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ON p-ADIC ANALOGUES OF THE BIRCH AND SWINNERTON-DYER CONJECTURE FOR GARRETT L-FUNCTIONS

**by Massimo BERTOLINI,
Marco Adamo SEVESO & Rodolfo VENERUCCI**

ABSTRACT. — This article formulates a p-adic analogue of the Birch and Swinnerton-Dyer conjecture for a p-adic L-function associated to a triple of Hida families of modular forms. This involves the construction of a p-adic regulator, obtained by building on Nekovář theory of Selmer complexes. Moreover, it is proved that our conjectures imply the "Elliptic Stark Conjectures" of Darmon, Lauder and Rotger.

RÉSUMÉ. — Cet article formule un analogue p-adique de la conjecture de Birch et Swinnerton-Dyer pour une fonction L p-adique associée à un triplet de familles de Hida de formes modulaires. Cela nécessite la construction d'un régulateur p-adique, obtenu en s'appuyant sur la théorie des complexes de Selmer de Nekovář. De plus, il est prouvé que nos conjectures impliquent les « Elliptic Stark Conjectures » de Darmon, Lauder et Rotger.

Introduction

Let A be an elliptic curve over the field $\mathbf{Q}$ of rational numbers and let ϱ_1, ϱ_2 be a pair of two-dimensional odd Artin representations of the absolute Galois group of $\mathbf{Q}$. Set $\varrho = \varrho_1 \otimes \varrho_2$ and denote by K_ϱ the extension of $\mathbf{Q}$ cut out by ϱ. Assume the self-duality hypothesis $\det(\varrho_1) = \det(\varrho_2)^{-1}$. The equivariant Birch and Swinnerton-Dyer conjecture aims at understanding the ϱ-component $A(K_\varrho)^\varrho$ of the Mordell–Weil group of A/K_ϱ in terms of the complex L-function $L(A, \varrho, s)$ of A twisted by ϱ.

This question is better understood, if one assumes that ϱ_1 and ϱ_2 are induced by finite order Hecke characters of the same imaginary quadratic

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field K. In this case ϱ factors as the direct sum of the Artin representations induced from K to $\mathbf{Q}$ by two ring class characters χ and ψ. This is mirrored in the decomposition of Mordell–Weil groups

$$A(K_\varrho)^\varrho = A(K_\chi)^\chi \oplus A(K_\psi)^\psi,$$

where $A(K_\chi)^\chi$ denotes the χ-component of the Mordell–Weil group of A over the extension of K cut out by χ, and likewise for $A(K_\psi)^\psi$. On the level of L-functions one has the corresponding factorisation

$$L(A, \varrho, s) = L(A/K, \chi, s) \cdot L(A/K, \psi, s).$$

As a consequence, one is led to consider the two factors separately. Focusing on the terms corresponding to χ, if the order of vanishing of $L(A/K, \chi, s)$ at $s = 1$ is at most one, the fundamental work of Gross–Zagier [19] and Kolyvagin [23], as complemented by [4, 24], establishes most of the Birch and Swinnerton-Dyer conjecture. In particular, the presence of points of infinite order in $A(K_\chi)^\chi$ is accounted for by the Heegner point construction. In the case of the Mordell–Weil group of $A/\mathbf{Q}$, Kato's Euler system [22] provides an alternate approach to the above results in analytic rank zero. For rank one, specialisations of Kato's class can be related to Heegner points, via a somewhat indirect method as in [6]. In higher rank situations, a canonical construction of rational points is not generally available. As a partial replacement, techniques of Iwasawa theory can be invoked, in order to construct non-trivial classes in the p-adic Selmer group of A/K_χ and hence of A/K_ϱ. These classes can sometimes be obtained as universal norms of Heegner points, and otherwise their existence may be seen as a consequence of the anticyclotomic main conjecture. See for example [3, 14, 20, 31] for the first type of construction, and [5, 7] for the second approach. It should be noted that these Selmer classes do not in general bear an explicit connection to algebraic points.

Needless to say, the situation is even more mysterious for a representation ϱ which does not factor as above as a direct sum of 2-dimensional representations. All known results rely heavily on p-adic methods, and notably on the arithmetic theory of triple product p-adic L-functions, which has received considerable attention in recent years. If $L(A, \varrho, 1)$ is non-zero, the article [17] by Darmon–Rotger has obtained the finiteness of $A(K_\varrho)^\varrho$. The proof is based on an explicit reciprocity law, of the kind evisaged by Perrin-Riou [28], which relates a *big* diagonal class to a triple product p-adic L-function. In particular, the specialisation of this class at the triple of weights $(2, 1, 1)$ corresponding to the pair (A, ϱ) encodes the special value $L(A, \varrho, 1)$, and can be used to bound the Mordell–Weil group.

More generally, assume in the rest of this discussion that the order of vanishing of $L(A, \varrho, s)$ at $s = 1$ is even. In the rank 2 case, [17] and the work [13] by Castella–Hsieh establish different relations between the structure of the p-Selmer group of A and the non-vanishing of the specialisation of the big diagonal class in the weights $(2, 1, 1)$. (This class is Selmer as a consequence of the explicit reciprocity law.) This poses the challenge of elucidating the relation between the resulting Selmer classes and rational points. As mentioned in the last part of this Introduction, this is a major theme of this paper, where the role of certain canonical p-adic regulators is made explicit. Furthermore, under the assumption that ϱ decomposes as above as the sum of the Artin representations induced by two Hecke characters of a quadratic number field (real or imaginary), the work [9] relates this specialisation to explicit logarithmic expressions involving Heegner and Stark–Heegner points on A. More germane to the setting of this paper are the results of [8, 10], which prove low rank cases of our p-adic equivariant Birch and Swinnerton-Dyer conjecture, under the assumption that p is a multiplicative prime for A. See also Remarks 1.3 for additional information.

The purpose of this article is twofold. The first objective is to formulate a p-adic analogue of the above equivariant Birch and Swinnerton-Dyer conjecture. Assume for simplicity (but see Section 1.1 for generalisations) that p is an ordinary prime for A and that ϱ_1 and ϱ_2 are irreducible. Let (f, g, h) be the triple of cuspidal modular forms associated to $(A, \varrho_1, \varrho_2)$ by the modularity theorems. Hida's theory associates to (f, g, h) a triple $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ of p-adic families of ordinary cuspidal modular forms, where $\boldsymbol{f}$ specialises in weight 2 to the unique ordinary p-stabilisation of f, while $\boldsymbol{g}_\alpha$ and $\boldsymbol{h}_\alpha$ specialise in weight 1 to a choice of p-stabilisations g_α and h_α of g and h respectively. Our conjecture replaces $L(A, \varrho, s)$ with a p-adic L-function $L_p^{\alpha\alpha}(A, \varrho)$ arising from the triple of p-adic families $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$. The L-function $L_p^{\alpha\alpha}(A, \varrho)$ interpolates the central critical values of the complex L-functions of $f_k \otimes g_l \otimes h_m$ at triples of classical weights (k, l, m) such that $k \geqslant l+m$, where f_k, g_l and h_m denotes the specialisation of $\boldsymbol{f}$, $\boldsymbol{g}_\alpha$ and $\boldsymbol{h}_\alpha$ at k, l and m respectively. A p-adic avatar of the Birch and Swinnerton-Dyer conjecture suggests that the behaviour of $L_p^{\alpha\alpha}(A, \varrho)$ at the triple of weights $(2, 1, 1)$ should reflect the arithmetic of A over K_ϱ. This is the content of our Conjecture 1.1, which states that the order of vanishing of $L_p^{\alpha\alpha}(A, \varrho)$ at $(2, 1, 1)$ is equal to the rank of the ϱ-component $A^\dagger(K_\varrho)^\varrho$ of the extended Mordell–Weil group of A/K_ϱ. Furthermore, it relates the leading term of $L_p^{\alpha\alpha}(A, \varrho)$ to the regulator $R_p^{\alpha\alpha}(A, \varrho)$ of a p-adic height pairing on this extended Mordell–Weil group, constructed in Section 2 by exploiting

Nekovář's theory of Selmer complexes associated to Hida's deformation of the Galois representations of (f, g_α, h_α).

The second objective of this article is to understand the Elliptic Stark Conjectures of Darmon, Lauder and Rotger [15, 16] within the conceptual framework of the p-adic variants of the Birch and Swinnerton-Dyer conjecture. Under the assumption that the Mordell–Weil rank is equal to 2, the above mentioned works obtained experimentally a relation between an iterated p-adic integral associated to the triple (f, g_α, h_α) and certain combinations of p-adic logarithms of rational points in the ϱ-component of the Mordell–Weil group of A. Section 3 (see in particular Conjecture 3.5 and Remarks 3.6) shows that these conjectural relations are a consequence of Conjecture 1.1, combined with a formula, established in Theorem 3.3 by building on the ideas and methods of [28, 30, 34], for the derivatives of the *big* diagonal class encoding $L_p^{\alpha\alpha}(A, \varrho)$ via the explicit reciprocity law.

1. The p-adic Birch and Swinnerton-Dyer conjecture

This section states the main conjecture of this paper, assuming the precise definition of the Garrett–Nekovář p-adic height pairings given in Section 2 below. To ease the exposition we state our conjecture for p-ordinary elliptic curves over $\mathbf{Q}$, i.e. p-stabilised ordinary weight-two newforms with trivial character and rational Fourier coefficients. See Section 1.1 below for possible generalisations.

Let $\overline{\mathbf{Q}} \subset \mathbf{C}$ be the field of algebraic numbers. Fix a rational prime $p > 3$, an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ and an embedding of $\overline{\mathbf{Q}}$ into $\overline{\mathbf{Q}}_p$. For positive integers k and m, a Dirichlet character $\chi : (\mathbf{Z}/m\mathbf{Z})^* \to \overline{\mathbf{Q}}^*$ and a subfield F of $\overline{\mathbf{Q}}_p$, denote by $M_k(m, \chi)_F$ the F-module of modular forms of weight k, level $\Gamma_1(m)$, character χ and Fourier coefficients contained in F, and by $S_k(m, \chi)_F$ its subspace of cuspidal modular forms. When χ is the trivial character, we omit it from the notation.

Let A be an elliptic curve defined over $\mathbf{Q}$ and let

$$\varrho = \varrho_1 \otimes \varrho_2$$

be the tensor product of two odd, two-dimensional Artin representations

$$\varrho_i : G_{\mathbf{Q}} \longrightarrow \mathrm{GL}(V_{\varrho_i}) \simeq \mathrm{GL}_2(\mathbf{Q}(\varrho))$$

of $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with coefficients in a number field $\mathbf{Q}(\varrho)$ (contained in $\overline{\mathbf{Q}}$), satisfying the *self-duality condition*

$$\det(\varrho_1) = \det(\varrho_2)^{-1}. \tag{1.1}$$

According to the modularity theorem of Wiles, Taylor–Wiles et al., the p-adic Tate module of $A/\mathbf{Q}$ with $\mathbf{Q}_p$-coefficients is isomorphic to the dual of the p-adic Deligne representation of the weight-two cuspidal newform

$$f = \sum_{n \geqslant 1} a_n(f) \cdot q^n \in S_2(N_f)_{\mathbf{Q}},$$

where N_f is the conductor of $A/\mathbf{Q}$ and $a_\ell(f) = 1 + \ell - |A(\mathbf{Z}/\ell\mathbf{Z})|$ for each prime $\ell \nmid N_f$. Similarly, the Serre conjecture, proved by Khare and Wintenberger, implies that ϱ_1 and ϱ_2 are isomorphic respectively to the duals of the Deligne–Serre representations associated with weight-one normalised Hecke eigenforms

$$g = \sum_{n \geqslant 0} a_n(g) \cdot q^n \in M_1(N_g, \chi_g)_{\mathbf{Q}(\varrho)}$$

and

$$h = \sum_{n \geqslant 0} a_n(h) \cdot q^n \in M_1(N_h, \chi_h)_{\mathbf{Q}(\varrho)}$$

of conductors N_g and N_h equal to those of ϱ_1 and ϱ_2 respectively and characters χ_g and $\chi_h = \chi_g^{-1}$ (cf. Equation (1.1)). The form g (resp., h) is cupidal precisely if the Artin representation ϱ_1 (resp., ϱ_2) is irreducible.

Assume that A has good ordinary or multiplicative reduction at p, so that N_f is of the form $M_f \cdot p^{r_f}$ with $r_f \leqslant 1$ and M_f coprime with p. The p-th Hecke polynomial $X^2 - a_p(f) \cdot X + 1_{N_f}(p) \cdot p$ of f has a unique root α_f which is a p-adic unit, the other root being $\beta_f = 1_{N_f}(p)p/\alpha_f$. (Here 1_{N_f} is the trivial character modulo N_f.) By Hida theory, the ordinary p-stabilisation

$$f_\alpha(q) = f(q) - \beta_f \cdot f(q^p) \in S_2(M_f p)_{\mathbf{Q}(\alpha_f)}$$

is the specialisation at weight two of a unique cuspidal Hida family

$$\boldsymbol{f} = \boldsymbol{f}_\alpha = \sum_{n \geqslant 1} a_n(\boldsymbol{f}) \cdot q^n \in \mathcal{O}(U_{\boldsymbol{f}})[\![q]\!]$$

for a suitable connected open disc $U_{\boldsymbol{f}}$ centred at 2 in the weight space $\mathcal{W}$ over $\mathbf{Q}_p$. Here $\mathcal{W}$ is the rigid analytic space over $\mathbf{Q}_p$ whose $\overline{\mathbf{Q}}_p$-points $\mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p^*, \overline{\mathbf{Q}}_p^*)$ contain $\mathbf{Z}$ via the embedding sending k to $t \mapsto t^{k-2}$. Moreover $\mathcal{O}(U_{\boldsymbol{f}})$ is the ring of analytic functions on $U_{\boldsymbol{f}}$. For each classical weight k in $U_{\boldsymbol{f}} \cap \mathbf{Z}_{>2}$, the weight-$k$ specialisation $\boldsymbol{f}_k = \sum_{n \geqslant 1} a_n(\boldsymbol{f})(k) \cdot q^n$ of $\boldsymbol{f}$ is (the q-expansion of) the ordinary p-stabilisation of a p-ordinary newform f_k of weight k and level $\Gamma_0(M_f)$.

Let ξ denote either g or h, and let α_ξ and $\beta_\xi = \chi_\xi(p)/\alpha_\xi$ be the roots of its pth Hecke polynomial $X^2 - a_p(\xi) \cdot X + \chi_\xi(p)$. Fix a finite extension L of $\mathbf{Q}_p$ which contains the Fourier coefficients of ξ, the roots α_f and α_ξ

(for $\xi = g, h$), and the N-th roots of unity, where N is the least common multiple of N_f, N_g and N_h. We assume that p does not divide N_ξ and that ξ is *cuspidal* and *p-regular* (viz. the roots α_ξ and β_ξ are distinct). Moreover we assume that ξ is not the theta series associated with a ray class character of a *real* quadratic field in which *p splits*. Under these assumptions the p-stabilisation

$$\xi_\alpha(q) = \xi(q) - \beta_\xi \cdot \xi(q^p) \in S_1(N_\xi p, \chi_\xi)_L$$

is the weight-one specialisation of a unique cuspidal Hida family

$$\boldsymbol{\xi}_\alpha = \sum_{n \geqslant 1} a_n(\boldsymbol{\xi}_\alpha) \cdot q^n \in \mathcal{O}(U_{\boldsymbol{\xi}})[\![q]\!],$$

where $U_{\boldsymbol{\xi}}$ is a connected open disc in $\mathcal{W} \otimes_{\mathbf{Q}_p} L$ centred at 1. For each classical weight u in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}_{\geqslant 1}$, the weight-$u$ specialisation $\boldsymbol{\xi}_{\alpha,u} = \sum_{n \geqslant 1} a_n(\boldsymbol{\xi})(u) \cdot q^n$ is the ordinary p-stabilisation of a p-ordinary newform ξ_u of weight u, level $\Gamma_1(N_\xi)$ and character χ_ξ. We refer the reader to [11] (especially the discussion following Assumption 1.1, Remark 1.4 and Section 5) and the references therein for more details.

Let Σ^{cl} denote the set of classical triples, namely the intersection of $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$ with $\mathbf{Z}^3_{\geqslant 1}$. Under the self-duality assumption (1.1), for each (k, l, m) in Σ^{cl} the complex Garrett L-function $L(f_k \otimes g_l \otimes h_m, s)$ admits an analytic continuation to all of $\mathbf{C}$ and satisfies a functional equation with sign $+1$ or -1 relating its values at s and $k+l+m-2-s$. Assume from now on that the conductors N_g and N_h of g and h are coprime to the conductor N_f of the elliptic curve A:

$$(N_g \cdot N_h, N_f) = 1. \tag{1.2}$$

Assumption (1.2) guarantees that the signs in the above functional equations are equal to $+1$ for all classical triples (k, l, m) in the *f-unbalanced region*, i.e. triples (k, l, m) in Σ^{cl} such that $k \geqslant l + m$. In particular the complex Garrett L-function

$$L(A, \varrho, s) = L(f \otimes g \otimes h, s)$$

vanishes to *even* order at the central critical point $s = 1$. Set $\mathscr{O}_{\boldsymbol{fgh}} = \mathscr{O}_{\boldsymbol{f}} \widehat{\otimes}_{\mathbf{Q}_p} \mathscr{O}_{\boldsymbol{g}} \widehat{\otimes}_L \mathscr{O}_{\boldsymbol{h}}$, where $\mathscr{O}_\cdot$ denotes the ring of bounded functions on $U_\cdot$. The article [21] associates to the triple of Hida families $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ a *square-root Garrett–Hida p-adic L-function*

$$\mathscr{L}_p^{\alpha\alpha}(A, \varrho) = \mathscr{L}_p(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha) \in \mathscr{O}_{\boldsymbol{fgh}},$$

whose square, the *Garrett–Hida p-adic L-function of* (A, ϱ),

$$L_p^{\alpha\alpha}(A, \varrho) = \mathscr{L}_p^{\alpha\alpha}(A, \varrho)^2$$

interpolates the central critical values

$$L\left(f_k \otimes g_l \otimes h_m, \frac{k+l+m-2}{2}\right)$$

of the complex Garrett L-functions $L(f_k \otimes g_l \otimes h_m, s)$ at classical triples (k, l, m) in the f-unbalanced region. We refer to [11, Section 6.1] (where $L_p^{\alpha\alpha}(A, \varrho)$ is denoted by $L_p(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$ for the precise interpolation property (see in particular equation (132) of loc. cit.). The L-function $L_p^{\alpha\alpha}(A, \varrho)$ is symmetric in the families $\boldsymbol{g}_\alpha$ and $\boldsymbol{h}_\alpha$.

Enlarging $\mathbf{Q}(\varrho)$ if necessary, we assume it contains α_ξ for ξ equal to f, g and h. The weight-one specialisation (cf. Section 2.1 below)

$$V(\xi) = V(\boldsymbol{\xi}_\alpha) \otimes_1 L$$

of the Galois representation $V(\boldsymbol{\xi}_\alpha)$ associated with $\boldsymbol{\xi}_\alpha$ affords the dual of the p-adic Deligne–Serre representation of ξ with coefficients in L. The $G_{\mathbf{Q}}$-representation $V(\boldsymbol{\xi}_\alpha)$ is a free rank-two $\mathscr{O}_{\boldsymbol{\xi}}$-module and the tensor product $\cdot \otimes_1 L = \cdot \otimes_{\mathscr{O}_{\boldsymbol{\xi}},1} L$ is taken with respect to evaluation at 1 in $U_{\boldsymbol{\xi}}$. The global p-adic representation $V(\xi)$ is equipped with a canonical, $G_{\mathbf{Q}}$-equivariant, perfect, skew-symmetric pairing

$$\pi_\xi : V(\xi) \otimes_L V(\xi) \longrightarrow L(\chi_\xi), \tag{1.3}$$

arising as the weight-one specialisation of a suitably twisted Poincaré duality on $V(\boldsymbol{\xi}_\alpha)$ (cf. Section 2.1). Enlarging L if necessary, fix isomorphisms

$$\gamma_g : V_{\varrho_1} \otimes_{\mathbf{Q}(\varrho)} L \simeq V(g) \quad \text{and} \quad \gamma_h : V_{\varrho_2} \otimes_{\mathbf{Q}(\varrho)} L \simeq V(h) \tag{1.4}$$

of $L[G_{\mathbf{Q}}]$-modules such that the perfect dualities $\pi_g \circ \gamma_g \otimes \gamma_g$ and $\pi_h \circ \gamma_h \otimes \gamma_h$ map the $\mathbf{Q}(\varrho)$-structures $V_{\varrho_1} \otimes_{\mathbf{Q}(\varrho)} V_{\varrho_1}$ and $V_{\varrho_2} \otimes_{\mathbf{Q}(\varrho)} V_{\varrho_2}$ into the $\mathbf{Q}(\varrho)$-structures $\mathbf{Q}(\varrho)(\chi_g)$ and $\mathbf{Q}(\varrho)(\chi_h)$ of $L(\chi_g)$ and $L(\chi_h)$ respectively.

Let $V(f) = \mathrm{Ta}_p(A/\mathbf{Q}) \otimes_{\mathbf{Z}_p} L$ be the p-adic Tate module $A/\mathbf{Q}$ with coefficients in L, and let $V(f)^-$ be the maximal unramified quotient of the restriction of $V(f)$ to $G_{\mathbf{Q}_p}$. It is a 1-dimensional L-module, on which an arithmetic Frobenius in $G_{\mathbf{Q}_p}$ acts as multiplication by α_f. Set $V_\varrho = V_{\varrho_1} \otimes_{\mathbf{Q}(\varrho)} V_{\varrho_2}$, $V(f, \varrho) = V(f) \otimes_{\mathbf{Q}(\varrho)} V_\varrho$ and $V(f, \varrho)^- = V(f)^- \otimes_{\mathbf{Q}(\varrho)} V_\varrho$, so that $V(f, \varrho)^-$ is the maximal $G_{\mathbf{Q}_p}$-unramified quotient of $V(f, \varrho)$, on which an arithmetic Frobenius acts with eigenvalues $\alpha_f\alpha_g\alpha_h$, $\alpha_f\beta_g\alpha_h$, $\alpha_f\alpha_g\beta_h$ and $\alpha_f\beta_g\beta_h$. Define the module of *p-adic periods* of (A, ϱ):

$$\mathscr{Q}_p(A, \varrho)_L = H^0(\mathbf{Q}_p, V(f, \varrho)^-)$$

to be the space of $G_{\mathbf{Q}_p}$-invariants of $V(f, \varrho)^-$. As suggested by the notation

$$\mathscr{Q}_p(A, \varrho)_L = \mathscr{Q}_p(A, \varrho) \otimes_{\mathbf{Q}(\varrho)} L$$

for a canonical $\mathbf{Q}(\varrho)$-submodule $\mathscr{Q}_p(A,\varrho)$ defined as follows. Note first that $\mathscr{Q}_p(A,\varrho)_L$ is zero if A has good reduction at p. In this case set $\mathscr{Q}_p(A,\varrho)=0$. If A has multiplicative reduction at p, Tate's theory gives a rigid analytic isomorphism

$$\wp_{\mathrm{Tate}} : \mathbf{G}^{\mathrm{an}}_{m,\mathbf{Q}_{p^2}}/q_A^{\mathbf{Z}} \simeq A_{\mathbf{Q}_{p^2}},$$

unique up to sign. Here $A_{\mathbf{Q}_{p^2}}$ is the base change of A to the quadratic unramified extension $\mathbf{Q}_{p^2}$ of $\mathbf{Q}_p$ and q_A in $p\mathbf{Z}_p$ is the Tate period of A. Taking the p-adic Tate modules $\wp_{\mathrm{Tate}}$ induces a (canonical up to sign) isomorphism of $G_{\mathbf{Q}_{p^2}}$-modules $V(f)^- \simeq L$. Write $q(A)$ in $V(f)^-$ for the element corresponding to the identity of L under this isomorhism and define

$$\mathscr{Q}_p(A,\varrho) = \big(\mathbf{Q}(\varrho)\cdot q(A) \otimes_{\mathbf{Q}(\varrho)} V_\varrho\big)^{G_{\mathbf{Q}_p}}.$$

Let $X_1(N_f,p)$ be the compact modular curve of level $\Gamma_1(N_f,p)=\Gamma_1(N_f)\cap\Gamma_0(p)$ over $\mathbf{Q}$. Fix a modular parametrisation (viz. a non-constant map of $\mathbf{Q}$-schemes)

$$\wp_\infty : X_1(N_f,p) \longrightarrow A.$$

Let K_ϱ be a finite Galois extension of $\mathbf{Q}$ such that ϱ_1 and ϱ_2 factor through $\mathrm{Gal}(K_\varrho/\mathbf{Q})$. Define the *p-extended Mordell–Weil group* of (A,ϱ) by

$$A^\dagger(K_\varrho)^\varrho = \big(A(K_\varrho)\otimes_{\mathbf{Z}} V_\varrho\big)^{\mathrm{Gal}(K_\varrho/\mathbf{Q})} \oplus \mathscr{Q}_p(A,\varrho).$$

Section 2 below associates with the triple (f,g_α,h_α), the modular parametrisation $\wp_\infty$, and the isomorphisms γ_g and γ_h a *Garrett–Nekovář p-adic height pairing*

$$\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} : A^\dagger(K_\varrho)^\varrho \times A^\dagger(K_\varrho)^\varrho \longrightarrow \mathscr{I}/\mathscr{I}^2, \tag{1.5}$$

where $\mathscr{I}$ is the ideal of analytic functions in $\mathscr{O}_{\boldsymbol{fgh}}$ vanishing at $w_o=(2,1,1)$. The pairing $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ is skew-symmetric and associated by cohomological means to an appropriate self-dual twist of the representation $V(\boldsymbol{f})\widehat{\otimes}_{\mathbf{Q}_p}V(\boldsymbol{g}_\alpha)\widehat{\otimes}_L V(\boldsymbol{h}_\alpha)$, viewed as a p-adic deformation of $V(f,g,h)=V(f)\otimes_L V(g)\otimes_L V(h)$. Its construction grounds on Nekovář's theory of Selmer complexes and generalised Poitou–Tate duality [26]. More precisely, after identifying $V(f)$ with the f_α-isotypic component of the cohomology group $H^1_{\text{ét}}(X_1(N_f,p)_{\overline{\mathbf{Q}}},L(1))$ via the fixed modular parametrisation $\wp_\infty$, Section 2 below defines a *canonical* Garrett–Nekovář p-adic height pairing

$$\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} : \mathrm{Sel}^\dagger(\mathbf{Q},V(f,g,h))\otimes_L \mathrm{Sel}^\dagger(\mathbf{Q},V(f,g,h)) \longrightarrow \mathscr{I}/\mathscr{I}^2, \tag{1.6}$$

where the *(naive) extended Selmer group*

$$\mathrm{Sel}^\dagger(\mathbf{Q},V(f,g,h)) = \mathrm{Sel}(\mathbf{Q},V(f,g,h)) \oplus H^0(\mathbf{Q}_p,V(f,g,h)^-) \tag{1.7}$$

is the direct sum of the Bloch–Kato Selmer group of $V(f,g,h)$ over $\mathbf{Q}$ and the module of $G_{\mathbf{Q}_p}$-invariants of the maximal p-unramified quotient $V(f,g,h)^-$ of $V(f,g,h)$. The global Kummer map $A(K_\varrho) \to H^1(K_\varrho, V(f))$ and the fixed isomorphisms γ_g and γ_h give rise to an embedding γ_{gh} : $A^\dagger(K_\varrho)^\varrho \hookrightarrow \mathrm{Sel}^\dagger(\mathbf{Q}, V(f,g,h))$, and one defines (1.5) as the restriction of the canonical height pairing (1.6) along γ_{gh}.

Set

$$r^\dagger(A,\varrho) = \dim_{\mathbf{Q}(\varrho)} A^\dagger(K_\varrho)^\varrho$$

and define the *Garrett–Nekovář regulator*

$$R_p^{\alpha\alpha}(A,\varrho) \in \Big(\mathscr{I}^{r^\dagger(A,\varrho)}/\mathscr{I}^{r^\dagger(A,\varrho)+1}\Big)/\mathbf{Q}(\varrho)^{*2}$$

to be the discriminant of the Garrett–Nekovář p-adic height pairing:

$$R_p^{\alpha\alpha}(A,\varrho) = \det\Big(\langle\langle P_i, P_j\rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}\Big)_{1\leqslant i,j\leqslant r^\dagger(A,\varrho)},$$

where $P_1,\dots,P_{r^\dagger(A,\varrho)}$ is a $\mathbf{Q}(\varrho)$-basis of the p-extended Mordell–Weil group $A^\dagger(K_\varrho)^\varrho$. In view of the normalisation of the isomorphisms γ_g and γ_h fixed in (1.4), the regulator $R_p^{\alpha\alpha}(A,\varrho)$ is independent of the choice of γ_g and γ_h. Moreover, it does not depend on the modular parametrisation $\wp_\infty$.

If $\mathcal{Q}_p(A,\varrho)$ is non-zero —the *exceptional case*— assume that either

$$\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \neq \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}_\alpha} \quad \text{or} \quad \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \neq \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}_\alpha}, \tag{1.8}$$

where the *analytic $\mathscr{L}$-invariants* of $\boldsymbol{f}$ and $\boldsymbol{\xi}_\alpha = \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha$ are defined respectively as the logarithmic derivatives

$$\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} = -2\cdot d\log(a_p(\boldsymbol{f}))_{\boldsymbol{k}=2} \quad \text{and} \quad \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{\xi}_\alpha} = -2\cdot d\log(a_p(\boldsymbol{\xi}_\alpha))_{\boldsymbol{u}=1} \tag{1.9}$$

of -2 times the p-th Fourier coefficients of $\boldsymbol{f}$ and $\boldsymbol{\xi}_\alpha$ at $\boldsymbol{k}=2$ and $\boldsymbol{u}=1$. Here $\mathscr{O}_{\boldsymbol{f}}$ and $\mathscr{O}_{\boldsymbol{\xi}}$ are identified with subrings of the power series rings $L[\![\boldsymbol{k}-2]\!]$ and $L[\![\boldsymbol{u}-1]\!]$, where $\boldsymbol{k}-2$ and $\boldsymbol{u}-1$ are uniformisers at the centres 2 and 1 of $U_{\boldsymbol{f}}$ and $U_{\boldsymbol{\xi}}$ respectively.

Conjecture 1.1.

(1) *The Garrett–Hida p-adic L-function $L_p^{\alpha\alpha}(A,\varrho)$ belongs to $\mathscr{I}^{r^\dagger(A,\varrho)}$. Denote by $L_p^{\alpha\alpha}(A,\varrho)^*$ the image of $L_p^{\alpha\alpha}(A,\varrho)$ in*

$$\Big(\mathscr{I}^{r^\dagger(A,\varrho)}/\mathscr{I}^{r^\dagger(A,\varrho)+1}\Big)\Big/\mathbf{Q}(\varrho)^{*2}.$$

Then

$$L_p^{\alpha\alpha}(A,\varrho)^* = R_p^{\alpha\alpha}(A,\varrho).$$

(2) *$L_p^{\alpha\alpha}(A,\varrho)^*$ is non-zero if and only if $L_p^{\alpha\alpha}(A,\varrho)$ is not identically zero.*

Remark 1.2. — In special cases where the forms g and h are theta series associated with the same imaginary quadratic field, it may occur that $L_p^{\alpha\alpha}(A,\varrho)$ vanishes identically, as a consequence of the fact that $V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$ splits as the direct sum of two non-trivial factors. It follows in these cases from Conjecture 1.1 that the Garrett–Nekovář p-adic height $\langle\langle\cdot,\cdot\rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ is necessarily degenerate.

Remarks 1.3.

(1) Under the current assumptions, the module $\mathcal{Q}_p(A,\varrho)$ is non-zero precisely if

$$\alpha_f = \alpha_g\cdot\alpha_h \quad \text{or} \quad \alpha_f = \beta_g\cdot\alpha_h,$$

in which case $\dim_{\mathbf{Q}(\varrho)}\mathcal{Q}_p(A,\varrho) = 2$ and one says that (A,ϱ) is *exceptional at* p. Since by assumption g is p-regular, only one of the displayed equalities can be satisfied. Moreover, as α_ξ and β_ξ are roots of unity for $\xi = g,h$, if (A,ϱ) is exceptional at p, then $\alpha_f^2 = 1$ and either $\alpha_g\cdot\alpha_h = \alpha_f = \beta_g\cdot\beta_h$ or $\alpha_g\cdot\beta_h = \alpha_f = \beta_g\cdot\alpha_h$ by the self-duality assumption (1.1).

(2) The value of $L_p^{\alpha\alpha}(A,\varrho)$ at $w_0 = (2,1,1)$ is a non-zero complex multiple of

$$\left(1-\frac{\alpha_g\alpha_h}{\alpha_f}\right)^2\left(1-\frac{\beta_g\alpha_h}{\alpha_f}\right)^2\left(1-\frac{\alpha_g\beta_h}{\alpha_f}\right)^2\left(1-\frac{\beta_g\beta_h}{\alpha_f}\right)^2\cdot L(A,\varrho,1).$$

It follows that (A,ϱ) is exceptional at p precisely if $L_p^{\alpha\alpha}(A,\varrho)$ has an exceptional zero in the sense of [25], viz. one of the Euler factors which appear in the previous expression is equal to zero. In this case $r^\dagger(A,\varrho) = \dim_{\mathbf{Q}(\varrho)} A(K_\varrho)^\varrho + 2$, hence Conjecture 1.1 and the classical Birch and Swinnerton-Dyer conjecture predict that the order of vanishing of $L_p^{\alpha\alpha}(A,\varrho)$ at w_o equals $\mathrm{ord}_{s=1} L(A,\varrho,s) + 2$.

(3) Since $\langle\langle\cdot,\cdot\rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ is skew-symmetric, the regulator $R_p^{\alpha\alpha}(A,\varrho)$ vanishes if $r^\dagger(A,\varrho)$ is odd. On the other hand, the assumption (1.2) implies that the order of vanishing of $L(A,\varrho,s)$ at $s=1$ is even, hence $r^\dagger(A,\varrho)$ should also be even by the classical Birch and Swinnerton-Dyer conjecture (and the first remark).

(4) If $L(A,\varrho,s)$ does not vanish at $s=1$ and (A,ϱ) is not exceptional at p, then $L_p^{\alpha\alpha}(A,\varrho)(w_o)$ is the square of a non-zero element of $\mathbf{Q}(\varrho)^*$. In this case Conjecture 1.1 is a consequence of the classical Birch and Swinnerton-Dyer conjecture.

(5) Assume that (A,ϱ) is exceptional at p. The article [10] proves Conjecture 1.1 when $L(A,\varrho,s)$ does not vanish at $s=1$. It also shows

the equality

$$\langle\langle q,q'\rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} = \left(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}}\right)\cdot(\boldsymbol{l}-1) + \varepsilon\cdot\left(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}}\right)\cdot(\boldsymbol{m}-1)$$

in $(\mathscr{I}/\mathscr{I}^2)/\mathbf{Q}(\varrho)^*$ (cf. Equation (1.9)), where (q,q') is a $\mathbf{Q}(\varrho)$-basis of $\mathcal{Q}_p(A,\varrho)$ and $\varepsilon=+1$ if $\alpha_f=\alpha_g\cdot\alpha_h$ while $\varepsilon=-1$ if $\alpha_f=\beta_g\cdot\alpha_h$. (Recall that $\langle\langle\cdot,\cdot\rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ is skew-symmetric, and that by assumption either $\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\neq\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}}$ or $\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\neq\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}}$, hence $\langle\langle q,q'\rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ is a non-zero square root of $R_p^{\alpha\alpha}(A,\varrho)$.)

(6) Assume that (A,ϱ) is exceptional and that $L(A,\varrho,s)$ vanishes at $s=1$. Let (q,q') be a $\mathbf{Q}(\varrho)$-basis of $\mathcal{Q}_p(A,\varrho)$. Conjecture 1.1 predicts the equality

$$\frac{\partial^2\mathscr{L}_p^{\alpha\alpha}(A,\varrho)}{\partial\boldsymbol{k}^2}(w_o) = \log_q(P)\cdot\log_{q'}(Q) - \log_{q'}(P)\cdot\log_q(Q)$$

in $L/\mathbf{Q}(\varrho)^*$ for two rational points P and Q in $A(K_\varrho)^\varrho$, where $\log_{q^\cdot}(\cdot)$ is the evaluation at $q^\cdot$ of the Bloch–Kato p-adic logarithm for $q^\cdot=q,q'$. The reader is referred to [10, Section 2.2] for details.

1.1. Generalisations

1.1.1. The semi-stable case

Assume that A has semi-stable reduction at p, and let α_f be a non-zero root of the p-th Hecke polynomial $h_{f,p}=X^2-a_p(f)\cdot X+1_{N_f}(p)\cdot p$ of f. If A has good ordinary reduction at p and α_f is the root of $h_{f,p}$ with positive p-adic valuation, assume in addition that A does not have complex multiplication. Under these assumptions, there exists a unique Coleman family (of slope $\mathrm{ord}_p(\alpha_f)$) which specialises to $f_\alpha=f(q)-\beta_f\cdot f(q^p)$ in weight 2, where $\beta_f\cdot\alpha_f=1_{N_f}(p)\cdot p$. By combining the results of [1, 21], one should be able to associate to the triple $(\boldsymbol{f}_\alpha,\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$ a canonical p-adic L-function $L_p^{\alpha\alpha}(f_\alpha,\varrho)=\mathscr{L}_p(\boldsymbol{f}_\alpha,\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)^2$ (generalising the construction of $L_p^{\alpha\alpha}(A,\varrho)=L_p^{\alpha\alpha}(f_\alpha,\varrho)$ when A is p-ordinary and α_f is the unit root of $h_{f,p}$). On the algebraic side of the matter, (while not necessarily ordinary) the Galois representation $V(\boldsymbol{f}_\alpha)$ associated with $\boldsymbol{f}_\alpha$ is trianguline at p. In light of the extension of Nekovář's theory to families of trianguline representations obtained in [2, 29], the construction of $\langle\langle\cdot,\cdot\rangle\rangle_{\boldsymbol{f}_\alpha\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$, given in Section 2 below when A is p-ordinary and α_f is the unit root of $h_{f,p}$, easily generalises to the present setting. Conjecture 1.1 should then extend to the semi-stable setting.

1.1.2. The reducible case

The formalism leading to the definition of the p-adic regulator $R_p^{\alpha\alpha}(A,\varrho)$ extends to the case in which one or both the Artin representations ϱ_1 and ϱ_2 is reducible and p-irregular, i.e. of the form $\chi \oplus \chi'$ for Dirichlet characters satisfying $\chi(p) = \chi'(p)$. Let $\xi = g$ or h be the associated weight-one Eisenstein series $\mathrm{Eis}_1(\chi, \chi')$. According to the main result of [12] there exists a unique cuspidal Hida family $\boldsymbol{\xi}_\alpha$ specialising in weight one to the (unique) p-stabilisation ξ_α of ξ. The construction of $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ given in Section 2 carries over to this setting, if $V(\boldsymbol{\xi}_\alpha)$ is replaced by its parabolic counterpart (cf. Section 2.3 and [6, Proposition 2.2]). This guarantees the freeness of $V(\boldsymbol{\xi}_\alpha)$ and of its maximal p-unramified quotient. Note that the p-regular reducible cases would involve the Hida–Rankin p-adic L-functions associated to $\boldsymbol{f}$ and one or two families of Eisenstein series.

1.1.3. The higher-weight case

One can formulate a higher-weight analogue of Conjecture 1.1, in which the weight-2 newform associated with A is replaced by a newform

$$f = \sum_{n \geqslant 1} a_n(f) \cdot q^n \in S_k(N_f)_L$$

of even weight $k \geqslant 2$ and trivial character. Assume for simplicity that p does not divide the conductor N_f of f, and that $a_p(f)$ is a p-adic unit (under the embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ fixed at the outset). Let $\boldsymbol{f} = \boldsymbol{f}_\alpha$ be the unique Hida family specialising to the ordinary p-stabilisation f_α of f at weight k. The article [21] associates to $(\boldsymbol{f}_\alpha, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ a p-adic L-function $L_p^{\alpha\alpha}(f_\alpha, \varrho) = \mathscr{L}_p^f(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)^2$. Let $\mathcal{E}^{k-2}$ be the $(k-2)$-fold fibre product of the universal generalised elliptic curve $\mathcal{E} \to X_1(N_f)$ over the modular curve $X_1(N_f)$ of level $\Gamma_1(N_f)$ over $\mathbf{Q}$. The self-dual twist V_f of the Deligne representation of f is a direct summand of $H^{k-1}_{\text{ét}}(\mathcal{E}^{k-2} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, L(k/2))$, hence the p-adic Abel–Jacobi map yields a morphism (cf. [27])

$$r_{\text{ét}} : \Big(\mathrm{CH}^{k/2}\big(\mathcal{E}^{k-2} \otimes_{\mathbf{Q}} K_\varrho\big)_0 \otimes_{\mathbf{Q}} V_\varrho\Big)^{\mathrm{Gal}(K_\varrho/\mathbf{Q})} \longrightarrow \mathrm{Sel}\big(\mathbf{Q}, V_f \otimes_{\mathbf{Q}(\varrho)} V_\varrho\big),$$

where $\mathrm{CH}^i(\cdot)_0$ is the Chow group of homologically trivial codimension i cycles in $\cdot$ with $\mathbf{Q}$-coefficients, and $\mathrm{Sel}(\mathbf{Q}, \cdot)$ is the Bloch–Kato Selmer group of $\cdot$ over $\mathbf{Q}$. Define $A_f(K_\varrho)^\varrho$ to be the image of the Abel–Jacobi map $r_{\text{ét}}$:

$$A_f(K_\varrho)^\varrho = \mathrm{Image}(r_{\text{ét}}).$$

The constructions of Section 2 below readily generalise to give a pairing

$$\langle\langle\cdot,\cdot\rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} : A_f(K_\varrho)^\varrho \otimes_{\mathbf{Q}(\varrho)} A_f(K_\varrho)^\varrho \longrightarrow \mathscr{I}_k/\mathscr{I}_k^2,$$

where $\mathscr{I}_k$ is the ideal of functions in $\mathscr{O}_{\boldsymbol{fgh}}$ which vanish at $(k,1,1)$. The pairing $\langle\langle\cdot,\cdot\rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ is skew-symmetric, and canonical up to the choice of the isomorphisms γ_g and γ_h fixed in (1.4). The Bloch–Kato conjecture predicts that $r_{\text{ét}}$ is injective, and that the dimension $r(f_\alpha,\varrho)$ of $A_f(K_\varrho)^\varrho$ over $\mathbf{Q}(\varrho)$ is finite. Generalising Conjecture 1.1, we expect that $L_p^{\alpha\alpha}(f_\alpha,\varrho)$ belongs to $\mathscr{I}_k^{r(f_\alpha,\varrho)} - \mathscr{I}_k^{r(f_\alpha,\varrho)+1}$ when non-identically zero, and moreover its image in $(\mathscr{I}^{r(f_\alpha,\varrho)}/\mathscr{I}^{r(f_\alpha,\varrho)+1})/\mathbf{Q}(\varrho)^{*2}$ is equal to the regulator of the pairing $\langle\langle\cdot,\cdot\rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$.

2. Garrett–Nekovářp-adic height pairings

Notation. — In this section we set $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = (\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$. We denote by G_{Np} the Galois group of the maximal algebraic extension of $\mathbf{Q}$ which is uramified at all the rational primes not dividing Np.

2.1. Galois representations (cf. [11])

Let $\boldsymbol{\xi}$ be one of $\boldsymbol{f},\boldsymbol{g}$ and $\boldsymbol{h}$, and let $V(\boldsymbol{\xi})$ be the Galois representation introduced in [11, Section 5]. Under the current assumptions it is a free $\mathscr{O}_{\boldsymbol{\xi}}$-module of rank two, equipped with a linear action of G_{Np}. (Recall that $\mathscr{O}_{\boldsymbol{\xi}}$ denotes the ring of bounded functions on $U_{\boldsymbol{\xi}}$, cf. Section 1.) For each classical point u in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}_{\geqslant 2}$, there is a natural specialisation isomorphism

$$\rho_u : V(\boldsymbol{\xi}) \otimes_u L \simeq V(\boldsymbol{\xi}_u)$$

between the base change of $V(\boldsymbol{\xi})$ along evaluation at u on $\mathscr{O}_{\boldsymbol{\xi}}$ and the homological Deligne representation $V(\boldsymbol{\xi}_u)$ of $\boldsymbol{\xi}_u$. (We refer to Equation (106) of loc. cit. for more details.) Moreover, if $\boldsymbol{\xi} = \boldsymbol{g},\boldsymbol{h}$, the base change of $V(\boldsymbol{\xi})$ along evaluation at 1 on $U_{\boldsymbol{\xi}}$ yields a canonical model of the (homological) Deligne–Serre representation associated with the weight-one cuspidal eigenform $\boldsymbol{\xi}_1$. In this case we set (cf. Section 1) $V(\xi) = V(\boldsymbol{\xi}_1) = V(\boldsymbol{\xi}) \otimes_1 L$ and denote by $\rho_1 : V(\boldsymbol{\xi}) \otimes_1 L \simeq V(\boldsymbol{\xi}_1)$ the identity.

The representation $V(\boldsymbol{f}_2)$ is the f-isotypic component of $H^1_{\text{ét}}(X_1(N_f,p), L(1))$ and the modular parametrisation $\wp_\infty : X_1(N_f,p) \to A$ fixed in Section 1 induces an isomorphism $\wp_{\infty *} : V(\boldsymbol{f}_2) \simeq V(f)$.With a slight abuse of notation we write again

$$(2.1) \qquad \rho_2 : V(\boldsymbol{f}) \otimes_2 L \simeq V(f)$$

for the composition of $\wp_{\infty *}$ with the specialisation isomorphism ρ_2.

The restriction of $V(\boldsymbol{\xi})$ to $G_{\mathbf{Q}_p}$ is nearly-ordinary: let $\chi_{\mathrm{cyc}}^{\boldsymbol{u}-1} : G_{\mathbf{Q}} \to \mathscr{O}_{\boldsymbol{\xi}}^*$ be the character whose composition with evaluation at u in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}$ is the $(u-1)$-th power of the p-adic cyclotomic character $\chi_{\mathrm{cyc}} : G_{\mathbf{Q}} \to \mathbf{Z}_p^*$, and let $\breve{a}_p(\boldsymbol{\xi}) : G_{\mathbf{Q}_p} \to \mathscr{O}_{\boldsymbol{\xi}}^*$ be the unramified character sending an arithmetic Frobenius to the p-th Fourier coefficient $a_p(\boldsymbol{\xi})$ of $\boldsymbol{\xi}$. Then there exists a natural short exact sequence of $\mathscr{O}_{\boldsymbol{\xi}}[G_{\mathbf{Q}_p}]$-modules

$$0 \longrightarrow V(\boldsymbol{\xi})^+ \xrightarrow{i^+} V(\boldsymbol{\xi}) \xrightarrow{p^-} V(\boldsymbol{\xi})^- \longrightarrow 0$$

with

$$(2.2) \qquad V(\boldsymbol{\xi})^+ \simeq \mathscr{O}_{\boldsymbol{\xi}}\big(\chi_{\mathrm{cyc}}^{\boldsymbol{u}-1} \cdot \chi_\xi \cdot \breve{a}_p(\boldsymbol{\xi})^{-1}\big) \quad \text{and} \quad V(\boldsymbol{\xi})^- \simeq \mathscr{O}_{\boldsymbol{\xi}}(\breve{a}_p(\boldsymbol{\xi})).$$

According to [11, Equations (103) and (114)], there exists a natural skew-symmetric $G_{\mathbf{Q}}$-equivariant perfect pairing

$$\pi_{\boldsymbol{\xi}} : V(\boldsymbol{\xi}) \otimes_{\mathscr{O}_{\boldsymbol{\xi}}} V(\boldsymbol{\xi}) \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}\big(\chi_\xi \cdot \chi_{\mathrm{cyc}}^{\boldsymbol{u}-1}\big).$$

For each u in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}_{\geqslant 2}$, the base change of $\pi_{\boldsymbol{\xi}}$ along evaluation at u and the specialisation isomorphism ρ_u yield a perfect pairing $\pi_{\boldsymbol{\xi}_u} : V(\boldsymbol{\xi}_u) \otimes_E V(\boldsymbol{\xi}_u) \to L(\chi_\xi + u - 1)$. If $\boldsymbol{\xi} = \boldsymbol{f}$ and $u = 2$, then $\pi_{\boldsymbol{f}_2}$ is equal, up to sign, to the pairing arising from the Poincaré duality $H^1(X_1(N_f, p), \mathbf{Q}_p(1))^{\otimes 2} \to \mathbf{Q}_p(1)$ (cf. loco citato), hence its composition

$$\pi_f : V(f) \otimes_L V(f) \longrightarrow L(1)$$

with the inverse of $\wp_{\infty *}^{\otimes 2}$ is a rational multiple of the Weil pairing. If $\boldsymbol{\xi}$ equals either $\boldsymbol{g}$ or $\boldsymbol{h}$, then the base change of $\pi_{\boldsymbol{\xi}}$ along evaluation at $u = 1$ on $\mathscr{O}_{\boldsymbol{\xi}}$ yields the perfect pairing $\pi_\xi : V(\xi) \otimes_L V(\xi) \to L(\chi_\xi)$ introduced in Equation (1.3).

As in Section 1, set $\mathscr{O}_{\boldsymbol{fgh}} = \mathscr{O}_{\boldsymbol{f}} \widehat{\otimes}_{\mathbf{Q}_p} \mathscr{O}_{\boldsymbol{g}} \widehat{\otimes}_L \mathscr{O}_{\boldsymbol{h}}$ and define

$$V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = V(\boldsymbol{f}) \widehat{\otimes}_{\mathbf{Q}_p} V(\boldsymbol{g}) \widehat{\otimes}_L V(\boldsymbol{h}) \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}},$$

where $\Xi_{\boldsymbol{fgh}} : G_{\mathbf{Q}} \to \mathscr{O}_{\boldsymbol{fgh}}^*$ is the character satisfying

$$\Xi_{\boldsymbol{fgh}}(g)(w) = \chi_{\mathrm{cyc}}(g)^{(4-k-l-m)/2}$$

for each g in $G_{\mathbf{Q}}$ and $w = (k, l, m)$ in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}} \cap \mathbf{Z}^3$. The $G_{\mathbf{Q}}$-representation $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ is a free $\mathscr{O}_{\boldsymbol{fgh}}$-module of rank eight. Moreover $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ is Kummer self-dual: because $\chi_g = \chi_h^{-1}$ (cf. Equation (1.1)), the product of the perfect pairings $\pi_{\boldsymbol{\xi}}$ (for $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$) define a $G_{\mathbf{Q}}$-equivariant and skew-symmetric perfect pairing

$$\pi_{\boldsymbol{fgh}} : V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}(1). \tag{2.3}$$

Set $w_o = (2, 1, 1)$. Then the specialisation map (2.1) induces an isomorphism

$$\rho_{w_o} : V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \otimes_{w_o} L \simeq V(f, g, h) \tag{2.4}$$

between the base change of $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ along evaluation at w_o on $\mathscr{O}_{\boldsymbol{fgh}}$ and

$$V(f, g, h) = V(f) \otimes_L V(g) \otimes_L V(h).$$

The pairing $\pi_{\boldsymbol{fgh}}$ and ρ_{w_o} yield a $G_{\mathbf{Q}}$-equivariant, skew-symmetric and perfect duality

$$\pi_{fgh} : V(f, g, h) \otimes_L V(f, g, h) \longrightarrow L(1),$$

which by construction equals the product of the dualities π_f, π_g and π_h.

2.2. Selmer complexes (cf. [26])

For $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$, denote by $\Lambda_{\boldsymbol{\xi}}$ the ring of analytic functions on $U_{\boldsymbol{\xi}}$ bounded by 1, and set $\Lambda_{\boldsymbol{fgh}} = \Lambda_{\boldsymbol{f}} \widehat{\otimes}_{\mathbf{Z}_p} \Lambda_{\boldsymbol{g}} \widehat{\otimes}_{\mathcal{O}_L} \Lambda_{\boldsymbol{h}}$, so that $\mathscr{O}_{\boldsymbol{fgh}} = \Lambda_{\boldsymbol{fgh}}[1/p]$. The $\mathcal{O}_L$-algebra $\Lambda_{\boldsymbol{fgh}}$ is isomorphic to a three-variable power series ring with coefficients in $\mathcal{O}_L$. In particular it is a regular local complete Noetherian ring with finite residue field. Let G denote either G_{Np} or $G_{\mathbf{Q}_\ell}$, for a rational prime ℓ dividing Np, and let $(\mathtt{B}, \mathtt{M})$ denote one of the pairs $(\mathcal{O}_L, \mathtt{V}(f, g, h))$ and $(\Lambda_{\boldsymbol{fgh}}, \mathtt{V}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$, where $\mathtt{V}(f, g, h)$ (resp., $\mathtt{V}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$) is an $\mathcal{O}_L$-lattice (resp., a $\Lambda_{\boldsymbol{fgh}}$-lattice) in $V(f, g, h)$ (resp., $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$) preserved by the action of G_{Np}. Equip G with the profinite topology and $\mathtt{M}$ with the $m_{\mathtt{B}}$-adic topology, where $m_{\mathtt{B}}$ is the maximal ideal of $\mathtt{B}$. Set $(B, M) = (\mathtt{B}[1/p], \mathtt{M}[1/p])$ and

$$\mathrm{C}^\bullet_{\mathrm{cont}}(G, M) = \mathrm{C}^\bullet_{\mathrm{cont}}(G, \mathtt{M}) \otimes_{\mathtt{B}} B,$$

where $\mathrm{C}^\bullet_{\mathrm{cont}}(G, \mathtt{M})$ is the complex of non-homogeneous continuous cochains of G with values in $\mathtt{M}$. If $G = G_{\mathbf{Q}_\ell}$, we also write $\mathrm{C}^\bullet_{\mathrm{cont}}(\mathbf{Q}_\ell, M)$ as a shorthand for $\mathrm{C}^\bullet_{\mathrm{cont}}(G_{\mathbf{Q}_\ell}, M)$.

Recall the $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}_p}]$-submodule $V(\boldsymbol{f})^+$ of $V(\boldsymbol{f})$ introduced in Section 2.1 and set

$$V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^+ = V(\boldsymbol{f})^+ \widehat{\otimes}_{\mathbf{Q}_p} V(\boldsymbol{g}) \widehat{\otimes}_L V(\boldsymbol{h}) \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}}.$$

Define $V(f)^+$ to be the image of $V(\boldsymbol{f})^+ \otimes_2 L$ under the specialisation isomorphism $\rho_2 : V(\boldsymbol{f}) \otimes_2 L \simeq V(f)$ (cf. Equation (2.1)), and set

$$V(f,g,h)^+ = V(f)^+ \otimes_L V(g) \otimes_L V(h).$$

Denote by $i^+ : M^+ \hookrightarrow M$ the natural inclusion, fix a $G_{\mathbf{Q}_p}$-stable $\mathtt{B}$-lattice $\mathtt{M}^+$ mapping into $\mathtt{M}$ under i^+, and define $\mathrm{C}^\bullet_{\mathrm{cont}}(G_{\mathbf{Q}_p}, M^+) = \mathrm{C}^\bullet_{\mathrm{cont}}(\mathbf{Q}_p, M^+)$ to be the base change to B of the complex $\mathrm{C}^\bullet_{\mathrm{cont}}(G_{\mathbf{Q}_p}, \mathtt{M}^+)$ of continuous non-homogeneous cochains of $G_{\mathbf{Q}_p}$ with values in $\mathtt{M}^+$. The inclusion i^+ induces a morphism of complexes

$$i^+ : \mathrm{C}^\bullet_{\mathrm{cont}}\big(\mathbf{Q}_p, M^+\big) \longrightarrow \mathrm{C}^\bullet_{\mathrm{cont}}(\mathbf{Q}_p, M),$$

which we call the *f-Greenberg local condition* on the $G_{\mathbf{Q}_p}$-representation M.

The *Nekovář–Selmer complex*

$$\widetilde{\mathrm{C}}^\bullet_f(G_{Np}, M)$$

relative to the form f of the G_{Np}-representation M is the complex of B-modules

$$\mathrm{Cone}\left(\mathrm{C}^\bullet_{\mathrm{cont}}(G_{Np}, M) \oplus \mathrm{C}^\bullet_{\mathrm{cont}}\big(\mathbf{Q}_p, M^+\big) \xrightarrow{\mathrm{res}_{Np} - i^+} \bigoplus_{\ell | Np} \mathrm{C}^\bullet_{\mathrm{cont}}(\mathbf{Q}_\ell, M)\right)[-1],$$

where $\mathrm{res}_{Np} = \bigoplus_{\ell|Np} \mathrm{res}_\ell$ is the direct sum over the primes dividing Np of the restriction morphisms $\mathrm{res}_\ell : \mathbf{R}\Gamma_{\mathrm{cont}}(G_{Np}, M) \to \mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_\ell, M)$ associated with fixed embeddings $i_\ell : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_\ell$ (with i_p the embedding fixed at the outset.) Denote by

$$\mathbf{R}\widetilde{\Gamma}_f(\mathbf{Q}, M) \in \mathrm{D}^b_{\mathrm{ft}}(B)$$

the image of $\widetilde{\mathrm{C}}^\bullet_f(G_{Np}, M)$ in the derived category $\mathrm{D}^b_{\mathrm{ft}}(B)$ of bounded complexes of B-modules with cohomology of finite type over B and by

$$\widetilde{H}^\cdot_f(\mathbf{Q}, M) = H^\cdot\Big(\mathbf{R}\widetilde{\Gamma}_f(\mathbf{Q}, M)\Big)$$

its cohomology. (The complex $\mathbf{R}\widetilde{\Gamma}_f(\mathbf{Q}, M)$ is indeed a perfect complex of perfect amplitude contained in $[0,3]$, cf. [26].) Similarly denote by

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G_{Np}, M), \quad \mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_\ell, M) \quad \text{and} \quad \mathbf{R}\Gamma_{\mathrm{cont}}\big(\mathbf{Q}_p, M^+\big)$$

the images in $\mathrm{D}^b_{\mathrm{ft}}(B)$ of $\mathrm{C}^\bullet_{\mathrm{cont}}(G_{Np}, M)$, $\mathrm{C}^\bullet_{\mathrm{cont}}(\mathbf{Q}_\ell, M)$ and $\mathrm{C}^\bullet_{\mathrm{cont}}(\mathbf{Q}_p, M^+)$, and by

$$H^\cdot(G_{Np}, M), \quad H^\cdot(\mathbf{Q}_\ell, M) \quad \text{and} \quad H^\cdot\big(\mathbf{Q}_p, M^+\big)$$

their cohomology.

The specialisation isomorphism (2.4) induces isomorphisms in $\mathrm{D}^b_{\mathrm{ft}}(L)$:

$$\rho_{w_o} : \mathbf{R}\Gamma_{\mathrm{cont}}(G, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}},w_o} L \simeq \mathbf{R}\Gamma_{\mathrm{cont}}(G, V(f,g,h)) \tag{2.5}$$

and

$$\rho_{w_o} : \mathbf{R}\Gamma_{\mathrm{cont}}\big(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})^+\big) \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}},w_o} L \simeq \mathbf{R}\Gamma_{\mathrm{cont}}\big(\mathbf{Q}_p, V(f,g,h)^+\big),$$

which in turn induce on f-Selmer complexes an isomorphism

$$\rho_{w_o} : \mathbf{R}\widetilde{\Gamma}_f(\mathbf{Q}, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}},w_o} L \simeq \mathbf{R}\widetilde{\Gamma}_f(\mathbf{Q}, V(f,g,h)). \tag{2.6}$$

(This follows easily by the fact the kernel of evaluation at w_o on $\mathscr{O}_{\boldsymbol{fgh}}$ is generated by an $\mathscr{O}_{\boldsymbol{fgh}}$-regular sequence.)

The local Tate duality implies that for each prime ℓ dividing N the complex $\mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_\ell, V(f,g,h))$ is isomorphic to zero, hence so is $\mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_\ell, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$ by Equation (2.5). It then follows from the definition of the Selmer complex $\widetilde{\mathrm{C}}^\bullet_f(G_{Np}, M)$ that one has a distinguished triangle in $\mathrm{D}^b_{\mathrm{ft}}(R)$:

$$\mathbf{R}\widetilde{\Gamma}_f(\mathbf{Q}, M) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G_{Np}, M) \xrightarrow{p^-\circ \mathrm{res}_p} \mathbf{R}\Gamma_{\mathrm{cont}}\big(\mathbf{Q}_p, M^-\big), \tag{2.7}$$

where M^- is the quotient of M by M^+ and p^- is the map induced on complexes by the the projection $p^- : M \to M^-$.

2.3. The extended Selmer group

The exact triangle (2.7) gives rise to a long exact cohomology sequence

$$\widetilde{H}^i_f(\mathbf{Q}, M) \longrightarrow H^i(G_{Np}, M) \longrightarrow H^i(\mathbf{Q}_p, M^-) \xrightarrow{\jmath} \widetilde{H}^{i+1}_f(\mathbf{Q}, M). \tag{2.8}$$

As easily checked

$$\mathrm{Sel}(\mathbf{Q}, V(f,g,h)) = \ker\Big(H^1(G_{Np}, V(f,g,h)) \xrightarrow{p^-\circ \mathrm{res}_p} H^1(\mathbf{Q}_p, V(f,g,h)^-)\Big),$$

hence one can extract from the previous sequence the short exact sequence

$$0 \longrightarrow H^0(\mathbf{Q}_p, V(f,g,h)^-) \longrightarrow \widetilde{H}^1_f(\mathbf{Q}, V(f,g,h)) \longrightarrow \mathrm{Sel}(\mathbf{Q}, V(f,g,h)) \longrightarrow 0. \tag{2.9}$$

The projection in the previous equation has a natural splitting

$$\imath_{\mathrm{ur}} : \mathrm{Sel}(\mathbf{Q}, V(f,g,h)) \hookrightarrow \widetilde{H}^1_f(\mathbf{Q}, V(f,g,h)), \tag{2.10}$$

characterised by the following property. Denote by

$$\cdot^{+} : \widetilde{H}^1_f(\mathbf{Q}, V(f,g,h)) \longrightarrow H^1\big(\mathbf{Q}_p, V(f,g,h)^+\big) \tag{2.11}$$

the morphism induced by the natural map of complexes (i.e. projection)

$$\widetilde{\mathrm{C}}^{\bullet}_f(G_{Np}, V(f,g,h)) \longrightarrow \mathrm{C}^{\bullet}_{\mathrm{cont}}\big(\mathbf{Q}_p, V(f,g,h)^+\big).$$

Then for any Selmer class $\mathfrak{x}$ in $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ one has

$$\imath_{\mathrm{ur}}(\mathfrak{x})^+ \in H^1_{\mathrm{fin}}\big(\mathbf{Q}_p, V(f,g,h)^+\big),$$

where $H^1_{\mathrm{fin}}(\mathbf{Q}_p, \cdot)$ denotes the Bloch–Kato finite subspace of $H^1(\mathbf{Q}_p, \cdot)$. We often identify the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ with a subgroup of the Nekovář extended Selmer group $\widetilde{H}^1_f(\mathbf{Q}, V(f,g,h))$ via the splitting $\imath_{\mathrm{nr}}$. In other words, we use the splitting $\imath_{\mathrm{nr}}$ to identify the Nekovář extended Selmer group $\widetilde{H}^1_f(\mathbf{Q}, V(f,g,h))$ with the naive extended Selmer group $\mathrm{Sel}^{\dagger}(\mathbf{Q}, V(f,g,h))$ introduced in Equation (1.7):

$$\widetilde{H}^1_f(\mathbf{Q}, V(f,g,h)) = \mathrm{Sel}(\mathbf{Q}, V(f,g,h)) \oplus H^0(\mathbf{Q}_p, V(f,g,h)^-). \tag{2.12}$$

The Kummer map and the Shapiro isomorphism yield an injective morphism

$$(A(K_\varrho) \otimes_{\mathbf{Z}} V_\varrho)^{\mathrm{Gal}(K_\varrho/\mathbf{Q})} \otimes_{\mathbf{Q}(\varrho)} L \hookrightarrow \mathrm{Sel}\big(\mathbf{Q}, V(f) \otimes_{\mathbf{Q}(\varrho)} V_\varrho\big).$$

Together with the isomorphism of $L[G_{\mathbf{Q}}]$-modules

$$\gamma_g \otimes \gamma_h : V_\varrho \otimes_{\mathbf{Q}(\varrho)} L \simeq V(g) \otimes_L V(h)$$

(cf. Equation (1.4)), it entails an injective morphism of L-vector spaces

$$\gamma_{gh} : A^{\dagger}(K_\varrho)^{\varrho} \otimes_{\mathbf{Q}(\varrho)} L \hookrightarrow \widetilde{H}^1_f(\mathbf{Q}, V(f,g,h)), \tag{2.13}$$

which is an isomorphism precisely if the p-part of the ϱ-isotypic component of the Shafarevich–Tate group of A over K_ϱ is finite.

2.4. Generalised Poitou–Tate duality (cf. [26])

[26, Section 6.3] (see also Proposition 1.3.2) associates to the Kummer duality

$$\pi_{fgh} : V(f,g,h) \otimes_L V(f,g,h) \longrightarrow L(1)$$

(satisfying $\pi_{fgh}(V(f,g,h)^+ \otimes_L V(f,g,h)^+) = 0$) a global cup-product pairing

$$\cup_{\mathrm{Nek}} : \mathbf{R}\widetilde{\Gamma}_f(\mathbf{Q}, V(f,g,h)) \otimes^{\mathbf{L}}_L \mathbf{R}\widetilde{\Gamma}_f(\mathbf{Q}, V(f,g,h)) \longrightarrow \mathbf{R}\widetilde{\Gamma}_{\emptyset}(\mathbf{Q}, L(1)),$$

where $\mathbf{R}\widetilde{\Gamma}_{\emptyset}(\mathbf{Q}, L(1))$ denotes the complex

$$\mathrm{Cone}\left(\mathbf{R}\Gamma_{\mathrm{cont}}(G_{Np}, L(1)) \xrightarrow{\mathrm{res}_{Np}} \bigoplus_{\ell \mid Np} \mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_\ell, L(1))\right)[-1].$$

Let $\widetilde{H}^{\cdot}_{\emptyset}(\mathbf{Q}, L(1))$ be the cohomology of $\mathbf{R}\widetilde{\Gamma}_{\emptyset}(\mathbf{Q}, L(1))$. The fundamental exact sequence of global class field theory yields a canonical isomorphism

$$\mathrm{Tr}_L : \widetilde{H}^3_{\emptyset}(\mathbf{Q}, L(1)) \simeq \bigoplus_{\ell \mid Np} H^2(\mathbf{Q}_\ell, L(1)) \big/ \mathrm{res}_{Np}\big(H^2(G_{Np}, L(1))\big) \simeq L,$$

arising from the sum of the invariant maps $\mathrm{inv}_\ell : H^2(\mathbf{Q}_\ell, L(1)) \simeq L$ of local class field theory, for ℓ dividing Np (cf. [26, Equation (5.3.1.3.2)]). Define

$$\begin{aligned}(2.14)\quad \langle \cdot, \cdot \rangle_{\mathrm{Nek}} : \widetilde{H}^2_f(\mathbf{Q}, V(f,g,h)) \otimes_L \widetilde{H}^1_f(\mathbf{Q}, V(f,g,h)) \\ \longrightarrow \widetilde{H}^3_{\emptyset}(\mathbf{Q}, L(1)) \simeq L.\end{aligned}$$

to be the composition of the map $H^{2,1}(\cup_{\mathrm{Nek}})$ induced on $(2,1)$-cohomology by Nekovář's global cup-product $\cup_{\mathrm{Nek}}$ with the trace isomorphism Tr_L.

2.5. The p-adic height pairing

To lighten the notation, we abbreviate $V(f,g,h), V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}), \mathbf{R}\widetilde{\Gamma}_f(\mathbf{Q}, \cdot)$ and $\widetilde{H}(\mathbf{Q}, \cdot)$ with $V, \boldsymbol{V}, \mathbf{R}\widetilde{\Gamma}_f(\cdot)$ and $\widetilde{H}_f(\cdot)$ respectively.

Applying $\mathbf{R}\widetilde{\Gamma}_f(\boldsymbol{V}) \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}}} \cdot$ to the exact triangle

$$(2.15)\qquad \mathscr{I}/\mathscr{I}^2 \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}/\mathscr{I}^2 \longrightarrow L \xrightarrow{\delta} \mathscr{I}/\mathscr{I}^2[1]$$

arising from evaluation at w_o on $\mathscr{O}_{\boldsymbol{fgh}}$, yields a morphism in $\mathrm{D}^b_{\mathrm{ft}}(\mathscr{O}_{\boldsymbol{fgh}})$:

$$(2.16)\qquad \mathbf{R}\widetilde{\Gamma}_f(\boldsymbol{V}) \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}}, w_o} L \longrightarrow \mathbf{R}\widetilde{\Gamma}_f(\boldsymbol{V}) \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}}} \mathscr{I}/\mathscr{I}^2[1].$$

The specialisation map ρ_{w_o} gives rise to isomorphisms (cf. Equation (2.6))

$$\rho_{w_o} : \mathbf{R}\widetilde{\Gamma}_f(\boldsymbol{V}) \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}}, w_o} L \simeq \mathbf{R}\widetilde{\Gamma}_f(V)$$

and

$$\rho_{w_o} \otimes \mathrm{id} : \mathbf{R}\widetilde{\Gamma}_f(\boldsymbol{V}) \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}}} \mathscr{I}/\mathscr{I}^2 \simeq \mathbf{R}\widetilde{\Gamma}_f(V) \otimes_L \mathscr{I}/\mathscr{I}^2,$$

which together with (2.16) induce a *derived Bockstein map*

$$\mathbf{R}\widetilde{\beta}_{\boldsymbol{fgh}} : \mathbf{R}\widetilde{\Gamma}_f(\mathbf{Q}, V(f,g,h)) \longrightarrow \mathbf{R}\widetilde{\Gamma}_f(\mathbf{Q}, V(f,g,h))[1] \otimes_L \mathscr{I}/\mathscr{I}^2.$$

The *Garrett–Nekovář canonical p-adic height pairing*

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}} : \widetilde{H}^1_f(\mathbf{Q}, V(f,g,h)) \otimes_L \widetilde{H}^1_f(\mathbf{Q}, V(f,g,h)) \longrightarrow \mathscr{I}/\mathscr{I}^2$$

is the composition of the Nekovář cup-product pairing (cf. Equation (2.14))

$$\langle\cdot,\cdot\rangle_{\mathrm{Nek}}\otimes\mathscr{I}/\mathscr{I}^2:\widetilde{H}^2_f(V)\otimes_L\widetilde{H}^1_f(V)\otimes_L\mathscr{I}/\mathscr{I}^2\longrightarrow\mathscr{I}/\mathscr{I}^2$$

with the morphism

$$\widetilde{H}^1_f(V)\otimes_L\widetilde{H}^1_f(V)\longrightarrow\widetilde{H}^2_f(V)\otimes_L\widetilde{H}^1_f(V)\otimes_L\mathscr{I}/\mathscr{I}^2,$$

arising from the *Bockstein map*

$$\widetilde{\beta}_{\boldsymbol{fgh}}=H^1\Big(\mathbf{R}\widetilde{\beta}_{\boldsymbol{fgh}}\Big):\widetilde{H}^1_f(\mathbf{Q},V(f,g,h))\longrightarrow\widetilde{H}^2_f(\mathbf{Q},V(f,g,h))\otimes_L\mathscr{I}/\mathscr{I}^2. \tag{2.17}$$

Proposition 2.1. — *The p-adic height $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\boldsymbol{fgh}}$ is skew-symmetric.*

Proof. — As explained in Section 2.1, the Kummer self-duality π_{fgh} on $V(f,g,h)$ lifts (under ρ_{w_o}) to a skew-symmetric, $G_{\mathbf{Q}}$-equivariant perfect pairing

$$\pi_{\boldsymbol{fgh}}:V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\otimes_{\mathscr{O}_{\boldsymbol{fgh}}}V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\longrightarrow\mathscr{O}_{\boldsymbol{fgh}}(1),$$

under which the $G_{\mathbf{Q}_p}$-submodule $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})^+$ of $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ is its own orthogonal complement. The proposition then follows from the results of [32, Appendix C]. □

The p-adic height pairing (cf. Equation (1.5))

$$\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\boldsymbol{fg}_\alpha\boldsymbol{h}_\alpha}:A^\dagger(K_\varrho)^\varrho\otimes_{\mathbf{Q}(\varrho)}A^\dagger(K_\varrho)^\varrho\longrightarrow\mathscr{I}/\mathscr{I}^2$$

which appears in Conjecture 1.1 is defined to be restriction of the canonical height pairing $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\boldsymbol{fg}_\alpha\boldsymbol{h}_\alpha}:\widetilde{H}^1_f(\mathbf{Q},V(f,g,h))^{\otimes 2}\to\mathscr{I}/\mathscr{I}^2$ to the p-extended Mordell–Weil group $A^\dagger(K_\varrho)^\varrho$ along the injective morphism γ_{gh} introduced in Equation (2.13).

3. Diagonal classes and rational points

As proved in [11, Theorem A] and [18, Theorem 5.1], the square root p-adic L-function $\mathscr{L}_p^{\alpha\alpha}(A,\varrho)$ is the image of a *big diagonal class* $\kappa(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$ in $H^1(\mathbf{Q},V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha))$ under an appropriate branch of the Perrin-Riou big logarithm. The leading term of $L_p^{\alpha\alpha}(A,\varrho)$ at $w_o=(2,1,1)$ is then intimately connected to the derivatives of the class $\kappa(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$ at w_o. This section exploits this connection and its relation with Conjecture 1.1.

To simplify the exposition, we assume in this section that

$$\alpha_f\neq\alpha_g\cdot\alpha_h\quad\text{and}\quad\alpha_f\neq\beta_g\cdot\alpha_h. \tag{3.1}$$

This condition is equivalent to the vanishing of the module of p-adic periods $\mathscr{Q}_p(A,\varrho)$ of (A,ϱ) (or equivalently of the module $H^0(\mathbf{Q}_p, V(f,g,h)^-)$), and is satisfied when A has good (ordinary) reduction at p (cf. Remark 1.3.1). In particular, in this section, the Nekovář extended Selmer group and the Bloch–Kato Selmer group of $V(f,g,h)$ over $\mathbf{Q}$ are equal to each other (cf. Equation (2.12)):

$$\widetilde{H}^1_f(\mathbf{Q}, V(f,g,h)) = \mathrm{Sel}(\mathbf{Q}, V(f,g,h)).$$

3.1. Differentials and logarithms

Let $\boldsymbol{\xi}$ denote one of $\boldsymbol{f}$, $\boldsymbol{g}_\alpha$ or $\boldsymbol{h}_\alpha$, and recall the short exact sequence of $\mathscr{O}_{\boldsymbol{\xi}}$-modules $V(\boldsymbol{\xi})^+ \hookrightarrow V(\boldsymbol{\xi}) \twoheadrightarrow V(\boldsymbol{\xi})^-$ (cf. Section 2.1).

If $\boldsymbol{\xi} = \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha$ define

$$V(\xi)_\alpha = V(\boldsymbol{\xi})^- \otimes_1 L \quad \text{and} \quad V(\xi)_\beta = V(\boldsymbol{\xi})^+ \otimes_1 L.$$

Equation (2.2) implies that

$$V(\xi)_\alpha = V(\xi)^{\mathrm{Frob}_p = \alpha_\xi} \quad \text{and} \quad V(\xi)_\beta = V(\xi)^{\mathrm{Frob}_p = \beta_\xi}$$

are the subspaces of $V(\xi)$ on which an arithmetic Frobenius Frob_p in $G_{\mathbf{Q}_p}$ acts as multiplication by α_ξ and β_ξ respectively. In particular one has the decomposition

$$V(\xi) = V(\xi)_\alpha \oplus V(\xi)_\beta$$

of $L[G_{\mathbf{Q}_p}]$-modules. (Recall that by assumption the roots α_ξ and $\beta_\xi = \chi_\xi(p)\cdot\alpha_\xi^{-1}$ of the p-th Hecke polynomial of ξ are distinct, cf. Section 1.)

Set $D(\boldsymbol{\xi})^- = H^0(\mathbf{Q}_p, V(\boldsymbol{\xi})^- \widehat{\otimes}_{\mathbf{Q}_p} \widehat{\mathbf{Q}}_p^{\mathrm{nr}})$, where $\widehat{\mathbf{Q}}_p^{\mathrm{nr}}$ is the p-adic completion of the maximal unramified extension of $\mathbf{Q}_p$ (equipped with its natural $G_{\mathbf{Q}_p}$-action). As explained in [11, Section 5], the $\mathscr{O}_{\boldsymbol{\xi}}$-module $D(\boldsymbol{\xi})^-$ is free of rank one, and its base change $D(\boldsymbol{\xi})^-_u = D(\boldsymbol{\xi})^- \otimes_u L$ along evaluation at a classical weight u in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}_{\geqslant 2}$ on $\mathscr{O}_{\boldsymbol{\xi}}$ is canonically isomorphic to the $\boldsymbol{\xi}_u$-isotypic component $L\cdot\boldsymbol{\xi}_u$ of $S_u(pN_\xi, \chi_\xi)_L$. Moreover, there exists an $\mathscr{O}_{\boldsymbol{\xi}}$-basis

$$\omega_{\boldsymbol{\xi}} \in D(\boldsymbol{\xi})^-$$

whose image $\omega_{\boldsymbol{\xi}_u}$ in $D(\boldsymbol{\xi})^-_u$ corresponds to $\boldsymbol{\xi}_u$ under the aforementioned isomorphism for each classical weight u in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}_{\geqslant 2}$ (cf. [11, Equations (117)–(119)]).

Remark 3.1. — We caution the reader that the notation used here differ from that of [11]. Precisely, Section 5 of loc. cit. introduces a differential $\omega_{\boldsymbol{\xi}} = \omega_{\boldsymbol{\xi}}^{\mathrm{BSV}}$ in a suitable dual $D^*(\boldsymbol{\xi})^-$ of $D(\boldsymbol{\xi})^-$. Here we denote by $\omega_{\boldsymbol{\xi}}$

the image of $\omega_{\boldsymbol{\xi}}^{\mathrm{BSV}}$ under the isomorphism $w_{Np}^{-}: D^{*}(\boldsymbol{\xi})^{-} \simeq D(\boldsymbol{\xi})^{-}$ induced by the Atkin–Lehner isomorphism $w_{Np}^{-}: V^{*}(\boldsymbol{\xi})^{-}(1+\boldsymbol{\kappa}_{U_{\boldsymbol{f}}}) \simeq V(\boldsymbol{\xi})^{-}$ defined in [11, Equation (114)]. Accordingly the canonical isomorphism $D(\boldsymbol{\xi})_{u}^{-} \simeq L\cdot \boldsymbol{\xi}_u$ mentioned above arises from the specialisation isomorphism $D^{*}(\boldsymbol{\xi})^{-} \otimes_u L \simeq \mathrm{Fil}^1 V_{\mathrm{dR}}^{*}(\boldsymbol{\xi}_u)$ defined in [11, Equation (116)] and the Atkin–Lehner operator (cf. Equation (29) of loc. cit.).

If $\boldsymbol{\xi}$ is either $\boldsymbol{g}_\alpha$ or $\boldsymbol{h}_\alpha$, the weight-one specialisation of $\omega_{\boldsymbol{\xi}}$ yields canonical elements

$$\omega_{\xi_\alpha} \in D(\boldsymbol{\xi})_1^{-} = D_{\mathrm{cris}}(V(\xi)_\alpha).$$

In this case, let η_{ξ_α} in $D_{\mathrm{cris}}(V(\xi)_\beta)$ be the class satisfying

$$\langle \eta_{\xi_\alpha}, \omega_{\xi_\alpha} \rangle_{\xi} = 1,$$

where

$$\langle \cdot,\cdot \rangle_{\xi} : D_{\mathrm{cris}}(V(\xi)_\alpha) \otimes_L D_{\mathrm{cris}}(V(\xi)_\beta) \longrightarrow D_{\mathrm{cris}}(L(\chi_\xi)) \simeq L$$

is the perfect pairing induced by the duality π_ξ introduced in Equation (1.3). (The crystalline module $D_{\mathrm{cris}}(\chi_\xi) = H^0(\mathbf{Q}_p, L(\chi_\xi) \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}})$ of the one-dimensional representation $L(\chi_\xi)$ is generated over L by the Gauß sum

$$G(\chi_\xi) = \sum_{a \,\in\, (\mathbf{Z}/c(\chi_\xi)\mathbf{Z})^{*}} \chi_\xi(a) \otimes e^{2\pi i a/c(\chi_\xi)}$$

in $L \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(\mu_{N_\xi})$ of the primitive character $\chi_\xi : (\mathbf{Z}/c(\chi_\xi)\mathbf{Z})^{*} \to L^{*}$ associated with χ_ξ. Since by assumption L contains $\mathbf{Q}(\mu_{N_\xi})$, here we identify $G(\chi_\xi)$ with the element $\sum_a \chi_\xi(a)\cdot e^{2\pi i a/c(\chi_\xi)}$ of L, hence $D_{\mathrm{cris}}(\chi_\xi)$ with L.)

Identify $V(f) = \mathrm{Ta}_p(A) \otimes_{\mathbf{Z}_p} L$ with the f-isotypic component of the étale cohomology group $H^1_{\text{ét}}(X_1(N_f,p)_{\overline{\mathbf{Q}}}, \mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} L$ under the modular parametrisation $\wp_\infty$ fixed in Section 1. The modular form f in

$$\mathrm{Fil}^0 H^1_{\mathrm{dR}}\big(X_1(N_f,p)_{\mathbf{Q}_p}, \mathbf{Q}_p(1)\big)$$

then defines (via the comparison isomorphism between étale and de Rham cohomology) a class

$$\omega_f \in \mathrm{Fil}^0 D_{\mathrm{dR}}(V(f))$$

(where $D_{\mathrm{dR}}(\cdot) = H^0(\mathbf{Q}_p, \cdot \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})$ is Fontaine's de Rham functor). Define η_f in $D_{\mathrm{dR}}(V(f))/\mathrm{Fil}^0$ to be the de Rham class satisfying

$$\langle \eta_f, \omega_f \rangle_f = 1,$$

where $\langle \cdot,\cdot \rangle_f : D_{\mathrm{dR}}(V(f)) \otimes_L D_{\mathrm{dR}}(V(f)) \to L$ is the perfect pairing induced on the de Rham modules by the Weil pairing on $V(f)$.

Set $V_{\mathrm{dR}}(f,g,h) = D_{\mathrm{dR}}(V(f,g,h))$. The Bloch–Kato exponential map gives an isomorphism between $V_{\mathrm{dR}}(f,g,h)/\operatorname{Fil}^0$ and the finite subspace $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(f,g,h))$ of $H^1(\mathbf{Q}_p, V(f,g,h))$ (cf. Lemma [11, 9.1]). Denote by

$$\log_p : H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(f,g,h)) \longrightarrow V_{\mathrm{dR}}(f,g,h)/\operatorname{Fil}^0$$

the inverse of the Bloch–Kato exponential. Under the self-duality assumption (1.1), the product of the pairings $\langle\cdot,\cdot\rangle_\xi$, for $\xi = f,g,h$, yields a perfect duality

$$\langle\cdot,\cdot\rangle_{fgh} : V_{\mathrm{dR}}(f,g,h) \otimes_L V_{\mathrm{dR}}(f,g,h) \longrightarrow D_{\mathrm{dR}}(\mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} L = L.$$

(Here one identifies $V_{\mathrm{dR}}(f,g,h)$ with the tensor product of $D_{\mathrm{dR}}(V(f))$, $D_{\mathrm{cris}}(V(g))$ and $D_{\mathrm{cris}}(V(h))$ under the natural isomorphism.) Define the *$\alpha\alpha$-logarithm*

$$\log_{\alpha\alpha} = \left\langle \log_p(\cdot), \omega_f \otimes \eta_{g_\alpha} \otimes \eta_{h_\alpha} \right\rangle_{fgh} : H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(f,g,h)) \longrightarrow L.$$

to be the composition of the Bloch–Kato p-adic logarithm with evaluation on the class $\omega_f \otimes \eta_{g_\alpha} \otimes \eta_{h_\alpha}$ in $\operatorname{Fil}^0 V_{\mathrm{dR}}(f,g,h)$ under the duality $\langle\cdot,\cdot\rangle_{fgh}$. If κ is a global Selmer class in $\operatorname{Sel}(\mathbf{Q}, V(f,g,h))$, we often write $\log_{\alpha\alpha}(\kappa)$ as a shorthand for $\log_{\alpha\alpha}(\operatorname{res}_p(\kappa))$.

3.2. Diagonal classes

Following [11, Section 7.2] define (cf. Section 2.1)

$$\mathscr{F}^2 V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha) = \left[\sum_{p+q+r=2} \mathscr{F}^p V(\boldsymbol{f}) \widehat{\otimes}_L \mathscr{F}^q V(\boldsymbol{g}_\alpha) \widehat{\otimes}_L \mathscr{F}^r V(\boldsymbol{h}_\alpha) \right] \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}},$$

where for $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha$ one sets $\mathscr{F}^i V(\boldsymbol{\xi}) = V(\boldsymbol{\xi})$ for $i \leqslant 0$, $\mathscr{F}^1 V(\boldsymbol{\xi}) = V(\boldsymbol{\xi})^+$ and $\mathscr{F}^j V(\boldsymbol{\xi}) = 0$ for $j \geqslant 2$. It is an $\mathscr{O}_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-submodule of $V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$, free of rank four over $\mathscr{O}_{\boldsymbol{fgh}}$. We call the image of the injective natural map

$$H^1\left(\mathbf{Q}_p, \mathscr{F}^2 V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)\right) \longrightarrow H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$$

the *balanced local condition*, and denote it by $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$. The *balanced Selmer group* $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$ is the module of global cohomology classes in $H^1(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$ which are unramified at every prime $\ell \neq p$ and whose restriction at p belongs to the balanced local condition. For each classical triple $w = (k,l,m)$ in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}} \cap \mathbf{Z}^3_{\geqslant 2}$, one defines similarly the balanced local condition $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V_w)$, where

$V_w = V(\boldsymbol{f}_k, \boldsymbol{g}_{\alpha,l}, \boldsymbol{h}_{\alpha,m})$ is the self-dual Tate twist of the tensor product of the homological Deligne representations $V(\boldsymbol{\xi}_u)$ of $\boldsymbol{\xi}_u = \boldsymbol{f}_k, \boldsymbol{g}_{\alpha,l}, \boldsymbol{h}_{\alpha,m}$. If w is balanced (id est $k < l+m$, $l < k+m$ and $m < k+l$), then $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V_w)$ equals the Bloch–Kato finite subspace of $H^1(\mathbf{Q}_p, V_w)$ (cf. [11, Lemma 7.2]). The work of Perrin-Riou et alii yields a big logarithm map

$$\mathscr{L}_{\boldsymbol{f}} : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)) \longrightarrow \mathscr{O}_{\boldsymbol{fgh}},$$

satisfying the following interpolation property. Let $\mathfrak{Z}$ be a local balanced class in $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$, and let $w = (k, l, m)$ be a balanced classical triple. Denote by $\mathfrak{Z}_w$ in $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V_w)$ the image of $\mathfrak{Z}$ under the map induced in cohomology by the specialisation isomorphism $\rho_w : V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha) \otimes_w L \simeq V_w$ (the latter being defined as the tensor product of the specialisation isomorphisms $\rho_u : V(\boldsymbol{\xi}) \otimes_u L \simeq V(\boldsymbol{\xi}_u)$, for $\boldsymbol{\xi}_u = \boldsymbol{f}_k, \boldsymbol{g}_{\alpha,l}, \boldsymbol{h}_{\alpha,m}$, cf. Section 2.1). Set $c_w = (k + l + m - 2)/2$, $\alpha_k = a_p(\boldsymbol{f})(k)$, $\alpha_l = a_p(\boldsymbol{g}_\alpha)(l)$, $\alpha_m = a_p(\boldsymbol{h}_\alpha)(m)$, and define β_ξ by the identities $\alpha_k \cdot \beta_k = p^{k-1}$, $\alpha_l \cdot \beta_l = \chi_g(p) \cdot p^{l-1}$ and $\alpha_m \cdot \beta_m = \chi_h(p) \cdot p^{m-1}$. Then one has

$$\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z})(w) = \frac{(-1)^{c_w - k}}{(c_w - k)!} \cdot \frac{\left(1 - \frac{\beta_k \alpha_l \alpha_m}{p^{c_w}}\right)}{\left(1 - \frac{\alpha_k \beta_l \beta_m}{p^{c_w}}\right)} \cdot \left\langle \log_p(\mathfrak{Z}_w), \mho_w \right\rangle_w,$$

where $\log_p$ is the Bloch–Kato logarithm map, $\mho_w$ in $\mathrm{Fil}^0 D_{\mathrm{dR}}(V_w)$ denotes the differential $\eta_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_{\alpha,l}} \otimes \omega_{\boldsymbol{h}_{\alpha,m}}$ (defined similarly as in Section 3.1), and the pairing $\langle \cdot, \cdot \rangle_w : D_{\mathrm{dR}}(V_w)/\,\mathrm{Fil}^0 \otimes_L \mathrm{Fil}^0 D_{\mathrm{dR}}(V_w) \to L$ is the one induced by the specialisation at w of the perfect duality $\pi_{\boldsymbol{fgh}}$ (cf. Equation (2.3)). We refer to [11, Proposition 7.3] for a proof of the existence of $\mathscr{L}_{\boldsymbol{f}}$.

[11, Theorem A] constructs a canonical *big balanced diagonal class*

$$\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha) \in H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$$

such that

$$\mathscr{L}_{\boldsymbol{f}}\left(\mathrm{res}_p\left(\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)\right)\right) = \mathscr{L}_p^{\alpha\alpha}(A, \varrho). \tag{3.2}$$

One defines the *(balanced) diagonal class*

$$\kappa(f, g_\alpha, h_\alpha) \in H^1(\mathbf{Q}, V(f, g, h))$$

to be the image of $\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ under the map induced in cohomology by the specialisation isomorphism ρ_{w_o} defined in Equation (2.4). Note that w_o lies outside the balanced region, hence the class $\kappa(f, g_\alpha, h_\alpha)$ is not necessarily crystalline at p. Indeed, under the current assumption (3.1), it follows from the explicit reciprocity law (3.2) and Perrin-Riou's reciprocity law for big dual exponentials that $\kappa(f, g_\alpha, h_\alpha)$ is crystalline at p (hence a Selmer class) precisely if the the complex Garrett L-function

$L(A, \varrho, s) = L(f \otimes g \otimes h, s)$ vanishes at the central point $s = 1$. (Cf. [11, Theorem B], proved in the present setting in Section 9.1 of loco citato.)

Lemma 3.2. — *Assume that $L(A, \varrho, s)$ vanishes at $s = 1$, so that the diagonal class $\kappa(f, g_\alpha, h_\alpha)$ is crystalline at p. Then*

$$\log_{\alpha\alpha}(\kappa(f, g_\alpha, h_\alpha)) = 0.$$

Proof. — Set $V = V(f, g, h)$, $\boldsymbol{V} = V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$, $\kappa = \kappa(f, g_\alpha, h_\alpha)$ and $\kappa_p = \mathrm{res}_p(\kappa)$. Define $\mathscr{F}^2 V = \mathscr{F}^2 \boldsymbol{V} \otimes_{w_o} L$ and $V_{ij}^{\cdot} = V(f)^{\cdot} \otimes_L V(g)_i \otimes_L V(h)_j$. By construction κ is the specialisation at w_o of a balanced class, hence κ_p belongs to the kernel of

$$\eta : H^1(\mathbf{Q}_p, V) \longrightarrow H^1\big(\mathbf{Q}_p, V/\mathscr{F}^2 V\big).$$

Consider the following commutative diagram of $L[G_{\mathbf{Q}_p}]$-modules with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V^+ & \longrightarrow & V & \longrightarrow & V^- & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V^+_{\alpha\alpha} & \longrightarrow & V/\mathscr{F}^2 V & \longrightarrow & V^-_{\alpha\beta} \oplus V^-_{\beta\alpha} \oplus V^-_{\alpha\alpha} & \longrightarrow & 0
\end{array}$$

where the vertical maps are the natural projections. The non-exceptionality assumption (3.1) implies that $H^0(\mathbf{Q}_p, V_{ij}^-) = 0$ for each $(i, j) \in \{\alpha, \beta\}^2$. Moreover the inclusion $V^+ \hookrightarrow V$ induces an isomorphism (cf. Section 2.3)

$$H^1\big(\mathbf{Q}_p, V^+\big) \simeq H^1_{\mathrm{fin}}(\mathbf{Q}_p, V).$$

We then obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1(\mathbf{Q}_p, V^+) & \longrightarrow & H^1_{\mathrm{fin}}(\mathbf{Q}_p, V) & \longrightarrow & 0 \\
 & & \gamma \downarrow & & \downarrow \eta & & \\
0 & \longrightarrow & H^1(\mathbf{Q}_p, V^+_{\alpha\alpha}) & \longrightarrow & H^1\big(\mathbf{Q}_p, V/\mathscr{F}^2 V\big) & &
\end{array}$$

By definition $\log_{\alpha\alpha}$ factors through γ, hence the statement follows from the previous diagram and the identity $\eta(\kappa) = 0$. □

When $L(f \otimes g \otimes h, s)$ vanishes at $s = 1$, the following theorem relates the linear form $\langle\langle \kappa(f, g_\alpha h_\alpha), \cdot \rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ on $\mathrm{Sel}(\mathbf{Q}, V(f, g, h))$ and the derivative of $\mathscr{L}_p^{\alpha\alpha}(A, \varrho)$.

Theorem 3.3. — *Assume that the complex Garrett L-function $L(A, \varrho, s)$ vanishes at $s = 1$, so that $\kappa(f, g_\alpha, h_\alpha)$ is a Selmer class. Then*

$$\frac{\left(1 - \frac{\alpha_g \alpha_h}{\alpha_f}\right)}{\left(1 - \frac{\alpha_f}{p\alpha_g \alpha_h}\right)} \cdot \langle\langle \kappa(f, g_\alpha, h_\alpha), \cdot \rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha \boldsymbol{h}_\alpha} = \log_{\alpha\alpha}\big(\mathrm{res}_p(\cdot)\big) \cdot \mathscr{L}_p^{\alpha\alpha}(A, \varrho) \pmod{\mathscr{I}^2}$$

as $\mathscr{I}/\mathscr{I}^2$-valued linear maps on the Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f, g, h))$.

Theorem 3.3 is proved in Section 3.4 below.

Remark 3.4. — *The construction of the class $\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ and the proof of the reciprocity law* (3.2) *given in* [11] *work also when the assumption* (3.1) *is not satisfied, id est if A has multiplicative reduction at p and α_f equals either $\alpha_g \cdot \alpha_h$ or $\beta_g \cdot \alpha_h$. (Since g is p-regular by an assumption of Section 1, one has $\alpha_g \cdot \alpha_h \neq \beta_g \cdot \alpha_h$.) Assume that $\alpha_f = \alpha_g \cdot \alpha_h$ and that $L(A, \varrho, s)$ vanishes at $s = 1$, so that $\kappa(f, g_\alpha, h_\alpha)$ is crystalline at p by* [11, Theorem B]. *Let q and q' be generators of $\mathcal{Q}_p(A, \varrho)$. For Selmer classes x and y in* $\mathrm{Sel}(\mathbf{Q}, V(f, g, h))$, *denote by $\widetilde{h}_p^{\alpha\alpha}(x \otimes y)$ the square-root of the discriminant of $\langle\langle \cdot, \cdot \rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha \boldsymbol{h}_\alpha}$ computed on the $\mathbf{Q}(\varrho)$-submodule of $\widetilde{H}^1_f(\mathbf{Q}, V(f, g, h))$ generated by x, y, q and q'. The article* [8] *proves the equality*

$$\widetilde{h}_p^{\alpha\alpha}\big(\kappa(f, g_\alpha, h_\alpha) \otimes y\big) = \log_{\alpha\alpha}\big(\mathrm{res}_p(y)\big) \cdot \mathscr{L}_p^{\alpha\alpha}(A, \varrho) \pmod{\mathscr{I}^3}$$

in $(\mathscr{I}^2/\mathscr{I}^3)/\mathbf{Q}(\varrho)^*$ for each Selmer class y.

3.3. Perrin-Riou conjecture for diagonal classes

Recall the map

$$\gamma_{gh} : A(K_\varrho)^\varrho \otimes_{\mathbf{Q}(\varrho)} L \hookrightarrow \mathrm{Sel}(\mathbf{Q}, V(f, g, h))$$

defined in Equation (2.13), arising from the Kummer map on $A(K_\varrho)$ and the isomorphisms γ_g and γ_h fixed in (1.4). Assume that $A(K_\varrho)^\varrho$ has dimension 2 over $\mathbf{Q}(\varrho)$ and that $L_p^{\alpha\alpha}(A, \varrho)$ is not identically zero. The classical Birch and Swinnerton-Dyer conjecture predicts that the Shafarevich–Tate group of A over K_ϱ is finite, hence that γ_{gh} is an isomorphism. Moreover, it implies in this case the vanishing of $L(E, \varrho, s)$ at $s = 1$, which combined with the explicit reciprocity law shows that $\kappa(f, g_\alpha, h_\alpha)$ is a Selmer class. Then, if

(P,Q) is a $\mathbf{Q}(\varrho)$-basis of $A(K_\varrho)^\varrho$, one has $\kappa(f,g_\alpha,h_\alpha)=a\cdot\gamma_{gh}(P)+b\cdot\gamma_{gh}(Q)$ with a and b in L. After setting

$$\mathscr{E}=\left(1-\frac{\alpha_g\alpha_h}{\alpha_f}\right)\cdot\left(1-\frac{\alpha_f}{p\alpha_g\alpha_h}\right)^{-1},$$

Theorem 3.3 and Proposition 2.1 yield the identities

$$\mathscr{E}\cdot a\cdot\langle\langle P,Q\rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}=\log_{\alpha\alpha}(\gamma_{gh}(Q))\cdot\mathscr{L}_p^{\alpha\alpha}(A,\varrho)\pmod{\mathscr{I}^2}$$

and

$$-\mathscr{E}\cdot b\cdot\langle\langle P,Q\rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}=\log_{\alpha\alpha}(\gamma_{gh}(P))\cdot\mathscr{L}_p^{\alpha\alpha}(A,\varrho)\pmod{\mathscr{I}^2}$$

Moreover, Conjecture 1.1 predicts that $\langle\langle P,Q\rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ and $\mathscr{L}_p^{\alpha\alpha}(A,\varrho)\pmod{\mathscr{I}^2}$ are non-zero, and equal up to multiplication by a non-zero algebraic scalar in $\mathbf{Q}(\varrho)^*$. To sum up, when $\dim_{\mathbf{Q}(\varrho)}A(K_\varrho)^\varrho=2$, one expects that $\kappa(f,g_\alpha,h_\alpha)$ is equal to $\log_{\alpha\alpha}(\gamma_{gh}(Q))\cdot\gamma_{gh}(P)-\log_{\alpha\alpha}(\gamma_{gh}(P))\cdot\gamma_{gh}(Q)$ up to multiplication by a non-zero scalar in $\mathbf{Q}(\varrho)^*$. When $\dim_{\mathbf{Q}(\varrho)}A(K_\varrho)^\varrho>2$, Conjecture 1.1 predicts that $\mathscr{L}_p^{\alpha\alpha}(A,\varrho)$ belongs to $\mathscr{I}^2$ and that $\langle\langle\cdot,\cdot\rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ is non-degenerate, hence that $\kappa(f,g_\alpha,h_\alpha)$ is zero by Theorem 3.3 and the conjectural finiteness of the relevant Shafarevich–Tate group. Under the running assumptions of this section, the above discussion shows that the next conjecture is a direct consequence of Conjecture 1.1 combined with Theorem 3.3.

CONJECTURE 3.5.

(1) *Assume that the $\mathbf{Q}(\varrho)$-vector space $A(K_\varrho)^\varrho$ has dimension 2. Then, for each $\mathbf{Q}(\varrho)$-basis (P,Q) of $A(K_\varrho)^\varrho$, the equality*

$$\kappa(f,g_\alpha,h_\alpha)=\log_{\alpha\alpha}(\gamma_{gh}(Q))\cdot\gamma_{gh}(P)-\log_{\alpha\alpha}(\gamma_{gh}(P))\cdot\gamma_{gh}(Q)$$

holds in the Selmer group $\mathrm{Sel}(\mathbf{Q},V(f,g,h))$ up to multiplication by a non-zero element of $\mathbf{Q}(\varrho)^*$.

(2) *If $A(K_\varrho)^\varrho$ has dimension greater than 2 over $\mathbf{Q}(\varrho)$, then the diagonal class $\kappa(f,g_\alpha,h_\alpha)$ is equal to zero.*

Remarks 3.6.

(1) The equality displayed in Part 1 of Conjecture 3.5 is independent of the choice of the isomorphisms γ_g and γ_h fixed in Equation (1.4).

(2) Assume that both $r_{\mathrm{MW}}=\dim_{\mathbf{Q}(\varrho)}A(K_\varrho)^\varrho$ and $r_{\mathrm{S}}=\dim_L\mathrm{Sel}(\mathbf{Q},V(f,g,h))$ are equal to 2, and let (P,Q) be a $\mathbf{Q}(\varrho)$-basis of $A(K_\varrho)^\varrho$. If $\log_{\alpha\alpha}$ is not identically zero on (the image under res_p of) $\mathrm{Sel}(\mathbf{Q},V(f,g,h))$, then Lemma 3.2 implies

$$\kappa(f,g_\alpha,h_\alpha)=\lambda\cdot\big(\log_{\alpha\alpha}(\gamma_{gh}(Q))\cdot\gamma_{gh}(P)-\log_{\alpha\alpha}(\gamma_{gh}(P))\cdot\gamma_{gh}(Q)\big)\tag{3.3}$$

for some constant λ in L. In this case, the actual content of Conjecture 3.5 is then the non-vanishing and rationality statement *λ belongs to $\mathbf{Q}(\varrho)^*$*.

(3) Assume $r_{\mathrm{MW}} = r_{\mathrm{S}} = 2$ and that $\log_{\alpha\alpha}$ is not identically zero on the Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$. Fix a $\mathbf{Q}(\varrho)$-basis (P,Q) of $A(K_\varrho)^\varrho$. Equation (3.3), Proposition 2.1, Theorem 3.3 and the non-triviality of $\log_{\alpha\alpha}$ give the identity

$$\mathscr{L}_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^2} = \lambda \cdot \langle\!\langle P, Q\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$$

in $(\mathscr{I}/\mathscr{I}^2)/\mathbf{Q}(\varrho)^*$. According to Proposition 2.1 and the current assumption (3.1) (which implies $A(K_\varrho)^\varrho = A^\dagger(K_\varrho)^\varrho$), the square of $\langle\!\langle P, Q\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ equals the regulator $R_p^{\alpha\alpha}(A,\varrho)$, hence the previous equation yields the equality

$$L_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^3} = \lambda^2 \cdot R_p^{\alpha\alpha}(A,\varrho)$$

in $(\mathscr{I}^2/\mathscr{I}^3)/\mathbf{Q}(\varrho)^{*2}$. As a consequence Conjecture 3.5, namely the statement λ belongs to $\mathbf{Q}(\varrho)^*$, and the non-degeneracy of $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ on the Mordell–Weil group $A(K_\varrho)^\varrho$, is equivalent to Conjecture 1.1.

(4) Since by assumption the forms g and h are p-regular (cf. Section 1), one can actually consider the four diagonal classes $\kappa(f, g_\alpha, h_\alpha)$, $\kappa(f, g_\alpha, h_\beta)$, $\kappa(f, g_\beta, h_\alpha)$ and $\kappa(f, g_\beta, h_\beta)$ arising from the different choices of the roots of the pth Hecke polynomials of g and h. Conjecture 3.5, combined with standard conjectures, predicts that these classes generate a non-trivial submodule of $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ precisely when $r_{\mathrm{MW}} = 2$. Assuming $r_{\mathrm{MW}} = 2$, one has that res_p is not identically zero on $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$, hence one of the logarithms $\log_{\alpha\alpha}$, $\log_{\alpha\beta}$, $\log_{\beta\alpha}$ and $\log_{\beta\beta}$ (defined similarly as in Section 3.1) is not identically zero on $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$. Reordering the roots (α_g, β_g) and (α_h, β_h) if necessary, one can assume that $\log_{\alpha\alpha}$ is not identically zero. It follows from Conjecture 3.5 that the class $\kappa(f, g_\alpha, h_\alpha)$ is non-zero. Conversely, assume that $\kappa(f, g_\alpha, h_\alpha)$ is non-zero. According to the parity conjecture and the conjectural finiteness of the p-primary part of the ϱ-component of the Shafarevich–Tate group of A over K_ϱ one has $r_{\mathrm{MW}} \geqslant 2$. Conjecture 3.5 implies the equality $r_{\mathrm{MW}} = 2$.

(5) Conjecture 3.5 is a reformulation of [16, Conjecture 3.12], which (together with Conjecture 2.1 of loc. cit.) is a refinement of the Elliptic Stark Conjecture formulated in [15] (cf. [16, Proposition 3.13 and Remark 3.14]). The above discussion then gives a conceptual explanation of the conjectures formulated in [15, 16] in the framework of

the p-adic analogues of the Birch and Swinnerton-Dyer conjecture. Theoretical evidence for these conjectures is obtained in [9, 13, 17]

(6) Assume in this remark that (A,ϱ) is exceptional at p. When $\alpha_f = \alpha_g \cdot \alpha_h$ we expect that Conjecture 3.5 holds verbatim in light of Remark 3.4. By contrast, if $\alpha_f = \beta_g \cdot \alpha_h$, then the specialisation $\kappa(f, g_\alpha, h_\alpha)$ of $\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ at $w_o = (2,1,1)$ is equal to zero, independently on whether $L(f \otimes g \otimes h, s)$ vanishes or not at $s = 1$. In this case, we expect that Conjecture 3.5 holds after replacing $\kappa(f, g_\alpha, h_\alpha)$ with the *improved diagonal class* $\kappa^*(f, g_\alpha, h_\alpha)$ defined in [11, Section 1.2] (cf. Theorem B of loco citato).

3.4. Proof of Theorem 3.3

This section proves Theorem 3.3.

Under the running assumption (3.1), the module $H^0(\mathbf{Q}_p, V(f,g,h)^-)$ is equal to zero and we identify the Block–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f, g, h))$ with Nekovář's extended Selmer group $\widetilde{H}^1_f(\mathbf{Q}, V(f,g,h))$ under the isomorphism (2.9). Fix a 1-cocycle

$$\widetilde{z} = (z, z^+, a) \in \widetilde{\mathrm{C}}^1_f(G_{Np}, V(f,g,h))$$

which represents the diagonal class $\kappa(f, g_\alpha, h_\alpha)$ in $\widetilde{H}^1_f(\mathbf{Q}, V(f,g,h))$. Then

$$z \in \mathrm{C}^1_{\mathrm{cont}}(G_{Np}, V(f,g,h)), \quad z^+ \in \mathrm{C}^1_{\mathrm{cont}}\big(\mathbf{Q}_p, V(f,g,h)^+\big)$$

and

$$a = (a_v)_{v|Np} \in \bigoplus_{v|Np} V(f,g,h)$$

satisfy the relations

$$dz = 0, \quad \kappa(f, g_\alpha, h_\alpha) = \mathrm{cl}(z), \quad dz^+ = 0 \quad \text{and} \quad \mathrm{res}_{Np}(z) = i^+\big(z^+\big) - da,$$

where d denotes the differentials of the complexes $\mathrm{C}^\bullet_{\mathrm{cont}}$ and $\mathrm{cl}(\cdot)$ denotes the cohomology class represented by $\cdot$. Let

$$Z \in \mathrm{C}^1_{\mathrm{cont}}(G_{Np}, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$$

be a 1-cocycle representing $\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ and specialising to z at w_o:

$$dZ = 0, \quad \kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha) = \mathrm{cl}(Z) \quad \text{and} \quad \rho_{w_o}(Z) = z$$

(cf. Equation (2.5)). The 1-cocycle $\widetilde{z}$ is then lifted by a 1-cochain of the form

$$\widetilde{Z} = (Z, Z^+, A) \in \widetilde{\mathrm{C}}^1_f(G_{Np}, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$$

under the morphism of complexes

$$\rho_{w_o}:\widetilde{\mathrm{C}}^{\bullet}_{f}(G_{Np},V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha))\longrightarrow\widetilde{\mathrm{C}}^{\bullet}_{f}(G_{Np},V(f,g,h))$$

induced by ρ_{w_o} (cf. Equation (2.6)), where the cochains

$$Z^+\in\mathrm{C}^1_{\mathrm{cont}}(\mathbf{Q}_p,V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha))\quad\text{and}\quad A=(A_v)_{v|Np}\in\bigoplus_{v|Np}V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$$

are lifts of z^+ and a respectively under the map induced by ρ_{w_o}.

In the rest of the proof, let $\boldsymbol{u}$ denote one of $\boldsymbol{k},\boldsymbol{l}$ and $\boldsymbol{m}$. As $\widetilde{z}$ is a 1-cocycle, the differential $d\widetilde{Z}$ of $\widetilde{Z}$ in $\widetilde{\mathrm{C}}^2_f(G_{Np},V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha))$ can be written as

$$d\widetilde{Z}=(\boldsymbol{k}-2)\cdot\widetilde{Z}_{\boldsymbol{k}}+(\boldsymbol{l}-1)\cdot\widetilde{Z}_{\boldsymbol{l}}+(\boldsymbol{m}-1)\cdot\widetilde{Z}_{\boldsymbol{m}}\tag{3.4}$$

with 2-cochains $\widetilde{Z}_{\boldsymbol{u}}$ in $\widetilde{\mathrm{C}}^2_f(G_{Np},V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha))$ of the form

$$\widetilde{Z}_{\boldsymbol{u}}=\big(Z_{\boldsymbol{u}},Z^+_{\boldsymbol{u}},W_{\boldsymbol{u}}\big),\tag{3.5}$$

where the 1-cochains $W_{\boldsymbol{u}}=(W_{\boldsymbol{u},v})_{v|Np}$ in $\bigoplus_{v|Np}\mathrm{C}^1_{\mathrm{cont}}(\mathbf{Q}_v,V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha))$ satisfy

$$(\boldsymbol{k}-2)\cdot W_{\boldsymbol{k}}+(\boldsymbol{l}-1)\cdot W_{\boldsymbol{l}}+(\boldsymbol{m}-1)\cdot W_{\boldsymbol{m}}=i^+\big(Z^+\big)-\mathrm{res}_{Np}(Z)-dA.\tag{3.6}$$

A slight extension of [33, Lemma 5.5] (cf. [34, Appendix C]) proves that

$$\widetilde{z}_{\boldsymbol{u}}=\rho_{w_o}(\widetilde{Z}_{\boldsymbol{u}})$$

are 2-cocycles in $\widetilde{\mathrm{C}}^2_f(G_{Np},V(f,g,h))$ and (cf. Equation (2.17))

$$\begin{aligned}-\widetilde{\beta}_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}&(\kappa(f,g_\alpha,h_\alpha))\\&=(\boldsymbol{k}-2)\cdot\,\mathrm{cl}(\widetilde{z}_{\boldsymbol{k}})+(\boldsymbol{l}-1)\cdot\,\mathrm{cl}(\widetilde{z}_{\boldsymbol{l}})+(\boldsymbol{m}-1)\cdot\,\mathrm{cl}(\widetilde{z}_{\boldsymbol{m}}).\end{aligned}\tag{3.7}$$

For $V=V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}),V(f,g,h)$, denote by

$$p^-:\mathrm{C}^{\bullet}_{\mathrm{cont}}(\mathbf{Q}_p,V)\longrightarrow\mathrm{C}^{\bullet}_{\mathrm{cont}}(\mathbf{Q}_p,V^-)$$

the morphism of complexes induced by the projection $p^-:V\to V^-$. Define

$$\begin{aligned}X_{\boldsymbol{u}}&=p^-(W_{\boldsymbol{u},p})\in\mathrm{C}^1_{\mathrm{cont}}(\mathbf{Q}_p,V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)^-);\\x_{\boldsymbol{u}}&=\rho_{w_o}(X_{\boldsymbol{u}})=p^-\circ\rho_{w_o}(W_{\boldsymbol{u},p})\in\mathrm{C}^1_{\mathrm{cont}}(\mathbf{Q}_p,V(f,g,h)^-).\end{aligned}\tag{3.8}$$

After setting $A_p^-=p^-(A_p)$, Equation (3.6) yields

$$(\boldsymbol{k}-2)\cdot X_{\boldsymbol{k}}+(\boldsymbol{l}-1)\cdot X_{\boldsymbol{l}}+(\boldsymbol{m}-1)\cdot X_{\boldsymbol{m}}=-p^-\big(\mathrm{res}_p(Z)\big)-dA_p^-.\tag{3.9}$$

As Z is a 1-*cocycle*, this implies that the 1-cochains $x_{\boldsymbol{u}}$ are 1-cocycles, and one sets

$$\mathfrak{x}_{\boldsymbol{u}}=\mathrm{cl}(x_{\boldsymbol{u}})\in H^1(\mathbf{Q}_p,V(f,g,h)^-).$$

Similarly, as Z is a 1-cocycle, Equations (3.4) and (3.5) imply that $\rho_{w_o}(Z_{\boldsymbol{u}}) = 0$, hence

$$\widetilde{z}_{\boldsymbol{u}} = \big(0, \rho_{w_o}\big(Z_{\boldsymbol{u}}^+\big), \rho_{w_o}(W_{\boldsymbol{u}})\big).$$

Because $\mathrm{C}^\bullet_{\mathrm{cont}}(\mathbf{Q}_v, V(f,g,h))$ is acyclic for $v \neq p$, this implies

$$\mathrm{cl}(\widetilde{z}_{\boldsymbol{u}}) = \jmath(\mathfrak{x}_{\boldsymbol{u}}) \tag{3.10}$$

(cf. Equations (2.8) and (3.8)). After recalling the definition of Garrett–Nekovář p-adic height $\langle\langle \cdot, \cdot \rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ given in Section 2.5, Equations (3.7) and (3.10) yield

$$\begin{aligned} \langle\langle \kappa(f, g_\alpha, h_\alpha), \widetilde{s} \rangle\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} &\\ &= \langle \cdot, \cdot \rangle_{\mathrm{Nek}} \otimes \mathscr{I}/\mathscr{I}^2 \Big(\widetilde{\beta}_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}(\kappa(f, g_\alpha, h_\alpha)) \otimes \widetilde{s}\Big) \\ &= -\sum_{\boldsymbol{u}} \langle \jmath(\mathfrak{x}_{\boldsymbol{u}}), \widetilde{s} \rangle_{\mathrm{Nek}} \cdot (\boldsymbol{u} - u_o) \\ &= -\sum_{\boldsymbol{u}} \big\langle \mathfrak{x}_{\boldsymbol{u}}, \widetilde{s}^+ \big\rangle_{\mathrm{Tate}} \cdot (\boldsymbol{u} - u_o) \end{aligned} \tag{3.11}$$

for each $\widetilde{s}$ in $\widetilde{H}^1_f(\mathbf{Q}, V(f,g,h))$. Here $(\boldsymbol{u}, u_o)$ denotes one of the pairs $(\boldsymbol{k}, 2)$, $(\boldsymbol{l}, 1)$ and $(\boldsymbol{m}, 1)$, where

$$\langle \cdot, \cdot \rangle_{\mathrm{Tate}} : H^1(\mathbf{Q}_p, V(f,g,h)^-) \otimes_L H^1\big(\mathbf{Q}_p, V(f,g,h)^+\big) \longrightarrow L$$

is the local Tate duality induced by the perfect pairing π_{fgh} (cf. Section 2.1), and where $\cdot^+$ is the morphism introduced in Equation (2.11). The last equality in Equation (3.11) follows from the adjointness of the maps $\jmath$ and $\cdot^+$ with respect to the pairings $\langle \cdot, \cdot \rangle_{\mathrm{Nek}}$ and $\langle \cdot, \cdot \rangle_{\mathrm{Tate}}$ (cf. [33, Lemma 5.7].)

To conclude the proof we will need the following lemma. Set

$$V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f = V(\boldsymbol{f})^- \widehat{\otimes}_L V(\boldsymbol{g}_\alpha)^+ \widehat{\otimes}_L V(\boldsymbol{h}_\alpha)^+ \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}}.$$

The projection $p^- : V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha) \to V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)^-$ maps $\mathscr{F}^2 V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ onto $V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f$, hence induces in cohomology a morphism

$$p_f : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)) \longrightarrow H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f). \tag{3.12}$$

(Recall that the natural map $H^1(\mathbf{Q}_p, \mathscr{F}^2 V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)) \to H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$ is injective, hence identifies its source with $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$.)

Lemma 3.7. — *There exist $\mathfrak{Y}_{\boldsymbol{k}}, \mathfrak{Y}_{\boldsymbol{l}}$ and $\mathfrak{Y}_{\boldsymbol{m}}$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f)$ such that*

$$p_f(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))) = (\boldsymbol{k} - 2) \cdot \mathfrak{Y}_{\boldsymbol{k}} + (\boldsymbol{l} - 1) \cdot \mathfrak{Y}_{\boldsymbol{l}} + (\boldsymbol{m} - 1) \cdot \mathfrak{Y}_{\boldsymbol{m}}.$$

Moreover, if the previous equation is satisfied, then for $\boldsymbol{u} = \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}$ *one has*

$$\mathfrak{x}_{\boldsymbol{u}} = -\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}}).$$

Proof. — Set $V(f)^{-}_{\beta\beta} = V(f)^{-} \otimes_{\mathbf{Q}_p} V(g)_{\beta} \otimes_L V(h)_{\beta}$. It is an $L[G_{\mathbf{Q}_p}]$-direct summand of $V(f,g,h)^{-}$, and the specialisation map ρ_{w_o} induces an isomorphism

$$\rho_{w_o} : V(\boldsymbol{f}, \boldsymbol{g}_{\alpha}, \boldsymbol{h}_{\alpha})_f \otimes_{w_o} \mathscr{O}_{\boldsymbol{fgh}} \simeq V(f)^{-}_{\beta\beta}.$$

Since the kernel of evaluation at w_o on $\mathscr{O}_{\boldsymbol{fgh}}$ is generated by a regular sequence and $H^2(\mathbf{Q}_p, V(f)^{-}_{\beta\beta})$ is equal to zero, the specialisation isomorphism ρ_{w_o} induces in cohomology an isomorphism (denoted by the same symbol)

$$\rho_{w_o} : H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_{\alpha}, \boldsymbol{h}_{\alpha})_f) \otimes_{w_o} L \simeq H^1\Big(\mathbf{Q}_p, V(f)^{-}_{\beta\beta}\Big). \tag{3.13}$$

As explained in [11, Section 9.1], the Bloch–Kato finite subspace of the local cohomology group $H^1(\mathbf{Q}_p, V(f,g,h))$ is equal to the kernel of

$$p^{-} : H^1(\mathbf{Q}_p, V(f,g,h)) \longrightarrow H^1\big(\mathbf{Q}_p, V(f,g,h)^{-}\big)$$

(cf. Section 9.1). Because $\kappa(f, g_{\alpha}, h_{\alpha}) = \rho_{w_o}(\kappa(\boldsymbol{f}, \boldsymbol{g}_{\alpha}, \boldsymbol{h}_{\alpha}))$ is a Selmer class (under the current assumption $L(A, \varrho, 1) = 0$), it follows that the local class

$$\boldsymbol{\kappa}_f = p_f(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}_{\alpha}, \boldsymbol{h}_{\alpha})))$$

belongs to the kernel of (3.13), thus proving the first statement.

Let $\mathfrak{Y}_{\boldsymbol{u}}$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_{\alpha}, \boldsymbol{h}_{\alpha})_f)$ be local classes satisfying

$$\boldsymbol{\kappa}_f = \sum_{\boldsymbol{u}} \mathfrak{Y}_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o).$$

We prove that $\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}})$ is equal to $-\mathfrak{x}_{\boldsymbol{u}}$ for $\boldsymbol{u} = \boldsymbol{k}$, the cases $\boldsymbol{u} = \boldsymbol{l}, \boldsymbol{m}$ being similar. Since by construction $\mathrm{cl}(Z) = \kappa(\boldsymbol{f}, \boldsymbol{g}_{\alpha}, \boldsymbol{h}_{\alpha})$, according to Equation (3.9) one has

$$\mathrm{cl}\left(\sum_{\boldsymbol{u}} X_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o)\right) = -\sum_{\boldsymbol{u}} i_f(\mathfrak{Y}_{\boldsymbol{u}}) \cdot (\boldsymbol{u} - u_o) \in H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_{\alpha}, \boldsymbol{h}_{\alpha})^{-}), \tag{3.14}$$

where i_f denotes both the inclusion $V(\boldsymbol{f}, \boldsymbol{g}_{\alpha}, \boldsymbol{h}_{\alpha})_f \hookrightarrow V(\boldsymbol{f}, \boldsymbol{g}_{\alpha}, \boldsymbol{h}_{\alpha})^{-}$ and the morphism it induces in cohomology. Let $\nu : \mathscr{O}_{\boldsymbol{fgh}} \to \mathscr{O}_{\boldsymbol{f}}$ be the surjective morphism of rings sending the analytic function $F(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m})$ to $F(\boldsymbol{k}, 1, 1)$, and set

$$V(\boldsymbol{f}, g, h)^{-} = V(\boldsymbol{f}, \boldsymbol{g}_{\alpha}, \boldsymbol{h}_{\alpha})^{-} \otimes_{\nu} \mathscr{O}_{\boldsymbol{f}}$$

and

$$V(\boldsymbol{f})^-_{\beta\beta} = V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f \otimes_\nu \mathscr{O}_{\boldsymbol{f}}.$$

(Note that $V(\boldsymbol{f})^-_{\beta\beta} = V(\boldsymbol{f})^- \otimes_L V(g)_\beta \otimes_L V(h)_\beta \otimes_{\mathscr{O}_{\boldsymbol{f}}} \chi_{\mathrm{cyc}}^{1-\boldsymbol{k}/2}$ is an $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}_p}]$-direct summand of $V(\boldsymbol{f}, g, h)^-$ and $i_f \otimes_\nu \mathscr{O}_{\boldsymbol{f}}$ is the natural inclusion.) If one denotes by ν also the morphisms induced in cohomology (resp., on continuous cochains) by the projections $V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)^- \to V(\boldsymbol{f}, g, h)^-$ and $V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f \to V(\boldsymbol{f})^-_{\beta\beta}$, then $\nu(X_{\boldsymbol{k}})$ is a 1-cocycle in $\mathrm{C}^1_{\mathrm{cont}}(\mathbf{Q}_p, V(\boldsymbol{f}, g, h)^-)$ (cf. Equation (3.9)) and Equation (3.14) gives

$$(\boldsymbol{k}-2) \cdot \big(\mathrm{cl}(\nu(X_{\boldsymbol{k}})) + \nu(\mathfrak{Y}_{\boldsymbol{k}})\big) = 0.$$

On the other hand, the $(\boldsymbol{k}-2)$-torsion of $H^1(\mathbf{Q}_p, V(\boldsymbol{f}, g, h)^-)$ is a quotient of $H^0(\mathbf{Q}_p, V(f, g, h)^-)$, which is zero by assumption (viz. (A, ϱ) is not exceptional at p). Then $\nu(\mathfrak{Y}_{\boldsymbol{k}}) = -\mathrm{cl}(\nu(X_{\boldsymbol{k}}))$, hence by construction $\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{k}}) = -\mathfrak{x}_{\boldsymbol{k}}$. □

Let $\mathfrak{Y}_{\boldsymbol{u}}$ be as in the statement of Lemma 3.7, and let $\widetilde{y}$ be an element of $\widetilde{H}^1_f(\mathbf{Q}, V(f, g, h))$. Equation (3.11) and Lemma 3.7 give the identity

$$\langle\!\langle \kappa(f, g_\alpha, h_\alpha), \widetilde{y} \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} = \sum_{\boldsymbol{u}} \big\langle \rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}}), \widetilde{y}^+ \big\rangle_{\mathrm{Tate}} \cdot (\boldsymbol{u} - u_o). \tag{3.15}$$

If $\widetilde{y} = \imath_{\mathrm{ur}}(y)$ corresponds to the Selmer class y in $\mathrm{Sel}(\mathbf{Q}_p, V(f, g, h))$, then the image of $\widetilde{y}^+$ under the map induced in cohomology by the inclusion $V(f, g, h)^+ \hookrightarrow V(f, g, h)$ is equal to the restriction of y at p. In this case we claim that

$$\begin{aligned}\big\langle \rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}}), \widetilde{y}^+ \big\rangle_{\mathrm{Tate}} &\\ = \log_{\alpha\alpha}(\mathrm{res}_p(y)) &\cdot \big\langle \exp_p^*(\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}})), \eta_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha} \big\rangle_{fgh},\end{aligned} \tag{3.16}$$

where $\exp_p^* \colon H^1(\mathbf{Q}_p, V(f, g, h)^-) \to D_{\mathrm{dR}}(V(f, g, h)^-)$ is the Bloch–Kato dual exponential. Indeed, note that the projection $p^- : V(f, g, h) \twoheadrightarrow V(f, g, h)^-$ and the inclusion $i^+ : V(f, g, h)^+ \hookrightarrow V(f, g, h)$ induce natural isomorphisms

$$\mathrm{Fil}^0 V_{\mathrm{dR}}(f, g, h) \simeq D_{\mathrm{dR}}(V(f, g, h)^-)$$

and

$$D_{\mathrm{dR}}(V(f, g, h)^+) \simeq V_{\mathrm{dR}}(f, g, h)/\,\mathrm{Fil}^0,$$

which we consider as equalities. Moreover, since by assumption (A, ϱ) is not exceptional at p, the Bloch–Kato exponential map gives an isomorphism between $D_{\mathrm{dR}}(V(f,g,h)^+)$ and $H^1(\mathbf{Q}_p, V(f,g,h)^+)$. As $i^+(\widetilde{y}^+) = \mathrm{res}_p(y)$, it follows that

$$\langle \rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}}), \widetilde{y}^+\rangle_{\mathrm{Tate}} = \langle \exp_p^*(\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}})), \log_p(\mathrm{res}_p(y))\rangle_{fgh}. \tag{3.17}$$

For (i,j) in $\{\alpha,\beta\}^2$ and $\cdot = \emptyset, \pm$, define

$$V(f)_{ij}^{\cdot} = V(f)^{\cdot} \otimes_{\mathbf{Q}_p} V(g)_i \otimes_L V(h)_j$$

(so that $V(f,g,h)^{\cdot}$ is the direct sum of the submodules $V(f)_{ij}^{\cdot}$). Then $\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}})$ belongs to $H^1(\mathbf{Q}_p, V(f)_{\beta\beta}^-)$ (cf. the proof of Lemma 3.7), hence the linear form

$$\langle \exp_p^*(\rho_{w_o}(\mathfrak{Y})_{\boldsymbol{u}}), \cdot\rangle_{fgh} : V_{\mathrm{dR}}(f,g,h)/\,\mathrm{Fil}^0 \longrightarrow L$$

factors through the map $\mathrm{pr}_{\alpha\alpha} : V_{\mathrm{dR}}(f,g,h)/\,\mathrm{Fil}^0 \to D_{\mathrm{dR}}(V(f)_{\alpha\alpha})/\,\mathrm{Fil}^0$ induced by the projection $V(f,g,h) \twoheadrightarrow V(f)_{\alpha\alpha}$. Since by definition (cf. Section 3.1)

$$\mathrm{pr}_{\alpha\alpha}(\log_p(\mathrm{res}_p(y))) = \log_{\alpha\alpha}(\mathrm{res}_p(y)) \cdot \eta_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha}$$

the claim Equation (3.16) follows from Equation (3.17).

After setting

$$\exp_{\alpha\alpha}^*(\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}})) = \langle \exp_p^*(\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}})), \eta_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha}\rangle_{fgh},$$

Equations (3.15) and (3.16) prove the equality

$$\begin{aligned} &\langle\!\langle \kappa(f, g_\alpha, h_\alpha), \cdot\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} \\ &\qquad = \log_{\alpha\alpha}(\mathrm{res}_p(\cdot)) \cdot \sum_{\boldsymbol{u}} \exp_{\alpha\alpha}^*(\rho_{w_o}(\mathfrak{Y})_{\boldsymbol{u}}) \cdot (\boldsymbol{u} - u_o) \end{aligned} \tag{3.18}$$

of $\mathscr{I}/\mathscr{I}^2$-valued L-linear forms on the Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$.

By [11, Proposition 7.3], the Perrin-Riou logarithm $\mathscr{L}_{\boldsymbol{f}}$ introduced in Section 3.2 factors through the map p_f defined in Equation (3.12), and hence gives rise to a morphism (denoted again by the same symbol)

$$\mathscr{L}_{\boldsymbol{f}} : H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f) \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}.$$

Moreover, for each local class $\mathfrak{Z}$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f)$ one has (cf. loc. cit.)

$$\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z})(w_o) = \frac{\left(1 - \frac{\alpha_g \alpha_h}{\alpha_f}\right)}{\left(1 - \frac{\alpha_f}{p\alpha_g\alpha_h}\right)} \cdot \exp_{\alpha\alpha}^*(\rho_{w_o}(\mathfrak{Z})).$$

Applying $\mathscr{L}_{\boldsymbol{f}}$ to both sides of the identity

$$p_f\big(\mathrm{res}_p\big(\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)\big)\big) = \sum_{\boldsymbol{u}} \mathfrak{Y}_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o),$$

and using the explicit reciprocity law Equation (3.2), one then gets the identity

$$\mathscr{L}_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^2} = \frac{\left(1-\frac{\alpha_g\alpha_h}{\alpha_f}\right)}{\left(1-\frac{\alpha_f}{p\alpha_g\alpha_h}\right)} \cdot \sum_{\boldsymbol{u}} \exp^*_{\alpha\alpha}(\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}})) \cdot (\boldsymbol{u}-u_o).$$

Theorem 3.3 follows from the previous equation and Equation (3.18).

BIBLIOGRAPHY

[1] F. Andreatta & A. Iovita, "Triple product p-adic L-functions associated to finite slope p-adic families of modular forms", *Duke Math. J.* **170** (2021), no. 9, p. 1989-2083.

[2] D. Benois, *p-adic heights and p-adic Hodge theory*, Mémoires de la Société Mathématique de France. Nouvelle Série, vol. 167, Société Mathématique de France, 2020.

[3] M. Bertolini, "Selmer groups and Heegner points in anticyclotomic $\mathbf{Z}_p$-extensions", *Compos. Math.* **99** (1995), no. 2, p. 153-182.

[4] M. Bertolini & H. Darmon, "A rigid analytic Gross–Zagier formula and arithmetic applications", *Ann. Math. (2)* **146** (1997), no. 1, p. 111-147, with an appendix by Bas Edixhoven.

[5] ———, "Iwasawa's Main Conjecture for Elliptic Curves over Anticyclotomic $\mathbb{Z}_p$-extensions", *Ann. Math.* **162** (2005), no. 1, p. 1-64.

[6] M. Bertolini, H. Darmon & R. Venerucci, "Heegner points and Beilinson–Kato elements: a conjecture of Perrin–Riou", *Adv. Math.* **398** (2022), article no. 108172 (50 pages).

[7] M. Bertolini, M. Longo & R. Venerucci, "The anticyclotomic main conjectures for elliptic curves", 2023, `https://arxiv.org/abs/2306.17784v1`.

[8] M. Bertolini, M. A. Seveso & R. Venerucci, "Heegner points and exceptional zeros of Garrett p-adic L-functions", *Milan J. Math.* **89** (2021), no. 1, p. 241-261.

[9] ———, "Balanced diagonal classes and rational points on elliptic curves", in *Heegner points, Stark–Heegner points, and diagonal classes*, Astérisque, vol. 434, Société Mathématique de France, 2022, p. 175-201.

[10] ———, "On exceptional zeros of Garrett–Hida p-adic L-functions", *Ann. Math. Qué.* **46** (2022), no. 2, p. 303-324.

[11] ———, "Reciprocity laws for balanced diagonal classes", in *Heegner points, Stark–Heegner points, and diagonal classes*, Astérisque, vol. 434, Société Mathématique de France, 2022, p. 77-171.

[12] A. Betina, M. Dimitrov & A. Pozzi, "On the failure of Gorensteinness at weight 1 Eisenstein points of the eigencurve", *Am. J. Math.* **144** (2022), no. 1, p. 227-265.

[13] F. Castella & M.-L. Hsieh, "On the nonvanishing of generalised Kato classes for elliptic curves of rank 2", *Forum Math. Sigma* **10** (2022), article no. e12 (32 pages).

[14] C. Cornut, "Mazur's conjecture on higher Heegner points", *Invent. Math.* **148** (2002), no. 3, p. 495-523.

[15] H. Darmon, A. Lauder & V. Rotger, "Stark points and p-adic iterated integrals attached to modular forms of weight one", *Forum Math. Pi* **3** (2015), article no. e8 (95 pages).

[16] H. Darmon & V. Rotger, "Elliptic curves of rank two and generalised Kato classes", *Res. Math. Sci.* **3** (2016), article no. 27 (32 pages).

[17] ———, "Diagonal cycles and Euler systems. II: The Birch and Swinnerton-Dyer conjecture for Hasse–Weil–Artin L-functions", *J. Am. Math. Soc.* **30** (2017), no. 3, p. 601-672.

[18] ———, "p-adic families of diagonal cycles", preprint, 2020, `https://arxiv.org/abs/2207.01314`.

[19] B. H. Gross & D. B. Zagier, "Heegner points and derivatives of L-series", *Invent. Math.* **84** (1986), no. 2, p. 225-320.

[20] B. Howard, "The Heegner point Kolyvagin system", *Compos. Math.* **140** (2004), no. 6, p. 1439-1472.

[21] M.-L. Hsieh, "Hida families and p-adic triple product L-functions", *Am. J. Math.* **143** (2021), no. 2, p. 411-532.

[22] K. Kato, "p-adic Hodge theory and values of zeta functions of modular forms", in *Cohomologies p-adiques et applications arithmétiques. III*, Astérisque, vol. 295, Société Mathématique de France, 2004, p. 117-290.

[23] V. A. Kolyvagin, "Euler systems", in *The Grothendieck Festschrift. Vol. II*, Progress in Mathematics, vol. 87, Birkhäuser, 1990, p. 435-483.

[24] M. Longo & S. Vigni, "On the vanishing of Selmer groups for elliptic curves over ring class fields", *J. Number Theory* **130** (2010), no. 1, p. 128-163.

[25] B. Mazur, J. T. j. Tate & J. T. Teitelbaum, "On p-adic analogues of the conjectures of Birch and Swinnerton–Dyer", *Invent. Math.* **84** (1986), no. 1, p. 1-48.

[26] J. Nekovář, *Selmer complexes*, Astérisque, vol. 310, Société Mathématique de France, 2006.

[27] J. Nekovář & W. Nizioł, "Syntomic cohomology and p-adic regulators for varieties over p-adic fields", *Algebra Number Theory* **10** (2016), no. 8, p. 1695-1790.

[28] B. Perrin-Riou, "Fonctions L p-adiques d'une courbe elliptique et points rationnels", *Ann. Inst. Fourier* **43** (1993), no. 4, p. 945-995.

[29] J. Pottharst, "Analytic families of finite-slope Selmer groups", *Algebra Number Theory* **7** (2013), no. 7, p. 1571-1612.

[30] K. Rubin, "Abelian varieties, p-adic heights and derivatives", in *Algebra and number theory (Essen, 1992)*, Walter de Gruyter, 1994, p. 247-266.

[31] V. Vatsal, "Uniform distribution of Heegner points", *Invent. Math.* **148** (2002), no. 1, p. 1-46.

[32] R. Venerucci, "p-adic regulators and p-adic analytic families of modular forms", PhD Thesis, University of Milan, Italy, 2013.

[33] ———, "Exceptional zero formulae and a conjecture of Perrin-Riou", *Invent. Math.* **203** (2016), no. 3, p. 923-972.

[34] ———, "p-adic regulators and p-adic families of modular representations", preprint, 2016.

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