'(t)\mathrm{d}t$ that are Lipschitz-continuous on $B_1(0)$ and vanish at the boundary.

Finally, we set

$$\lambda_\infty(x) = \operatorname{div} f(|x|)\frac{x}{|x|} \in \mathcal{C}^\infty(B_1(0)).$$

Hence, by Proposition 2.11, u and v are solutions of the same problem $\mathcal{P}_{\lambda_\infty}$. Moreover, a direct computation shows that $\lambda_\infty(x) > 0$ on $\overline{B_1(0)}$. □

We can give an explicit counter-example where $\mathcal{P}_\lambda$ has more than one solution with $\lambda > 0$ and $U \neq \emptyset$ with g is as in (1.2):

$$g(t) = \begin{cases} \frac{1}{2}|t|^2 & \text{if } |t| \leqslant 1, \\ |t| - \frac{1}{2} & \text{if } 1 < |t| < 2, \\ \frac{1}{4}|t|^2 + \frac{1}{2} & \text{if } 2 \leqslant |t|. \end{cases} \tag{2.3}$$

PROPOSITION 2.13. — *There exists $\lambda \in \mathcal{C}^\infty(\overline{B_1(0)})$, $\min_{\overline{B_1(0)}} \lambda > 0$ such that $\mathcal{P}_\lambda$ has more than one solution on $W_0^{1,2}(B_1(0))$ and $U \neq \emptyset$.*

Proof. — We take the same notations as in the previous proof. In this case, we have

$$g^*(t) = \begin{cases} \frac{1}{2}|t|^2 & \text{if } |t| \leqslant 1, \\ |t|^2 - \frac{1}{2} & \text{if } 1 < |t|. \end{cases}$$

For $t \in \mathbb{R}$, we define:

$$\theta(t) = \begin{cases} t & \text{if } t \leqslant \frac{1}{4}, \\ 3t - \frac{1}{2} & \text{if } \frac{1}{4} < t < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leqslant t. \end{cases}$$

Hence, we can take f as the convolution of θ with a smooth standard mollifier. For instance,

$$f(t) := C \int_{t-\frac{1}{8}}^{t+\frac{1}{8}} \theta(s) \exp\left(-\frac{1}{\frac{1}{64} - |t-s|^2} \right) \mathrm{d}s$$

with

$$C^{-1} = \int_{-\frac{1}{8}}^{\frac{1}{8}} \exp\left(-\frac{1}{\frac{1}{64} - |s|^2} \right) \mathrm{d}s.$$

Then $\widetilde{u}' = -f$ on $[0,1]$, $\widetilde{v}' = -f$ on $[0, \frac{5}{8})$ and $\widetilde{v}' = -2$ on $(\frac{5}{8}, 1]$. Thus, $u(x) := \int_{|x|}^{1} -\widetilde{u}'(t)\mathrm{d}t$ and $v(x) := \int_{|x|}^{1} -\widetilde{v}'(t)\mathrm{d}t$ are two solutions on $W_0^{1,2}(B_1(0))$ with $\lambda(x) := \frac{N-1}{|x|} f(|x|) + f'(|x|)$.

Moreover, $U \neq \emptyset$ since $|\nabla u| < 1$ on $B_{\frac{1}{2}}(0)$. □

2.4. BV functions

We start by giving the definitions of functions of bounded variation and sets of finite perimeter:

Definition 2.14. — *A function $f \in L^1(\Omega)$ has bounded variation in Ω if*

$$\int_\Omega |Df| := \sup\left\{ \int_\Omega f \operatorname{div} \theta \mathrm{d}x \,\middle|\, \theta \in \mathcal{C}_c^1\left(\Omega; \mathbb{R}^2\right),\ |\theta(x)| \leqslant 1,\ \forall\, x \in \Omega \right\} < \infty.$$

We denote by $BV(\Omega)$ the set of functions in $L^1(\Omega)$ having bounded variation in Ω.

If $f \in BV(\Omega)$, the distributional gradient of f is a vector valued Radon measure that we denote by Df and $|Df|$ is the total variation of Df.

Definition 2.15. — *Let E be a Borel set. We say that E has finite perimeter in Ω if the characteristic function χ_E of E belongs to $BV(\Omega)$. The perimeter* $\operatorname{Per}(E, \Omega)$ *is defined as:*

$$\operatorname{Per}(E, \Omega) = \int_\Omega |D\chi_E| = \sup\left\{ \int_E \operatorname{div} \theta \,\middle|\, \theta \in C_c^1\left(\Omega; \mathbb{R}^2\right),\ \|\theta\|_{L^\infty(\Omega)} \leqslant 1 \right\}.$$

Definition 2.16. — *For a set E of finite perimeter in $\mathbb{R}^2$, we define the reduced boundary $\partial^* E$ of E as the subset of supp $|D\chi_E|$ such that for every $x \in \partial^* E$,*

$$\nu_E(x) := \lim_{r\to 0} \frac{\int_{B_r(x)} D\chi_E}{\int_{B_r(x)} |D\chi_E|}$$

exists and $|\nu_E(x)| = 1$.

Remark 2.17. — The reduced boundary $\partial^* E$ is a subset of ∂E and we have that $\operatorname{Per}(E,\Omega) = \mathcal{H}^1(\partial^* E \cap \Omega)$.

We recall the coarea formula for Lipschitz continuous functions from [11, Theorem 3.4.2.1, p. 112] that will be useful throughout the article.

Proposition 2.18 (Coarea formula). — *Let u be a Lipschitz continuous function with compact support and f be a nonnegative measurable function. Then*

$$\int_{\mathbb{R}^2} f|\nabla u|\mathrm{d}x = \int_{\mathbb{R}} \int_{L_s(u)} f(x)\mathrm{d}\mathcal{H}^1(x)\mathrm{d}s$$

where $L_s(u) := u^{-1}(s) \subset \mathbb{R}^2$.

Remark 2.19. — By replacing f by the indicator function

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

we observe that if $|A| = 0$ then for a.e. $s \in \mathbb{R}$, $\mathcal{H}^1(L_s(u) \cap A) = 0$.

Proposition 2.20. — *Let v be a Lipschitz continuous function with compact support in $\mathbb{R}^2$. For a.e. $s \in \mathbb{R}$ we have*

$$\frac{\nabla v(x)}{|\nabla v(x)|} = \frac{D\mathbb{1}_{[v>s]}(x)}{|D\mathbb{1}_{[v>s]}|(x)} \text{ for } \mathcal{H}^1 \text{ a.e. } x \in L_s(v).$$

Proof. — By the vector valued coarea formula [2, Theorem 3.40] we have that

$$\int_A \nabla v = \int_{\mathbb{R}} \int_A D\mathbb{1}_{[v>s]}\mathrm{d}s$$

for every Borel set A.

By linearity, for every linear combination of indicator functions χ, we have

$$\int_{\operatorname{supp} v} \langle \chi, \nabla v\rangle = \int_{\mathbb{R}} \int_{\operatorname{supp} v} \big\langle \chi, D\mathbb{1}_{[v>s]}\big\rangle \mathrm{d}s.$$

By density, for every $\theta \in L^\infty(\operatorname{supp} v; \mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^2} \langle \theta, \nabla v\rangle = \int_{\mathbb{R}} \int_{\mathbb{R}^2} \big\langle \theta, D\mathbb{1}_{[v>s]}\big\rangle \mathrm{d}s.$$

We fix $\theta := \frac{\nabla v}{|\nabla v|}$ when $\nabla v \neq 0$, $\theta := 0$ when $\nabla v = 0$ and obtain

$$\int_{\mathbb{R}^2} |\nabla v| = \int_{\mathbb{R}} \int_{\mathbb{R}^2} \left\langle \frac{\nabla v}{|\nabla v|}, D\mathbb{1}_{[v>s]} \right\rangle \mathrm{d}s. \tag{2.4}$$

But, by [13, Theorem 4.4] we have for a.e. $s \in \mathbb{R}$ that

$$\left|D\mathbb{1}_{[v>s]}\right| = \mathcal{H}^1 \llcorner \partial^*[v > s].$$

Hence,

$$\begin{aligned} \int_{\mathbb{R}^2} \langle \theta, D\mathbb{1}_{[v>s]} \rangle &= \int_{\mathbb{R}^2} \left\langle \theta, \frac{D\mathbb{1}_{[v>s]}}{\left|D\mathbb{1}_{[v>s]}\right|} \right\rangle \mathrm{d}\left|D\mathbb{1}_{[v>s]}\right| \\ &= \int_{\partial^*[v>s]} \left\langle \theta, \frac{D\mathbb{1}_{[v>s]}}{\left|D\mathbb{1}_{[v>s]}\right|} \right\rangle \mathrm{d}\mathcal{H}^1. \end{aligned} \tag{2.5}$$

Since $\langle \theta, \frac{D\mathbb{1}_{[v>s]}}{|D\mathbb{1}_{[v>s]}|} \rangle \leqslant 1$ for a.e. $s \in \mathbb{R}$, with (2.4) and (2.5) we get

$$\int_{\mathbb{R}^2} |\nabla v| \leqslant \int_{\mathbb{R}} \mathcal{H}^1(\partial^*[v > s])\mathrm{d}s.$$

By Remark 2.17, we have

$$\int_{\mathbb{R}^2} |\nabla v| \leqslant \int_{\mathbb{R}} \mathcal{H}^1(\partial^*[v > s])\mathrm{d}s \leqslant \int_{\mathbb{R}} \mathcal{H}^1(L_s(v))\mathrm{d}s.$$

By the coarea formula given in Proposition 2.18, the following equalities hold true:

$$\int_{\mathbb{R}^2} |\nabla v| = \int_{\mathbb{R}} \mathcal{H}^1(\partial^*[v > s])\mathrm{d}s = \int_{\mathbb{R}} \mathcal{H}^1(L_s(v))\mathrm{d}s.$$

Hence, we get $\langle \theta, \frac{D\mathbb{1}_{[v>s]}}{|D\mathbb{1}_{[v>s]}|} \rangle = 1$ for $\mathcal{H}^1$ a.e. $x \in \partial^*[v > s]$ and

$$\mathcal{H}^1(L_s(v) \backslash \partial^*[v > s]) = 0 \quad \text{for a.e.} \quad s \in \mathbb{R}.$$

Thus, $\theta = \frac{D\mathbb{1}_{[v>s]}}{|D\mathbb{1}_{[v>s]}|}\mathcal{H}^1$ a.e. on $L_s(v)$ for a.e $s \in \mathbb{R}$, as desired. □

3. Relation between the level lines and U

In this section, we use the Lipschitz regularity of the level lines of a minimizer u to prove that they are level sets for the other minimizers in a generic sense. Subsequently, we study the case where a level line intersects the set U, which implies that the gradient of another solution is equal to ∇u on that particular level line.

3.1. Equality on level lines

We first prove that the difference between two minimizers is constant on every connected component of almost every level sets.

Proposition 3.1. — *Let u and v be two minimizers of the same problem $\mathcal{P}_\lambda$. There exists a negligible subset N_0 of $\mathbb{R}$ such that for every $s \in S_0 := \mathbb{R}\backslash N_0$, for every connected component $l_s(u)$ of $L_s(u)$, the map $u - v$ is constant on $l_s(u)$.*

Proof. — We consider that u and v are extended by ψ outside of Ω. By Proposition 2.9 there is a negligible set N_∞ such that for every $s \in \mathbb{R}\backslash N_\infty$, every connected component $l_s(u)$ of $L_s(u) \subset \mathbb{R}^2$ that is not a point has a parametrization that is Lipschitz continuous.

Since ∇u and ∇v are defined and collinear a.e. on $\mathbb{R}^2$, by the coarea formula we obtain that there exists a negligible set N'_∞ such that for every $s \in \mathbb{R}\backslash N'_\infty$, ∇u and ∇v are defined and collinear $\mathcal{H}^1$ a.e. on $L_s(u)$. We set $N_0 := N_\infty \cup N'_\infty$.

Hence, for every $s \in \mathbb{R}\backslash N_0$, we have that ∇v is orthogonal to each Lipschitz connected curve $l_s(u)$. We introduce $\gamma_s : [0, \text{length}(l_s(u))) \mapsto l_s(u)$ a Lipschitz-continuous parametrization of $l_s(u)$. Then, by the chain rule, we have $(v \circ \gamma_s)' = \langle \nabla v(\gamma_s), \gamma'_s \rangle$ a.e. on $[0, \text{length}(l_s(u)))$. By orthogonality of ∇v to $l_s(u)$, we have that v is constant on $l_s(u)$. □

The following proposition is the first step to prove the uniqueness result.

Proposition 3.2. — *For $s \in S_0$, if $l_s(u) \cap (\mathbb{R}^2\backslash\Omega) \neq \emptyset$ then $u = v$ on $l_s(u)$.*

Proof. — Thanks to the previous proposition, we know that $u - v$ is constant on $l_s(u)$. Since u and v are extended by ψ outside Ω, we have $u \equiv v$ on $\mathbb{R}^2\backslash\Omega$. By assumption, we have $l_s(u) \cap \mathbb{R}^2\backslash\Omega \neq \emptyset$ then $u = v$ on $l_s(u)$. □

3.2. Relation between U and the level lines

In this section, we consider two minimizers u and v of $\mathcal{I}_\lambda$ with the same boundary condition. According to Proposition 2.4, $\nabla u = \nabla v$ on U. We will extend this result to the level lines that intersect U.

Notation 3.3. — We denote by $S \subset \mathbb{R}$ the set of these s that satisfy the two following conditions:

- $s \in S_0$ with S_0 defined in Proposition 3.1,
- $\nabla u \neq 0\, \mathcal{H}^1$ a.e. on $L_s(u)$.

We introduce the following set:

$$\Gamma := \left\{ l_s^i(u), s \in S \text{ and } i \in I_s \right\}$$

where the index set I_s corresponds to the non-constant curves $l_s^i(u)$ among the connected components of $L_s(u)$ inside Ω which do not intersect $\partial\Omega$.

Proposition 3.4. — *Given $s \in S$ and $i \in I_s$, let F_s^i be the bounded connected component of $\mathbb{R}^2 \backslash l_s^i(u)$ given by the Jordan curve theorem. Then, for every $i \in I_s$, $u > s$ on F_s^i.*

Proof. — We fix $s \in S$. Let $l_s^i(u)$ a connected component of $L_s(u)$ such that $l_s^i(u) \Subset \Omega$. Since Ω is simply connected, $F_s^i \subset \Omega$ and by Proposition 2.1, we have that $u \geqslant s$ on F_s^i. If $l_s'(u)$ is a connected component of $L_s^*(u)$ that is inside F_s^i then for $\mathcal{H}^1$ a.e. $y \in l_s'(u)$, $\nabla u(y)$ is defined and $\nabla u(y) \neq 0$. Since $u \geqslant s$ on F_s^i, every point y in $l_s'(u)$ is a local minimum on F_s^i and hence, either $\nabla u(y) = 0$ or $\nabla u(y)$ is not defined. Thus, there is no such $l_s'(u)$ in F_s^i. By assumptions on s, we have that $\mathcal{H}^1(L_s \backslash L_s^*) = 0$. Therefore, $\mathcal{H}^1(\{u = s\} \cap F_s^i) = 0$.

Let us assume that there exists $x \in F_s^i$ such that $u(x) = s$. Then by Proposition 2.1 for every $\epsilon < \operatorname{dist}(x, l_s^i(u))$ there exists $y_\epsilon \in \partial B_\epsilon(x)$ such that $u(y_\epsilon) = s$ and we define the following set $Y := \{y_\epsilon, 0 < \epsilon < \operatorname{dist}(x, l_s^i(u))\}$. We have that

$$\mathcal{H}^1(Y) := \lim_{\delta \to 0} \mathcal{H}^1_\delta(Y)$$

with

$$\mathcal{H}^1_\delta(Y) := \inf \left\{ \sum_{n \in \mathbb{N}} \operatorname{diam}(V_n) \right\}$$

where the infimum is taken over the families of sets $(V_n)_{n\in\mathbb{N}}$ such that $Y \subset \bigcup_n V_n$ and $\operatorname{diam}(V_n) < \delta$ for every $n \in \mathbb{N}$.

For every admissible $(V_n)_{n\in\mathbb{N}}$, we define E_n as the set of those ϵ such that $y_\epsilon \in V_n \cap Y$. We define $e_m := \inf\{\epsilon \in E_n\}$, $e^M := \sup\{\epsilon \in E_n\}$ and $\widetilde{V}_n := [e_m, e^M]$. We have that $\operatorname{diam}(\widetilde{V}_n) = e^M - e_m \leqslant \operatorname{diam}(V_n) < \delta$ and $]0, \operatorname{dist}(x, l_s^i(u))[\subset \cup \widetilde{V}_n$.

Hence, $(\widetilde{V}_n)_{n\in\mathbb{N}}$ is admissible for $\mathcal{H}^1_\delta((0, \operatorname{dist}(x, l_s^i(u))))$ and

$$\mathcal{H}^1_\delta(Y) \geqslant \mathcal{H}^1_\delta\big((0, \operatorname{dist}(x, l_s^i(u)))\big).$$

By taking the limit when δ goes to 0, we obtain:

$$\mathcal{H}^1(Y) \geqslant \mathcal{H}^1\big((0, \operatorname{dist}(x, l_s^i(u))\big) = \operatorname{dist}\big(x, l_s^i(u)\big).$$

Thus $\mathcal{H}^1(\{u = s\} \cap F_s^i) \geqslant \operatorname{dist}(x, l_s^i(u)) > 0$. That is a contradiction. Hence, there is no such x. Therefore, $u > s$ on F_s^i. □

A direct consequence of that result is the following:

Proposition 3.5. — *For every $s \in S$, I_s is countable. Moreover, for every $i \in I_s$, $l_s^i(u)$ is the boundary of a connected component F_s^i of $E_s = \{u > s\}$.*

Proof. — By Theorem 2.8, I_s is countable. Let us consider $i \in I_s$. By the previous proposition, $l_s^i(u)$ is the boundary of a connected component of $E_s = \{u > s\}$. □

We also have that:

Proposition 3.6. — *For every $s \in S$, every connected component F_s of E_s, if $F_s \Subset \Omega$ then F_s is simply connected and its boundary is a closed simple curve $l_s(u)$ with Lipschitz parametrization.*

Proof. — Since F_s is bounded, $\mathbb{R}^2 \backslash F_s$ has only one unbounded connected component. We call $\widetilde{F}_s$ the complement of this unbounded set. We claim that $\widetilde{F}_s = F_s$. We have that $\partial \widetilde{F}_s \subset \partial F_s$. Hence, $u \equiv s$ on $\partial \widetilde{F}_s$. Since $\widetilde{F}_s$ is simply connected, $\partial \widetilde{F}_s$ is a connected set in $L_s(u)$ with $\mathcal{H}^1(\partial \widetilde{F}_s) > 0$. By Theorem 2.8, $\partial \widetilde{F}_s$ is a closed subset of a closed simple curve $l_s(u)$, which has a Lipschitz parametrization. Hence, $\widetilde{F}_s$ is a bounded set such that $\partial \widetilde{F}_s \subset l_s(u)$. We have that $\widetilde{F}_s$ is an open set in $\mathbb{R}^2 \backslash l_s(u)$. Since, $\partial \widetilde{F}_s \subset l_s(u)$, $\widetilde{F}_s$ is also closed in $\mathbb{R}^2 \backslash l_s(u)$. The fact that $\widetilde{F}_s$ is bounded and connected gives that $\widetilde{F}_s$ is the bounded connected component of $\mathbb{R}^2 \backslash l_s(u)$ and by the Jordan curve theorem we have $\partial \widetilde{F}_s = l_s(u)$. By Proposition 3.4, $\widetilde{F}_s \subset E_s$. Since $\widetilde{F}_s$ contains F_s, we get that $\widetilde{F}_s = F_s$. Moreover, we have proved that F_s is simply connected with $l_s(u)$ as boundary. □

The main result of this subsection is the following:

Proposition 3.7. — *For a.e. $s \in S$, for every $i \in I_s$, if $l_s^i(u) \cap U \neq \emptyset$ then $\nabla(u - v) = 0\,\mathcal{H}^1$ a.e. on $l_s^i(u)$.*

In order to prove this result, we state two technical lemmata:

Lemma 3.8. — *For a.e. $s \in S$, for every $i \in I_s$ there exists a decreasing sequence $(s_n)_{n \in \mathbb{N}}$ converging to s such that:*

- *There exists a simple connected curve $l_{s_n}(u)$ in L_{s_n} with Lipschitz parametrization that is inside F_s^i.*
- *$(F_{s_n})_{n \in \mathbb{N}}$ is an increasing sequence with $\bigcup_{n \in \mathbb{N}} F_{s_n} = F_s^i$.*

Here, F_t is the bounded connected component of $\mathbb{R}^2\backslash l_t(u)$ given by the Jordan curve theorem and F_s^i is the bounded connected component of $\mathbb{R}^2\backslash l_s^i(u)$.

Proof of Lemma 3.8. — By Proposition 3.4 we have that $u > s$ on F_s^i. By the coarea formula in Proposition 2.18 there exists $s_0 > s$, $s_0 \in S$ such that $\mathcal{H}^1(L_{s_0}(u) \cap F_s^i) > 0$. Moreover, by Theorem 2.8, $\mathcal{H}^1(L_{s_0}(u)\backslash L^*_{s_0}(u)) = 0$. Hence, there exists $l_{s_0}(u)$ in L_{s_0} satisfying the assumptions of Lemma 3.8.

Then, we next select $s < s_1 < s_0$ with $s_1 \in S$ such that $\mathcal{H}^1(L_{s_1}(u) \cap F_s^i) > 0$. We have that $F_{s_0} \subset E_{s_1}$. Hence, F_{s_0} is in one connected component of E_{s_1}, we call $l_{s_1}(u)$ the boundary of that connected component. By Proposition 3.6 we have that $l_{s_1}(u)$ is a connected simple curve with Lipschitz parametrization. We repeat this argument to find a sequence $(s_n)_{n\in\mathbb{N}}$ that satisfies the first part of the lemma. By construction $(F_{s_n})_{n\in\mathbb{N}}$ is an increasing sequence, it remains to prove that $\bigcup_{n\in\mathbb{N}} F_{s_n} = F_s^i$.

We introduce $F_\infty := \bigcup_{n\in\mathbb{N}} F_{s_n}$ a subset of F_s^i. If $y \in \partial F_\infty$ there exists a sequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \in \partial F_{s_n}$ and $y_n \to y$. By continuity of u and the fact that y_n converges to y we obtain that $u(y) = s$.

By Proposition 3.4 we have that $\partial F_\infty \subset \partial F_s^i$. We claim that $F_\infty = F_s^i$. Indeed, if those two sets are not equal, then there exists $x \in F_s^i\backslash F_\infty$. Since F_s^i is a connected open set, for every $y \in F_\infty$ there exists a continuous path from x to y included in F_s^i. By continuity, this path must intersect $\partial F_\infty \subset \partial F_s^i$, contradicting the fact that the path is in F_s^i. Hence, $F_\infty = F_s^i$. □

Lemma 3.9. — *Let u be a minimizer. For a.e. $s \in S$ and every $i \in I_s$, we consider a sequence $(l_{s_n})_{n\in\mathbb{N}}$ as in the previous lemma. Then, we have that $\lim_{n\to+\infty} D(l_{s_n}, l_s^i(u)) = 0$ where $D(E,F) := \sup_{e\in E}\ \inf_{f\in F} |e - f|$. Moreover, let v be another minimizer, if there exists a constant C such that $u - v \equiv C$ on every l_{s_n} then $\nabla(u - v) = 0\,\mathcal{H}^1$ a.e on $l_s^i(u)$.*

Proof of Lemma 3.9. — For every $\epsilon > 0$, we claim that there exists $N \in \mathbb{N}$ such that for every $n \geqslant N$, $D(l_{s_n}, l_s^i(u)) < \epsilon$. Indeed, assume by contradiction that there exists $\epsilon > 0$ such that $\forall\ N$, there exist $n \geqslant N$ and $y_n \in l_{s_n}(u)$ such that $\forall\ x \in l_s^i(u)$, $d(y_n, x) \geqslant \epsilon$. Since all these y_n are in Ω, there exists a sequence $(y_{g(n)})_{n\in\mathbb{N}}$ converging towards some $y \in F_s^i$. We have that $d(y,x) \geqslant \epsilon$ for every $x \in l_s^i(u)$. By continuity of u, we have that $u(y) = s$. Thus, by Proposition 3.4, $y \in \partial F_s^i = l_s^i(u)$. That is a contradiction.

For $\mathcal{H}^1$ a.e. $x \in l_s^i(u)$ we have that $\nabla u(x) \neq 0$ and $\nabla v(x)$ exist. Moreover, for $\mathcal{H}^1$ a.e. $x \in l_s^i(u)$ we have that $\nabla u(x)$ and $\nabla v(x)$ are orthogonal to $l_s^i(u)$ at x in the sense that $\langle \nabla u(x), \gamma_s'(\gamma_s^{-1}(x))\rangle = 0$ where γ_s is a Lipschitz parametrization of $l_s^i(u)$. We consider $d_x := x+\mathbb{R}\nabla u(x)$. Let us call H_- and H_+ the two half-planes of $\mathbb{R}^2\backslash d_x$. For every $r>0$, $H_\pm \cap l_s^i(u)\cap B_r(x)\neq\emptyset$, otherwise it would contradict the fact that $\nabla u(x)$ is orthogonal to $l_s^i(u)$. A direct consequence is the fact that $H_-\cap F_s^i\neq\emptyset$ and $H_+\cap F_s^i\neq\emptyset$.

We assume that for every $N\in\mathbb{N}$, there exists $\widetilde{n}\geqslant N$ such that $l_{s_{\tilde{n}}}(u)\cap d_x=\emptyset$. Thus, we can assume that $l_{s_{\tilde{n}}}(u)\cap H_-=\emptyset$. Hence, there exist $y\in l_s^i(u)$ and $\epsilon_0>0$ such that $d(y,F_{s_{\tilde{n}}})>\epsilon_0$. Since $l_{s_{\tilde{n}}(u)}\to l_s^i(u)$ in the sense of D that is a contradiction. Then, there exists $N\in\mathbb{N}$ such that for every $n\geqslant N$, $l_{s_n}(u)\cap d_x\neq\emptyset$. For every such n, we take x_n as a point that minimizes $d(x,y)$ on $l_{s_n}(u)\cap d_x$. Since this sequence $(x_n)_{n\in\mathbb{N}}$ is bounded, it converges, up to a subsequence, to a point $x'\in\overline{F_s^i}$ such that $u(x')=s$. By Proposition 3.4, $x'\in l_s^i(u)$. If $x'\neq x$ then there exists $r>0$ such that $d_x\cap B_r(x)\cap l_{s_n}(u)=\emptyset$ for every $n\in\mathbb{N}$ large enough. Hence, $d_x\cap B_r(x)\cap F_{s_n}=\emptyset$ for every $n\in\mathbb{N}$ large enough which contradicts the fact that $\bigcup_{n\in\mathbb{N}}F_{s_n}=F_s^i$. Thus, $x=x'$ and we can find a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n\in l_{s_n}(u)\cap d_x$ and $x_n\to x$. By assumption, $u-v\equiv C$ on $l_{s_n}(u)$ for n large enough. By continuity of $u-v$ we obtain that $u-v(x)=C$. Then, $\frac{(u-v)(x)-(u-v)(x_n)}{|x-x_n|}=\frac{C-C}{|x-x_n|}=0$. Moreover, $\nabla(u-v)(x)$ is collinear to $\nabla u(x)$, hence, we obtain $\nabla(u-v)(x)=0$. Since that is the case for $\mathcal{H}^1$ a.e. $x\in l_s^i(u)$, we have the desired conclusion. □

Proof of Proposition 3.7. — For every $s\in S$, $l_s^i(u)$ is a Lipschitz continuous closed curve such that ∇u and ∇v are defined and collinear $\mathcal{H}^1$ a.e. on $l_s^i(u)$ and $\nabla u\neq 0\mathcal{H}^1$ a.e. on $l_s^i(u)$. If $l_s^i(u)\cap U\neq\emptyset$ then there exists U_i a connected component of U such that $l_s^i(u)\cap U_i\neq\emptyset$. By Proposition 2.5 and Proposition 3.1 we have $u-v\equiv C_i$ on $l_s^i(u)$. We consider the sequences $(s_n)_{n\in\mathbb{N}}$, $(l_{s_n}(u))_{n\in\mathbb{N}}$ and $(F_{s_n})_{n\in\mathbb{N}}$ from Lemma 3.8.

Hence, by the first part of Lemma 3.9, there exists $N\in\mathbb{N}$ such that for every $n\geqslant N$, $l_{s_n}(u)\cap U_i\neq\emptyset$. Thus, by Proposition 2.5 and Proposition 3.1, $u-v\equiv C_i$ on $l_{s_n}(u)$ for every $n\geqslant N$. By the second part of Lemma 3.9, we have that $\nabla(u-v)=0\,\mathcal{H}^1$ a.e on $l_s^i(u)$. □

4. $W^{1,2}$ regularity of $|\sigma|$

In this section, we prove the following proposition on $\sigma=\nabla\varphi(\nabla u)$:

Proposition 4.1. — *For every $\alpha>0$, the function $f:=\max(\alpha,|\sigma|)$ is in $W^{1,2}(\Omega')$ for any $\Omega'\Subset\Omega$.*

Proof of Proposition 4.1. — We prove this result in four parts. In Step 1, we regularize our problem in order to work with smooth solutions $(u_n)_{n\in\mathbb{N}}$.

Then in Step 2, we prove that $\| \max(\alpha, |\nabla\varphi_n(\nabla u_n)|)\|_{W^{1,2}(\Omega')}$ is uniformly bounded in $n \in \mathbb{N}$. In the subsequent Step 3, we show that

$$\max(\alpha, |\nabla\varphi_n(\nabla u_n)|) \longrightarrow f$$

a.e. on Ω. We conclude that f is in $W^{1,2}(\Omega')$ in Step 4.

Step 1. — For every $n \in \mathbb{N}$, we introduce $(\rho_n)_{n\in\mathbb{N}}$ a standard mollifying sequence with $\operatorname{supp} \rho_n \subset B_{\frac{1}{n}}(0)$. If we set $g_n := g * \rho_n$ and $\lambda_n := \lambda * \rho_n$ then $(g_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ are sequences of smooth approximations of g and λ. We consider $\varphi_n := g_n(|\cdot|) + \frac{1}{n}\theta(|\cdot|)$. The function θ is smooth quadratic around the origin such that $0 < \theta''(x)$ for every $x \in \mathbb{R}$ and

$$C_-|x|^p \leqslant \theta(z) \leqslant C_+(|x|^p + 1)$$

for all $|x| \geqslant 1$ with $0 < C_- < C_+$.

Let u_n be the minimizer of:

$$\mathcal{I}_n : v \longrightarrow \int_\Omega \varphi_n(\nabla v(x)) - \lambda_n v(x) dx$$

on $W^{1,p}_\psi(\Omega)$.

Proposition 4.2. — *The sequence $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $W^{1,\infty}(\Omega)$. There exists a subsequence still denoted by $(u_n)_{n\in\mathbb{N}}$ that weakly converges in $W^{1,p}(\Omega)$ towards $\widetilde{u} \in W^{1,\infty}(\Omega)$. Moreover, $\widetilde{u}$ is a minimizer of $\mathcal{P}_\lambda$ on $W^{1,p}_\psi(\Omega)$.*

Proof. — By Proposition 2.7 we have that the sequence $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $W^{1,\infty}(\Omega)$. Hence, we can extract a subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$, that converges strongly in $L^p(\Omega)$ and weakly in $W^{1,p}_\psi(\Omega)$ towards $\widetilde{u} \in W^{1,\infty}(\Omega)$. It remains to prove that $\widetilde{u}$ is a minimizer of $\mathcal{P}_\lambda$ on $W^{1,p}_\psi(\Omega)$.

By Jensen's inequality, we have $\varphi_n \geqslant \varphi$. Hence

$$\liminf_{n\to+\infty} \int_\Omega \varphi_n(\nabla u_n) \geqslant \liminf_{n\to+\infty} \int_\Omega \varphi(\nabla u_n). \tag{4.1}$$

By weak lower semi-continuity of I_λ, (4.1) and the fact that u_n is the minimizer for $\mathcal{I}_n$ we have

$$\begin{aligned}\int_\Omega \varphi(\nabla\widetilde{u}) - \lambda\widetilde{u} &\leqslant \liminf_{n\to+\infty} \int_\Omega \varphi(\nabla u_n) - \lambda u_n \\ &\leqslant \liminf_{n\to+\infty} \int_\Omega \varphi_n(\nabla u_n) - \lambda_n u_n \\ &\leqslant \lim_{n\to+\infty} \int_\Omega \varphi_n(\nabla u) - \lambda_n u.\end{aligned} \tag{4.2}$$

By the dominated convergence theorem applied to the last quantity we obtain

$$\int_\Omega \varphi(\nabla\widetilde{u}) - \lambda\widetilde{u} \leqslant \int_\Omega \varphi(\nabla u) - \lambda u.$$

Hence, $\widetilde{u}$ is a minimizer on $W^{1,p}_\psi(\Omega)$. □

Step 2. — For every $n \in \mathbb{N}$, we introduce $\sigma_n := \nabla\varphi_n(\nabla u_n)$. In this part, we prove the following result on $f_n := \max(\alpha, |\sigma_n|)$:

Proposition 4.3. — *For every $\alpha > 0$ and every $\Omega' \Subset \Omega$, the functions $f_n := \max(\alpha, |\sigma_n|)$ are uniformly bounded in $W^{1,2}(\Omega')$.*

Proof. — By [20, Proposition 2.4], we have for every $b > 0$ and $k \in \{1,2\}$ that:

$$\int_{\Omega'\cap\{\partial_k u_n \geqslant b\}} |\nabla\sigma_n|^2 \leqslant C_1\left(b, \|\nabla u_n\|_{L^\infty(\Omega)}, \sup_{b\leqslant t\leqslant \|\nabla u_n\|_{L^\infty(\Omega)}} g_n''(t) + \frac{\theta''(t)}{n}\right).$$

Thus,

$$\int_{\Omega'\cap\{|\nabla u_n| \geqslant b\}} |\nabla\sigma_n|^2 \leqslant C_2\left(b, \|\nabla u_n\|_{L^\infty(\Omega)}, \sup_{b\leqslant t\leqslant \|\nabla u_n\|_{L^\infty(\Omega)}} g_n''(t) + \frac{\theta''(t)}{n}\right).$$

Finally,

$$\int_{\Omega'\cap\{|\sigma_n| \geqslant g_n'(b)\}} |\nabla\sigma_n|^2 \leqslant C_2\left(b, \|\nabla u_n\|_{L^\infty(\Omega)}, \sup_{b\leqslant t\leqslant \|\nabla u_n\|_{L^\infty(\Omega)}} g_n''(t) + \frac{\theta''(t)}{n}\right).$$

By Proposition 2.7 we have that $\|\nabla u_n\|_{L^\infty(\Omega)}$ can be bounded by L uniformly in $n \in \mathbb{N}$. Moreover, $g \in \mathcal{C}^{1,1}_{\mathrm{loc}}(\mathbb{R}\backslash\{0\})$ and g_n is a convolution of g. Hence, $\sup_{b\leqslant t\leqslant \|\nabla u_n\|_{L^\infty(\Omega)}} g_n''(t) + \frac{\theta''(t)}{n}$ can be bounded by $\sup_{\frac{b}{2}\leqslant t\leqslant L+\frac{b}{2}} g''(t) + 1$ for every $n \in \mathbb{N}$ such that g_n'' is close enough to g'' on $(\frac{b}{2}, +\infty)$ and larger than $\sup_{b\leqslant t\leqslant L} \theta''(t)$. Namely, every $n \in \mathbb{N}$ larger than $\max\{\frac{2}{b}, \sup_{b\leqslant t\leqslant L} \theta''(t)\}$. Thus, we get

$$\int_{\Omega'\cap\{|\sigma_n| \geqslant g_n'(b)\}} |\nabla\sigma_n|^2 \leqslant C_2\left(b, L, \sup_{\frac{b}{2}\leqslant t\leqslant L+\frac{b}{2}} g''(t) + 1\right)$$

for every $n \in \mathbb{N}$ larger than $\max\{\frac{2}{b}, \sup_{b\leqslant t\leqslant L+\frac{b}{2}} g''(t)\}$.

By growing assumptions on g, for every $\alpha > 0$ we can find $b > 0$ such that $g_n'(b) \leqslant \alpha$ for $n \in \mathbb{N}$ large enough. Hence, for every $\alpha > 0$ and every $n \in \mathbb{N}$ large enough we have:

$$\int_{\Omega'\cap[|\sigma_n|>\alpha]} |\nabla\sigma_n|^2 \leqslant C(\alpha, g'', L).$$

Thus, the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded in $W^{1,2}(\Omega')$. □

Remark 4.4. — In the case where g'' is also bounded around the origin, we can apply directly [20, Proposition 2.4] or [6, Theorem 2.1] to obtain that $f_n := |\sigma_n|$ are uniformly bounded in $W^{1,2}(\Omega')$.

Since $f_n := \max(\alpha, |\sigma_n|)$ are uniformly bounded in $W^{1,2}(\Omega')$, we can extract a subsequence which converges weakly.

Step 3. — We prove that $\sigma_n \to \sigma$ a.e. Ω up to a subsequence. To do so, we use the Young measures associated to $(\nabla u_n)_{n\in\mathbb{N}}$.

Proposition 4.5. — *We have the following equality*

$$\liminf_{n\to+\infty} \int_\Omega \varphi(\nabla u_n) = \int_\Omega \varphi(\nabla\widetilde{u}). \tag{4.3}$$

Proof. — If we replace u by $\widetilde{u}$ in the last term of (4.2) we obtain that

$$\begin{aligned} \int_\Omega \varphi(\nabla\widetilde{u}) - \lambda\widetilde{u} &\leqslant \liminf_{n\to+\infty} \int_\Omega \varphi(\nabla u_n) - \lambda u_n \\ &\leqslant \liminf_{n\to+\infty} \int_\Omega \varphi_n(\nabla u_n) - \lambda_n u_n \\ &\leqslant \liminf_{n\to+\infty} \int_\Omega \varphi_n(\nabla\widetilde{u}) - \lambda_n\widetilde{u}. \end{aligned} \tag{4.4}$$

Since $\widetilde{u} \in W^{1,\infty}(\Omega)$, the uniform convergence of $(\varphi_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ over compact sets t'he last term is equal to the first term. Hence, all those inequalities are equalities, in particular:

$$\liminf_{n\to+\infty} \int_\Omega \varphi(\nabla u_n) - \lambda u_n = \int_\Omega \varphi(\nabla\widetilde{u}) - \lambda\widetilde{u}.$$

Since $u_n \to \widetilde{u}$ in $L^p(\Omega)$ we have that

$$\lim_{n\to+\infty} \int_\Omega \lambda u_n = \int_\Omega \lambda\widetilde{u}.$$

Hence,

$$\liminf_{n\to+\infty} \int_\Omega \varphi(\nabla u_n) = \int_\Omega \varphi(\nabla\widetilde{u}).$$ □

We consider $(u_{\psi(n)})_{n\in\mathbb{N}}$ a subsequence such that

$$\liminf_{n\to+\infty}\int_\Omega \varphi(\nabla u_n) = \lim_{n\to+\infty}\int_\Omega \varphi(\nabla u_{\psi(n)}). \tag{4.5}$$

In order to simplify the notations, we still denote $(u_{\psi(n)})_{n\in\mathbb{N}}$ by $(u_n)_{n\in\mathbb{N}}$.

Proposition 4.6. — *For a.e. $x\in\Omega$ we have*

$$\varphi(\nabla\widetilde{u}(x)) = \bar{\varphi}(x) := \int_{\mathbb{R}^2}\varphi(y)\mathrm{d}\nu_x(y)$$

where ν_x is a probability measure that depends on x and on the weak convergence of $(\nabla u_n)_{n\in\mathbb{N}}$ towards $\nabla\widetilde{u}$. Moreover, $\operatorname{supp}\nu_x \subset \{y\in\mathbb{R}^2, \nabla\varphi(y) = \nabla\varphi(\nabla\widetilde{u}(x))\}$ for a.e. $x\in\Omega$.

Proof. — Let $(\nu_x)_{x\in\Omega}$ be the Young measures associated to a subsequence of $(\nabla u_n)_{n\in\mathbb{N}}$ given by [4, Theorem 2]. For every Carathéodory function F such that $\{F(\cdot,\nabla u_n(\cdot))\}_{n\in\mathbb{N}}$ is uniformly integrable, we have:

$$\lim_{n\to+\infty}\int_\Omega F(x,\nabla u_n(x))\mathrm{d}x = \int_\Omega \overline{F}(x)\mathrm{d}x \tag{4.6}$$

with $\overline{F}(x) = \int_{\mathbb{R}^2} F(x,y)\mathrm{d}\nu_x(y)$. Moreover, for a.e. $x\in\Omega$,

$$\nabla\widetilde{u}(x) = \int_{\mathbb{R}^2} y\mathrm{d}\nu_x(y). \tag{4.7}$$

Since u_n is uniformly bounded in $W^{1,\infty}(\Omega)$,

$$\lim_{n\to+\infty}\int_\Omega \varphi(\nabla u_n(x))\mathrm{d}x = \int_\Omega \bar{\varphi}(x)\mathrm{d}x \text{ where } \bar{\varphi}(x) = \int_{\mathbb{R}^2}\varphi(y)\mathrm{d}\nu_x(y). \tag{4.8}$$

If we combine this last equation with (4.5) we get

$$\int_\Omega \varphi(\nabla\widetilde{u}) = \lim_{n\to+\infty}\int_\Omega \varphi(\nabla u_n(x))\mathrm{d}x = \int_\Omega \bar{\varphi}(x)\mathrm{d}x.$$

If we apply the triangle inequality and Jensen's inequality to (4.7) we obtain for a.e. $x\in\Omega$,

$$\varphi(\nabla\widetilde{u}(x)) \leqslant \int_{\mathbb{R}^2}\varphi(y)\mathrm{d}\nu_x(y)\mathrm{d}x = \bar{\varphi}(x). \tag{4.9}$$

If we combine the two last equations, we obtain for a.e. $x\in\Omega$ that

$$\varphi(\nabla\widetilde{u}(x)) = \bar{\varphi}(x). \tag{4.10}$$

By Jensen's inequality, φ is affine on $\operatorname{supp}\nu_x$ and thus, for a.e. $x\in\Omega$ we have that $\operatorname{supp}\nu_x \subset \{y\in\mathbb{R}^2, \nabla\varphi(y) = \nabla\varphi(\nabla\widetilde{u}(x))\}$. □

Then, we can prove the following convergence result:

Proposition 4.7. — *We have that $\sigma_n\to\sigma$ in $L^1(\Omega)$ when $n\to+\infty$. Here, $\sigma_n = \nabla\varphi_n(\nabla u_n)$.*

Proof. — If we set $F(x,y) = |\nabla\varphi(y) - \sigma(x)|$ in (4.6), we obtain that

$$\lim_{n\to+\infty}\int_\Omega |\nabla\varphi(\nabla u_n(x)) - \sigma(x)|\mathrm{d}x = \int_\Omega\int_{\mathbb{R}^2} |\nabla\varphi(y) - \sigma(x)|\mathrm{d}\nu_x(y)\mathrm{d}x = 0.$$

Since ∇u_n is uniformly bounded in $L^\infty(\Omega)$, we have that

$$\lim_{n\to+\infty}\int_\Omega |\nabla\varphi_n(\nabla u_n) - \nabla\varphi(\nabla u_n)| = 0.$$

Hence, by the triangle inequality,

$$\lim_{n\to+\infty}\int_\Omega |\sigma_n(x) - \sigma(x)|\mathrm{d}x = 0.$$

Hence, $\sigma_n \to \sigma$ in $L^1(\Omega)$ when $n \to +\infty$. □

Thanks to the previous Proposition, we can extract a subsequence, we do not relabel, such that $\sigma_n \to \sigma$ a.e. on Ω when $n \to +\infty$.

Step 4. — Since $\sigma_n \to \sigma$ a.e. on Ω when $n \to +\infty$, we have that $f_n \to \max(\alpha, |\sigma|)$ a.e. on Ω. By Proposition 4.3, we have that $\max(\alpha, |\sigma|) \in W^{1,2}(\Omega')$. □

5. Continuity of $|\sigma|$ on the level lines and a maximum principle

In this section, we prove that, generically, $\max(d_0, |\sigma|)$ is continuous on the level lines of u and also satisfies a maximum principle.

For u and v two minimizers, we introduce

$$\Gamma' := \left\{l_s^i(u), s \in S \text{ and } i \in I_s\right\}$$

where $S \subset \mathbb{R}$ is the set of those s that satisfy the conclusion of Theorem 2.8 and such that ∇u, ∇v are defined, $\nabla u \neq 0$, ∇u and ∇v are collinear $\mathcal{H}^1$ a.e. on $L_s(u)$. The index set I'_s corresponds to the non-constant curves $l_s^i(u)$ among the connected components of $L_s(u)$ such that $l_s^i(u) \Subset \Omega$.

Proposition 5.1. — *There exists a representative f_0 of $\max(d_0, |\sigma|)$ that is absolutely continuous on $l_s^i(u)$ for a.e. $s \in S$ and every $i \in I_s$.*

Proof. — We consider the sequence $(\sigma_n)_{n\in\mathbb{N}}$ introduced in the previous section. We have that $\sigma_n \to \sigma$ a.e. on Ω when $n \to +\infty$. By Proposition 4.1, we have $\|\nabla\max(d_0, |\sigma_n|)\|_{L^2(\Omega_1)} \leqslant C_1$ with $\Omega_1 \Subset \Omega$ and C_1 independent of $n \in \mathbb{N}$. Thus, there exists a constant C_2 independent of n such that

$$C_2 \geqslant \int_{\Omega_1} |\nabla\max(d_0, |\sigma_n|)|^2|\nabla u| = \int_{\mathbb{R}}\int_{L_s(u)\cap\Omega_1} |\nabla\max(d_0, |\sigma_n|)|^2\mathrm{d}\mathcal{H}^1\mathrm{d}s$$

where the equality is given by Proposition 2.18. With Fatou's lemma we obtain that for a.e. $s \in S$,

$$\liminf_{n\to+\infty} \int_{L_s(u)\cap\Omega_1} |\nabla \max(d_0, |\sigma_n|)|^2 \mathrm{d}\mathcal{H}^1 \leqslant C_3(s). \tag{5.1}$$

We define S_1 as the subset of S such that (5.1) holds. We have $|S\backslash S_1| = 0$. We introduce the index I_s^1 that corresponds to the non-constant curves $l_s^i(u)$ among the connected components of $L_s(u)$ such that $l_s^i(u) \Subset \Omega_1$.

Now, we fix $s \in S_1$ and $i \in I_s^1$. We can extract a subsequence such that for every $n \in \mathbb{N}$:

$$\int_{l_s^i(u)} |\nabla \max(d_0, |\sigma_n|)|^2 \mathrm{d}\mathcal{H}^1 \leqslant 2C_3(s).$$

Let us call $\gamma_s^i : [0, \mathrm{length}(\mathrm{l_s^i(u)})) \to \mathrm{l_s^i(u)}$ a Lipschitz parametrization of $l_s^i(u)$. We have that $\max(d_0, |\sigma_n|) \circ \gamma_s^i$ is uniformly bounded in $W^{1,2}([0, \mathrm{length}(\mathrm{l_s^i(u)})))$. By the Arzelà–Ascoli theorem,there exists a subsequence of $\max(d_0, |\sigma_n|) \circ \gamma_s^i$ converging uniformly to $v \in \mathcal{C}^0([0, \mathrm{length}(\mathrm{l_s^i(u)})))$.

Since $\max(d_0, |\sigma_n|) \to \max(d_0, |\sigma|)\mathcal{H}^1$ a.e. on $l_t^j(u)$ for a.e. $t \in \mathbb{R}$ and every $j \in I_t$, we choose f_0 as a representative of $\max(d_0, |\sigma|)$ such that $f_0 = v \circ (\gamma_s^i)^{-1}$ on $l_s^i(u)$.

Now, we introduce an increasing sequence of open sets $(\Omega_k)_{k\in\mathbb{N}}$ such that $\chi_{\Omega_k} \to \chi_\Omega$ in $L^1(\mathbb{R}^2)$. For a.e. $s \in S_1$ and for every $i \in I_s^2\backslash I_s^1$, we can define f_0 as we did on Ω_1. Hence, there exists $S_2 \subset S_1$ satisfying $|S\backslash S_2| = 0$ such that for every $s \in S_2$ and every $i \in I_s^2$, we have that f_0 absolutely continuous on $l_s^i(u)$.

Thus, we can select by induction a representative of $\max(d_0, |\sigma|)$ that is absolutely continuous on $l_s^i(u)$ for a.e. $s \in S$ and every $i \in I_s$. □

For a.e. $s \in S$, if $l_s^i(u) \cap U = \emptyset$ we have some additional information that will be useful in the final proof.We recall that g is strictly convex on $\mathcal{SC} = \bigcup_{n=1}^N \mathcal{SC}_n$ with $\mathcal{SC}_n = (a_n, b_n)$ for every $n \in \mathbb{N}^*$, $n < N$ and $d_n = g'(b_n) = g'(a_{n+1})$.

Proposition 5.2. — *For a.e. $s \in S$ and for every $i \in I_s$, if $l_s^i(u)\cap U = \emptyset$, then $f_0 = C_s^i$ is constant on $l_s^i(u)$ with $C_s^i \in \{d_n, n \in \mathbb{N}, 0 \leqslant n < N\}$.*

Proof. — For a.e. $x \in \Omega\backslash U$ we have $|\sigma(x)| \in \{0\} \cup g'(\mathbb{R}\backslash\mathcal{SC})$. By the coarea formula for a.e. $s \in \mathbb{R}$, for $\mathcal{H}^1$ a.e. $x \in (\Omega \cap L_s(u))\backslash U$ we have $|\sigma(x)| \in g'(\mathbb{R}\backslash\mathcal{SC})$. Hence, for a.e. $s \in \mathbb{R}$ if $l_s^i(u) \cap U = \emptyset$ then $f_0(l_s^i(u)) \subset g'(\mathbb{R}\backslash\mathcal{SC}) \cup f_0(X)$ for some $X \subset l_s^i(u)$ with $\mathcal{H}^1(X) = 0$.

Moreover, for a.e. $s \in \mathbb{R}$ and every $i \in I_s$, $l_s^i(u)$ is a Lipschitz continuous curve such that f_0 is absolutely continuous on $l_s^i(u)$. Since $g'(\mathbb{R}\backslash\mathcal{SC})$ is finite and $f_0(X)$ is the image of a negligible set by an absolutely continuous function, we have $|g'(\mathbb{R}\backslash\mathcal{SC})\cup f_0(X)|=0$. The continuity of f_0 on $l_s^i(u)$ implies that f_0 is constant on $l_s^i(u)$. Since $l_s^i(u)\cap U=\emptyset$ we obtain that

$$f_0|_{l_s^i(u)}\equiv C_s^i\in\{d_n,n\in\mathbb{N},0\leqslant n<N\}.$$

□

We adopt the notations of Section 4, where $(\sigma_n)_{n\in\mathbb{N}}$ is a smooth approximation that converges a.e. on Ω to σ. We prove the following maximum principle on $\max(d_0,|\sigma_n|)$:

Proposition 5.3. — *We assume that Ω has a $\mathcal{C}^{1,1}$ boundary, $\psi\in\mathcal{C}^{1,1}(\mathbb{R}^2)$, λ is globally Lipschitz continuous on $\overline{\Omega}$ and $\lambda>0$. There exists*

$$\Upsilon:=\Upsilon\left(|\Omega|,\max_{\overline{\Omega}}\lambda,\min_{\overline{\Omega}}\lambda,\quad\|\psi\|_{\mathcal{C}^{1,1}(\mathbb{R}^2)},\kappa\right)>0$$

where κ is the essential infimum of the curvature of $\partial\Omega$, such that if $\|\nabla\lambda\|_{L^\infty(\Omega)}\leqslant\Upsilon$ then for $n\in\mathbb{N}$ large enough, for a.e. every $s\in S$, for every $i\in I_s$, if $l_s^i(u)\cap U=\emptyset$ we have

$$\sup_{F_s^i}|\sigma_n|\leqslant\sup_{l_s^i(u)}\max(d_0,|\sigma_n|)$$

where F_s^i is the bounded connected component of $\mathbb{R}^2\backslash l_s^i(u)$.

Remark 5.4. — When λ is constant, this result is true even if Ω and ψ are only Lipschitz continuous.

Proof. — By the coarea formula in Proposition 2.18, $\sigma_n\to\sigma\mathcal{H}^1$ a.e. on $l_s^i(u)$ for a.e. $s\in S$ and every $i\in I_s$. We apply the maximum principle from [12, Theorem 15.1] to $|\nabla u_n|$ on F_s^i. To do so, we assume that

$$\|\nabla\lambda\|_{L^\infty(\Omega)}\leqslant\frac{\min\limits_{\overline{\Omega}}\lambda^2}{2\times L\times\sup\limits_{x\in\left[\frac{b_0}{2},L\right]}g''(x)+\frac{g'(x)}{x}}.$$

Here L is the Lipschitz constant introduced in Proposition 2.7. For every $n\in\mathbb{N}$, there exists b_n such that $g_n'(b_n)=d_0$.

Hence, for $n\in\mathbb{N}$ large enough, $b_n\geqslant\frac{b_0}{2}$ and

$$\|\nabla\lambda_n\|_{L^\infty(\Omega)}\leqslant\frac{\min\limits_{\overline{\Omega}}\lambda_n^2}{\|\nabla u_n\|_{L^\infty(\Omega)}\times\sup\limits_{x\in\left[\frac{b_n}{2},L\right]}g_n''(x)+\frac{g_n'(x)}{x}}.$$

Thus, thanks to [12, Theorem 15.1, Eq. (15.15)] for a.e. $s\in\mathbb{R}$ we have

$$\sup_{F_s}|\nabla u_n|\leqslant\sup_{l_s^i(u)}\max\left(\frac{b_0}{2},|\nabla u_n|\right)\leqslant\sup_{l_s^i(u)}\max(b_n,|\nabla u_n|)$$

for $n \in \mathbb{N}$ large enough. Since g_n' is increasing, we obtain $\sup_{F_s} |\sigma_n| \leqslant \sup_{l_s^i(u)} \max(d_0, |\sigma_n|)$. □

We have the following partial maximum principal for $|\sigma|$:

Proposition 5.5. — *Let us consider $s \in S$ and $i \in I_s$ such that $l_s^i(u) \cap U = \emptyset$ and $f_0 = C_s^i$ on $l_s^i(u)$. Then for a.e. $t > s$, for every $j \in I_t$ such that $l_t^j(u) \Subset F_s^i$ and $l_t^j(u) \cap U = \emptyset$, we have $\max(d_0, |\sigma|) = C_t^j \mathcal{H}^1$ a.e. on $l_t^j(u)$ with $C_t^j \in \{d_n, n \in \mathbb{N}, 0 \leqslant n < N\}$ not larger than C_s^i.*

Proof. — By construction of f_0 in the proof of Proposition 5.1 and Proposition 5.2, we can construct a subsequence $\max(d_0, |\sigma_{\psi(n)}|)$ converging uniformly to C_s^i on $l_s^i(u)$ that also converges uniformly to C_t^j on $l_t^j(u)$. Hence, with the previous proposition we get:

$$\begin{aligned} C_t^j &\leqslant \limsup_{n\to+\infty} \sup_{l_t^j(u)} \max\big(d_0, |\sigma_{\psi(n)}|\big) \\ &\leqslant \limsup_{n\to+\infty} \sup_{F_s^i(u)} \max\big(d_0, |\sigma_{\psi(n)}|\big) \\ &\leqslant \lim_{n\to+\infty} \sup_{l_s^i(u)} \max\big(d_0, |\sigma_{\psi(n)}|\big) = C_s^i. \end{aligned}$$

Thus, we have that $C_t^j \leqslant C_s^i$. □

6. Proof of the main theorem

6.1. Pseudo Cheeger problem

In this part, we combine the maximum principle for $|\sigma|$ and the Euler–Lagrange equation to prove that the level sets are almost Cheeger sets.

We recall the definition of the Cheeger constant of a set:

Definition 6.1. — *The Cheeger constant of Ω is defined as:*

$$h_\Omega = \inf_{D \subset \overline{\Omega}} \frac{\mathrm{Per}(D, \mathbb{R}^2)}{|D|}$$

A set $D \subset \overline{\Omega}$ of finite perimeter with $|D| > 0$ is said to be a Cheeger set *if* $\mathrm{Per}(D, \mathbb{R}^2) = h_\Omega |D|$.

Remark 6.2. — There is no Cheeger set D of Ω such that $D \Subset \Omega$ because the function $t \mapsto \frac{\mathrm{Per}(tD,\mathbb{R}^2)}{|tD|}$ is -1-homogeneous.

The following equality is a consequence of Proposition 5.2:

PROPOSITION 6.3. — *For a.e. $s \in S$, for every $i \in I_s$ if $l_s^i(u) \cap U = \emptyset$ we have*

$$\int_{F_s^i} \lambda = C_s^i \operatorname{Per}(F_s^i)$$

where F_s^i is the bounded connected component of $\mathbb{R}^2 \backslash l_s^i(u)$ and C_s^i is the constant introduced Proposition in Section 5.2.

Proof. — By [11, Section 5.11, Theorem 1], for a.e. $s \in S$ and for every $i \in I_s$, we have that $D\mathbb{1}_{F_s^i} \in BV(\Omega)$.

We consider the sequence $(\sigma_n)_{n\in\mathbb{N}}$ that converges a.e. on Ω towards σ introduced in Section 4. By [11, Section 5.8, Theorem 1] we obtain:

$$\begin{aligned}\int_{F_s^i} \operatorname{div}(\sigma_n)\mathrm{d}x &= \int_\Omega \mathbb{1}_{F_s^i}(x) \operatorname{div}(\sigma_n)\mathrm{d}x \\ &= -\int_\Omega \left\langle \sigma_n, \frac{D\mathbb{1}_{F_s^i}}{\left|D\mathbb{1}_{F_s^i}\right|} \right\rangle \mathrm{d}\left|D\mathbb{1}_{F_s^i}\right| \\ &= -\int_{\partial^* F_s^i} \left\langle \sigma_n, \frac{D\mathbb{1}_{F_s^i}}{\left|D\mathbb{1}_{F_s^i}\right|} \right\rangle \mathrm{d}\mathcal{H}^1.\end{aligned}$$

The set $\partial^* F_s^i$ is introduced in Definition 2.16. We can use Proposition 2.20 that gives:

$$-\int_{F_s^i} \operatorname{div}(\sigma_n)\mathrm{d}x = \int_{\partial^* F_s^i} \left\langle \sigma_n, \frac{\nabla u}{|\nabla u|} \right\rangle \mathrm{d}\mathcal{H}^1.$$

But by the coarea formula $\sigma_n \to \sigma\mathcal{H}^1$ a.e. on $\partial^* F_s^i \subset l_s^i(u)$ for a.e. $s \in \mathbb{R}$ and every $i \in I_s$. By Proposition 5.2 and since σ is collinear to $\frac{\nabla u}{|\nabla u|}\mathcal{H}^1$ a.e. on $l_s^i(u)$, we get for such an s:

$$\lim_{n\to+\infty} -\int_{F_s^i} \operatorname{div}(\sigma_n)\mathrm{d}x = \int_{\partial^* F_s^i} |\sigma|\mathrm{d}\mathcal{H}^1 = C_s^i \operatorname{Per}(F_s^i). \tag{6.1}$$

Moreover,

$$-\int_{F_s^i} \operatorname{div}(\sigma_n) = \int_{F_s^i} \lambda_n \longrightarrow \int_{F_s^i} \lambda \tag{6.2}$$

when $n \to +\infty$, where $\lambda_n := \lambda * \rho_n$. Hence, with (6.1) and (6.2), we have the desired result:

$$\int_{F_s^i} \lambda = C_s^i \operatorname{Per}(F_s^i)$$

for a.e. $s \in S$, for every $i \in I_s$ when $l_s^i(u) \cap U = \emptyset$. □

We also have that:

PROPOSITION 6.4. — *For every set $F \subset \overline{F_s^i}$ of finite perimeter, we have*

$$\int_F \lambda \leqslant C_s^i \operatorname{Per}(F).$$

Proof. — We follow the same ideas developed in the previous proof. We have:

$$-\int_F \operatorname{div}(\sigma_n) = \int_{\partial^* F} \langle \sigma_n, \nu_F \rangle \mathrm{d}\mathcal{H}^1.$$

The term in the left-hand side tends to $\int_F \lambda$ when $n \to +\infty$. For the term in the right-hand side, we get:

$$\int_{\partial^* F} \langle \sigma_n, \nu_F \rangle \mathrm{d}\mathcal{H}^1 \leqslant \int_{\partial^* F} |\sigma_n| \mathrm{d}\mathcal{H}^1.$$

By Proposition 5.3, $\sup_{F_s^i} |\sigma_n| \leqslant \sup_{l_s^i} \max(d_0, |\sigma_n|)$. By Proposition 5.1 and Proposition 5.2 we have $\max(d_0, |\sigma_n|) \to C_s^i \mathcal{H}^1$ a.e. on $l_s^i(u)$ when $n \to +\infty$. Hence,

$$\int_F \lambda \leqslant C_s^i \operatorname{Per}(F).$$

□

6.2. Main proof

We first prove Theorem 1.5:

Proof of Theorem 1.5. — For a.e. $s \in \mathbb{R}$, for every $i \in I_s$ if $l_s^i(u) \cap (U \cup \partial\Omega) = \emptyset$ then by Proposition 6.3,

$$\int_{F_s^i} \lambda = C_s^i \operatorname{Per}\big(F_s^i\big).$$

We assume that such an $l_s^i(u)$ exists. Since $F_s^i \Subset \Omega$ by Remark 6.2 and the previous equality, we have

$$h_\Omega < \frac{\operatorname{Per}\big(F_s^i\big)}{|F_s^i|} = \frac{1}{C_s^i |F_s^i|} \int_{F_s^i} \lambda.$$

We have that $\|\lambda\|_{L^\infty(\Omega)} \leqslant d_0 h_\Omega$. Thus,

$$h_\Omega < \frac{d_0 h_\Omega}{C_s^i}.$$

Hence, $C_s^i < d_0$ which is a contradiction. Thus, for a.e. $s \in \mathbb{R}$ we have $l_s^i(u) \cap (U \cup \partial\Omega) \neq \emptyset$.

Let v be another minimizer. By Proposition 3.2 and Proposition 3.7, for a.e. $s \in \mathbb{R}$, on every connected component $l_s^i(u)$ of $L_s(u)$ that is not a point we have $u = v$ on $l_s^i(u)$ or $\nabla(u - v) = 0\mathcal{H}^1$ a.e. on $l_s^i(u)$.

By the coarea formula:

$$\int_{\mathbb{R}^2\cap\{u\neq v\}} |\nabla(u-v)||\nabla u| = \int_{\mathbb{R}}\int_{L_s(u)\cap\{u\neq v\}} |\nabla(u-v)|\mathrm{d}\mathcal{H}^1\mathrm{d}s.$$

By Theorem 2.8, for a.e. $s\in\mathbb{R}$, $\mathcal{H}^1(L_s\backslash L_s^*)=0$ and L_s^* is composed by a countable number of curves $l_s^i(u)$. For every $i\in I_s$, we have that:

$$\int_{l_s^i(u)\cap\{u\neq v\}} |\nabla(u-v)|\mathrm{d}\mathcal{H}^1 = 0.$$

By Proposition 3.2, we get that

$$\int_{L_s(u)\cap\{u\neq v\}} |\nabla(u-v)|\mathrm{d}\mathcal{H}^1 = 0.$$

Hence,

$$\int_{\mathbb{R}^2\cap\{u\neq v\}} |\nabla(u-v)||\nabla u|\mathrm{d}x = 0.$$

For the same reasons,

$$\int_{\mathbb{R}^2\cap\{u\neq v\}} |\nabla(u-v)||\nabla v|\mathrm{d}x = 0.$$

Hence we have $\nabla(u-v)=0$ a.e. on $\{u\neq v\}$. This implies that the map $u-v$ is constant on $\mathbb{R}^2$. Since $u=v$ on $\partial\Omega$, we have that $u=v$ on Ω. ☐

Now, we are ready to prove the main theorem:

Proof of Theorem 1.1. — Let u and v be two minimizers of $\mathcal{P}_\lambda$. We introduce

$$\Theta := \min\left\{\frac{\min_{\overline{\Omega}}\lambda}{\operatorname{diam}\Omega}, \Upsilon\right\}$$

where Υ comes from Proposition 5.3 and we assume that $\|\nabla\lambda\|_{L^\infty(\Omega)}\leqslant\Theta$. For a.e. $s\in\mathbb{R}$, every $i\in I_s$, if $l_s^i(u)\cap(U\cup\partial\Omega)=\emptyset$ by Proposition 5.2, $|\sigma|=C_s^i\mathcal{H}^1$ a.e. on $l_s^i(u)$ with $C_s^i\in\{d_n,\,0\leqslant n<N\}$.

We prove by induction on $0\leqslant n<N$ that if $|\sigma|=d_n$ on $l_s^i(u)$ then $\nabla(u-v)=0\mathcal{H}^1$ a.e. on l_s^i.

Step 1. — As an initialization step, we assume that $C_s^i=d_0$. By Proposition 3.4, $u>s$ on F_s^i. By the coarea formula, for a.e. $t>s$, t belongs to S. We assume that there exists $t>s$ and $j\in I_t$ such that $l_t^j(u)\cap U=\emptyset$ and $F_t\Subset F_s^i$. By Proposition 5.3, $|\sigma|=d_0$ a.e. on $l_t^j(u)$. Thus, by Proposition 6.3 we have that $\int_{F_t}\lambda = d_0\operatorname{Per}(F_t)$. For $r>1$ close to 1 and $x_0\in\Omega$, we introduce $F_t^r = r(F_t-x_0)+x_0\Subset F_s$.

Hence, by Proposition 5.3 we have $|\sigma| \leqslant d_0$ on ∂F_t^r. Then, by Proposition 6.4,

$$r^2 \int_{F_t} \lambda(r(x-x_0)+x_0)\mathrm{d}x = \int_{F_t^r} \lambda(y)\mathrm{d}y \leqslant d_0 \operatorname{Per}\big(F_t^r\big).$$

But,

$$d_0 \operatorname{Per}\big(F_t^r\big) = r d_0 \operatorname{Per}(F_t) = r \int_{F_t} \lambda(x)\mathrm{d}x.$$

Thus,

$$\int_{F_t} r\lambda\big(r(x-x_0)+x_0\big) - \lambda(x) \leqslant 0.$$

Since $\|\nabla\lambda\|_{L^\infty(\Omega)} < \frac{\min_{\bar{\Omega}} \lambda}{\operatorname{diam}(\Omega)}$, for $r > 1$ small enough, we have that $r\lambda(r(x - x_0) + x_0) - \lambda(x) > 0$ for every $x \in F_s^i$. That is a contradiction. Hence, for a.e. $t > s$ and every $j \in I_t$ such that $l_t^j(u) \Subset F_s^i$ we have $l_t^j(u) \cap U \neq \emptyset$. By Proposition 3.7, $\nabla(u-v) = 0\mathcal{H}^1$ a.e. on $l_t^j(u)$. By the coarea formula, $\nabla(u-v) = 0$ a.e. in F_s^i. By Lemma 3.9, we have that $\nabla(u-v) = 0\mathcal{H}^1$ a.e. on $l_s^i(u)$.

Step 2. — Now, we prove the induction part. We consider $1 \leqslant n < N$. Let us assume that for every $k < n$, for a.e. $t \in \mathbb{R}$ and every $j \in I_t$ if $l_t^j(u) \cap (U \cup \partial\Omega) = \emptyset$ and $C_t^j = d_k$ then $\nabla(u-v) = 0\mathcal{H}^1$ a.e. on $l_t^j(u)$.

If $l_s^i(u)$ is such that $l_s^i(u) \cap (U \cup \partial\Omega) = \emptyset$ and $C_s^i = d_n$, we consider $t > s$ such that $l_t^j(u) \cap U = \emptyset$ and $F_t \Subset F_s^i$. Hence, by Proposition 5.5, either $C_t^j = d_n$ or $C_t^j < d_n$. If $C_t^j = d_n$, then as in Step 1 we construct $F_t^r \Subset F_s^i$ and we prove that $\nabla(u-v) = 0\mathcal{H}^1$ a.e. on $l_t^j(u)$. In the second case, by induction, we have $\nabla(u-v) = 0\mathcal{H}^1$ a.e. on $l_t^j(u)$. Hence, $\nabla(u-v) = 0\mathcal{H}^1$ a.e. on $l_t^j(u)$. We can conclude as in Step 1 that $\nabla(u-v) = 0\mathcal{H}^1$ a.e. on $l_s^i(u)$.

Step 3. — For a.e. $s \in S$, we consider $l_s(u)$ a connected component of $L^*(u)$. If $l_s(u) \cap (\mathbb{R}^2 \backslash \Omega) \neq 0$ then by Proposition 3.2, $u = v$ on $l_s(u)$. If $l_s(u) \subset \Omega$ and $l_s(u) \cap U \neq \emptyset$ then by Proposition 3.7, $\nabla(u-v) = 0\,\mathcal{H}^1$ a.e. on $l_s(u)$. Finally, thanks to Step 2 if $l_s(u) \Subset \Omega$ and $l_s(u) \cap U = \emptyset$ then we have $\nabla(u-v) = 0\mathcal{H}^1$ a.e. on $l_s(u)$. Hence, as in the proof of Theorem 1.5, we can prove with the coarea formula that $u = v$. □

7. Extensions

In this section, we present an extension of the main theorem when $\mathcal{SC}$ has a countable number of connected components. We assume that the convex function g is $\mathcal{C}^2$ and $g'' > 0$ on $\mathrm{int}(\mathcal{SC})\backslash\{0\}$ and:

$$\mathcal{SC} \cap \mathbb{R}_+ = \mathcal{SC}_\infty \cup \left(\bigcup_{n\in\mathbb{N}} \mathcal{SC}_n \right)$$

with $\mathcal{SC}_0 := [0, b_0)$, $\mathcal{SC}_n := (a_n, b_n)$ for every $n \in \mathbb{N}^*$ and $\mathcal{SC}_\infty$ is defined below. We assume that $(a_n)_{n\in\mathbb{N}^*}$ and $(b_n)_{n\in\mathbb{N}}$ are strictly increasing sequences with $a_{n+1} < b_n$. Moreover, the sequence $(a_n)_{n\in\mathbb{N}^*}$ is bounded and $\lim_{n\to+\infty} a_n = \alpha$. For every $n \in \mathbb{N}$, $d_n := g'(b_n) = g'(a_{n+1})$ is an increasing sequence.

The connected component $\mathcal{SC}_\infty$ is exceptional because it can have two different shapes. We introduce $\alpha_\infty := \inf\{t \in \mathcal{SC}_\infty\}$. If $\alpha < a_\infty$ then $\mathcal{SC}_\infty := (a_\infty, +\infty)$ and $\mathcal{SC}_\infty := [\alpha, +\infty)$ if $\alpha = a_\infty$.

Proposition 7.1. — *In that case, Theorem 1.1 is still valid.*

Proof. — With this new *structural assumptions*, the minimizers are still globally Lipschitz-continuous on Ω. We can define U as previously with $\mathrm{int}(\mathcal{SC})$ instead of $\mathcal{SC}$. The function $\max(d_0, g'(\nabla u))$ is still in $W^{1,2}_{\mathrm{loc}}(\Omega)$. Since $|g'(\mathbb{R}\backslash\mathcal{SC})| = 0$, Proposition 5.2 remains valid. Hence, the last crucial point is the end of the induction argument in Step 2 of the proof of Theorem 1.1. We assume that there exists $l^i_s(u) \Subset \Omega$ such that $l^i_s(u) \cap U = \emptyset$ and $C^i_s = g'(a_\infty)$. Then for every $l_t \Subset F^i_s$, we either have that $C_t = g'(a_\infty)$ or $\nabla(u - v) = 0\mathcal{H}^1$ a.e. on l_t. Hence, we have that $\nabla(u - v) = 0\mathcal{H}^1$ a.e. on $l^i_s(u)$. Thus, $u = v$ on Ω. □

Remark 7.2. — The sets

$$\left[0, \frac{1}{2}\right) \cup \left(\bigcup_{n\in\mathbb{N}^*} \left(\frac{2^{2n}-1}{2^{2n}}, \frac{2^{2n+1}-1}{2^{2n+1}} \right) \right) \cup [1, +\infty)$$

and

$$\left[0, \frac{1}{2}\right) \cup \left(\bigcup_{n\in\mathbb{N}^*} \left(\frac{2^{2n}-1}{2^{2n}}, \frac{2^{2n+1}-1}{2^{2n+1}} \right) \right) \cup (2, +\infty)$$

satisfy the new *structural assumptions* made on $\mathcal{SC}$.

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