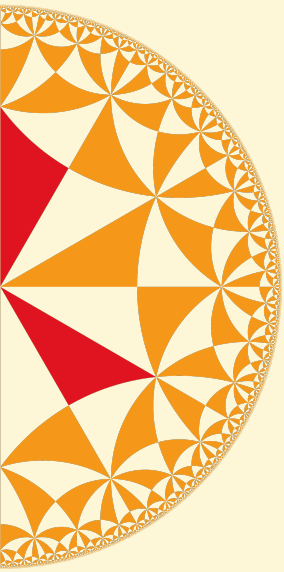
Université Grenoble Alpes



ANNALES DE
L'INSTITUT FOURIER

Mikael de la Salle

A duality operators/Banach spaces

Article à paraître, mis en ligne le 1er août 2025, 43 p.

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
Article à paraître
Mis en ligne le 1er août 2025.

A DUALITY OPERATORS/BANACH SPACES

by Mikael DE LA SALLE (*)

ABSTRACT. — To a pair (T, X) of an operator T between subspaces of L_p spaces and a Banach space X we can associate a finite or infinite number, the norm $\|T_X\|$ of T between the subspaces of the X-valued L_p spaces. Given such an operator T, we characterize all the operators S for which the implication $\|T_X\| < \infty \Rightarrow \|S_X\| < \infty$ holds.

This is a form of the bipolar theorem for a duality between the class of Banach spaces and the class of operators between subspaces of L_p spaces, essentially introduced by Pisier. The methods we introduce allow us to recover also the other direction – characterizing the bipolar of a set of Banach spaces –, which had been obtained by Hernandez in 1983.

RÉSUMÉ. — À une paire (T, X) d'un opérateur T entre sous-espaces d'espaces L_p et d'un espace de Banach X on peut associer un nombre, potentiellement infini: la norm $\|T_X\|$ de T entre sous-espaces d'espaces L_p à valeurs dans X. Étant donné un tel opérateur T, nous caractérisons tous les autres opérateurs S pour lesquels l'implication $\|T_X\| < \infty \Rightarrow \|S_X\| < \infty$ est vraie.

Cet énoncé est une forme du théorème du bipolaire pour une dualité entre classes d'espaces de Banach et classes de sous-espaces d'espaces L_p, essentiellement introduite par Gilles Pisier. Nos méthodes permettent également de retrouver l'autre direction – caractériser le bipolaire d'une classe d'espaces de Banach –, qui a été obtenue par Hernandez en 1983.

1. Introduction

All the Banach spaces appearing in this paper will be assumed to be separable, and will be over the field $\mathbf{K}$ of real or complex numbers.

The *local theory of Banach spaces* studies infinite dimensional Banach spaces through their finite-dimensional subspaces. For example it cannot distinguish between the (non linearly isomorphic if $p \neq 2$, see [3, Theorem XII.3.8]) spaces $L_p\big([0,1]\big)$ and $\ell_p(\mathbf{N})$, as they can both be written as the

Keywords: duality, geometry of Banach space, bipolar theorem, L_p spaces.
2020 *Mathematics Subject Classification:* 46A22, 46B20, 46B50.
(*) Research supported by the Charles Simonyi Endowment at the Institute for Advanced Study, and the ANR project ANCG Project-ANR-19-CE40-0002.

closure of an increasing sequence of subspaces isometric to $\ell_p\big(\{1,\dots,2^n\}\big)$: the subspace of $L_p\big([0,1]\big)$ made functions that are constant on the intervals $\big(\frac{k}{2^n},\frac{k+1}{2^n}\big]$, and the subspace of $\ell_p(\mathbf{N})$ of sequences that vanish oustide of $\{0,\dots,2^n-1\}$ respectively.

The relevant notions in the local theory of Banach spaces are the properties of a Banach space that depend only on the collection of his finite dimensional subspaces and not on the way they are organized. Said differently, the properties that are inherited by finite representability. Such properties are called *super-properties*. The central question is to understand whether one super-property implies another, see Section 2 for terminology, details and examples.

Given a real number $p \geqslant 1$ and a linear map T between subspaces $\mathrm{dom}(T)$ and $\mathrm{ran}(T)$ of L_p spaces $L_p(\Omega_1,m_1)$ and $L_p(\Omega_2,m_2)$, we can define a super-property $P(T)$ as follows: a Banach space X has $P(T)$ if and only $\|T_X\| \leqslant 1$. Here $\|T_X\|$ denotes the (possibly infinite) norm of $T\otimes \mathrm{id}_X$ between the subspaces $\mathrm{dom}(T)\otimes X$ and $\mathrm{ran}(T)\otimes X$ of $L_p(\Omega_i,m_i;X)$. It is the smallest real number such that

$$\int_{\Omega_2}\Big\|\sum_i (Tf_i)(\omega_2)x_i\Big\|^p \,\mathrm{d}m_2(\omega_2) \leqslant \|T_X\|^p \int_{\Omega_1}\Big\|\sum_i f_i(\omega_1)x_i\Big\|^p \,\mathrm{d}m_1(\omega_1)$$

for every n, every $f_1,\dots,f_n$ in the domain of T and every $x_1,\dots,x_n\in X$.

Estimating $\|T_X\|$ in terms of the properties of T and the geometric properties of X is a central aspect in the geometry of Banach spaces, as most classical super-properties are of this form $P(T)$: for example being a Hilbert space, a subspace of an L_p-space, a subquotient of an L_p space, type, cotype, UMD property, the maximal distance to Euclidean space of n-dimensional subspaces [15]... Moreover, many celebrated results can be expressed in the form "$\|T_X\| \leqslant 1 \Rightarrow \|S_X\| \leqslant 1$ for some concrete operators S,T" [5, 6, 13, 22, 29, 30]. We also present in some details in Section 2 two explicit open problems from group theory and metric geometry that can also be expressed in this way. In fact the ubiquity of super-properties of this form is not an accident, as a result by Hernandez [11, 12] (Theorem 1.3 below) asserts that a superproperty is of the form $P(T)$ for some operator T between subspaces of L_p spaces if and only if it is stable by ℓ_p-direct sums.

Our main result, Theorem 1.6, characterizes, for two operators S and T between subspaces of L_p spaces, when $\|T_X\| \leqslant 1$ implies $\|S_X\| \leqslant 1$. So it provides a theoretical characterization when a super-property that

is stable by ℓ_p-direct sums implies another. We leave for future investigations whether this theoretical characterization can be applied in concrete situations.

This is a form of the bipolar theorem for a duality between the set $\mathcal{X}$ of separable Banach spaces up to isometry and the set $\mathcal{T}$ of linear operators between subspaces of L_p spaces defined by the assignement $(T, X) \mapsto \|T_X\|$. Indeed, adapting the standard terminology for locally convex topological vector spaces (see [4, II Section 6]), we define:

DEFINITION 1.1. — *If $A \subset \mathcal{X}$ is a class of Banach spaces, then its* polar A° *is the class of operators $T \in \mathcal{T}$ such that $\|T_X\| \leqslant 1$ for every X in A.*

DEFINITION 1.2. — *If $B \subset \mathcal{T}$, then its polar ${}^\circ B$ is the class of Banach spaces $X \in \mathcal{X}$ such that $\|T_X\| \leqslant 1$ for every T in B.*

This duality is a variant of the one considered in [30], where Pisier restricts to operators between L_p spaces (and not subspaces of L_p spaces). If one is interested in the bipolar of a class of Banach spaces, the two dualities are very different. On the other hand, a description of the bipolar for a class of operators for Pisier's duality can be obtained from our result, see Section 5 for details.

In a locally convex topological space, the bipolar theorem ([4, II Section 6]) states that the bipolar of a set C is equal to the closed convex hull of $C \cup \{0\}$. The inclusion of the closed convex hull of $C \cup \{0\}$ in the bipolar of C is obvious; the content of the theorem is the other inclusion, which follows from the Hahn–Banach theorem. The aim of this paper is to state and prove a version of the bipolar theorem in this setting, for the correct definition of "closed convex hull". For the bipolar of a class of Banach spaces, this is due to Hernandez. The methods we introduce allow us to give a new proof of it (see Section 5 for the duality involving operators between L_p spaces).

THEOREM 1.3 ([11]). — *The bipolar ${}^\circ(A^\circ)$ of a class of Banach spaces $A \subset \mathcal{X}$ is the class of Banach spaces finitely representable in the class of all finite ℓ_p-direct sums of elements in A.*

There is also an isomorphic version of the previous result.

THEOREM 1.4 ([11]). — *Let $A \subset \mathcal{X}$ and $X \in \mathcal{X}$. The following are equivalent:*

- *$\|T_X\| < \infty$ for every $T \in A^\circ$;*

- *X is isomorphic to a space finitely representable in the class of finite ℓ_p direct sums of spaces in A, i.e. to a space in ${}^\circ A^\circ$.*

In that case, the Banach–Mazur distance from X to a space in ${}^\circ A^\circ$ is equal to $\sup_{T\in A^\circ}\|T_X\|$.

Our main result is the bipolar theorem for sets of operators. To state it we have to introduce some definition.

Definition 1.5. — *A* spatial isometry *between finite dimensional subspaces of L_p spaces is a composition of isometries of the form:*

- *(change of density) restriction to a subspace of $L_p(\Omega, m)$ of the multiplication by a nonvanishing measurable function $h\colon \Omega \to \mathbf{K}^*$, i.e. $f \in L_p(\Omega, m) \mapsto hf \in L_p\big(\Omega, |h|^{-p}m\big)$;*
- *(equidistribution outside of 0) maps of the form $T\colon \mathrm{dom}(T) \subset L_p(\Omega, m) \to L_p(\Omega', m')$ such that for every finite family $f_1, \dots, f_n \in \mathrm{dom}(T)$ and every Borel subset $E \subset \mathbf{K}^n \setminus \{0\}$,*

$$m\Big(\big\{x \mid \big(f_1(x), \dots, f_n(x)\big) \in E\big\}\Big) = m'\Big(\big\{x \mid \big(Tf_1(x), \dots, Tf_n(x)\big) \in E\big\}\Big).$$

It is not hard to prove (see Lemma 4.11 and Remark 4.12) that every spatial isometry is of the form C_1EC_2 for C_1, C_2 changes of density and E an equidistribution outside of 0.

It is important that we require $0 \notin E$, as we want for example that $f \in L_p\big([0,1]\big) \mapsto f\chi_{[0,1]} \in L_p\big([0,2]\big)$ is a spatial isometry.

When p is not an even integer, it is known that every isometry between separable subspaces of L_p spaces is a spatial isometry [9]. The idea developped in this article allows to recover this result, and to generalize it to arbitrary p: a linear map T is a spatial isometry if and only if it is a regular isometry, i.e. $\|T\|_r = \|T^{-1}\|_r = 1$, where $\|T\|_r = \sup_X \|T_X\|$ is the regular norm of T (see Remark A.2 and Corollary A.3).

We can now state the version of the bipolar theorem for sets of operators.

Theorem 1.6. — *Let $T\colon \mathrm{dom}(T) \subset L_p(\Omega_1, m_1) \to L_p(\Omega_2, m_2)$ be a linear map, $B \subset \mathcal{T}$, and $f_1, f_2, \dots,$ be a sequence generating a dense subspace of* $\mathrm{dom}(T)$. *The following are equivalent:*

- *For every $X \in$ Banach, $\sup_{S\in B}\|S_X\| \leqslant 1 \Rightarrow \|T_X\| \leqslant 1$.*
- *For every n and $\varepsilon > 0$, there exist:*
  - *an operator $S = S_0 \oplus S_1 \oplus \cdots \oplus S_k$ with S_0 of regular norm 1 and $S_1, \dots, S_k \in B$;*

– *spatial isometries*

$$U\colon \operatorname{dom}(U) \subset L_p\big(\Omega_1 \times [0,1]\big) \oplus_p L_p\big([0,1]\big) \to \operatorname{dom}(S),$$
$$V\colon \operatorname{dom}(V) = S(\operatorname{ran} U) \to L_p\big(\Omega_2 \times [0,1]\big) \oplus_p L_p\big([0,1]\big);$$

– *for every* $i = 1, \dots, n$ *there are* $g_i \in L_p\big(\Omega_1 \times [0,1]\big)$, $g_i' \in L_p\big(\Omega_2 \times [0,1]\big)$ *and* $h_i \in L_p\big([0,1]\big)$;

such that $(g_i, h_i) \in \operatorname{dom}(U)$, $V \circ S \circ U(g_i, h_i) = (g_i', h_i)$ *and*

$$\left(\int_{\Omega_1 \times [0,1]} \big|f_i(\omega) - g_i(\omega, s)\big|^p \,\mathrm{d}m_1(\omega)\,\mathrm{d}s\right)^{\frac{1}{p}} \leqslant \varepsilon,$$

$$\left(\int_{\Omega_2 \times [0,1]} \big|(Tf_i)(\omega) - g_i'(\omega, s)\big|^p \,\mathrm{d}m_2(\omega)\,\mathrm{d}s\right)^{\frac{1}{p}} \leqslant \varepsilon.$$

It is instructing to work out explicitly a very simple case of this theorem, namely for the obvious implication $\max\big(\|S_X\|, \|T_X\|\big) \leqslant 1 \Rightarrow \big\|(T \circ S)_X\big\| \leqslant 1$. See Example 3.6.

In particular, we obtain the following characterization of the bipolar of a set of operators.

Corollary 1.7. — *The bipolar* $({}^\circ B)^\circ$ *of a class* $B \subset \mathcal{T}$ *is the smallest class* $B' \subset \mathcal{T}$ *containing* B *and satisfying the following properties:*

(1) B' *contains* $\big\{T \in \mathcal{T},\ \sup_{X \in \mathcal{X}} \|T_X\| \leqslant 1\big\}$.
(2) B' *is stable under finite* ℓ_p*-direct sums.*
(3) *If* $T \in B'$ *and* U, V *are spatial isometries then* $U \circ T \circ V \in B'$.
(4) *Let* $T \in B'$ *such that* $T\colon \operatorname{dom}(T) \subset L_p(\Omega_1, m_1) \oplus L_p(\Omega, m) \to L_p(\Omega_1, m_1) \oplus L_p(\Omega, m)$ *is of the form* $(f, g) \mapsto (Sf, g)$ *for some* $S \in \mathcal{T}$ *with domain equal to the image of* $\operatorname{dom}(T)$ *by the first coordinate projection. Then* $S \in B'$.
(5) *If* $T \in \mathcal{T}$ *is an operator between respectively subspaces of* $L_p(\Omega, m)$ *and* $L_p(\Omega', m')$ *and if, for every finite family* $f_1, \dots, f_n$ *in the domain of* T *and every* $\varepsilon > 0$, *there is* $S \in B'$ *with domain contained in* $L_p(\Omega, m)$ *and range contained in* $L_p(\Omega', m')$ *and elements* $g_1, \dots, g_n \in \operatorname{dom}(S)$ *such that* $\|f_i - g_i\| \leqslant \varepsilon$ *and* $\|Tf_i - Sg_i\| \leqslant \varepsilon$, *then* $T \in B'$.

Note however that Theorem 1.6 is in a sense more precise than Corollary 1.7, as it almost says that to obtain the bipolar of B from B, it is enough to apply the operations (1)–(5) only once, and in that order. Almost because we obtain in this way all operators of the form $T \otimes \mathrm{id}_{L_p([0,1])}$ with domain $\big\{(\omega, s) \mapsto f(\omega) \mid f \in \operatorname{dom}(T)\big\}$ for $T \in ({}^\circ B)^\circ$, so one needs

to apply one last time (3) to obtain all of $({}^\circ B)^\circ$. This improvement is not formal. For a long time, the author was only able to prove Corollary 1.7, and actually expected that to construct $({}^\circ B)^\circ$ out of B, it was necessary to iterate these operations (and in particular (4) and (5)) a large number of times (even an arbitrarily large countable ordinal of times), and this ordinal number was a measurement of the difficulty of computing the bipolar of a B. This is closely related to the classical fact, essentially due to Banach, that, to obtain the weak-* closure of a convex subset in the dual of a separable Banach space, the number of times one needs to take limits of weak-* convergent sequences can be an arbitrary countable ordinal. That this is not the case will rely on a particularly strong form of the bipolar theorem (in the linear setting) for the weak-* topology that we prove in Proposition 3.3. See the discussion in subsection 3.1.

As for the usual bipolar theorem, the main content of the theorem is the inclusion ${}^\circ B^\circ \subset B'$. The reverse inclusion is rather obvious because it is rather clear that ${}^\circ B^\circ$ contains B and satisfies all the properties (1)–(5).

So one can reformulate the nontrivial part of Corollary 1.7 as follows: if $T \notin B'$, then there is a Banach space X such that $X \in {}^\circ B$ but $\|T_X\| > 1$. Constructing Banach spaces with prescribed properties is a notoriously difficult problem in general. What saves us here is that we do not construct X explicitly, but we let the Hahn–Banach theorem construct it for us. This is achieved by suitably encoding the class of Banach spaces in a locally convex topological vector space H and the class of operators between subspaces of L_p spaces in its dual H^*, in such a way that the polarity between $\mathcal{X}$ and $\mathcal{T}$ corresponds to the usual polarity in topological vector spaces. So, once these two encodings are well understood, both Theorems 1.3 and 1.6 are just an application of the bipolar theorem in H and H^*. When X is a Banach space of dimension n, X will be encoded inside the real Banach space $C(\mathbf{KP}^{n-1})$ of real-valued continuous functions on the projective space of dimension $n-1$. Similarly an operator T with a domain of dimension n will be encoded inside the dual of $C(\mathbf{KP}^{n-1})$. The space H evoked would then be the projective limit of a suitable system of the spaces $C(\mathbf{KP}^{n-1})$. But since the study of the polarity between $\mathcal{X}$ and $\mathcal{T}$ readily reduces to finite dimensional Banach spaces and operators with finite dimensional domains, we prefer to work directly with $C(\mathbf{KP}^{n-1})$ and never even formally introduce H.

Organization of the paper

The first section presents some necessary background and some motivation for studying this polarity. Section 3 contains various preliminaries, including basic reminders on measure theory and on the linear bipolar theorem, as well as one result on which the rest will rely: Proposition 3.3. Section 4 contains the proof of the main theorem. It starts by defining the encoding of spaces and operators in a linear duality, and then studies this encoding. Section 5 contains a discussion of variants of the duality presented in the introduction. In an appendix we present a new proof and a generalization, in the context of Section 4, of Hardin's theorem [9]. Hardin's theorem appears as a direct corollary of the study of the invariant subspaces for some families of representations of $\mathrm{GL}_n(\mathbf{K})$ on $C(\mathbf{KP}^{n-1})$.

Some of the results have been announced in the Oberwolfach report *Group actions on Banach spaces and a duality spaces/operators* (see [31, pp. 2304–2307]).

Acknowledgments

The author thanks Jean-Christophe Mourrat for interesting discussions that lead to the proof of Proposition A.4, and Alexandros Eskenazis, Mikhail Ostrovskii and Ignacio Vergara for useful comments, suggestions and corrections. Special thanks are due to Gilles Pisier for all that, and also for encouraging the author to think about the content of Section 5. Finally, the author thanks the referee for their numerous and wise suggestions, including for the presentation of the proof of Proposition 3.3.

2. Background and motivation

2.1. Measure spaces and operators

To avoid any set-theoretical problem (of $\mathcal{T}$ not being a set), all the measure spaces appearing here will be standard measure spaces taken in some fixed set containing $[0,1]$ with the Lebesgue measure and that is stable by taking equivalent measures, measurable subsets with restriction of the measure, and finite direct sums. By direct sum of a finite sequence $(\Omega_1, m_1), \dots, (\Omega_n, m_n)$ we mean the space $(\Omega_1 \cup \dots \cup \Omega_n, m_1 \oplus \dots \oplus m_n)$

where $\Omega_1 \cup \cdots \cup \Omega_n$ is the disjoint union and the measure is $A \mapsto \sum_i m_i(A \cap \Omega_i)$. None of the results depend on the choice.

Throughout the text, $p \in [1, \infty)$ will be a fixed number. The ℓ_p-direct sum of a finite family $T_1, \ldots, T_n$ of operators from $\mathrm{dom}(T_i) \subset L_p(\Omega_i, m_i)$ to $\mathrm{ran}(T_i) \subset L_p(\Omega_i', m_i')$ is the operator $T_1 \oplus \cdots \oplus T_n$ from $\mathrm{dom}(T_1) \oplus \cdots \oplus \mathrm{dom}(T_n) \subset L_p(\Omega_1 \cup \cdots \cup \Omega_n, m_1 \oplus \cdots \oplus m_n)$ to $\mathrm{ran}(T_1) \oplus \cdots \oplus \mathrm{ran}(T_n) \subset L_p(\Omega_1' \cup \cdots \cup \Omega_n', m_1' \oplus \cdots \oplus m_n')$.

An operator $T \in \mathcal{T}$ is called regular if $\|T_X\| < \infty$ for every Banach space X, or equivalently if $\|T_{\ell_\infty}\| < \infty$. In that case the quantity $\|T_{\ell_\infty}\| = \sup_X \|T_X\|$ is called the regular norm of T and denoted $\|T\|_r$. We will denote by REG the set of operators $T \in \mathcal{T}$ such that $\|T\|_r \leqslant 1$.

2.2. Banach space geometry

If n is an integer, the set $Q(n)$ of all n-dimensional normed space up to isometry, equipped with the Banach–Mazur distance

$$d(E, F) = \inf\big\{ \|u\| \|u^{-1}\| \mid u \colon E \to F \text{ linear invertible} \big\},$$

becomes a compact metric space, *the Banach–Mazur compactum*. Beware that it is not d but $\log d$ which is a distance in the usual way (d is submultiplicative $d(E, G) \leqslant d(E, F) d(F, G)$ rather than subadditive, and two isometric spaces are at Banach–Mazur distance 1), but following the tradition we still call d the Banach–Mazur distance.

We say that a Banach space X is finitely representable in another Banach space Y if for every finite-dimensional space $E \subset X$ and every $\varepsilon > 0$ there is a subspace $F \subset Y$ of same dimension as E such that $d(E, F) \leqslant 1 + \varepsilon$. In other words, if for every n, the closure in $Q(n)$ of the space of n-dimensional subspaces of X is contained in the same closure but for Y. This is equivalent to X being isometrically a subspace of an ultraproduct of Y [10].

More generally, we say that a Banach space X is finitely representable in a class B of Banach spaces if for every finite-dimensional space $E \subset X$ and every $\varepsilon > 0$ there is a subspace F of a space in B of same dimension as E such that $d(E, F) \leqslant 1 + \varepsilon$. We can therefore view *a class of Banach spaces up to finite representability* as a collection A_n of closed subsets of $Q(n)$ such that for every $n > m$, every m-dimensional subspace of every $E \in A_n$ belongs to A_m. In this representation, finite representability corresponds to inclusion.

The ℓ_p-direct sum of a finite family $X_1, \ldots, X_n$ of Banach spaces is the space $X_1 \oplus X_2 \oplus \cdots \oplus X_n$ for the norm $\big\|(x_1, \ldots, x_n)\big\| = \big(\|x_1\|^p + \cdots + \|x_n\|^p\big)^{\frac{1}{p}}$.

The next two subsections contain some very concrete questions of the form "does one operator belong to the bipolar of another?", that played an essential rôle motivating the author's investigations. We leave it to further research whether Theorem 1.6 and Corollary 1.7 can be applied to answer these questions.

2.3. Group representations on Banach spaces

The author's first main motivation comes from the study of representations of groups on Banach spaces. Let G be a locally compact topological group with a fixed left Haar measure. We recall that every strong-operator-topology (SOT) continuous representation π of G on a Banach space X extends to a representation of the convolution algebra $C_c(G)$ of compactly supported continuous functions on G by setting $\pi(f)x = \int f(g)\pi(g)x \,\mathrm{d}g$ for every $x \in X$.

For example, if λ_p denotes the left-regular representation on $L_p(G)$:

$$\lambda_p(g)f = f(g^{-1}\cdot),$$

then $\lambda_p(f)$ is the convolution operator $\xi \mapsto f * \xi$.

When A is a class of Banach spaces, denote by $C_A(G)$ the completion of $C_c(G)$ for the norm

$$\|f\|_{C_A(G)} = \sup \big\|\pi(f)\big\|_{B(X)},$$

where the supremum is over all SOT-continuous continuous isometric representations π of G on a space X in A.

The following result, which generalizes the classical fact that the full and reduced C^*-algebras of an amenable group coincide, reduces the understanding the representation theory of G on a Banach space X to the understanding of $\|T_X\|$ for convolution operators T. This known fact has already appeared in several unpublished texts (for example in the author's habilitation thesis), but seems to be missing from the published literature.

PROPOSITION 2.1. — *If G is amenable and π is an isometric representation of G on a Banach space X, then for every $f \in C_c(G)$,*

$$\big\|\pi(f)\big\|_{B(X)} \leqslant \big\|\lambda_p(f)_X\big\|.$$

In particular, if a class of Banach spaces A has the property that $L_p(G;X) \in A$ for every $X \in A$, then

$$\|f\|_{C_A(G)} = \sup_{X\in A} \big\|\lambda_p(f)_X\big\|.$$

Proof. — Fix a norm 1 element $\xi \in L_p(G)$ and define an isometric linear map $\alpha\colon X \to L_p(G;X)$ by $\alpha(x)(g) = \xi(g)\pi(g^{-1})x$.

Then for $h \in G$, $\big(\alpha\big(\pi(h)x\big) - \lambda(h)\alpha(x)\big)(g) = \big(\xi(g) - \xi(h^{-1}g)\big)\pi(g^{-1}h)x$, and

$$\big\|\alpha\big(\pi(h)x\big) - \lambda(h)\alpha(x)\big\| = \|x\|\big\|\xi - \lambda(h)\xi\big\|_{L_p(G)}.$$

By the triangle inequality

$$\big\|\alpha(\pi(f)x) - \lambda(f)\alpha(x)\big\| \leqslant \|f\|_{L_1(G)}\|x\| \sup_{h\in\mathrm{supp}(f)} \big\|\xi - \lambda(h)\xi\big\|_{L_p(G)},$$

and using that α is isometric we obtain

$$\big\|\pi(f)x\big\| \leqslant \big\|\lambda(f)_X\big\|\|x\| + \|f\|_{L_1(G)}\|x\| \sup_{h\in\mathrm{supp}(f)} \big\|\xi - \lambda(h)\xi\big\|_{L_p(G)}.$$

We deduce

$$\big\|\pi(f)\big\| \leqslant \big\|\lambda(f)_X\big\| + \|f\|_{L_1(G)} \sup_{h\in\mathrm{supp}(f)} \big\|\xi - \lambda(h)\xi\big\|_{L_p(G)}.$$

When G is amenable, the last term can be made arbitrarily small, which proves the proposition. □

In the particular case of a compact group, this result lies at the heart of the proofs of Lafforgue's strong property (T) for higher-rank algebraic groups. For example, thanks to the techniques of strong property (T) and in particular Proposition 2.1, the conjecture[(1)] [2] that any action by isometries of a lattice in a connected higher-rank simple Lie group on a super-reflexive Banach space has been reduced to the following conjecture, see [14, 18, 19], see also [34]. Denote, for any $\delta \in [-1,1]$, by T_δ the operator on $L_2(\mathbf{S}^2)$ mapping f to the fonction $(T_\delta f)(x) =$ the average of f on the circle $\big\{y \in \mathbf{S}^2 \mid \langle x,y\rangle = \delta\big\}$. For any $\theta \in \mathbf{R}/2\pi\mathbf{Z}$, denote by S_θ the operator on $L_2(\mathbf{S}^3)$ mapping f to the fonction $(S_\theta f)(z) =$ the average of f on the circle $\big\{\frac{1}{\sqrt{2}}(e^{i\theta} + e^{i\varphi}j)z \mid \varphi \in \mathbf{R}/2\pi\mathbf{Z}\big\}$ (where we identify $\mathbf{S}^3$ with the norm 1 quaternions in the usual way). The conjecture is that for every super-reflexive Banach space, there exist $\alpha > 0$ and $C \in \mathbf{R}_+$ such that for every $\delta \in [-1,1]$ and $\theta \in \mathbf{R}$,

$$\big\|(T_\delta - T_0)_X\big\| \leqslant C|\delta|^\alpha \qquad \text{and} \qquad \big\|(S_\theta - S_{\pi/2})_X\big\| \leqslant C|\theta - \pi/2|^\alpha.$$

It is even conjectured that the same holds for every space with nontrivial type. If U is any operator on L_2 quantifying the fact that a Banach space

(1) (Added in January 2024) This conjecture for superreflexive space has been recently settled by de Laat and the author [17], following a breakthrough by Oppenheim [25], and with different methods than the ones suggested here. The question raised here remains however interesting, as it implies strong Banach property, a stronger property than the fixed point property and for a wider class of Banach spaces.

has nontrivial type (for example a multiple of the orthogonal projection on the space spanned by a sequence of iid Rademacher random variables [29]), this conjecture is equivalent to saying

$$\left\{ \frac{1}{C|\delta|^\alpha}(T_\delta - T_0) \;\middle|\; \delta \in (-1,1) \right\} \bigcup \left\{ \frac{1}{C|\theta - \pi/2|^\alpha}(S_\theta - S_{\pi/2}) \;\middle|\; \theta \in \mathbf{R} \right\}$$

is contained in the bipolar of $\{U\}$ for some C and $\alpha > 0$.

2.4. Super-expanders and embeddability of graphs in Banach spaces

Another motivation for studying the quantity $\|T_X\|$ is its well-known connection with Poincaré inequalities and embeddability of expanders in X. If $\mathcal{G} = (V, E)$ is a finite connected graph, we may define[(2)] its X-valued p-Poincaré constant $\pi_{p,\mathcal{G}}(X)$ as the smallest constant π such that for every $f : V \to X$ satisfying $\sum_{v\in V} \deg(v) f(v) = 0$,

$$\left(\sum_{v\in V} \deg(v) \|f(v)\|^p \right)^{\frac{1}{p}} \leqslant \pi \left(\sum_{(v,w)\in E} \big\|f(v) - f(w)\big\|^p \right)^{\frac{1}{p}}.$$

Note that $\pi_{p,\mathcal{G}}(X) = \|T_X\|$ for T the inverse of the linear map $f \in \ell_p^0(V, \deg) \mapsto \big(f(v) - f(w)\big)_{(v,w)\in E} \in \ell_p(E)$.

A sequence $\mathcal{G}_n = (V_n, E_n)$ of bounded degree graphs is called *a sequence of expanders with respect to* X if $\lim_n |V_n| = \infty$ and $\sup_n \pi_{p,\mathcal{G}_n}(X) < \infty$. This does not depend on p [7, 23, 24], see also [16, Proposition 3.9].

For example, if $p = 2$ and $X = \mathbf{K}$ (or a Hilbert space), then $\pi_{p,2}(\mathbf{K})$ is equal to $(2 - 2\lambda_2)^{-\frac{1}{2}}$, for λ_2 the second largest eigenvalue of the random walk operator on $\mathcal{G}$. So being a sequence of expanders with respect to $\mathbf{K}$, or to an L^p space for some $p < \infty$, is the same as the usual definition of expander graphs.

According to [21], a sequence $\mathcal{G}_n$ is called a sequence of super-expanders if they are expanders with respect to all uniformly convex Banach spaces. The existence of super-expanders is a difficult result. Essentially two classes of examples have been obtained, by Lafforgue [19] and by Mendel and Naor [21]. Lafforgue's examples are even expanders with respect to all Banach spaces of type > 1. All these results are therefore results of the form "T belongs to be bipolar of S", where S is any of the operators quantifying

(2) There are many small variants of the definition. But they do not matter for the discussion here, though they do matter for other issues, see for example [16].

the fact that a Banach space has nontrivial type or is super-reflexive, and T are correctly scaled operators in the definition of the p-Poincaré constant. Many intriguing questions remain open, which can all be formulated in the same way. For example,

QUESTION 2.2 ([21]). — *Are all expander sequences super-expanders? Expanders with respect to all spaces of nontrivial type?*

QUESTION 2.3 ([19, 21]). — *Do super-expanders of girth going to infinity exist? And of logarithmic girth in the number of vertices? Are the expanders coming from higher-rank simple Lie groups super-expanders?*

QUESTION 2.4 ([19, 21]). — *Does there exist a sequence of expanders with respect to all Banach spaces of nontrivial cotype?*

A positive answer to this question is conjectured in [21], and Lafforgue even suggests that the super-expanders coming from lattices in $\mathrm{SL}_3(\mathbf{Q}_p)$ (or other higher-rank simple algebraic groups over non-archimedean local fields) as in [19] are such examples. But this is wide open, as is the following.

QUESTION 2.5 ([30]). — *Are all expander sequences expanders with respect to all spaces of nontrivial cotype?*

One of the reasons for the interest in expanders with respect to Banach spaces is the well-known fact, which essentially goes back to Gromov, that a sequence of expanders with respect to X does not coarsely embed into X. See for example [30, Section 3]. Being an expander with respect to X is much stronger than non coarse embeddability (a striking example is given in [1]), but by [37] (see also [28] for L_1 spaces) there is equivalence between non-coarse embeddability into families of Banach spaces under closed finite representability and ℓ_p direct sums and some *other* forms of Poincaré inequalities.

3. Preliminaries

3.1. On the bipolar in a dual Banach space

In the whole paper, for a subset C of a real Banach space E with dual E^*, we denote its polar

$$C^\circ = \big\{x^* \in E^*,\ \langle x^*, x\rangle \geqslant -1 \text{ for all } x \in C\big\}.$$

When $C \subset E$ is a cone (that is $x \in C$ implies $\{tx \mid t \in [0,\infty)\} \subset C$), then its polar C° coincides with $\{x^* \in E^*, \langle x^*, x\rangle \geqslant 0 \text{ for all } x \in C\}$. It is also a cone.

Similarily, when $C \subset E^*$ we denote its polar for the weak-* topology by

$$^\circ C = \{x \in E, \langle x^*, x\rangle \geqslant -1 \text{ for all } x^* \in C\}.$$

Again, if C is a cone, $^\circ C$ coincides with $\{x \in E, \langle x^*, x\rangle \geqslant 0 \text{ for all } x^* \in C\}$ and is again a cone.

It should be always clear from the context whether the polarity is considered in this linear setting of two vector spaces in duality or between $\mathcal{X}$ and $\mathcal{T}$ as in Definition 1.1 and 1.2.

The classical bipolar theorems in this setting take the following forms:

THEOREM 3.1. — *Let E be a real Banach space.*

If $C \subset E$, then its bipolar $^\circ(C^\circ)$ is equal to the norm closure of the convex hull of $C \cup \{0\}$.

If $C \subset E^*$, then its bipolar $(^\circ C)^\circ$ is equal to the weak- closure of the convex hull of $C \cup \{0\}$.*

The second statement is not so useful for our purposes because taking the weak-* closure can be quite complicated, as we shall soon recall. Fortunately, there is an interesting consequence of the Krein–Smulian theorem [8, Theorem V.12.1], which asserts that a convex subset of E^* for a separable Banach space E is weak-* closed if and only if it is sequentially weak-* closed, see [8, Theorem V.12.10]. This allows to significantly strengthen the result for separable Banach spaces as follows.

If C is a subset of a dual E^*, let us define an increasing family of subsets $C_\alpha \subset E^*$ indexed by the ordinals α by letting $C_0 = C$, C_α be the set of all weak-* limits of sequences in $C_{\alpha-1}$ if α is a successor and $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$ if α is a limit ordinal. The smallest ordinal α such that $C_\alpha = C_{\alpha+1}$ (that is C_α is weak-* sequentially closed) is sometimes called the order of C. When E is separable, the order of C is countable, see for example the argument in the proof of [8, Theorem V.12.10]. Moreover, if C is convex, then so is C_α for every α. It follows from [8, Theorem V.12.10] that, for the order of C, C_α coincides with the weak-* closure of C. Let us summarize this discussion.

PROPOSITION 3.2. — *Let E be a real separable Banach space and C be a subset of E^*. There is a countable ordinal α such that the bipolar of C coincides with $\big(\mathrm{conv}(C)\big)_\alpha$.*

The smallest ordinal α in the previous proposition measures the difficulty to construct the bipolar of C out of C.

The order has been more studied for linear subspaces C. It is known that for many cases, every countable ordinal appears as the order of a linear subspace of E^*. This was stated by Banach [20], and later examples such as $E^* = \ell_1 = (c_0)^*$, ℓ_∞, H_∞ were provided together will full proofs [35, 36]. It is now known that this holds whenever E is not quasi-reflexive, that is when the canonical image of E has infinite codimension in its bidual [26]. See also the survey [27] for more information on this.

It turns out that, for our applications, the order will always be equal to 1. This will follow from the following result.

PROPOSITION 3.3. — *Let E be a real Banach space and $C \subset E^*$. Assume that there is a convex subset $A \subset E$ containing 0 such that $A \cap \{x \in E \mid \|x\| \leqslant r\}$ is norm-compact for every $r > 0$ and $A^\circ \subset C$.*

Then the bipolar $({}^\circ C)^\circ$ of C is equal to the norm closure of the convex hull of C.

Proof. — Note that our assumption implies that $0 \in C$ (as $0 \in A^\circ$). The set A is a norm convex subset containing 0, so $A = {}^\circ(A^\circ)$ by the bipolar theorem in E. From the assumption $A^\circ \subset C$, we deduce that ${}^\circ C \subset {}^\circ A^\circ = A$. Hence ${}^\circ C$ satisfies the same assumption as A. It follows from the Krein–Smulian theorem that its image by the canonical injection $j\colon E \to E^{**}$ is weak-* closed (as norm-compact sets are weak-* closed), and hence $j({}^\circ C) = C^\circ$ by Hahn–Banach. We deduce $({}^\circ C)^\circ = {}^\circ j({}^\circ C) = {}^\circ(C^\circ)$, which is the norm-closed convex hull of C by the bipolar theorem in E^*. This concludes the proof. □

3.2. Reminders on the Jordan decomposition of measures

Recall that any signed measure m on a Borel space has a unique decomposition $m = m_+ - m_-$ for two positive measures satisfying $\|m\| = \|m_+\| + \|m_-\|$ (where the norm is the total variation norm). This is the Jordan decomposition of m. If $m = m_1 - m_2$ is any other decomposition with m_1, m_2 positive measures, then $m_1 - m_+ = m_2 - m_-$ is a positive measure. We will use the following elementary fact.

LEMMA 3.4. — *Let m and m' be any signed measure, and let m_1, m_2 be any positive finite measures such that $m = m_1 - m_2$. There is a decomposition $m' = m'_1 - m'_2$ with*

$$\|m_1 - m'_1\| + \|m_2 - m'_2\| = \|m - m'\|.$$

Proof. — Let $m = m_+ - m_-$ and $m' = m'_+ - m'_-$ be the Jordan decompositions. A small computation gives that $\|m - m'\| = \|m_+ - m'_+\| + \|m_- - m'_-\|$.

By the property of the Jordan decomposition just recalled, $m'' \coloneqq m_1 - m_+ = m_2 - m_-$ is a positive measure. Define $m'_1 = m'_+ + m''$ and $m'_2 = m'_2 + m''$, so that $m' = m'_1 - m'_2$ and

$$\|m_1 - m'_1\| + \|m_2 - m'_2\| = \|m_+ - m'_+\| + \|m_- - m'_-\| = \|m - m'\|. \quad \square$$

3.3. On (4) in Corollary 1.7

This short subsection is not needed anywhere else in the paper, but it hopefully illustrates some basic things about Theorem 1.6 and Corollary 1.7. We start by a lemma which clarifies in which situation an operator T is of the form (4) in Corollary 1.7.

LEMMA 3.5. — *Let T be an operator between subspaces*

$$\mathrm{dom}(T), \mathrm{ran}(T) \subset L_p(\Omega, m)$$

of norm $\leqslant 1$ and $A \subset \Omega$ measurable. The following are equivalent.

- *$Tf(x) = f(x)$ for almost every $x \in \Omega \setminus A$ and every $f \in \mathrm{dom}(T)$.*
- *If we write $L_p(\Omega, m) = L_p(A, m) \oplus_p L_p(\Omega \setminus A, m)$, then there is an operator S with domain equal to the image of $\mathrm{dom}(T)$ by the first coordinate projection such that $T(f_1, f_2) = (Sf_1, f_2)$ for all $(f_1, f_2) \in \mathrm{dom}(T)$.*

In that case, S is unique, $\mathrm{dom}(S) = \{f\big|_A,\ f \in \mathrm{dom}(S)\}$ and $S(f\big|_A) = (Tf)\big|_A$ for all $f \in \mathrm{dom}(T)$.

Proof. — Clearly, the assumption that $Tf(x) = f(x)$ for almost every $x \in \Omega \setminus A$ and every $f \in \mathrm{dom}(T)$ is equivalent to the existence of a linear map $S\colon \mathrm{dom}(T) \to L_p(A, m)$ such that $T(f_1, f_2) = \big(S(f_1, f_2), f_2\big)$. So to prove the equivalence stated in the lemma, we have to observe that, in this situation, $S(f_1, f_2)$ depends only on f_1, i.e. (by linearity) that $S(f_1, f_2) = 0$ if $f_1 = 0$. For $(0, f_2) \in \mathrm{dom}(T)$ we have $\big\|T(0, f_2)\big\|_p^p = \big\|S(0, f_2)\big\|_p^p + \|f_2\|^p$, which (by the assumption that $\|T\| \leqslant 1$) is less than $\|f_2\|^p$. This proves that $\big\|S(0, f_2)\big\|_p^p = 0$, as requested.

The last assertion is a tautology. $\square$

Finally, we provide an example that illustrates the main result.

Example 3.6. — The inequality $\big\|(T\circ S)_X\big\| \leqslant \|T_X\|\|S_X\|$ is clear for every Banach space X and every operators T, S such that $T\circ S$ makes sense. So it follows from Corollary 1.7 that, with the notation therein, if $S, T \in B$ then $T\circ S$ belongs to B'. We prove this directly, because it illustrates the subtle property (4).

So let $S, T \in B$ such that $\mathrm{ran}(S) \subset \mathrm{dom}(T)$. By (2) the operator $S \oplus T\colon \mathrm{dom}(S)\oplus\mathrm{dom}(T) \to \mathrm{ran}(S)\oplus\mathrm{ran}(T)$ belongs to B'. By composing by the spatial isometry $(f,g) \in \mathrm{ran}(S)\oplus\mathrm{ran}(T) \mapsto (g,f) \in \mathrm{ran}(T)\oplus\mathrm{ran}(S)$ (which is allowed by (3)) and restricting to the subspace $D = \big\{(f, Sf) \mid f \in \mathrm{dom}(S)\big\} \subset \mathrm{dom}(S)\oplus\mathrm{dom}(T)$ (which is allowed by (5)), we obtain that the map $(f,Sf)\in D \mapsto (T\circ Sf, Sf)$ belongs to B'. By (4), we conclude that $T\circ S$ belongs to B' as required.

4. The space of degree p homogeneous functions on $\mathbf{K}^n$

Let n be a positive integer. Denote by $|z|$ the ℓ_p-norm on $\mathbf{K}^n$,

$$|z| = \big(|z_1|^p + \cdots + |z_n|^p\big)^{\frac{1}{p}}.$$

In the rare occasions when we want to insist on p, we write $|z|_p$ for this quantity.

A function $\varphi\colon \mathbf{K}^n \to \mathbf{R}$ is called homogeneous of degree p if $\varphi(\lambda z) = |\lambda|^p\varphi(z)$ for all $z \in \mathbf{K}^n$ and $\lambda \in \mathbf{K}$. The space H_n of continuous homogeneous of degree p functions on $\mathbf{K}^n$ is a Banach space over the field of real numbers for the topology of uniform convergence on compact subsets on $\mathbf{K}^n$. A particular choice of norm is $\|\varphi\| = \sup_{|z|\leqslant 1}\big|\varphi(z)\big|$, so that for this norm H_n is isometrically isomorphic to the space of real-valued continuous functions on $\mathbf{KP}^{n-1}$ through the identification of $\varphi \in H_n$ with the function $\mathbf{K}z \in \mathbf{KP}^{n-1} \mapsto \varphi\big(\frac{z}{|z|}\big)$. An equivalent definition of the norm of $\varphi \in H_n$ is the smallest number such that for every $z \in \mathbf{K}^n$

$$\big|\varphi(z)\big| \leqslant \big(|z_1|^p + \cdots + |z_n|^p\big)\|\varphi\|. \tag{4.1}$$

We encode a class $A \subset \mathcal{X}$ of Banach spaces by the cone $N(A,n) \subset H_n$ (N for norms) of functions of the form $z \mapsto \big\|\sum_{i=1}^n z_i x_i\big\|^p$ for $X \in A$ and $x_1, \ldots, x_n \in X$.

When (Ω, m) is a measure space and $f = (f_1, \ldots, f_n)$ is an n-tuple of elements of $L_p(\Omega, m)$, we can define a continuous linear form μ_f on H_n by

$$\langle \mu_f, \varphi\rangle = \int \varphi\big(f_1(\omega), \ldots, f_n(\omega)\big)\,\mathrm{d}m(\omega). \tag{4.2}$$

Indeed, it follows from (4.1) that the integral is well-defined and that $\mu_f \in H_n^*$ with norm equal to $\|f_1\|_p^p + \cdots + \|f_n\|_p^p$ (the inequality $\leqslant$ is immediate from (4.1), and the equality follows by evaluating μ_f at the norm 1 element $z \mapsto |z|^p$ in H_n).

We encode a class $B \subset \mathcal{T}$ of operators by the cone $P(B,n) \subset H_n^*$

$$P(B,n) = \left\{\mu_f - \mu_{Tf},\, T \in B \text{ and } f \in \mathrm{dom}(T)^n\right\}$$

where for $f = (f_1, \ldots, f_n) \in \mathrm{dom}(T)^n$, we denote $Tf = (Tf_1, \ldots, Tf_n)$. It is a cone because for every $t \geqslant 0$, $t(\mu_f - \mu_{Tf}) = \mu_{t^{\frac{1}{p}} f} - \mu_{Tt^{\frac{1}{p}} f}$.

The crucial but obvious property motivating these definitions is that, if $\varphi(z) = \left\|\sum_{i=1}^n z_i x_i\right\|_X^p$ for elements $x_1, \ldots, x_n$ in a Banach space X, then $\langle \mu_f, \varphi\rangle = \left\|\sum_i f_i x_i\right\|^p_{L_p(\Omega,m;X)}$. As a consequence,

$$\langle \mu_f - \mu_{Tf}, \varphi\rangle = \left\|\sum_i f_i x_i\right\|^p_{L_p(\Omega,m;X)} - \left\|\sum_i (Tf_i) x_i\right\|^p_{L_p(\Omega,m;X)}.$$

In particular, we have

LEMMA 4.1. — *Let $A \subset \mathcal{X}$ be a class of Banach spaces and $B \subset \mathcal{T}$ a class of operators.*

(1) *$B \subset A^\circ$ if and only if for every n, $P(B,n) \subset N(A,n)^\circ$.*
(2) *$A \subset {}^\circ B$ if and only if for every n, $N(A,n) \subset {}^\circ P(B,n)$.*

4.1. Polarity in H_n

We start by improving Lemma 4.1. The next result expresses that the polarity in $\langle \mathcal{X}, \mathcal{T}\rangle$ (see Definitions 1.1 and 1.2) is well encoded by the polarity $\langle H_n, H_n^*\rangle$ (see Section 3.1). Recall that REG is the class of all operators $T \in \mathcal{T}$ with regular norm $\|T\|_r \coloneqq \sup_{X \in \mathcal{X}} \|T_X\| \leqslant 1$.

PROPOSITION 4.2. — *Let $A \subset \mathcal{X}$ be a class of Banach spaces and $B \subset \mathcal{T}$ a class of operators. Then*

(1) *$P(A^\circ, n) = N(A,n)^\circ$.*
(2) *$N({}^\circ B, n) \subset {}^\circ P(B \cup \mathrm{REG}, n)$.*

In the proof, we need a description of the dual of H_n.

LEMMA 4.3. — *Every continuous linear form l on H_n is of the form $\mu_f - \mu_g$ for some measure spaces (Ω, m) and (Ω', m') and n-tuples $f \in L_p(\Omega, m)^n$ and $g \in L_p(\Omega', m')^n$. Moreover Ω, m, f and Ω', m', g can be chosen so that f and g take almost surely their values in $\left\{z \in \mathbf{K}^n,\, |z| = 1\right\}$ and so that $m(\Omega) + m'(\Omega')$ is equal to the norm of l.*

Proof. — By the identification of H_n with $C(\mathbf{KP}^{n-1})$ and by the Riesz representation theorem, every continuous linear form l on H_n is of the form

$$\varphi \mapsto \int_{\mathbf{KP}^{n-1}} \varphi\Big(\frac{z}{|z|}\Big)\,\mathrm{d}\nu(\mathbf{K}z)$$

for a unique signed measure ν on $\mathbf{KP}^{n-1}$, and the norm of l is the total variation of ν. Let $\nu = \nu_+ - \nu_-$ be the Jordan decomposition of ν and $s\colon \mathbf{KP}^{n-1} \to \big\{z \in \mathbf{K}^n,\ |z| = 1\big\}$ a measurable section. Define $(\Omega, m) = (\mathbf{KP}^{n-1}, \nu_+)$ and $f \in L_p(\Omega, m)^n$ by $s(\omega) = \big(f_1(\omega), \dots, f_n(\omega)\big)$. Similarly define $(\Omega', m') = (\mathbf{KP}^{n-1}, \nu_-)$ and $g \in L_p(\Omega', m')^n$ by

$$s(\omega) = \big(g_1(\omega), \dots, g_n(\omega)\big).$$

Then we have

$$\int_{\mathbf{KP}^{n-1}} \varphi\Big(\frac{z}{|z|}\Big)\,\mathrm{d}\nu(\mathbf{K}z) = \langle \mu_f - \mu_g, \varphi\rangle.$$

This proves the lemma, because by construction f, g both take values in $\big\{z \in \mathbf{K}^n,\ |z| = 1\big\}$ and $m(\Omega) + m'(\Omega) = (\nu_+ + \nu_-)(\mathbf{KP}^{n-1})$ is the norm of l. □

Proof of Proposition 4.2. — We start by (1). If every space in A is trivial (of dimension 0), we have $N(A,n)^\circ = H_n^*$, $A^\circ = \mathcal{T}$, and the result is easy. We can therefore assume that A contains a space of dimension $\geqslant 1$. Let f, g be n-tuples in L_p spaces. Note that if $\varphi(z) = \big\|\sum_{i=1}^n z_i x_i\big\|^p$ then

$$\langle \mu_f - \mu_g, \varphi\rangle = \Big\|\sum_i f_i x_i\Big\|_{L_p(X)}^p - \Big\|\sum_i g_i x_i\Big\|_{L_p(X)}^p.$$

So the linear form $\mu_f - \mu_g \in H_n^*$ belongs to $N(A,n)^\circ$ if and only if for every $X \in A$ and $x_1, \dots, x_n \in X$, $\big\|\sum f_i x_i\big\|_{L_p(X)}^p \geqslant \big\|\sum g_i x_i\big\|_{L_p(X)}^p$. Using that there is a nonzero $X \in A$, this holds if and only if there is a linear map T sending f_i to g_i such that $T \in A^\circ$. This shows that $\mu_f - \mu_g$ belongs to $N(A,n)^\circ$ if and only if it belongs to $P(A^\circ, n)$. By Lemma 4.3 every element of H_n^* is of this form, which proves (1).

We move to (2). Denote by C_n the closed convex cone $C_n = N(\mathcal{X}, n)$. We first prove that $N({}^\circ B, n) = {}^\circ P(B,n) \cap C_n$. By definition $N({}^\circ B, n) \subset C_n$. So we have to prove that for $\varphi \in C_n$, $\varphi \in N({}^\circ B, n)$ if and only if $\varphi \in {}^\circ P(B,n)$. But if $\varphi(z) = \big\|\sum_{i=1}^n z_i x_i\big\|^p$ and $X = \mathrm{span}(x_1, \dots, x_n)$, then we have that $\varphi \in N(B^\circ, n)$ if and only if $\|T \otimes \mathrm{id}_X\| \leqslant 1$ for all $T \in B$, if and only if for all $T \in B$ and $f_1, \dots, f_n \in \mathrm{dom}(T)$, $\big\|\sum_i T f_i x_i\big\|^p \leqslant \big\|\sum_i f_i x_i\big\|^p$, if and only if $\varphi \in P(B,n)^\circ$.

We can now conclude with (2). By (1) for $A = \mathcal{X}$, we have $C_n^\circ = P(\mathrm{REG}, n)$. On the other hand, since C_n is a closed convex cone, the bipolar

theorem implies that $C_n = {}^\circ C_n^\circ$, and hence $C_n = {}^\circ P(\mathrm{REG}, n)$. We therefore get

$$\begin{aligned} N({}^\circ B, n) &= {}^\circ P(B, n) \cap {}^\circ P(\mathrm{REG}, n) \\ &= {}^\circ\big(P(B, n) \cup P(\mathrm{REG}, n)\big) \\ &= {}^\circ P(B \cup \mathrm{REG}, n). \end{aligned}$$

This proves (2). □

By the bipolar theorem in H_n and H_n^*, we obtain

COROLLARY 4.4. — *Let $A \subset \mathcal{X}$ be a class of Banach spaces and $B \subset \mathcal{T}$ a class of operators. Then*

(1) $N({}^\circ A^\circ, n) = \overline{\mathrm{conv}}\, N(A, n)$.
(2) $P({}^\circ B^\circ, n) = \overline{\mathrm{conv}}^{w*}\, P(B \cup \mathrm{REG}, n)$.

The rest of this section consists in understanding the closed convex hulls of $N(A, n)$ and $P(B, n)$.

4.2. Understanding the encoding of Banach spaces in H_n

The following easy fact will be important later.

LEMMA 4.5. — *For every integer n, bounded subsets of $N(\mathcal{X}, n)$ are relatively norm-compact.*

Proof. — By the Arzelà–Ascoli theorem, we have to prove that bounded subsets of $N(\mathcal{X}, n)$ are equicontinuous, seen in $C(\mathbf{KP}^{n-1})$. This follows from the triangle inequality. For example for $p = 1$ and $\varphi(z) = \big\|\sum_i z_i x_i\big\|$, then we have

$$\big|\varphi(z) - \varphi(z')\big| \leqslant \Big\|\sum_i (z_i - z_i')x_i\Big\| \leqslant \sum_i |z_i - z_i'|\varphi(e_i).$$

The case of arbitrary p is similar. Alternatively, it follows from the case $p = 1$ by continuity of the map $t \mapsto t^p$. □

Lemma 4.5 allows to considerably strengthen the second statement in Corollary 4.4, replacing weak-* closure by norm closure.

COROLLARY 4.6. — *Let $B \subset \mathcal{T}$ a class of operators. Then*

$$P({}^\circ B^\circ, n) = \overline{\mathrm{conv}}^{\|\cdot\|}\, P(B \cup \mathrm{REG}, n).$$

Proof. — The set $N(\mathcal{X},n)$ is a closed convex cone in H_n, so by Lemma 4.5 $N(\mathcal{X},n)\cap\{\varphi\in H_n \mid \|\varphi\|\leqslant r\}$ is norm-compact for every r. Moreover, we have that $N(\mathcal{X},n)^\circ = P(\mathrm{REG},n)$ by Proposition 4.2. So, since $P(B\cup\mathrm{REG},n)$ contains $N(\mathcal{X},n)^\circ$, Proposition 3.3 implies that its bipolar is equal to the norm closure of its convex hull. □

Let us list elementary properties of N.

LEMMA 4.7. — *Let $A,A_1,A_2\subset\mathcal{X}$ be classes of Banach spaces.*

(1) *$N(A_1,n)\subset N(A_2,n)$ if and only if, for every $X\in A_1$, every subspace of dimension $\leqslant n$ of X is isometric to a subspace of a space in A_2.*
(2) *The convex hull of $N(A,n)$ is equal to $N(\oplus_{\ell_p}A,n)$, where $\oplus_{\ell_p}A$ denotes the set of all finite ℓ_p-direct sums of Banach spaces in A.*
(3) *The norm closure of $N(A,n)$ in H_n coincides with $N(\overline{A},n)$ where $\overline{A}$ denotes the set of Banach spaces finitely represented in A.*

As a consequence of (1) and (3), if two classes of Banach spaces A_1,A_2 are closed under finite representability, then $A_1=A_2$ if and only if $N(A_1,n)=N(A_2,n)$ for all n.

Proof. — The first point is obvious from the following observation: if $x_1,\dots,x_n$ (respectively $y_1,\dots,y_n$) are elements in a Banach space X (respectively in a Banach space Y), then the functions $z\mapsto\left\|\sum_{i=1}^n z_ix_i\right\|^p$ and $z\mapsto\left\|\sum_{i=1}^n z_iy_i\right\|^p$ coincide if and only if there is an isometry from the linear span of $\{x_1,\dots,x_n\}$ to the linear span of $\{y_1,\dots,y_n\}$ sending x_i to y_i.

If $\varphi_1,\dots,\varphi_k\in N(A,n)$ are given by $\varphi_j(z)=\left\|\sum_{i=1}^n z_ix_i^{(j)}\right\|_{X_j}^p$ then by the definition of the ℓ_p-direct sum $X_1\oplus_p\cdots\oplus_p X_k$ we can write

$$\sum_{j=1}^k\varphi_j(z)=\left\|\sum_{i=1}^n z_i(x_i^{(j)})_{1\leqslant j\leqslant k}\right\|_{X_1\oplus_p\cdots\oplus_pX_k}^p.$$

This shows that $N(\oplus_{\ell_p}A,n)$ coincides with

$$\left\{\varphi_1+\cdots+\varphi_k,\ k\in\mathbf{N},\ \varphi_j\in N(A,n)\right\}.$$

This is the convex hull of $N(A,n)$ because $N(A,n)$ is a cone.

We move to (3). If a sequence $\varphi_k\in N(A,n)$ converges uniformly on compact subsets to $\varphi\in H_n$, then $\varphi^{\frac{1}{p}}$ is the uniform limit on compact sets of the seminorms $\varphi_k^{\frac{1}{p}}$, so it is a seminorm on $\mathbf{K}^n$. This means that there is a Banach space $X\in\mathcal{X}$ and $x_1,\dots,x_n$ spanning X such that $\varphi(z)=\left\|\sum_{i=1}^n z_ix_i\right\|^p$. The family $x_1,\dots,x_n$ might not be linearly independent, so

we extract from it a basis of X. Without loss of generality we can assume that this basis is $x_1, \dots, x_m$ for some $m \leqslant n$. Write

$$\varphi_k(z) = \left\| \sum_{i=1}^{n} z_i x_i^{(k)} \right\|_{X_k}^{p}$$

for some $X_k \in A$ and $x_1^{(k)}, \dots, x_n^{(k)} \in X_k$. From the assumption that φ_k converges uniformly on compacta to φ and the assumption that $x_1, \dots, x_m$ is linearly independent, we get that for every $\varepsilon > 0$ there is k such that

$$(1-\varepsilon)\varphi(z,0) \leqslant \varphi_k(z,0) \leqslant (1+\varepsilon)\varphi(z,0)$$

for all $z \in \mathbf{K}^m$. This means that the linear map $u\colon X \to X_k$ sending x_i to $(1-\varepsilon)^{-\frac{1}{p}} x_i^{(k)}$ for $i \leqslant m$ satisfies

$$\|x\| \leqslant \big\|u(x)\big\| \leqslant \Big(\frac{1+\varepsilon}{1-\varepsilon}\Big)^{\frac{1}{p}} \|x\| \quad \text{for all } x \in X.$$

Since $\varepsilon > 0$ was arbitrary we have proved that X is finitely representable in A, i.e. that $\varphi \in N(\overline{A}, n)$. This proves that $\overline{N(A,n)} \subset N(\overline{A}, n)$.

The converse inclusion is proved by reading the preceding argument backwards. □

We can conclude our proof of Hernandez' theorem.

Proof of Theorem 1.3. — Let A' be the class of Banach spaces which are finitely representable in the class of ℓ_p-direct sums of spaces in A. It follows from Corollary 4.4 and Lemma 4.7 that for every integer n,

$$N({}^{\circ}A^{\circ}, n) = N(A', n).$$

Since both ${}^{\circ}A^{\circ}$ and A' are closed under finite representability, we get the equality ${}^{\circ}A^{\circ} = A'$ by the remark following Lemma 4.7. □

4.3. Proof of Theorem 1.4

If X is at Banach–Mazur $\leqslant C$ from ${}^{\circ}A^{\circ}$, then the inequality $\|T_X\| \leqslant C$ for every $T \in A^{\circ}$ is clear.

For the converse, we will need the following consequence of the Hahn–Banach theorem.

LEMMA 4.8. — *Let K be a compact Hausdorff topological space, and $C(K)$ the space of real-valued continuous functions on K. Let A be a closed convex cone in the positive cone of $C(K)$ such that $A \cap B(0,1)$ is compact. Let $s \geqslant 1$. Then for every $\psi \in C(K)$, the following are equivalent*

- $\exists\, \varphi \in A,\ \psi \leqslant \varphi \leqslant s\psi$.

- $\langle s\mu-\nu,\psi\rangle \geqslant 0$ *for every positive measures* μ,ν *on* K *such that* $\langle \mu-\nu,\varphi\rangle \geqslant 0$ *for all* $\varphi\in A$.

Proof. — $\Rightarrow$ is easy because the inequality $\psi\leqslant\varphi\leqslant s\psi$ implies $\langle s\mu-\nu,\psi\rangle \geqslant \langle\mu-\nu,\varphi\rangle$.

For the converse, since A is a convex cone, the set B of ψ satisfying $\exists\,\varphi\in A$, $\psi\leqslant\varphi\leqslant s\psi$ is a convex cone. Moreover, the compactness assumption on A implies that B is also closed. Assume that $\psi\notin B$. By Hahn–Banach there is a linear form on $C(K)$ which is nonnegative on B and negative at ψ. By the Riesz representation theorem and the Hahn decomposition, this linear form can be written as $f\mapsto\int f\,\mathrm{d}(\mu-\nu)$ for positive measures μ,ν such that there is a Baire measurable subset $E\subset K$ satisfying $\nu(E)=0$ and $\mu(K\setminus E)=0$.

Let $\varphi\in A$. Let $f_n\colon K\to[0,1]$ be a sequence of continuous functions converging in $L^1(K,\mu+\nu)$ to the indicator function of E. Then for every n, the function $\big(\frac{1}{s}+\big(1-\frac{1}{s}\big)f_n\big)\varphi$ belongs to B so $\big\langle\mu-\nu,\big(\frac{1}{s}+\big(1-\frac{1}{s}\big)f_n\big)\varphi\big\rangle>0$. By making $n\to\infty$ we get $\big\langle\mu-\nu,\frac{1}{s}\varphi 1_{K\setminus E}+\varphi 1_E\big\rangle\geqslant 0$, which can be written as $\big\langle\frac{1}{s}\mu-\nu,\varphi\big\rangle\geqslant 0$.

So we have $\big\langle\frac{1}{s}\mu-\nu,\varphi\big\rangle\geqslant 0$ for every $\varphi\in A$, whereas $\langle\mu-\nu,\psi\rangle<0$. This proves the lemma. □

We can now prove the converse implication in Theorem 1.4. Assume that $\|T_X\|\leqslant C$ for every $T\in A^\circ$. Let $x_1,\dots,x_n\in X$. Define $\psi\in H_n$ by $\psi(z)=\big\|\sum_{i=1}^n z_ix_i\big\|^p$, and view ψ in $C(\mathbf{KP}^{n-1})$. The assumption that $\|T_X\|\leqslant C$ for every $T\in A^\circ$ implies that $\langle C^p\mu-\nu,\psi\rangle\geqslant 0$ for every positive measures μ,ν on $\mathbf{KP}^{n-1}$ such that $\langle\mu-\nu,\varphi\rangle\geqslant 0$ for all $\varphi\in A$. By Lemma 4.8 (remember Lemma 4.5) this implies that there is φ in the closed convex hull of $N(A,n)$ such that $\psi\leqslant\varphi\leqslant C^p\psi$. By the proof of Theorem 1.3, there is a space $Y\in{}^\circ A^\circ$ and $y_1,\dots,y_n\in Y$ such that $\varphi(z)=\big\|\sum_i z_iy_i\big\|^p$. By taking the $1/p$-th power in the inequality $\psi\leqslant\varphi\leqslant C^p\psi$ we get that $\big\|\sum z_ix_i\big\|\leqslant\big\|\sum z_iy_i\big\|\leqslant C\big\|\sum z_ix_i\big\|$ for every $y\in\mathbf{K}^n$. This means that the linear span of $x_1,\dots,x_n$ is at Banach–Mazur distance $\leqslant C$ from the linear span on $\{y_1,\dots,y_n\}$ and concludes the proof.

4.4. Understanding the encoding of operators in H_n^*

The main result of this subsection is Proposition 4.9, which describes how operators are encoded in H_n^*. If $B\subset\mathcal{T}$, we define new (larger) classes as follows:

- $\Lambda_1(B)$ is the set of operators

$$(T, \mathrm{id})\colon \mathrm{dom}(T) \oplus_p L_p(\Omega,\mu) \to \mathrm{ran}(T) \oplus L_p(\Omega,\mu)$$

  for $T \in B$ and a measure space (Ω,μ);
- $\Lambda_2(B) = \big\{U \circ T \circ V \mid U, V \text{ spatial isometries}, T \in B\big\}$;
- $\Lambda_3(B)$ is the set of all $S\colon \mathrm{dom}(S) \subset L_p(\Omega_1, m_1) \to L_p(\Omega_2, m_2)$ such that there is $T \in B$ where $\mathrm{dom}(T) \subset L_p(\Omega_1, m_1) \oplus L_p(\Omega, m)$, $\mathrm{ran}(T) \subset L_p(\Omega_2, m_2) \oplus L_p(\Omega, m)$, $\mathrm{dom}(S)$ is the image of $\mathrm{dom}(T)$ by the first coordinate projection and $T(f \oplus g) = Sf \oplus g$ for every $f \oplus g \in \mathrm{dom}(T)$;
- $\Lambda_4(B)$ is the set of all $S\colon \mathrm{dom}(S) \subset L_p(\Omega, m) \to L_p(\Omega', m')$ such that for every finite family $f_1, \dots, f_n$ in the domain of T and every $\varepsilon > 0$, there is $T \in B$ with domain contained in $L_p(\Omega, m)$ and range contained in $L_p(\Omega', m')$ and elements $g_1, \dots, g_n \in D(S)$ such that $\|f_i - g_i\| \leqslant \varepsilon$ and $\|Tf_i - Sg_i\| \leqslant \varepsilon$.

To save place, we denote $\Lambda_{123}(B) = \Lambda_3\big(\Lambda_2\big(\Lambda_1(B)\big)\big)$.

Proposition 4.9. — *For every $T \in \mathcal{T}$ and $B \subset \mathcal{T}$, the following are equivalent:*

- *for every n, $P(T,n) \subset P(B,n)$;*
- *the restriction of T to every finite dimensional subspace of* $\mathrm{dom}(T)$ *belongs to $\Lambda_{123}(B)$.*

The start with a couple of lemmas that will be used in the proof.

Lemma 4.10. — *Let $f, g, \widetilde{f}, \widetilde{g}$ be n-tuples of elements of L_p spaces. Then $\mu_f - \mu_g = \mu_{\tilde f} - \mu_{\tilde g}$ if and only if there is $h \in L_p(\Omega, m)^n, \widetilde{h} \in L_p(\widetilde{\Omega}, \widetilde{m})^n$ such that $\mu_{(f_i \oplus h_i)_{i=1}^n} = \mu_{(\tilde f_i \oplus \tilde h_i)_{i=1}^n}$ and $\mu_{(g_i \oplus h_i)_{i=1}^n} = \mu_{(\tilde g_i \oplus \tilde h_i)_{i=1}^n}$.*

Proof. — The if direction is easy, because $\mu_{(f_i \oplus h_i)_{i=1}^n} = \mu_f + \mu_h$.

For the converse, assume that $\mu_f - \mu_g = \mu_{\tilde f} - \mu_{\tilde g}$. Let ν_f be the positive measure on $\mathbf{KP}^{n-1}$ such that, for every $\varphi \in H_n$

$$\langle \mu_f, \varphi \rangle = \int_{\mathbf{KP}^{n-1}} \varphi\Big(\frac{z}{|z|}\Big)\, \mathrm{d}\nu_f(\mathbf{K}z). \tag{4.3}$$

Define similarly $\nu_g, \nu_{\tilde f}, \nu_{\tilde g}$. Then $\nu_f - \nu_g = \nu_{\tilde f} - \nu_{\tilde g}$ is a signed measure on $\mathbf{KP}^{n-1}$. Let $\nu_+ - \nu_-$ be its Jordan decomposition. By the properties of the Jordan decomposition, $\nu_f - \nu_+ = \nu_g - \nu_-$ is a positive measure on $\mathbf{KP}^{n-1}$, and therefore by the proof of Lemma 4.3 it is of the form to $\nu_{\tilde h}$ for some n-tuple $\widetilde{h} \in L_p(\widetilde{\Omega}, \widetilde{m})$. Similarly, there is a $h \in L_p(\Omega, m)^n$ such that $\nu_{\tilde f} - \nu_+ = \nu_{\tilde g} - \nu_- = \nu_h$.

We can rewrite these equalities as

$$\nu_+ = \nu_f - \nu_{\tilde{h}} = \nu_{\tilde{f}} - \nu_h$$

and

$$\nu_- = \nu_f - \nu_{\tilde{h}} = \nu_{\tilde{f}} - \nu_h.$$

This implies that $\mu_f + \mu_h = \mu_{\tilde{f}} + \mu_{\tilde{h}}$ and $\mu_g + \mu_h = \mu_{\tilde{g}} + \mu_{\tilde{h}}$ and proves the lemma. □

LEMMA 4.11. — *For two families $f \in L_p(\Omega, m)^n$ and $g \in L_p(\Omega', m')^n$, $\mu_f = \mu_g$ if and only if there is a spatial isometry* $\operatorname{span}\{f_1, \dots, f_n\} \to \operatorname{span}\{g_1, \dots, g_n\}$ *sending f_i to g_i.*

Proof. — The if direction is easy: firstly if there is a measurable function $h\colon \Omega \to \mathbf{K} \setminus \{0\}$, if $(\Omega', m') = \big(\Omega, |h|^{-p} m\big)$ and $g_i = hf_i$ for all i, then for every $\varphi \in H_n$, $\varphi(g_1, \dots, g_n) = |h|^p \varphi(f_1, \dots, f_n)$ and therefore $\langle \mu_g, \varphi \rangle = \langle \mu_f, \varphi \rangle$. Secondly if $f_1, \dots, f_n$ and $g_1, \dots, g_n$ are equidistributed outside of 0 in the sense of Definition 1.5, then $\int \varphi(f_1, \dots, f_n)\, \mathrm{d}m = \int \varphi(g_1, \dots, g_n)\, \mathrm{d}m'$ for every Borel function φ vanishing at 0 and such that the integrals are defined. In particular $\mu_f = \mu_g$.

For the converse, assume that $\mu_f = \mu_g$. Take a measurable section $s\colon \mathbf{KP}^{n-1} \to \mathbf{K}^n$ with values in $\big\{z \in \mathbf{K}^n,\ |z| = 1\big\}$. Then there are measurable nonvanishing functions $h\colon \Omega \to \mathbf{K}^*$ and $h'\colon \Omega' \to \mathbf{K}^*$ such that $f(\omega) = h(\omega) s\big(\mathbf{K} f(\omega)\big)$ for every $\omega \in \Omega$ such that $f(\omega) \neq 0$, and similarly $g(\omega') = h'(\omega') s\big(\mathbf{K} g(\omega')\big)$ if $g(\omega') \neq 0$. By replacing m by $|h|^{-1/p} m$ and f_i by $h_i f_i$ and similarly for g we can assume that $h = 1$ and $h' = 1$, and we shall prove that f and g are equidistributed outside of 0. By this we mean that for every Borel $E \subset \mathbf{K}^n \setminus \{0\}$, $m\big(\big\{\omega, \big(f_1(\omega), \dots, f_n(\omega)\big) \in E\big\}\big) = m'\big(\big\{\omega', \big(g_1(\omega'), \dots, g_n(\omega')\big) \in E\big\}\big)$. This clearly implies that, for every matrix $A \in M_{m,n}(\mathbf{K})$, Af and Ag are equidistributed outside of 0. Hence the linear map sending f_i to g_i is well-defined, and an equidistribution outside 0 (see Definition 1.5).

By the identification of H_n with $C(\mathbf{KP}^{n-1})$, using that $|f| \in \{0, 1\}$ we have

$$\int_{\Omega \setminus f^{-1}(0)} \psi(\mathbf{K} f)\, \mathrm{d}m = \int_{\Omega' \setminus g^{-1}(0)} \psi(\mathbf{K} g)\, \mathrm{d}m'$$

for every continuous function $\psi\colon \mathbf{KP}^{n-1} \to \mathbf{K}$, and therefore also for every bounded Borel function $\varphi\colon \mathbf{KP}^{n-1} \to \mathbf{K}$. This implies, since f and g take values in $\{0\} \cup s(\mathbf{KP}^{n-1})$, that $\int \varphi(f)\, \mathrm{d}m = \int \varphi(g)\, \mathrm{d}m'$ for every Borel function $\mathbf{K}^n \to \mathbf{K}$ vanishing at 0. Equivalently, f and g are equidistributed outside of 0. □

Remark 4.12. — The same proof shows actually a bit more: if a linear map $T\colon E \subset L_p(\Omega, m) \to L_p(\Omega', m')$ satisfies $\mu_f = \mu_{Tf}$ for every n and every $f \in E^n$, then T is a spatial isometry. Indeed, since by our standing assumption E (as every other space considered in this paper) is separable, we can find a sequence $(f_i)_{i\geqslant 0}$ generating a dense subspace of E and satisfying $\sum_i \|f_i\|^p < \infty$, and in particular $\big(f_i(\omega)\big)_{i\geqslant 0}$ belongs to ℓ_p for almost every ω. Then the same proof applies, except that we replace $\mathbf{K}^n$ by ℓ_p and $\mathbf{KP}^{n-1}$ by its projectivization $\ell_p/\mathbf{K}^*$.

We shall also need the following variant:

LEMMA 4.13. — *For two families $f \in L_p(\Omega, m)^n$ and $g \in L_p(\Omega', m')^n$ and $\varepsilon > 0$, $\|\mu_f - \mu_g\| < \varepsilon$ if and only if there are spatial isometries*

$$U\colon \operatorname{span}\{f_1, \dots, f_n\} \to L_p(\Omega'', m'')$$
$$V\colon \operatorname{span}\{g_1, \dots, g_n\} \to L_p(\Omega'', m'')$$

such that

$$\int_{\Omega''} \big(|Uf|^p + |Vg|^p\big)\chi_{Uf\neq Vg} < \varepsilon.$$

Proof. — We prove the slightly stronger statement with $< \varepsilon$ replaced by $\leqslant \varepsilon$.

The if direction is easy: by Lemma 4.11 we have $\mu_f = \mu_{Uf}$ and $\mu_g = \mu_{Vg}$, and therefore for every $\varphi \in H_n$,

$$\begin{aligned}\langle \mu_f - \mu_g, \varphi\rangle &= \int_{\Omega''} \varphi(Uf) - \varphi(Vg)\\ &\leqslant \int_{\Omega''} \Big(\big|\varphi(Uf)\big| + \big|\varphi(Vg)\big|\Big)\chi_{f\neq Ug}\\ &\leqslant \int_{\Omega''} \big(|Uf|^p + |Vg|^p\big)\|\varphi\|\\ &\leqslant \varepsilon\|\varphi\|.\end{aligned}$$

Taking the supremum over φ we get $\|\mu_f - \mu_g\| \leqslant \varepsilon$.

The converse follows from a coupling argument. Assume that $\|\mu_f - \mu_g\| \leqslant \varepsilon$. Let ν_f and ν_g be the measures on $\mathbf{KP}^{n-1}$ given by (4.3), so that the total variation norm of $\nu_f - \nu_g$ is at most ε. This means that we can decompose $\nu_f = \nu_0 + \nu_1$ and $\nu_g = \nu_0 + \nu_2$ for positive measures with $(\nu_1 + \nu_2)(\mathbf{KP}^{n-1}) \leqslant \varepsilon$. As in the proof of Lemma 4.3, each ν_k corresponds by (4.3) to μ_{h^k} for an n-tuple $h^k \in L_p(\Omega_k, m_k)^n$ with $\sum_i \|h_i^k\|_p^p = \nu_k(\mathbf{KP}^{n-1})$. In particular, we have $\mu_f = \mu_{h^0} + \mu_{h^1}$ and $\mu_g = \mu_{h^0} + \mu_{h^2}$.

Let us define Ω'' as the disjoint union $\Omega_0 \cup \Omega_1 \cup \Omega_2$, m'' as $m_0 + m_1 + m_2$, and $f' = h^0 \oplus h^1 \oplus 0$ and $g' = h^0 \oplus 0 \oplus h^1$, so that $\mu_{f'} = \mu_{h^0} + \mu_{h^1} = \mu_f$ and

$\mu_{g'} = \mu_g$. By Lemma 4.11, there are spatial isometries U and V sending f to f' and g to g' respectively, and we have

$$\int_{\Omega''} \big(|f'|^p + |g'|^p\big)\chi_{f'\neq g'} = \int_{\Omega_1} |h^1|^p + \int_{\Omega_2} |h^2|^p \leqslant \varepsilon.$$

This proves the lemma. □

We will also need the following asymmetric variant of the previous lemma.

Remark 4.14. — In Lemma 4.13, we can moreover assume that (Ω'', m'') is $\big(\Omega \times [0,1], m \otimes \mathrm{d}\lambda\big)$ (for λ the Lebesgue measure), and that the spatial isometry U is simply $U\xi(\omega, s) = \xi(\omega)$.

Proof. — Let μ_f, μ_g, $\nu_f = \nu_0 + \nu_1$, $\nu_g = \nu_0 + \nu_2$ be as in the proof of Lemma 4.13, where $\|\nu_1 + \nu_2\| < \varepsilon$. We can even assume that $\nu_1 \neq 0$ (this is where the strict inequality $< \varepsilon$ is used). Denote by $\frac{\mathrm{d}\nu_0}{\mathrm{d}\nu_f} : \mathbf{KP}^{n-1} \to [0,1]$ the Radon–Nikodym derivative. Define $A \subset \Omega \times [0,1] = \Omega''$ by $A = \big\{(\omega, s) \mid s \leqslant 0 \leqslant s \leqslant \frac{\mathrm{d}\nu_0}{\mathrm{d}\nu_f}\big(\mathbf{K}^* f(x)\big)\big\}$, so that $\mu_{f\chi_A} = \nu_0$ and $\mu_{f\chi_{\Omega''\setminus A}} = \nu_1$. In particular, $\Omega'' \setminus A$ has positive measure and is therefore an atomless standard measure space, and we can find $h \in L_p(\Omega'', m'')^n$ that vanishes on A such that μ_h corresponds to ν_2. We then have $\mu_g = \mu_h + \mu_{f\chi_A} = \mu_{h+f\chi_A}$. The last equality is because h and $f\chi_A$ are disjointly supported. By Lemma 4.11, there is a spatial isometry V sending g to $h + f\chi_A$. Moreover, we have

$$\int_{\Omega''} \big(|f|^p + |h + f\chi_A|^p\big)\chi_{f\neq h+f\chi_A} \leqslant \int_{\Omega''\setminus A} \big(|f|^p + |h|^p\big) < \varepsilon. \qquad \square$$

We can now prove Proposition 4.9.

Proof of Proposition 4.9. — Assume that, for a fixed n, $P(T,n) \subset P(B,n)$. This means that, for every $f \in \mathrm{dom}(T)^n$, there exist $S \in B$ and $g \in \mathrm{dom}(S)^n$ such that $\mu_f - \mu_{Tf} = \mu_g - \mu_{S_g}$. By Lemmas 4.10 and 4.11, there are $h \in L_p(\Omega, m)^n$ and $\overline{h} \in L_p(\Omega', m')^n$ and spatial isometries $U\colon \mathrm{span}\big\{Sg_i \oplus \overline{h}_i\big\} \to \mathrm{span}\big\{Tf_i \oplus h_i\big\}$ sending $Sg_i \oplus \overline{h}_i$ to $Tf_i \oplus h_i$ and $V\colon \mathrm{span}\big\{f_i \oplus h_i \to g_i \oplus \overline{h}_i\big\}$ sending $f_i \oplus h_i$ to $g_i \oplus \overline{h}_i$. The operator $S_1 = (S, \mathrm{id})$ on $\mathrm{dom}(T) \oplus L_p(\Omega', m')$ belongs to $\Lambda_1(B)$, so the operator $S_2 = U \circ S_1 \circ V$, which sends $f_i \oplus h_i$ to $Tf_i \oplus h_i$ belongs to $\Lambda_2\big(\Lambda_1(B)\big)$, and therefore the restriction of T to $\mathrm{span}\{f_1, \dots, f_n\}$ belongs to $\Lambda_3\big(\Lambda_2\big(\Lambda_1(B)\big)\big)$. This proves one direction.

The converse is simpler: it follows from the easy directions in Lemmas 4.10 and 4.11 that $P\big(\Lambda_i(B), n\big) = P(B,n)$ for $i = 1, 2, 3$. In particular, if the restriction of T to every $\leqslant n$-dimensional subspace of $\mathrm{dom}(T)$ belongs to $\Lambda_{123}(B)$, then $P(T,n) \subset P(B,n)$. □

4.5. Convergences in H_n^*

This section is devoted to the understanding of the encoding of both weak-* sequential convergence and norm convergence in H_n^*. Our first result asserts that weak-* convergence of sequences corresponds to the operation Λ_4 we just defined.

PROPOSITION 4.15. — *Let $B \subset \mathcal{T}$. The smallest class containing B and stable by all operations $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ coincides with the set of $T \in \mathcal{T}$ such that for every n, $P(T,n)$ is contained in the sequential weak-* closure of $P(B,n)$.*

Proof. — We define by transfinite induction, for every ordinal α, a class B_α as follows. B_0 is $\Lambda_{123}(B)$. Set $B_\alpha = \Lambda_{123}\big(\Lambda_4(B_{\alpha-1})\big)$ if α is a successor ordinal, and $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$ if α is a limit ordinal.

Similarly, we define, for every integer n and every ordinal α, a subset $C_\alpha^n \subset H_n^*$ by $C_0^n = P(B,n)$, for a successor ordinal C_α^n is the set of all limits of weak-* converging sequences of elements of $C_{\alpha-1}^n$. If α is a limit ordinal we set $C_\alpha^n = \bigcup_{\beta<\alpha} C_\beta^n$.

We claim that, for every $T \in \mathcal{T}$ with finite-dimensional domain, $P(T,n)$ is contained in C_α^n for every n if and only if T belongs to B_α. We prove it by transfinite induction. If $\alpha = 0$, this is Proposition 4.9. Let $\alpha > 0$ and assume that the claim holds for all $\beta < \alpha$. If α is a limit ordinal, the claim is clear.

So assume that α is a successor. Assume first that $P(T,n) \subset C_\alpha^n$ for every n. Let n be the dimension of $\mathrm{dom}(T)$ and $f = (f_1, \dots, f_n)$ a basis. Then $\mu_f - \mu_{Tf}$ is a limit of a weak-* converging sequence ν_k of elements of $C_{\alpha-1}^n$. By Lemma 4.3 there are $f^{(k)} \in L_p(\Omega_k, m_k)^n$ and $g^{(k)} \in L_p(\Omega_k', m_k')^n$ with values in $\big\{z \in \mathbf{K}^n, |z| = 1\big\}$ such that $\nu_k = \mu_{f^{(k)}} - \mu_{g^{(k)}}$ and $m_k(\Omega_k) + m_k'(\Omega_k')$ is the norm of the corresponding linear form, which is bounded by Banach–Steinhaus. For simplicity of the exposition assume that $m_k(\Omega_k) + m_k'(\Omega_k') \leqslant 1$. By the induction hypothesis, there is an operator $S_k \in B_{\alpha-1}$ such that $f^{(k)} \in D(S_k)^n$ and $S_k f^{(k)} = g^{(k)}$. We have two sequences of probability measures, $f_*^{(k)} m_k + \big(1 - m_k(\Omega_k)\big)\delta_0$ and $g_*^{(k)} m_k' + \big(1 - m_k'(\Omega_k')\big)\delta_0$, on $\{0\} \cup \big\{z \in \mathbf{K},\ |z| = 1\big\} \subset \mathbf{K}^n$. By compactness, up to an extraction we can assume that both sequences converge weak-*, and by Skorohod's representation theorem we can assume that (Ω_k, m_k) does not depend on k and that $f^{(k)}$ converges almost surely to some $f^{(\infty)} \in L_p^n$ and similarly $g^{(k)}$ converges almost surely, and in particular in L_p, to $g^{(\infty)}$ (this modifies the operators S_k, but they still satisfy $\nu_k = \mu_{f^{(k)}} - \mu_{Sf^{(k)}}$ and therefore still belong to $B_{\alpha-1}$). In particular, the operator $S^{(\infty)}$ from

$\mathrm{dom}(S^{(\infty)}) = \mathrm{span}\{f_1^{(\infty)}, \dots, f_n^{(\infty)}\} \to \mathrm{span}\{g_1^{(\infty)}, \dots, g_n^{(\infty)}\}$ sending $f_i^{(\infty)}$ to $g_i^{(\infty)}$ belongs to $\Lambda_4(B_{\alpha-1})$ and it satisfies

$$\mu_{f^{(\infty)}} - \mu_{S^{(\infty)} f^{(\infty)}} = \lim_k \mu_{f^{(k)}} - \mu_{S^{(k)} f^{(k)}} = \mu_f - \mu_{Tf}.$$

By Proposition 4.9 again, the restriction of T to $\mathrm{span}\{f_1, \dots, f_n\} = E$ belongs to $\Lambda_{123}(S^{(\infty)}) \subset B_\alpha$. This concludes the proof that $P(T,n) \subset C_\alpha^n$ for all n implies that the restriction of T belongs to B_α. The converse is similar but easier and left to the reader. □

Similarly, norm convergence is well encoded.

Lemma 4.16. — *Let $B \subset \mathcal{T}$, $T\colon \mathrm{dom}(T) \subset L_p(\Omega_1, m_1) \to L_p(\Omega_2, m_2)$ a linear map with domain of finite dimension n, and $(f_1, \dots, f_n)$ be a basis of $\mathrm{dom}(T)$. Then $P(T,n)$ is contained in the norm-closure of $P(B,n)$ if and only if for every $\varepsilon > 0$, there is $S \in \Lambda_{123}(B)$ with $\mathrm{dom}(S) \subset L_p\big(\Omega_1 \times [0,1], m_1 \otimes \mathrm{d}\lambda\big)$ and $\mathrm{ran}(S) \subset L_p\big(\Omega_2 \times [0,1], m_2 \otimes \mathrm{d}\lambda\big)$, there are $g_1, \dots, g_n \in \mathrm{dom}(S)$ such that*

$$\int_{\Omega_1 \times [0,1]} \Big(\big|f(\omega)\big|^p + \big|g(\omega,s)\big|^p \Big) \chi_{f(\omega) \neq g(\omega,s)} \, \mathrm{d}m_1(\omega) \, \mathrm{d}s \leqslant \varepsilon$$

and

$$\int_{\Omega_2 \times [0,1]} \Big(\big|Tf(\omega)\big|^p + \big|Sg(\omega,s)\big|^p \Big)^{\frac{1}{p}} \chi_{Tf(\omega) \neq Sg(\omega,s)} \, \mathrm{d}m_2(\omega) \, \mathrm{d}s \leqslant \varepsilon.$$

Proof. — Assume that $P(T,n)$ is contained in the closure of $P(B,n)$. This means that for every $\varepsilon > 0$, there is $\mu' \in P(B,n)$ such that $\|\mu_f - \mu_{Tf} - \mu'\| < \varepsilon$. By Lemma 3.4, we can write $\mu' = \mu_g - \mu_h$ for n-tuples of elements of L^p spaces g, h where $\|\mu_f - \mu_g\| + \|\mu_{Tf} - \mu_h\| < \varepsilon$. Since we have some room ($< \varepsilon$), we can even assume that $\{g_1, \dots, g_n\}$ are linearly independent, so that we can define a linear map S sending g_i to h_i. By Proposition 4.9, S belongs to $\Lambda_{123}(B)$, and so does S composed with any spatial isometry. So the only if direction follows from Lemma 4.13 and its improvement in Remark 4.14.

The converse is proved the same way. □

4.6. Proof of the main theorem

We are also ready to prove our main theorem (Theorem 1.6). Before we do so, we only need to understand the operation of taking convex hulls.

LEMMA 4.17. — *Let $B \subset \mathcal{T}$ and $n \in \mathbf{N}$. The convex hull of $P(B,n)$ is equal to $P\big(\oplus_{\ell_p}(B), n\big)$ where $\oplus_{\ell_p}(B)$ is the class of all finite ℓ_p-direct sums of operators in B.*

Proof. — This is clear: if $T_1, \dots, T_k \in B$ and $f^{(j)} \in D(T_j)^n$ for all j, then

$$\sum_j \mu_{f^{(j)}} - \mu_{Tf^{(j)}} = \mu_f - \mu_{(T_1 \oplus \cdots \oplus T_k)f}$$

where $f_i = f_i^{(1)} \oplus \cdots \oplus f_i^{(k)} \in D(T_1) \oplus \dots D(T_k)$ and $f = (f_1, \dots, f_n) \in \big(D(T_1) \oplus \cdots \oplus D(T_k)\big)^n$. □

We can conclude.

Proof of Theorem 1.6. — We start with the easy direction. Assume that for every n and ε, the assumption in the second bullet point holds. Let X be a Banach space such that $\sup_{S \in B} \|S_X\| \leqslant 1$. We have to prove that $\|T_X\| \leqslant 1$. That is, for every integer n and every $x_1, \dots, x_n$,

$$\Big\|\sum_i (Tf_i)x_i\Big\|_{L_p(\Omega_2;X)} \leqslant \Big\|\sum_i f_i x_i\Big\|_{L_p(\Omega_1,X)}. \tag{4.4}$$

Let $\varepsilon > 0$, and $S = S_0 \oplus S_1 \oplus \cdots \oplus S_k$, U, V, g_i, g_i', h_i given by the assumption. In the following computation, we view Tf_i as an element of $L_p\big(\Omega_2 \times [0,1]\big)$ that does not depend on the second variable in $[0,1]$, and similarly for f_i. We denote simply by $\|\cdot\|_p$ the norm in $L_p\big(\Omega_i \times [0,1]; X\big)$ or $L_p\big(\Omega_i \times [0,1]\big)$. We can bound

$$\begin{aligned}
\Big\|\sum_i (Tf_i)x_i\Big\|_{L_p(\Omega_2;X)} &\leqslant \sum_i \|Tf_i - g_i'\|_p \|x_i\| + \Big\|\sum_i g_i' x_i\Big\|_p \\
&\leqslant \varepsilon \sum_i \|x_i\| + \Big\|\sum_i g_i' x_i\Big\|_p \\
&= \varepsilon \sum_i \|x_i\| + \left(\Big\|\sum_i (g_i', h_i)x_i\Big\|_p^p - \Big\|\sum_i h_i x_i\Big\|_p^p\right)^{\frac{1}{p}}.
\end{aligned}$$

The quantity inside the parenthesis is equal to

$$\Big\|\sum_i S(g_i, h_i)x_i\Big\|_p^p - \Big\|\sum_i h_i x_i\Big\|_p^p,$$

so using that $\|S_X\| = \max_{0 \leqslant i \leqslant k} \big\|(S_i)_X\big\| \leqslant 1$, we obtain that it is bounded above by

$$\Big\|\sum_i (g_i, h_i)x_i\Big\|_p^p - \Big\|\sum_i h_i x_i\Big\|_p^p = \Big\|\sum_i g_i x_i\Big\|_p^p.$$

We can therefore go on with our computation and get

$$\begin{aligned}\Big\|\sum_i (Tf_i)x_i\Big\|_{L_p(\Omega_2;X)} &\leqslant \varepsilon\sum_i \|x_i\| + \Big\|\sum_i g_i x_i\Big\|_p \\ &\leqslant \varepsilon\sum_i \|x_i\| + \sum_i \|f_i - g_i\|_p \|x_i\| + \Big\|\sum_i f_i x_i\Big\|_p \\ &\leqslant 2\varepsilon\sum_i \|x_i\| + \Big\|\sum_i f_i x_i\Big\|_p .\end{aligned}$$

Making $\varepsilon \to 0$, we obtain (4.4) as required.

The converse direction relies on everything we have obtained so far. Assume that $T \in ({}^\circ B)^\circ$. We know from Corollary 4.6 that for every integer n, $P(T,n) \subset \overline{\mathrm{conv}}^{\|\cdot\|} P(B \cup \mathrm{REG}, n)$, which is the same as the norm-closure of $P\big(\oplus_{\ell_p}(B \cup \mathrm{REG})\big)$ by Lemma 4.17. So by Lemma 4.16, for every $\varepsilon > 0$ there is $S \in \Lambda_{123}(B \cup \mathrm{REG})$ with $\mathrm{dom}(S) \subset L_p\big(\Omega_1 \times [0,1], m_1 \otimes \mathrm{d}\lambda\big)$ and $\mathrm{ran}(S) \subset L_p\big(\Omega_2 \times [0,1], m_2 \otimes \mathrm{d}\lambda\big)$, there are $g_1, \ldots, g_n \in \mathrm{dom}(S)$ such that

$$\int_{\Omega_1\times[0,1]} \Big(\big|f(\omega)\big|^p + \big|g(\omega,s)\big|^p\Big)\chi_{f(\omega)\neq g(\omega,s)}\,\mathrm{d}m_1(\omega)\,\mathrm{d}s \leqslant \varepsilon^p$$

and

$$\int_{\Omega_2\times[0,1]} \Big(\big|Tf(\omega)\big|^p + \big|Sg(\omega,s)\big|^p\Big)^{\frac{1}{p}}\chi_{Tf(\omega)\neq Sg(\omega,s)}\,\mathrm{d}m_2(\omega)\,\mathrm{d}s \leqslant \varepsilon^p.$$

In particular, using that $\int \big(|a|^p + |b|^p\big)\chi_{a\neq b} \geqslant \int |a-b|^p = \sum_i \int |a_i - b_i|^p$ for every $a, b \in (L_p)^n$, we have for every i,

$$\left(\int_{\Omega_1\times[0,1]} \big|f_i(\omega) - g_i(\omega,s)\big|^p\,\mathrm{d}m_1(\omega)\,\mathrm{d}s\right)^{\frac{1}{p}} \leqslant \varepsilon$$

and

$$\left(\int_{\Omega_2\times[0,1]} \big|Tf_i(\omega) - Sg_i(\omega,s)\big|^p\,\mathrm{d}m_2(\omega)\,\mathrm{d}s\right)^{\frac{1}{p}} \leqslant \varepsilon.$$

Also, using that REG contains the identity and is stable by ℓ_p-direct sums, $\Lambda_1\big(\oplus_{\ell_p}(B\cup\mathrm{REG})\big)$ is the set of all operators of the form $S_0 \oplus S_1 \oplus \cdots \oplus S_k$ for $S_0 \in \mathrm{REG}$ and $S_1, \ldots, S_k \in B$. So the fact that S belongs to $\Lambda_{123}(B\cup\mathrm{REG})$ means that there exist $S_0 \in \mathrm{REG}$, $S_1, \ldots, S_k \in B$, spatial isometries U, V and a measure space (Ω, m) such that $V \circ (S_0 \oplus \cdots \oplus S_k) \circ V$ contains elements of the form (g_i, h_i) in its domain for all $1 \leqslant i \leqslant n$ and some $h_i \in L_p(\Omega, m)$ and so that $V \circ (S_0 \oplus \cdots \oplus S_k) \circ V(g_i, h_i) = (Sg_i, h_i)$. We can of course replace (Ω, m) by any standard measure space as this amounts to conjugating by another spatial isometry. So we have obtained the conclusion of the theorem. □

5. Variants of the duality

The duality defined in the introduction was implicit in many early works on the geometry of Banach spaces, and was essentially present in [30], where Pisier explicitly considered a duality that is very close to ours. He defines the polars by the same formulas as in Definitions 1.1 and 1.2, but with different classes of operators instead of $\mathcal{T}$.

DEFINITION 5.1. — *Denote by $\mathcal{T}_f$ the class of all linear operators between (full, as opposed to subspace of) L_p spaces. If n is an integer, denote $\mathcal{T}_{f,n}$ the class of all linear operators $\ell_p^n \to L_p$ and $\mathcal{T}_{f,<\infty} = \bigcup_{n\in\mathbf{N}} \mathcal{T}_{f,n}$.*

In particular, for the duality between $\mathcal{X}$ and $\mathcal{T}_f$, the polar of a set B of Banach spaces is smaller than for the duality between $\mathcal{X}$ and $\mathcal{T}$, and therefore its bipolar is larger. Hernandez also obtained a description of the bipolar B for this duality: it is the set of Banach spaces that are finitely representable in subspaces of quotients of finite ℓ_p direct sums of spaces in B, see Theorem 5.5. This is quite different from Theorem 1.3. For example for the duality considered here, every Banach space in the bipolar of ℓ_1 has cotype $\max(p,2)$ (this is immediate from Hernandez's Theorem 1.3 and the fact that $\ell_p(\ell_1)$ has cotype $\max(p,2)$), whereas for the duality in [30], the bipolar of ℓ_1 contains every space finitely representable in a quotient of ℓ_1, i.e. every Banach space.

However, as far as the bipolar of a set of operators is concerned, the two dualities are very related: if $B \subset \mathcal{T}_f$, then its bipolar for the polarity between $\mathcal{X}$ and $\mathcal{T}_f$ is the set of operators between L_p spaces which belong to ${}^\circ B^\circ$ (for the polarity between $\mathcal{X}$ and $\mathcal{T}$). So our Theorem 1.6 also provides an answer to [30, Problem 4.1].

The dualities discussed so far are isometric variants of two other isomorphic forms of the duality in [30], where A° is the class of operators such that $\|T_X\| < \infty$ for all $X \in A$, and ${}^\circ B$ is the class of Banach space such $\|T_X\| < \infty$ for all $T \in B$. But, if B is finite, the bipolar of B for this "isomorphic" duality coincides with $\bigcup_{R>0} R\big({}^\circ(R^{-1}B)^\circ\big)$. If B is infinite, the "isomorphic" bipolar of B is $\bigcup_{R>0}\bigcup_{c\in(0,1]^B} R({}^\circ cB)^\circ$, where for $c = (c_T)_{T\in B} \in (0,1]^B$, $cB = \big\{c_T T \mid T \in B\big\}$. So our bipolar Theorem 1.6 also allows to describe the bipolar for the isomorphic forms of the duality.

It turns out that the methods of this paper also allow us to recover Hernandez's characterization of the bipolar of a set of Banach spaces for the duality between $\mathcal{X}$ and $\mathcal{T}_f$. Let us start with an easy fact, which allows us to restrict our attention to $\mathcal{T}_{f,n}$.

LEMMA 5.2. — *The bipolar of a subset $A \subset \mathcal{X}$ for the duality between $\mathcal{X}$ and $\mathcal{T}_f$ coincides with its bipolar for the duality between $\mathcal{X}$ and $\mathcal{T}_{f,<\infty}$.*

Proof. — Any L_p space can be written as the closure of an increasing net of finite dimensional L_p spaces, which are isomorphic to ℓ_p^n. □

Denote by $e = (e_1, \dots, e_n)$ the standard basis of ℓ_p^n. We encode a subset $B \subset \mathcal{T}_{f,n}$, as the cone $\widetilde{P}(B,n) \subset H_n^*$

$$\widetilde{P}(B,n) = \big\{ s(\mu_e - \mu_{Te}) \mid T \in B,\, s > 0 \big\}.$$

We warn the reader that $\widetilde{P}(B,n)$ is strictly smaller than $P(B,n)$ even for $B \subset \mathcal{T}_{f,n}$. Note however that the duality is still efficiently encoded. The following result is the analogue of Lemma 4.1 and is proved identically.

LEMMA 5.3. — *Let $n \in \mathbf{N}$ be an integer, $A \subset \mathcal{X}$ be a class of Banach spaces and $B \subset \mathcal{T}_{f,n}$ a class of operators $\ell_p^n \to L_p$.*

(1) *$B \subset A^\circ$ if and only if $\widetilde{P}(B,n) \subset N(A,n)^\circ$.*
(2) *$A \subset {}^\circ B$ if and only if $N(A,n) \subset {}^\circ\widetilde{P}(B,n)$.*

Moreover, we have the following.

LEMMA 5.4. — *The subset $\widetilde{P}(\mathcal{T}_{f,n}, n) \subset H_n^*$ is a weak-* closed convex cone. Its polar is*

$$C_n \coloneqq \big\{ \varphi \in H_n \mid \varphi \leqslant 0,\, \varphi(e_1) = \dots = \varphi(e_n) = 0 \big\}.$$

Proof. — The convexity of the cone $\widetilde{P}(\mathcal{T}_{f,n}, n)$ is clear, as $\theta(\mu_e - \mu_{Te}) + (1-\theta)(\mu_e - \mu_{Se})$ can be written as $\mu_e - \mu_{Re}$ for the linear map $R\colon \ell_p^n \to L_p \oplus L_p$ given in matrix form by $R = \begin{pmatrix} \theta^{\frac{1}{p}} T \\ (1-\theta)^{\frac{1}{p}} S \end{pmatrix}$. For the weak-* closedness, by the Krein–Smulian theorem [8, Theorem V.12.1], we have to show that the intersection of $\widetilde{P}(\mathcal{T}_{f,n}, n)$ with the closed unit ball $B_{H_n^*}$ is weak-* closed. Recall that every element of H_n^* can be regarded as a signed measure on $\mathbf{KP}^{n-1}$. If it belongs to $\widetilde{P}(\mathcal{T}_{f,n}, n) \cap B_{H_n^*}$, then its positive part in the Jordan decomposition has total mass $\leqslant 1$ and has support contained in $\{\mathbf{K}e_1, \dots, \mathbf{K}e_n\}$. In particular, it is less than μ_e. It follows that it can be written as $\mu_e - \mu_f$ for some $f \in (L_p)^n$. So we are reduced to showing that $\big\{\mu_e - \mu_f \mid f \in L_p^n\big\}$ is weak-* closed in H_n^*, which is clear.

It remains to identify the polar of $\widetilde{P}(\mathcal{T}_{f,n}, n)$. If $\varphi \in C_n$, and $T \in \mathcal{T}_{f,n}$, we have

$$\langle \mu_e, \varphi \rangle = \sum_i \varphi(e_i) = 0,$$

so

$$\langle \mu_e - \mu_{Te}, \varphi \rangle = -\langle \mu_{Te}, \varphi \rangle \geqslant 0.$$

This shows the inclusion $C_n \subset {}^\circ\widetilde{P}(\mathcal{T}_{f,n}, n)$.

For the converse inclusion, consider $\varphi \in {}^\circ\widetilde{P}(\mathcal{T}_{f,n}, n)$. Then for every $f \in (L_p)^n$, we have

$$\langle \mu_e - \mu_f, \varphi \rangle \geqslant 0. \tag{5.1}$$

In particular, replacing f by sf and making $s \to \infty$, we obtain

$$-\langle \mu_f, \varphi \rangle \geqslant 0$$

for every $f \in (L_p)^n$. Taking for f a constant z, this forces $\varphi(z) \leqslant 0$ for every $z \in \mathbf{K}^n$. Taking $f = 0$ in (5.1) leads to

$$\sum_i \varphi(e_i) = \langle \mu_e, \varphi \rangle \geqslant 0.$$

This implies that $\varphi(e_i) = 0$ for every i, and that φ belongs to C_n. This concludes the proof of the inclusion ${}^\circ\widetilde{P}(\mathcal{T}_{f,n}, n) \subset C_n$ and of the lemma. □

We can now reprove Hernandez' Theorem.

THEOREM 5.5 ([11]). — *Let $A \subset \mathcal{X}$ and $X \in \mathcal{X}$. The following are equivalent.*

1. *For every operator $T\colon L_p \to L_p$, $\|T_X\| \leqslant \sup_{Y \in A} \|T_Y\|$.*
2. *X is finitely representable in the class of all quotients of finite ℓ_p-direct sums of elements in A.*

Proof. — The interesting direction is (1)⇒(2). So assume that (1) holds. Equivalently by Lemma 5.2 X belongs to the bipolar of A for the duality between $\mathcal{X}$ and $\mathcal{T}_{f,<\infty}$. By Lemma 5.3, this holds if and only if for every n,

$$N(X, n) \subset {}^\circ\big(\widetilde{P}(\mathcal{T}_{f,n}, n) \cap N(Y, n)^\circ\big).$$

By Lemma 5.4 and the bipolar theorem, $\widetilde{P}(\mathcal{T}_{f,n}, n)$ coincides with C_n°, so the previous inclusion becomes

$$N(X, n) \subset {}^\circ\big(\big(C_n \cup N(Y, n)\big)^\circ\big).$$

By the bipolar theorem again, we obtain that $N(X, n)$ belongs to the closed convex hull of $C_n \cup N(Y, n)$, which is nothing but $C_n + \overline{\operatorname{conv}}\big(N(Y, n)\big)$ (use compactness as in Lemma 4.5 to see that $C_n + \overline{\operatorname{conv}}\big(N(Y, n)\big)$ is closed). By Lemma 4.7, we obtain that for every $n \in \mathbf{N}$ and every $x_1, \dots, x_n \in X$, there is a space Y finitely representable in the finite ℓ_p-direct sums of elements in A and elements $y_1, \dots, y_n$ spanning Y such that

$$\forall\, i, \quad \|x_i\| = \|y_i\| \qquad \text{and} \qquad \forall\, z \in \mathbf{K}^n, \Big\|\sum_i z_i x_i\Big\|_X \leqslant \Big\|\sum_i z_i y_i\Big\|_Y.$$

Now if E is any finite dimensional subspace of X, and $\varepsilon > 0$, we can pick a finite family $x_1, \dots, x_n$ in its unit sphere whose convex hull contains the ball of radius $(1+\varepsilon)^{-1}$. Applying the preceding to these x_i's, we obtain a space Y finitely representable in $\oplus_{\ell_p} A$ and a linear map $u\colon Y \to E$ of norm 1 such that the image of the unit ball contains the ball of radius $(1+\varepsilon)^{-1}$ of E. In other words, E is at Banach–Mazur distance $\leqslant 1+\varepsilon$ from a quotient of Y. But a subspace of a quotient is the same as a quotient of a subspace, so we have obtained that X is finitely representable in the quotients of spaces in $\oplus_{\ell_p} A$. This is (2).

The converse implication (2)⇒(1) can be proved using the same arguments, but it is easy and classical to check it directly. The point that perhaps deserves a small justification is why $\|T_X\| \leqslant \|T_Y\|$ if X is a quotient of Y and $T\colon L_p \to L_p$ is an operator. One argument is by duality. Indeed, X^* identifies then as a subspace of Y^*, and if $T^*\colon L_q \to L_q$ denotes the dual of T (for $\frac{1}{q} + \frac{1}{p} = 1$), then

$$\|T_X\| = \|T^*_{X^*}\| \leqslant \|T^*_{Y^*}\| = \|T_Y\|. \qquad \square$$

Appendix A. On the $\mathrm{GL}(n, \mathbf{K})$-invariant subspaces of homogeneous functions

Let $0 < p < \infty$. We recall some definition that already appeared in the body of the paper for $p \geqslant 1$.

Let n be a positive integer. Denote by $|z|$ the ℓ_p-"norm" on $\mathbf{K}^n$

$$|z| = \big(|z_1|^p + \cdots + |z_n|^p\big)^{\frac{1}{p}}.$$

A function $\varphi\colon \mathbf{K}^n \to \mathbf{R}$ is called homogeneous of degree p if $\varphi(\lambda z) = |\lambda|^p \varphi(z)$ for all $z \in \mathbf{K}^n$ and $\lambda \in \mathbf{K}$. The space H_n of real continuous homogeneous of degree p functions on $\mathbf{K}^n$ is a Banach space for the topology of uniform convergence on compact subsets on $\mathbf{K}^n$. A particular choice of norm is $\|\varphi\| = \sup_{|z|\leqslant 1} \big|\varphi(z)\big|$, so that for this norm H_n is isometrically isomorphic to the space of continuous functions on $\mathbf{KP}^{n-1}$ through the identification of $\varphi \in H_n$ with the function $\mathbf{K}z \in \mathbf{KP}^{n-1} \mapsto \varphi\big(\frac{z}{|z|}\big)$. For this identification, the natural action of $\mathrm{GL}_n(\mathbf{K})$ on H_n corresponds to the action of $\mathrm{GL}_n(\mathbf{K})$ on $C(\mathbf{KP}^{n-1})$ given by

$$A \cdot \varphi(\mathbf{K}z) = \frac{|A^{-1}z|^p}{|z|^p} \varphi(A^{-1}\mathbf{K}z).$$

THEOREM A.1. — *The $\mathrm{GL}_n(\mathbf{K})$-invariant closed subspaces of the Banach space H_n of continuous p-homogeneous functions $\mathbf{K}^n \to \mathbf{R}$ are:*

- $\{0\}$ *and* H_n *if* p *is not an even integer;*
- $\{0\}$, H_n *and the subspace of degree* p *homogeneous polynomials if* p *is an even integer.*

Remark A.2. — This theorem allows to reprove the result [9] that if p is not an even integer, then every isometry between subspaces of L_p spaces is a spatial isometry. Indeed, if T is such an isometry, n is an integer, $f \in D(T)^n$ and $\varphi(z) = |z_1|^p$, then we get for every $A \in \mathrm{GL}_n(\mathbf{K})$ (with the notation of (4.2))

$$\langle \mu_f - \mu_{Tf}, \varphi \circ A \rangle = \Big\| \sum_j a_{1,j} f_j \Big\|^p - \Big\| \sum_j a_{1,j} T f_j \Big\|^p = 0.$$

The linear form $\mu_f - \mu_{Tf}$ therefore vanishes on the $\mathrm{GL}_n(\mathbf{K})$-invariant subspace spanned by $\big\{\varphi \circ A,\ A \in \mathrm{GL}_n(\mathbf{K})\big\}$. By Theorem A.1 this subspace is dense, which implies that $\mu_f - \mu_{Tf} = 0$. One concludes by Remark 4.12 that T is a spatial isometry.

When p is an even integer, the same argument shows that if X is a Banach space and $x, y \in X$ are so that $(z_1, z_2) \mapsto \|z_1 x + z_2 y\|^p$ is not a polynomial in $z_1, z_2, \bar{z}_1, \bar{z}_2$ (for example if $X = \mathbf{K}^2$ with the ℓ_q norm for q which is not an even divisor of p), then every operator T between subspaces of L_p spaces such that $\|T_X\| = \|T_X^{-1}\| = 1$ is a spatial isometry. In particular we have the following.

Corollary A.3. — *For any* $0 < p < \infty$ *(even integer or not) a linear map* T *between subspaces of* L_p *spaces is a spatial isometry if and only if* T *is a regular isometry.*

Rudin's proof in [32] relied on the Wiener Tauberian theorem. In the proof of Theorem A.1, we shall need the following variant.

Proposition A.4. — *Let* $f, g \colon \mathbf{R}^d \to \mathbf{C}$ *be two measurable functions and* $C > 0$ *such that* $\big|f(x)\big| \leqslant C\big(1 + |x|\big)^p$ *and* $\big|g(x)\big| \leqslant C\big(1 + |x|\big)^{-p-d-1}$ *for all* $x \in \mathbf{R}^d$. *Assume that* $g * f = 0$. *Then the support of the tempered distribution* $\widehat{f}$ *is contained in* $\big\{\xi \in \mathbf{R}^d,\ \widehat{g}(\xi) = 0\big\}$.

Proof. — First observe that $g \in L_1(\mathbf{R}^d)$, by the assumption on g.

If g belongs to $\mathcal{D}(\mathbf{R}^d)$ (the space of compactly supported C^∞ functions), then the proposition is easy: by taking Fourier transform we have $\widehat{g}\widehat{f} = 0$ (multiplication of a distribution by a C^∞ function), from which the conclusion follows. The strategy will be to approximate g by compactly supported C^∞ functions.

We have to prove that for every $\xi \in \mathbf{R}^d$ with $\widehat{g}(\xi) \neq 0$, there is a neighbourhood V of ξ such that $\langle \widehat{f}, \varphi \rangle = 0$ for every $\varphi \in \mathcal{D}(V)$. By standard

translation/convolution/dilation arguments, we can assume that $\xi = 0$, g is C^∞, and that $\widehat{g}$ does not vanish on the closure of $B(0,1)$. We will prove that $\langle \widehat{f}, \varphi\rangle = 0$ for every $\varphi \in \mathcal{D}\big(B(0,1)\big)$.

Let $\rho\colon \mathbf{R}^d \to [0,1]$ be a compactly supported C^∞ function, equal to 1 on $B(0,1)$, and define a sequence of functions $g_n \in \mathcal{D}(\mathbf{R}^d)$ by $g_n(x) = g(x)\rho\big(\frac{x}{n}\big)$. By the dominated convergence theorem, $\|g_n - g\|_{L_1(\mathbf{R}^d)} \to 0$, and so $\|\widehat{g}_n - \widehat{g}\|_{L_\infty} \to 0$. In particular there exists n_0 such that $\widehat{g}_n$ does not vanish on $B(0,1)$ for all $n \geqslant n_0$.

Let $\varphi \in \mathcal{D}\big(B(0,1)\big)$. Then $\frac{\varphi}{\widehat{g}_n}$ belongs to $\mathcal{D}\big(B(0,1)\big)$, so we can write

$$\langle \widehat{f}, \varphi\rangle = \Big\langle \widehat{g}_n \widehat{f}, \frac{\varphi}{\widehat{g}_n}\Big\rangle = \Big\langle g_n * f, \mathcal{F}^{-1}\Big(\frac{\varphi}{\widehat{g}_n}\Big)\Big\rangle$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. Using that $g_n * f(x) = (g_n - g) * f(x) = O\big(\frac{1}{n}\big(1+\frac{|x|}{n}\big)^p\big)$ (this inequality will be explained below), we get

$$\big|\langle \widehat{f}, \varphi\rangle\big| \leqslant \frac{C}{n}\int \Big(1 + \frac{|x|}{n}\Big)^p \Big|\mathcal{F}^{-1}\Big(\frac{\varphi}{\widehat{g}_n}\Big)\Big|\,\mathrm{d}x. \tag{A.1}$$

To justify to domination of $h_n(x) \coloneqq (g_n - g) * f(x) = \int (g_n - g)(y) f(x-y)\,\mathrm{d}y$, use that

$$\big|(g_n - g)(y)\big| \lesssim \big(1 + |y|\big)^{-p-d-1} 1_{|y|>n}$$

and

$$\big|f(x-y)\big| \lesssim \big(1 + |x-y|\big)^p \lesssim \Big(1 + \max\big(|x|, |y|\big)\Big)^p$$

to obtain

$$\big|h_n(x)\big| \lesssim \int_{|y|>n} \big(1 + |y|\big)^{-p-d-1}\Big(1 + \max\big(|x|, |y|\big)\Big)^p \,\mathrm{d}y.$$

If $|x| \leqslant n$, then the preceding inequality becomes

$$\big|h_n(x)\big| \lesssim \int_{|y|>n} \big(1 + |y|\big)^{-d-1}\,\mathrm{d}y \lesssim \frac{1}{n}.$$

If $|x| \geqslant n$, then we cut the integral as $\int_{n<|y|\leqslant|x|} + \int_{|x|<|y|}$ and get

$$\begin{aligned}\big|h_n(x)\big| &\lesssim \int_{n<|y|\leqslant|x|} \frac{\big(1+|x|\big)^p}{\big(1+|y|\big)^{p+d+1}}\,\mathrm{d}y + \int_{|y|>|x|} \big(1+|y|\big)^{-d-1}\,\mathrm{d}y \\ &\lesssim |x|^p \frac{1}{n^{p+1}} + \frac{1}{|x|} \\ &\lesssim \frac{|x|^p}{n^{p+1}}.\end{aligned}$$

This proves the announced inequality.

In view of (A.1), we see that our goal is to prove good integrability properties on the function $\mathcal{F}^{-1}\big(\frac{\varphi}{\widehat{g}_n}\big)$, i.e. good regularity properties of its

Fourier transform $\frac{\varphi}{\widehat{g}_n}$. To achieve this, we denote by $A(\mathbf{R}^d)$ the Fourier algebra of $\mathbf{R}^d$, i.e. the Banach space $\mathcal{F}\big(L_1(\mathbf{R}^d)\big)$ for the norm $\|h\|_{A(\mathbf{R}^d)} = \|\mathcal{F}^{-1}h\|_{L_1(\mathbf{R}^d)}$. The inequality

$$\|h_1h_2\|_{A(\mathbf{R}^d)} \leqslant \|h_1\|_{A(\mathbf{R}^d)}\|h_2\|_{A(\mathbf{R}^d)} \tag{A.2}$$

is the reason for the term "algebra" and is clear from the usual properties of convolution and Fourier transform. We have the following lemmas.

LEMMA A.5. — *For every $\varphi \in \mathcal{D}\big(B(0,1)\big)$, there is a constant $C = C(\varphi)$ such that $\frac{\varphi}{\widehat{g}_n}$ belongs to $A(\mathbf{R}^d)$ with norm $\leqslant C$ for all $n \geqslant n_0$.*

LEMMA A.6. — *There is a constant C' such that $D^\alpha \widehat{g}_n$ belongs to $A(\mathbf{R}^d)$ with norm $\leqslant C'$ for all $n \in \mathbf{N}$ and $\alpha \in \mathbf{N}^d$, $|\alpha| < p+1$.*

These two lemmas, together with the Leibniz derivation rule and the fact that $A(\mathbf{R}^d)$ is a Banach algebra (A.2), imply that, for every $\varphi \in \mathcal{D}\big(B(0,1)\big)$, there is a constant C such that $D^\alpha \frac{\varphi}{\widehat{g}_n}$ belongs to $A(\mathbf{R}^d)$ with norm less than C for all $n \geqslant n_0$ and $\alpha \in \mathbf{N}^d$, $|\alpha| < p+1$. Therefore, for every such n and α we have

$$\int \Big| x^\alpha \mathcal{F}^{-1}\Big(\frac{\varphi}{\widehat{g}_n}\Big)\Big| \,\mathrm{d}x \leqslant C.$$

This implies that, if k is the unique integer in the interval $[p, p+1)$, then for every $n \geqslant n_0$

$$\int \big(1+|x|\big)^p \Big|\mathcal{F}^{-1}\Big(\frac{\varphi}{\widehat{g}_n}\Big)\Big| \,\mathrm{d}x \leqslant \int \big(1+|x|\big)^k \Big|\mathcal{F}^{-1}\Big(\frac{\varphi}{\widehat{g}_n}\Big)\Big| \,\mathrm{d}x \leqslant C'.$$

A fortiori, by (A.1) we have

$$\big|\langle \widehat{f}, \varphi\rangle\big| \leqslant \frac{C'}{n},$$

so making $n \to \infty$ we obtain $\langle \widehat{f}, \varphi\rangle = 0$. This concludes the proof. □

We have to prove the two lemmas used above.

Proof of Lemma A.5. — Let $\rho \in D\big(B(0,1)\big)$ which is equal to 1 on the support of φ. The fact that $\frac{\rho}{\widehat{g}}$ (and $\frac{\rho}{\widehat{g}_n}$ for every $n \geqslant n_0$) belongs to $A(\mathbf{R}^d)$ is essentially the Wiener Tauberian theorem. Indeed, the proof in [33, Theorem 9.3] shows that for every $x \in \mathbf{C}$ such that $\widehat{g}(x) \neq 0$, there is $\varepsilon > 0$ such that $\frac{\rho}{\widehat{g}} \in A(\mathbf{R}^d)$ for every $\rho \in D\big(B(x,\varepsilon)\big)$. The claimed result follows by a partition of unity argument. To obtain a bound on $\frac{\rho}{\widehat{g}_n}$ independent from n, we write

$$\frac{\varphi}{\widehat{g}_n} = \frac{\varphi}{\widehat{g}} \frac{1}{1 - \frac{\rho}{\widehat{g}}(\widehat{g} - \widehat{g}_n)}.$$

Since $\frac{\rho}{\widehat{g}}$ belongs to $A(\mathbf{R}^d)$ and $\|\widehat{g}-\widehat{g}_n\|_{A(\mathbf{R}^d)} = \|g-g_n\|_{L_1(\mathbf{R}^d)} \to 0$, there is $n_1 \geqslant n_0$ such that $\frac{\rho}{\widehat{g}}(\widehat{g}-\widehat{g}_n)$ has $A(\mathbf{R}^d)$-norm less than $\frac{1}{2}$ for all $n \geqslant n_1$. This implies that for $n \geqslant n_1$

$$\frac{\varphi}{\widehat{g}_n} = \sum_{k\geqslant 0} \frac{\varphi}{\widehat{g}}\Big(\frac{\rho}{\widehat{g}}(\widehat{g}-\widehat{g}_n)\Big)^k$$

belongs to $A(\mathbf{R}^d)$ with norm less than $2\big\|\frac{\varphi}{\widehat{g}}\big\|_{A(\mathbf{R}^d)}$. The lemma follows with

$$C = \max\left(2\Big\|\frac{\varphi}{\widehat{g}}\Big\|_{A(\mathbf{R}^d)}, \max_{n_0\leqslant n<n_1}\Big\|\frac{\varphi}{\widehat{g}_n}\Big\|_{A(\mathbf{R}^d)}\right). \qquad \square$$

Proof of Lemma A.6. — We have

$$\|D^\alpha \widehat{g}_n\|_{A(\mathbf{R}^d)} = \|x^\alpha g_n\|_{L_1(\mathbf{R}^d)} \leqslant \|x^\alpha g\|_{L_1(\mathbf{R}^d)}$$

because $g_n(x) = g(x)\rho\big(\frac{x}{n}\big)$ and $0 \leqslant \rho \leqslant 1$. The quantity $\|x^\alpha g\|_{L_1(\mathbf{R}^d)}$ is finite because $g(x) = O\big(|x|^{-p-d-1}\big)$ and $|\alpha| < p+1$. $\square$

We can now prove the main result on $\mathrm{GL}_n(\mathbf{K})$-invariant subspaces of H_n.

Proof of Theorem A.1. — For simplicity we write the proof for $\mathbf{K} = \mathbf{C}$. The real case is similar, see Remark A.9. Let $f_0 \in H_n$ be a nonzero function such that the space spanned by the functions $f_0 \circ A$ for $A \in \mathrm{GL}_n(\mathbf{C})$ is not dense in H_n. We will prove that p is an even integer and that f_0 is a homogeneous polynomial. By the Hahn–Banach theorem, there is a nonzero linear form φ on H_n which vanishes on $f_0 \circ A$ for all A. By the Riesz representation theorem, there is a unique nonzero signed measure μ on $\mathbf{CP}^{n-1}$ such that $\varphi(f) = \int f\big(\frac{z}{|z|}\big)\,\mathrm{d}\mu(\mathbf{C}z)$. We can assume that μ is absolutely continuous with respect to the Lebesgue measure (= the unique $\mathrm{U}(n)$-invariant probability measure) on $\mathbf{CP}^{n-1}$, with a C^∞ Radon–Nikodym derivative. Indeed, if ρ is a C^∞ function on $\mathrm{U}(n)$, then the measure $\rho * \mu = \int (u_*\mu)\rho(u)\,\mathrm{d}u$ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{CP}^{n-1}$, has a C^∞ density, and still satisfies $\int f_0 \circ A\big(\frac{z}{|z|}\big)\,\mathrm{d}(h*\mu)(\mathbf{C}z) = 0$ for every $A \in \mathrm{GL}_n(\mathbf{C})$. Moreover if $\rho \geqslant 0$ has a support which is a small enough neighbourhood of the identity, then $\rho * \mu \neq 0$.

So in particular, μ has a nonzero bounded Radon–Nikodym derivative h with respect to the Lebesgue measure. By Lemma A.8 we can write

$$\int_{\mathbf{CP}^{n-1}} F(\mathbf{C}z)\,\mathrm{d}\mu(z) = \int_{\mathbf{C}^{n-1}} (Fh)\big(\mathbf{C}(1,z)\big)\frac{c}{\big(1+|z_1|^2+\cdots+|z_{n-1}|^2\big)^n}\,\mathrm{d}z.$$

Taking $F(\mathbf{C}z) = (f_0\circ A)\big(\frac{z}{|z|}\big)$, we get $F\big(\mathbf{C}(1,z)\big) = \frac{f_0\circ A(1,z)}{1+|z|^p}$ and

$$0 = \int_{\mathbf{C}^{n-1}} f_0 \circ A(1,z) g(z)\,\mathrm{d}z \tag{A.3}$$

for the nonzero function $g(z) = \frac{1}{1+|z|^p}\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\big(\mathbf{C}(1,z)\big)\frac{c}{\big(1+|z|_2^2\big)^n}$, which satisfies

$$g(z) = O\big(\big(1+|z|\big)^{-p-2n}\big). \tag{A.4}$$

Now if we take for $A = \left(\begin{smallmatrix} 1 & 0 \\ b & -A'^{-1} \end{smallmatrix}\right)$ for $A' \in \mathrm{GL}_{n-1}(\mathbf{C})$, then (A.3) becomes

$$0 = \int f_0(1, b - A'^{-1}z)g(z)\,\mathrm{d}z = |\det A'| \int (g \circ A')(z) f_0(1, b-z)\,\mathrm{d}z.$$

The second equality is a change of variable. In other words, if $f\colon \mathbf{C}^{n-1} \to \mathbf{R}$ is the function $f(z) = f_0(1,z)$, then f is a continuous function satisfying $f(z) = O\big(1+|z|^p\big)$ as $z \to \infty$, and such that $(g \circ A') * f = 0$ for every $A' \in \mathrm{GL}_{n-1}(\mathbf{C})$.

Viewing $\mathbf{C}^{n-1}$ as a real vector space $\mathbf{R}^d$ with $d = 2n-2$, we see that we are in the setting of Proposition A.4 ((A.4) indeed implies that $(g \circ A')(z) = O\big(\big(1+|z|\big)^{-p-d-2}\big)$). So the proposition implies that the support of $\widehat{f}$ is contained in $\big\{\xi \in \mathbf{C}^{n-1},\, \mathcal{F}(g \circ A')(\xi) \neq 0\big\}$. But, g being nonzero, there exists $\xi \neq 0$ such that $\widehat{g}(\xi) \neq 0$. Since $\mathrm{GL}_{n-1}(\mathbf{C})$ acts transitively on $\mathbf{C}^{n-1} \setminus \{0\}$, we get that the support of $\widehat{f}$ is contained in $\{0\}$. This implies that f is a polynomial function in $z, \overline{z}$. So we have proved that (A.3) implies that the function $z \mapsto f_0(1,z)$ is a polynomial function in $z, \overline{z}$. But since (A.3) for f_0 clearly implies (A.3) for $f_0 \circ A$ for every $A \in \mathrm{GL}_n(\mathbf{C})$, we get that $z \mapsto f_0 \circ A(1,z)$ is a polynomial for every A. This implies that p is an even integer and that f_0 is a homogeneous polynomial, see Lemma A.7.

This shows that if p is not an even integer, then $\{0\}$ and H_n are the only closed $\mathrm{GL}_n(\mathbf{C})$-invariant subspaces of H_n, and that otherwise all other invariant closed subspaces are contained in the space of degree p homogeneous polynomials. It remains to show that for every nonzero degree p homogeneous polynomial, every other such polynomial belongs to the linear space spanned by its $\mathrm{GL}_n(\mathbf{C})$ orbit. This is not difficult. □

Lemma A.7. — *Let $f_0 \in H_n$ be a nonzero function such that, for every $A \in \mathrm{GL}_n(\mathbf{C})$, $z \in \mathbf{C}^{n-1} \mapsto f_0 \circ A(1,z)$ is a polynomial in $z, \overline{z}$. Then p is an even integer and f_0 is a homogeneous polynomial of degree p.*

Proof. — Let $P \in \mathbf{C}[X_1, \dots, X_{2n-2}]$ such that $f_0(1,z) = P(z, \overline{z})$. Using that $f_0 \in H_n$, we have that $\big|P(z,\overline{z})\big| = O\big(\big(1+|z|\big)^p\big)$, and in particular $\deg(P) \leqslant p$, so we can write

$$P(z,\overline{z}) = \sum_{\substack{\alpha,\beta \in \mathbf{N}^d \\ |\alpha|+|\beta| \leqslant p}} a_{\alpha,\beta} z^\alpha \overline{z}^\beta.$$

Let $c \in \mathbf{C}^{n-1}$ and $A = \left(\begin{smallmatrix} 1 & c^* \\ 0 & 1 \end{smallmatrix}\right)$. Similarly there is a polynomial $P_c \in \mathbf{C}[X_1, \dots, X_{2n-2}]$ of degree $\leqslant p$ such that $f \circ A(1,z) = P_c(z)$. Then

$$P_c(z) = \big|1+\langle z,c\rangle\big|^p f\Big(1, \frac{z}{1+\langle z,c\rangle}\Big) = \big|1+\langle z,c\rangle\big|^p P\Big(\frac{z}{1+\langle z,c\rangle}, \frac{\overline{z}}{1+\overline{\langle z,c\rangle}}\Big).$$

We can rewrite this quantity as

$$\sum_{\alpha,\beta} a_{\alpha,\beta}\big(1+\langle z,c\rangle\big)^{\frac{p}{2}-|\alpha|}\big(1+\overline{\langle z,c\rangle}\big)^{\frac{p}{2}-|\beta|} z^\alpha \overline{z}^\beta.$$

By expanding $(1+t)^l = \sum_{n\geqslant 0}\binom{l}{n}t^n$, for small z the preceding sum is

$$\sum_{\alpha,\beta,n,m} a_{\alpha,\beta}\binom{\frac{p}{2}-|\alpha|}{n}\binom{\frac{p}{2}-|\beta|}{m}\langle z,c\rangle^n\overline{\langle z,c\rangle}^m z^\alpha\overline{z}^\beta.$$

Since P_c is a polynomial of degree $\leqslant p$, we get that for every $N > p$,

$$\sum_{|\alpha|+|\beta|+n+m=N} a_{\alpha,\beta}\binom{\frac{p}{2}-|\alpha|}{n}\binom{\frac{p}{2}-|\beta|}{m}\langle z,c\rangle^n\overline{\langle z,c\rangle}^m z^\alpha\overline{z}^\beta = 0.$$

Since this is valid for every c, we get

$$a_{\alpha,\beta}\binom{\frac{p}{2}-|\alpha|}{n}\binom{\frac{p}{2}-|\beta|}{m} = 0$$

for every $\alpha,\beta \in \mathbf{N}^d$ and $n,m \in \mathbf{N}$ such that $|\alpha|+|\beta|+n+m > p$.

Let α,β such that $a_{\alpha,\beta} \neq 0$ (such α,β exist by the assumption that f_0 is nonzero). Then taking $n=0$ and m very large, we find that $\binom{\frac{p}{2}-|\beta|}{m} = 0$, which implies that $\frac{p}{2}-|\beta|$ is a nonnegative integer. Similarly $\frac{p}{2}-|\alpha|$ is a nonnegative integer. This proves that p is an even integer and

$$f_0(1,z) = \sum_{|\alpha|,|\beta|\leqslant\frac{p}{2}} a_{\alpha,\beta} z^\alpha \overline{z}^\beta.$$

By homogeneity we get

$$f_0(z_1,z) = \sum_{|\alpha|,|\beta|\leqslant\frac{p}{2}} z_1^{\frac{p}{2}-|\alpha|} z^\alpha \overline{z}_1^{\frac{p}{2}-|\beta|}\overline{z}^\beta.$$

This is the lemma. □

LEMMA A.8. — *The Lebesgue measure λ on $\mathbf{CP}^{n-1}$ is given by*

$$\int_{\mathbf{CP}^{n-1}} F(\mathbf{C}z)\,\mathrm{d}\lambda(z) = c\int_{\mathbf{C}^{n-1}} F\big(\mathbf{C}(1,z)\big)\frac{1}{\big(1+|z_1|^2+\cdots+|z_{n-1}|^2\big)^n}\,\mathrm{d}z$$

for some number $c > 0$.

Proof. — One checks by a change of variable that the finite measure

$$F \in C(\mathbf{CP}^{n-1}) \mapsto \int_{\mathbf{C}^{n-1}} F\big(\mathbf{C}(1,z)\big)\,\frac{1}{\big(1+|z_1|^2+\cdots+|z_{n-1}|^2\big)^n}\,\mathrm{d}z$$

is invariant by $\mathrm{U}(n)$. □

Remark A.9. — We did not use the full strength of Proposition A.4 for $\mathbf{K}=\mathbf{C}$, as we used it for a function g satisfying $g(z)=O\big(\big(1+|z|\big)^{-p-d-2}\big)$, which is strictly stronger that the required $g(z)=O\big(\big(1+|z|\big)^{-p-d-1}\big)$. The reason for this 2 is that the real dimension drops by 2 between $\mathbf{C}^n$ and $\mathbf{CP}^{n-1}$. In the real case, the dimension drops by 1, and the same proof (using all the assumptions of Proposition A.4 this time) leads to the following: the $\mathrm{GL}_n(\mathbf{R})$-invariant closed subspace of the Banach space $H_{n,\mathbf{R}}$ of continuous p-homogeneous functions $\mathbf{R}^n \to \mathbf{R}$ are (1) $\{0\}$ and $H_{n,\mathbf{R}}$ if p is not an even integer (2) $\{0\}$, $H_{n,\mathbf{R}}$ and the space of homogeneous degree p polynomials if p is an even integer.

As a consequence, the conclusion of Remark A.2 holds also over $\mathbf{R}$.

BIBLIOGRAPHY

[1] G. Arzhantseva & R. Tessera, "Relative expanders", *Geom. Funct. Anal.* **25** (2015), no. 2, p. 317-341.

[2] U. Bader, A. Furman, T. Gelander & N. Monod, "Property (T) and rigidity for actions on Banach spaces", *Acta Math.* **198** (2007), no. 1, p. 57-105.

[3] S. Banach, *Théorie des opérations linéaires*, Monografie Matematyczne, vol. 1, Panstwowe Wydawnictwo Naukowe, 1932, vii+254 pages.

[4] N. Bourbaki, *Espaces vectoriels topologiques. Chapitres 1 à 5*, new ed., Masson, 1981, vii+368 pages.

[5] J. Bourgain, "Some remarks on Banach spaces in which martingale difference sequences are unconditional", *Ark. Mat.* **21** (1983), no. 2, p. 163-168.

[6] D. L. Burkholder, "A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions", in *Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981)*, The Wadsworth Mathematics Series, Wadsworth, 1983, p. 270-286.

[7] Q. Cheng, "Sphere equivalence, property H, and Banach expanders", *Stud. Math.* **233** (2016), no. 1, p. 67-83.

[8] J. B. Conway, *A course in functional analysis*, second ed., Graduate Texts in Mathematics, vol. 96, Springer, 1990, xvi+399 pages.

[9] C. D. Hardin, Jr., "Isometries on subspaces of L^p", *Indiana Univ. Math. J.* **30** (1981), no. 3, p. 449-465.

[10] S. Heinrich, "Ultraproducts in Banach space theory", *J. Reine Angew. Math.* **313** (1980), p. 72-104.

[11] R. Hernandez, "Espaces L^p, factorisation et produits tensoriels dans les espaces de Banach", *C. R. Math.* **296** (1983), no. 9, p. 385-388.

[12] ———, "Espaces L^p, factorisations et produits tensoriels", PhD Thesis, Université Pierre et Marie Curie (France), 1983.

[13] S. Kwapień, "Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients", *Stud. Math.* **44** (1972), p. 583-595.

[14] T. de Laat & M. de la Salle, "Strong property (T) for higher-rank simple Lie groups", *Proc. Lond. Math. Soc. (3)* **111** (2015), no. 4, p. 936-966.

[15] ———, "Approximation properties for noncommutative L^p-spaces of high rank lattices and nonembeddability of expanders", *J. Reine Angew. Math.* **737** (2018), p. 49-69.

[16] ———, "Banach space actions and L^2-spectral gap", *Anal. PDE* **14** (2021), no. 1, p. 45-76.

[17] ———, "Actions of higher rank groups on uniformly convex Banach spaces", 2023, `https://arxiv.org/abs/2303.01405`.

[18] V. Lafforgue, "Un renforcement de la propriété (T)", *Duke Math. J.* **143** (2008), no. 3, p. 559-602.

[19] ———, "Propriété (T) renforcée banachique et transformation de Fourier rapide", *J. Topol. Anal.* **1** (2009), no. 3, p. 191-206.

[20] O. C. McGehee, "A proof of a statement of Banach about the weak* topology", *Mich. Math. J.* **15** (1968), p. 135-140.

[21] M. Mendel & A. Naor, "Nonlinear spectral calculus and super-expanders", *Publ. Math., Inst. Hautes Étud. Sci.* **119** (2014), p. 1-95.

[22] V. D. Milman & H. J. Wolfson, "Minkowski spaces with extremal distance from the Euclidean space", *Isr. J. Math.* **29** (1978), no. 2-3, p. 113-131.

[23] M. Mimura, "Sphere equivalence, Banach expanders, and extrapolation", *Int. Math. Res. Not.* (2015), no. 12, p. 4372-4391.

[24] A. Naor, "Comparison of metric spectral gaps", *Anal. Geom. Metr. Spaces* **2** (2014), no. 1, p. 1-52.

[25] I. Oppenheim, "Banach property (T) for $\mathrm{SL}_n(\mathbb{Z})$ and its applications", *Invent. Math.* **234** (2023), no. 2, p. 893-930.

[26] M. I. Ostrovskii, "w*-derivatives of transfinite order for subspaces of a conjugate Banach space", *Dokl. Akad. Nauk Ukr. SSR, Ser. A* **1987** (1987), no. 10, p. 9-12.

[27] ———, "Weak* sequential closures in Banach space theory and their applications", in *General topology in Banach spaces*, Nova Science Publishers, 2001, p. 21-34.

[28] ———, "Coarse embeddability into Banach spaces", *Topol. Proc.* **33** (2009), p. 163-183.

[29] G. Pisier, "Holomorphic semigroups and the geometry of Banach spaces", *Ann. Math. (2)* **115** (1982), no. 2, p. 375-392.

[30] ———, "Complex interpolation between Hilbert, Banach and operator spaces", *Mem. Am. Math. Soc.* **208** (2010), no. 978, p. vi+78.

[31] M. Rørdam, A. Thom, S. Vaes & D.-V. Voiculescu, "C*-Algebras", *Oberwolfach Rep.* **13** (2016), no. 3, p. 2269-2345.

[32] W. Rudin, "L^p-isometries and equimeasurability", *Indiana Univ. Math. J.* **25** (1976), no. 3, p. 215-228.

[33] ———, *Functional analysis*, second ed., International Series in Pure and Applied Mathematics, McGraw-Hill, 1991, xviii+424 pages.

[34] M. de la Salle, "Towards strong Banach property (T) for $\mathrm{SL}(3,\mathbf{R})$", *Isr. J. Math.* **211** (2015), no. 1, p. 105-145.

[35] D. Sarason, "On the order of a simply connected domain", *Mich. Math. J.* **15** (1968), p. 129-133.

[36] ———, "A remark on the weak-star topology of l^∞", *Stud. Math.* **30** (1968), p. 355-359.

[37] R. Tessera, "Coarse embeddings into a Hilbert space, Haagerup property and Poincaré inequalities", *J. Topol. Anal.* **1** (2009), no. 1, p. 87-100.

Manuscrit reçu le 9 mars 2021,
révisé le 13 janvier 2024,
accepté le 21 mars 2024.

Mikael DE LA SALLE
Universite Claude Bernard Lyon 1,
CNRS, Ecole Centrale de Lyon, INSA Lyon,
Université Jean Monnet
I CJ UMR5208
69622 Villeurbanne, France
delasalle@math.univ-lyon1.fr