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COUNTER-EXAMPLES TO A CONJECTURE OF KARPENKO FOR SPIN GROUPS

by Sanghoon BAEK & Rostislav DEVYATOV (*)

Abstract. — Consider the canonical morphism from the Chow ring of a smooth variety X to the associated graded ring of the topological filtration on the Grothendieck ring of X. In general, this morphism is not injective. However, Nikita Karpenko conjectured that these two rings are isomorphic for a generically twisted flag variety X of a semisimple group G. The conjecture was first disproved by Nobuaki Yagita for $G = \mathrm{Spin}(2n+1)$ with $n = 8, 9$. Later, another counter-example to the conjecture was given by Karpenko and the first author for $n = 10$. In this note, we provide an infinite family of counter-examples to Karpenko's conjecture for any 2-power integer n greater than 4. This generalizes Yagita's counter-example and its modification due to Karpenko for $n = 8$.

Résumé. — Considérons le morphisme canonique de l'anneau de Chow d'une variété lisse X à l'anneau gradué associé à la filtration topologique sur l'anneau de Grothendieck de X. En général, ce morphisme n'est pas injectif. Cependant, Nikita Karpenko a supposé que ces deux anneaux sont isomorphes pour une variété de drapeaux génériquement tordue X d'un groupe semi-simple G. La conjecture a été réfutée pour la première fois par Nobuaki Yagita pour $G = \mathrm{Spin}(2n+1)$ avec $n = 8, 9$. Plus tard, un autre contre-exemple à la conjecture a été donné par Karpenko et le premier auteur pour $n = 10$. Dans cette note, nous fournissons une famille infinie de contre-exemples à la conjecture de Karpenko pour tout entier n égal à une puissance de 2 et supérieur à 4. Ceci généralise le contre-exemple de Yagita et sa modification due à Karpenko pour $n = 8$.

1. Introduction

For a smooth variety X over a field k, let $\mathrm{CH}(X)$ and $K(X)$ denote the Chow and Grothendieck rings of X, respectively. Consider the associated

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graded ring $GK(X)$ of $K(X)$ with respect to the topological filtration, i.e.,

$$GK(X) = \bigoplus_{i=0}^{\dim X} K(X)^{(i)}/K(X)^{(i+1)},$$

where $K(X)^{(i)}$ denotes the i^{th} term of the topological filtration of $K(X)$.

The canonical morphism

$$\varphi : \mathrm{CH}(X) \longrightarrow GK(X) \tag{1.1}$$

sending the class of a closed subvariety of X in $\mathrm{CH}^i(X)$ to the class of its structure sheaf in $K(X)^{(i)}/K(X)^{(i+1)}$, is surjective but not injective in general. By Riemann–Roch theorem, for all $i \geqslant 1$, the kernel of the i^{th} homogeneous component

$$\varphi^i : \mathrm{CH}^i(X) \longrightarrow GK^i(X) := K(X)^{(i)}/K(X)^{(i+1)}$$

is annihilated by $(i-1)!$. Hence, the morphism φ becomes an isomorphism after tensoring with $\mathbb{Q}$. In particular, if X is a flag variety, that is, the quotient G/P of a split semisimple group G by a parabolic subgroup P, then φ is an isomorphism as $\mathrm{CH}(X)$ is torsion-free. In [6], Nikita Karpenko conjectured that the morphism φ is still injective for a generic flag variety X, namely:

Conjecture 1.1. — *The morphism in* (1.1) *is injective for a generic flag variety $X = E/P$ of a split semisimple group G, where E denotes a generic G-torsor given by the generic fiber of a G-torsor* $\mathrm{GL}(N) \to \mathrm{GL}(N)/G$ *induced by an embedding of G into a general linear group* $\mathrm{GL}(N)$ *for some $N \geqslant 1$ and P denotes a parabolic subgroup of G.*

This conjecture has been verified in a number of cases, including simple groups G of type A and C (see [7, Theorem 1.2]), special orthogonal groups G, the simply connected groups G of type G_2, F_4, and E_6 (see [6, Theorem 3.3]).

Now we consider the split spin group $G = \mathrm{Spin}(N)$ of a non-degenerate quadratic form of dimension N over a field k. Let P denote a maximal parabolic subgroup whose conjugacy class is obtained by the subset of the Dynkin diagram of G corresponding to removing the last vertex. Then, a generic G-torsor E gives rise to an N-dimensional generic quadratic form q whose discriminant and Clifford invariant are trivial. The generic flag variety $X = E/P$ becomes a maximal orthogonal grassmannian of q. By [1, Proposition 2.16], Conjecture 1.1 with $N = 2n+1$ is equivalent to the same conjecture with $N = 2n+2$. Thus, in this paper, we shall only consider the maximal orthogonal grassmannian X with $N = 2n+1$.

Conjecture 1.1 holds for $1 \leqslant n \leqslant 5$ (see [8]). On the other hand, the conjecture was first disproved for $n = 8, 9$ by Yagita [15]. Later, the counter-examples due to Yagita were extended to $n = 8, 9, 10$ over the base field of arbitrary characteristic in [1, 10]. In the present paper, we generalize the proof for $n = 8$ due to Karpenko and construct an infinite family of counterexamples over a field of any characteristic:

Theorem 1.2. — *Let $n \geqslant 8$ be a power of 2 and let X be the maximal orthogonal grassmannian of a generic n-dimensional quadratic form with trivial discriminant and Clifford invariant. Then, the canonical epimorphism $\varphi : \mathrm{CH}(X) \to GK(X)$ is not injective.*

For each 2-power $n \geqslant 8$, we construct an explicit element $x \in \mathrm{CH}(X)$ (see (4.3) below), which is not divisible by 2 in $\mathrm{CH}(X)$, but $\varphi(x)$ is divisible by 2 in $GK(X)$. In the following part of introduction, we sketch the proof that x has these properties and provide some ideas behind the construction of such an element x. The detailed proof is given in later Sections 3 and 4.

First, by [8, Proposition 2.1] the Chow ring $\mathrm{CH}(X)$ is generated by the Chern classes $c(1), \ldots, c(n)$ and an additional element $e \in \mathrm{CH}^1(X)$ (see Section 2.3). Since the Chern classes satisfy the relations ([9, Theorem 2.1]):

$$c(i)^2 = (-1)^{i+1} 2c(2i) + 2\sum_{k=1}^{i-1}(-1)^{k+1}c(i-k)c(i+k), \tag{1.2}$$

we can rewrite any polynomial in $c(i)$ as a square-free polynomial. Hence, together with relation $c(1) = 2e$, it suffices to consider an element x of the form

$$x = e^s \prod_{j \in J} c(j) \in \mathrm{CH}^l(X), \text{ where } s \geqslant 0 \text{ and } J \text{ is a subset of } [2, n] \tag{1.3}$$

for some $l \geqslant 1$ or a linear combination of elements of this form. In this note, we focus on an element of the form (1.3).

In the proof of non-2-divisibility of x, we make use of the degree map $\deg : \mathrm{CH}(X) \to \mathbb{Z}$ and the Steenrod operation S on $\mathrm{Ch}(X) := \mathrm{CH}(X)/2\,\mathrm{CH}(X)$ following [1, 10]. In general, it is quite difficult to check the divisibility of an element in $\mathrm{CH}(X)$. However, the exact value of the index $\operatorname{ind} X$ (i.e., the torsion index of $\mathrm{Spin}(2n+1)$) of X is available. Let $r = \dim X = \frac{n(n+1)}{2}$. Then

$$\operatorname{ind} X = 2^m, \text{ where } m = n - \lfloor \log_2(1+r) \rfloor \text{ or } m = n - \lfloor \log_2(1+r) \rfloor + 1$$

(depending on n, see [14] for details). In particular, if n is a power of 2, then the second formula for m holds. So, it is often possible to determine the non-divisibility of an element in $\mathrm{CH}_0(X) = \mathrm{CH}^r(X)$ by 2. Namely, since the image of the degree map is equal to $2^m\mathbb{Z}$, we get a well-defined homomorphism $2^{-m}\deg : \mathrm{Ch}(X) \to \mathbb{Z}/2\mathbb{Z}$. Moreover, non-2-divisibility of an element x of the form (1.3) immediately follows from the non-triviality of the image $S(\bar{x})$ under the map $2^{-m}\deg$, where $\bar{x}$ denote the image of x in $\mathrm{Ch}(X)$. Here, the use of Steenrod operations gives us more flexibility to find such an element x that is non-divisible by 2, while $\varphi(x)$ is divisible by 2: using Steenrod operations, one can try to find such an element x in an arbitrary graded component of $\mathrm{CH}(X)$.

The degree map is determined by the restriction map $\mathrm{res} : \mathrm{CH}(X) \to \mathrm{CH}(\overline{X})$, where $\overline{X}$ denotes the base change of X to an algebraic closure of k. Hence, to show $2^{-m}\deg(S(\bar{x})) \neq 0$, it suffices to prove that

(1.4) the image of an integral representative x' of $S(\bar{x})$ under the restriction map is congruent to $2^m p$ modulo 2^{m+1},

where p denotes the class of a rational point. In fact, the congruence relation (1.4) is the key step for the proof of non-2-divisibility of x. For each 2-power $n \geqslant 8$, this is proven in Proposition 3.10 by considering the element x of the form (1.3), where $l = r-3$, $s = n(\frac{n}{4}-1)+n-1$, and

$$J = \left(\left[2, \frac{n}{4}+1\right] \cup \left[\frac{3n}{4}-1, n-1\right]\right) \setminus \left(\{5\} \cup \left\{2^i \,\middle|\, 2 \leqslant i \leqslant \log_2(n)-2\right\}\right)$$

as in (4.1).

From the formula (2.20) ignoring the quadratic part and (2.21), we can find an integral representative x', which is a sum of elements of the same form as in (1.3), but with various numbers $s' \geqslant n(\frac{n}{4}-1)+n-1$ instead of s, and with various multi-subsets J' of $[2,n]$ with $|J| = |J'|$ instead of J. Then, we check the divisibility of $\mathrm{res}(x') \in \mathrm{CH}(\overline{X})$ by 2. The Chow ring $\mathrm{CH}(\overline{X})$ is generated by the special Schubert classes $e(1), \dots, e(n)$ with the relations (2.18) and the generators $c(i)$ and e of $\mathrm{CH}(X)$ map to $2e(i)$ and $e(1)$ in $\mathrm{CH}(\overline{X})$, respectively, under the restriction map. Note that the relation (2.18), as well as its powers, become simpler if considered modulo powers of 2, which makes it easier to check the non-divisibility of an element of $\mathrm{CH}_0(\overline{X})$ by a power of 2 compared to non-divisibility by other numbers.

Since $e(1)^n \equiv e(\frac{n}{2})^2 \mod 4$ (here we use the assumption that n is a power of 2), a direct calculation using a multinomial expansion of the

$(\frac{n}{4}-1)^{\text{th}}$ power of the right-hand side of (2.18) with $i=\frac{n}{2}$ yields that

$$\begin{aligned}(1.5)\quad & e(1)^{n\left(\frac{n}{4}-1\right)} \\ & \equiv -\left(\frac{n}{4}-1\right) e(n)\cdot 2^{\frac{n}{4}-2}\left(\sum_{k=1}^{\frac{n}{2}-1} e\left(\frac{n}{2}-k\right) e\left(\frac{n}{2}+k\right)\right)^{\frac{n}{4}-2} \\ & \qquad \bmod 2^{\frac{n}{2}-\log_2(n)},\end{aligned}$$

and $e(1)^{n(\frac{n}{4}-1)} \equiv 0 \mod 2^{\frac{n}{2}-\log_2(n)-1}$ (see Lemma 3.3).

As for each J' above we have $|J'|=|J|=\frac{n}{2}-\log_2(n)+3$, and $m=n-2\log_2(n)+2$, we see that each summand of $\operatorname{res}(x')$ becomes a multiple of 2^m. Now, to conclude the proof, a careful computation is required to see from which multi-subsets J' (and for which exponents s') an extra multiple of 2 arises. This is done in Corollary 3.9 by multiplying (1.5) by the Chern classes with indexes in $[\frac{3n}{4}-1, n-1]$, in Proposition 3.10, in Remark 3.11, and in Lemma 4.3.

Now, to show that $\varphi(x)$ is divisible by 2 in $GK(X)$, we use the Rees ring $\widetilde{K}(X)$ and its ideal $I(X)$ generated by 2 and t that are surjectively mapped onto $GK(X)$ and $2GK(X)$, respectively, by the map ξ (see Section 2.2). Since x is of the form (1.3), by (2.16) and Lemma 2.2, we have a standard preimage w of $\varphi(x)$ in $\widetilde{K}(X)$ under ξ. By replacing the Chern classes $\mathbf{c}(i)$ in w with the element $2\mathbf{e}(i)-t\mathbf{e}(i+1)$ (see Lemma 2.1), we obtain another preimage $y\in\widetilde{K}(X)$ of $\varphi(x)$ under ξ (i.e., $\xi(y)=\xi(w)=\varphi(x)$) and show $y\in I(X)$.

In order to prove $y\in I(X)$, we view y as contained in $\widetilde{K}(\overline{X})$ via the embedding $\widetilde{K}(X)\subset\widetilde{K}(\overline{X})$ and adopt an inductive argument as in [1, 10]. For any integers l with $m>r-l\geqslant 0$ and $j\geqslant 1$, write

$$\begin{aligned}&\widetilde{K}^l(\overline{X})\cap I(\overline{X})^{m+j} \\ &\qquad = 2^{m+j}\widetilde{K}^l(\overline{X}) + 2^{m+j-1}t\widetilde{K}^{l+1}(\overline{X}) + \cdots + 2^{m+j+l-r}t^{r-l}\widetilde{K}^r(\overline{X}).\end{aligned}$$

Then, by the restriction-corestriction formula, $\operatorname{ind} X\cdot I(\overline{X})\subset I(X)$ (see (2.10)). Hence, if $j<r-l$, we have modulo $I(X)$:

$$\begin{aligned}&\widetilde{K}^l(\overline{X})\cap I(\overline{X})^{m+j} \\ &\equiv 2^{m-1}t^{j+1}\widetilde{K}^{j+1}(\overline{X}) + 2^{m-2}t^{j+2}\widetilde{K}^{j+2}(\overline{X}) + \cdots + 2^{m+j+l-r}t^{r-l}\widetilde{K}^r(\overline{X}).\end{aligned}$$

For $j=r-l$, we simply get $\widetilde{K}^l(\overline{X})\cap I(\overline{X})^{m+r-l}\subset I(X)$.

In the proof of Theorem 1.2, we consider the case $l = r-3$ and $y \in \widetilde{K}^l(X)$ so that by (2.5) we get three congruence equations:

$$\begin{aligned}\widetilde{K}^{r-3}(\overline{X}) \cap I(\overline{X})^{m+1} &\equiv \mathbb{Z}\cdot(2^{m-1}\mathbf{l})u^{r-3} + \mathbb{Z}\cdot(2^{m-2}\mathbf{p})u^{r-3} \quad \operatorname{mod} I(X),\\ \widetilde{K}^{r-3}(\overline{X}) \cap I(\overline{X})^{m+2} &\equiv \mathbb{Z}\cdot(2^{m-1}\mathbf{p})u^{r-3} \quad \operatorname{mod} I(X),\end{aligned}$$

and $\widetilde{K}^{r-3}(\overline{X}) \cap I(\overline{X})^{m+3} \subset I(X)$, where $\mathbf{p}$ and $\mathbf{l}$ denote the classes of a point and a line in $K(\overline{X})$. If the generators $(2^{m-1}\mathbf{l})u^{r-3}$ and $(2^{m-2}\mathbf{p})u^{r-3}$ are contained in $I(X) + I(\overline{X})^{m+2}$, then

$$\begin{aligned}\widetilde{K}^{r-3}(\overline{X}) \cap I(\overline{X})^{m+1} &\subset \widetilde{K}^{r-3}(\overline{X}) \cap (I(\overline{X})^{m+2} + I(X))\\ &\subset \widetilde{K}^{r-3}(\overline{X}) \cap (I(\overline{X})^{m+3} + I(X)) \subset I(X).\end{aligned} \tag{1.6}$$

In addition, if y is contained in $I(\overline{X})^{m+1}$, then by (1.6) we conclude that $y \in I(X)$.

Alternatively, if $(2^{m-1}\mathbf{l})u^{r-3}, (2^{m-2}\mathbf{p})u^{r-3} \in I(X)$, then we could immediately conclude that

$$\widetilde{K}^{r-3}(\overline{X}) \cap I(\overline{X})^{m+1} \subset I(X). \tag{1.7}$$

Consequently, the proof of 2-divisibility of $\varphi(x)$ is based on two main ingredients. The first one is to check that y is contained in $I(\overline{X})^{m+1}$ (or a higher power of $I(\overline{X})$), which is proven in Proposition 3.7 (a). This part is similar to the proof, as mentioned above, of the divisibility of each summand of $\operatorname{res}(x') \in \mathrm{CH}(\overline{X})$ by 2^m. Indeed, some parts of the proof for $\operatorname{res}(x')$ even directly follow from the proof for y because of a surjective morphism (2.11) from $\widetilde{K}(\overline{X})$ to $\mathrm{CH}(\overline{X})$.

The second ingredient is to show that some product of the class of a line or a point by a strict divisor of the torsion index is contained in $I(X) + I(\overline{X})^{m+2}$ i.e., in our case $(2^{m-1}\mathbf{l})u^{r-3}, (2^{m-2}\mathbf{p})u^{r-3} \in I(X) + I(\overline{X})^{m+2}$. This is proven in Proposition 3.7 (b) by slightly modifying y into an element $z \in \widetilde{K}^{r-3}(X)$, which is congruent to $(2^{m-2}\mathbf{l})u^{r-3}$ modulo $I(\overline{X})^{m+1}$. As an additional consequence of Proposition 3.7 (b), we indeed have $(2^{m-1}\mathbf{l})u^{r-3}, (2^{m-2}\mathbf{p})u^{r-3} \in I(X)$ (see Remark 4.2). Therefore, we obtain (1.6) and (1.7).

In this note, we focus on values of n that are powers of 2. This choice is advantageous for some arguments, such as the congruence relation $f(1)^n \equiv f(n) \mod I(X)$ given by (2.15) and the property that the factorial $(\frac{n}{4})!$ is significantly more divisible by powers of 2 than $(\frac{n}{4} - 1)!$. However, the restriction to powers of 2 is not always necessary for all arguments. We expect that the arguments requiring n to be a power of 2 can be extended to other values of n, and we plan to present generalizations in future publications, using examples from [14] of elements of $\mathrm{CH}(X)$ of top degree (i.e., of dimension 0) that become divisible by $\operatorname{ind} X$ but not by $2 \operatorname{ind} X$ in $\mathrm{CH}(\overline{X})$.

So, throughout this note, n is a power of 2 and is bigger than 4. We denote the integer interval $\{a, a+1, \ldots, b\}$ by $[a, b]$ for any $a \leqslant b$. If $b < a$, then $[a, b]$ denotes the empty set.

2. Grothendieck and Chow rings of orthogonal grassmannians

Throughout this paper, let X denote the maximal orthogonal grassmannian (i.e., the variety of n-dimensional totally isotropic subspaces) of a generic $(2n+1)$-dimensional quadratic form q of trivial discriminant and Clifford invariant. The index of X, denoted by $\operatorname{ind} X$, is defined as the greatest common divisor of the degrees of closed points on X. Indeed, the index of X is equal to the torsion index of $\mathrm{Spin}(2n+1)$, which is computed as follows (see [14]):

$$\operatorname{ind} X = 2^{n-2v(n)+2}, \tag{2.1}$$

where n is a power of 2 and $v(n)$ denotes the exponent of 2 in n.

2.1. Grothendieck ring of orthogonal grassmannians

Let $\overline{X}$ denote X over an algebraic closure of k. In general, since $K(X)$ is torsion-free [12, Theorem 4.2], the ring $K(X)$ is identified with a subring of $K(\overline{X})$. As the Clifford invariant of q is trivial, by [5, Lemma 4.1], [12], we have an isomorphism

$$K(X) = K(\overline{X}). \tag{2.2}$$

The restriction map $K(X)^{(i)} \to K(\overline{X})^{(i)}$ is injective so that we view it as an inclusion:

$$K(X)^{(i)} \subset K(\overline{X})^{(i)} \tag{2.3}$$

for any $i \geqslant 1$. In particular, we have $K(X)^{(1)} = K(\overline{X})^{(1)}$. On the other hand, it follows by a restriction-corestriction argument that

$$\operatorname{ind} X \cdot K(\overline{X})^{(i)} \subset K(X)^{(i)} \tag{2.4}$$

for $i \geqslant 1$.

Write $\mathbf{c}(i) \in K(X)^{(i)}$ for the K-theoretic Chern class of the dual of the (rank n) tautological vector bundle $\mathcal{T}$ on X. Note that $\mathbf{c}(i) = 0$ for $i > n$. Let $\overline{Y}$ denote the quadric Y of q over an algebraic closure of k. We write $\mathbf{e}(i) \in K(\overline{X})^{(i)}$ for the image of the class of a projective $(n-i)$-dimensional subspace l_{n-i} on $\overline{Y}$ under the composition $(\pi_1)_* \circ (\pi_2)^*$ of the projective bundle $\pi_1\colon \mathcal{P} \to \overline{X}$ given by the tautological vector bundle on $\overline{X}$ and the projection $\pi_2\colon \mathcal{P} \to \overline{Y}$. We also set $\mathbf{e}(i) = 0$ for $i > n$. Then, the following relations hold.

Lemma 2.1 ([1, Lemma 2.12]). — *For any $i \geqslant 0$, the element*

$$2\mathbf{e}(i) - \mathbf{e}(i+1) - \mathbf{c}(i)$$

is a sum of monomials in $\mathbf{c}(1), \dots, \mathbf{c}(n)$ of degrees greater than or equal to $i+1$, where the degree of $\mathbf{c}(j)$ for any $j \geqslant 0$ is defined to be j. In particular, $2\mathbf{e}(i) - \mathbf{e}(i+1) = \mathbf{c}(i)$ in $GK^i(X)$.

Let us denote by $\mathbf{p}$ and $\mathbf{l}$ the classes of $\prod_{i=1}^{n} \mathbf{e}(i)$ and $\prod_{i=2}^{n} \mathbf{e}(i)$ in $K(\overline{X})^{(\dim \overline{X})}$ and $K(\overline{X})^{(\dim \overline{X}-1)}$, respectively. Then, we have

$$K(\overline{X})^{(\dim \overline{X})} = \mathbb{Z} \cdot \mathbf{p} \quad \text{and} \quad K(\overline{X})^{(\dim \overline{X}-1)} = \mathbb{Z} \cdot \mathbf{p} \oplus \mathbb{Z} \cdot \mathbf{l}. \tag{2.5}$$

2.2. Rees ring associated to the topological filtration

Consider the extended Rees ring $\widetilde{K}(X)$ of the Grothendieck ring $K(X)$ with respect to the topological filtration on $K(X)$, i.e.,

$$\widetilde{K}(X) = \bigoplus_{i \in \mathbb{Z}} \widetilde{K}^i(X), \text{ where } \widetilde{K}^i(X) = K(X)^{(i)} t^{-i} \tag{2.6}$$

for a variable t. Here we set $K(X)^{(i)} = K(X)$ for $i < 0$. Note also that $K(X)^{(i)} = 0$ for $i > \dim X$. We view $\widetilde{K}(X)$ as a subring of the Laurent polynomial ring $K(X)[t, t^{-1}]$. For notational simplicity, we write u for t^{-1}. Observe that $t \in \widetilde{K}(X)$, while $u \notin \widetilde{K}(X)$.

Let $I(X)$ denote the ideal of $\widetilde{K}(X)$ generated by t and 2. Then, we have an isomorphism $\widetilde{K}(X)/t\widetilde{K}(X) \xrightarrow{\sim} GK(X)$. Denote the composition of the projection $\widetilde{K}(X) \to \widetilde{K}(X)/t\widetilde{K}(X)$ and this isomorphism by

$$\xi\colon \widetilde{K}(X) \longrightarrow GK(X). \tag{2.7}$$

Note that then

$$\xi(I(X)) = 2GK(X). \tag{2.8}$$

We define $\widetilde{K}(\overline{X})$ and $\bar{\xi}\colon \widetilde{K}(\overline{X}) \to GK(\overline{X})$ in a similar way as in (2.6) and (2.7), respectively. By (2.3), we will treat $\widetilde{K}(X)$ as a subring of $\widetilde{K}(\overline{X})$. Moreover, by (2.4) we have

$$\operatorname{ind} X \cdot \widetilde{K}(\overline{X}) \subset \widetilde{K}(X). \tag{2.9}$$

In particular,

$$2 \operatorname{ind} X \cdot \widetilde{K}(\overline{X}),\ \ t \operatorname{ind} X \cdot \widetilde{K}(\overline{X}) \subset I(X). \tag{2.10}$$

Let $\bar{\varphi}$ denote the morphism in (1.1) for $\overline{X}$. As $\overline{X}$ is a flag variety, $\bar{\varphi}$ is becomes an isomorphism. We shall denote by ψ the composition

$$\psi : \widetilde{K}(\overline{X}) \xrightarrow{\bar{\xi}} GK(\overline{X}) \xrightarrow{\bar{\varphi}^{-1}} \mathrm{CH}(\overline{X}). \tag{2.11}$$

We write

$$f(i) = \mathbf{e}(i)u^i \in \widetilde{K}^i(\overline{X}) \text{ and } g(i) = 2f(i) - tf(i+1) \in \widetilde{K}^i(\overline{X}) \cap I(\overline{X})$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.1

$$g(i) \in \widetilde{K}^i(X) \ \text{ and } \ \xi(g(i)) = \xi(\mathbf{c}(i)u^i). \tag{2.12}$$

Moreover, by Corollary A.6, we have $f(n)^2 = 0$ and by Proposition A.10, the following relations hold modulo $I(\overline{X})^2$:

$$f(i)^2 \equiv \begin{cases} (-1)^{i-1}f(2i) + tf(2i+1) + \sum\limits_{k=1}^{i-1} f(i+k)g(i-k) & \text{if } i \text{ is even,} \\ (-1)^{i-1}f(2i) + \sum\limits_{k=1}^{i-1} f(i+k)g(i-k) & \text{if } i \text{ is odd.} \end{cases} \tag{2.13}$$

Instead of using this formula in full generality, we shall use it either for $i \geqslant \frac{n}{2}$, or modulo $I(\overline{X})$. If $i \geqslant \frac{n}{2}$, then, since $f(2i+1) = 0$, the relations (2.13) become

$$f(i)^2 \equiv (-1)^{i-1}f(2i) + \sum_{k=1}^{i-1} f(i+k)g(i-k) \mod I(\overline{X})^2, \tag{2.14}$$

regardless of the parity of i. Modulo $I(\overline{X})$, we simply have

$$f(i)^2 \equiv f(2i) \mod I(\overline{X}) \tag{2.15}$$

for any $i \in [1, n]$.

2.3. Chow ring of orthogonal grassmannians

Let $c(i) \in \mathrm{CH}^i(X)$ denote the Chern class of the dual of the tautological vector bundle $\mathcal{T}$ and let $e(i) \in \mathrm{CH}^i(\overline{X})$ denote the image of the class $l_{n-i} \in \mathrm{CH}^{n-i}(\overline{X})$ of a projective $(n-i)$-dimensional subspace on $\overline{Y}$ under the composition $(\pi_1)_* \circ (\pi_2)^*$. Since the morphism φ in (1.1) commutes with Chern classes, we have

$$\varphi\big(c(i)\big) = \mathbf{c}(i) + K(X)^{(i+1)}. \tag{2.16}$$

Moreover, the image of $e(i)$ under the isomorphism $\bar{\varphi}$ is given by

$$\bar{\varphi}\big(e(i)\big) = \mathbf{e}(i) + K(\overline{X})^{(i+1)}. \tag{2.17}$$

As an abelian group, $\mathrm{CH}(\overline{X})$ is freely generated by the set of all products of the form $\prod_{i\in I} e(i)$, where I is an arbitrary subset of $[1, n]$. The Chow ring $\mathrm{CH}(\overline{X})$ is generated by $e(1), \dots, e(n)$ subject to the relations

$$e(i)^2 = (-1)^{i+1} e(2i) + 2\sum_{k=1}^{i-1} (-1)^{k+1} e(i-k)e(i+k) \tag{2.18}$$

for all $i \geqslant 1$, where we set $e(i) = 0$ for $i > n$. In particular, we shall denote by p the class of a rational point, i.e., $p = \prod_{i=1}^{n} e(i) \in \mathrm{CH}(\overline{X})^{(\dim \overline{X})}$. By [4, Proposition 86.13], we have

$$\mathrm{res}\,\big(c(i)\big) = 2e(i) \tag{2.19}$$

for all $1 \leqslant i \leqslant n$, where $\mathrm{res} : \mathrm{CH}(X) \to \mathrm{CH}(\overline{X})$ denotes the restriction map.

By [11, Section 2], there is an exact sequence of abelian groups:

$$0 \longrightarrow \mathrm{CH}^1(X) \xrightarrow{\mathrm{res}} \mathrm{CH}^1(\overline{X}) \longrightarrow \mathrm{Br}(k),$$

where the second map sends the generator $e(1)$ to the Brauer class of the even Clifford algebra of q. Since the Clifford invariant of q is trivial, i.e., the Brauer class of the even Clifford algebra of q is trivial, the restriction map is an isomorphism so that

$$\mathrm{res}(e) = e(1)$$

for some $e \in \mathrm{CH}^1(X)$. As $\mathrm{res}(c(1)) = 2e(1)$, we have $c(1) = 2e$.

Since $K(X)^{(1)} = K(\overline{X})^{(1)}$, the element $\mathbf{e}(1) \in K(X)^{(1)}$ defines a class $\mathbf{e}(1) + K(X)^{(2)}$ in $GK^1(X)$. In particular, we have

Lemma 2.2. — $\varphi(e) = \mathbf{e}(1) + K(X)^{(2)}$.

Proof. — Since φ and $\bar{\varphi}$ commute with the field extension, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{CH}^1(X) & \xrightarrow{\varphi^1} & GK^1(X) \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathrm{CH}^1(\overline{X}) & \xrightarrow{\bar{\varphi}^1} & GK^1(\overline{X}). \end{array}$$

Since all maps except the right vertical map are isomorphisms, the right vertical map $\mathrm{res} : GK^1(X) \to GK^1(\overline{X})$ is an isomorphism as well.

As $\mathrm{res}(\mathbf{e}(1)+K(X)^{(2)}) = \mathbf{e}(1)+K(\overline{X})^{(2)}$ and $\bar{\varphi}(\mathrm{res}(e)) = \mathbf{e}(1)+K(\overline{X})^{(2)}$ by (2.17), both $\mathbf{e}(1) + K(X)^{(2)}$ and $\varphi(e)$ have the same image under $\mathrm{res} : GK^1(X) \to GK^1(\overline{X})$, whence the proof follows. □

Let $\mathrm{Ch}(X)$ denote the modulo 2 Chow group, i.e.,

$$\mathrm{Ch}(X) := \mathrm{CH}(X)/2\,\mathrm{CH}(X).$$

For any $x \in \mathrm{CH}(X)$, we write $\bar{x}$ for the image of x in $\mathrm{Ch}(X)$. Consider the total cohomological Steenrod operation $S : \mathrm{Ch}(X) \to \mathrm{Ch}(X)$ as in [4] ($\mathrm{char}\, k \neq 2$) and in [13] ($\mathrm{char}\, k = 2$). The operation commutes with pull-back morphisms, so it can be viewed as a contravariant functor from the category of smooth varieties to the category of abelian groups. Moreover, the Steenrod operation satisfies Cartan formula ([4, Corollary 61.15] for characteristic $\neq 2$ and [13, Proposition 6.1] for characteristic 2), i.e., it is a ring homomorphism.

For any $j \geqslant 0$, we denote by $S^j : \mathrm{Ch}^*(X) \to \mathrm{Ch}^{*+j}(X)$ the j^{th} component of S. In particular, S^0 is the identity map. A formula for the values of S^j on the Chern classes is given in [2, Théorème 7.1] (see also [1, Lemma 2.5]):

$$S^j\big(\bar{c}(i)\big) = \binom{i-1}{j}\bar{c}(i+j) + Q(i,j) \tag{2.20}$$

for any $i \geqslant 0$ and $j \geqslant 1$, where $Q(i,j)$ denotes a linear combination of $\bar{c}(1)\bar{c}(i+j-1), \ldots, \bar{c}(i)\bar{c}(j)$. We also have

$$S(\bar{e}) = \bar{e} + \bar{e}^2. \tag{2.21}$$

3. Congruence relations for split orthogonal grassmannians

In this section, we shall compute some basic congruence relations in both the extended Rees ring $\widetilde{K}(\overline{X})$ and the Chow ring $\mathrm{CH}(\overline{X})$.

Let us recall some basic notions concerning multisets and introduce some specific notations. A *multiset* is an unordered collection of elements with duplicates allowed. The *cardinality* of a multiset J is the sum of the multiplicities of all its elements and is denoted by $|J|$. The *sum* of two multisets J and L, denoted by $J+L$, is the multiset such that the multiplicity of an element is equal to the sum of the multiplicities of the element in J and L. We say that a multiset J is a *multi-subset* of a set S and write $J \subset S$, if every element of J is an element of S (note that we allow multiplicities greater than 1 in J here). For any multi-subset J of $[1,n]$, we write

$$\mathbf{e}(J) = \prod_{j \in J} \mathbf{e}(j) \in K(\overline{X}) \quad \text{and} \quad e(J) = \prod_{j \in J} e(j) \in \mathrm{CH}(\overline{X}).$$

Similarly, we write

$$f(J) = \prod_{j \in J} f(j) \in \widetilde{K}^{|J|}(\overline{X}) \quad \text{and} \quad g(J) = \prod_{j \in J} g(j) \in \widetilde{K}^{|J|}(\overline{X}).$$

For a nonzero element $a \in \widetilde{K}(\overline{X})$, we write $\mathbf{v}(a)$ for the highest power of $I(\overline{X})$ containing a. Similarly, for a nonzero element b in $\mathbb{Z}$ or $\mathrm{CH}(\overline{X})$ we write $v(b)$ for the highest power of 2 dividing b.

We shall write

$$\begin{aligned} I_0 := \left[\frac{n}{2}+1, n-1\right] \\ &= \left[\frac{4n}{8}+1, \frac{5n}{8}\right] \cup \left[\frac{5n}{8}+1, \frac{6n}{8}-2\right] \cup \left[\frac{6n}{8}-1, n-1\right] \\ &=: I_1 \cup I_2 \cup I_3 \end{aligned}$$

and $\bar{I}_3 = I_3 \cup \{n\}$. We set $I_1 = \emptyset$ for $n = 8$.

In the following, we find some congruence relations modulo certain powers of $I(\overline{X})$ and 2, respectively, for some elements of $\widetilde{K}(\overline{X})$ and $\mathrm{CH}(\overline{X})$ that can be written as products of $f(i)$'s and $g(i)$'s, and of $e(i)$'s, respectively. We start with powers of a single factor $f(i)$ or $e(i)$.

Lemma 3.1. — *Let $i \in I_0$ and $j \in \mathbb{N}$ be integers. Then,*

$$\begin{aligned} f(i)^j &\equiv \sum a(J) f(J) \mod I(\overline{X})^{v(j!)+1} \quad \text{and} \\ e(i)^j &\equiv 2^{v(j!)} \sum e(J) \mod 2^{v(j!)+1}, \end{aligned} \tag{3.1}$$

where $a(J) \in I(\overline{X})^{v(j!)}$ and the sums range over some multi-subsets $J \subset [1,n]$ such that $|J| = j$. In particular,

$$\mathbf{v}\big(f(i)^j\big),\ v\big(e(i)^j\big) \geqslant v(j!).$$

Moreover, if $j \geqslant 2$, then the multisets J above satisfy $J \cap [i+1,n] \neq \emptyset$.

Furthermore, if $i \in I_2$ and $j \geqslant 2$, then the same relations (3.1) *hold, where the sum ranges over some multi-subsets J with $|J| = j$, $J \cap [i+1, n] \neq \emptyset$, and such that either $J \subset I_1 \cup I_2$ or $J \cap \bar{I}_3 \neq \emptyset$.*

Proof. — Instead of proving the lemma as it is stated, claiming simply that $a(J) \in I(\overline{X})^{v(j!)}$, let us prove a stronger statement: $a(J)$ is a sum of terms of the form

$$2^q t^{v(j!)-q} \text{ for some } q \in [0, v(j!)]. \tag{3.2}$$

If $j = 1$, then the statement is trivial. For the case of arbitrary $j \geqslant 2$ we show the first equation in (3.1). Let $i \in I_0$. By the binary expansion of j, it suffices to prove the statement for any integer j that is a power of 2. We prove by induction on $j \geqslant 2$. Assume that $j = 2$. Then, as $\mathbf{e}(2i) = \mathbf{e}(2i+1) = 0$ for any $i \in I_0$, we have $f(2i) = f(2i+1) = 0$, thus the statement follows by (2.14). Assume that the statement holds for j. Then, modulo $I(\overline{X})^{2v(j!)+2}$ we have

$$\begin{aligned} f(i)^{2j} &\equiv \left(\sum_J a(J)f(J)\right)^2 \\ &= \sum_J a(J)^2 f(J)^2 + \sum_{J \neq J'} 2a(J)a(J')f(J+J'). \end{aligned} \tag{3.3}$$

Let $k \in J \cap [i+1, n]$ and $J^c = J - \{k\}$. Then, the case $j = 2$ implies that

$$f(J)^2 = f(k)^2 f(J^c + J^c) = \sum_L b(L)f(L)f(J^c + J^c) = \sum_L b(L)f(L + J^c + J^c),$$

where L denotes a multi-subset such that $|L| = 2$ and $L \cap [k+1, n] \neq \emptyset$, and $b(L) = 2$ or t. Since $2v(j!) + 1 = v\big((2j)!\big)$, each summand in (3.3) satisfies the statement. The same proof works in the case $i \in I_2$.

Furthermore, since $a(J)$ is a sum of terms of the form (3.2), we have $\psi(a(J)) \equiv 2^{v(j!)} \mod 2^{v(j!)+1}$ or $\psi(a(J)) \equiv 0 \mod 2^{v(j!)+1}$, where ψ denotes the morphism in (2.11), so the second equation in (3.1) follows. □

As a corollary of this lemma, we can observe the behavior of powers of $f(i)g(n-i)$ after the multiplication by $f(\bar{I}_3)$.

Corollary 3.2. — *Let $j \geqslant 2$ be an integer. Then, modulo $I(\overline{X})^{v(j!)+j+1}$ we have*

$$f(i)^j \cdot g(n-i)^j \cdot f(\bar{I}_3) \equiv \begin{cases} 0 & \text{if } i \in I_1, \\ \sum a(J)f(J)f(\bar{I}_3) & \text{if } i \in I_2 \end{cases}$$

for some $a(J) \in I(\overline{X})^{v(j!)+j}$, where the sum ranges over some multi-subsets $J \subset I_1 \cup I_2$ with $|J| = j + 1$.

Proof. — For $j \geqslant 2$, the binomial expansion of $g(n-i)^j$ tells us that each summand (modulo $I(\overline{X})^{j+1}$) is divisible by $f(n-i)^2$ or $f(n-i+1)^2$ and moreover, it can be written in the form $bf(n-i)^2$ or $bf(n-i+1)^2$ for some $b \in I(\overline{X})^j$. It follows by (2.15) that

$$f(n-i)^2 \equiv f(2n-2i),\ f(n-i+1)^2 \equiv f(2n-2i+2) \mod I(\overline{X})$$

for any $i \in I_1 \cup I_2$.

Assume that $i \in I_1$. Then, $2n-2i$, $2n-2i+2 \in \bar{I}_3$, thus we get

$$bf(n-i)^2 \cdot f(\bar{I}_3) \equiv bf(n-i+1)^2 \cdot f(\bar{I}_3) \equiv 0 \mod I(\overline{X})^{j+1}.$$

Hence, by Lemma 3.1 each summand of $f(i)^j g(n-i)^j f(\bar{I}_3)$ is contained in $I(\overline{X})^{v(j!)+j+1}$.

Now we assume that $i \in I_2$. Then $2n-2i \in I_1 \cup I_2$ and $2n-2i+2 \in I_1 \cup I_2 \cup \{\frac{6n}{8}\}$. If $2n-2i+2 = \frac{6n}{8}$, then again, by Lemma 3.1, the summands of $f(i)^j g(n-i)^j f(\bar{I}_3)$ divisible by $f(n-i+1)^2$ are contained in $I(\overline{X})^{v(j!)+j+1}$.

Consider a summand of $g(n-i)^j$ of the form $bf(n-i)^2$ or $bf(n-i+1)^2$ with $2n-2i \in I_1 \cup I_2$ or $2n-2i+2 \in I_1 \cup I_2$, respectively. Let us still rewrite it (modulo $I(\overline{X})^{j+1}$) as $bf(2n-2i)$ or $bf(2n-2i+2)$. By Lemma 3.1, we get

$$f(i)^j \equiv \sum a(J')f(J') \mod I(\overline{X})^{v(j!)+1}$$

for some $a(J') \in I(\overline{X})^{v(j!)}$, where the sum ranges over some multi-subsets J' with $|J'| = j$ such that either $J' \subset I_1 \cup I_2$ or $J' \cap \bar{I}_3 \neq \emptyset$. Set $J = J' + \{2n-2i\}$ or $J' + \{2n-2i+2\}$ and $a(J) = b \cdot a(J')$. Then, as $f(J') \cdot f(\bar{I}_3) \equiv 0 \mod I(\overline{X})$ for any J with $J \cap I_3 \neq \emptyset$, the statement follows. ☐

Let us use these results to express powers of $f(1)^n$ in terms of powers of $f(i)$ and $g(i)$, and powers of $e(1)^n$ in terms of powers of $e(i)$ with different values of i.

Lemma 3.3. — *For any $j \geqslant 2$, we have $f(1)^{nj} \in I(\overline{X})^{j+v(j!)-1}$ and*

$$f(1)^{nj} \equiv -j \cdot \left(\sum_{i \in I_0} f(i)g(n-i)\right)^{j-1} \cdot f(n) \mod I(\overline{X})^{j+v(j!)}. \tag{3.4}$$

Also, we have $e(1)^{nj} \equiv 0 \mod 2^{j+v(j!)-1}$ and

$$e(1)^{nj} \equiv -j \cdot 2^{j-1} \left(\sum_{i \in I_0} e(i)e(n-i)\right)^{j-1} \cdot e(n) \mod 2^{j+v(j!)}.$$

In particular,

$$\mathbf{v}\left(f(1)^{\frac{n^2}{4}}\right),\ v\left(e(1)^{\frac{n^2}{4}}\right) \geqslant \frac{n}{2} - 2$$

and

$$\mathbf{v}\left(f(1)^{\frac{n^2}{4}-n}\right),\ v\left(e(1)^{\frac{n^2}{4}-n}\right) \geqslant \frac{n}{2} - 1 - v(n).$$

Proof. — Let $h = \sum_{i \in I_0} f(i)g(n-i)$. For any $k \geqslant 1$, consider the multinomial expansion $h^k = \sum C(k_{\frac{n}{2}+1}, \ldots, k_{n-1})$, where the sum runs over $(k_{\frac{n}{2}+1}, \ldots, k_{n-1}) \in (\mathbb{N} \cup \{0\})^{\frac{n}{2}-1}$ with $\sum_{i \in I_0} k_i = k$ and

$$C\left(k_{\frac{n}{2}+1}, \ldots, k_{n-1}\right) = \binom{k}{k_{\frac{n}{2}+1}, \ldots, k_{n-1}} \prod_{i \in I_0} f(i)^{k_i} g(n-i)^{k_i}. \tag{3.5}$$

Then, by Lemma 3.1

$$v\left(\binom{k}{k_{\frac{n}{2}+1}, \ldots, k_{n-1}}\right) + \sum_{i \in I_0} \mathbf{v}\left(f(i)^{k_i}\right) \geqslant v(k!), \tag{3.6}$$

$$\text{thus } \mathbf{v}(h^k) \geqslant v(k!) + k.$$

By (2.15), $f(1)^{\frac{n}{2}} \equiv f(\frac{n}{2}) \mod I(\overline{X})$ and by (2.14), $f(\frac{n}{2})^2 \equiv h - f(n) \mod I(\overline{X})^2$, thus

$$f(1)^n \equiv h - f(n) \mod I(\overline{X})^2.$$

Write $f(1)^n = a + h - f(n)$ for some $a \in I(\overline{X})^2$. As $f(n)^2 = 0$, we have

$$f(1)^{nj} = \sum_{k=0}^{j} \binom{j}{j-k, k} a^{j-k} h^k - f(n) \sum_{k=0}^{j-1} \binom{j}{j-k-1, 1, k} a^{j-k-1} h^k. \tag{3.7}$$

Since $j - k \geqslant v((j-k)!)$ for $j - k \geqslant 0$, it follows from (3.6) that each summand of the first sum in (3.7) is contained in $I(\overline{X})^{j+v(j!)}$. Similarly, as $j - k - 2 \geqslant v((j-k-1)!)$ for $k < j - 1$, each summand of the second sum in (3.7) is contained in $I(\overline{X})^{j+v(j!)}$ except for the last term, which completes the proof of the equation (3.4).

After we get (3.4), it follows from (3.6) with $k = j - 1$ that

$$f(1)^{nj} \in I(\overline{X})^{v(j)+v((j-1)!)+j-1} = I(\overline{X})^{j+v(j!)-1}.$$

The statements for $\mathrm{CH}(\overline{X})$ are obtained from the statements for $\widetilde{K}(\overline{X})$ by applying ψ in (2.11). The last statement immediately follows from

$$\begin{aligned} v\left(\left(\frac{n}{4}\right)!\right) &= \frac{n}{4} - 1, \\ v\left(\left(\frac{n}{4} - 1\right)!\right) &= \frac{n}{4} - 1 - v\left(\frac{n}{4}\right) \\ &= \frac{n}{4} + 1 - v(n). \quad \square \end{aligned} \tag{3.8}$$

Now let us obtain an expression for the product of certain high powers of $f(1)$ and $f(I_3)f(\frac{n}{2})$ in $\widetilde{K}(\overline{X})$, and a similar result in $\mathrm{CH}(\overline{X})$. Roughly speaking, what we are going to do is to express (modulo powers of $I(\overline{X})$ and of 2) the powers of sums in right-hand sides of the formulas in Lemma 3.3 as square-free products of $f(i)$'s and $g(i)$'s and of $e(i)$'s, respectively.

Proposition 3.4. — *For any $n \geqslant 8$, we have $f(1)^{\frac{n^2}{4}-n} \cdot f(I_3) \cdot f(\frac{n}{2}) \in I(\overline{X})^{\frac{n}{2}-v(n)-1}$ and*

$$f(1)^{\frac{n^2}{4}-n} \cdot f(I_3) \cdot f\left(\frac{n}{2}\right) \equiv -\left(\frac{n}{4}-1\right)! \cdot f\left(\left[\frac{n}{2}, n\right]\right) \cdot g\left(\left[\frac{n}{4}+2, \frac{n}{2}-1\right]\right) \mod I(\overline{X})^{\frac{n}{2}-v(n)}.$$

Also, we have $e(1)^{\frac{n^2}{4}-n} \cdot e(I_3) \cdot e(\frac{n}{2}) \equiv 0 \mod 2^{\frac{n}{2}-v(n)-1}$ and

$$e(1)^{\frac{n^2}{4}-n} \cdot e(I_3) \cdot e\left(\frac{n}{2}\right) \equiv -\left(\frac{n}{4}-1\right)! \cdot 2^{\frac{n}{4}-2} e\left(\left[\frac{n}{4}+2, n\right]\right) \mod 2^{\frac{n}{2}-v(n)}.$$

Proof. — Let $k = \frac{n}{4} - 2$. Consider a summand $C(k_{\frac{n}{2}+1}, \dots, k_{n-1})$ of the multinomial expansion of h^k as in (3.5). We first show that

$$C(k_{\frac{n}{2}+1}, \dots, k_{n-1}) \cdot f(\bar{I}_3) \equiv 0 \mod I(\overline{X})^{v(k!)+k+1} \tag{3.9}$$

for all $k_{\frac{n}{2}+1}, \dots, k_{n-1}$ except for

$$k_i = \begin{cases} 1 & \text{if } i \in I_1 \cup I_2, \\ 0 & \text{if } i \in I_3. \end{cases}$$

If $k_l > 0$ for some $l \in I_3$, then by Lemma 3.1

$$\begin{aligned} f(l)^{k_l} f(\bar{I}_3) &= f(l)^{k_l+1} f\left(\bar{I}_3 - \{l\}\right) \\ &\equiv \sum_J a(J) f(J) f\left(\bar{I}_3 - \{l\}\right) \mod I(\overline{X})^{v((k_l+1)!)+1}, \end{aligned}$$

where $|J| = k_l + 1$, $J \cap [l+1, n] \neq \emptyset$, and $a(J) \in I(\overline{X})^{v((k_l+1)!)}$. Since $f(J)f(\bar{I}_3 - \{l\}) \equiv 0 \mod I(\overline{X})$ by (2.15), we get

$$f(l)^{k_l} f(\bar{I}_3) \equiv 0 \mod I(\overline{X})^{v(k_l!)+1},$$

thus again by Lemma 3.1

$$\begin{aligned} v\left(\binom{k}{k_{\frac{n}{2}+1}, \dots, k_{n-1}}\right) &+ \mathbf{v}\left(f(l)^{k_l} f(\bar{I}_3)\right) \\ &+ \sum_{i \in I_0 \setminus \{l\}} \mathbf{v}\left(f(i)^{k_i}\right) \geqslant v(k!) + 1. \end{aligned} \tag{3.10}$$

Hence,

$$v\left(\binom{k}{k_{\frac{n}{2}+1},\ldots,k_{n-1}}\right)+\mathbf{v}\left(f(l)^{k_l}f(\bar{I}_3)g(l)^{k_l}\right) + \sum_{i\,\in\,I_0\setminus\{l\}}\mathbf{v}\left(f(i)^{k_i}g(i)^{k_i}\right)\geqslant v(k!)+k+1, \tag{3.11}$$

and the congruence (3.9) holds. Therefore, we may assume that $k_i=0$ for all $i\in I_3$ and $\sum_{i\,\in\,I_1\cup I_2}k_i=k$.

Similarly, if $k_l\geqslant 2$ for some $l\in I_1$, then by Corollary 3.2 and Lemma 3.1, we get (3.11), thus the congruence (3.9) follows.

Now, if $k_l\geqslant 2$ for some $l\in I_2$, then again by Lemma 3.1 and Corollary 3.2

$$\left(\prod_{i\,\in\,I_1\,\cup\,I_2}f(i)^{k_i}\right)\cdot g(n-l)^{k_l}\cdot f(\bar{I}_3) \equiv f(\bar{I}_3)\prod_{i\,\in\,I_1\,\cup\,I_2}\sum_{J_i}a(J_i)f(J_i)\mod I(\overline{X})^{s+1}, \tag{3.12}$$

where $s=\sum_{i\in I_1\cup I_2}v(k_i!)+k_l$, $J_i\subset I_1\cup I_2$ or $J_i\cap\bar{I}_3\neq\emptyset$,

$$|J_i|=\begin{cases}k_i & \text{if } i\in(I_1\cup I_2)\setminus\{l\},\\ k_l+1 & \text{if } i=l,\end{cases}$$

and

$$a(J_i)\in\begin{cases}I(\overline{X})^{v(k_i!)} & \text{if } i\in(I_1\cup I_2)\setminus\{l\},\\ I(\overline{X})^{v(k_l!)+k_l} & \text{if } i=l.\end{cases}$$

Since for each k-tuple $(J_i)_{i\,\in\,I_i\,\cup\,I_2}$

$$\sum_{i\,\in\,I_1\,\cup\,I_2}|J_i|=1+\sum_{i\,\in\,I_1\,\cup\,I_2}k_i>|I_1\cup I_2|=k,$$

by (2.15) we obtain

$$f(\bar{I}_3)\prod_{i\,\in\,I_1\,\cup\,I_2}f(J_i)\in I(\overline{X}).$$

Thus, the product of the sums on the right-hand side of (3.12) belongs to $I(\overline{X})^{s+1}$. As

$$\prod_{i\,\in\,(I_1\,\cup\,I_2)\setminus\{l\}}g(n-i)^{k_i}\in I(\overline{X})^{k-k_l},$$

the congruence (3.9) follows. Therefore,

$$h^k\cdot f(\bar{I}_3)\cdot f\left(\frac{n}{2}\right)\equiv k!\cdot f\left(\left[\frac{n}{2},n\right]\right)\cdot g\left(\left[\frac{n}{4}+2,\frac{n}{2}-1\right]\right)\mod I(\overline{X})^{v(k!)+k+1}.$$

Hence, the second equation in the statement follows from Lemma 3.3 and (3.8) with the equality $v((\frac{n}{4}-2)!) = v((\frac{n}{4}-1)!)$. Since

$$\left|\left[\frac{n}{4}+2, \frac{n}{2}-1\right]\right| = \frac{n}{4}-2 \quad \text{and} \quad g\left(\left[\frac{n}{4}+2, \frac{n}{2}-1\right]\right) \in I(\overline{X})^{\frac{n}{4}-2},$$

the first equation in the statement follows from the second equation.

The remaining equations in the statement follow from the first and second equations by applying ψ in (2.11) together with (2.17). □

As a corollary, we also obtain an expression for the product of $f(1)^{\frac{n^2}{4}-n}$ and $g(I_3)f(\frac{n}{2})$.

Corollary 3.5. — *For any $n \geqslant 8$, we obtain*

$$f(1)^{\frac{n^2}{4}-n} \cdot g(I_3) \in I(\overline{X})^{\frac{3n}{4}-v(n)}$$

and

$$\begin{aligned} &f(1)^{\frac{n^2}{4}-n} \cdot g(I_3) \cdot f\left(\frac{n}{2}\right) \\ &\equiv 2^{\frac{n}{2}-v(n)+2} \cdot f\left(\left[\frac{n}{2}, n\right]\right) \cdot g\left(\left[\frac{n}{4}+2, \frac{n}{2}-1\right]\right) \mod I(\overline{X})^{\frac{3n}{4}-v(n)+1}. \end{aligned}$$

Proof. — The first statement follows immediately from the last statement of Lemma 3.3. For the second statement, we show that

$$(3.13) \quad f(1)^{\frac{n^2}{4}-n} g(I_3) \equiv f(1)^{\frac{n^2}{4}-n} \cdot 2^{\frac{n}{4}+1} \cdot f(I_3) \mod I(\overline{X})^{\frac{3n}{4}-v(n)+1}.$$

Then, the second statement immediately follows by Proposition 3.4 and (3.8).

Write the left-hand side of the equation (3.13) as

$$f(1)^{\frac{n^2}{4}-n} g(I_3) = f(1)^{\frac{n^2}{4}-n} \cdot g(I_3 \backslash \{n-1\}) \cdot \big(2f(n-1) - tf(n)\big).$$

Since $f(n) \equiv f(1)^n \mod I(\overline{X})$ by (2.15), it follows by Lemma 3.3 that

$$f(1)^{\frac{n^2}{4}-n} f(n) \equiv 0 \mod I(\overline{X})^{\frac{n}{2}-v(n)}.$$

As $|I_3| = \frac{n}{4}+1$, we have $t \cdot g(I_3 \backslash \{n-1\}) \in I(\overline{X})^{\frac{n}{4}+1}$, thus

$$f(1)^{\frac{n^2}{4}-n} \cdot g(I_3) \equiv 2f(1)^{\frac{n^2}{4}-n} \cdot g(I_3 \backslash \{n-1\}) f(n-1) \mod I(\overline{X})^{\frac{3n}{4}-v(n)+1}.$$

Let us expand the term $g(I_3 \backslash \{n-1\})f(n-1)$. Then, each summand has a factor of the form $f(J)$ for some multi-subset $J \subset I_3$ with $|J| = |I_3|$. Since $f(j)^2 \equiv 0 \mod I(\overline{X})$ for any $j \in I_3$,

$$g(I_3 \backslash \{n-1\}) f(n-1) \equiv 2^{\frac{n}{4}} \cdot f(I_3) \mod I(\overline{X})^{\frac{n}{4}+1},$$

whence the equation (3.13) follows. □

We shall need the following lemma in the proofs of Propositions 3.7 and 3.10.

Lemma 3.6. — *Let J be a multi-subset of $[1,n]$ satisfying the following conditions:*

(∗) *There exists a number $k \in J$ with multiplicity at least 2 and every number j with $k < j \leqslant n$ is contained in J with multiplicity 1.*

Then, we have

$$f(J) \equiv 0 \mod I(\overline{X}) \ \ \text{and} \ \ e(J) \equiv 0 \mod 2. \tag{3.14}$$

Proof. — Let $k \in J$ denote a number with multiplicity at least 2 as in (∗). We prove by decreasing induction on k. If $2k > n$, then the first equation in (3.14) follows directly from (2.15) since $f(2k) = 0$. Otherwise, by (2.15) again, we have $f(J) \equiv f(J') \mod I(\overline{X})$, where $J' = J + \{2k\} - \{k,k\}$. Since $2k \in J'$ has multiplicity 2 and every j with $2k < j \leqslant n$ is contained in J', it follows by the induction that $f(J') \equiv 0 \mod I(\overline{X})$, whence the first equation follows. The second equation in (3.14) follows from the first one by applying ψ in (2.11) together with (2.17). □

We denote

$$I_4 = \left[6, \frac{n}{4}+1\right] \setminus \left\{2^i \,\middle|\, 3 \leqslant i \leqslant v(n)-2\right\}.$$

Now we will prove the main result of this section, which plays a key role in the proof of 2-divisibility of $\varphi(x)$ (see Proposition 4.1). For $n = 8$, an analogue of the following proposition is proved inside the proof of [10, Proposition 4.4] (see Remark 3.8 below).

Proposition 3.7. — *Let $n \geqslant 16$. Then, the following equations hold modulo $I(\overline{X})^{v(\operatorname{ind} X)+1}$.*

(a) $f(1)^{\frac{n^2}{4}-1} \cdot g(I_3 \cup I_4 \cup \{2,3\}) \equiv 0,$
(b) $f(1)^{\frac{n^2}{4}-2} \cdot g(I_3 \cup I_4 \cup \{2,4\}) \equiv 2^{v(\operatorname{ind} X)-2} t^2 \cdot f([2,n]).$

Proof. — By (2.15), $f(1)^m \equiv f(m) \mod I(\overline{X})$ for any 2-power integer m, thus

$$\begin{gathered} f(1)^{n-2} \equiv \prod_{k=1}^{v(n)-1} f(2^k) \\ \text{and} \ \ f(2^i) f\left(\left[\frac{n}{2}, n\right]\right) \prod_{k=1}^{v(n)-2} f(2^k) \equiv 0 \mod I(\overline{X}) \end{gathered} \tag{3.15}$$

for any $1 \leqslant i \leqslant v(n)$. Since

$$\frac{3n}{4} - v(n) + 1 + |I_4| = n - 2v(n) + 1, \tag{3.16}$$

by Corollary 3.5 and the first equation in (3.15), the following equation holds modulo $I(\overline{X})^{n-2v(n)+1}$:

$$f(1)^{\frac{n^2}{4}-2} g(I_3 \cup I_4) \\ \equiv 2^{\frac{n}{2}-v(n)+2} \cdot g\left(\left[\frac{n}{4}+2, \frac{n}{2}-1\right] \cup I_4\right) f\left(\left[\frac{n}{2}, n\right]\right) \prod_{k=1}^{v(n)-2} f(2^k).$$

Let A denote the right hand side of the preceding equation without the factor $2^{\frac{n}{2}-v(n)+2}$. Thus, to prove (*a*) and (*b*), by (2.1) it suffices to show the following congruences modulo $I(\overline{X})^{\frac{n}{2}-v(n)+1}$:

$$\big(2f(2) - tf(3)\big)\big(2f(3) - tf(4)\big)A \equiv 0,$$
$$\big(2f(2) - tf(3)\big)\big(2f(4) - tf(5)\big)A \equiv 2^{\frac{n}{2}-v(n)-2} t^2 f([2, n]),$$

respectively.

Since $|[\frac{n}{4}+2, \frac{n}{2}-1]| = \frac{n}{4} - 2$ and $|I_4| = \frac{n}{4} - v(n)$, we have

$$g\left(\left[\frac{n}{4}+2, \frac{n}{2}-1\right] \cup I_4\right) \in I(\overline{X})^{\frac{n}{2}-v(n)-2}, \tag{3.17}$$

thus by the second equation in (3.15) and $f(3)^2 \equiv f(6) \mod I(\overline{X})$ it is enough to show that the following hold modulo $I(\overline{X})^{\frac{n}{2}-v(n)-1}$:

$$f(6) \cdot A \equiv 0 \quad \text{and} \quad f(\{3, 5\}) \cdot A \equiv 2^{\frac{n}{2}-v(n)-2} f([2, n]),$$

respectively. By (3.17), the left-hand sides of these congruences can be expanded as

$$\sum_J a(J) f(J) \;\; \text{and} \;\; 2^{\frac{n}{2}-v(n)-2} f([2, n]) + \sum_{L \neq [2,n]} b(L) f(L),$$

respectively, where $a(J), b(L) \in I(\overline{X})^{\frac{n}{2}-v(n)-2}$, J denotes a multi-subset of $\{2, 4\} \cup [6, n]$, and L denotes a multi-subset of $[2, n]$. Since each of J and L satisfies the condition $(*)$ in Lemma 3.6, the statement follows. □

Remark 3.8. — For $n = 8$, the congruences (a) and (b) in Proposition 3.7 still hold if $I_3 = [5, 7]$ is replaced by $I_3' = [6, 7]$, i.e.,

$$f(1)^{15} \cdot g([6, 7] \cup \{2, 3\}) \equiv 0 \mod I(\overline{X})^5, \tag{3.18}$$

$$f(1)^{14} \cdot g([6, 7] \cup \{2, 4\}) \equiv 2^2 t^2 f([2, 8]) \mod I(\overline{X})^5. \tag{3.19}$$

Indeed, since $f(1)^8 \equiv f(8)$ and $f(8)^2 \equiv f(7)^2 \equiv 0 \mod I(\overline{X})$,

$$f(1)^8 g([6, 7]) \equiv 2^2 f([6, 8]) \mod I(\overline{X})^3.$$

Hence, the congruences (3.18) and (3.19) follow from

$$f(2)^2 f(4) f(8) \equiv f(4)^2 f(8) \equiv f(8)^2 \equiv f(3)^2 f(6) \equiv f(6)^2 \equiv 0 \mod I(\overline{X}).$$

Below we provide analogues of Corollary 3.5 and Proposition 3.7 (a) in $\mathrm{CH}(\overline{X})$. In the analogue (Proposition 3.10) of Proposition 3.7 (a), the element of $\mathrm{CH}(\overline{X})$ is not divisible by the same power of 2 as the power of $I(\overline{X})$ in Proposition 3.7 (a) itself anymore. This difference will enable us to prove that the element x in (4.3) is not divisible by 2 in $\mathrm{CH}(X)$. For each $i \in [1, n]$, we denote

$$\widehat{S}^j(i) = \binom{i-1}{j} c(i+j), \quad \widehat{S}(i) = \sum_{j=0}^{i-1} \binom{i-1}{j} c(i+j). \tag{3.20}$$

In other words, $\widehat{S}(i)$ is an integral representative of the sub-linear combination of $S(\bar{c}(i))$ that consists of multiples of single $c(j)$'s only, not of their products. For a subset $L \subset [1, n]$, denote $\widehat{S}(L) = \prod_{l \in L} \widehat{S}(l)$.

Corollary 3.9. — *For any $n \geqslant 8$, we have $e(1)^{\frac{n^2}{4}-n} \cdot \mathrm{res}(\widehat{S}(I_3)) \equiv 0$ $\mod 2^{\frac{3n}{4}-v(n)}$ and*

$$e(1)^{\frac{n^2}{4}-n} \cdot \mathrm{res}\left(\widehat{S}(I_3)\right) \cdot e\left(\frac{n}{2}\right) \equiv 2^{\frac{3n}{4}-v(n)} \cdot e\left(\left[\frac{n}{4}+2, n\right]\right) \mod 2^{\frac{3n}{4}-v(n)+1}.$$

Proof. — The proof is similar to the proof of Corollary 3.5. Again, the first statement follows immediately from the last statement of Lemma 3.3. For the second statement, we show that

$$e(1)^{\frac{n^2}{4}-n} \,\mathrm{res}\left(\widehat{S}(I_3)\right) \equiv e(1)^{\frac{n^2}{4}-n} \cdot 2^{\frac{n}{4}+1} \cdot e(I_3) \mod 2^{\frac{3n}{4}-v(n)+1}. \tag{3.21}$$

Then, the second statement immediately follows by Proposition 3.4 and (3.8).

The term $\widehat{S}(I_3)$ is expanded as

$$\widehat{S}(I_3) = c(I_3) + \sum_L a(L) c(L),$$

where $a(L) \in \mathbb{N}$ and the sum ranges over some multi-subsets $L \subset \overline{I}_3$ such that L contains either n or a multiple element. Then, by (2.19) we get

$$\mathrm{res}\left(\widehat{S}(I_3)\right) = 2^{\frac{n}{4}+1} e(I_3) + 2^{\frac{n}{4}+1} \sum_L a(L) e(L). \tag{3.22}$$

If $n \in L$, then since $e(n) \equiv e(1)^n \mod 2$ by (2.18), we get by Lemma 3.3

$$\begin{aligned} e(1)^{\frac{n^2}{4}-n} \cdot 2^{\frac{n}{4}+1} e(L) &\equiv e(1)^{\frac{n^2}{4}} \cdot 2^{\frac{n}{4}+1} e(L - \{n\}) \\ &\equiv 0 \mod 2^{\frac{3n}{4}-v(n)+1}. \end{aligned} \tag{3.23}$$

If L has a multiple element $i \in \bar{I}_3$, then $e(i)^2 \equiv 0 \mod 2$ by (2.18), thus by Lemma 3.3 again, we get

$$(3.24)\quad e(1)^{\frac{n^2}{4}-n} \cdot 2^{\frac{n}{4}+1} e(L) = e(1)^{\frac{n^2}{4}-n} \cdot 2^{\frac{n}{4}+1} e(L - \{i,i\}) e(i)^2 \equiv 0 \mod 2^{\frac{3n}{4}-v(n)+1}.$$

Hence, the equation in (3.21) immediately follows from (3.22), (3.23), and (3.24). □

For $n = 8$, an analogue of the following proposition is proved inside the proof of [10, Proposition 3.3] (see Remark 3.11 below).

Proposition 3.10. — *Let $n \geqslant 16$. Then, we have*

$$e(1)^{\frac{n^2}{4}-1} \cdot \operatorname{res}\left(\widehat{S}(I_3 \cup I_4 \cup \{2,3\})\right) \equiv \operatorname{ind} X \cdot e([1,n]) \mod 2 \operatorname{ind} X.$$

Proof. — The proof is similar to the proof of Proposition 3.7. By (2.18), $e(1)^m \equiv e(m) \mod 2$ for any 2-power integer m, thus

$$(3.25)\quad e(1)^{n-1} \equiv \prod_{k=0}^{v(n)-1} e(2^k) \quad \text{and} \quad e(2^i) e(1) e(2) e(4) e([6,n]) \equiv 0 \mod 2$$

for any $0 \leqslant i \leqslant v(n)$. By Corollary 3.9, (3.16), and the first equation in (3.25) we get a congruence modulo $2^{n-2v(n)+1}$:

$$(3.26)\quad e(1)^{\frac{n^2}{4}-1} \operatorname{res}\left(\widehat{S}(I_3 \cup I_4)\right) \equiv 2^{\frac{3n}{4}-v(n)} \cdot \operatorname{res}\left(\widehat{S}(I_4)\right) \cdot e\left(\left[\frac{n}{4}+2, n\right]\right) \prod_{k=0}^{v(n)-2} e(2^k).$$

The term $\widehat{S}(I_4)$ is expanded as

$$\widehat{S}(I_4) = c(I_4) + \sum_L a(L) c(L),$$

where $a(L) \in \mathbb{N}$ and the sum ranges over some multi-subsets $L \subset [1,n]$ such that the multiset $L + [\frac{n}{4}+2, n] + \{2^i \,|\, 3 \leqslant i \leqslant v(n)-2\}$ satisfies the condition $(*)$ in Lemma 3.6. Since $|I_4| = \frac{n}{4} - v(n)$, by (2.19) we get

$$(3.27)\qquad \operatorname{res}(\widehat{S}(I_4)) = 2^{\frac{n}{4}-v(n)} e(I_4) + 2^{\frac{n}{4}-v(n)} \sum_L a(L) e(L).$$

By Lemma 3.6, $e(L) e([\frac{n}{4}+2, n]) \prod_{k=3}^{v(n)-2} e(2^k)$ is divisible by 2, thus by (3.26) and (3.27) we have a congruence modulo $2^{n-2v(n)+1}$

$$e(1)^{\frac{n^2}{4}-1} \operatorname{res}\left(\widehat{S}(I_3 \cup I_4)\right) \equiv 2^{n-2v(n)} e(1) e(2) e(4) e([6,n]).$$

Now let us multiply both sides of the last congruence by

$$\operatorname{res}\left(\widehat{S}(\{2,3\})\right) = 2^2(e(2)+e(3))(e(3)+2e(4)+e(5)). \tag{3.28}$$

Since $2^{n-2v(n)+3} = 2\operatorname{ind} X$, by the second equation in (3.25) it is enough to show that the following holds modulo 2

$$e(3)^2 \cdot e(1)e(2)e(4)e([6,n]) \equiv 0.$$

Since $e(3)^2 \equiv e(6) \mod 2$ by (2.18), and the multiset $\{6\}+[6,n]$ satisfies the condition $(*)$ in Lemma 3.6, the term $e(\{6\}+[6,n])$ is divisible by 2, which completes the proof. □

Remark 3.11. — Let $n=8$ and $I_3' = [6,7]$. Then, the statement of Proposition 3.10 becomes

$$e(1)^{15} \cdot \operatorname{res}\left(\widehat{S}(I_3' \cup \{2,3\})\right) \equiv 2^4 \cdot e([1,8]) \mod 2^5. \tag{3.29}$$

Since $e(1)^8 \equiv e(8), \quad e(8)^2 \equiv e(7)^2 \equiv 0 \mod 2$, and

$$\widehat{S}(6) = c(6) + 5c(7) + 10c(8), \quad \widehat{S}(7) = c(7) + 6c(8),$$

we have

$$e(1)^8 \operatorname{res}(\widehat{S}(I_3')) \equiv 2^2 e([6,8]) \mod 2^3.$$

Hence, the congruence (3.29) follows from (3.28) and

$$e(2)^2 e(4)e(8) \equiv e(4)^2 e(8) \equiv e(8)^2 \equiv e(3)^2 e(6) \equiv e(6)^2 \equiv 0 \mod 2.$$

4. Proof of Theorem 1.2

In this section, we set

$$J = \begin{cases} [2,3] \cup I_3 \cup I_4 & \text{if } n \geqslant 16, \\ [2,3] \cup I_3' \cup I_4 = \{2,3,6,7\} & \text{if } n = 8. \end{cases} \tag{4.1}$$

Then, a direct computation shows that

$$|J| = \frac{n}{2} - v(n) + 3. \tag{4.2}$$

Consider the following element

$$x = e^{\frac{n^2}{4}-1} \cdot \prod_{j \in J} c(j) \in \mathrm{CH}(X). \tag{4.3}$$

Since $\dim X = \frac{n^2+n}{2}$ and $(\frac{n^2}{4}-1) + \sum_{j \in J} j = \dim X - 3$, we have $x \in \mathrm{CH}_3(X)$ and $\varphi(x) \in K(X)^{(\dim X - 3)}$. We first prove the 2-divisibility of the image of x under the map in (1.1).

Proposition 4.1. — *Let $y = f(1)^{\frac{n^2}{4}-1} \cdot g(J)$, where J denotes the set in* (4.1)*. Then, $y \in I(X)$ and $\varphi(x)$ is divisible by 2 in $GK(X)$.*

Proof. — Let $z = f(1)^{\frac{n^2}{4}-2} \cdot g(J')$, where J' denotes the set obtained from J by replacing the element 3 with 4. Then, by (2.12) both y and z belong to $\widetilde{K}^{\dim X-3}(X)$. We first show that $y \in I(X)$. By Proposition 3.7 (a) for $n \geqslant 16$ and by (3.18) for $n = 8$, we can write

$$y = 2^{m+1} \cdot y_0 + 2^m t \cdot y_1 + 2^{m-1}t^2 \cdot y_2 + 2^{m-2}t^3 \cdot y_3,$$

where $m = v(\operatorname{ind} X)$ and $y_i \in \widetilde{K}^{\dim X-3+i}(\overline{X})$. By (2.10), it suffices to prove that

$$y' := 2^{m-1}t^2 \cdot y_2 + 2^{m-2}t^3 \cdot y_3 \in I(X).$$

We simply write $\mathbf{p}$ and $\mathbf{l}$ for the classes of $\prod_{i=1}^n \mathbf{e}(i)$ and $\prod_{i=2}^n \mathbf{e}(i)$ in $K(\overline{X})^{(\dim \overline{X})}$ and $K(\overline{X})^{(\dim \overline{X}-1)}$, respectively, as in Section 2.1. Since $2^{m-1}t^2 \cdot t\mathbf{p} = 2^{m-2}t^3 \cdot 2\mathbf{p}$, by (2.5), y' can be written as

$$y' = a\left(2^{m-1}t^2\right) \cdot \mathbf{l}u^{\dim X-1} + b\left(2^{m-2}t^3\right) \cdot \mathbf{p}u^{\dim X} \tag{4.4}$$

for some $a, b \in \mathbb{Z}$. On the other hand, by Proposition 3.7 (b) for $n \geqslant 16$ and by (3.19) for $n = 8$, we have

$$\begin{aligned} 2z &\equiv \left(2^{m-1}t^2 \cdot \mathbf{l}\right) u^{\dim X-1} \\ \text{and} \quad tf(1)z &\equiv \left(2^{m-2}t^3 \cdot \mathbf{p}\right) u^{\dim X} \quad \bmod I(\overline{X})^{m+2}, \end{aligned} \tag{4.5}$$

thus it follows by (4.4) that

$$y' - 2az - btf(1)z \in I(\overline{X})^{m+2} \cap \widetilde{K}^{\dim X-3}(X).$$

Hence, we can write

$$y' - 2az - btf(1)z = 2^{m+2} \cdot z_0 + 2^{m+1}t \cdot z_1 + 2^m t^2 \cdot z_2 + 2^{m-1}t^3 \cdot z_3,$$

where $z_i \in \widetilde{K}^{\dim X-3+i}(\overline{X})$, thus by (2.10), it suffices to prove that

$$y'' := 2^{m-1}t^3 \cdot z_3 \in I(X).$$

By (2.5), y'' can be written as

$$y'' = b'\left(2^{m-1}t^3\right) \cdot \mathbf{p}u^{\dim X} \tag{4.6}$$

for some $b' \in \mathbb{Z}$. Since

$$\left(2^{m-1}t^3 \cdot \mathbf{p}\right) u^{\dim X} \equiv 2tf(1)z \mod I(\overline{X})^{m+3},$$

we obtain

$$y'' - 2b'tf(1)z \in I(\overline{X})^{m+3} \cap \widetilde{K}^{\dim X-3}(X).$$

As every element in $I(\overline{X})^{m+3} \cap \widetilde{K}^{\dim X-3}(X)$ belongs to $I(X)$ by (2.10), we get $y'' \in I(X)$, and therefore $y \in I(X)$.

For the statement about x, note that by the second equation in (2.12), we have

$$f(1)^{\frac{n^2}{4}-1} \cdot \prod_{j \in J} \mathbf{c}(j) u^j \in I(X).$$

Also, by (2.16) and Lemma 2.2, we get

$$\varphi(x) = \xi \left(f(1)^{\frac{n^2}{4}-1} \cdot \prod_{j \in J} \mathbf{c}(j) u^j \right),$$

where ξ denotes the morphism in (2.7). Hence, by (2.8) $\varphi(x)$ is divisible by 2. □

Remark 4.2. — In the proof of Proposition 4.1, we have shown that $(2^{m-1}\mathbf{l})u^{\dim X-3}$ and $(2^{m-2}\mathbf{p})u^{\dim X-3}$ are contained in $I(X) + I(\overline{X})^{m+2}$. In fact, we could alternatively show that the following slightly stronger statement holds:

$$(4.7) \qquad \left(2^{m-1}\mathbf{l}\right) u^{\dim X-3},\ \left(2^{m-2}\mathbf{p}\right) u^{\dim X-3} \in I(X).$$

Since $z-(2^{m-2}\mathbf{l})u^{\dim X-3} \in I(\overline{X})^{m+1} \cap \widetilde{K}^{\dim X-3}(\overline{X})$ (Proposition 3.7 (b)), the same argument as in the beginning of the proof of Proposition 4.1 shows that

$$(4.8) \quad z - \left(2^{m-2}\mathbf{l}\right) u^{\dim X-3} \equiv \left(a2^{m-1}\mathbf{l} + b2^{m-2}\mathbf{p}\right) u^{\dim X-3} \mod I(X)$$

for some $a, b \in \mathbb{Z}$. Since $2z$, $\mathbf{e}(1)z \in I(X)$, multiplying the congruence in (4.8) by 2 and $\mathbf{e}(1)$, we have

$$\left(2^{m-1}\mathbf{l} + b2^{m-1}\mathbf{p}\right) u^{\dim X-3},\ \left(2^{m-2}\mathbf{p} + a2^{m-1}\mathbf{p}\right) u^{\dim X-3} \in I(X).$$

As $(2^{m-1}\mathbf{p})u^{\dim X-3} = \mathbf{e}(1)(2^{m-1}\mathbf{l} + b2^{m-1}\mathbf{p})u^{\dim X-3} \in I(X)$, the statement in (4.7) follows.

Recall from (2.20) that $Q(i,j)$ denotes a linear combination of $\bar{c}(k)\bar{c}(i+j-k)$ for $1 \leqslant k \leqslant i$. We write $\widehat{Q}(i,j)$ for the linear combination obtained from $Q(i,j)$ by replacing every term $\bar{c}(k)\bar{c}(i+j-k)$ with $c(k)c(i+j-k)$. Set

$$\widetilde{S}^j(i) = \widehat{S}^j(i) + \widehat{Q}(i,j), \quad \widetilde{S}(i) = \sum_{j \geqslant 0} \widetilde{S}^j(i),$$

where $\widehat{S}^j(i)$ denotes the term in (3.20), i.e., $\widetilde{S}(i)$ is an integral representative of $S(\bar{c}(i))$. For a subset $L \subset [1,n]$, denote $\widetilde{S}(L) = \prod_{l \in L} \widetilde{S}(l)$.

Before we prove that $x \in \mathrm{CH}(X)$ is not divisible by 2, we shall need the following lemma.

LEMMA 4.3. — *For any $n \geqslant 8$, we have*

$$e(1)^{\frac{n^2}{4}} \operatorname{res}\left(\widetilde{S}(J)\right) \equiv 0 \mod 2 \operatorname{ind} X$$

and

$$e(1)^{\frac{n^2}{4}-1} \operatorname{res}\left(\widetilde{S}(J)\right) \equiv e(1)^{\frac{n^2}{4}-1} \operatorname{res}\left(\widehat{S}(J)\right) \mod 2 \operatorname{ind} X.$$

Proof. — Note that $v(\operatorname{ind} X) = n - 2v(n) + 2$. By Lemma 3.3 and (4.2), we have

$$v\left(2^{|J|}e(1)^{\frac{n^2}{4}}\right) \geqslant \left(\frac{n}{2} - v(n) + 3\right) + \left(\frac{n}{2} - 2\right) > v(\operatorname{ind} X)$$

and

$$\begin{aligned} v\left(2^{|J|+1}e(1)^{\frac{n^2}{4}-1}\right) &\geqslant v\left(2^{|J|+1}e(1)^{\frac{n^2}{4}-n}\right) \\ &\geqslant \left(\frac{n}{2} - v(n) + 4\right) + \left(\frac{n}{2} - 1 - v(n)\right) > v(\operatorname{ind} X). \end{aligned}$$

Hence, the first statement follows from (2.19). For the second statement, note additionally that $\widetilde{S}(J) - \widehat{S}(J)$ is the sum of several products of the form $\prod_{1 \leqslant k \leqslant |J|} A_k$, where each A_k can be either $\widehat{S}(i)$, or $\widehat{Q}(i,j)$, and at least one factor $\widehat{Q}(i,j)$ is present. So, by (2.19), $\operatorname{res}(\prod_{1 \leqslant k \leqslant |J|} A_k)$ is divisible by $2^{|J|+1}$, and the second statement follows. □

Finally, let us prove the non-2-divisibility in $\mathrm{CH}(X)$.

PROPOSITION 4.4. — *For any $n \geqslant 8$, the element x in* (4.3) *is not divisible by* 2.

Proof. — Let $w = (e + e^2)^{\frac{n^2}{4}-1}\widetilde{S}(J)$. Then, the $(\dim X)^{\text{th}}$ degree homogeneous part of w is an integral representative of $S^3(\bar{x})$, i.e., the $(\dim X)^{\text{th}}$ degree homogeneous part of $\overline{w} \in \mathrm{Ch}(X)$ is equal to $S^3(\bar{x})$. We show that

$$\operatorname{res}(w) \equiv \operatorname{ind} X \cdot p \mod 2 \operatorname{ind} X, \tag{4.9}$$

where p denotes the class of a rational point as in Section 2.3. Since

$$w = \left(e^{\frac{n^2}{4}-1} + e^{\frac{n^2}{4}}\alpha(e)\right)\widetilde{S}(J),$$

where $\alpha(e)$ is a polynomial in e with integer coefficients, by Lemma 4.3, we have modulo $2 \operatorname{ind} X$:

$$\operatorname{res}(w) \equiv e(1)^{\frac{n^2}{4}-1} \operatorname{res}(\widetilde{S}(J)) \equiv e(1)^{\frac{n^2}{4}-1} \operatorname{res}(\widehat{S}(J)).$$

Hence, by Proposition 3.10 we obtain (4.9).

Let $\deg : \mathrm{CH}(X) \to \mathbb{Z}$ denote the degree homomorphism. Then, it induces the morphism

$$\frac{\deg}{\operatorname{ind} X} : \mathrm{Ch}(X) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

sending the class of a closed point v of X to the class of $\deg(v)/\operatorname{ind} X$. Since the restriction map commutes with the degree homomorphism, by (4.9) we have

$$\frac{\deg}{\operatorname{ind} X}(\overline{w}) = \frac{\deg}{\operatorname{ind} X}(S^3(\bar{x})) = 1.$$

Therefore, $\bar{x}$ is nonzero in $\mathrm{Ch}(X)$, thus x is not divisible by 2 in $\mathrm{CH}(X)$. □

Theorem 4.5. — *φ is not injective.*

Proof. — Follows from Proposition 4.1, Proposition 4.4, and the surjectivity of φ. □

Appendix A. Pieri formula in the Grothendieck ring of $\overline{X}$

In this section, we give a proof of the congruence relations in (2.13). Using the Pieri-type formula in Lemma A.4, we first compute the products $\mathbf{e}_i\mathbf{e}_m$ in terms of the Schubert classes (Lemmas A.5 and A.7). Then, we derive the formulas for the square of $f(i) \in \widetilde{K}^i(\overline{X})$ in Proposition A.10.

Recall that the group $K(\overline{X})$ is free abelian with basis the set of all products $\prod_{i\in[1,n]} \mathbf{e}(i)$ (including the empty product, the unit). Recall also that a *strict partition* in $[1,n]$ is a sequence $\lambda = (\lambda_1, \dots, \lambda_m)$ such that $n \geqslant \lambda_1 > \cdots > \lambda_m \geqslant 1$. The size of λ is denoted by $|\lambda| = \lambda_1 + \cdots + \lambda_m$. Then, the group $K(\overline{X})$ has another basis $\mathbf{e}_\lambda \in K(\overline{X})^{(|\lambda|)}$, where λ ranges over all strict partitions in $[1,n]$ including the empty partition, given by the Schubert classes. Note that if λ consists of a single element $\{i\}$, then $\mathbf{e}_i = \mathbf{e}(i)$. We allow notation $\mathbf{e}_\lambda$ with λ an arbitrary finite decreasing sequence of natural numbers: if λ contains numbers bigger than n, we set $\mathbf{e}_\lambda = 0$.

We shall first recall some basic notions from [3]. Let λ be a finite decreasing sequence of natural numbers. The *shifted diagram* of λ is an array of boxes in which the i^{th} row has λ_i boxes, and is shifted $i-1$ units to the right with respect to the top row. We denote the number of rows of λ by $l(\lambda)$. A *skew shifted diagram* (or shape) ν/λ is obtained by removing a shifted diagram λ from a larger shifted diagram ν containing λ. The number of boxes in ν/λ is denoted by $|\nu/\lambda|$. A skew shifted diagram is called *connected* if all boxes share an edge. A skew shifted diagram is called a *rim* if it does not contain a pair of boxes one of which is located strictly to the right (east) and strictly to the bottom (south) of the other one.

Definition A.1 ([3, Section 4]). — *Let θ be a rim. A* KOG-tableau *of θ is a labeling of the boxes of θ with positive integers such that*

(i) *each row (resp. column) of θ is strictly increasing from left (resp. top) to right (resp. bottom); and*
(ii) *each box is either smaller than or equal to all the boxes south-west of it, or it is greater than or equal to all the boxes south-west of it.*

If θ is not a rim, then there are no KOG-tableaux with shape θ. The content *of a KOG-tableau is the set of integers contained in its boxes.*

Remark A.2. — Let B be a box in a KOG-tableau of shape θ. If there is a box in θ located directly to the left of B, then B is actually greater than or equal to all the boxes south-west of it. If there is a box in θ directly below B, then B is actually less than or equal to all the boxes south-west of it.

Example A.3.

(1) Let us consider the following rim with two rows

$$\begin{array}{r} \cdots\boxed{a_1} \\ \boxed{b_1}\boxed{\cdots}\boxed{b_r}\qquad \end{array} \tag{A.1}$$

such that the two rows of the rim are disconnected[(1)], where the top row consists of only one box and the bottom row consists of r boxes. Then, for any $r \geqslant 1$, the number of KOG-tableaux of shape (A.1) with content $[1, r+1]$ is equal to 2. This can be verified in the following way. As the number of boxes of (A.1) is equal to $r+1$, $a_1, b_1, \ldots, b_r$ are distinct numbers of $[1, r+1]$. If $a_1 > b_r$, then by Definition A.1 (i) we have the unique KOG-tableau with labeling $(a_1, b_1, \ldots, b_r) = (r+1, 1, \ldots, r)$. Otherwise, by Definition A.1 (i)-(ii), we see that $b_r > b_{r-1} > \cdots b_1 > a_1$, thus we also have the unique KOG-tableau with labeling $(a_1, b_1, \ldots, b_r) = (1, 2, \ldots, r+1)$.

(2) Now consider the following diagram, which is the same as the diagram above, but with one more cell added to the first row.

$$\begin{array}{r} \cdots\boxed{a_1}\boxed{a_2} \\ \boxed{b_1}\boxed{\cdots}\boxed{b_r}\qquad\quad \end{array} \tag{A.2}$$

(1) Here and further, dots outside boxes denote empty space of any nonnegative length, in particular, length 0 is possible. In other words, it is possible that in the tableau (A.1), the cells a_1 and b_r share a vertex (but not an edge).

Then, for any $r \geqslant 2$ the number of KOG-tableaux of shape (A.2) with content $[1, r+1]$ is equal to 3: by Remark A.2, we obviously get $a_2 = r+1$. If $a_1 \geqslant b_r$, then there is a unique KOG-tableau with labeling $(a_1, a_2, b_1, \dots, b_r) = (r, r+1, 1, \dots, r)$. Otherwise, we have $a_1 \leqslant b_1 < \dots < b_r \leqslant r+1$, thus there are exactly two KOG-tableaux with labelings $(a_1, a_2, b_1, \dots, b_r) = (1, r+1, 1, \dots, r)$ and $(1, r+1, 2, \dots, r+1)$.

We shall make use of the following combinatorial Pieri-type formula due to Buch and Ravikumar:

Lemma A.4 ([3, Corollary 4.8]). — *Let $1 \leqslant i \leqslant n$ be an integer. For strict partitions λ and ν in $[1, n]$, we denote by $C^{\nu}_{\lambda,i}$ the number of KOG-tableaux of shape ν/λ with content $[1, i]$. Then,*

$$\mathbf{e}_i \mathbf{e}_\lambda = \sum_{\nu \subseteq [1,n]} (-1)^{|\nu/\lambda| - i} \cdot C^{\nu}_{\lambda,i} \cdot \mathbf{e}_\nu .$$

To be precise, in view of our convention that $\mathbf{e}_\nu$ is defined and equals zero for any finite decreasing sequence of natural numbers ν containing numbers bigger than n, we will use this formula with the sum over "strict partitions ν" replaced with the sum over "decreasing sequences of natural numbers ν". All extra summands appearing this way are zeros, even if the coefficients $C^{\nu}_{\lambda,i}$ alone are not zeros.

Now we compute the coefficients (modulo terms in $K(\overline{X})^{(2i+2)}$) in the Pieri formula (Lemma A.4) for $\lambda = (i)$.

Lemma A.5. — *We have $\mathbf{e}_1^2 = \mathbf{e}_2$ and $\mathbf{e}_n^2 = 0$ in $K(\overline{X})$, and the following relations hold modulo $K(\overline{X})^{(2i+2)}$:*

$$\mathbf{e}_i^2 \equiv \mathbf{e}_{2i} + 2\left(\sum_{k=1}^{i-1} \mathbf{e}_{i+k,\, i-k}\right) - \mathbf{e}_{i+1,\, i} - 3\left(\sum_{k=2}^{i-1} \mathbf{e}_{i+k,\, i-k+1}\right) - 2\mathbf{e}_{2i,\, 1}$$

for any $1 < i < n$.

Proof. — For now, in addition to $1 < i < n$, let us also allow $i = 1$ and $i = n$. Let us use Lemma A.4 for this i and for $\lambda = (i)$. First, note that if $l(\nu) \geqslant 3$, then the leftmost box of the third row of ν/λ is strictly below and strictly to the right of the leftmost box of the second row of ν/λ, thus ν/λ is not a rim. Hence, we may assume that $l(\nu) \leqslant 2$.

Since we consider the number of KOG-tableaux of shape ν/λ with content $[1, i]$, it suffices to consider ν with $|\nu/\lambda| \geqslant i$ (i.e., $|\nu| \geqslant 2i$).

If $l(\nu) = 1$, then by Definition A.1 (i) $C^{\nu}_{\lambda,i} \neq 0$ if and only if $|\nu/\lambda| = i$. In this case, ν/λ is simply ν without the first leftmost i boxes, thus $C^{\nu}_{\lambda,i} = 1$, i.e., $\mathbf{e}_{2i}$ occurs in $\mathbf{e}_i^2$ with coefficient 1.

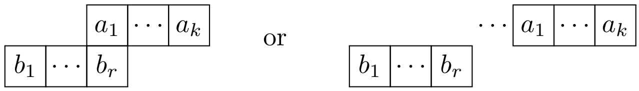
Let $i = 1$. Then, again by Definition A.1 (i) $C^{\nu}_{\lambda,i} = 0$ for any ν with $l(\nu) = 2$, thus $\mathbf{e}_2$ is the only summand in $\mathbf{e}_1^2$.

Let $i = n$. Since ν is a strict partition of $[1, n]$, the condition $l(\nu) = 2$ implies that $|\nu| \leqslant 2n - 1$. As $|\nu| \geqslant 2n$ and $\mathbf{e}_{2n} = 0$ by definition, we obtain the relation $\mathbf{e}_n^2 = 0$.

From now on, we assume that $2 \leqslant i \leqslant n-1$, and we compute $\mathbf{e}_i^2$ modulo $K(\overline{X})^{(2i+2)}$. If $|\nu/\lambda| \geqslant i + 2$, then $|\nu| \geqslant 2i + 2$, thus $\mathbf{e}_\nu \in K(\overline{X})^{(2i+2)}$. Therefore, we may assume that $|\nu/\lambda| = i$ or $i + 1$.

Let $\nu = (j, r)$ be such that $j > r$ and $j + r = 2i$ or $j + r = 2i + 1$. By the definition of a rim, there cannot be more than one box in the top row of ν/λ located directly above cells of the bottom row. Hence, it suffices to compute $C^{\nu}_{\lambda,i}$ for the following tableaux of shape ν/λ:

(A.3)



where $k = j - i$, so $k = i - r$ or $k = i + 1 - r$ (note that the two rows of the second tableau are disconnected but they can share a vertex – see Example A.3).

Assume that $k = i - r$. As $i + 1 \leqslant j \leqslant 2i - 1$, $1 \leqslant r \leqslant i - 1$, we have $1 \leqslant k \leqslant i - 1$. As $r < i$, two rows of the tableau (A.3) are disconnected. We show by induction that $C^{\nu}_{\lambda,k} = 2$ for any $1 \leqslant k \leqslant i - 1$. The case $k = 1$ follows from Example A.3 (1). Assume $k \geqslant 2$. Then, by Remark A.2 we have $a_k = i$, thus the statement follows by induction. Hence, for any $1 \leqslant k \leqslant i - 1$ the term $\mathbf{e}_{i+k,\, i-k}$ occurs in $\mathbf{e}_i^2$ with coefficient 2.

Now we assume that $k = i + 1 - r$. As $i + 1 \leqslant j \leqslant 2i$, $1 \leqslant r \leqslant i$, we have $1 \leqslant k \leqslant i$. We shall consider three subcases: $k = 1$, $k = i$, and $2 \leqslant k \leqslant i - 1$. If $k = 1$, then $r = i$ and the first row of (A.3) consists of a single element a_1 just above b_r. Hence, by Remark A.2, we get $C^{\nu}_{\lambda,i} = 1$ with a unique labeling $(a_1, b_1, b_2 \dots, b_r) = (1, 1, 2, \dots, i)$, i.e., the term $\mathbf{e}_{i+1,\, i}$ occurs in $\mathbf{e}_i^2$ with coefficient -1. If $k = i$, then $r = 1$ and two rows of the tableau (A.3) are disconnected. By Definition A.1 (i), we see that $a_m = m$ for any $1 \leqslant m \leqslant k$. By Remark A.2 applied to a_2, we have $b_1 = 1$ or 2, thus $C^{\nu}_{\lambda,i} = 2$, i.e., the term $\mathbf{e}_{2i,\, 1}$ occurs in $\mathbf{e}_i^2$ with coefficient -2.

Finally, let $2 \leqslant k \leqslant i - 1$. We show by induction that $C^{\nu}_{\lambda,i} = 3$. The case $k = 2$ immediately follows from Example A.3 (2). Assume that $k \geqslant 3$. Then, by Remark A.2, $a_k = i$. If we remove the box a_k from the diagram, all numbers from $[1, i-1]$ must be present in the remaining boxes. But the content of the remaining boxes cannot be $[1, i]$ since otherwise we would have $a_{k-1} = a_k = i$ by Remark A.2, which contradicts Definition A.1 (i).

So, we can proceed by induction on k and get $C^{\nu}_{\lambda,i} = 3$ for $2 \leqslant k \leqslant i-1$, i.e., for any $2 \leqslant k \leqslant i-1$ the term $\mathbf{e}_{i+k,\, i-k+1}$ occurs in $\mathbf{e}_i^2$ with coefficient -3. □

Recall that we have denoted $f(i) = \mathbf{e}(i)u^i = \mathbf{e}_i u^i \in \widetilde{K}(\overline{X})$. We also simply denote $f_{m,\,i} = \mathbf{e}_{m,\,i}\, u^{m+i} \in \widetilde{K}(\overline{X})$. We deduce some formulas for $\widetilde{K}(\overline{X})$. The proof immediately follows from Lemma A.5.

Corollary A.6. — *We have $f(1)^2 = f(2)$ and $f(n)^2 = 0$ in $\widetilde{K}(\overline{X})$, and the following relations hold modulo $I(\overline{X})^2$:*

$$f(i)^2 \equiv f(2i) + 2\left(\sum_{k=1}^{i-1} f_{i+k,\, i-k}\right) - t\left(\sum_{k=1}^{i-1} f_{i+k,\, i-k+1}\right)$$

for any $1 < i < n$.

In the following, we compute the coefficients (modulo terms in $K(\overline{X})^{(i+m+2)}$) in the Pieri formula (Lemma A.4) for any $\lambda = (m)$ with $m > i$.

Lemma A.7. — *Let $m > 1$. Then, we have $\mathbf{e}_1\mathbf{e}_m = \mathbf{e}_{m+1} + \mathbf{e}_{m,\,1} - \mathbf{e}_{m+1,\,1}$ in $K(\overline{X})$, and the following relations hold modulo $K(\overline{X})^{(i+m+2)}$:*

$$\mathbf{e}_i\mathbf{e}_m \equiv \mathbf{e}_{m+i} + \mathbf{e}_{m,\,i} + 2\left(\sum_{k=1}^{i-1} \mathbf{e}_{m+k,\, i-k}\right) - 2\mathbf{e}_{m+1,\,i} - 3\left(\sum_{k=2}^{i-1} \mathbf{e}_{m+k,\, i-k+1}\right) - 2\mathbf{e}_{m+i,\,1}$$

for any $1 < i < m$.

Proof. — The proof is similar to the proof of Lemma A.5 above. For now, in addition to $1 < i < m$, let us also allow $i = 1$. Let us use Lemma A.4 for this i and for $\lambda = (m)$. Then, the arguments of the first two paragraphs of the proof of Lemma A.5 show that $l(\nu) \leqslant 2$ and $|\nu/\lambda| \geqslant i$.

If $l(\nu) = 1$, then it follows from Definition A.1 (i) that $C^{\nu}_{\lambda,i} = 1$ if $|\nu/\lambda| = i$, and $C^{\nu}_{\lambda,i} = 0$ otherwise. So, $\mathbf{e}_{i+m}$ occurs in $\mathbf{e}_i\mathbf{e}_m$ with coefficient 1, and there are no terms $\mathbf{e}_k$ with $k \neq i + m$ in the decomposition of $\mathbf{e}_i\mathbf{e}_m$ from Lemma A.4.

From now on, let $l(\nu) = 2$. Let us consider the case $i = 1$ first. Then the content of the KOG-tableau is simply $\{1\}$. By Definition A.1 (i), each row of ν/λ can have at most 1 box. There are only two partitions ν that contain λ and satisfy these conditions: $\nu = (m, 1)$ and $\nu = (m + 1, 1)$. So, $C^{(m,1)}_{\lambda,1} = C^{(m+1,1)}_{\lambda,1} = 1$, thus the first equation in the statement of the lemma immediately follows.

Let $2 \leqslant i \leqslant m-1$. If $|\nu/\lambda| \geqslant i+2$, then $|\nu| \geqslant i+m+2$, thus $\mathbf{e}_\nu \in K(\overline{X})^{(i+m+2)}$. Therefore, we may assume that $|\nu/\lambda| = i$ or $i+1$.

Let $\nu = (j,r)$, where $j > r$ and $j+r = m+i$ or $j+r = m+i+1$. If there is a box in the top row of ν/λ located directly above boxes of the bottom row, then $r = m \geqslant i+1$, thus $|\nu/\lambda| \geqslant i+2$. Hence, it suffices to compute $C^\nu_{\lambda,i}$ for the following tableaux of shape ν/λ:

$$\text{(A.4)}\qquad \begin{array}{|c|c|c|}\hline b_1 & \cdots & b_r \\ \hline\end{array} \qquad \text{or} \qquad \begin{array}{cccccc} & & & \cdots & \boxed{a_1} \boxed{\cdots} \boxed{a_k} \\ \boxed{b_1} \boxed{\cdots} \boxed{b_r} & & & & \end{array}$$

where $k = j-m$, so $k = i-r$ or $k = i+1-r$.

Assume that $k = i-r$. As $m \leqslant j \leqslant i+m-1$, $1 \leqslant r \leqslant i$, we get $0 \leqslant k \leqslant i-1$. We have two subcases: $k=0$ and $1 \leqslant k < i$. If $k=0$, then by Definition A.1 (i) $C^\nu_{\lambda,i} = 1$. If $1 \leqslant k < i$, then we use induction on k. For $k=1$, we get $C^\nu_{\lambda,i} = 2$ by Example A.3 (1). For $k \geqslant 2$, we have $a_k = i$ by Remark A.2, thus $C^\nu_{\lambda,i} = 2$ by induction.

Now assume that $k = i+1-r$. As the content of ν/λ should be $[1,i]$, it follows from Definition A.1 (i) that $r \leqslant i$, and the top row in the tableau (A.4) is non-empty. So, $1 \leqslant r \leqslant i$, $m+1 \leqslant j \leqslant i+m$, and $1 \leqslant k \leqslant i$. We consider three subcases: $k=1$, $k=i$, and $2 \leqslant k \leqslant i-1$.

If $k=1$, then $r=i$. By Definition A.1 (i), there is only one option for the bottom row: $(b_1,\dots,b_i) = (1,\dots,i)$. By Definition A.1 (ii), we have two options for a_1: $a_1 = 1$ or $a_1 = i$. Hence, we get $C^\nu_{\lambda,i} = 2$, i.e., the term $\mathbf{e}_{m+1,\,i}$ occurs in $\mathbf{e}_i\mathbf{e}_m$ with coefficient -2.

If $k=i$, then $r=1$. By Definition A.1 (i), we get $(a_1,\dots,a_i) = (1,\dots,i)$. By Definition A.1 (ii) applied to a_2, we have two options for b_1: $b_1 = 1$ or $b_1 = 2$. So, $C^\nu_{\lambda,i} = 2$, and the term $\mathbf{e}_{m+i,\,1}$ occurs in $\mathbf{e}_i\mathbf{e}_m$ with coefficient -2.

Finally, let $2 \leqslant k \leqslant i-1$. Then, by exactly the same argument as in the last paragraph of the proof of Lemma A.5, we have $C^\nu_{\lambda,i} = 3$, thus the term $\mathbf{e}_{m+k,\,i-k+1}$ occurs in $\mathbf{e}_i\mathbf{e}_m$ with coefficient -3 for any $2 \leqslant k \leqslant i-1$. ☐

Lemma A.7 directly implies the following equations in $\widetilde{K}(\overline{X})$.

Corollary A.8. — *Let $m > 1$. Then, we have $f(m)f(1) = f(m+1) + f_{m,\,1} - tf_{m+1,\,1}$ in $\widetilde{K}(\overline{X})$, and the following relations hold modulo $I(\overline{X})^2$:*

$$f(m)f(i) \equiv f(m+i) + f_{m,\,i} + 2\left(\sum_{k=1}^{i-1} f_{m+k,\,i-k}\right) + t\left(\sum_{k=2}^{i-1} f_{m+k,\,i-k+1}\right)$$

for any $1 < i < m$.

Now using Corollary A.8, we get the following intermediate result.

Lemma A.9. — *For any $m > 1$, we have the following relation modulo $I(\overline{X})^2$.*

$$f_{m,1} - f(m)f(1) \equiv f(m+1) - tf(1)f(m+1) + tf(m+2).$$

For any $1 < i < m$, the difference $f_{m,i} - f(m)f(i)$ is congruent modulo $I(\overline{X})^2$ to

$$(-1)^i f(m+i) - 2\left(\sum_{k=1}^{i-1} f(m+k)f(i-k)\right) - t\left(\sum_{k=2}^{i-1} f(m+k)f(i-k+1)\right)$$

if i is even, and is congruent modulo $I(\overline{X})^2$ to

$$(-1)^i f(m+i) - 2\left(\sum_{k=1}^{i-1} f(m+k)f(i-k)\right) - t\left(\sum_{k=2}^{i-1} f(m+k)f(i-k+1)\right) - tf(m+i+1)$$

if i is odd.

Proof. — We first observe that $2 \equiv -2$, $t \equiv -t \mod I(\overline{X})^2$. The formula for $f_{m,1} - f(m)f(1)$ is obtained from the formulas for $f(m)f(1)$ and for $f(m+1)f(1)$ (multiplied by t) in Corollary A.8.

Let us prove the formula for $f_{m,i} - f(m)f(i)$ with $1 < i < m$. We show by induction on i for all values of $m > i$ together. If $i = 2$, then the formula for $f_{m,2} - f(m)f(2)$ is obtained from the formulas for $f(m)f(2)$ and for $f(m+1)f(1)$ (multiplied by 2) in Corollary A.8. Now we assume that the formulas for $f_{m',i'} - f(m')f(i')$ hold for any $2 \leqslant i' < i$ and any $m' > i'$. Let us multiply the formulas by 2 and t, respectively. Then, we have the following congruences modulo $I(\overline{X})^2$:

$$2f_{m',i'} - 2f(m')f(i') \equiv 2f(m'+i') \quad \text{and} \quad tf_{m',i'} - tf(m')f(i') \equiv tf(m'+i'), \tag{A.5}$$

respectively. Note that by the first formula in Lemma A.9, the first formula in (A.5) still holds for $i' = 1$ and any $m' > 1$.

Taking the sum of the first formulas (A.5) for $m' = m + k$, $i' = i - k$, and $1 \leqslant k \leqslant i - 1$, we get

$$2\left(\sum_{k=1}^{i-1} f_{m+k,i-k}\right) \equiv 2\left(\sum_{k=1}^{i-1} f(m+k)f(i-k)\right) + 2(i-1)f(m+i) \mod I(\overline{X})^2. \tag{A.6}$$

Similarly, taking the sum of the second formulas (A.5) for $m' = m + k$, $i' = i - k + 1$, and $2 \leqslant k \leqslant i - 1$, we get

$$\text{(A.7)}\quad t\left(\sum_{k=2}^{i-1} f_{m+k,\, i-k+1}\right) \equiv t\left(\sum_{k=2}^{i-1} f(m+k)f(i-k)\right) + t(i-2)f(m+i+1) \mod I(\overline{X})^2.$$

Let us plug the formulas (A.6) and (A.7) into the second formula in Corollary A.8. Then, the difference $f_{m,\,i} - f(m)f(i)$ is congruent modulo $I(\overline{X})^2$ to

$$-(2i-1)f(m+i) - t(i-2)f(m+i+1) - 2\left(\sum_{k=1}^{i-1} f(m+k)f(i-k)\right) - t\left(\sum_{k=2}^{i-1} f(m+k)f(i-k+1)\right).$$

Since the following congruences hold modulo $I(\overline{X})^2$

$$\text{(A.8)}\qquad -(2i-1) \equiv (-1)^i \quad\text{and}\quad -t(i-2) \equiv \begin{cases} 0 & \text{if } i \text{ is even,} \\ t & \text{if } i \text{ is odd,} \end{cases}$$

the formula follows. □

Combining Corollary A.6 and Lemma A.9, we obtain the following main result of this section.

Proposition A.10. — *For any $1 < i < n$, the following relations hold modulo $I(\overline{X})^2$:*

$$f(i)^2 \equiv (-1)^{i-1}f(2i) + 2\left(\sum_{k=1}^{i-1} f(i+k)f(i-k)\right) - t\left(\sum_{k=1}^{i-1} f(i+k)f(i-k+1)\right) + tf(2i+1)$$

for even i, and

$$f(i)^2 \equiv (-1)^{i-1}f(2i) + 2\left(\sum_{k=1}^{i-1} f(i+k)f(i-k)\right) - t\left(\sum_{k=1}^{i-1} f(i+k)f(i-k+1)\right)$$

for odd i.

Proof. — Let us rewrite the formulas from Lemma A.9 as follows:

$$f_{m',\,i'} \equiv f(m')f(i') + f(m'+i') - tf(i')f(m'+i') + tf(m'+i'+1)$$

for $m' > 1, i' = 1$,

$$\begin{aligned} f_{m',\,i'} &\equiv f(m')f(i') + (-1)^{i'} f(m'+i') \\ &\quad - 2\left(\sum_{k=1}^{i'-1} f(m'+k)f(i'-k)\right) - t\left(\sum_{k=2}^{i'-1} f(m'+k)f(i'-k+1)\right) \end{aligned}$$

for $m' > i' > 1$, i' even, and

$$\begin{aligned} f_{m',\,i'} \equiv f(m')f(i') + (-1)^{i'} f(m'+i') - 2\left(\sum_{k=1}^{i'-1} f(m'+k)f(i'-k)\right) \\ - t\left(\sum_{k=2}^{i'-1} f(m'+k)f(i'-k+1)\right) - tf(m'+i'+1) \end{aligned}$$

for $m' > i' > 1$, i' odd, where all congruences are modulo $I(\overline{X})^2$.

Multiplying each of these formulas by 2, for any $i' \geqslant 1$ and any $m' > i'$ we have

$$2f_{m',\,i'} \equiv 2f(m')f(i') + 2f(m'+i') \mod I(\overline{X})^2. \tag{A.9}$$

Similarly, multiplying by t, for any $m' > i' \geqslant 1$ we get

$$tf_{m',\,i'} \equiv tf(m')f(i') + tf(m'+i') \mod I(\overline{X})^2. \tag{A.10}$$

For any $1 < i < n$, let us take the sum of (A.9) for $m' = i+k$ and $i' = i-k$ over $1 \leqslant k \leqslant i-1$. Then, we get

$$2\left(\sum_{k=1}^{i-1} f_{i+k,\,i-k}\right) \equiv 2\left(\sum_{k=1}^{i-1} f(i+k)f(i-k)\right) + 2(i-1)f(2i) \mod I(\overline{X})^2.$$

Similarly, we take the sum of (A.10) for $m' = i+k$ and $i' = i-k+1$ over $1 \leqslant k \leqslant i-1$. Then, we have

$$\begin{aligned} &t\left(\sum_{k=1}^{i-1} f_{i+k,\,i-k+1}\right) \\ &\qquad \equiv t\left(\sum_{k=1}^{i-1} f(i+k)f(i-k+1)\right) + t(i-1)f(2i+1) \mod I(\overline{X})^2. \end{aligned}$$

Now let us plug these formulas into the statement of Corollary A.6 for $1 < i < n$, thus we have the following congruence modulo $I(\overline{X})^2$

$$f(i)^2 \equiv 2\left(\sum_{k=1}^{i-1} f(i+k)f(i-k)\right) - t\left(\sum_{k=1}^{i-1} f(i+k)f(i-k+1)\right) + (2i-1)f(2i) - t(i-1)f(2i+1).$$

Since

$$-t(i-1) \equiv \begin{cases} t & \mod I(\overline{X})^2 \text{ if } n \text{ is even,} \\ 0 & \mod I(\overline{X})^2 \text{ otherwise,} \end{cases}$$

the statement follows from the first congruence equation in (A.8). □

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