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Homoclinic bifurcation in Morse–Novikov theory, a doubling phenomenon

HOMOCLINIC BIFURCATION IN MORSE–NOVIKOV THEORY, A DOUBLING PHENOMENON

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ABSTRACT. — We consider a compact manifold of dimension greater than 2 with a differential form of degree one which is closed but non-exact. This form, viewed as a multi-valued real function has a gradient vector field with respect to any Riemannian metric. After S. Novikov's work and a complement by J.-C. Sikorav, under some genericness assumptions, this data yields a complex, called today the Morse–Novikov complex. Due to the non-exactness of the form, its gradient may have a homoclinic orbit. The one-form being fixed, we investigate the codimension-one stratum, in the space of gradients, formed by those having only one *simple* homoclinic orbit. The crossing of such a stratum has a subtle effect on the Morse–Novikov complex: some crossing may creates infinitely many new heteroclinic orbits; and some simple homoclinic orbit may be approached by *simple* homoclinic orbits of double energy. These latter two phenomena are linked.

RÉSUMÉ. — On considère une variété compacte de dimension plus grande que 2 munie d'une 1-forme différentielle fermée non-exacte. Cette forme, vue comme une fonction réelle multivaluée, a un champ de gradient pour toute métrique riemannienne. D'après S. Novikov et un complément de J.-C. Sikorav, sous des hypothèses convenables de généricité, il découle de ces données un complexe appelé aujourd'hui le complexe de Morse–Novikov. La non-exactitude de la 1-forme permet des orbites homoclines. Nous étudions la strate, de codimension-un dans l'espace des gradients, formée de ceux qui ont une orbite homocline *simple*. Son croisement a un effet subtil sur le complexe de Morse–Novikov : il peut créer une infinité d'orbites hétéroclines ; par ailleurs, une orbite homocline simple peut être approchée pa des orbites homoclines *simples* d'énergie double. Ces deux phénomènes sont reliés.

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1. Introduction

1.1. Morse–Novikov Theory setup

We consider a closed connected n-dimensional manifold M equipped with a closed differential form α which is of degree one and of Morse-type, meaning that its zeroes are non-degenerate. In other words, the local primitives f_{loc} of this 1-form are Morse functions. Morse–Novikov Theory deals with the case where α is non-exact, that is, the cohomology class u of α is non-zero in $H^1(M; \mathbb{R})$. The set of zeroes of α will be denoted $Z(\alpha)$; each zero $p \in Z(\alpha)$ has a Morse index $i(p) \in \{0, \ldots, n\}$. The set of zeroes of index k is denoted by $Z_k(\alpha)$.

Since the indeterminacy of f_{loc} is just an additive constant, for any Riemannian metric the *descending* or *negative* gradient $-\nabla f_{loc}$ is globally defined. Such a vector field X will be said to be an *α-gradient*. The zeroes of X coincide with the zeroes of α and are hyperbolic. Therefore, each zero p of α has a stable manifold $W^s(p, X)$ and an unstable manifold $W^u(p, X)$. Both are manifolds which are injectively immersed[1] in M; the unstable (resp. stable) manifold is diffeomorphic to $\mathbb{R}^{i(p)}$ (resp. $\mathbb{R}^{n-i(p)}$). In the present article, we will point out some dynamical particularities of gradients of *multivalued Morse functions*, according to the terminology introduced by S. Novikov for speaking of Morse closed 1-forms [8].

By Kupka–Smale's theorem [10, 9], generically among the α-gradients the invariant manifolds $W^u(p, X)$ and $W^s(q, X)$ are mutually transverse for every $p, q \in Z(\alpha)$. Note that this property is not open in general. A vector field whose zeroes are hyperbolic and which fulfils this transversality property will be named a *Kupka–Smale* vector field in what follows though the classical definition is more restrictive;[2] for brevity, we shall speak of KS vector fields. In that case, if an orbit of X is a *connecting orbit* going from p to q then, as in Morse Theory, we have

$$i(p) > i(q).$$

In particular, as $p \neq q$ such an orbit is heteroclinic.

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[1] When α is exact (case of Morse Theory), the stable and unstable maniolds are embedded.

[2] In the literature, a vector field is said to be Kupka–Smale if the periodic orbits are all hyperbolic and their center-stable and center-unstable manifolds are mutually transverse.

1.2. A key fact

Let us define the α-*length* of an α-gradient orbit ℓ by

$$\mathcal{L}(\ell) := -\int_\ell \alpha.$$

This positive number is nothing but the Riemannian *energy* of ℓ. It may be infinite which is the case for almost every orbit when the local primitive has no critical points of extremal Morse index. Here is a key fact which makes Morse–Novikov gradient dynamics very special; its proof, actually elementary, will be given in Appendix A (also [3, Proposition 2.8]).

> Assume X is KS. Let $p \in Z_k(\alpha)$ and $q \in Z_{k-1}(\alpha)$. Then, for every $L > 0$, the number of connecting orbits from p to q whose α-length is bounded by L is finite.

This fact might be the basic observation when Novikov discovered his famous *Novikov ring*. It gives an algebraic "counting" of the gradient orbits which connect two zeroes of α when the difference of their Morse indices is exactly one. We are going to sketch how this counting is carried out. This will be relevant for dynamical information.

1.3. Introduction to the Morse–Novikov complex

In our paper, the Morse–Novikov complex is just a tool for *encoding* a part of the dynamics of α-gradients: namely, the count of orbits connecting two zeroes whose Morse indices differ by one. We adopt a point of view which is due to J.-C. Sikorav [12], namely, the *universal* Novikov ring Λ_u is some *completion* of the group ring $\mathbb{Z}[\pi_1(M, *)]$ of the fundamental group associated with the cohomology class $[\alpha] = u$ (see Novikov Condition (3.1)).

The Morse–Novikov complex $NC_*(\alpha, X)$ is only defined when the α-gradient X is Kupka–Smale. As a graded module, in degree k is the free module generated by $Z_k(\alpha)$ over the ring Λ_u. Each unstable manifold $W^u(p, X)$ is oriented arbitrarily. The differential $\partial^X : NC_k(\alpha, X) \to NC_{k-1}(\alpha, X)$ has the following form on a generator $p \in Z_k(\alpha)$:

$$\partial^X p = \sum_{q \in Z_{k-1}(\alpha)} \left(\sum_{\gamma \in \Gamma_p^q} n_\gamma(p, q) \gamma \right) q.$$

Here, $n_\gamma(p, q)$ is a relative integer, Γ_p^q denotes the set of homotopy classes of paths from p to q. As the unstable manifolds are oriented, and hence, the stable manifolds are co-oriented, each connecting orbit from p to q

carries a sign (once some conventions are fixed). The integer $n_\gamma(p,q)$ is the total number of the signs carried by the connecting orbits in the homotopy class γ. If $n_\gamma(p,q) \neq 0$, there is at least one connecting orbit from p to q in the class γ. One checks that $\partial^X \circ \partial^X = 0$ which justifies the term "complex" in the algebraic sense.

By Stokes' formula, all connecting orbits in a given homotopy class γ have the same α-length. Thus, by the *key fact*, $n_\gamma(p,q)$ is finite. Moreover, by definition of the Novikov ring Λ_u (see [1] or (3.1)), the *incidence coefficient*

$$(1.1) \qquad n(p,q) := \sum_{\gamma \in \Gamma_p^q} n_\gamma(p,q)\gamma$$

is an element of Λ_u, and hence, ∂^X is a morphism of graded Λ_u-rings of degree -1. Note the following: if $n(p,q)$ is an infinite series then there is a sequence (γ_j) of homotopy classes in Γ_p^q whose α-lengths go to $+\infty$ and such that each γ_j contains at least one connecting orbit.

1.4. Homoclinic bifurcation

The complement of the set of KS gradients is stratified thanks to a measure of the default to be a Kupka–Smale vector field. Here, we list the strata[3] of "codimension one":

(1) There is a unique pair (p,q) with $i(p) = i(q)+1$ and a unique X-orbit from p to q along which $W^u(p,X)$ shows a minimal transversality defect to $W^s(q,X)$. Crossing this stratum consists of either creating or cancelling a pair of orbits of opposite sign. Such a pair does not appear in the counting that we have explained in Subsection 1.3. So, we neglect the strata of this type.

(2) There is a unique pair (p,p'), with $i(p) = i(p')$ and $p \neq p'$, and a unique connecting orbit from p to p'. Crossing such a stratum consists of making a *handle slide* in the sense of Morse Theory. This is out of the scope of our paper.

(3) There is a unique orbit γ, *broken*[4] or not, from p to itself. In this paper, we only study the case where γ is *simple*[5] and *non-broken*.

[3] At the beginning, the stratification in question is just a collection of disjoint subsets of the space of gradients we are considering. Theorem 1.1 gives some of them the status of genuine *strata*.

[4] A broken orbit is the concatenation of orbits, one being linked to the next one by some zero.

[5] 'Simple' in the sense of multiplicity (see Subsection (2.2)).

This bifurcation will be named *homoclinic bifurcation*; it was also considered by M. Hutchings [2].

If the gradient X belongs to the stratum $\mathcal{S}$ under consideration in (3), the unique homoclinic orbit ℓ of X forms a loop with the zero $p \in Z(\alpha)$ as a base point. As previously said, the α-length is positive and by Stokes' formula, the loop ℓ is not homotopic to zero. Let g denote its homotopy class in $\pi_1(M,p)$ and let denote by $\mathcal{S}_g \subset \mathcal{S}$ the stratum made of the α-gradients whose homoclinic orbit belongs to the homotopy class g. Actually, $\mathcal{S}$ is the disjoint union $\coprod_{g \in \pi_1(M,p)} \mathcal{S}_g$.

From now on, we are going to impose restrictions on the underlying Riemannian metric: we require that, for any considered metric, the α-gradient is *adapted*, meaning that this vector field is linearizable at every zero $p \in Z(\alpha)$ with a spectrum in $\{-1,1\}$ (See Definition 2.1). Up to rescaling, this spectrum condition means that the linearizing coordinates about p make the α-gradient lines *radial* in both the local stable and unstable manifolds. Let $\mathcal{F}_\alpha$ denote the space of adapted α-gradients.

This constraint on the spectrum (not at all generic) is more informative than a metric yielding a *non-resonant* spectrum.[6] Our constraint, for which Kupka–Smale's Theorem still applies (among the adapted α-gradients with given germs), *enriches* the holonomy of ℓ so much that there is a well-defined real function $\chi : \mathcal{S}_g \to \mathbb{R}$ which depends only on the linearized holonomy of ℓ from p to itself and is continuous. This function will be constructed in Section 2) and will play the main role in our paper. We name χ the *character function*. We have the following statement.

THEOREM 1.1.

(1) *The stratum $\mathcal{S}_g$ is a codimension-one, co-oriented submanifold of $\mathcal{F}_\alpha$ of class C^∞.*
(2) *Assume $n > 2$ and $\mathcal{S}_g \neq \emptyset$. Then, the vanishing locus $\mathcal{S}_g^0$ of χ is a non-empty co-oriented codimension-one submanifold of class C^∞ in $\mathcal{S}_g$ meeting each of its connected components.*

Set $\mathcal{S}_g^+ := \{X \in \mathcal{S}_g \mid \chi(X) > 0\}$ and $\mathcal{S}_g^- := \{X \in \mathcal{S}_g \mid \chi(X) < 0\}$. We shall see that crossing $\mathcal{S}_g$ positively through $\mathcal{S}_g^+$ or $\mathcal{S}_g^-$ changes the Morse–Novikov complex in a completely different manner.

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[6] This brings us back to a principle that R. Thom strongly defended in the seventies: a non-generic object is richer than one of its generic approximations since it contains the information of its *universal unfolding*.
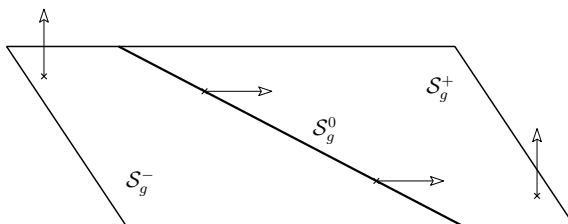
Figure 1.1. Local situation in $\mathcal{F}_\alpha$ of every connected component of $\mathcal{S}_g$ near $\mathcal{S}_g^0$. The arrows indicate the co-orientation of $\mathcal{S}_g$ in $\mathcal{F}_\alpha$ and $\mathcal{S}_g^0$ in $\mathcal{S}_g$.

1.5. Bifurcation by crossing $\mathcal{S}_g$

Since $\mathcal{S}_g$ is co-oriented, we can study the generic one-parameter families $(X_s)_{s \in \mathcal{O}p(0)}$ which intersect $\mathcal{S}_g$ positively at X_0; here, we use Gromov's notation: $\mathcal{O}p(0)$ stands for an open interval which contains 0 and whose size is chosen as small as desired[7] In other words, the path $(X_s)_{s \in \mathcal{O}p(0)}$ has to be thought of as a germ of path at X_0.

We recall p, the zero of α which is involved in the homoclinic orbit of X_0. Let $q \in Z(\alpha)$ be any zero whose Morse index satisfies $i(q) = i(p) - 1$.

The next theorem requires some genericity assumption, namely the property for $X \in \mathcal{S}_g$ to be *almost Kupka–Smale* (Definition 3.5). The corresponding residual set is denoted by $\mathcal{S}_{g,\infty}$. Let $X_0 \in \mathcal{S}_{g,\infty}$. Take any $L > 0$. By Proposition 3.6, for every $s \neq 0$ close enough to 0 the vector field X_s is *Kupka–Smale up to L*, that is, the algebraic number of connecting orbits from p to q with α-length smaller than L is finite and locally constant. Denote it by $n(p,q)_L^-$ or $n(p,q)_L^+$ depending on the sign of s; these numbers are called *truncated incidence coefficients*.

Theorem 1.2 states how these truncated incidence coefficients change through crossing $\mathcal{S}_g$.[8] More explanation of the formulas will be given just after the statement.

THEOREM 1.2. — *Let $(X_s)_{s \in \mathcal{O}p(0)}$ be a path crossing $\mathcal{S}_g$ positively at time $s = 0$. If $X_0 \in \mathcal{S}_{g,\infty}$ the following holds for every $L > 0$:*

(1) *when X_0 belongs to $\mathcal{S}_g^-$, then $n(p,q)_L^+ = (1+g) \cdot n(p,q)_L^-$ (mod. L),*

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[7] More generally, if A is a closed subset of B, $\mathcal{O}p(A)$ stands for an open neighborhood of A in B which is not specified.

[8] This makes more precise the statement of [7, Proposition 2.2.36] whose proof was inaccurate.

(2) when X_0 belongs to $\mathcal{S}_g^+$, then $n(p,q)_L^+ = (1 + g + g^2 + \cdots) \cdot n(p,q)_L^-$ (mod. L).

Of course, in order to keep the squared differential equal to zero there are similar formulas for the change of the incidence $n(q',p)_L^{\pm}$ when Morse indices satisfy $i(q') = i(p) + 1$. More precisely, we have:

(3) when $X_0 \in \mathcal{S}_g^-$, then $n(q',p)_L^+ = n(q',p)_L^- \cdot (1 - g + g^2 - g^3 + \cdots)$ (mod. L),

(4) when $X_0 \in \mathcal{S}_g^+$, then $n(q',p)_L^+ = n(q',p)_L^- \cdot (1 - g)$ (mod. L).

The explanation for the product in Formulas (1) and (2) goes as follows (and similarly for (3) and (4)). On the one hand, recall Formula (1.1); namely, $n(p,q) = \sum_{\gamma} n_{\gamma}(p,q)\gamma$ where $\gamma \in \Gamma_p^q$ is a homotopy class of paths from p to q and $n_{\gamma}(p,q)$ is a relative integer which gives the total algebraic number of connecting orbits in that class. On the other hand g is a homotopy class of loops based at p. So, the concatenation $g^j \cdot \gamma$ makes sense and yields a new element in Γ_p^q. This product is distributive with respect to the sum.

Remark 1.3. — We have examples described in [5] where the homoclinic bifurcation is isolated. In that case, truncation by L becomes useless. There are a pair p, q of zeroes with $i(p) = i(q) + 1$ and a one-parameter family of adapted α-gradients $(X_s)_{s \in [-\varepsilon, +\varepsilon]}$ crossing $\mathcal{S}_g$ positively such that:

- for every $s < 0$ there is a unique heteroclinic orbit ℓ from p to q;
- for every $s > 0$ there are infinitely many heteroclinic orbits from p to q.

This infinity which appears at once reads as follows: one heteroclinic orbit in each homotopy class $g^j \cdot [\ell]$, for $j = 0, 1, 2, \ldots$; here, [-] stands for the homotopy class.

1.6. The doubling phenomenon

This relates the strata $\mathcal{S}_g$ and $\mathcal{S}_{g^2}$. Look for instance at the above-mentioned example and consider a small generic loop in $\mathcal{F}_{\alpha}$ going around the codimension-two stratum $\mathcal{S}_g^0$, beginning by crossing $\mathcal{S}_g^-$ positively and returning by crossing $\mathcal{S}_g^+$ negatively. Then the cumulative factor after one turn is $(1 - g^2)$ while it should be equal to 1; this is a contradiction. The next theorem solves this contradiction. We emphasize that its statement does not need the presence of any other zero of α than the base point of the homoclinic orbit.

THEOREM 1.4. — *Again, assume $n > 2$ and $\mathcal{S}_g \neq \emptyset$. Then, there exists a codimension-two stratum $\mathcal{S}_g^{0,0}$ in $\mathcal{S}_g$, contained in $\mathcal{S}_g^0$, such that $\mathcal{S}_g^0 \setminus \mathcal{S}_g^{0,0}$ adheres to $\mathcal{S}_{g^2}$ as a boundary[9] of class C^1, more precisely of its positive part $\mathcal{S}_{g^2}^+$.*

In dynamical language, if X is an adapted α-gradient in $\mathcal{S}_g^0$ (that is, $\chi(X) = 0$) whose homoclinic orbit is ℓ, then X can be approximated by an adapted α-gradient X' having a unique homoclinic orbit ℓ' turning twice along ℓ. In particular, $[\ell'] = [\ell]^2$ in $\pi_1(M, p)$.

The precise definition of $\mathcal{S}_g^{0,0}$ yields a decomposition $\mathcal{S}_g^0 \setminus S_g^{0,0} = \mathcal{S}_g^{0,-} \sqcup \mathcal{S}_g^{0,+}$ (see Definition 4.1). Locally along $\mathcal{S}_g^0 \setminus \mathcal{S}_g^{0,0}$, the stratum $\mathcal{S}_{g^2}$ approaches from one side of $\mathcal{S}_g$ only. As a matter of fact, $\mathcal{S}_{g^2}$ approaches $\mathcal{S}_g^{0,+}$ (resp. $\mathcal{S}_g^{0,-}$) from the positive (resp. negative side) of the co-oriented stratum $\mathcal{S}_g$. A precise statement about the latter facts is given in Theorem 4.2 and may be illustrated by Figure 1.2.
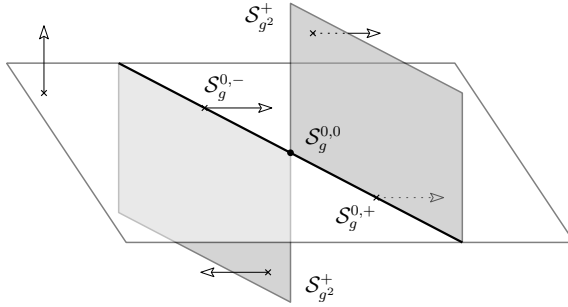
Figure 1.2. The stratum $\mathcal{S}_g^0 \setminus \mathcal{S}_g^{0,0}$ as a boundary of $\mathcal{S}_{g^2}$ which is grayed.

Remarks 1.5.

(1) Our doubling phenomenon evokes the period doubling bifurcation, also called Andronov–Hopf's bifurcation. With this aim, it would be good to know that crossing $\mathcal{S}_g$ creates (or destroys) a periodic orbit in the free homotopy class g. Shilnikov's theorem [11] deals with this question. Unfortunately, it is not applicable here because we are going to use very non-generic Morse charts (only $+1$ and -1 as eigenvalues of the Hessian at critical points). In counterpart, such charts offer very nice advantages.

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[9] This does not contradict the fact that, by their very definition, $\mathcal{S}_g$ and $\mathcal{S}_{g^2}$ are disjoint.

(2) We were asked the question whether our results depend on the assumption that the α-gradient X is *adapted* in the sense of Definition 2.1. Most probably, if this assumption is not fulfilled, there is no way to define something which gives so much information as the *character function*. In this case, all of the three above-stated theorems disappear.[10]

(3) We were also suggested to find a more general setting for our results. For instance, one could fix a Riemannian metric and look at vector fields with given hyperbolic zeroes. In this setting, the key fact 1.2 still holds true if the α-length is replaced with the Riemannian energy. Unfortunately, there is no natural stratification of the complement of Kupka–Smale vector fields. Namely, there is no "grading" of the homoclinic orbits based at a zero of the vector field under consideration. For instance, one faces, in general, a defect of equicontinuity of sequences of the homoclinic orbits in a given homotopy class. Therefore, we do not see any natural generalization of our study.

Acknowledgements

2. Homoclinic bifurcation, orientation and character

We focuse on homoclinic bifurcations even though some of the statements hold true for other bifurcations (see the list in Subsection 1.4). We consider an α-gradient X with a simple homoclinic orbit ℓ based at some zero $p \in Z(\alpha)$ in the homotopy class $g \in \pi_1(M, p)$. In this section, we are going to show that if the Morse coordinates around p are *simple* in the sense of Definition 2.1 then the Morse model $\mathcal{M}_p$ in these coordinates allows us to enrich the holonomy of ℓ with some specific information. From this latter we deduce the *character function* which is the key new tool of the paper. Finally, we prove Theorem 1.1.

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[10] This confirms Thom's principle which we have mentioned in Footnote 6.

Definition 2.1.

(1) *For each zero p of α of index $i(p) = i$, simple Morse coordinates around p are coordinates where the form α is equal to the differential of the standard quadratic form*

$$Q_i := \frac{1}{2}\left[-x_1^2 + \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_n^2\right].$$

(2) *An α-gradient X is said to be* adapted *if for every $p \in Z(\alpha)$ there are simple Morse coordinates around p such that X coincides with the standard descending gradient X_i of Q_i, where $i = i(p)$, that is:*

$$X_i = \sum_{1}^{i} x_k \partial_{x_k} - \sum_{i+1}^{n} x_k \partial_{x_k}.$$

Such Morse coordinates are also said to be adapted *to X.*

The property for X to be adapted depends only on the germ of X near $Z(\alpha)$. We recall that, for simplicity, we fix the germ of adapted α-gradients once and for all at every zero of α; the set of such adapted α-gradients is denoted by $\mathcal{F}_\alpha$.

Remark 2.2. — The simplicial group $\mathcal{G} := \mathrm{Diff}(Q_i)$ of germs of diffeomorphisms of $(\mathbb{R}^n, 0)$ preserving Q_i retracts by deformation to $O(i, n-i)$, the linear group of isometries of Q_i. Indeed, if $\varphi \in \mathcal{G}$, the Alexander isotopy $\varphi_t : x \mapsto \frac{1}{t}\varphi(tx)$ is made of elements in $\mathcal{G}$ for every $t \in (0,1]$ and tends to the derivative $\varphi'(0)x$ as t goes to 0. Moreover, $O(i, n-i)$ retracts by deformation to its maximal compact subgroup $G := O(i) \times O(n-i)$ which is the isometry group of the pair (Q_i, X_i).

As a consequence, the space of germs of adapted α-gradients is made of a unique element up to the action of $\mathrm{Diff}_0(M, \alpha)$, the neutral component in the group of diffeomorphisms preserving α. For this reason, without loss of generality, we may fix for every $p \in Z(\alpha)$ the germ at p of all adapted α-gradients considered in what follows. This choice will be done for all bifurcation families in Sections 3 and 4.

2.1. Morse model

Given $p \in Z(\alpha)$ of Morse index i, a *Morse model* $\mathcal{M}_p \subset M$ with positive parameters (δ, δ^*) (which we do not make explicit in notation) is diffeomorphic to the subset of $\mathbb{R}^i \times \mathbb{R}^{n-i}$ made of pairs (x^-, x^+) such that $Q_i(x^-, x^+) \in [-\delta^*, +\delta^*]$, $|x^-|^2|x^+|^2 \leqslant \delta\delta^*$ and $\alpha|_{\mathcal{M}_p} = dQ_i$. The bottom

of $\mathcal{M}_p$, that is its intersection with $\{Q_i = -\delta^*\}$ is denoted by $\partial^- \mathcal{M}_p$; similarly, the top is denoted by $\partial^+ \mathcal{M}_p$. The rest of the boundary of $\mathcal{M}_p$ is denoted by $\partial^\ell \mathcal{M}_p$ and X_i is tangent to it. Note that:

- the group G preserves $\mathcal{M}_p$ for every parameters (δ, δ^*);
- the set of Morse models, as compact subsets of $\mathbb{R}^n$, is contractible.

The flow of X_i is denoted by $(X_i^t)_{t \in \mathbb{R}}$. The *local unstable* (resp. *local stable*) manifold is formed by the points $x \in \mathcal{M}_p$ whose negative (resp. positive) flow line $X_i^t(x)$ goes to p when t goes to $-\infty$ (resp. $+\infty$) without getting out of $\mathcal{M}_p$. Denote by Σ^- the $(i-1)$-sphere which is formed by the points in the bottom of $\mathcal{M}_p$ which belong to $W_{loc}^u(p, X_i)$; that is

$$\Sigma^- = \left\{ (x^-, x^+) \,\middle|\, |x^-|^2 = 2\delta^*, |x^+| = 0 \right\}.$$

This is called the *attaching sphere*. Similarly, Σ^+ denotes the *co-sphere*, the $(n - i - 1)$-sphere which is contained in the top of $\mathcal{M}_p$ and made of points belonging to $W_{loc}^s(p, X_i)$, that is

$$\Sigma^+ = \left\{ (x^-, x^+) \,\middle|\, |x^-| = 0, |x^+|^2 = 2\delta^* \right\}.$$

We will use the two projections associated with these coordinates:

$$\pi^+ : \partial^+ \mathcal{M}_p \longrightarrow \Sigma^+ \quad \text{and} \quad \pi^- : \partial^- \mathcal{M}_p \longrightarrow \Sigma^-.$$

2.2. Simple homoclinic orbit, tube and orientation.

Let X be an adapted α-gradient and let $\mathcal{M}_p$ be a Morse model adapted to X around $p \in Z(\alpha)$. A homoclinic orbit ℓ of X based at p is said to be *simple* when, at any point $m \in \ell$, the span $T_m W^u(p, X) + T_m W^s(p, X)$ is of codimension one in $T_m M$. When X is said to have a *unique* homoclinic orbit it will be meant that this orbit is simple, that is, unique with multiplicity.

In this setting, denote by $\underline{\ell}$ the closure of $\ell \smallsetminus \mathcal{M}_p$; it will be named the *restricted* homoclinic orbit. The end points of $\underline{\ell}$ are denoted respectively $a^- \in \Sigma^-$ and $a^+ \in \Sigma^+$. We also introduce a *compact tube* T around $\underline{\ell}$ made of X-trajectories from $\partial^- \mathcal{M}_p$ to $\partial^+ \mathcal{M}_p$. As ℓ is simple, if the tube is small enough there are coordinates on T that we note $(x, y, v, z) \in \mathbb{R}^{i-1} \times \mathbb{R}^{n-i-1} \times [-1, 1] \times [0, 1]$ with the following properties:

- X is positively colinear to ∂_z,
- $\{z = 0\} = T \cap \partial^- \mathcal{M}_p$ and $\{z = 1\} = T \cap \partial^+ \mathcal{M}_p$;
- $T \cap \Sigma^- = \{y = 0, v = 0, z = 0\}$ and $T \cap \Sigma^+ = \{x = 0, v = 0, z = 1\}$;
- $\underline{\ell} = \{x = 0, y = 0, v = 0\}$;
- the frame $(\partial_x, \partial_y, \partial_v)$ is tangent to the leaves of α.

In what follows, $\{z = 0\}$ (resp. $\{z = 1\}$) will stand for $T \cap \partial^- \mathcal{M}_p$ (resp. $T \cap \partial^+ \mathcal{M}_p$).

Orient the unstable $W^u(p, X)$. Thus, the stable manifold $W^s(p, X)$ is co-oriented. Therefore, we can choose the coordinate v in the tube so that, for every $z_0 \in [0, 1]$, the following holds:

$$(2.1) \qquad \partial_v \wedge \text{or} \left(W^u(p, X) \cap \{z = z_0\} \right) = \text{co-or}(W^s(p, X)).$$

If the orientation of $W^u(p, X)$ is changed, then the co-orientation of $W^s(p, X)$ is also changed and the above equation shows that the positive direction of v remains unchanged.

Remark 2.3. — It is important to notice that (2.1) tells us nothing about the holonomy along $\underline{\ell}$ of the foliation defined by X (see the next subsection). Therefore, for a given $\partial_v \in T_{a^+}(\partial^+ \mathcal{M}_p)$, the tangent vector $\partial_v \in T_{a^-}(\partial^- \mathcal{M}_p)$ may have any position not contained in the hyperplane $\mathbb{R}\{\partial_x, \partial_y\}$, depending on X.

2.3. Holonomy and perturbed holonomy

The foliation of $M \smallsetminus Z(\alpha)$ by the orbits of X together with its two transversals $\partial^\pm \mathcal{M}_p$ defines a *holonomy* diffeomorphism $H_X : \mathcal{N}_X^- \to \mathcal{N}_X^+$, that is:

- $\mathcal{N}_X^-$ is an open connected neighborhood of $\{z = 0\}$ in $\partial^- \mathcal{M}_p$;
- $\{z = 1\} \subset \mathcal{N}_X^+ \subset \partial^+ \mathcal{M}_p$;
- the restriction of H_X to $\{z = 0\}$ is defined by $(x, y, v, 0) \mapsto (x, y, v, 1)$;
- for every $a \in \mathcal{N}_X^-$, the image $H_X(a)$ belongs to the X-orbit of a.

The time of the flow X^t for going from a to $H_X(a)$ is smooth.

The existence of such holonomy diffeomorphism is an open property with respect to X. More precisely, if X' is a close enough approximation of X in the C^1-topology, there is a *perturbed holonomy* diffeomorphism $H_{X'}$ from an open neighborhood $\mathcal{N}_{X'}^-$ of $\{z = 0\}$ in $\partial^- \mathcal{M}_p$ to an open neighborhood $\mathcal{N}_{X'}^+$ of $\{z = 1\}$ in $\partial^+ \mathcal{M}_p$.

Remark 2.4. — It makes sense to speak of $H_{X'}^{-1}(\Sigma^+) \cap \{z = 0\}$. It is an $(n - i - 1)$-disc C^1-close to the y-axis in $\{z = 0\}$. Similarly, it makes sense to speak of $H_{X'}(\Sigma^-) \cap \{z = 1\}$. It is an $(i - 1)$-disc close to the x-axis in $\{z = 1\}$.

We now state and prove the first item of Theorem 1.1. Let $p \in Z(\alpha)$. For every $g \in \pi_1(M, p)$, we consider $\mathcal{S}_g \subset \mathcal{F}_\alpha$, the set of adapted α-gradients

that have a unique homoclinic orbit forming a loop based at p in the homotopy class g. The existence of a broken homoclinic orbit is excluded from $\mathcal{S}_g$. Recall u, the cohomology class of the closed form α; if the evaluation $u(g)$ is non-negative then $\mathcal{S}_g$ is empty.

PROPOSITION 2.5. — *For every $g \in \pi_1(M, p)$, the subset $\mathcal{S}_g \subset \mathcal{F}_\alpha$ is a C^∞ codimension-one submanifold of $\mathcal{F}_\alpha$, that is, $\mathcal{S}_g$ is locally defined by a regular real valued equation. This stratum has a canonical co-orientation according to formula (2.1).*

Proof. — Let X_0 be any point in $\mathcal{S}_g$. Let ℓ denote the homoclinic orbit which forms a loop in the class $g \in \pi_1(M, p)$. We intend to find a regular real valued smooth equation for $\mathcal{S}_g$ near X_0. From Remark 2.2 we have the two following properties:

- the local C^∞ stability near p of the adapted α-gradients ;
- the acyclicity of the space of Morse models adapted to X_0 near p.

Therefore, the action of the group $\mathrm{Diff}_0(M, \alpha)$ on $\mathcal{F}_\alpha$ reduces us to consider a local *slice* $S \subset \mathcal{F}_\alpha$ for this action and to look for the smoothness of $\mathcal{S}_g \cap S$. Namely, choose a Morse model $\mathcal{M}_p$ adapted to X_0 and define $S := \{X \in \mathcal{F}_\alpha \mid X = X_0 \text{ in } \mathcal{M}_p\}$. Thanks to the above-mentioned stability property, this S is indeed a local slice for the action of $\mathrm{Diff}_0(M, \alpha)$.

We use the tube T and its coordinates as introduced in Subsection 2.2. The Implicit Function Theorem allows us to follow C^∞-continuously, for X close to X_0, a connected component $D(X)$ of $W^u(p, X) \cap \{z = 1\}$ which coincides with $\{y = 0, v = 0, z = 1\}$ when $X = X_0$. Let

$$p_v : \{z = 1\} \longrightarrow \{v = 0, z = 1\}$$

denote the projection parallel to ∂_v onto the (x, y)-space. The image $p_v(D(X))$ is transverse to Σ^+ in $\{v = 0, z = 1\}$ by C^∞-closeness to $p_v(D(X_0))$. The intersection is a point $a^+(X)$ which depends C^∞ on X. Let $b(X)$ be the point of $D(X)$ which has the same coordinates as $a^+(X)$ except the coordinate v. Thus, the desired equation is

$$(2.2) \qquad\qquad v(b(X)) = 0.$$

This is clearly a C^∞ equation. To prove that equation is *regular* it is sufficient to exhibit a one-parameter family $(X_s)_{s \in \mathcal{O}p(0)}$ passing through X_0 and satisfying the following inequality:

$$(2.3) \qquad\qquad \partial_s \left(v(b(X_s)) \right)_{|s=0} > 0.$$

This is easy to perform by taking

$$X_s = X_0 + s\, h(z, v)\, \partial_v$$

where h is a small non-negative function, supported in the interior of the tube T associated with X_0, which has a positive integral along $\underline{\ell}$. Let us check that such (X_s) fulfils (2.3). Indeed, at every point of the tube T the vector X_s is in span$\{\partial_v, \partial_z\}$. Therefore, if H_s denotes the perturbed holonomy along $\underline{\ell}$ of the flow of X_s then $b(X_s) = H_s(a^-)$ and $v(b(X_s)) = s \int_0^1 h \, dz$. Hence, (2.3) is fulfilled.

Right after (2.1) we noticed that the positive direction of v does not depend on the chosen orientation of the unstable manifolds. Therefore, (2.3) defines a canonical co-orientation of $\mathcal{S}_g$.

What we have done is not sufficient for proving the statement. Equation (2.2) solves only the question of existence of a homoclinic orbit at p close to ℓ for X close to X_0 in $\mathcal{S}_g$. We have still to prove that $\mathcal{S}_g$ does not accumulate to itself[11] near X_0. More precisely, there does not exist a sequence $X_k \in \mathcal{S}_g$ converging to X_0 (in C^0-topology) such that

$$(2.4) \qquad v(b(X_k)) \neq 0 \text{ for every large enough } k.$$

Assume such a sequence exists. Let ℓ_k be the unique homoclinic orbit of X_k based at p. As the sequence (X_k) is close to X_0, the C^0-norm of X_k is uniformly bounded and then the family (ℓ_k) is equicontinuous. By Ascoli's Theorem, there is a sub-sequence, still denoted by (ℓ_k), that C^0-converges to some line ℓ_∞ from p to p, possibly distinct from ℓ.

Let us show that ℓ_∞ is a (possibly broken) homoclinic orbit of X_0. To this end, consider $x_\infty \in \ell_\infty$ and a sequence of points $x_k \in \ell_k$ which converges to x_∞. Let X_k^t be the flow of X_k. For every given t, the sequence $(X_k^t(x_k))$ converges to $X_0^t(x_\infty)$; for a given $t_0 \in \mathbb{R}$, this convergence is uniform for $t \in [0, t_0]$. In particular, the piece of ℓ_∞ between x_∞ and $X_0^t(x_\infty)$ is contained in the X_0-orbit of x_∞.

If x_k is close enough to ℓ, then the restricted homoclinic orbit $\underline{\ell}_k$ lies in the tube T. Moreover, by choosing the above-mentioned t_0 conveniently (negative and positive), we get that $a^-(X_k) \in \Sigma^-$ and $a^+(X_k) \in \Sigma^+$. Hence, $v(b(X_k)) = 0$ follows and contradicting our assumption (2.4). Therefore, the sequence (x_k) does not approach ℓ and there exists a small tube T' around ℓ such that ℓ_k avoids T' for every k and the C^0-limit ℓ_∞ as well. Finally, we get two distinct homoclinic orbits of X_0 based at p, one of them being possibly broken. This is excluded by the very definition of $\mathcal{S}_g$. $\square$

DEFINITION 2.6. — *Let $(X_s)_{s \in \mathcal{O}_{p(0)}}$ be a one-parameter family of adapted α-gradients with $X_0 \in \mathcal{S}_g$. This family is said to be* positively transverse *to the stratum S_g if it satisfies (2.3).*

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[11] as does a leaf of the irrational linear foliation on the 2-torus,

Let $(X_s)_{s \in \mathcal{O}p(0)}$ be such a one-parameter family and let H_s be the perturbed holonomy along ℓ of the flow of X_s. Below, we use the coordinates (x, y, v) both in $\{z = 0\}$ and in $\{z = 1\}$.

For further use, we are interested in the local solution x_s of the equation

$$(x \circ H_s)(x, 0, 0) = 0$$

which is equal to 0 when $s = 0$. In this equation, the unknown is the unique point of the x-space in $\{z = 0\}$ whose image through H_s is in the $\{y, v\}$-space of $\{z = 1\}$. And similarly, we consider the solution y_s of the equation

$$(y \circ H_s^{-1})(0, y, 0) = 0$$

which is equal to 0 when $s = 0$.

LEMMA 2.7. — *With the above data and notations, the following equality holds:*

$$\partial_s \left(v \circ H_s\right) (x_s, 0, 0)_{|s=0} + \partial_s \left(v \circ H_s^{-1}\right) (0, y_s, 0)_{|s=0} = 0 \,.$$

Proof. — The left summand is equal to

$$dv \left(\partial_s H_s(0, 0, 0)\right)_{|s=0} + dv \cdot dH_0(0, 0, 0)(\partial_s x_s)_{|s=0}$$

in which $dH_0 = \mathrm{Id}$ as $H_0(x, y, v) = (x, y, v)$ and $dv(\partial_s x_s)_{s=0}) = 0$, since the velocity of x_s is tangent to the x-space, and hence, in the kernel of dv. And similarly for the second summand. Differentiating the composed map $H_s^{-1} \circ H_s = \mathrm{Id}$ with respect to s at $s = 0$ yields:

$$\partial_s H_s^{-1}(0, 0, 0)_{|s=0} + \partial_s H_s(0, 0, 0)_{|s=0} = 0 \,.$$

Altogether, we get the desired formula. □

2.4. Equators, signed hemispheres and latitudes

We introduce some useful notations. Let $\mathbb{D}^k$, $k \geqslant 1$, be the closed Euclidean disc of dimension k and radius 1 equipped with spherical coordinates $(r, \theta) \in [0, 1] \times \mathbb{S}^{k-1}$. A point $\theta \in \mathbb{S}^{k-1}$ will also be viewed as unit vector $\theta \in T_0\mathbb{D}^k$.

Suppose that we are given a co-oriented hyperplane $\Delta \subset T_0\mathbb{D}^k$. It determines a *preferred co-oriented equator* $E^\Delta \subset \mathbb{S}^{k-1}$. The oriented normal to Δ determines two poles on the sphere: the *North pole* ν_Δ on the positive side of $\mathbb{D}^k$ and the *South pole* σ_Δ on the negative side; and two open hemispheres of $\mathbb{S}^{k-1}$ respectively noted $\mathcal{H}^+(\mathbb{S}^{k-1})$ and $\mathcal{H}^-(\mathbb{S}^{k-1})$.

Any point $\theta \in \mathbb{S}^{k-1}$ determines an angle with respect to the North pole ν_Δ. The cosinus of this angle defines a *latitude* $\cos_\Delta : \mathbb{S}^{k-1} \to [-1, 1]$ by the scalar product formula

$$(2.5) \qquad\qquad \cos_\Delta(\theta) := \langle \nu_\Delta, \theta \rangle.$$

PROPOSITION 2.8. — *Every X in $\mathcal{S}_g$ defines a preferred latitude on both the attaching sphere Σ^- and the co-sphere Σ^+.*

For this purpose, we choose once and for all multispherical coordinates[12] $(\phi, r, \psi) \in \mathbb{S}^{i-1} \times [0, 1] \times \mathbb{S}^{n-i-1}$ on each level set of $\mathcal{M}_p$ (not well defined on the local stable/unstable manifolds). We recall the map

$$(2.6) \qquad\qquad \mathrm{Desc} : \partial^+ \mathcal{M}_p \smallsetminus \Sigma^+ \longrightarrow \partial^- \mathcal{M}_p \smallsetminus \Sigma^-$$

obtained by descending the flow lines in $\mathcal{M}_p$. This map reads Id in these coordinates as X is an adapted gradient.

The preferred latitude that we are going to define on Σ^- and Σ^+ will be called respectively the *ϕ-latitude* and the *ψ-latitude*. We insist that these functions depend on $X \in \mathcal{S}_g$. We will denote them by

$$(2.7) \qquad \cos_\phi^X : \Sigma^- \longrightarrow [-1, 1] \quad \text{and} \quad \cos_\psi^X : \Sigma^+ \longrightarrow [-1, 1].$$

When the vector field is clear from the context, these functions will just be denoted $\cos_\phi$ and $\cos_\psi$. We shall decorate all the data related to $\cos_\phi$ or $\cos_\psi$ by using the letter ϕ or ψ respectively; namely, the preferred hyperplane Δ^ϕ, the preferred equator $E^\phi \subset \Sigma^-$, the pole ν_ϕ, and so on.

Proof. — Take any X in $\mathcal{S}_g$ and denote by ℓ its unique homoclinic orbit from p to itself. The end point a^+ of $\underline{\ell}$ has coordinates $a^+ = (-, 0, \psi_0)$; as usual with polar coordinates, when the radius is 0 the spherical coordinate is not defined. Let

$$\pi^{\psi_0} : \partial^+ \mathcal{M}_p \longrightarrow \{\psi = \psi_0\}, \quad \pi^{\psi_0}(\phi, r, \psi) = (\phi, r, \psi_0),$$

be the projection onto the meridian i-disc.

Let $H : \{z = 0\} \to \{z = 1\}$ denote the holonomy diffeomorphism defined by the vector field X in the tube T. The image of $T \cap \Sigma^-$ through H is a $(i-1)$-disc $D \subset \partial^+ \mathcal{M}_p$. Due to the simplicity condition of ℓ, this disc is a graph over its projection $D_{\psi_0} := \pi^{\psi_0}(D)$ if the tube is small enough around $\underline{\ell}$. Then,

$$(2.8) \qquad \Delta^\phi := T_{a^+} D_{\psi_0} \subset T_{a^+}\{\psi = \psi_0\} \text{ is the preferred hyperplane.}$$

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[12] Speaking here of coordinates is somehow abusive and confusing since a point on a sphere is not a coordinate in the usual sense, in contrast with the latitude. It does not require any North pole.

As we noticed in Remark 2.3, the vector $\partial_v \in T_{a^+}\partial^+\mathcal{M}_p$ is neither tangent to Σ^+ nor to D, which implies that

$$\left(d\pi^{\psi_0}\right)_{a^+}(\partial_v) \text{ defines a co-orientation of } \Delta^\phi \text{ in } T_{a^+}\{\psi = \psi_0\}.$$

This provides us with a preferred latitude on the ϕ-sphere $\partial\{\psi = \psi_0\} \cong \mathbb{S}^{i-1} \times \{1\} \times \{\psi_0\}$. By the canonical isomorphism

$$(2.9) \qquad \mathbb{S}^{i-1} \times \{1\} \times \{\psi_0\} \cong \mathbb{S}^{i-1} \times \{0\} \times \{-\} \cong \Sigma^-,$$

the preferred latitude on $\partial\{\psi = \psi_0\}$ descends to some ϕ-latitude which defines the announced $\cos_\phi^X : \Sigma^- \to [-1,1]$ in (2.7). The ϕ-equator is the locus $\{\cos_\phi^X = 0\}$, the North pole is $\nu_\phi = (\cos_\phi^X)^{-1}(1)$.

For the ψ-latitude on the co-sphere Σ^+, we do the same construction by using the reversed flow and its holonomy H^{-1}. More precisely, take the image of $T \cap \Sigma^+$ through H^{-1}; it is a $(n-i-1)$-disc $D' \subset \partial^-\mathcal{M}_p$ centered in a^- whose spherical coordinates are $a^- = (\phi_0, 0, -)$. Let

$$\pi^{\phi_0} : \partial^-\mathcal{M}_p \longrightarrow \{\phi = \phi_0\}, \quad \pi^{\phi_0}(\phi, r, \psi) = (\phi_0, r, \psi),$$

be the projection onto the meridian $(n-i)$-disc and let D_{ϕ_0} be the image $\pi^{\phi_0}(D')$. Hence,

$$\Delta^\psi := T_{a^-}D_{\phi_0} \subset T_{a^-}\{\phi = \phi_0\} \text{ is the preferred hyperplane.}$$

Moreover, $\partial_v \in T_{a^-}\partial^-\mathcal{M}_p$ is neither tangent to Σ^- nor to D', which implies that

$$(2.10) \quad \left(d\pi^{\phi_0}\right)_{a^-}(\partial_v) \text{ defines a co-orientation of } \Delta^\psi \text{ in } T_{a^-}\{\phi = \phi_0\}.$$

This yields a preferred latitude on the sphere $\partial\{\phi = \phi_0\}$, that can be induced on Σ^+ by means of the canonical isomorphism

$$\{\phi_0\} \times \{1\} \times \mathbb{S}^{n-i-1} \cong \{-\} \times \{0\} \times \mathbb{S}^{n-i-1} \cong \Sigma^+.$$

This defines the ψ-latitude $\cos_\psi^X : \Sigma^+ \to [-1, +1]$ announced in (2.7). $\qquad \square$

2.5. Holonomic factor and character function

By construction of the ψ-latitude and the ϕ-latitude, we have the following splittings:

$$(2.11) \qquad \begin{cases} T_{a^-}\{z = 0\} = T_{a^-}\Sigma^- \oplus (\mathbb{R}\nu_\psi \oplus \mathbb{R}\Delta^\psi), \\ T_{a^+}\{z = 1\} = (\mathbb{R}\Delta^\phi \oplus \mathbb{R}\nu_\phi) \oplus T_{a^+}\Sigma^+. \end{cases}$$

For $X \in \mathcal{S}_g$, we recall the homoclinic orbit ℓ whose homotopy class is $g \in \pi_1(M, p)$ and all associated objects that we introduced in Subsection 2.2:

the tube T, its coordinates (x, y, v, z) and the holonomy diffeomorphism $H : \{z = 0\} \to \{z = 1\}$. This latter reads Id in the (x, y, v)-coordinates and $H(a^-) = a^+$. We are free to choose the coordinates of the tube such that the unit tangent vector $\partial_v^1 := \partial_v \in T_{a^+}\{z = 1\}$ verifies

$$(2.12) \qquad\qquad \partial_v^1 = \nu_\phi.$$

The *linearized holonomy* $T_{a^+}H^{-1}$ maps ∂_v^1 to $\partial_v^0 := \partial_v \in T_{a^-}\{z = 0\}$. By (2.11), ∂_v^0 decomposes as

$$(2.13) \qquad \partial_v^0 = v_x + \bar\eta\, \nu_\psi + v_y, \text{ where } v_x \in T_{a^-}\Sigma^-,\ v_y \in \Delta^\psi,\ \bar\eta \in \mathbb{R}.$$

As we pointed out in Remark 2.3, the only restriction on the holonomy of X along $\underline{\ell}$ is that $\bar\eta \neq 0$. Moreover, according to (2.10), the vector ∂_v^0 defines the positive side of the preferred hyperplane Δ^ψ. As a consequence, $\bar\eta$ must be positive.

DEFINITION 2.9. — *The holonomic factor associated with X is the positive real number given by*

$$\eta(X) := \frac{1}{\bar\eta} > 0\,.$$

The following subsets of $\mathcal{S}_g$ are respectively called the ϕ-*axis* and the ψ-*axis* of $\mathcal{S}_g$:

$$(2.14)\quad \mathcal{S}_g^\phi := \{X \in \mathcal{S}_g \mid a^+(X) \in E^\psi\}\ \&\ \mathcal{S}_g^\psi := \{X \in \mathcal{S}_g \mid a^-(X) \in E^\phi\},$$

that we also call the *spherical axes*. Here, E^ψ and E^ϕ denote the respective equators; $a^+(X)$ and $a^-(X)$ stand for the extremities of the restricted orbit $\underline{\ell}$, where ℓ is the unique homoclinic orbit of X in the homotopy class g. Denote the intersection of the axes by:

$$\mathcal{S}_g^{0,0} := \mathcal{S}_g^\phi \cap \mathcal{S}_g^\psi$$

which is empty when one axis is so.

Remark 2.10. — When the Morse index of $\mathcal{S}_g$ is equal to 1, then the ϕ-equator is empty but there are still signed poles. In that case, the ϕ-latitude takes only the values $\{-1, +1\}$ and the ψ-axis is empty. When the Morse index of $\mathcal{S}_g$ equals $n - 1$, then the ψ-equator and the ϕ-axis are empty and the ψ-latitude is valued in $\{-1, +1\}$. If $n > 2$, these two events do not happen simultaneously. This is the reason for the dimension assumption in Theorem 1.1 (2).

We are now ready for defining the important notion of *character function* together with its following ingredients. In order to simplify notations,

we introduce the *extended ϕ- and ψ-latitudes* to $\mathcal{S}_g$ by setting for every $X \in \mathcal{S}_g$:

$$(2.15) \qquad \omega_\phi(X) := \cos_\phi^X(a^-(X)) \quad \text{and} \quad \omega_\psi(X) := \cos_\psi^X(a^+(X)).$$

DEFINITION 2.11. — *The character function $\chi : \mathcal{S}_g \to \mathbb{R}$ is defined by:*

$$\chi(X) := \eta(X)\,\omega_\psi(X) + \omega_\phi(X).$$

Define $\mathcal{S}_g^0$ (resp. $\mathcal{S}_g^+$, $\mathcal{S}_g^-$) as the locus where χ vanishes (resp. is positive, is negative).

By the very definition of the latitudes, it is clear that each axis intersects $\mathcal{S}_g^0$ along $\mathcal{S}_g^{0,0}$, as Figure 2.1 suggests (compare Figure 1.1).
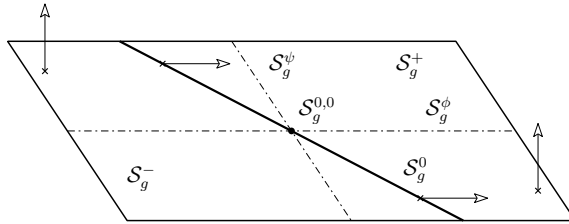

Figure 2.1. *The substratum $\mathcal{S}_g^{0,0} \subset \mathcal{S}_g^0$ as the intersection of the ϕ-axis with the ψ-axis.*

Below, we start giving some information about $\mathcal{S}_g^{0,0}$ and $\mathcal{S}_g^0$ from which Theorem 1.1 will be completely proved.

PROPOSITION 2.12.

(1) *The axes $\mathcal{S}_g^\phi$ and $\mathcal{S}_g^\psi$ are C^∞ submanifolds of codimension 1 in $\mathcal{S}_g$. Moreover, when they are both non-empty their intersection $\mathcal{S}_g^{0,0}$ is non-empty and transverse. Hence, $\mathcal{S}_g^{0,0}$ is a C^∞ submanifold of codimension 2 in $\mathcal{S}_g$.*

(2) *If $n > 2$, the zero set $\mathcal{S}_g^0 = \chi^{-1}(0)$ of the character function is a non-empty co-oriented C^∞ submanifold of codimension 1 in each connected component of $\mathcal{S}_g$.*

Proof. — Let $i = i(p)$ denote the Morse index of the zero $p \in Z(\alpha)$ where g is based.

(1). — The equation of the ψ-axis $\mathcal{S}_g^\psi$ in $\mathcal{S}_g$ reads with the notations introduced in (2.15) and (2.14):

$$\omega_\phi(X) = 0.$$

If the index i is equal to 1, by Remark 2.10, the ψ-axis is empty and there is nothing to prove. If not, let $X_0 \in \mathcal{S}_g^\psi$. We have to exhibit a germ $(X_s)_{s \in \mathcal{O}p(0)}$ of path in $\mathcal{S}_g$ passing through X_0 such that the s-derivative of $\omega_\phi(X_s)$ at $s = 0$ is non-zero. Let H_s be the perturbed holonomy diffeomorphism of X_s from a neighborhood of $\{z = 0\}$ in $\partial^- \mathcal{M}_p$ to a neighborhood of $\{z = 1\}$ in $\partial^+ \mathcal{M}_p$. Let $a^- = (\phi_0, 0, -)$ and $a^+ = (-, 0, \psi_0)$ be the end points of the restricted homoclinic orbit $\underline{\ell}$ of X_0. We arrange that H_s keeps the π^{ψ_0}-projection of $H_s(\Sigma^-)$ into the meridian $\{\psi = \psi_0\} \subset \partial^+ \mathcal{M}_p$ independent of s. Thus, the equator E^ϕ is so and the ϕ-latitude $\cos_\phi$ does not depend on s. Therefore, we are reduced to control the s-derivative of $\cos_\phi(a^-(X_s))$.

We recall that every germ of isotopy of the holonomy H_0 lifts to a deformation of X_0. Then, we are free to choose the holonomy so that $s \mapsto a^-(X_s) \in \Sigma^-$ crosses the non-empty equator E^ϕ transversely at time $s = 0$. Thus, we are done. For a similar reason, the equation $\omega_\psi(X) = 0$ of the ϕ-axis $\mathcal{S}_g^\phi$ is regular.

Let us show the property of $\mathcal{S}_g^{0,0}$ when both axes are non-empty. In that case, the Morse index verifies $1 < i < n - 1$, and we could have the simultaneous vanishing of $\cos_\phi(a^-)$ and $\cos_\psi(a^+)$, that is $\mathcal{S}_g^{0,0} \neq \emptyset$. Consider any $X_{0,0} \in \mathcal{S}_g^{0,0}$. We choose a 2-parameter family $X_{s,u} \in \mathcal{S}_g$ whose holonomy $H_{s,u}$ satisfies the following conditions:

  (a) the equator E^ϕ is independent of s when $u = 0$ and
      $\partial_s \cos_\phi(a^-(X_{s,0})) > 0$;
  (b) the equator E^ψ is independent of u when $s = 0$ and
      $\partial_u \cos_\psi(a^+(X_{0,u})) > 0$:
  (c) for every (s, u) close to $(0, 0)$, we have $a^-(X_{s,u}) \in \Sigma^-$ and
      $a^+(X_{s,u}) \in \Sigma^+$.

Condition (c) guarantees that $X_{s,u}$ runs in $\mathcal{S}_g$. Thanks to (a) and (b), the evaluation map $(s, u) \mapsto (a^-(X_{s,u}), a^+(X_{s,u})) \in \Sigma^- \times \Sigma^+$ is transverse to the submanifold $E^\phi \times E^\psi$. This proves that the system of equations defining $\mathcal{S}_g^{0,0}$ near $X_{0,0}$, namely $\{\omega_\phi(X) = 0, \ \omega_\psi(X) = 0\}$, is of rank 2.

(2). — First, let us prove that the equation $\chi(X) = 0$ has a solution in each connected component of $\mathcal{S}_g$. Let $X \in \mathcal{S}_g$ and let a^+ and a^- be the corresponding end point of its restricted homoclinic orbit $\underline{\ell}$. Any move of these points in their respective sphere lifts to a deformation of X in the space of adapted α-gradients. If Σ^+ and Σ^- are both connected, there is such a move until a^- and a^+ lie in the equator of their respective sphere. This move lifts to a deformation of X in $\mathcal{S}_g$ until it lies in $\mathcal{S}_g^{0,0} \subset \mathcal{S}_g^0$; these facts answer the question in this case.

As $n > 2$, one of the spheres Σ^- and Σ^+ is not 0-dimensional, say $\Sigma^- \neq \mathbb{S}^0$. Then, one can move X in $\mathcal{S}_g$ and modify the holonomic factor by some homothety for making it less than 1; secondly, knowing that $\omega_\psi(X) = \pm 1$, one moves a^- in Σ^- and changes X accordingly, keeping the holonomic factor constant, up to reach the locus $\chi(X) = 0$. Hence, $\mathcal{S}_g^0$ is visible in each connected component of $\mathcal{S}_g$.

It remains to prove that the equation $\chi(X) = 0$ is regular everywhere. For every $X_0 \in \mathcal{S}_g^0$ we have to exhibit a germ of path $(X_s)_s$ in $\mathcal{S}_g$ passing through X_0 such that $\partial_s \chi(X_s) > 0$. Let $\underline{\ell}$ be the restricted homoclinic orbit of X_0; let $a^+ \in \Sigma^+$ and $a^- \in \Sigma^-$ be its end points. First, we arrange that the equators E^ϕ and E^ψ do not depend on s by requiring that the holonomy H_s along the homoclinic orbit ℓ of X_0 fulfils the following conditions:

- for every s, there is a homoclinic orbit ℓ_s (the end points of the restricted orbit $\underline{\ell}_s$ are noted $a^-(X_s)$ and $a^+(X_s)$);
- the π^{ψ_0}-projection of $H_s(\Sigma^-)$ into the meridian $\{\psi = \psi_0\}$ is independent of s;
- the π^{ϕ_0}-projection of $(H_s)^{-1}(\Sigma^+)$ into the meridian $\{\phi = \phi_0\}$ is independent of s.

Now, there are two cases depending on whether $\cos_\psi(a^+(X_0))$ is equal to 0 or not. If $\cos_\psi(a^+(X_0))$ is not 0, the germ of H_s at $a^-(X_0)$ is chosen to be a contraction: its center is $a^+(X_0)$ and its factor is e^s (in the coordinate (x, y, v) of the extremity $\{z = 1\}$ of the tube T around $\underline{\ell}$). Notice that such a contraction preserves the above requirements for the constancy of the equators. Then, a calculation shows that the holonomic factor is multiplied by the same factor, which implies that $\partial_s \chi(X_s) > 0$ since $a^\pm(X_s)$ is constant.

Finally, we have to solve the case when $\cos_\psi(a^+(X_0)) = 0$. Here, we arrange the holonomy H_s so that $\partial_s \cos_\psi(a^+(X_s)) > 0$ and $\partial_s \cos_\phi(a^-(X_s)) = 0$, which again implies $\partial_s \chi(X_s) > 0$ since $\eta(X_s) > 0$. This finishes the proof of Proposition 2.12.

After Proposition 2.5 and Proposition 2.12, we have a complete proof of Theorem 1.1. $\qquad \square$

2.6. Normalization of crossing path

The normalization in question will be used for proving Theorem 1.2 and Theorem 1.4. The normalization is achieved by making some group act on M. At the end of the present subsection, it will be proved that the stratification $(\mathcal{S}_g, \mathcal{S}_g^0, \mathcal{S}_g^{0,0})$ is invariant under this action.

In this subsection we use notations as $D_1(0), C_1(0)$ which will be used repeatedly in Section 3 (see Notation 3.9). Consider the image $H_0(\Sigma^-\cap\{z=0\})$ $\subset \partial^+\mathcal{M}_p$ by the holonomy map of X_0 along its homoclinic orbit ℓ in the homotopy class $g \in \pi_1(M,p)$. Let us define

$$D_1(0) := H_0(\Sigma^-) \cap \{z = 1\} \text{ and } C_1(0) := \text{Desc}(D_1(0))$$

where Desc is the descent map defined in (2.6). We recall $a^+ = \ell \cap \Sigma^+$ (resp. $a^- = \ell \cap \Sigma^-$) whose spherical coordinate is denoted by ψ_0 (resp. ϕ_0).

DEFINITION 2.13. — *A crossing path $(X_s)_{s \in \mathcal{O}p(0)}$ of $\mathcal{S}_g$ is said to be normalized if it fulfills the following requirements for every s, where H_s denotes the perturbed holonomy of X_s.*

  (1) *$D_1(0)$ has to be contained in the preferred hyperplane Δ^ϕ of the meridian i-disc $\{\psi = \psi_0\}$.*
  (2) *Let R_{a^-} denote the half ray $\{\phi = \frac{\pi}{2}\text{sign}(\cos_\phi(a^-)), r \in [0,1], \psi = \psi_0\}$. The curve $J_0 := H_0^{-1}(R_{a^-})$ has to be contained in the meridian disc $\{\phi = \phi_0\}$ of $\partial^-\mathcal{M}_p$.*
  (3) *The disc $D'(s) := H_s^{-1}(\Sigma^+) \cap \{z = 0\}$ has also to move in the meridian disc $\{\phi = \phi_0\}$.*

Only the first item will be used to prove Theorem 1.2; the two other items enter the proof of Theorem 1.4. The main tool for *normalization by conjugation* (see Proposition 2.15) is given by the next lemma about diffeomorphisms of $\mathcal{M}_p$. Its proof by Taylor expansion is detailed in the Appendix to [4, Chapter 6].

LEMMA 2.14. — *Let K be a C^1-diffeomorphism of $\partial^+\mathcal{M}_p$ of the form $(\phi, r, \psi) \mapsto (\phi, r, k(\phi, r, \psi))$ with $k(\phi, 0, \psi) = \psi$. Then, K uniquely extends to $\mathcal{M}_p$ as a C^1-diffeomorphism which is the identity on both stable and unstable local manifolds and which keeps the standard vector field X_i invariant. Moreover, the extension $\overline{K}$ is C^1-tangent to Id along the attaching sphere Σ^-.*

It is worth noting that the extension cannot be C^2 in general, even if K is C^∞. This lemma can be also used by interchanging the roles of $\partial^+\mathcal{M}_p$ and $\partial^-\mathcal{M}_p$ and simultaneously the roles of ϕ and ψ.

PROPOSITION 2.15. — *Given a positive crossing path $(X_s)_s$ of the stratum $\mathcal{S}_g$, there exists a C^1-diffeomorphism $\overline{K}$ of M, isotopic to Id_M among the C^1-diffeomorphisms keeping α and $\mathcal{M}_p$ invariant, such that the crossing path $(\overline{K}_*X_s)_s$ induced by $\overline{K}$ is normalized. Moreover, $\overline{K}$ may be chosen so that it preserves ℓ pointwise.*

Notice that the vector field $(\overline{K}_* X_s)$ might only be C^0. But it is uniquely integrable and the associate foliation is transversely C^1; its holonomy is changed by C^1-conjugation.

Proof of Definition 2.13.

(1). — We first look for a diffeomorphism $\overline{K}$ of M mapping $(X_s)_s$ to a crossing path which fulfils the first item of Definition 2.13. If the tube T around ℓ is small enough $D_1(0)$ is nowhere tangent to the fibres of the projection π^{ψ_0} onto the meridian disc $\{\psi = \psi_0\}$. As a consequence, its projected disc D_{ψ_0} is smooth and there exists a smooth map $\overline{k} : D_{\psi_0} \to \Sigma^+$ such that $D_1(0)$ reads

$$D_1(0) = \left\{ \left(\phi, r, \overline{k}(\phi, r)\right) \,\middle|\, (\phi, r) \in D_{\psi_0} \right\}.$$

Since its source is contractible, $\overline{k}$ is homotopic to the constant map valued in a^+. By isotopy extension preserving the fibres of π^{ψ_0}, there exists some diffeomorphism K_1 of $\partial^+ \mathcal{M}_p$ of the form assumed in Lemma 2.14 which maps the given $D_1(0)$ to D_{ψ_0}. Therefore, this K_1 extends to $\mathcal{M}_p$. Since K_1 is isotopic to Id through diffeomorphisms of the same type, its extension $\overline{K}_1$ to $\mathcal{M}_p$ also extends to M with the same name. Moreover, the isotopy of $\overline{K}_1$ to Id_M is supported in a neighborhood of $\mathcal{M}_p$ and preserves each level set of a local primitive of α. Since $\Sigma^\pm$ are kept fixed by $\overline{K}_1$, it is easy to get that ℓ is fixed by $\overline{K}_1$.

After this $\overline{K}_1$, we are reduced to the case where $D_1(0)$ is contained in the meridian disc $\{\psi = \psi_0\}$. Therefore, the tangent space $T_{a^+} D(0)$ is the preferred hyperplane Δ^ϕ in $T_{a^+}\{\psi = \psi_0\}$. Decreasing the radius of the tube T if necessary, the tangent plane $T_m D_1(0)$ is close to be orthogonal to the pole axis directed by ν_ϕ at each $m \in D_1(0)$. This implies that, for every $r \in (0, 1)$, the disc $D_1(0)$ is transverse to the $(i-1)$-sphere of radius r in $\{\psi = \psi_0\}$.

The image $C_1(0)$ of $D_1(0)$ by Desc is diffeomorphic to $S^{i-2} \times (0, 1]$ and contained in the spherical annulus $\mathbb{A}_{\psi_0} := \{(\phi, r, \psi_0) \mid \phi \in \Sigma^-, r \in (0, 1]\}$. By tangency of $D_1(0)$ with the preferred hyperplane Δ^ϕ, the *end* of $C_1(0)$ when $r \to 0$ compactifies as the ϕ-equator $E^\phi \subset \Sigma^-$. Moreover, $C_1(0)$ is transverse (inside $\mathbb{A}_{\psi_0}$) to the sphere $\Sigma^- \times \{(r, \psi_0)\}$ for every $r \in (0, 1)$, since the corresponding assertion holds in $\partial^+ \mathcal{M}_p$. Thus, there is an annulus $C_{\mathrm{eq}} \subset E^\phi \times [0, 1] \times \{\psi_0\}$ such that $C_1(0)$ reads as the graph of some map $\overline{\kappa} : C_{\mathrm{eq}} \to \Sigma^-$ valued in the complement the poles. Then, $\overline{\kappa}$ is homotopic to the map $(\phi, r) \mapsto \phi$ from C_{eq} to the equator E^ϕ of Σ^-.

By isotopy extension preserving each sphere $\Sigma^- \times \{(r, \psi)\}$, we have some diffeomorphism K_2 of $\partial^- \mathcal{M}_p$ of the form $(\phi, r, \psi) \mapsto (\kappa(\phi, r, \psi), r, \psi)$ which

pushes $C_1(0)$ to its *flat* position C_{eq} and satisfies $\kappa(\phi, 0, \psi) = \phi$. By apply-ing Lemma 2.14 "up side down", K_2 extends to some diffeomorphism $\overline{K}_2$ of M preserving $\mathcal{M}_p$ with its standard gradient. On the upper boundary of the Morse model, this means that $\overline{K}_2$ pushes $K_1(D_1(0))$ to an $(i-1)$-disc in the hyperplane Δ^ϕ by some diffeomorphism tangent to Id in a_+. As for $\overline{K}_1$, this $\overline{K}_2$ may be chosen so that ℓ is fixed pointwise. The composed diffeomorphism $\overline{K}_2 \circ \overline{K}_1$ is as desired.

(2). — We now prove the last two items. It consists just in an easy addition to what we have done above. The diffeomorphism $\overline{K}_1$ keeps ℓ fixed pointwise. Let π^{ϕ_0} be the projection of $\partial^- \mathcal{M}_p \cong (\Sigma^- \times \{\phi = \phi_0\})$ onto its second factor which is a meridian disc. Consider the three linear spaces: $T_{a^-}\Sigma^-$, $T_{a^-}D'(0)$ and J_0. They are linearly independent and their span is equal to $T_{a^-}\partial^- \mathcal{M}_p$. Then, after shrinking the tube T if necessary, π^{ϕ_0} embeds the union $(\bigcup_s D'(s)) \cup J_0$ into $\{\phi = \phi_0\}$. Due to the non-vanishing of the velocity with respect to s, the union $(\bigcup_s D'(s))$ is an $(n-i)$-disc transverse to the fibres of π^{ϕ_0}.

Even when a^- lies in the equator E^ϕ, it is easy to find a common diffeo-morphism K_2 of $\partial^- \mathcal{M}_p$ preserving the coordinates (r, ψ) such that it maps $C_1(0)$ to the equatorial annulus C_{eq} (the move required for item (1)) and simultaneously $(\bigcup_s D'(s)) \cup J_0$ to its π^{ϕ_0}-image in the meridian $\{\phi = \phi_0\}$ by diffeomorphism.[13] This completes the proof of Definition 2.13.   □

Remark 2.16. — Proposition 2.15 holds true for a finite dimensional fam-ily. For instance, if we are given a two-dimensional germ $(X_{s,t})_{s,t}$ adapted to the pair $(\mathcal{S}_g, \mathcal{S}_g^0)$ in $X_{0,0}$ (in the sense of Definition 4.1) then each cross-ing path $\gamma_t := (s \mapsto X_{s,t})$ of $\mathcal{S}_g$ has a normalization by some C^1-diffeo-morphism $\overline{K}_t$ depending continuously on t in the C^1-topology.

Notation 2.17. — Let $\mathcal{G}^\pm$ be the groups of diffeomorphisms of M isotopic to Id_M among those which fix the homoclinic orbit ℓ pointwise, preserve the closed one-form α and its standard gradient in $\mathcal{M}_p$, (or, equivalently, the foliation by gradient lines in $\mathcal{M}_p$), and have the following form:

- the restriction of every element in $\mathcal{G}^+$ to $\partial^+ \mathcal{M}_p$ reads
  $(\phi, r, \psi) \mapsto (\phi, r, k^+(\phi, r, \psi))$ with $k^+(\phi, 0, \psi) = \psi$;
- the restriction of every element in $\mathcal{G}^-$ to $\partial^- \mathcal{M}_p$ reads
  $(\phi, r, \psi) \mapsto (k^-(\phi, r, \psi), r, \psi)$ with $k^-(\phi, 0, \psi) = \phi$.

---

[13] In case $a^- \in E^\phi$, Lemma 2.14 tells us that $\overline{K}_1(C_1(0))$ is tangent to C_{eq} along E^ϕ. Then, the above-mentioned linear independence makes K_2 still easy to find by applying the Whitney extension theorem, [6, 13].

PROPOSITION 2.18. — *The action of the groups $\mathcal{G}^+$ and $\mathcal{G}^-$ on the space of adapted α-gradients preserves the strata $\mathcal{S}_g$, $\mathcal{S}_g^0$ and $\mathcal{S}_g^{0,0}$.*

Proof. — We do it for $\mathcal{G}^+$. Let $G \in \mathcal{G}^+$ and $X_0 \in \mathcal{S}_g$. Since G fixes the homoclinic orbit ℓ pointwise, the carried vector field $G_*(X_0)$ has the same homoclinic orbit. According to the form of the restriction of G to the upper boundary of $\mathcal{M}_p$, the projection of $D_1(0)$ to the meridian disc is unchanged. Therefore, the ϕ-equator is preserved. Looking in the lower boundary, one derives that the ϕ-latitude of $a^-(X_0) = \Sigma^- \cap \ell$ is preserved.

Consider the disc $D_1'(0) := H_{X_0}^{-1}(\Sigma^+)$. Recall from Lemma 2.14 that $G|_{\partial^- \mathcal{M}_p}$ is tangent to Id at every point of Σ^-. Therefore, the tangent space $T_{a^-} D_1'(0)$ remains invariant by G. It follows that the ψ-equator is not changed, and hence, the ψ-latitude of a^+ is preserved. Thus we have the invariance of the spherical axes and of their intersection $\mathcal{S}_g^{0,0}$.

It remains to show that the character function is invariant. We already have seen the invariance of the two latitudes. The last term to control is the holonomic factor $\eta(X_0)$ (resp. $\eta(G_*(X_0))$) defined in (2.13). This factor remains unchanged by the action of G thanks to the invariance of:

- ∂_v^1 since the ϕ-latitude is invariant (see (2.12));
- ∂_v^0 since $DG_{a^-} = \mathrm{Id}$;
- the framing in which ∂_v^0 decomposes (this framing is preserved by invariance of the ψ-latitude). $\qquad\square$

3. Change in the Morse–Novikov complex

3.1. A groupoid approach

A groupoid $\mathcal{G}$ is a *small category* where every arrow is invertible. The set of objects in $\mathcal{G}$ is denoted by $\mathcal{G}^0$ and the set of arrows (or morphisms) is denoted by $\mathcal{G}^1$. Given two objects $p, q \in \mathcal{G}^0$, the set of arrows from p to q is denoted by $\mathrm{Hom}(p, q)$. The *identity* arrow at p is denoted by $1_p \in \mathrm{Hom}(p, p)$. The map $p \mapsto 1_p$ embeds $\mathcal{G}^0$ into $\mathcal{G}^1$; and hence, $\mathcal{G}$ may be identified with its set of arrows endowed with its subset of identity arrows. The maps *source* and *target*, $s, t : \mathcal{G}^1 \to \mathcal{G}^0$, are defined by $s(g) = p$ and $t(g) = q$ for every morphism $g \in \mathrm{Hom}(p, q)$.

Remark 3.1. — We denote by $\mathbb{Z}[\![\mathcal{G}]\!]$ the set of formal series of the arrows of $\mathcal{G}$. An element $\lambda \in \mathbb{Z}[\![\mathcal{G}]\!]$ is usually written as $\lambda = \sum_{g \in \mathcal{G}} n_g(\lambda) g$ where

$n_g(\lambda) \in \mathbb{Z}$. Define the *support* of λ as the set $\operatorname{supp}(\lambda) := \{g \in \mathcal{G} \mid n_g(\lambda) \neq 0\}$. Consider the set

$$\mathbb{Z}[\mathcal{G}] := \{\lambda \in \mathbb{Z}[\![\mathcal{G}]\!] \mid \operatorname{supp}(\lambda) \text{ is finite}\}.$$

Given two arrows g, h (seen as elements of $\mathbb{Z}[\mathcal{G}]$) the product $gh \in \mathbb{Z}[\mathcal{G}]$ is defined by their composition in $\mathcal{G}^1$ when $t(g) = s(h)$ and by 0 otherwise. Extending the previous rule distributively with respect to the sum, we obtain a ring structure for $\mathbb{Z}[\mathcal{G}]$. Moreover, when $\mathcal{G}^0$ is finite, the element $1 := \sum_{p \in \mathcal{G}^0} 1_p \in \mathbb{Z}[\mathcal{G}]$ is an *identity element* of this product. We call $\mathbb{Z}[\mathcal{G}]$ the *groupoid ring* associated with $\mathcal{G}$.

DEFINITION 3.2. — *The fundamental groupoid Π of the manifold M is defined as follows: its objects are the points of M and if (p, q) is a pair of points $\operatorname{Hom}(p, q)$ is the set of homotopy classes of paths from p to q. If γ is a such a path, its homotopy class $[\gamma]$ will be called the Π-value of γ.*

The closed 1-form α (whose cohomology class is denoted by u) defines a groupoid morphism

$$u_\alpha : \Pi \longrightarrow \mathbb{R}, \quad g \longmapsto \int_\gamma \alpha,$$

where g is the Π-value of a path γ in M and $\mathbb{R}$ is seen as a groupoid with a single object 0. The restriction of any such u_α to the fundamental group $\pi_1(M, p)$ clearly coincides with the group morphism associated with u.

Let Π_α denote the full subcategory of Π whose set of objects is the set $Z(\alpha)$, the zero set of α. By Remark 3.1, when α is Morse and $Z(\alpha)$ is non-empty, we may consider the groupoid ring $\mathbb{Z}[\Pi_\alpha]$.

A formal series $\lambda \in \mathbb{Z}[\![\Pi_\alpha]\!]$ fulfills the *Novikov Condition* if

(3.1)   for every $L \in \mathbb{R}$, the set $\operatorname{supp}(\lambda) \cap u_\alpha^{-1}(L, +\infty)$ is finite.

Denote by $\Lambda_u \subset \mathbb{Z}[\![\Pi_\alpha]\!]$ the subset of formal series satisfying the Novikov Condition. It can be easily checked that the product rule given in Remark 3.1 also endows Λ_u with a ring structure, having the same identity element as $\mathbb{Z}[\Pi_\alpha]$. We call Λ_u the *Novikov ring associated with α*.

Example 3.3. — Let $p \in Z(\alpha)$ and $g \in \pi_1(M, p)$ with $u(g) < 0$ (for instance, the Π-value of a homoclinic orbit of some α-gradient). The following formal series are elements of the Novikov ring:

$$\sum_{j=1}^{\infty} g^j \quad \text{and} \quad \sum_{j=1}^{\infty} (-1)^j g^j$$

Indeed, the Novikov Condition (3.1) is fulfilled since $u_\alpha(g^j) = j \cdot u_\alpha(g)$ which goes to $-\infty$ as $j \to +\infty$. Thus, these two series belong to the Novikov ring Λ_u. In particular $1 + g + g^2 + \ldots$ is inversible and its inverse is $1 - g$.

3.2. The Morse–Novikov complex

Let X be an α-gradient which is assumed KS (Kupka–Smale). An orientation is arbitrarily chosen on the unstable manifold $W^u(p, X)$ for each zero $p \in Z(\alpha)$. We are going to define a chain complex $(C_*(\alpha), \partial^X)$ of Λ_u-modules; it is graded by the integers $i \in \{0, 1, \ldots, n = \dim M\}$. This complex will be called the *Morse–Novikov* complex associated with the pair (α, X).

For each degree i, the module $C_i(\alpha)$ is the left Λ_α-module freely generated by $Z_i(\alpha)$, the finite set of zeroes of α of Morse index i. The Λ_u-morphism $\partial^X_* : C_*(\alpha) \to C_{*-1}(\alpha)$ must have the following form on each generator of $C_i(\alpha)$:

$$(3.2) \qquad \partial^X_*(p) = \sum_{q \in Z_{*-1}(\alpha)} n(p, q)^X q,$$

where the coefficient of q has to be an element of Λ_u (called the *incidence* of p to q). This coefficient $n(p, q)^X$ is the algebraic count which we are going to define.

Let $\mathrm{Orb}^X(p, q)$ denote the set of connecting orbits of X from p to q. First, we define the sign of a connecting orbit $\ell \in \mathrm{Orb}^X(p, q)$. Given a point $x \in \ell$, the sign ε_ℓ is defined by the following equation:

$$\varepsilon_\ell \, X(x) \wedge \text{co-or}\left(W^s(q, X)\right) = \text{or}\left(W^u(p, X)\right).$$

This definition is clearly indepedent of $x \in \ell$.

DEFINITION 3.4. — *Assume the α-gradient X is KS. The* Morse–Novikov incidence *associated with the data (p, q, X), $p \in Z_i(\alpha)$, $q \in Z_{i-1}(\alpha)$, is defined by:*

$$(3.3) \qquad n(p, q)^X := \sum_{\ell \in \mathrm{Orb}^X(p,q)} \varepsilon_\ell \, g_\ell$$

where g_ℓ denotes the Π-value of the connecting orbit ℓ.

By Proposition A.1 from Appendix A, this coefficient fulfills the Novikov Condition (3.1). So, it is an element of Λ_u. Moreover, the map ∂^X as in (3.2) is indeed a differential; this can be found in [3]. The resulting $(C_*(\alpha), \partial^X)$ is known as the Morse–Novikov complex (see [8, 12]).

We denote by $\mathrm{Orb}_L^X(p,q)$ the set of connecting orbits from p to q whose α-length $\mathcal{L}(\ell)$ is less than a fixed $L > 0$. Since these orbits verify the inequality $u_\alpha(g_\ell) > -L$, we are led to define a *L-truncation* map $\mathcal{T}_L : \Lambda_u \to \mathbb{Z}[\Pi_\alpha]$ by:

$$\mathcal{T}_L(\lambda) := \sum_{u_\alpha(g) > -L} n_g(\lambda) g.$$

Two elements $\lambda, \mu \in \Lambda_\alpha$ are said to be equal modulo L if $\mathcal{T}_L(\lambda - \mu) = 0$. Finally, the *L-incidence* is defined as follows:

$$n(p,q)_L^X := \sum_{\ell \in \mathrm{Orb}_L^X(p,q)} \varepsilon_\ell \, g_\ell \in \mathbb{Z}[\Pi_\alpha].$$

Of course, we have $\mathcal{T}_L\left(n(p,q)^X\right) = n(p,q)_L^X$.

3.3. Effect of homoclinic bifurcation on the incidence

Consider a generic one-parameter family of adapted α-gradients $(X_s)_s$ such that $X_0 \in \mathcal{S}_g$. By definition, X_0 has a unique homoclinic orbit ℓ connecting $p \in Z(\alpha)$ to itself whose Π-value is g. Denote the Morse index of p by i.

The next definition specifies some genericity conditions that will be needed to prove Theorem 3.8. The remainder of this section is devoted to its proof and consequences.

DEFINITION 3.5. — *Let $L > 0$.*

(1) *The α-gradient X is said to be* Kupka–Smale up to L *if, for every pair of zeroes $p, q \in Z(\alpha)$ and every X-orbit ℓ from p to q with $-\int_\ell \alpha < L$, the unstable and stable manifolds, $W^u(p, X)$ and $W^s(q, X)$, are transverse along ℓ. The subset of $\mathcal{F}_\alpha$ formed with such α-gradients is denoted by $\mathcal{KS}_L$.*

(2) *An α-gradient $X \in \mathcal{S}_g$ is said to be* almost Kupka–Smale up to L, *if the preceding transversality condition is fulfilled except for the unique homoclinic orbit whose Π-value is g. The subset of $\mathcal{S}_g$ formed with such elements is denoted by $\mathcal{S}_{g,L}$.*

(3) *The α-gradient $X \in \mathcal{S}_g$ is said to be* almost Kupka–Smale *if it is KS up to L for every $L > 0$. These gradients are the elements of $\mathcal{S}_{g,\infty} := \bigcap_L \mathcal{S}_{g,L}$.*

PROPOSITION 3.6.

(1) *The subspace $\mathcal{S}_{g,L}$ is open and dense in $\mathcal{S}_g$. Moreover, there exists an open set $\mathcal{W}_L$ in $\mathcal{F}_\alpha$ such that $\mathcal{W}_L \smallsetminus \mathcal{S}_g$ is contained in $\mathcal{KS}_L$.*
(2) *The subspace $\mathcal{S}_{g,\infty}$ is residual in $\mathcal{S}_g$.*

Proof.

(1). — One checks that the constraint of having a unique homoclinic orbit with a given Π-value does not prevent us from arguing as Peixoto [10].

(2). — This item is a little more subtle since $\mathcal{S}_g$ is not a complete metric space. But it is separable. Then, it is sufficient to prove that, for any closed ball B in $\mathcal{S}_g$, the intersection $\mathcal{S}_{g,\infty} \cap B$ is residual in $\mathcal{S}_g \cap B$. And now, we are allowed to follow Peixoto word for word. More details are left to the reader. □

Notation 3.7. — It is easily seen that $\mathcal{S}_{g,L}$ is open in $\mathcal{S}_g$; and, if $X \in \mathcal{S}_{g,L}$ there is an arbitrarily small neighborhood U of X in $\mathcal{F}_\alpha$ such that $U \smallsetminus \mathcal{S}_g$ is made of two connected components in $\mathcal{KS}_L$. In particular, if $(X_s)_s$ is a path which intersects $\mathcal{S}_g$ transversely at $X_0 \in \mathcal{S}_{g,L}$ the gradient X_s lies in $\mathcal{KS}_L$ for every $s \neq 0$ close enough to 0.

Therefore, for every $q \in Z_{i-1}(\alpha)$ the L-incidence $n(p,q)_L^{X_s}$ is well defined and independent of s when $s \in \mathcal{O}_+$ (resp. $s \in \mathcal{O}_-$); it is denoted by $n(p,q)_L^{\pm}$ respectively. Here, the symbol $\mathcal{O}_-$ stands for an open interval $(-\epsilon, 0)$ whose size is not specified and which is as small as needed by the statement; and similarly for $\mathcal{O}_+$.

THEOREM 3.8. — *Let $(X_s)_s$ be a path of adapted α-gradients that intersects $\mathcal{S}_g$ transversely at X_0 and let $L > -u_\alpha(g)$. Assume X_0 is almost Kupka–Smale up to L, that is $X_0 \in \mathcal{S}_{g,L}$. Then the following holds. When $(X_s)_s$ intersects the stratum $\mathcal{S}_g$ positively, we have the following relations in Λ_u:*

(1) *if $X_0 \in \mathcal{S}_g^+$, then $n(p,q)_L^+ = (1+g+g^2+g^3+\ldots)\cdot n(p,q)_L^-$ (mod. L),*
(2) *if $X_0 \in \mathcal{S}_g^-$, then $n(p,q)_L^+ = (1+g)\cdot n(p,q)_L^-$ (mod. L).*

When $(X_s)_s$ intersects the stratum $\mathcal{S}_g$ negatively, we have:

(1′) *if $X_0 \in \mathcal{S}_g^+$, then $n(p,q)_L^+ = (1-g)\cdot n(p,q)_L^-$ (mod. L),*
(2′) *if $X_0 \in \mathcal{S}_g^-$, then $n(p,q)_L^+ = (1-g+g^2-g^3+\ldots)\cdot n(p,q)_L^-$ (mod. L).*

It is worth noticing the reason why the truncation: in general, the bifurcation at $s = 0$ is not isolated among bifurcation times of the path $(X_s)_s$. When it is isolated, the truncation is not needed any longer; this will be the case in [5].

Proof of $(1) \Longleftrightarrow (1') \Longleftrightarrow (2) \Longleftrightarrow (2')$. The first equivalence is obvious since $1 - g$ is the inverse of $1 + \sum_{j=1}^{\infty} g^j$. The last equivalence follows similarly.

Let us show the middle equivalence. It is obtained by changing the vector ∂_v into its opposite in the coordinates of the tube around the homoclinic orbit of X_0. This amounts to put a sign in Formula (2.1). The latter change has three effects:

(i) It reverses the co-orientation of $\mathcal{S}_g$. Hence, positive and negative crossings are exchanged.

(ii) The character is changed into its opposite since the ϕ- and ψ-latitudes are so. Thus, both sides of $\mathcal{S}_g^0$ in $\mathcal{S}_g$ are exchanged.

(iii) The homoclinic orbit becomes negative in the following sense: if the ϕ-sphere is seen as the boundary of the meridian disc at a^+, the new positive hemisphere projects to the preferred hyperplane Δ^ϕ (whose orientation is unchanged) by reversing the orientation. This implies that, in the algebraic count of connecting orbits from p to q (where $i(p) = i(q) + 1$), the coefficient g has to be changed into $-g$ (see the orientation claim in Lemma 3.10).

We are left to prove Theorem 3.8 in case (1). This will be done in Subsection 3.5. $\qquad\square$

According to Proposition 2.18, the statement of Theorem 3.8 is invariant by the groups $\mathcal{G}^\pm$ introduced in Notation 2.17. By Proposition 2.15, it is sufficient to consider the case where the crossing path in question is *normalized* in the sense of Definition 2.13. This assumption is done in what follows. We need some more notations and lemmas. The setting of Theorem 3.8 is still assumed.

Notation 3.9.

(1) Recall from Subsection 2.3 that H_s denotes the perturbed holonomy diffeomorphism along the homoclinic orbit ℓ. For $s \in \mathcal{O}p\,(0)$, it maps $\mathcal{O}p(\{z = 0\}) \subset \partial^- \mathcal{M}_p$ to an open set of $\partial^+ \mathcal{M}_p$ containing $\{z = 1\}$.

(2) For $s \in \mathcal{O}p(0)$, let $D_1(s)$ denote the image $H_s(\Sigma^-) \cap \{z = 1\}$. Consider its projection $\pi^{\psi_0}(D_1(s))$ to the meridian disc $\{\psi = \psi_0\}$ and define
$$a^+(s) := \pi^{\psi_0}(D_1(s)) \cap \mathbb{R}\partial_v^1.$$

The *crossing velocity* of the crossing path is

(3.4)
$$V_1 := \frac{d}{ds}a^+(s)_{|s=0}.$$

After reparametrization of the crossing path, we may assume $V_1 = 1$.

(3) For $s \in \mathcal{O}_\pm$, by definition of a crossing path, $D_1(s)$ avoids Σ^+. Therefore we are allowed to define $C_1(s) := \mathrm{Desc}(D_1(s)) \subset \partial^- \mathcal{M}_p$. It is still an $(i-1)$-disc.

LEMMA 3.10. — *Recall the natural projection $\pi^- : \partial^- \mathcal{M}_p \to \Sigma^-$. Let K be any compact disc in the open hemisphere $\mathcal{H}^-(\Sigma^-)$ (as in Subsection 2.4). Then, for $s \in \mathcal{O}_-$, the disc $C_1(s) \cap (\pi^-)^{-1}(K)$ is a graph over K of a section of π^-. This section goes to the zero-section 0_K of π^- over K in the C^1-topology as s goes to 0. Moreover, $\pi^- : C_1(s) \to \Sigma^-$ is orientation reversing.*

A similar statement holds when $K \subset \mathcal{H}^+(\Sigma^-)$ and $s \in \mathcal{O}_+$, except that $\pi^- : C_1(s) \to \Sigma^-$ is orientation preserving in this case.

Proof. — Consider the case $s \in \mathcal{O}_-$, the other case being similar. Recall the normalization assumption: the disc $D_1(0)$ is contained in the meridian disc $\{\psi = \psi_0\}$. Recall the projection π^{ψ_0} of $\partial^+ \mathcal{M}_p$ onto this meridian disc. The normalization implies that the projected discs $\pi^{\psi_0}(D_1(s))$ tend to $D_1(0)$ in the C^1-topology.

Recall the identification $\partial(\{\psi = \psi_0\}) \cong \Sigma^-$ of (2.9) and think of K as a compact subset of the South hemisphere in the boundary of the meridian disc $\{\psi = \psi_0\}$. For every such a K, the following property holds:

> For every s close enough to 0 and for every $\phi \in K$, the disc $\pi^{\psi_0}(D_1(s))$ intersects the ray directed by ϕ in one point only and transversely in $\{\psi = \psi_0\}$.

This point is denoted by $m_s(\phi)$; it is the image of some $\widetilde{m}_s(\phi) \in D_1(s)$ through π^{ψ_0}. We have $m_0(\phi) = \widetilde{m}_0(\phi) = a^+$, but when $s \neq 0$, the point $\widetilde{m}_s(\phi)$ has well-defined multi-spherical coordinates $(\phi, r_s(\phi), \psi_s(\phi))$ where r_s and ψ_s depend smoothly on s.

Going back to $\partial^- \mathcal{M}_p$ by the map Desc, we see that $C_1(s)$ is the image of a section of the projection π^- over K. Indeed, if $s \in \mathcal{O}_-$ goes to 0 then $D_1(s) \cap \{(\phi, r, \psi) \in \partial^+ \mathcal{M}_p \mid \phi \in K\}$ goes to a^+ in the metric sense. In particular, $\max_{\phi \in K}\{r \mid (\phi, r, \psi) \in D_1(s)\}$ goes to 0. Therefore, $C_1(s) \cap (\pi^-)^{-1}(K)$ goes to 0_K in the C^0-topology when s goes to 0 negatively.

For the C^1-convergence, we use that K is far from the ϕ-equator of Σ^-. Therefore, the angle in the meridian disc $\{\psi = \psi_0\}$ between the ray directed by ϕ and the tangent plane to $\pi^{\psi_0}(D_1(s))$ at $m_s(\phi)$ is bounded from below. Including the fact that $s \to 0_-$ implies $r \to 0$, it follows that the smooth

functions $r_s(\phi)$ and $\psi_s(\phi)$ satisfy

$$\begin{cases} |dr| = O_s(r)|d\phi|, \\ |d\psi| = O_s(r)|d\phi|, \end{cases}$$

where $O_s(r)$ stands for any quantity which is uniformly bounded by a positive multiple of r when s goes to 0. This yields the claimed C^1-convergence of the part of $C_1(s)$ over $K \subset \mathcal{H}^-(\Sigma^-)$ to $0_K = K$.

The statement about orientation follows directly from this C^1-convergence. $\qquad\square$

3.4. Geometric interpretation of the character function.

We still consider a germ of *normalized* positive crossing path $(X_s)_s$. Let $D'_1(0)$ be the image of Σ^+ by the inverse holonomy diffeomorphism H_0^{-1} along ℓ. Remark that $D'_1(0)$ is an $(n - i - 1)$-disc located in the connected component of $W^s(p, X_0) \cap \{z = 0\}$ which contains a^-. For every $s \in \mathcal{O}p(0)$, consider now $D'_1(s) := H_s^{-1}(\Sigma^+) \cap \{z = 0\}$.

Recall from Subsection 2.2 that $\Sigma^- \cap \{z = 0\}$ is identified with the x-axis whereas $D'_1(0)$ is identified with the y-axis. Let also $p_v : \{z = 0\} \to \{v = 0, z = 0\}$ denote the projection parallel to ∂_v onto the (x, y)-space. When $s \in \mathcal{O}p(0)$ goes to 0, the family $D'_1(s)$ accumulates to the y-axis in the C^1-topology.

Until Remark 3.14, we assume $\cos_\phi(a^-) \neq 0$. Lemma 3.10 tells us that the family $C_1(s) \cap \{z = 0\}$ accumulates to the x-axis in the C^1-topology if and only if $s \cos_\phi(a^-)$ goes to 0_+. In particular, when $s \cos_\phi(a^-) > 0$ the projections $p_v(C_1(s))$ and $p_v(D'_1(s))$ intersect in a unique point $b_1(s)$ and transversely. If $s \cos_\phi(a^-)$ is negative, then $C_1(s) \cap \{z = 0\}$ is empty.

Denote by $c_1(s)$ and $d'_1(s)$ the only points in $C_1(s) \cap \{z = 0\}$ and in $D'_1(s)$ respectively such that $p_v(c_1(s)) = b_1(s) = p_v(d'_1(s))$. Consider the real number

(3.5) $\quad v_1(s) := v(c_1(s)) - v(d'_1(s))$ for every s such that $s \cos_\phi(a^-) \in \mathcal{O}_+$.

This function $v_1(s)$ depends smoothly on s. Its derivative with respect to s is denoted by $\dot{v}_1(s)$.

Remark 3.11. — By construction, $v_1(s) = 0$ implies $c_1(s) = d'_1(s)$ which in turn implies the existence of an orbit $\ell_s \in \mathrm{Orb}^{X_s}(p, p)$ passing through $c_1(s)$ such that $[\ell_s] = g^2$. This remark will be used when analysing the doubling phenomenon in Section 4.

Lemma 3.12 will show the kinematic meaning of the character function χ at X_0.

LEMMA 3.12. — *Let $(X_s)_s$ be a normalized positive crossing path of $\mathcal{S}_g$ whose crossing velocity (3.4) is equal to $+1$. If $\cos_\phi(a^-) \neq 0$ then the following relation holds:*

$$(3.6) \qquad \cos_\phi(a^-)\, \dot{v}_1(0) = \chi(X_0).$$

Proof. — Let us study the v-coordinate of $c_1(s)$ first. We notice that, if $\bar{c}_1(s)$ is another point of $C_1(s)$ depending smoothly on s and such that $\bar{c}_1(0) = a^- = (\phi_0, 0, -)$, we have the same velocity at $s = 0$:

$$(3.7) \qquad \frac{d}{ds} v\,(\bar{c}_1(s))_{|s=0} = \frac{d}{ds} v\,(c_1(s))_{|s=0}.$$

Indeed, $C_1(s)$ accumulates C^1 to $\Sigma^- \cap \{z = 0\}$ (Lemma 3.10), then the difference $\dot{c}_1(0) - \dot{\bar{c}}_1(0)$ is a vector in $T_{a^-}\Sigma^-$.[14] Apply this remark to the point $\bar{c}_1(s) := C_1(s) \cap \{\phi = \phi_0\}$. Let $d_1(s) \in D_1(s)$ the lift of $\bar{c}_1(s)$ by Desc^{-1}.

Since Desc preserves the (ϕ, r, ψ)-coordinates, both paths $s \mapsto \bar{c}_1(s)$ and $s \mapsto d_1(s)$ have the same coordinates $(\phi(s) = \phi_0, r(s), \psi(s))$ when $s \neq 0$. As $\bar{c}_1(s) \in \{\phi = \phi_0\}$, the vector $\dot{d}_1(0)$ belongs to the $(n-i)$-plane which is the span of $\{\mathbb{R}\,\phi_0, T_{a^+}\Sigma^+\}$. Let $\widehat{d}_1(0)$ be its projection to the line $\mathbb{R}\,\phi_0$ in the meridian disc $\{\psi = \psi_0\}$. Then,

$$\widehat{d}_1(0) = \rho\,\phi_0 \quad \text{where} \quad \rho = \frac{d}{ds} r(s)_{|s=0}.$$

By definition of the ϕ-latitude (Proposition 2.8) we have:

$$\left\langle \nu_\phi, \widehat{d}_1(0) \right\rangle = \rho\langle \nu_\phi, \phi_0 \rangle = \rho\,\cos_\phi(a^-).$$

By definition, the hyperplane Δ^ϕ is tangent to $D_1(0)$ at a^+. Therefore, some calculus of Taylor expansion tells us that

$$\left\langle \nu_\phi, \widehat{d}_1(0) \right\rangle = \frac{d}{ds} a^+(s)_{|s=0} = +1.$$

We derive:

$$(3.8) \qquad \rho = \frac{1}{\cos_\phi(a^-)}.$$

Since $d_1(s)$ goes to $a^+ = (-, 0, \psi_0)$ as s goes to 0 and since the radial velocity is preserved by Desc, then we have:

$$\dot{\bar{c}}_1(0) = \rho\,\psi_0 \in T_{a^-}\{\phi = \phi_0\}.$$

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[14] As $C_1(s)$ is the graph over $C_1(0)$ of some C^1 map q_s, the point $\bar{c}_1(s)$ reads $q_s(\overline{m}_s)$ and similarly for $c_1(s)$. Differentiating at $s = 0$ proves the claim.

Using again (2.5), but relatively to the preferred hyperplane Δ^ψ which defines the ψ-latitude we obtain

$$\langle \nu_\psi, \dot{\tilde{c}}_1(0) \rangle = \rho \, \cos_\psi(a^+).$$

This together with the decomposition of $T_{a^-}\{z = 0\}$ of (2.11) says that there are two vectors $w_x \in T_{a^-}\Sigma^-$ and $w_y \in \Delta^\psi$ such that

$$(3.9) \qquad \begin{aligned} \dot{\tilde{c}}_1(0) &= w_x + \rho \, \omega_\psi(a^+)\nu_\psi + w_y \\ &= w_x + \rho \cos_\psi(a^+)\nu_\psi + w_y. \end{aligned}$$

By (3.9), we have $v(\dot{\tilde{c}}_1(0))\, \partial_v^0 = \rho\,\omega_\psi(a^+)\langle \partial_v^0, \nu_\psi \rangle$. On the other hand, (2.13) tells us that:

$$(3.10) \qquad v(\nu_\psi) = \frac{1}{\bar\eta} = \eta.$$

Putting together (3.7), (3.8), (3.9) and (3.10) we obtain:

$$(3.11) \qquad v(\dot{c}_1(0)) = v\left(\dot{\tilde{c}}_1(0)\right) = \rho\,\omega_\psi(a^+)\,v(\nu_\psi) = \frac{\cos_\psi(a^+)}{\cos_\phi(a^-)}\,\eta.$$

We now come to estimate the term $v(\dot{d'}_1(0))$. We apply Lemma 2.7 for comparing velocities associated with the holonomy H_s and its inverse. From Formula (3.4) we derive that $\frac{d}{ds}(v \circ H_s)(a^-)_{|s=0} = +1$. Then, the inverse holonomy satisfies

$$\frac{d}{ds}\left(v \circ H_s^{-1}\right)(a^+)_{|s=0} = -1$$

from which it is easily seen that $v(\dot{d'}_1(0)) = -1$. Therefore, using (3.5) we have

$$(3.12) \qquad \dot{v}_1(0) = \eta\frac{\cos_\psi(a^+)}{\cos_\phi(a^-)} + 1$$

which is a reformulation of the desired formula. ∎

Lemma 3.13 right below is the last tool that we need for proving Theorem 3.8. This lemma extracts the geometric information contained in Equation (3.6). The setting is the same as in the previous lemma. We are only looking at normalized paths $(X_s)_s$ which cross $\mathcal{S}_g$ positively at a point $X_0 \in S_g^+$.

Lemma 3.13.

   (1) *Suppose the ϕ-latitude $\cos_\phi(a^-)$ is positive (and hence $\dot{v}_1(0) > 0$). Then, for $s \in \mathcal{O}_+$ there are sequences of non-empty $(i-1)$-discs*

$(D_k(s))_{k>1}$ and $(C_k(s))_{k>1}$ *inductively defined from the previous* $D_1(s)$ *and* $C_1(s)$ *by*

(3.13)
$$\begin{cases} D_k(s) := H_s\left(C_{k-1}(s)\right) \cap \{z = 1\} \\ C_k(s) := \mathrm{Desc}\left(D_k(s)\right) \subset \partial^- \mathcal{M}_p \end{cases}$$

Moreover, as s goes to 0, the disc $C_k(s)$ tends to the North hemisphere $\mathcal{H}^+(\Sigma^-)$ in the C^1-topology, uniformly over every compact set of $\mathcal{H}^+(\Sigma^-)$. When $s \in \mathcal{O}_-$, both previous sequences are empty when $k > 1$.

(2) *If $\dot{v}_1(0)$ and $\cos_\phi(a^-)$ are negative, then for $s \in \mathcal{O}_-$ the disc $C_2(s)$ is well defined as in (3.13) and the subsequent discs, $D_3(s), \ldots$, are empty. Moreover, $C_2(s)$ tends to $\mathcal{H}^+(\Sigma^-)$ in the C^1-topology with the reversed orientation. When $s \in \mathcal{O}_+$, all discs in (3.13) are empty when $k > 1$.*

Proof.

(1). — When $s \in \mathcal{O}_-$, the disc $C_1(s)$ does not meet the tube T around the homoclinic orbit ℓ. Then $D_2(s)$ is empty and hence, all further discs are so.

Assume now that $s \in \mathcal{O}_+$. In that case, $C_1(s)$ goes to $\mathcal{H}^+(\Sigma^-)$ (Lemma 3.10) and therefore meets the set $\{z = 0\}$. Then, the discs $D_2(s)$ and $C_2(s)$ defined in (3.13) are non-empty. We are going to compute the position of $C_2(s)$ with respect to $D'_1(s)$ measured by some $v_2(s)$ in the direction of the v-coordinate. We shall check the positivity of $\dot{v}_2(0)$ which will allow us to pursue the induction.

Recall the projection $\pi^{\psi_0} : \partial^+ \mathcal{M}_p \to \{\psi = \psi_0\}$ and define the spherical annulus $A := (\pi^{\psi_0})^{-1}(\mathbb{R}\partial_v^1)$. Consider the point $\widetilde{c}_1(s)$ which is the transverse intersection $C_1(s) \cap H_s^{-1}(A)$. By projecting to the v-axis we find a function $v(\widetilde{c}_1(s))$ which satisfies

$$\frac{d}{ds} v(\widetilde{c}_1(s))_{|s=0} = \frac{d}{ds} v(c_1(s))_{|s=0}.$$

Indeed, if s goes to 0_+, $\lim \widetilde{c}_1(s) = \lim c_1(s) = a^-$ (see footnote 14).

Recall the definition of $d'_1(s) \in D'_1(s)$ from (3.5). Compute the derivative V_2 at $s = 0$ of $v\left[H_s(\widetilde{c}_1(s))\right] - v\left[H_s(d'_1(s))\right]$, which is nothing but the velocity of the projection of $H_s(\widetilde{c}_1(s)) \in D_2(s)$ onto the v-axis of $\{z = 1\}$ at $s = 0$. Using $\widetilde{c}_1(0) = d'_1(0) = a^-$ and $dH_0(a^-) = \mathrm{Id}$ in the coordinates of the tube T, we find:

(3.14)
$$V_2 = \dot{v}_1(0)$$

which is positive by assumption. This V_2 will play the same role as the crossing velocity (3.4).

Since $V_2 > 0$, Lemma 3.10 tells us that $C_2(s)$ meets $\{z = 0\}$ when $s \in \mathcal{O}_+$. Therefore, we choose points $c_2(s) \in C_2(s) = \mathrm{Desc}(D_2(s))$ and $d_2'(s) \in D_1'(s)$ which form the unique pair of points of the respective subsets which have the same p_v-projection. We define

$$v_2(s) := v(c_2(s)) - v(d_2'(s)).$$

The computation of $\dot{v}_2(0)$ is exactly the same as in (3.12) except that we have to replace $V_1 = 1$ with V_2. The result is:

$$\dot{v}_2(0) = \eta \frac{\cos_\psi(a^+)}{\cos_\phi(a^-)} V_2 + 1.$$

Here, some discussion is needed according to the sign of $\cos_\psi(a^+)$:

  (i) if $\cos_\psi(a^+)$ is positive, then $\dot{v}_2(0)$ is larger than $V_1 = +1$. In that case the induction goes on with $V_k > V_{k-1} > \ldots > 1$.
  (ii) if $\cos_\psi(a^+)$ is negative, then $0 < V_2 = \dot{v}_1(0) < 1$, where the last inequality comes from (3.12). Therefore, $\dot{v}_2(0) - 1$ is the product of two numbers of opposite signs, one of them being V_2, and whose absolute values are smaller than 1. Thus, $\dot{v}_2(0)$ belongs to $(0, 1)$. Such a fact is preserved at each step of the induction.

The induction can be carried on.

(2). — Take $s \in \mathcal{O}_-$. The calculation yielding the equality (3.14) still holds and tells us that V_2 is negative. Note that $H_s(d_1'(s)) \in \Sigma^+$. As $s < 0$, one derives:

$$v\left[H_s(\tilde{c}_1(s))\right] = v\left[H_s(\tilde{c}_1(s))\right] - v\left[H_s(d_1'(s))\right] > 0.$$

Thus, Lemma 3.10 says that $C_2(s)$ tends to $\mathcal{H}^+(\Sigma^+)$ in the C^1-topology. As $\cos_\phi(a^-) < 0$, $C_2(s)$ does not meet $\{z = 0\}$ and the next discs are empty. Concerning the orientation, we check that $D_2(s)$ tends to $-D_1(0)$ in $\partial^+ \mathcal{M}_p$. Then, $C_2(s)$ tends to $-\mathcal{H}^+(\Sigma^-)$.

When $s \in \mathcal{O}_+$, we have $s \cos_\phi(a^-) < 0$ which implies the statement, as mentioned right above Remark 3.11. □

Remark 3.14. — In the previous analysis, from Notation 3.9 to Lemma 3.13, we have given the leading role to the bottom of the Morse model, the attaching sphere Σ^-, the perturbed holonomy and the map Desc. Here, the non-vanishing of the ϕ-latitude is required.

One can make a similar analysis with the top of the Morse model, the co-sphere Σ^+, the inverse of perturbed holonomy and Desc^{-1}. There, the non-vanishing of the ψ-latitude would be needed. But the statements of

the lemmas are analogous. As a consequence, if the proof of Theorem 3.8 can be completed under the assumption $\omega_\phi(X_0) \neq 0$, then it can also be completed when $\omega_\psi(X_0) \neq 0$.

3.5. Proof of Theorem 3.8 continued

We continue the proof which begins right after the statement of this theorem. After a series of equivalences, we are left to prove the case (1) of a positive crossing of the stratum $\mathcal{S}_g$ at a point $X_0 \in \mathcal{S}_{g,L}$ where the character funcion is positive. We recall that the statement of Theorem 3.8 is preserved under the action of the groups $\mathcal{G}^\pm$ (see Notation 2.17). Therefore, we may assume that $X_0 \in \mathcal{S}_{g,L}$ is normalized. Moreover, as $\chi(X_0) \neq 0$, one of the extended ϕ-latitude and ψ-latitude is non-zero.[15] By Remark 3.14, it is sufficient to complete the proof when $\omega_\phi(X_0) \neq 0$.

The element $g \in \Pi_\alpha$ is thought of as an arrow from the set of zeroes $Z(\alpha)$ into itself. Then g determines its origin p which is also its end point. Recall that the Morse index of p is i. We look at any zero $q \in Z(\alpha)$ of Morse index $i - 1$. We have to compute the change of $n(p,q)_L^X$ when X changes from X_{0_-} to X_{0_+} in the given crossing path $(X_s)_s$. It is useful to make some partition, adapted to g, of the set of connecting orbits from p to q for the gradient X_{0_-}.

Partition of the connecting orbits. We may assume that each connecting orbit of X_{0_-} from p to q is the unique one in its homotopy class. In general, one would take the multiplicity into account. Recall Γ_p^q, the set of homotopy classes of paths from p to q. The equivalence relation defining the partition of Γ_p^q is the following: $[\gamma_0] \sim [\gamma_1]$ if and only if the homotopy class of γ_1 reads $[\gamma_1] = g^k \cdot [\gamma_0]$ with some $k \in \mathbb{Z}$.

Consider $[\gamma]_\sim$, the $\sim$-class of a fixed connecting orbit γ. Since the α-lengths of connecting orbits are positive, we have $u_\alpha([\gamma']) < 0$ for every $\gamma' \in [\gamma]_\sim$. Therefore, since $u(g) < 0$ there are only finitely many connecting orbits $\gamma' \in [\gamma]_\sim$ verifying $u_\alpha([\gamma']) \geqslant u_\alpha([\gamma])$. Let γ_0 be a connecting orbit in $[\gamma]_\sim$ such that $u_\alpha([\gamma_0])$ is maximal. Then, any element of $[\gamma]_\sim$ reads $g^k \cdot [\gamma_0]$ for some $k \geqslant 0$.

End of the proof. — By Λ_u-linearity of the Morse–Novikov differential, we may assume, without loss of generality, that the above partition has only one $\sim$-class and that the maximal element γ is a positive connecting

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[15] The extended ϕ- and ψ-latitudes are defined in (2.15).

orbit (with respect to the chosen orientations). Let $b := \gamma \cap \Sigma^-$ and Δ_s, $s = 0_-$, be the connected component of $W^s(q, X_s) \cap \partial^- \mathcal{M}_p$ containing b. After shrinking the parameter δ of $\mathcal{M}_p$ if necessary (see Subsection 2.1), Δ_s is an $(n-i)$-disc which intersects Σ^- transversely and at b only.

We are looking for the change formula up to α-lenght L (for every $L > -u(g)$). Let $(X_s)_{s \in \mathcal{O}p(0)}$ be a crossing path with X_0 in $\mathcal{S}_g^+$. We do it first in the case where the ϕ-latitude $\omega_\phi(X_0) \neq 0$. There are still four cases to consider where a^- stands for $a^-(X_0)$ and $\mathcal{H}^\pm$ stands for $\mathcal{H}^\pm(\Sigma^-)$:

   (a.1) The ϕ-latitude $\cos_\phi(a^-)$ is positive and b belongs to $\mathcal{H}^+$.
   (a.2) The ϕ-latitude $\cos_\phi(a^-)$ is positive and b belongs to $\mathcal{H}^-$.
   (b.1) The ϕ-latitude $\cos_\phi(a^-)$ is negative and b belongs to $\mathcal{H}^+$.
   (b.2) The ϕ-latitude $\cos_\phi(a^-)$ is negative and b belongs to $\mathcal{H}^-$.

The proof consists of applying Lemma 3.13. It is convenient to use the following definitions.

DEFINITIONS.
   (1) The *positive* (resp. *negative*) part of $W^u(p, X_0)$ is the union of all X_0-orbits passing through the positive (resp. negative) hemisphere $\mathcal{H}^+(\Sigma^-)$ (resp. $\mathcal{H}^-(\Sigma^-)$). It will be denoted by $W^u(p, X_0)^\pm$.
   (2) For a given $k > 0$, we say that the unstable manifolds $W^u(p, X_s)$ *accumulate* to $g^k \cdot W^u(p, X_0)^\pm$ when s goes to 0_- (resp. 0_+) if it is true when lifting to the universal cover, that is: if $\widetilde{p}$ (resp. $\widetilde{X}_s$) is a lift of p (resp. X_s), the unstable manifolds $W^u(\widetilde{p}, \widetilde{X}_s)$ accumulate to $W^u(g^k \widetilde{p}, \widetilde{X}_0)^\pm$.

Here, it is worthy to note that, when a point lies in the accumulation set, its whole X_0-orbit is also accumulated. As a consequence, Lemma 3.10 tells us that $W^u(p, X_s)$ accumulate to $g \cdot W^u(p, X_0)^\pm$ in the C^1-topology when s goes to $0_\pm$. Thanks to this C^1-convergence, it makes sense to compare the orientations. The result is the following: when $s \to 0_\pm$, then $W^u(p, X_s)$ accumulate to $\pm g \cdot W^u(p, X_0)^\pm$.

Accumulation to $g^k \cdot W^u(p, X_0)^\pm$ for some $k > 1$ is dictated by Lemma 3.13 depending on the sign of the ϕ-latitude (knowing $\chi(X_0) > 0$). We are now ready for proving Theorem 3.8 (1) in each of the above-enumerated cases.

Let $\lambda_-(\gamma)$ (resp. $\lambda_+(\gamma)$) denote the element of the Novikov ring Λ_u which is the contribution of $[\gamma]_\sim$ in $n(p, q)^{X_s}$ when $s < 0$ (resp. $s > 0$). We have to check the next formula up to the given $L > 0$ in each case (a.1)... (b.2).

$$(3.15) \qquad \lambda_+(\gamma) = (1 + g + g^2 + \cdots) \cdot \lambda_-(\gamma)$$

Case (a.1). — When $s \to 0_-$, the oriented unstable manifolds $W^u(p, X_s)$ accumulate to $-g \cdot W^u(p, X_s)^-$ and nothing else. Therefore, as $b \in \mathcal{H}^+$, we have $\lambda_-(\gamma) = [\gamma]$.

When $s \to 0_+$, then $W^u(p, X_s)$ accumulate to $+g^k \cdot W^u(p, X_0)^+$ for every $k > 0$ and will intersect $g^k \cdot \Delta_0$ transversely at a single point. Thus, we have $\lambda_+(\gamma) = (1 + g + g^2 + \ldots) \cdot [\gamma]$. The change of $\lambda_\pm(\gamma)$ from $s < 0$ to $s > 0$ is then given by Formula (3.15).

Case (a.2). — As $b \in \mathcal{H}^-$ and taking into account the accumulation described right above, we have: $\lambda_-(\gamma) = (1 - g) \cdot [\gamma]$ and $\lambda_+(\gamma) = [\gamma]$. Formula (3.15) is still fulfilled.

Case (b.1). — Here, the accumulation of $W^u(p, X_s)$ is dictated by Lemma 3.13 (2) and the reason for Formula (3.15) is more surprising than in the previous cases. When $s \to 0_-$, the manifolds $W^u(p, X_s)$ accumulate to $-g \cdot W^u(p, X_0)^-$ and to $-g^2 \cdot W^u(p, X_0)^+$ and then nothing else. When $s \to 0_+$, the manifolds $W^u(p, X_s)$ accumulate to $+g \cdot W^u(p, X_0)^+$ and nothing else.

As $b \in \mathcal{H}^+$, we have $\lambda_-(\gamma) = (1 - g^2) \cdot [\gamma]$ and $\lambda_+(\gamma) = (1 + g) \cdot [\gamma]$. Formula (3.15) is right since the identity $(1 + g + g^2 + \cdots)(1 - g^2) = 1 + g$ holds in the Novikov ring.

Case (b.2). — Accumulation is as right above. One derives that $\lambda_-(\gamma) = (1 - g) \cdot [\gamma]$ and $\lambda_+(\gamma) = [\gamma]$. The desired formula is still satisfied.

The proof of Theorem 3.8 is now complete since only one L is involved. □

Remark 3.15. — One could ask what happens when there is no critical points q of index $i(p) - 1$. The answer is the following. The dichotomy $\mathcal{S}_g^+, \mathcal{S}_g^-$ still exists by the sign of the character function. Since the *bifurcation factors* $(1 + g + g^2 + \cdots)$ and $(1 + g)$ do not depend on any q, one can associate them with each part of $\mathcal{S}_g \smallsetminus \mathcal{S}_g^0$, even if there is no q.

Proof of Theorem 1.2. — Here, the statement claims something to hold *for every* $L > -u(g)$ instead of *for a given* L. In that case, it is natural that some genericity condition should be required. The condition in question (that is, a subset of $\mathcal{S}_g$) is the intersection of all conditions: $X_0 \in \mathcal{S}_{g,L}$ for $L \to +\infty$, each of them being the condition which makes Theorem 3.8 hold. A priori, this intersection could be empty. But thanks to Proposition 3.6, this intersection is a residual set in $\mathcal{S}_g$ and we are done. □

4. The doubling phenomenon. Proof of Theorem 1.4

4.1. Notations and statement

In this section, we state and prove the refined version of Theorem 1.4 which is given right after specifying some definitions and notations. It is about the local structure of $\mathcal{S}_g^0$ (the co-oriented locus in $\mathcal{S}_g$ where the character function χ vanishes) in the complement of $\mathcal{S}_g^{0,0}$, the latter being the locus where both of the extended ϕ-latitude and ψ-latitude vanish.[16]

DEFINITION 4.1.

(1) *Let $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) be the set of positive (resp. negative) real numbers. The open set $\mathcal{S}_g^{0,\pm} \subset \mathcal{S}_g^0$ is defined by the sign of the extended ϕ-latitude, that is: $X \in \mathcal{S}_g^{0,\pm} \Leftrightarrow \omega_\phi(X) \in \mathbb{R}_\pm$.*
(2) *Let $X_{0,0} \in \mathcal{S}_g^0 \setminus \mathcal{S}_g^{0,0}$. Let $(\mathcal{D}(s,t) := X_{s,t})$ be a germ at $X_{0,0}$ of a two-parameter family in $\mathcal{F}_\alpha$, the space of adapted α-gradients. This germ is said to be* adapted to the pair $(\mathcal{S}_g, \mathcal{S}_g^0)$ *if the following conditions are fulfilled:*
  (a) *The one-parameter family $(\mathcal{D}(0,t))_{t \in \mathcal{O}p(0)}$ is contained in $\mathcal{S}_g$, transverse to $\mathcal{S}_g^0$ and $\partial_t \mathcal{D}(0,0)$ is non-zero and points towards $\mathcal{S}_g^+$.*
  (b) *The partial derivative $\partial_s \mathcal{D}(0,0)$ is transverse to $\mathcal{S}_g$ and points towards its positive side.*

In particular, such a $\mathcal{D}$ is transverse to $\mathcal{S}_g^0$.

THEOREM 4.2. — *Let $\mathcal{D}$ be a germ of 2-discs transverse to $\mathcal{S}_g^0 \setminus \mathcal{S}_g^{0,0}$ and adapted to the pair $(\mathcal{S}_g, \mathcal{S}_g^0)$. Then $\mathcal{D}$ intersects $\mathcal{S}_{g^2}$ transversely along an arc of $\mathcal{S}_{g^2}^+$. The trace on $\mathcal{D}$ of the strata $\{\mathcal{F}_\alpha, \mathcal{S}_g \cup \mathcal{S}_{g^2}^+, \mathcal{S}_g^{0,\pm}\}$ is C^1-diffeomorphic to*

$$\{\mathbb{R}^2, \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}_\pm, \{(0,0)\}\}.$$

Moreover, the natural co-orientation of $\mathcal{S}_{g^2}$ restricts to the natural co-orientation of $\mathcal{S}_g^0$ in $\mathcal{S}_g$ or to its opposite depending upon $\mathcal{S}_{g^2}$ approaches $\mathcal{S}_g^{0,+}$ or $\mathcal{S}_g^{0,-}$ respectively (Figure 2.1).

Finally, if $X_{0,0}$ also fulfills the generic property $\mathcal{S}_{g,\infty}$ (Definition 3.5) then the germ $\mathcal{D}$ does not meet $\mathcal{S}_{g^k}$ for $k > 2$.

Actually, the proof of Proposition 3.6 yields that the last property is generic in $\mathcal{S}_g^0$. Indeed, the new constraint $\chi(X) = 0$ involves a compact domain of $W^u(p, X)$ only.

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[16] The extended latitudes are defined in (2.15).

We first prove Theorem 4.2 for particular germs $\mathcal{D}$ that we call *elementary*. Such a germ consists of a one-parameter family of positive *normalized* crossing paths of $\mathcal{S}_g$ in the sense of Definition 2.13 with some additional requirements. The definition of elementary crossing path looks a bit strange, but it is suggested by the *toy model* of crossing $\mathcal{S}_g$ in which all moving objects are domains of affine subspaces in the coordinates of $\mathcal{M}_p$.

4.2. Elementary crossing path

Let $(X_s)_s$ be a normalized positive crossing path of $\mathcal{S}_g$. In addition to the normalization (Proposition 2.15), we are still allowed to prescribe more special dynamics of X_s; the *perturbed holonomy* will be specified near the respective homoclinic orbit ℓ of $X_0 \in \mathcal{S}_g$

Let $a^\pm = \ell \cap \partial_\pm \mathcal{M}_p$; let $(\phi_0, 0, -)$ and $(-, 0, \psi_0)$ be the respective mutispherical coordinates of a^- and a^+. Consider the spherical annulus $\mathbb{A}_{\psi_0} := \Sigma^- \times (0,1) \times \{\psi_0\} \subseteq \partial^- \mathcal{M}_p$. Assume the ψ-latitude of a^+ different from zero (by its very definition it is always the case when $X_0 \in \mathcal{S}_g^{0,\pm}$). Therefore, whatever the perturbed holonomy H_s along ℓ, the inverse image $D_1'(s)$ of Σ^+ by H_s is transverse to $\mathbb{A}_{\psi_0}$ for every s close to 0. Call $b(s)$ the intersection point $D_1'(s) \cap \mathbb{A}_{\psi_0}$ when the intersection is non-empty; this is the case either when $s < 0$ or $s > 0$ depending on whether the ψ-latitude $\cos_\psi(a^+)$ is *negative* or *positive*. By normalization, $b(s)$ belongs to the ray $\{(\phi_0, r, \psi_0) \mid r \geqslant 0\}$. Below, we use Notation 3.9.

DEFINITION 4.3. — *The germ $(X_s)_s$ is said to be* elementary *if it is normalized (Definition 2.13) and the following conditions are fulfilled:*

(1) *The disc $D_1(s) := H_s(\Sigma^-) \cap \{z = 1\}$ moves in the meridian disc $\{\psi = \psi_0\}$ while remaining parallel to the preferred hyperplane Δ^ϕ (2.8).*

(2) *Let $a^+(s)$ be the intersection of $D_1(s)$ with the pole axis. For every s, we have*

$$(4.1) \qquad\qquad \partial_s a^+(s) = 1.$$

(3) *For every s close to 0 the velocity of $b(s)$ is*

$$(4.2) \qquad\qquad \partial_s b(s) = -\frac{1}{\eta \cos_\psi(a^+)} \ .$$

Here, η stands for the holonomic factor of X_0 (Definition 2.9).

This definition makes sense only when the ψ-latitude of $a^+(X_0)$ is not 0, that is, when X_0 does not lie on the ϕ-axis $\mathcal{S}_g^\phi$ (2.14). This is always the case when X_0 belongs to $\mathcal{S}_g^0 \smallsetminus \mathcal{S}_g^{0,0}$.

LEMMA 4.4. — *Let $X_0 \in \mathcal{S}_g \smallsetminus \mathcal{S}_g^\phi$ be an α-gradient in normal form. Then there exists a germ of elementary path (X_s) passing through X_0 and depending smoothly on s in the C^1-topology.*

Proof. — Recall the tube T with coordinates (x, y, v, z) around the restricted homoclinic orbit ℓ of X_0. The holonomy H_0 is defined on a neighborhood of $\{z = 0\}$ in $\partial^- \mathcal{M}_p$ and valued in a neighborhood of $\{z = 1\}$ in $\partial^+ \mathcal{M}_p$. Since we are looking for a one-parameter perturbation (X_s) of X_0 whose properties are expressed in T, it is sufficient to describe it near T.

For $|s|$ small enough, the perturbed holonomy always reads $H_s = H_0 \circ K_s$ where K_s is a diffeomorphism of $\partial^- \mathcal{M}_p$ supported in its interior with $K_0 = \mathrm{Id}$. In order to satisfy conditions (4.1) and (4.2) of Definition 4.3, we first choose $a^+(s)$ and $D_1(s)$ before choosing K_s. For $a^+(s)$ we take the point in $\{\psi = \psi_0\}$ moving in the oriented pole axis with velocity $+1$ such that $a^+(0) = a^+$. For $D_1(s)$ we take the paralell disc to $D_1(0)$ passing through $a^+(s)$. Noting $a^- := a^-(0)$, one takes K_s, smooth with respect to s, such that:

$$\begin{cases} K_s(a^-) = H_0^{-1}(a^+(s)) \\ K_s(\Sigma^-) = H_0^{-1}(D_1(s)) \text{ in } \mathcal{O}p\{z = 0\} \end{cases}$$

Thus, the first two items are fulfilled. Note that, by normalization of X_0, the point $K_s(a^-)$ moves in a prescribed curve in the meridian $\{\phi = \phi_0\}$, namely the curve $H_0^{-1}(\mathbb{R}\nu_\phi)$. Its velocity at time $s = 0$ is the vector ∂_v^0. By (2.13), we have

$$\langle \partial_v^0, \nu_\psi \rangle = \frac{1}{\eta}.$$

We are now dealing with the last item. We impose $D'(s) = K_s^{-1}(D'(0))$ to move in the meridian $\{\phi = \phi_0\}$; this is possible as the point $\kappa_s := K_s^{-1}(a^-)$ already moves in this meridian by normalization. There are two more contraints: the first one is $\partial_s \kappa_s|_{s=0} = -\partial_v^0$ by Lemma 2.7; the second one is (4.2). This two constraints are compatible since $\langle \partial_s b(s), \nu_\psi \rangle = -\frac{1}{\eta}$.

For having a one-parameter family (K_s) of diffeomorphisms of $\partial^- \mathcal{M}_p$ converging C^1 to identity when s goes to 0, one has to choose conveniently $\partial_s K_s$ at time $s = 0$. This is easy to achieve, since the velocity distribution is given along transverse submanifolds in the extended space $(\partial^- \mathcal{M}_p) \times \mathcal{O}p(0)$. □

Remark 4.5. — Note the great difference between the normalization process of a crossing path and the building of an elementary crossing path. The first one is achieved by an ambient C^1-conjugation; so, it does not change the dynamics. The second one is a genuine bifurcation.

Clearly, Lemma 4.4 holds with parameters, for instance when the data is a one-parameter family in $\mathcal{S}_g \smallsetminus \mathcal{S}_g^{\phi}$. Then, the next corollary follows.

COROLLARY 4.6. — *Let* $X_{0,0} \in \mathcal{S}_g^0 \smallsetminus \mathcal{S}_g^{0,0}$ *and let* $\gamma(t) = (X_{0,t})_t$ *be a germ of path in* $\mathcal{S}_g$ *passing through* $X_{0,0}$ *and crossing* $\mathcal{S}_g^0$ *transversely, such that* $\frac{\partial \gamma}{\partial t}(0)$ *points towards* $\mathcal{S}_g^+$. *Then, there exists a two-parameter family* $\mathcal{D} = (X_{s,t})$ *of pseudo-gradients of* α *adapted to* $(\mathcal{S}_g, \mathcal{S}_g^0)$ *such that, for every* t *close to 0, the path* $s \mapsto X_{s,t}$ *is elementary. Moreover, there are such* $X_{s,t}$ *which are smooth with respect to the parameters in the* C^1-*topology.*

DEFINITION 4.7. — *Let* $\mathcal{D}$ *be a 2-disc in* $\mathcal{F}_\alpha$ *transverse to* $\mathcal{S}_g^0 \smallsetminus \mathcal{S}_g^{0,0}$ *and adapted to the pair* $(\mathcal{S}_g, \mathcal{S}_g^0)$. *We say that* $\mathcal{D}$ *is elementary if it is made of a one-parameter family of elementary crossing paths as in* Corollary 4.6.

Proof of Theorem 4.2. — First, we prove the theorem in the particular case where the transverse disc $\mathcal{D}$ is elementary. Even in this particular case the proof is slightly different depending on where the base point $X_{0,0}$ lies, either in $\mathcal{S}_g^{0,-}$ or in $\mathcal{S}_g^{0,+}$. In each case, the proof has three items:

(1) What is the trace of $\mathcal{S}_{g^2}$ on $\mathcal{D}$? Is there a non-empty trace of $\mathcal{S}_{g^k}$ for $k \neq 1$ or 2?

(2) Is $\mathcal{D}$ transverse to $\mathcal{S}_{g^2}$? How is the positive co-orientation of $\mathcal{S}_{g^2}$?

(3) Which part of $\mathcal{S}_{g^2}$, either $\mathcal{S}_{g^2}^+$ or $\mathcal{S}_{g^2}^-$, is intersected by $\mathcal{D}$?

Case $X_{0,0} \in \mathcal{S}_g^{0,-}$. — In other words, $a^-(X_{0,0})$ has a negative ϕ-latitude.

(1). — The pseudo-gradient $X_{0,t}$ has a homoclinic orbit ℓ_t based in p, a zero of α, and the ϕ-latitude of $a^-(X_{0,t})$ lies in $[-1,0)$ for every $t \in \mathcal{O}p\,(0)$. Denote by ϕ_t the spherical coordinate of $a^-(X_{0,t})$. We use the tube T around ℓ_0 and its extremities: $\{z = 0\} \subset \partial^- \mathcal{M}_p$ and $\{z = 1\} \subset \partial^+ \mathcal{M}_p$.

For simplicity, we specify even more the path $(X_{0,t})_t$ by adding some assumptions (the discussion is similar with the other cases of latitudes by using other specifications [17]):

(i) The ϕ-equator of $X_{0,t}$ is fixed and the ϕ-latitude $\cos_\phi(a^-(X_{0,0}))$ is not equal to -1.

(ii) The point $a^+(X_{0,t}) = \Sigma^+ \cap \ell_t$ and the ψ-equator of $X_{0,t}$ are fixed.

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[17] If $\omega_\phi(X_{0,0}) = -1$, one makes $\frac{\partial \eta}{\partial t}(X_{0,t}) > 0$. Since $\omega_\psi(X_{0,0})$ must be positive, $\frac{\partial \chi}{\partial t}(X_{0,t}) > 0$.

(iii) The holonomic factor $\eta(X_{0,t})$ remains constant and is denoted by η.

Note that (i) allows one to take $(X_{0,t})_t$ positively transverse to $\mathcal{S}_g^0$ in $\mathcal{S}_g$ while satisfying (ii) and (iii). More precisely, one makes the ϕ-coordinate ϕ_t of $a^-(X_{0,t})$ vary on t by increasing the ϕ-latitude.

Denote the spherical coordinate of $a^+(X_{0,t})$ by ψ_0, independent of t. In this setting, as the paths $s \mapsto X_{s,t}$ are elementary the discs $D_1(s,t) \subset \{z = 1\}$, image of Σ^- by the perturbed holonomy, depend only on s and are denoted by $D_1(s)$. For every $s \neq 0$, their images by the descent map are discs $C_1(s)$ contained in the *spherical annulus*

$$\mathbb{A}_{\psi_0} := \{(\phi, r, \psi_0) \mid \phi \in \Sigma^-, r \in [0, 1]\}.$$

When s goes to 0_-, by Lemma 3.10 the discs $C_1(s)$ accumulate to the negative hemisphere $\mathcal{H}^-(\Sigma^-)$.

Since $s \mapsto X_{s,t}$ is elementary and $\cos_\psi(\psi_0) > 0$, the disc $D_1'(s,t)$, preimage in $\{z = 0\}$ of Σ^+ by the respective perturbed holonomy, intersects $\mathbb{A}_{\psi_0}$ in one point $b(s,t)$ when $s \leqslant 0$ and nowhere when $s > 0$, according to Definition 4.3. When t is fixed, $b(s,t)$ moves on the ray $\{(\phi_t, r, \psi_0) \mid r \geqslant 0\}$ and its velocity is given by the formula in Definition 4.3. According to Remark 3.11, we have:

$$(4.3) \qquad X_{s,t} \in \mathcal{S}_{g^2} \quad \text{if and only if} \quad b(s,t) \in C_1(s).$$

Denote by $c_1(s,t)$ the intersection point of $C_1(s)$ with the meridian disc $\{\phi = \phi_t\}$. When t is fixed, $c_1(s,t)$ also moves on the ray $\{(\phi_t, r, \psi_0) \mid r \geqslant 0\}$ and its radial velocity is the same as the one of its lift through Desc in $D_1(s) \subset \partial M_p^+$. Therefore,

$$\partial_s c_1(s,t) = \frac{1}{\cos_\phi(\phi_t)}.$$

As $X_{0,0}$ belongs to $\mathcal{S}_g^0$, that is $\chi(X_{0,0}) = 0$, and by formula (4.2), the curves $b(s,0)$ and $c_1(s,0)$, defined for $s < 0$, have the same radial velocities. Since both tend to $a^-(X_{0,0})$ on the same ray when s goes to 0_-, we have $b(s,0) = c_1(s,0)$ for every s. Then, (4.3) tells us that $X_{s,0} \in \mathcal{S}_{g^2}$ for every s close to 0 negatively.

For $t \neq 0$ and $s < 0$, the radial velocities of $c_1(s,t)$ and $b(s,t)$ are distinct while their limits when s goes to 0 coincide. Therefore, (4.3) tells us that $X_{s,t}$ never lies in $\mathcal{S}_{g^2}$ for $s < 0$.

When $s > 0$, the discs $C_1(s)$ accumulate to the positive hemisphere $\mathcal{H}^+(\Sigma^-)$. There is no chance for $C_1(s)$ to intersect $D_1'(s,t)$ which is far from any point in $\mathcal{H}^+(\Sigma^-)$.

What about $\mathcal{S}_{g^k}$? If $k \leqslant 0$, we have $u(g^k) \geqslant 0$ and there is no homo-clinic orbit in the homotopy class g^k. When $k > 2$, we have to discuss the successive passages of the unstable manifold $W^u(p, X_{s,t})$ in $\partial^- \mathcal{M}_p$, more precisely in $\{z = 0\}$.

By Lemma 3.13, if $t > 0$, that is $\chi(X_{0,t}) > 0$, and $s < 0$ only the discs $C_2(s,t)$ of the second passage are non-empty, but they accumulate to the positive hemisphere $\mathcal{H}^+(\Sigma^-)$. Therefore, no further passage could give rise to a homoclinic orbit. When $s > 0$, even the second passage does not exist.

If $t < 0$, one is able to see that there are infinitely many passages in $\{z = 0\}$. But, by velocity considerations $C_k(s,t)$ never meet $D_1'(s,t)$. We do not give more details here because this is similar to the symmetric case $X_{0,0} \in \mathcal{S}_g^{0,+}$ and $t > 0$ where the analysis of velocities will be completely achieved (see inequality (4.7)). Thus, the first item of case $X_{0,0} \in \mathcal{S}_g^{0,-}$ is proved.

(2). — The reason for transversality to $\mathcal{S}_{g^2}$ relies again on some com-putations of velocity. Define for $s \leqslant 0$:

$$\delta(s,t) := v\left(c_1(s,t)\right) - v\left(b_1(s,t)\right) \quad \text{and} \quad V(t) := \left.\frac{\partial \delta(s,t)}{\partial s}\right|_{s=0}.$$

Although points are not the same, this velocity $V(t)$ at $s = 0$ is easily checked to be the same as the velocity computed in Lemma 3.12. Then, for every t close to 0 we have:

$$(4.4) \qquad V(t) = \eta \, \frac{\cos_\psi(\psi_0)}{\cos_\phi(\phi_t)} + 1 \quad \text{that implies} \quad \frac{dV(t)}{dt} < 0 \,.$$

Here are a few words explaining this implication: by the choice of ∂_t tangent to $\mathcal{S}_g$ and pointing to $\chi > 0$, we have $d\chi/dt > 0$; by the above-mentioned specification (ii), $cos_\psi(\psi_t)$ is independent of t, and hence, $d(cos_\phi(\phi_t))/dt > 0$ and $dV/dt < 0$.

By definition of the character function, we have $V(0) = 0$ and hence, $V(t) < 0$ for $t > 0$. Define $V(s,t) := \partial_s \delta(s,t)$. By construction of $(X_{s,t})$, we have $V(s,0) = 0$ for every $s < 0$ close to 0. By (4.4), the second partial derivative $\partial_{ts}^2 \delta(s,t)$ is negative for every (s,t) close to $(0,0)$ with $s \leqslant 0$ (here we use the smoothness with respect to the parameters.)[18] By integrating

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[18] The vector fields in a normalized crossing path are not smooth with respect to (s,t). Their holonomy is C^1 only. Nevertheless, as the C^1-maps (of degree zero) $\partial^- \mathcal{M}_p \to \partial^+ \mathcal{M}_p$ form a Banach manifold it makes sense to consider a smooth family of such holonomies.

in the variable s from $s_0 < 0$ to 0 and noticing that $\delta(0,t) = 0$, we get:

$$(4.5) \quad \frac{\partial \delta}{\partial t}(s_0, t) = -\frac{\partial}{\partial t}\left(\int_{s_0}^0 \frac{\partial \delta}{\partial s}(s,t)\,ds\right) = -\left(\int_{s_0}^0 \partial_{ts}^2 \delta(s,t)\,ds\right) > 0.$$

For $t = 0$, this is exactly the transversality of $\mathcal{D}$ to $\mathcal{S}_{g^2}$ at $X_{s_0, 0}$.

We are now looking into orientation. Take $s_0 < 0$ such that $b(s_0, 0)$ lies in $\{z = 0\}$. It belongs to a homoclinic orbit ℓ' in the homotopy class g^2. There is a tube T' around ℓ' with coordinates (x', y', v', z'). The y'-axis is contained in $D_1'(s_0, 0)$ and is given a co-orientation which follows from the co-orientation of $D_1'(0,0)$ by continuity. The x'-axis is contained in $C_1(s_0)$. Its projection to the x-axis is orientation reversing (Lemma 3.10). Therefore:

$$v(\partial_{v'}) < 0.$$

By (4.5) we have: $\frac{\partial}{\partial t}[v(c_1(s_0, t)) - v(b_1(s_0, t))]_{|t=0} = \frac{\partial \delta}{\partial t}(s_0, 0) > 0$. By replacing v with v' in the last inequality, we get:

$$\frac{\partial}{\partial t}\left[v'\left(c_1(s,t)\right) - v'\left(b_1(s,t)\right)\right]_{|t=0} < 0\,.$$

This translates the fact that ∂_t points to the negative co-orientation of $\mathcal{S}_{g^2}$ for $s < 0$ while, for $s = 0$, ∂_t defines the positive co-orientation of $\mathcal{S}_g^0$ in $\mathcal{S}_g$.

(3). — Let $L > 0$. Consider a small circle $\gamma \subset \mathcal{D}$ centered at the origin of the coordinates (s,t) and turning positively with respect to the orientation given by these coordinates. If the radius of γ is small enough[19] and if $X_{0,0}$ is generic, γ avoids all codimension-one strata in $\mathcal{F}_\alpha$ except:

- $\mathcal{S}_g$ which is crossed once in $\mathcal{S}_g^-$ positively, and once in $\mathcal{S}_g^+$ negatively,
- $\mathcal{S}_{g^2}$ which is crossed once positively according to the above discussion.

As noted in Remark 3.15 each crossed signed stratum is endowed with a *bifurcation factor*. The product of these factors should be equal to 1 up to L in the Novikov ring after traversing γ once. The bifurcation factor of the a small sub-arc of γ crossing $\mathcal{S}_{g^2}$ is only known up to some ambiguity. Call $m(g)$ this bifurcation factor. The following (commutative) product is

$$m(g) \cdot (1 + g + g^2 + \cdots)^{-1} \cdot (1 + g) = 1 \pmod{L}.$$

Then, $m(g) = (1 - g^2)^{-1} \pmod{L}$, that can only happen if the crossing of $\mathcal{S}_{g^2}$ takes place in $\mathcal{S}_{g^2}^+$. The proof of Theorem 4.2 is complete for an elementary 2-disc in the case $X_{0,0} \in \mathcal{S}_g^{0,-}$.

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[19] The more L is large, the more this radius has to be small.

Case $X_{0,0} \in \mathcal{S}_g^{0,+}$. — In other words, $a^-(X_{0,0})$ has a positive ϕ-latitude.

The discussion is led in the same manner as in the previous case with same notation. We only mention the main differences. Here, $a^-(X_{0,0})$ belongs to the positive hemisphere of Σ^- while ψ_0 belongs to the negative hemisphere of Σ^+. The discs $C_1(s)$ intersect $\{z = 0\}$ only when $s > 0$. Therefore, for $s < 0$ there is no chance for meeting $\mathcal{S}_{g^k}$, for any $k \neq 0$.

By Lemma 3.13 there are infinitely many passages $C_k(s)$, $k \geqslant 1, s > 0$ of $W^u(p, X_{s,t})$ in $\partial^- \mathcal{M}_p$ meeting $\{z = 0\}$. Recall that the $(i-1)$-discs $C_k(s)$ do not depend on t. The fact that $X_{s,t}$ belongs to $\mathcal{S}_{g^2}$ if and only if $s > 0$ and $t = 0$ is proved exactly as in the previous case.

Then, we are left to show that for every $k > 2$, $\mathcal{S}_{g^k}$ does not intersect $\mathcal{D}$. Here it is important to think of $\mathcal{D}$ as a germ because for a given representative this result is not true; when k increases, the domain of the representative has to be restricted. Let $C_k(s,t)$ denote the $(i-1)$-disc in $\partial^- \mathcal{M}_p$ corresponding to the k-th passage of the unstable manifold $W^u(p, X_{s,t})$ (see Lemma 3.13); let $D_1'(s,t)$ denote the $(n-i-1)$-disc corresponding to the first passage of the stable manifold $W^s(p, X_{s,t})$. Observe that $\mathcal{D}$ intersects $\mathcal{S}_{g^{k+1}}$ if and only if, for (s,t) close to $(0,0)$, $C_k(s,t)$ intersects $D_1'(s,t)$. This translates into the next equation:

$$c_k(s,t) = d_k'(s,t)$$

where $c_k(s,t)$ and $d_k'(s,t)$ are the only two points of $\{z = 0\}$ lying respectively on $C_k(s,t)$ and $D_1'(s,t)$ which have the same (x,y)-coordinates. Then, the above equation becomes:

$$(4.6) \qquad v\big(c_k(s,t)\big) = v\big(d_k'(s,t)\big).$$

When s goes to 0, these two points go to the same point $a^-(X_{s,t}) \in \Sigma^-$. By computation from the proof of Lemma 3.13, we know that:

$$(4.7) \qquad \frac{\partial}{\partial s} \left[v\big(c_k(s,t)\big) - v\big(d_k'(s,t)\big) \right]_{|_{s=0}} \neq 0.$$

It follows that, for s close to 0 (closeness depending on t), the equation (4.6) cannot be fulfilled.

The answer to questions (2) and (3) are exactly as in the case $X_{0,0} \in \mathcal{S}_g^{0,-}$. Then, Theorem 4.2 is proved for elementary 2-discs as in Definition 4.7.

For finishing the proof of Theorem 4.2, it is suitable to use some C^1-topology. More precisely, we choose a system of finitely many closed flow boxes $(B_j)_{j \in J}$ of $X_{0,0}$ whose end faces $\partial_{\pm} B_j$ are tangent to $\ker \alpha$ and union covers M except a small open neighborhood N of the zero set of α. It is assumed that when slightly shrinking every B_j to B_j' tangentially to

ker α, the union $\bigcup_j B'_j$ still covers $M \smallsetminus N$. Fix a closed C^0-neighborhood U of $X_{0,0}$ among the *uniquely integrable* vector fields whose *transverse holonomy is well defined* for every $j \in J$ from $\partial_- B'_j$ to $\partial_+ B_j$ and of class C^1. This U, endowed with the C^0-topology of vector fields and the C^1-topology of holonomies $B'_j \to B_j$, $j \in J$, may be thought of as a closed ball in a Banach manifold.

By Proposition 2.15, there exists a neighborhood V of $X_{0,0}$ in $\mathcal{S}_g^0$ such that the following properties are fulfilled for every $Y \in V$:

- there exists a C^1-diffeomorphism Υ_Y of M which carries Y to a vector field in normal form, that is, $(\Upsilon_Y)_* Y$ is normalized;
- Υ_Y preserves the strata $\mathcal{S}_g$, $\mathcal{S}_g^0$ and $\mathcal{S}_g^{0,0}$ (Proposition 2.18).

It is easy to make Υ_Y depend continuously on Y in the C^1-topology with the property that $\Upsilon_Y = \mathrm{Id}$ when Y is already in normal form.

Let $\mathcal{D}$ be an elementary 2-disc centered at $X_{0,0}$. If $Y \in V$ is close enough to $X_{0,0}$ and normalized, one finds an elementary 2-disc $\mathcal{D}_Y$ centered at Y, depending C^1 on Y and equal to $\mathcal{D}$ if $Y = X_{0,0}$. If $Y \in V$ is not normalized, we still have a 2-disc centered at Y, namely

$$\mathcal{D}_Y := (\Upsilon_Y)_*^{-1} \, \mathcal{D}_{(\Upsilon_Y)_* Y}.$$

This $\mathcal{D}_Y$ is not elementary in the strict sense but it is conjugate to an elementary 2-disc in U. As Υ_Y preseves the stratification, in particular $\mathcal{S}_{g^2}$, the intersection of $\mathcal{D}_Y$ with the different strata under consideration is the same as in the elementary case. Finally, we have a C^1-map

$$F : V \times [-1, +1]^2 \longrightarrow U, \quad (Y, s, t) \longmapsto F(Y, s, t)$$

which meets $\mathcal{S}_{g^2}$ if and only if $t = 0$ and $s \in \mathbb{R}_{\pm}$ depending upon $X_{0,0} \in \mathcal{S}_g^{0,\pm}$. Moreover, the germ of F at $(s, t) = (0, 0)$ avoids all $\mathcal{S}_{g^k}$ for $k \neq 1, 2$.

One checks that $span\{\partial_s F, \partial_t F\}$ at $(s, t) = (0, 0)$ is transverse to $\mathcal{S}_g^0$ in U. The Inverse Function Theorem is available and states that, for V small enough, F is a C^1-diffeomorphism onto its image $\mathcal{N}$, an open set in U. Therefore $\mathcal{N}$ has a product structure and a projection $P : \mathcal{N} \to [-1, +1]^2$ such that, for every $X \in \mathcal{N}$, the following equivalences hold:

$$(4.8) \quad \begin{cases} X \in \mathcal{S}_g & \Longleftrightarrow & (s \circ P)(X) = 0 \\ X \in \mathcal{S}_g^0 & \Longleftrightarrow & P(X) = (0, 0) \\ X \in \mathcal{S}_{g^2} & \Longleftrightarrow & (t \circ P)(X) = 0 \text{ and } (s \circ P)(X) \in \mathbb{R}_{\pm} \\ & & \text{depending on } X_{0,0} \in \mathcal{S}_g^{0,\pm}. \end{cases}$$

Let $\mathcal{D}'$ be any germ of two-parameter family centered in $X_{0,0}$ transverse to $\mathcal{S}_g^0 \smallsetminus \mathcal{S}_g^{0,0}$ and contained in $\mathcal{N}$. Its projection $P \circ \mathcal{D}'$ is submersive. The equivalences (4.8) finish the proof of Theorem 4.2. □

Appendix A. Proof of the key Fact 1.2

Let us recall the statement in question.

PROPOSITION A.1. — *Let X be an α-gradient which is assumed Kupka–Smale. Let p and q be two zeroes of α of respective Morse indices k and $k-1$. Then, for every $L > 0$ the number of connecting orbits from p to q whose α-length is bounded by L is finite.*

Proof. — The proof mainly consists of comparing the α-length of any X-trajectory γ drawn on the unstable manifold $W^u(p, x)$ to the distance of its end points after lifting γ to the universal cover of M. By definition, the α-length $\mathcal{L}(\gamma)$ is additive with respect to any finite subivision of the considered trajectory γ.

First part. — For a zero z of α and a trajectory γ descending from the top of a Morse model $\mathcal{M}(z)$ to its bottom without getting out, $\mathcal{L}(\gamma)$ is equal to the oscillation of any local primitive of α on $\mathcal{M}(z)$. And hence, it does not depend on γ. Without loss of generality, we may assume that this oscillation is the same for every zero of α; it is denoted by h. Therefore, given a trajectory γ of α-length bounded by L, the number κ of segments traced on γ by the compact union $\mathcal{M} := \bigcup_{z \in Z(\alpha)} \mathcal{M}(z)$, excluding the two end segments of γ, fulfills

$$(A.1) \qquad\qquad \kappa \leqslant \frac{L}{h}\,.$$

In the complement of $\mathcal{M}$ in M denoted by $\mathcal{M}^*$, that is, away from the zeroes of α, there is some positive constant C such that we have $|X(x)| \geqslant C$ for every $x \in \mathcal{M}^*$. Denote by $\lambda(\gamma) := \int_\gamma |\dot\gamma|$ the length of a path γ and set $\gamma^* := \gamma \cap \mathcal{M}^*$. Then, for every X-trajectory γ whose α-length is bounded by L, we deduce

$$(A.2) \qquad C\lambda(\ell^*) = C\int_{\ell^*} |X(\gamma(t))\,dt| \leqslant \int_{\ell^*} |X(\gamma(t))|^2 = \left|\int_{\ell^*} \alpha\right| \leqslant L\,.$$

Here the variable t is the time of the flow of X.

Denote by $W_L(p)$ the union of the X-trajectories descending from p whose α-length is less than L. This is an open domain in the unstable manifold $W^u(p, X)$; it is homeomorphic to an open ball whose dimension is k. Let $\widetilde{M} \xrightarrow{\pi} M$ be the universal cover of M and let $\widetilde{p} \in \pi^{-1}(p)$. Let $\widetilde{X}$ be the lift of X to $\widetilde{M}$. It is a hyperbolic vector field and the unstable manifold $W^u(\widetilde{p}, \widetilde{X})$ is the lift of $W^u(p, X)$ through $\widetilde{p}$. Moreover, its truncation $W_L^u(\widetilde{p}, \widetilde{X})$ is a lift of $W_L^u(p, X)$.

Let ℓ be an X-trajectory descending from p in $W_L^u(p, X)$, let e be its end point. Take its lift $\widetilde{\ell}$ from $\widetilde{p}$ and denote by $\widetilde{e}$ its end point. One looks at the subdivision S of $\widetilde{\ell}$ marked by its crossings with $\pi^{-1}(\mathcal{M})$. From (A.1), (A.2) and the triangular inequality applied to the vertices of S, one deduces that the lifted distance satisfies

$$(A.3) \qquad d(\widetilde{p}, \widetilde{e}) \leqslant \frac{L}{C} + \frac{L}{h}\delta =: R(L)$$

where δ stands for the maximal diameter of $\mathcal{M}(z)$, $z \in Z(\alpha)$. Therefore, we have an inclusion

$$(A.4) \qquad W_L^u(\widetilde{p}, \widetilde{X}) \subset B(\widetilde{p}, R(L))$$

where $B(\widetilde{p}, R)$ stands for the ball of $\widetilde{M}$ centered at $\widetilde{p}$ of radius R and where $R(L)$ is the right hand side of (A.3). The consequence of these elementary estimations is that the closure $\mathrm{cl}_L(\widetilde{p})$ of $\widetilde{W}_L^u(\widetilde{p}, \widetilde{X})$ is compact. In particular, it contains finitely many lifts of the zero q that we are interested in. Indeed, these lifts cannot accumulate as their mutual distance is bounded from below.

Second part. — The end of the proof uses the Kupka–Smale assumption (KS). Let $\widetilde{f}$ be a global primitive of $\pi^*\alpha$; it exists since $\widetilde{M}$ is simply connected. The descending gradient of $\widetilde{f}$ with respect to the lifted metric is equal to $\widetilde{X}$. The truncation of $W^u(\widetilde{p}, \widetilde{X})$ to the upper level set $\{\widetilde{f} > \widetilde{f}(\widetilde{p}) - L\}$ is exactly the truncation $W_L^u(\widetilde{p}, \widetilde{X})$.

Let us enumerate $\widetilde{q}_1, \ldots, \widetilde{q}_m$ the lifts of q which belongs to $\mathrm{cl}_L(\widetilde{p})$. Now, we can argue as in usual Morse theory. For $1 \leqslant j \leqslant m$, consider the Morse model $\mathcal{M}(\widetilde{q}_j)$ and the so-called co-sphere Σ_j, a sphere of dimension $(n-k)$ in the top of $\mathcal{M}(\widetilde{q}_j)$. By the KS assumption, the following two properties hold for every $j = 1, \ldots, m$:

(1) the singular part (that is the frontier) of $\mathrm{cl}_L(\widetilde{p})$ avoids Σ_j;
(2) the regular part, that is $W_L^u(\widetilde{p}, \widetilde{X})$, is transverse to Σ_j.

It is classical that the compactness of $\mathrm{cl}_L(\widetilde{p})$ joined to these two properties implies the finiteness of $\Sigma_j \cap W_L^u(\widetilde{p}, \widetilde{X})$. Therefore, for every j, there are finitely many orbits of $\widetilde{X}$ descending from $\widetilde{p}$ and ending at $\widetilde{q}_j$. This is the desired finiteness. $\qquad\qquad\square$

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