Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Yuri G. Zarhin

Corrigendum to "Tate classes on self-products of Abelian varieties over finite fields"

Tome 75, n^o 6 (2025), p. 2317-2320.

https://doi.org/10.5802/aif.3724

CENTRE MERSENNE

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
75, 6 (2025) 2317-2320

CORRIGENDUM TO "TATE CLASSES ON SELF-PRODUCTS OF ABELIAN VARIETIES OVER FINITE FIELDS"

Annales de l'Institut Fourier, vol. 72 (2022), no 6, 2339-2383

by Yuri G. ZARHIN

Abstract. — We fix an inaccuracy in the proof of Proposition 6.6 in the original paper.

Résumé. — Nous corrigeons une imprécision dans la démonstration de la proposition 6.6 de l'article original.

Proposition 6.6 of [1] asserts the equivalence of certain two properties, (a) and (b). Its proof in [1, p. 2369–2371] consists of two parts. First, we proved that (b) $\Rightarrow$ (a). Second, we proved that not (a) $\Rightarrow$ not (b), which is actually the same! [(1)] The aim of this note is to provide a missing proof of the implication (a) $\Rightarrow$ (b).

We will freely use the notation of [1, §6.3]. In particular, $\mathbb{Z}_+$ is the set of all *nonnegative* integers, E is a field of characteristic zero, V_E is a nonzero finite-dimensional E-vector space, V_E^* is its dual, $A_E\colon V_E \to V_E$ is an invertible *diagonalizable* linear operator, $\mathrm{spec}(A) \subset E$ is the set of its *eigenvalues*, and $\mathrm{mult}_A\colon \mathrm{spec}(A) \to \mathbb{Z}_+$ is the map that assigns to each eigenvalue of A_E its multiplicity. Let us fix an eigenbasis B of V_E (with respect to A_E) and choose an order on B. We write $\pi\colon B \to \mathrm{spec}(A)$ for the (surjective) map that assigns to an eigenvector the corresponding eigenvalue; if $\alpha \in \mathrm{spec}(A)$ then $\pi^{-1}(\alpha)$ consists of $\mathrm{mult}_A(\alpha)$ elements [1, p. 2367].

If $j \leqslant \dim_E(V_E)$ is a positive integer then we assign to each j-element subset C of B a certain element $y_C \in \bigwedge_E^j(V_E^*)$ that is an eigenvector of $\bigwedge^j(A_E^*)$ with eigenvalue $\prod_{x\in C}\pi(x)$; all such y_C's consitute an eigenbasis

Keywords: Abelian varieties, Tate classes, Finite fields.
2020 *Mathematics Subject Classification:* 11G10, 11G25, 14G15.

(1) I am grateful to Sergey Rybakov for pointing it out.

of $\bigwedge_E^j(V_E^*)$ [1, p. 2367]. We will need functions

$$e_C\colon \operatorname{spec}(A) \longrightarrow \mathbb{Z}_+,\ \alpha \longmapsto \#\{x \in C \mid \pi(x) = \alpha\}.$$

If $\lambda \in E$ then $\bigwedge_E^j(V_E^*)(\lambda)$ stands for the eigenspace of $\bigwedge^j(A_E^*)$ attached to the eigenvalue λ. In order to describe explicitly a basis of $\bigwedge_E^j(V_E^*)(\lambda)$, let us consider functions $e\colon \operatorname{spec}(A) \to \mathbb{Z}_+$ that enjoy the following properties:

(i) $e(\alpha) \leqslant \operatorname{mult}_A(\alpha) \ \ \forall\, \alpha \in \operatorname{spec}(A)$;
(ii) $\sum_{\alpha\in\operatorname{spec}(A)} e(\alpha) = j$;
(iii) $\prod_{\alpha\in\operatorname{spec}(A)} \alpha^{e(\alpha)} = \lambda$.

Then the set of y_C's such that e_C enjoys the properties (i)–(iii), is a basis of $\bigwedge_E^j(V_E^*)(\lambda)$ [1, p. 2367].

Let $\lambda_1, \lambda_2 \in K$ where K is a subfield of E. Assuming that $j \geqslant 2$, let us consider the E-linear map

$$\bigwedge_E^{j-2}(V_E^*)(\lambda_1)\otimes_E\bigwedge_E^{2}(V_E^*)(\lambda_2) \longrightarrow \bigwedge_E^{j}(V_E^*)(\lambda_1\lambda_2), \quad \psi\otimes\phi \longmapsto \psi\wedge\phi. \tag{6.10}$$

By [1, Lem. 6.9], the image of (6.10) is generated by all y_C's where C is any j-element subset of B that enjoys the following properties.

The set C is a disjoint union of a $(j-2)$-element subset S and a 2-element subset T such that the corresponding functions

$$e_S\colon \operatorname{spec}(A) \longrightarrow \mathbb{Z}_+, \quad e_T\colon \operatorname{spec}(A) \longrightarrow \mathbb{Z}_+$$

enjoy the following properties.

$$\prod_{\alpha\in\operatorname{spec}(A)} \alpha^{e_S(\alpha)} = \lambda_1, \qquad \prod_{\alpha\in\operatorname{spec}(A)} \alpha^{e_T(\alpha)} = \lambda_2. \tag{6.11}$$

The condition (a) of [1, Prop. 6.6] is equivalent to the non-surjectiveness of (6.10).

The condition (b) of [1, Prop. 6.6] is the existence of a function $e\colon \operatorname{spec}(A) \to \mathbb{Z}_+$ such that:

(bi) $e(\alpha) \leqslant \operatorname{mult}_A(\alpha) \ \ \forall\, \alpha \in \operatorname{spec}(A)$;
(bii) $\sum_{\alpha\in\operatorname{spec}(A)} e(\alpha) = j$;
(biii) $\prod_{\alpha\in\operatorname{spec}(A)} \alpha^{e(\alpha)} = \lambda_1\lambda_2$;
(biv) If $\alpha \in \operatorname{spec}(A)$ and $e(\alpha) \neq 0$ then $e(\alpha) \geqslant 1$ and one of the following conditions holds.
  (1) $\lambda_2/\alpha \notin \operatorname{spec}(A)$;
  (2) $\lambda_2/\alpha \in \operatorname{spec}(A)$ but $e(\lambda_2/\alpha) = 0$;
  (3) $\alpha = \lambda_2/\alpha$ (i.e., $\alpha^2 = \lambda_2$) and $e(\alpha) = 1$.

Proposition 6.6 may be restated (see Remark 6.8 of [1]) as an equivalence of the *non-surjectiveness* of (6.10) and property (b). Our proof is based on the following assertion.

LEMMA 6.9 OF [1]. — *The image of the map* (6.10) *is generated by all* y_C*'s where* C *is any* j*-element subset of* B *that enjoys the following properties.*

The set C *is a disjoint union of a* $(j-2)$*-element subset* S *and a* 2*-element subset* T *such that the corresponding functions*

$$e_S\colon \operatorname{spec}(A) \longrightarrow \mathbb{Z}_+, \quad e_T\colon \operatorname{spec}(A) \longrightarrow \mathbb{Z}_+$$

enjoy the following properties.

$$\prod_{\alpha\in\operatorname{spec}(A)} \alpha^{e_S(\alpha)} = \lambda_1, \qquad \prod_{\alpha\in\operatorname{spec}(A)} \alpha^{e_T(\alpha)} = \lambda_2. \tag{6.11}$$

Proof of Proposition 6.6 of [1]. — The implication (b) $\Rightarrow$ (a) is proven in [1, p. 2370]. Conversely, suppose that (a) holds, i.e., (6.10) is *not* surjective. Then there is a *basis vector* $y_C \in \bigwedge_E^j(V_E^*)(\lambda_1\lambda_2)$ that does *not* belong to the image where C is a certain j-element subset of B such that

$$\prod_{x\in C} \pi(x) = \lambda_1\lambda_2.$$

I claim that the function $e := e_C$ enjoys the properties (b). Indeed, by definition, e_C enjoys the propetries (bi), (bii), (biii). In particular,

$$\prod_{\alpha\in C} \alpha^{e_C(\alpha)} = \prod_{x\in C} \pi(x) = \lambda_1\lambda_2.$$

Suppose that e_C does *not* enjoy the property (biv). This means that there exists $\alpha \in \operatorname{spec}(A)$ such that $e_C(\alpha) \neq 0$ (i.e., $\alpha \in \pi(C)$) and

$$\frac{\lambda_2}{\alpha} \in \operatorname{spec}(A), \quad e_C\left(\frac{\lambda_2}{\alpha}\right) \neq 0$$

(i.e., $\lambda_2/\alpha \in \pi(C) \subset \operatorname{spec}(A)$). In addition, if $\lambda_2/\alpha = \alpha$ then $e_C(\alpha) \geqslant 2$ (i.e., $\pi^{-1}(\alpha)$ contains at least two elements of C).

This implies that there are *distinct* elements x_1 and x_2 of C such that

$$\pi(x_1) = \alpha, \quad \pi(x_2) = \frac{\lambda_2}{\alpha}.$$

Then C coincides with the disjoint union of the 2-element subset $T = \{x_1, x_2\}$ and the $(j-2)$-element subset $S = C \setminus T$. By definition of T,

$$\prod_{\alpha\in T} \alpha^{e_T(\alpha)} = \prod_{x\in T} \pi(x) = \pi(x_1)\cdot\pi(x_2) = \alpha\cdot\frac{\lambda_2}{\alpha} = \lambda_2.$$

Hence,

$$\prod_{\alpha \in T} \alpha^{e_T(\alpha)} = \lambda_2.$$

Since C is the disjoint union of S and T, we get

$$\prod_{\alpha \in S} \alpha^{e_S(\alpha)} = \frac{\prod_{\alpha \in C} \alpha^{e_C(\alpha)}}{\prod_{\alpha \in T} \alpha^{e_T(\alpha)}} = \frac{\lambda_1 \lambda_2}{\lambda_2} = \lambda_1.$$

Hence,

$$\prod_{\alpha \in S} \alpha^{e_S(\alpha)} = \lambda_1.$$

It follows from Lemma 6.9 that y_C lies in the image of the map (6.10), which is not true. The obtained contradiction implies that e_C enjoys the property (biv). This ends the proof of Proposition 6.6. □

BIBLIOGRAPHY

[1] Y. G. Zarhin, "Tate classes on self-products of abelian varieties over finite fields", *Ann. Inst. Fourier* **72** (2022), no. 6, p. 2339-2383.

Manuscrit reçu le 7 janvier 2025,
révisé le 20 février 2025,
accepté le 20 mars 2025.

Yuri G. ZARHIN
Department of Mathematics
Pennsylvania State University
University Park, PA 16802, USA
zarhin@math.psu.edu