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GEOMETRIZATION OF SOLUTIONS OF THE GENERALIZED CLASSICAL YANG–BAXTER EQUATION AND A NEW PROOF OF THE BELAVIN–DRINFELD TRICHOTOMY

by Raschid ABEDIN (*)

Abstract. — We study the generalized classical Yang–Baxter equation (short: GCYBE) for central simple Lie algebras over fields of characteristic 0. Using a novel geometrization procedure, we assign to a solution of the GCYBE a cohomology free sheaf of Lie algebras on a projective curve. This assignment implies that all such solutions are algebraic in nature, i.e. they extend to rational functions on the product of two algebraic curves. Furthermore, the curves assigned to skew-symmetric solutions turn out to be either smooth, nodal, or cuspidal cubic plane curves. This results in a geometric trichotomy of skew-symmetric solutions of the GCYBE. Over the field of complex numbers, this geometric trichotomy implies the well-known Belavin–Drinfeld trichotomy, which states that skew-symmetric solutions of the GCYBE are either elliptic, trigonometric, or rational. As an interesting side result, we show that sheaves of Lie algebras with constant geometric fibers are locally free in the étale topology. This is used to classify such sheaves on the complex plane with at most one puncture.

Résumé. — Nous étudions l'équation de Yang–Baxter classique généralisée (abréviation anglaise: GCYBE) pour les algèbres de Lie simples centrales sur des corps de caractéristique 0. En utilisant une nouvelle procédure de géométrisation, nous attribuons à une solution de la GCYBE un faisceau sans cohomologie d'algèbres de Lie sur une courbe projective. Cette attribution implique que toutes ces solutions sont de nature algébrique, c'est-à-dire qu'elles s'étendent à des fonctions rationnelles sur le produit de deux courbes algébriques. De plus, les courbes assignées aux solutions antisymétrique s'avèrent être des courbes cubiques lisses, nodales ou cuspidales. Il en résulte une trichotomie géométrique des solutions antisymétrique de la GCYBE. Dans le domaine des nombres complexes, cette trichotomie géométrique implique la trichotomie bien connue de Belavin–Drinfeld, qui stipule que les solutions antisymétrique de la GCYBE sont soit elliptiques, trigonométriques ou rationnelles. Comme résultat secondaire intéressant, nous montrons que les faisceaux d'algèbres de Lie avec des fibres géométriques constantes sont localement libres dans la topologie étale. Ceci est utilisé pour classifier de telles faisceaux sur le plan complexe avec au plus un point supprimé.

Keywords: classical Yang–Baxter equation, sheaves of Lie algebras, infinite-dimensional Lie algebras.
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1. Introduction

Let $\mathfrak{g}$ be a simple complex Lie algebra and $U \subseteq \mathbb{C}$ be a connected open subset. A meromorphic function $r\colon U \times U \to \mathfrak{g} \otimes \mathfrak{g}$ is said to be a generalized r-matrix, if it solves the so-called generalized classical Yang–Baxter equation (GCYBE)

$$\left[r^{12}(x_1, x_2), r^{13}(x_1, x_3)\right] + \left[r^{12}(x_1, x_2), r^{23}(x_2, x_3)\right] + \left[r^{32}(x_3, x_2), r^{13}(x_1, x_3)\right] = 0. \tag{1.1}$$

Generalized r-matrices have many applications in mathematical physics: they appear in the r-matrix method and the Adler–Kostant–Symes scheme within the theory of integrable systems [5, 7, 51], in the construction of classical and quantum Gaudin-type models [50, 52], and in the definition of generalized Knizhnik–Zamolodchikov equation [20].

The skew-symmetric solutions of the GCYBE are of particular interest to applications. These solve the classical Yang–Baxter equation (CYBE)

$$\left[r^{12}(x_1, x_2), r^{13}(x_1, x_3)\right] + \left[r^{12}(x_1, x_2), r^{23}(x_2, x_3)\right] + \left[r^{13}(x_1, x_3), r^{23}(x_2, x_3)\right] = 0. \tag{1.2}$$

Solutions to the CYBE are simply called r-matrices. They are important in the study of Poisson–Lie groups as well as their infinitesimal counterparts, Lie bialgebras. The quantization of these structures is the backbone of the theory of quantum groups, where the CYBE appears as the classical counterpart to the quantum Yang–Baxter equation [13, 19, 20].

Remark 1.1. — Non-degenerate r-matrices are automatically skew-symmetric and therefore solve the GCYBE. In particular, a non-degenerate generalized r-matrix is skew-symmetric if and only if it solves the CYBE. This motivates the usage of the word "generalized".

Non-degenerate difference dependent r-matrices, i.e. the ones that can be written as

$$r(x, y) = s(x - y) \tag{1.3}$$

for some non-degenerate meromorphic function $s\colon D \to \mathfrak{g} \otimes \mathfrak{g}$ on an open disk $D \subseteq \mathbb{C}$, were classified in the fundamental work [9] by Belavin and Drinfeld. They proved, that the map s, up to a gauge transformation, is either

- an elliptic function,
- a rational function of exponentials or

- a rational function.

The associated r-matrix r is called elliptic, trigonometric or rational respectively. We refer to this result as the Belavin–Drinfeld (BD) trichotomy. Shortly later in [8], Belavin and Drinfeld showed that after certain transformations any r-matrix has the form (1.3).

Remark 1.2. — The transformations used in [8] are not uniquely determined and it is unclear whether they respect the type in the BD trichotomy. Therefore, the BD trichotomy of r-matrices which are not difference dependent remained ambiguous.

In this article, we use algebro-geometric methods in order to develop the structure theory of formal solutions of the GCYBE over any field $\Bbbk$ of characteristic 0. More precisely, we prove an algebraicity result for these objects which, if refined carefully in the skew-symmetric case for $\Bbbk = \mathbb{C}$, results in the BD trichotomy. In the process, we deduce two results in the theory of sheaves of (not-necessarily associative) algebras, which are of independent importance. Namely, the geometrization of lattices to sheaves of algebras and conditions for the local triviality of sheaves of algebras in the étale topology. The latter result is used to classify acyclic sheaves of Lie algebras on one-dimensional connected complex algebraic groups with constant fibers.

Remark 1.3. — The geometric approach developed in the paper does not distinguish difference dependent solutions of the GCYBE from arbitrary ones. Therefore, we do not encounter problems like the one described in Remark 1.2. In other words, the geometric version of the BD trichotomy derived in this work holds unambiguously without restricting to r-matrices of the form (1.3).

Moreover, non-skew-symmetric generalized r-matrices are not necessarily of the form (1.3), even up to equivalence. Therefore, it is unexpected that the above-mentioned algebraicity result for non-skew-symmetric generalized r-matrices can be deduced using the methods from [9], even for $\Bbbk = \mathbb{C}$.

The algebro-geometric methods introduced in this paper have been successfully applied to other structure theoretic problems in subsequent works [1, 4]. Moreover, the universal geometrization procedure for generalized r-matrices, which forms the basis of these methods, is precisely the inverse to the geometric construction of generalized r-matrices from [11, 15]. As such, it unifies the previous partial geometrization procedures from [2, 11, 12, 47].

Let us point out that solutions of the GCYBE over other fields than $\mathbb{C}$ are also interesting. For example, in the theory of integrable systems (resp. quantum groups), formal generalized r-matrices over $\mathbb{R}$ (resp. $\mathbb{C}((\hbar))$) are important; see e.g. [7] (resp. [32]). The structure theory established in this paper is applicable to these cases. Note that the methods used in [9] are complex analytic and hence cannot be extended to other fields in any obvious manner.

Results

Let $\mathfrak{g}$ be a central simple Lie algebra over a field $\Bbbk$ of characteristic 0. We want to study (normalized) formal generalized r-matrices r, i.e. formal solutions of the GCYBE (1.1) of the form

$$r(x,y) = \frac{\gamma}{x-y} + r_0(x,y), \tag{1.4}$$

where $r_0 \in (\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$ and $\gamma \in \mathfrak{g}\otimes\mathfrak{g}$ is the Casimir element. Let us point out that every non-degenerate complex analytic solution of the GCYBE is of this form for $\Bbbk = \mathbb{C}$ up to equivalence; see Remark A.12. The starting point of our study is the observation that such r correspond precisely to subalgebras $\mathfrak{g}(r) \subseteq \mathfrak{g}((z))$ satisfying

$$\mathfrak{g}[\![z]\!] \oplus \mathfrak{g}(r) = \mathfrak{g}((z)). \tag{1.5}$$

The results of [44] imply that for any unital subalgebra

$$O \subseteq \{f \in \Bbbk((z)) \mid f\mathfrak{g}(r) \subseteq \mathfrak{g}(r)\} \tag{1.6}$$

of finite codimension, the affine scheme $\mathrm{Spec}(O)$, whose algebra of global regular functions is O, is an irreducible affine algebraic curve. The Lie algebra $\mathfrak{g}(r)$ can then be naturally understood as sheaf of Lie algebras on $\mathrm{Spec}(O)$.

It turns out that the affine curve $\mathrm{Spec}(O)$ admits a natural compactification by a $\Bbbk$-rational smooth point p to an irreducible projective curve X. Moreover, $\mathfrak{g}(r)$ defines a torsion-free coherent sheaf $\mathcal{A}$ on X and the pair $(X,\mathcal{A})$ comes with formal trivializations

$$c\colon \widehat{\mathcal{O}}_{X,p} \xrightarrow{\cong} \Bbbk[\![z]\!] \quad\text{and}\quad \zeta\colon \widehat{\mathcal{A}}_p \xrightarrow{\cong} \mathfrak{g}[\![z]\!].$$

The geometrization, i.e. the assignment

$$\mathfrak{g}(r) \longmapsto ((X,\mathcal{A}),(p,c,\zeta)), \tag{1.7}$$

is a special instance of the general geometrization procedure Theorem 3.16 for so-called A-lattices, where A is any finite-dimensional not-necessarily associative central simple $\Bbbk$-algebra. The property (1.5) implies that $\mathrm{H}^0(\mathcal{A}) = 0 = \mathrm{H}^1(\mathcal{A})$ and we call $((X,\mathcal{A}),(p,c,\zeta))$ a geometric GCYBE datum.

Until now, the choice of a unital subalgebra O of finite codimension in (1.6) was arbitrary. In Theorem 4.10, we show that if r is a formal r-matrix, i.e. a skew-symmetric formal generalized r-matrix, O can be chosen in such a way that

$$O = \Bbbk[x, y]/\left(y^2 - x^3 - ax - b\right) \tag{1.8}$$

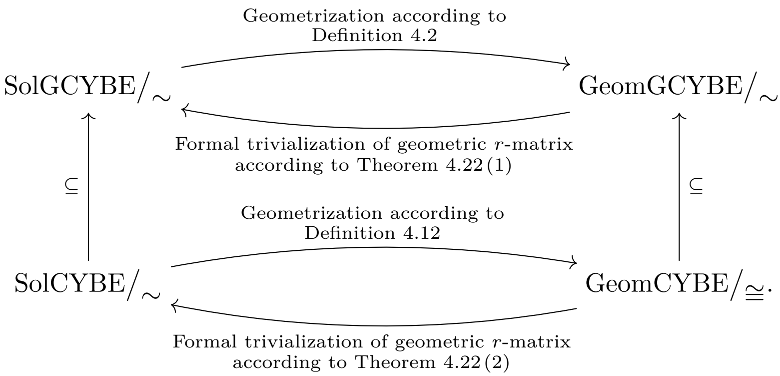
for appropriate $a, b \in \Bbbk$. In this case:

- X is an irreducible cubic plane curve;
- $\mathcal{A}$ is étale $\mathfrak{g}$-locally free on the subset $\breve{X}$ of smooth points of X and admits a global geometric analog of the Killing form.

We call a geometric GCYBE datum with these two additional properties a geometric CYBE datum.

Geometric GCYBE (resp. CYBE) data are generalizations of the data used in [11] in the geometric construction of solutions of the GCYBE (resp. CYBE). In fact, following [11] we can construct the so-called geometric r-matrix from any geometric GCYBE (resp. CYBE) datum, which is a specific rational section satisfying a geometric version of the GCYBE (resp. CYBE). In Theorem 4.22, we prove that we can recover the solution r from its geometric GCYBE datum $((X, \mathcal{A}), (p, c, \zeta))$ by trivializing the associated geometric r-matrix at the point (p, p) with respect to (c, ζ). In summary, we have the following correspondence.

Main Theorem A (Algebraic geometry of the GCYBE). — *Let us denote by* SolGCYBE *(resp.* SolCYBE*) the set of formal generalized r-matrices (resp. formal r-matrices) and by* GeomGCYBE *(resp.* GeomCYBE*) the set of geometric GCYBE (resp. geometric CYBE) data. The following diagram commutes:*



Here, the horizontal maps are mutually inverse bijections and the "$\sim$" in the left column denote an appropriate equivalence of formal generalized r-matrices, while "$\sim$" (resp. "$\cong$") in the right column denotes an appropriate equivalence (resp. isomorphism) of geometric GCYBE data.

In Theorem 3.10 and Theorem 3.11 we deduce general conditions for the étale local triviality of a sheaf of algebras. According to these conditions, the sheaf $\mathcal{A}$ is étale $\mathfrak{g}$-locally trivial at p, i.e. there exists an étale morphism $f\colon Y \to X$ such that

$$p \in f(Y) \quad \text{and} \quad f^*\mathcal{A} \cong \mathfrak{g} \otimes \mathcal{O}_Y.$$

Let S be the smooth projectification of Y. Then the pull-back of the geometric r-matrix ρ of $((X,\mathcal{A}),(p,c,\zeta))$ can be identified with a rational section of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathcal{O}_{S\times S}$. Combining this with the fact that r is a formal trivialization of ρ, we obtain the following result; see Theorem 4.27.

Main Theorem B (Algebraicity of formal generalized r-matrices). — *Every formal generalized r-matrix is equivalent to a formal trivialization of a rational section of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathcal{O}_{S\times S}$ for some irreducible smooth projective curve S over $\Bbbk$.*

By this point, we know how to associate to any formal r-matrix r an irreducible cubic plane curve X. This curve is defined by constants $a, b \in \Bbbk$ via (1.8). Moreover, X

- is smooth and hence elliptic when $4a^3 + 27b^2 \neq 0$,
- has a nodal singularity for $4a^3 = -27b^2 \neq 0$ or
- a cuspidal singularity if $a = b = 0$.

This yields a geometric trichotomy for formal r-matrices by the type of the corresponding curve.

Consider the special case $\Bbbk = \mathbb{C}$. We have already mentioned that the sheaf $\mathcal{A}$ associated to r is étale $\mathfrak{g}$-locally free on the subset $\breve{X}$ of smooth points of X. When X is elliptic we have $X = \breve{X}$. If X has a nodal (resp. cuspidal) singularity, the locus $\breve{X}$ is isomorphic to the punctured complex plane (resp. the complex plane). All possibilities for $\mathcal{A}|_{\breve{X}}$ are determined in Theorem 3.10 and Theorem 3.28. Using this classification results, we prove in Theorem 4.32 that the geometric trichotomy implies a stronger version of the BD trichotomy.

Main Theorem C (Geometric version of the BD trichotomy). — *Every formal r-matrix is associated to an irreducible cubic plane curve X and the type of X (i.e. elliptic, nodal, cuspidal) is invariant under equivalence.*

Furthermore, if $\Bbbk = \mathbb{C}$, r *is equivalent to the Taylor series expansion of a meromorphic function* $\mathbb{C} \times \mathbb{C} \to \mathfrak{g} \otimes \mathfrak{g}$ *which is*

- *elliptic in both variables if and only if* X *is elliptic,*
- *a rational function of exponentials if and only if* X *is nodal, and*
- *rational if and only if* X *is cuspidal.*

In Proposition A.13, we show that the equivalence classes of formal generalized r-matrices over $\mathbb{C}$ are in bijection with equivalence classes of nondegenerate analytic generalized r-matrices. In particular, all the results we prove for formal generalized r-matrices are valid for complex analytic ones. A similar result holds true over the field of real numbers.

Structure

In Section 2, we introduce the formal (generalized) r-matrices and the equivalence relations between them. Furthermore, we discuss the construction $r \mapsto \mathfrak{g}(r)$ in this formal setting.

In Section 3, we collect general results in the theory of sheaves of algebras, such as the aforementioned geometrization of A-lattices (see Subsection 3.3) and conditions for the étale local triviality for sheaves of algebras (see Subsection 3.2). Moreover, we deduce the classification of acyclic sheaves of Lie algebras with constant fiber $\mathfrak{g}$ over one-dimensional connected affine algebraic groups in Subsection 3.4.

In Section 4, we apply the results from Section 3 to formal generalized r-matrices in order to obtain the correspondence presented in Theorem A. We obtain Theorem B in Subsection 4.5. The main body of this paper is concluded in Subsection 4.7 with the derivation of the BD trichotomy Theorem C.

In Section A, we compare the formal and analytic theories of the GCYBE (1.1) for $\Bbbk = \mathbb{R}$ and $\Bbbk = \mathbb{C}$. Finally, we give an overview over our notation in Section B.

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2. Survey on formal generalized r-matrices

Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra over a field $\Bbbk$ of characteristic 0, $\{b_i\}_{i=1}^d \subseteq \mathfrak{g}$ be a basis, and $\{b_i^*\}_{i=1}^d \subseteq \mathfrak{g}$ be its dual basis with respect to the Killing form κ of $\mathfrak{g}$. In particular, $\kappa(b_i^*, b_j) = \delta_{ij}$ and the *Casimir element* $\gamma = \sum_{i=1}^d b_i^* \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$ is symmetric and $\mathfrak{g}$-invariant.

2.1. Formal generalized r-matrices

Consider $(x-y)^{-1}$ as the series $\sum_{k=0}^\infty x^{-k-1} y^k \in \Bbbk((x))[\![y]\!]$. Then

$$r_{\mathrm{Yang}}(x,y) := \frac{\gamma}{x-y} = \sum_{k=0}^{\infty}\sum_{i=1}^{d} x^{-k-1} b_i^* \otimes y^k b_i, \tag{2.1}$$

is an element of $(\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!] \cong (\mathfrak{g}((x))\otimes\mathfrak{g})[\![y]\!] \cong (\mathfrak{g}\otimes\mathfrak{g})\otimes\Bbbk((x))[\![y]\!]$. It is the formal version of Yang's r-matrix mentioned in the introduction.

Definition 2.1. — *A series $r \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ is said to be in* standard form *if*

$$r(x,y) = \frac{\lambda(y)}{x-y}\gamma + r_0(x,y) = \lambda(y) r_{\mathrm{Yang}}(x,y) + r_0(x,y) \tag{2.2}$$

for some $\lambda \in \Bbbk[\![z]\!]^\times$ and $r_0 \in (\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$. In this case

- *r is called* normalized *if $\lambda = 1$,*
- *$\overline{r}(x,y) := \lambda(x) r_{\mathrm{Yang}}(x,y) - \tau(r_0(y,x)) \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$, where τ is the $\Bbbk[\![x,y]\!]$-linear extension of the linear automorphism of $\mathfrak{g}\otimes\mathfrak{g}$ defined by $a\otimes b \mapsto b\otimes a$, and*
- *r is called* skew-symmetric *if $\overline{r} = r$.*

Remark 2.2. — For a general $r = r(x,y) \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ there is no appropriate $\overline{r}$ since switching x and y does not define a map of $\Bbbk((x))[\![y]\!]$ into itself. Therefore, a well-defined notion of skew-symmetry in this context needs additional assumptions, e.g. that r is in standard form.

Lemma 2.3. — *Let V be a $\Bbbk$-vector space and $f(x,y) \in V[\![x,y]\!]$ satisfy $f(z,z) = 0$ in $V[\![z]\!]$. Then f is divisible by $x-y$, i.e. $f(x,y) = (x-y)g(x,y)$ for some $g \in V[\![x,y]\!]$. Furthermore, if $f(x,y) = h(x) - h(y)$ for some $h \in \Bbbk[\![z]\!]$, the identity $g(z,z) = h'(z)$ holds.*

Proof. — Let $V[x,y]_\ell$ denote the homogeneous elements of total degree ℓ in $\Bbbk[x,y]$ for all $\ell \in \mathbb{N}_0$. Then $f = \sum_{\ell=0}^\infty f_\ell$, where $f_\ell \in V[x,y]_\ell$. Since $x-y$ is homogeneous of total degree one, it suffices to prove the claim for

$f = f_\ell \in V[x,y]_\ell$. The polynomial $x - y \in \Bbbk[x,y] = \Bbbk[y][x]$ is monic, so the polynomial division algorithm provides $g \in V[x,y]$ and $r \in V[y]$ such that

$$f(x,y) = (x-y)g(x,y) + r(y). \tag{2.3}$$

Therefore, $0 = f(z,z) = r(z)$ proves $f(x,y) = (x-y)g(x,y)$. Note that in the special case $f(x,y) = h(x) - h(y)$ for $h(z) = az^\ell$, a direct calculation verifies that

$$g(x,y) = a\sum_{k=0}^{\ell-1} x^k y^{\ell-1-k}. \tag{2.4}$$

Hence, $g(z,z) = \ell a z^{\ell-1} = h'(z)$ in this case, proving the second part of the statement. □

Remark 2.4. — Let $r = r(x,y) = f(x,y)r_{\mathrm{Yang}}(x,y) + r_0(x,y) \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ for some $f \in \Bbbk[\![x,y]\!]^\times$ and $r_0 \in (\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$. Then

$$r(x,y) = f(y,y)r_{\mathrm{Yang}}(x,y) + \frac{f(x,y)-f(y,y)}{x-y}\gamma + r_0(x,y) \tag{2.5}$$

combined with Lemma 2.3 shows that r is in standard form. In particular, $\bar{r}$ is also in standard form.

Notation 2.5. — Let $ij \in \{12,13,23\}$, $\mathrm{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and $R \coloneqq \Bbbk((x_1))((x_2))[\![x_3]\!]$. For $t = t_1 \otimes t_2 \in \mathfrak{g}\otimes\mathfrak{g}$ the assignments

$$t^{12} = t \otimes 1, \quad t^{13} = t_1 \otimes 1 \otimes t_2 \quad \text{and} \quad t^{23} = 1 \otimes t$$

define linear maps $(\cdot)^{ij}\colon \mathfrak{g}\otimes\mathfrak{g} \to \mathrm{U}(\mathfrak{g})\otimes\mathrm{U}(\mathfrak{g})\otimes\mathrm{U}(\mathfrak{g})$. The image of $s \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!] \cong (\mathfrak{g}\otimes\mathfrak{g})((x_i))[\![x_j]\!] \subseteq (\mathfrak{g}\otimes\mathfrak{g})\otimes R$ under

$$(\mathfrak{g}\otimes\mathfrak{g})\otimes R \xrightarrow{(\cdot)^{ij}\otimes \mathrm{id}_R} (\mathrm{U}(\mathfrak{g})\otimes\mathrm{U}(\mathfrak{g})\otimes\mathrm{U}(\mathfrak{g}))\otimes R$$

is also denoted by s^{ij}.

Definition 2.6. — *A series $r \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ is called* formal generalized r-matrix, *if it is in standard form and solves the* formal generalized classical Yang–Baxter equation

$$\mathrm{GCYB}(r) = 0, \text{ where } \mathrm{GCYB}(r) \coloneqq \left[r^{12}, r^{13}\right] + \left[r^{12}, r^{23}\right] + \left[r^{13}, \bar{r}^{23}\right]. \tag{2.6}$$

Here we used Notation 2.5 and note that the brackets in $\mathrm{GCYB}(r)$ *are the usual commutators in the associative R-algebra* $(\mathrm{U}(\mathfrak{g})\otimes\mathrm{U}(\mathfrak{g})\otimes\mathrm{U}(\mathfrak{g}))\otimes R$.

Example 2.7. — r_{Yang} is a skew-symmetric formal generalized r-matrix since

$$(2.7)\quad (x_1-x_2)(x_1-x_3)(x_2-x_3)\,\mathrm{GCYB}(r_{\mathrm{Yang}})(x_1,x_2,x_3) \\ = \big((x_2-x_3)-(x_1-x_3)+(x_1-x_2)\big)\left[\gamma^{12},\gamma^{13}\right]=0.$$

Here we used the fact that $[a\otimes 1+1\otimes a,\gamma]=0$ for all $a\in\mathfrak{g}$ implies that $[\gamma^{12},\gamma^{13}]=-[\gamma^{12},\gamma^{23}]=[\gamma^{13},\gamma^{23}]$ holds. It is easy to see that for all $\lambda\in\Bbbk[\![z]\!]$ and $\widetilde{r}(x,y):=\lambda(y)r_{\mathrm{Yang}}(x,y)$ we have:

$$(2.8)\quad \mathrm{GCYB}(\widetilde{r})(x_1,x_2,x_3)=\lambda(x_2)\lambda(x_3)\,\mathrm{GCYB}(r_{\mathrm{Yang}})(x_1,x_2,x_3)=0.$$

Therefore, $\widetilde{r}$ is a generalized r-matrix and this series is not skew-symmetric if $\lambda\notin\Bbbk^\times$.

Remark 2.8. — Defining $r^{32}(x_3,x_2):=-\bar{r}^{23}(x_2,x_3)$ in (2.6) results in the analog form of the GCYBE used in the introduction. Observe that e.g.

$$\left[(s_1\otimes s_2)^{13},(t_1\otimes t_2)^{23}\right]=s_1\otimes t_1\otimes s_2t_2-s_1\otimes t_1\otimes t_2s_2 \\ =s_1\otimes t_1\otimes[s_2,t_2]\in\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$$

for all $s_1,s_2,t_1,t_2\in\mathfrak{g}$. This and similar calculations show that $\mathrm{GCYB}(r)\in(\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})(\!(x_1)\!)(\!(x_2)\!)[\![x_3]\!]$ for every $r\in(\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$ in standard form. This can be further refined. Since $[a\otimes 1+1\otimes a,\gamma]=0$ for all $a\in\mathfrak{g}$, the same identity also holds for all $a\in\mathfrak{g}[\![z]\!]$. We can use this to derive e.g. that

$$(2.9)\quad \left[r_0^{12},\gamma^{13}\right]-\left[\gamma^{13},\tau(r_0)^{23}\right] \\ =\left[r_0^{12}+\tau(r_0)^{23},\gamma^{13}\right]\in(\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x_1,x_2,x_3]\!]$$

vanishes for $x_1=x_3$, where r_0 is determined by $r(x,y)=\lambda(y)r_{\mathrm{Yang}}(x,y)+r_0(x,y)$. Combining this and similar identities with Lemma 2.3 and the fact that $\lambda(y)r_{\mathrm{Yang}}(x,y)$ is a generalized r-matrix (see Example 2.7) implies that $\mathrm{GCYB}(r)\in(\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x_1,x_2,x_3]\!]$ for all $r\in(\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$ in standard form.

Remark 2.9. — Let $\Bbbk'$ be an arbitrary field extension of $\Bbbk$ and $r\in(\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$ be a formal generalized r-matrix. Then $\mathfrak{g}_{\Bbbk'}:=\mathfrak{g}\otimes\Bbbk'$ is a semi-simple Lie algebra over $\Bbbk'$ and the image $r_{\Bbbk'}$ of r under the canonical map

$$(\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]\longrightarrow(\mathfrak{g}_{\Bbbk'}\otimes_{\Bbbk'}\mathfrak{g}_{\Bbbk'})(\!(x)\!)[\![y]\!]$$

is again a formal generalized r-matrix.

Notation 2.10. — For $\varphi\in\mathrm{Aut}_{\Bbbk[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!])$ the image of φ under

$$\mathrm{Aut}_{\Bbbk[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!])\subseteq\mathrm{End}_{\Bbbk[\![z]\!]}(\mathfrak{g}[\![z]\!])\cong\mathrm{End}(\mathfrak{g})[\![z]\!]$$

is again denoted by $\varphi = \varphi(z)$. Furthermore,

$$\varphi \otimes \varphi \in \operatorname{End}(\mathfrak{g})[\![z]\!] \otimes_{\Bbbk[\![z]\!]} \operatorname{End}(\mathfrak{g})[\![z]\!] \cong (\operatorname{End}(\mathfrak{g}) \otimes \operatorname{End}(\mathfrak{g}))[\![z]\!]$$

is also denoted by $\varphi(z) \otimes \varphi(z)$ while $\varphi(x) \otimes \varphi(y)$ denotes the image of $\varphi \otimes \varphi \in \operatorname{End}(\mathfrak{g})[\![z]\!] \otimes \operatorname{End}(\mathfrak{g})[\![z]\!]$ under the canonical injection $\operatorname{End}(\mathfrak{g})[\![z]\!] \otimes \operatorname{End}(\mathfrak{g})[\![z]\!] \to (\operatorname{End}(\mathfrak{g}) \otimes \operatorname{End}(\mathfrak{g}))[\![x, y]\!]$.

Definition 2.11. — *A series* $\widetilde{r} \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$ *is called* equivalent *to* $r \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$ *if*

$$\widetilde{r}(x, y) = \mu(y)(\varphi(x) \otimes \varphi(y))r(w(x), w(y)), \tag{2.10}$$

where the triple (μ, w, φ) *is called an* equivalence *and consists of a series* $\mu \in \Bbbk[\![z]\!]^\times$ *called* rescaling, *a formal parameter* $w \in z\Bbbk[\![z]\!]^\times$ *called* coordinate transformation *and a map* $\varphi \in \operatorname{Aut}_{\Bbbk[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!])$ *called* gauge transformation. *Furthermore,* $\widetilde{r}$ *is called* gauge *(resp.* coordinate*)* equivalent *to* r *if* $\mu = 1$ *and* $w = z$ *(resp.* $\varphi = \operatorname{id}_{\mathfrak{g}[\![z]\!]}$*).*

Lemma 2.12. — *The following results are true.*

1. *Equivalences preserve the property of being a formal generalized r-matrix.*
2. *Equivalences with constant rescaling part preserve skew-symmetry.*
3. *Every series in standard form is coordinate equivalent to one in normalized standard form.*

Proof. — Let $r(x, y) = \lambda(y)r_{\text{Yang}}(x, y) + r_0(x, y)$ for $\mu \in \Bbbk[\![z]\!]^\times, r_0 \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$ and $\widetilde{r} \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$ be equivalent to r via an equivalence (μ, w, φ). Using Lemma 2.3 and $(\varphi(z) \otimes \varphi(z))\gamma = \gamma$ (see e.g. Remark 2.23 below), we can deduce that

$$(\varphi(x) \otimes \varphi(y))r_{\text{Yang}}(w(x), w(y)) - w'(y)^{-1}r_{\text{Yang}}(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!], \tag{2.11}$$

hence $\widetilde{r}$ is in standard form. It is easy to see that

$$\begin{aligned} &\operatorname{GCYB}(\widetilde{r})(x_1, x_2, x_3) \\ &= \mu(x_2)\mu(x_3)(\varphi(x_1) \otimes \varphi(x_2) \otimes \varphi(x_3)) \operatorname{GCYB}(r)(w(x_1), w(x_2), w(x_3)) \\ &= 0, \end{aligned}$$

since $\operatorname{GCYB}(r) = 0$. This proves (1). The proof of (2) is clear.

Let us put $\mu = 1$ and $\varphi = \operatorname{id}_{\mathfrak{g}[\![z]\!]}$. There exists a unique $u \in z\Bbbk[\![z]\!]^\times$ such that $u'(z) = \lambda(u(z))$ and setting $w = u$ proves (3) under consideration of (2.11). Indeed, if we write $u(z) = \sum_{k=1}^\infty u_k z^k$ and $\lambda(z) = \sum_{k=0}^\infty \lambda_k z^k$, it holds that $\lambda(u(z)) = \sum_{k=0}^{\infty} c_k z^k$, where $c_0 = \lambda_0$ and

$$c_k = \sum_{\ell=1}^{k} \sum_{\substack{(j_1,\dots,j_\ell)\in\mathbb{N}^\ell \\ j_1+\dots+j_\ell=k}} \lambda_\ell u_{j_1}\dots u_{j_\ell}. \tag{2.12}$$

for all $k \in \mathbb{N}$ (note that we exclude 0 from $\mathbb{N}$), hence the coefficients of u can be determined inductively by $u_{k+1} = \frac{c_k}{k+1}$ for all $k \in \mathbb{N}_0$. □

Remark 2.13. — The study of formal generalized r-matrices will be pursued up to equivalence, i.e. equivalent r-matrices will be treated as interchangeable, where in light of Lemma 2.12 (2) only equivalences with constant rescaling part will be used if skew-symmetry is relevant to the context. In particular, Lemma 2.12 (3) would permit us to restrict our attention to normalized generalized r-matrices. This was done in the introduction. Nevertheless, in the following this will be done only if necessary, since rescalings appear naturally in the algebro-geometric context; see e.g. Theorem 4.22 (1) below.

2.2. Lie subalgebras of $\mathfrak{g}((z))$ complementary to $\mathfrak{g}[\![z]\!]$

It was noted in e.g. [14, 51] that it is possible to assign a Lie subalgebra of $\mathfrak{g}((z))$ complementary to $\mathfrak{g}[\![z]\!]$ to each non-degenerated solution of GCYBE (1.1). In the following, we will discuss the formal analog of this construction. For a series

$$s = s(x, y) = \sum_{k=0}^{\infty}\sum_{i=1}^{d} s_{k,i}(x) \otimes b_i y^k \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$$

it is always possible to define the vector subspace

$$\mathfrak{g}(s) := \mathrm{Span}_{\Bbbk}\big\{s_{k,i}(z) \;\big|\; k \in \mathbb{N}_0, i \in \{1,\dots,d\}\big\} \tag{2.13}$$

of $\mathfrak{g}((z))$. It is the smallest subspace of $\mathfrak{g}((z))$ satisfying $s \in (\mathfrak{g}(s)\otimes\mathfrak{g})[\![y]\!]$ and does not depend on the choice of basis $\{b_i\}_{i=1}^d$.

Proposition 2.14. — *The assignment $r \mapsto \mathfrak{g}(r)$ gives a bijection between normalized formal generalized r-matrices and Lie subalgebras $W \subseteq \mathfrak{g}((z))$ satisfying $\mathfrak{g}((z)) = \mathfrak{g}[\![z]\!] \oplus W$.*

Proof. — Let $r(x, y) = \sum_{k=0}^{\infty}\sum_{i=1}^{d} r_{k,i}(x) \otimes y^k b_i$ be a normalized formal generalized r-matrix. The identity $\mathfrak{g}((z)) = \mathfrak{g}[\![z]\!] \oplus \mathfrak{g}(r)$ is a direct consequence of $r_{k,i} - z^{-k-1}b_i^* \in \mathfrak{g}[\![z]\!]$ for all $k \in \mathbb{N}_0, i \in \{1,\dots,d\}$. It remains to show that $\mathfrak{g}(r)$ is a subalgebra of $\mathfrak{g}((z))$. The fact that $[a\otimes 1+1\otimes a,\gamma]=0$ holds for all $a\in\mathfrak{g}$ forces

$$\begin{aligned}(2.14)\quad &\left[r^{12}+r^{13},\gamma^{23}\right](x_1,x_2,x_3)\\ &=\sum_{k=0}^{\infty}\sum_{i=1}^{d} r_{k,i}(x_1)\otimes\left[x_2^kb_i\otimes 1+1\otimes x_3^kb_i,\gamma\right]\in(\mathfrak{g}(r)\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x_2,x_3]\!]\end{aligned}$$

to vanish for $x_2=x_3$. Therefore, Lemma 2.3 implies that

$$\left[r^{12},r^{23}\right]+\left[r^{13},\bar{r}^{23}\right]\in(\mathfrak{g}(r)\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x_2,x_3]\!]. \tag{2.15}$$

This combined with $0=\mathrm{GCYB}(r)=[r^{12},r^{13}]+[r^{12},r^{23}]+[r^{13},\bar{r}^{23}]$ shows that

$$\sum_{k,\ell=0}^{\infty}\sum_{i,j=1}^{d}[r_{k,i}(x_1),r_{\ell,j}(x_1)]\otimes x_2^kb_i\otimes x_3^\ell b_j=\left[r^{12},r^{13}\right](x_1,x_2,x_3) \tag{2.16}$$

is an element of $(\mathfrak{g}(r)\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x_2,x_3]\!]$. We conclude that $[r_{k,i},r_{\ell,j}]\in\mathfrak{g}(r)$ for all $k,\ell\in\mathbb{N}_0, i,j\in\{1,\dots,d\}$. In particular, $\mathfrak{g}(r)$ is a subalgebra of $\mathfrak{g}((z))$.

Let us consider now a Lie subalgebra $W\subset\mathfrak{g}((z))$ satisfying $\mathfrak{g}((z))=\mathfrak{g}[\![z]\!]\oplus W$. For every $k\in\mathbb{N}_0, i\in\{1,\dots,d\}$ there is an unique element $r_{k,i}^W\in W$ such that $r_{k,i}^W-b_i^*z^{-k-1}\in\mathfrak{g}[\![z]\!]$. By construction,

$$r^W=r^W(x,y):=\sum_{k=0}^{\infty}\sum_{i=1}^{d}r_{k,i}^W(x)\otimes b_iy^k\in(\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!] \tag{2.17}$$

is in normalized standard form and satisfies $\mathfrak{g}(r^W)=W$. Furthermore, we can see that $r^{\mathfrak{g}(r)}=r$. Thus, it remains to show that $\mathrm{GCYB}(r^W)=0$. In Remark 2.8 it was noted that $\mathrm{GCYB}(r^W)\in(\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x_1,x_2,x_3]\!]$. Since $\mathfrak{g}(r^W)=W$ is closed under the Lie bracket, (2.15) and (2.16) show that $\mathrm{GCYB}(r^W)\in(W\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x_2,x_3]\!]$. Summarized, we obtain

$$\mathrm{GCYB}(r^W)\in(W\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x_2,x_3]\!]\cap(\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x_1,x_2,x_3]\!]=\{0\}, \tag{2.18}$$

since $\mathfrak{g}[\![z]\!]\cap W=\{0\}$, concluding the proof. □

Remark 2.15. — There are different methods to assign Lie algebras to certain classes of r-matrices, which should not be confused with the universal method described here. For example, in [33] the authors assign subalgebras of $\mathfrak{g}((z^{-1}))\times\mathfrak{g}$ to so-called quasi-trigonometric r-matrices and in [2] trigonometric solutions of the CYBE (1.2) are related to subalgebras of $\mathfrak{L}\times\mathfrak{L}$, where $\mathfrak{L}$ is a twisted loop algebra.

LEMMA 2.16. — *Let r be a normalized formal generalized r-matrix. Then $\mathfrak{g}(r)$ is generated as a Lie algebra by $\mathfrak{g}(r)\cap z^{-1}\mathfrak{g}[\![z]\!]=\{(1\otimes\alpha)r(z,0)\mid \alpha\in\mathfrak{g}^*\}$.*

Proof. — Let W be the Lie subalgebra of $\mathfrak{g}(r)$ generated by $\mathfrak{g}(r)\cap z^{-1}\mathfrak{g}[\![z]\!]$ and assume that for some $m \in \mathbb{N}$ we have $\mathfrak{g}(r) \cap z^{-m}\mathfrak{g}[\![z]\!] \subseteq W$. For every pair $a_1, a_2 \in \mathfrak{g}$ exist unique $\widetilde{a}_1, \widetilde{a}_2 \in W$ and $s \in \mathfrak{g}(r)$ such that

$$\widetilde{a}_1(z) - a_1 z^{-1}, \widetilde{a}_1(z) - a_2 z^{-m}, s - [a_1, a_2]z^{-m-1} \in \mathfrak{g}[\![z]\!]. \tag{2.19}$$

Since $[\widetilde{a}_1, \widetilde{a}_2] \in W$ and $s - [\widetilde{a}_1, \widetilde{a}_2] \in \mathfrak{g}(r) \cap z^{-m}\mathfrak{g}[\![z]\!] \subseteq W$, we see that $s \in W$. Therefore, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ implies that $\mathfrak{g}(r) \cap z^{-m-1}\mathfrak{g}[\![z]\!] \subseteq W$ and $W = \mathfrak{g}(r)$ is verified by induction on m. □

Lemma 2.17. — *Let $\widetilde{r} \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ be equivalent to a formal generalized r-matrix r via an equivalence (μ, w, φ). Then $\mathfrak{g}(\widetilde{r})$ is the image of $\mathfrak{g}(r)$ under the map $\varphi_w \in \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g}((z)))$ defined by $a(z) \mapsto \varphi(z)a(w(z))$.*

Proof. — First note that for any $s(x,y) = \sum_{k=0}^{\infty}\sum_{i=1}^{d} s_{k,i}(x) \otimes b_i y^k \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ and $\lambda(z) = \sum_{k=0}^{\infty} \lambda_k z^k \in \Bbbk[\![z]\!]^\times$ we have

$$\widetilde{s}(x,y) := \lambda(y)s(x,y) = \sum_{k=0}^{\infty}\sum_{i=1}^{d}\left(\sum_{\ell=0}^{k} \lambda_\ell s_{k-\ell,i}(x)\right) \otimes b_i y^k$$

and hence $\mathfrak{g}(\widetilde{s}) = \mathfrak{g}(s)$. Therefore, we may assume that r is normalized and $\mu(z) = w'(z)$. Then $\widetilde{r}$ is also a normalized generalized r-matrix, as can be seen in the proof of Lemma 2.12.

Since $\varphi_w \colon \mathfrak{g}((z)) \to \mathfrak{g}((z))$ defined by $a(z) \mapsto \varphi(z)a(w(z))$ is a $\Bbbk$-linear automorphism of Lie algebras and $\mathfrak{g}(r)\cap z^{-1}\mathfrak{g}[\![z]\!] = \{(1\otimes\alpha)r(z,0) \mid \alpha \in \mathfrak{g}^*\}$ generates $\mathfrak{g}(r)$ by Lemma 2.16, $\varphi_w(\mathfrak{g}(r))$ is generated by $\varphi_w(\mathfrak{g}(r)\cap z^{-1}\mathfrak{g}[\![z]\!])$. We have

$$\begin{aligned}\varphi_w((1 \otimes \alpha)r(z,0)) &= (\varphi(z) \otimes \alpha)r(w(z),0)\\ &= \left(1 \otimes \left(\mu(0)^{-1}\alpha\varphi(0)^{-1}\right)\right)\widetilde{r}(z,0),\end{aligned}$$

where $w(0) = 0$, $\mu(0) \in \Bbbk^\times$ and $\varphi(0) \in \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})$ was used. Since $\alpha \mapsto \mu(0)^{-1}\alpha\phi(0)^{-1}$ defines an linear automorphism of $\mathfrak{g}^*$, we see that

$$\varphi_w(\mathfrak{g}(r) \cap z^{-1}\mathfrak{g}[\![z]\!]) = \mathfrak{g}(\widetilde{r}) \cap z^{-1}\mathfrak{g}[\![z]\!]$$

and hence $\phi(\mathfrak{g}(r)) = \mathfrak{g}(\widetilde{r})$ by applying Lemma 2.16. □

Remark 2.18. — Lemma 2.17 implies that for a formal generalized r-matrix $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$, $\lambda \in \Bbbk[\![z]\!]^\times$ and $\widetilde{r}(x,y) = \lambda(y)r(x,y)$, we have $\mathfrak{g}(r) = \mathfrak{g}(\widetilde{r})$. In particular, $\mathfrak{g}(r)$ is a Lie subalgebra of $\mathfrak{g}((z))$ complementary to $\mathfrak{g}[\![z]\!]$ for all not necessarily normalized formal generalized r-matrices r by virtue of Proposition 2.14.

Example 2.19. — It is easy to see that $\mathfrak{g}(r_{\text{Yang}}) = z^{-1}\mathfrak{g}[z^{-1}]$. This subalgebra is stable under multiplication by z^{-1}. More generally, a subalgebra $W \subseteq \mathfrak{g}((z))$ satisfying $\mathfrak{g}((z)) = \mathfrak{g}[\![z]\!] \oplus W$ is called *homogeneous* if $z^{-1}W \subseteq W$. Such a subalgebra is automatically a deformation of $\mathfrak{g}(r_{\text{Yang}})$ in the sense that there exists $A \in \text{End}(\mathfrak{g})[\![z]\!]$ such that $A(0) = \text{id}_{\mathfrak{g}}$ and

$$W = A\mathfrak{g}(r_{\text{Yang}}) = \text{Span}_{\Bbbk}\left\{z^{-k-1}A(z)b_i^* \,\middle|\, k \in \mathbb{N}_0, i \in \{1,\dots,d\}\right\}.$$

The series A is thereby uniquely determined by $z^{-1}A(z)b_i^* \in W$ for all $i \in \{1,\dots,d\}$. The condition on A in order for W to be a Lie algebra is examined in [22] and turns out to be related to the notion of *compatible Lie brackets*.

Let r be the normalized formal generalized r-matrix such that $W = \mathfrak{g}(r)$. Since $\bar{r}$ is also in normalized standard form, we can see that $\mathfrak{g}((z)) = \mathfrak{g}[\![z]\!] \oplus \mathfrak{g}(\bar{r})$, hence there also exists a unique invertible series $\overline{A} \in \text{End}(\mathfrak{g})[\![z]\!]$ such that $\overline{A}(0) = \text{id}_{\mathfrak{g}}$ and $\mathfrak{g}(\bar{r}) = \overline{A}\mathfrak{g}(r_{\text{Yang}})$. We will use this in Example 2.25 below to show that the formula

$$r(x,y) = \frac{A(x) \otimes \overline{A}(y)}{x-y}\gamma, \tag{2.20}$$

from [50] holds, where $A(x) \otimes \overline{A}(y)$ is viewed as an element of $(\text{End}(\mathfrak{g}) \otimes \text{End}(\mathfrak{g}))[\![x,y]\!]$. In particular, r is skew-symmetric if and only if $A = \overline{A}$.

2.3. Skew-symmetry and formal r-matrices

This section is dedicated to the study of skew-symmetric formal generalized r-matrices. These obviously satisfy a formal version of the classical Yang–Baxter equation (1.2) and in fact turn out to be exactly solutions of this equation.

Definition 2.20. — *A series* $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ *is called* formal r-matrix *if it is in standard form and solves the* formal classical Yang–Baxter equation

$$\text{CYB}(r) = 0, \text{ where } \text{CYB}(r) := \left[r^{12}, r^{13}\right] + \left[r^{12}, r^{23}\right] + \left[r^{13}, r^{23}\right]. \tag{2.21}$$

Here Notation 2.5 were used and the brackets in $\text{CYB}(r)$ *are the usual commutators in the associative R-algebra* $(\text{U}(\mathfrak{g}) \otimes \text{U}(\mathfrak{g}) \otimes \text{U}(\mathfrak{g})) \otimes R$.

Proposition 2.21. — *A series $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ is a formal r-matrix if and only if it is a skew-symmetric formal generalized r-matrix.*

Proof. — It is obvious that a formal generalized r-matrix r solves the formal CYBE (2.21) if $\bar{r} = r$, so we have to prove the contrary, i.e. that each formal r-matrix solves the formal GCYBE (2.6) and satisfies $\bar{r} = r$. The equations (2.15), where the $\bar{r}$ is replaced by r, and (2.16) imply that $\mathfrak{g}(r) \subseteq \mathfrak{g}((z))$ is a Lie subalgebra since $\mathrm{CYB}(r) = 0$. Therefore, Proposition 2.14 states that $\mathrm{GCYB}(r) = 0$. In particular, we have:

$$0 = \mathrm{CYB}(r) - \mathrm{GCYB}(r) = \left[r^{13}, r^{23} - \bar{r}^{23}\right]. \tag{2.22}$$

Multiplying (2.22) with $x_1 - x_3$, setting $x_1 = x_3$ and probably multiplying with an element of $\Bbbk[\![x_3]\!]^\times$ results in $[\gamma^{13}, r^{23} - \bar{r}^{23}] = 0$. Application of the map $a_1 \otimes a_2 \otimes a_3 \mapsto a_2 \otimes [a_1, a_3]$ gives the desired $\bar{r} = r$. Here we used the following fact:

(2.23) The image of γ under $\mu\colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ defined by $a_1 \otimes a_2 \mapsto \mathrm{add}(a_1)\,\mathrm{add}(a_2)$ is $\mathrm{id}_{\mathfrak{g}}$.

Indeed, if $\mathfrak{g} = \bigoplus_{i=1}^k \mathfrak{g}_i$ is the decomposition of $\mathfrak{g}$ into simple ideals, we have $\gamma = \sum_{i=1}^k \gamma_i$, where γ_i is the Casimir element of $\mathfrak{g}_i$, so we may assume that $\mathfrak{g}$ is simple. Furthermore, an element $f \in \mathrm{End}(\mathfrak{g})$ with the property $f \otimes \mathrm{id}_{\bar{\Bbbk}} = \mathrm{id}_{\mathfrak{g}\otimes\bar{\Bbbk}}$, where $\bar{\Bbbk}$ is the algebraic closure of $\Bbbk$, already satisfies $f = \mathrm{id}_{\mathfrak{g}}$, hence we may assume $\Bbbk = \bar{\Bbbk}$. The endomorphism $\mu(\gamma) = \sum_{i=1}^d \mathrm{add}(b_i^*)\,\mathrm{add}(b_i)$ is the quadratic Casimir operator of the adjoint representation, which is a multiple of the identity due to Schur's Lemma and equals the identity since $\mathrm{Tr}(\mathrm{id}_{\mathfrak{g}}) = d = \sum_{i=1}^d \kappa(b_i^*, b_i) = \mathrm{Tr}(\mu(\gamma))$. □

Notation 2.22. — Let us denote the $\Bbbk((z))$-bilinear extension $\mathfrak{g}((z)) \times \mathfrak{g}((z)) \to \Bbbk((z))$ of the Killing form κ of $\mathfrak{g}$ with the same symbol. Then $\mathfrak{g}((z))$ is equipped with the $\Bbbk$-bilinear form κ_0 defined by

$$\kappa_0(s, t) := \mathrm{res}_0\, \kappa(s, t)\mathrm{d}z = \sum_{k+\ell=-1} \kappa(s_k, t_\ell) \tag{2.24}$$

for all $s = \sum_{k \gg -\infty} s_k z^k, t = \sum_{k \gg -\infty} t_k z^k \in \mathfrak{g}((z))$, where $\mathrm{res}_0\, f\mathrm{d}z = f_{-1}$ for any series $f = \sum_{k \gg -\infty} f_k z^k \in \Bbbk((z))$.

Remark 2.23. — The $\Bbbk((z))$-bilinear extension of κ is the Killing form of $\mathfrak{g}((z))$ as a Lie algebra over $\Bbbk((z))$. Therefore, γ can also be understood as the Casimir element of $\mathfrak{g}((z))$. In particular, $[a \otimes 1 + 1 \otimes a, \gamma] = 0$ for all $a \in \mathfrak{g}((z))$ and $(\varphi(z) \otimes \varphi(z))\gamma = \gamma$ for all

$$\varphi \in \mathrm{Aut}_{\Bbbk[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!]) \subseteq \mathrm{Aut}_{\Bbbk((z))\text{-alg}}(\mathfrak{g}((z))). \tag{2.25}$$

Moreover, κ_0 is symmetric, non-degenerate, and invariant. Recall that "invariant" means that the identity $\kappa_0([a,b],c) = \kappa_0(a,[b,c])$ holds for all $a,b,c \in \mathfrak{g}(\!(z)\!)$.

Lemma 2.24. — *Let r be a normalized formal generalized r-matrix. Then $\mathfrak{g}(r)^\perp = \mathfrak{g}(\bar{r})$ and r is skew-symmetric if and only if $\mathfrak{g}(r)^\perp = \mathfrak{g}(r)$.*

Proof. — Let us write

$$r(x,y) = \sum_{k=0}^\infty \sum_{i=1}^d r_{k,i}(x) \otimes y^k b_i \text{ and } \bar{r}(x,y) = \sum_{k=0}^\infty \sum_{i=1}^d \bar{r}_{k,i}(x) \otimes y^k b_i^* \tag{2.26}$$

as well as $r_{k,i}(z) = z^{-k-1}b_i^* + \sum_{\ell=0}^\infty \sum_{j=1}^d r_{k,i}^{\ell,j} z^\ell b_j^*$. Then we have

$$r(x,y) - r_{\mathrm{Yang}}(x,y) = \sum_{k,\ell=0}^\infty \sum_{i,j=1}^d r_{k,i}^{\ell,j} x^\ell b_j^* \otimes y^k b_i \tag{2.27}$$

and hence $\bar{r}_{\ell,j}(z) = z^{-\ell-1}b_j - \sum_{k=0}^\infty \sum_{i=1}^d r_{k,i}^{\ell,j} z^k b_i$. Therefore, we can deduce

$$\kappa_0(r_{k,i}, \bar{r}_{\ell,j}) = r_{k,i}^{\ell,j} - r_{k,i}^{\ell,j} = 0. \tag{2.28}$$

This implies that $\mathfrak{g}(\bar{r}) \subseteq \mathfrak{g}(r)^\perp$. Now $0 = \mathfrak{g}[\![z]\!] \cap \mathfrak{g}(r)^\perp = (\mathfrak{g}[\![z]\!] \oplus \mathfrak{g}(r))^\perp$ and $\mathfrak{g}(\!(z)\!) = \mathfrak{g}[\![z]\!] \oplus \mathfrak{g}(\bar{r})$, since $\bar{r}$ is in normalized standard form, show that $\mathfrak{g}(r)^\perp = \mathfrak{g}(\bar{r})$. In particular, we see that $r = \bar{r}$ implies $\mathfrak{g}(r)^\perp = \mathfrak{g}(r)$. On the other hand, since both r and $\bar{r}$ are of normalized standard form, $\mathfrak{g}(r) = \mathfrak{g}(r)^\perp = \mathfrak{g}(\bar{r})$ forces

$$r - \bar{r} \in (\mathfrak{g}(r) \otimes \mathfrak{g})[\![y]\!] \cap (\mathfrak{g} \otimes \mathfrak{g})[\![x,y]\!] = \{0\}. \tag{2.29}$$

We can conclude that $\mathfrak{g}(r)^\perp = \mathfrak{g}(r)$ if and only if $\bar{r} = r$. □

Example 2.25. — We now have all ingredients to prove formula (2.20) from [50]. Recall the setting in Example 2.19: we have invertible series $A, \overline{A} \in \mathrm{End}(\mathfrak{g})[\![z]\!]$ satisfying $A(0) = \overline{A}(0) = \mathrm{id}_\mathfrak{g}$ and $\mathfrak{g}(r) = z^{-1}A\mathfrak{g}[z^{-1}]$, $\mathfrak{g}(\bar{r}) = z^{-1}\overline{A}\mathfrak{g}[z^{-1}]$ for a normalized formal generalized r-matrix r. It is easy to see that $\mathfrak{g}(\bar{r}) = \mathfrak{g}(r)^\perp$ implies that

$$\mathrm{res}_0\, z^k \kappa(Aa_1, \overline{A}a_2)\mathrm{d}z = \begin{cases} \kappa(a_1,a_2) & k = 1 \\ 0 & k \neq 1 \end{cases}$$

for all $a_1, a_2 \in \mathfrak{g}$. From this we can deduce that $\kappa(Aa_1, \overline{A}a_2) = \kappa(a_1,a_2)$ for all $a_1, a_2 \in \mathfrak{g}(\!(z)\!)$ and as a consequence $(A(z) \otimes \overline{A}(z))\gamma = \gamma$. Therefore,

$$\widetilde{r}(x,y) := \frac{A(x) \otimes \overline{A}(y)}{x-y}\gamma \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$$

is in normalized standard form by virtue of Lemma 2.3. Furthermore, it is straight forward to verify that $\widetilde{r} \in (\mathfrak{g}(r) \otimes \mathfrak{g})[\![y]\!]$ and hence

$$r - \widetilde{r} \in (\mathfrak{g}(r) \otimes \mathfrak{g})[\![y]\!] \cap (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!] = \{0\},$$

where we used that r and $\widetilde{r}$ are both of normalized standard form.

Using Lemma 2.24 it is possible to equip the Lie algebra associated to normalized formal r-matrix with additional structure, namely a dual bracket defining a Lie bialgebra structure. Let us recall what this means.

Definition 2.26. — *A Lie algebra $\mathfrak{L}$ over $\Bbbk$ equipped with a skew-symmetric map $\delta\colon \mathfrak{L} \to \mathfrak{L} \otimes \mathfrak{L}$, called* Lie cobracket*, is called* Lie bialgebra *if $\delta^*\colon (\mathfrak{L} \otimes \mathfrak{L})^* \to \mathfrak{L}^*$ restricted to $\mathfrak{L}^* \otimes \mathfrak{L}^* \subseteq (\mathfrak{L} \otimes \mathfrak{L})^*$ defines a Lie algebra structure on $\mathfrak{L}^*$ and δ is a 1-cocycle, i.e.*

$$\delta([a, b]) = [a \otimes 1 + 1 \otimes a, \delta(b)] - [b \otimes 1 + 1 \otimes b, \delta(a)]$$

for all $a, b \in \mathfrak{L}$.

The following statement is a reformulation of [20, Proposition 6.2] for normalized formal r-matrices.

Proposition 2.27. — *For a normalized formal r-matrix $r \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$ the Lie algebra $\mathfrak{g}(r)$ equipped with the linear map $\delta\colon \mathfrak{g}(r) \to \mathfrak{g}(r) \otimes \mathfrak{g}(r)$ defined by*

$$\delta(a)(x, y) := [a(x) \otimes 1 + 1 \otimes a(y), r(x, y)]$$

for all $a \in \mathfrak{g}(r)$ is a Lie bialgebra.

Proof. — If $r(x, y) = \sum_{k=0}^{\infty} \sum_{i=1}^{d} r_{k,i}(x) \otimes b_i y^k$, $\mathrm{CYB}(r) = 0$ can be written as $[r^{13} + r^{23}, r^{12}] = [r^{13}, r^{23}]$, which is equivalent to

$$\sum_{k=0}^{\infty} \sum_{i=1}^{d} \delta(r_{k,i})(x_1, x_2) \otimes b_i x_3^k = \sum_{k,\ell=0}^{\infty} \sum_{i,j=1}^{d} r_{k,i}(x_1) \otimes r_{\ell,j}(x_2) \otimes \left[b_i x_3^k, b_j x_3^\ell\right]. \tag{2.30}$$

This shows that $\delta\colon \mathfrak{g}(r) \to \mathfrak{g}(r) \otimes \mathfrak{g}(r)$ is well-defined and the Jacobi-identity in $\mathfrak{g}(\!(z)\!)$ implies that δ is a 1-cocycle. Combining $\kappa_0(r_{k,i}(z), b_j z^\ell) = \delta_{ij}\delta_{k\ell}$ for all $i, j \in \{1, \ldots, d\}, k, \ell \in \mathbb{N}_0$ with the fact that $\mathfrak{g}(\!(z)\!) = \mathfrak{g}[\![z]\!] \oplus \mathfrak{g}(r)$ shows that $a \mapsto \kappa_0(a, \cdot)$ defines an isomorphism $\kappa_0^{\mathrm{a}}\colon \mathfrak{g}[\![z]\!] \to \mathfrak{g}(r)^*$. Applying $\kappa_0^{\otimes 3}(\cdot, b_i x_1^k \otimes b_j x_2^\ell \otimes r_{m,n}(x_3))$ to (2.30) yields

$$\kappa_0^{\otimes 2}\left(\delta(r_{m,n})(x, y), b_i x^k \otimes b_j y^\ell\right) = \kappa_0\left(r_{m,n}(z), \left[b_i z^k, b_j z^\ell\right]\right). \tag{2.31}$$

Here we wrote

$$B^{\otimes n}(v_1 \otimes \dots \otimes v_n, w_1 \otimes \dots \otimes w_n) = B(v_1, w_1) \dots B(v_n, w_n)$$

for a bilinear form B on a vector space V and $v_1, \dots, v_n, w_1, \dots, w_n \in V$. Equation (2.31) implies that δ^* is identified with the standard Lie bracket of $\mathfrak{g}[\![z]\!]$ after identifying $\mathfrak{g}(r)^*$ with $\mathfrak{g}[\![z]\!]$ via κ_0^{a}, hence $\mathfrak{g}(r)$ is a Lie bialgebra if equipped with δ. □

Remark 2.28. — For a normalized formal r-matrix r the triple

$$\big(\mathfrak{g}(\!(z)\!), \mathfrak{g}[\![z]\!], \mathfrak{g}(r)\big)$$

is a so-called *Manin triple* and the property (2.31) implies that this Manin triple *determines* the Lie bialgebra structure of δ.

The Lie bialgebra structure on the Lie algebra associated to a normalized formal r-matrix can be used to derive the following version of the result of [8].

Proposition 2.29. — *Let $r \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$ be a normalized formal r-matrix. There exists $s \in z^{-1}(\mathfrak{g} \otimes \mathfrak{g})[\![z]\!]$ and $\varphi \in \mathrm{Aut}_{\Bbbk[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!])$ such that*

$$s(x - y) = (\varphi(x) \otimes \varphi(y))r(x, y).$$

Furthermore, $\mathfrak{g}(s) := \varphi(\mathfrak{g}(r))$ is closed under the formal derivative $\mathrm{d}/\mathrm{d}z$: $\mathfrak{g}(\!(z)\!) \to \mathfrak{g}(\!(z)\!)$, $a(z) \mapsto a'(z)$.

Proof.

Step 1: Setup. — Let $\delta \colon \mathfrak{g}(r) \to \mathfrak{g}(r) \otimes \mathfrak{g}(r)$ be as in Proposition 2.27. Consider the *canonical derivation* $D \colon \mathfrak{g}(r) \to \mathfrak{g}(r)$ of the Lie bialgebra $\mathfrak{g}(r)$, i.e. the composition of δ with the Lie bracket $[\cdot,\cdot] \colon \mathfrak{g}(r) \otimes \mathfrak{g}(r) \to \mathfrak{g}(r)$. As the name suggests, the 1-cocycle condition of δ implies that D is a derivation of $\mathfrak{g}(r)$

Step 2: $D = \mathrm{d}/\mathrm{d}z - \mathrm{add}(h(z,z))$ for an appropriate $h(x,y) \in (\mathfrak{g} \otimes \mathfrak{g})[\![x,y]\!]$. — The linear map $\mathfrak{g} \otimes \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ defined by $a_1 \otimes a_2 \mapsto \mathrm{add}(a_1)\,\mathrm{add}(a_2)$ maps γ to $\mathrm{id}_{\mathfrak{g}}$; see (2.23). Combining this with the fact that $[a \otimes 1 + 1 \otimes a, \gamma] = 0$ for all $a \in \mathfrak{g}[\![z]\!]$ and Lemma 2.3 results in

$$\left[a(x) \otimes 1 + 1 \otimes a(y), \frac{\gamma}{x - y}\right] = \left[\frac{a(x) - a(y)}{x - y} \otimes 1, \gamma\right] \xmapsto{[\cdot,\cdot]} a'(z). \tag{2.32}$$

If we write $h(x, y) \in \mathfrak{g}[\![x, y]\!]$ for the image of $r(x, y) - r_{\mathrm{Yang}}(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$ under the $\Bbbk[\![x, y]\!]$-linear extension of the Lie bracket $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and use the fact that

$$(2.33)\quad [a\otimes 1 + 1\otimes a, c\otimes d] \\ = [a,c]\otimes d + c\otimes [a,d] \overset{[\cdot,\cdot]}{\longmapsto} [[a,c],d] + [c,[a,d]] \\ = [a,[c,d]]$$

holds for all $a,b,c\in\mathfrak{g}$, we can conclude that $D(a)(z) = a'(z) - [h(z,z),a(z)]$.

Step 3: There exists $\varphi\in\operatorname{Aut}_{\Bbbk[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!])$ such that $\varphi(\mathfrak{g}(r))$ is closed under the formal derivative. — It is not hard to see by induction on the coefficients that there exists a unique $\psi\in\operatorname{End}(\mathfrak{g})[\![z]\!]$ satisfying $\psi'(z) = \operatorname{add}(h(z,z))\psi(z)$ and $\psi(0) = \operatorname{id}_{\mathfrak{g}}$. For every $a_1,a_2\in\mathfrak{g}$ the series

$$(2.34)\qquad c_1(z) \coloneqq \psi(z)[a_1,a_2], c_2(z) \coloneqq [\psi(z)a_1,\psi(z)a_2]\in\mathfrak{g}[\![z]\!]$$

satisfy $c_i'(z) = [h(z,z),c_i(z)]$ and $c_i(0) = [a_1,a_2]$ for $i\in\{1,2\}$. Therefore, a coefficient comparison forces $c_1 = c_2$ and thus ψ defines an element of $\operatorname{Aut}_{\Bbbk[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!])$. Let $\varphi(z)\coloneqq\psi(z)^{-1}\in\operatorname{End}(\mathfrak{g})[\![z]\!]$, i.e. $\varphi(z)\psi(z) = \operatorname{id}_{\mathfrak{g}}$, and note that $\varphi(z)$ also defines an element of $\operatorname{Aut}_{\Bbbk[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!])$. Consider the normalized formal r-matrix $\widetilde{r}(x,y)\coloneqq(\varphi(x)\otimes\varphi(y))r(x,y)$ and let $\widetilde{\delta}$ be the Lie cobracket from Proposition 2.27 of $\mathfrak{g}(\widetilde{r})$. Clearly $\widetilde{\delta}(a)(x,y) = (\varphi(x)\otimes\varphi(y))\delta(\psi(a))(x,y)$ for all $a\in\mathfrak{g}(\widetilde{r})$. Therefore, the canonical derivation $\widetilde{D}$ of $\mathfrak{g}(\widetilde{r})$ reads

$$\begin{aligned}\widetilde{D}(a)(z) &= \varphi(z)D(\psi(a))(z)\\ &= \varphi(z)\big(\psi'(z)a(z) + \psi(z)a'(z) - [h(z,z),\psi(z)a(z)]\big)\\ &= a'(z) + \varphi(z)\big([h(z,z),\psi(z)a(z)] - [h(z,z),\psi(z)a(z)]\big) = a'(z).\end{aligned}$$

In particular, $\widetilde{D}\colon\mathfrak{g}(\widetilde{r})\to\mathfrak{g}(\widetilde{r})$ is the restriction of the formal derivative to $\mathfrak{g}(\widetilde{r})$.

Step 4: $s(z)\coloneqq\widetilde{r}(z,0)$ concludes the proof. — Since $\mathfrak{g}(\widetilde{r})$ is closed under $\widetilde{D}$, which is the restriction of the formal derivative to $\mathfrak{g}(\widetilde{r})$, we have

$$\big(1\otimes\kappa(b_i^*,\cdot)\big)\frac{(-1)^k}{k!}\widetilde{r}^{(k)}(z,0)\in\mathfrak{g}(\widetilde{r})\cap\big(b_i^*z^{-k-1} + \mathfrak{g}[\![z]\!]\big)$$

for all $i\in\{1,\dots,d\}$, $k\in\mathbb{N}_0$, where $\widetilde{r}^{(k)}(z,0)\coloneqq\widetilde{D}^k\widetilde{r}(z,0)$. The proof of Proposition 2.14 and the expansion of $\widetilde{r}(x-y,0)$ as a series imply that

$$\widetilde{r}(x,y) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\widetilde{r}^{(k)}(x,0)y^k = \widetilde{r}(x-y,0)$$

and we can see that putting $s(z)\coloneqq\widetilde{r}(z,0)$ concludes the proof of Proposition 2.29. □

3. Some results about sheaves of algebras

3.1. Definition and generalities

In Section 4 we will assign a geometric datum to any formal generalized r-matrix. The main ingredient of this datum consists of a sheaf of Lie algebras on a projective curve constructed from the subalgebra associated to that formal generalized r-matrix. In this section we will discuss all definitions and properties related to sheaves of Lie algebras that are required in the subsequent sections. Although we are interested in sheaves of Lie algebras, we will consider general sheaves of algebras as long as it does not obscure the constructions or restricts the scope of the statements.

Remark 3.1. — In this text, an R-algebra A over some unital commutative ring R does not necessarily satisfy any additional axioms, i.e. A is simply a pair (A, μ_A) consisting of a R-module A equipped with an R-linear map $\mu_A\colon A \otimes_R A \to A$ called multiplication. In particular, a Lie algebra over R is a R-algebra.

DEFINITION 3.2. — *Let $X = (X, \mathcal{O}_X)$ be a ringed space.*

- *A* sheaf of algebras $\mathcal{A}$ *on X is a pair $(\mathcal{A}, \mu_{\mathcal{A}})$ consisting of an $\mathcal{O}_X$-module $\mathcal{A}$ equipped with an $\mathcal{O}_X$-linear morphism $\mu_{\mathcal{A}}\colon \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{A}$ called* multiplication of $\mathcal{A}$.
- *A* morphism of sheaves of algebras $f\colon \mathcal{A} \to \mathcal{B}$ *is an $\mathcal{O}_X$-linear morphism satisfying $f\mu_{\mathcal{A}} = \mu_{\mathcal{B}}(f \otimes f)$.*
- *A sheaf of algebras $\mathcal{A}$ on X is called* sheaf of Lie algebras *if $[,]_{\mathcal{A}} := \mu_{\mathcal{A}}$ defines a Lie algebra structure on all local section. A* morphism of sheaves of Lie algebras *is simply a morphism of sheaves of algebras. The adjoint representations of the local sections of $\mathcal{A}$ induce an $\mathcal{O}_X$-linear morphism* $\mathrm{add}_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A})$ *called* adjoint representation of $\mathcal{A}$.

Remark 3.3. — Let $f\colon Y \to X$ be a morphism of locally ringed spaces, $\mathcal{A}$ (resp. $\mathcal{B}$) be a sheaf of algebras on X (resp. Y). The sheaf of algebras $f^*\mathcal{A}$ (resp. $f_*\mathcal{B}$) is naturally a sheaf of algebras on Y (resp. X) with the multiplication defined through

$$f^*\mathcal{A} \otimes_{\mathcal{O}_Y} f^*\mathcal{A} \xrightarrow{\cong} f^*(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}) \xrightarrow{f^*\mu_{\mathcal{A}}} \mathcal{A} \\ \Big(\text{resp. } f_*\mathcal{B} \otimes_{\mathcal{O}_X} f_*\mathcal{B} \longrightarrow f_*(\mathcal{B} \otimes_{\mathcal{O}_Y} \mathcal{B}) \xrightarrow{f_*\mu_{\mathcal{B}}} \mathcal{B}\Big), \tag{3.1}$$

respectively, where the unlabeled arrows are the canonical ones. If $\mathcal{A}$ (resp. $\mathcal{B}$) is a sheaf of Lie algebras, it is easy to see that $f^*\mathcal{A}$ (resp. $f_*\mathcal{B}$) is a sheaf of Lie algebras.

In particular, for any $p \in X$ with residue field $\Bbbk(p)$, the stalk $\mathcal{A}_p$ (resp. the fiber $\mathcal{A}|_p$) is naturally an $\mathcal{O}_{X,p}$-algebra (resp. $\Bbbk(p)$-algebra), which is a Lie algebra if $\mathcal{A}$ is a sheaf of Lie algebras. Indeed, this can be seen as a special case of the inverse image construction by choosing $Y = (\{p\}, \mathcal{O}_{X,p})$ (resp. $Y = (\{p\}, \Bbbk(p))$) and considering the inverse image with respect to the canonical morphism $f\colon Y \to X$.

It will turn out to be useful that the Killing form of a Lie algebra admits a geometric analog.

Definition 3.4. — *The Killing form $K_\mathcal{A}$ of a finite locally free sheaf $\mathcal{A}$ of Lie algebras on a ringed space $X = (X, \mathcal{O}_X)$ is the $\mathcal{O}_X$-bilinear morphism*

$$(3.2)\quad \mathcal{A} \times \mathcal{A} \xrightarrow{\operatorname{add}_\mathcal{A} \times \operatorname{add}_\mathcal{A}} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A}) \times \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A}) \longrightarrow \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A}) \xrightarrow{\operatorname{Tr}_\mathcal{A}} \mathcal{O}_X,$$

where $\operatorname{add}_\mathcal{A}$ *is the adjoint representation of $\mathcal{A}$, the second map is given by composition and* $\operatorname{Tr}_\mathcal{A}$ *is the sheaf trace of $\mathcal{A}$.*

Lemma 3.5. — *Let $f\colon Y \to X$ be a morphism of locally ringed spaces and $\mathcal{A}$ be a finite locally free sheaf of Lie algebras on $\mathcal{A}$ with Killing form $K_\mathcal{A}$. Then $f^*K_\mathcal{A}$ is the Killing form of the finite locally free sheaf of algebras $f^*\mathcal{A}$ on Y. In particular, for any point $p \in X$ the fiber $K_\mathcal{A}|_p$ coincides with the Killing form of $\mathcal{A}|_p$.*

Proof. — The canonical map $\chi\colon f^*\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A}) \longrightarrow \mathcal{E}nd_{\mathcal{O}_Y}(f^*\mathcal{A})$ coincides with the isomorphism

$$(3.3)\quad \operatorname{End}_{\mathcal{O}_{X,f(q)}}\left(\mathcal{A}_{f(q)}\right) \otimes_{\mathcal{O}_{X,f(q)}} \mathcal{O}_{Y,q} \cong \operatorname{End}_{\mathcal{O}_{Y,q}}\left(\mathcal{A}_{f(q)} \otimes_{\mathcal{O}_{X,f(q)}} \mathcal{O}_{Y,q}\right)$$

in the stalk in any point $q \in Y$, where we used that $\mathcal{A}$ is finite locally free. This shows that

$$(3.4)\quad f^*\mathcal{A} \xrightarrow{f^*\operatorname{add}_\mathcal{A}} f^*\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A}) \xrightarrow{\chi} \mathcal{E}nd_{\mathcal{O}_Y}(f^*\mathcal{A}) \quad \text{and} \quad f^*\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A}) \xrightarrow{\chi} \mathcal{E}nd_{\mathcal{O}_Y}(f^*\mathcal{A}) \xrightarrow{\operatorname{Tr}_{f^*\mathcal{A}}} \mathcal{O}_Y,$$

coincide with $\operatorname{add}_{f^*\mathcal{A}}$ and $f^*\operatorname{Tr}_\mathcal{A}$ respectively and that χ is compatible with the composition of endomorphisms of sheaves. Therefore, applying f^* to (3.2) and using χ implies that $f^*K_\mathcal{A}$ coincides with the Killing form of $f^*\mathcal{A}$. The observation that the functor $(\cdot)|_p$ can be realized by the inverse image via $(\{p\}, \Bbbk(p)) \to X$, where $\Bbbk(p)$ is the residue field of p, concludes the proof. □

3.2. Local triviality of sheaves of algebras

Let $\mathcal{A}$ be a coherent sheaf of algebras on a $\Bbbk$-scheme X of finite type. There are several different notions of local triviality for $\mathcal{A}$.

Definition 3.6. — *Let A be a finite-dimensional $\Bbbk$-algebra.*

- *$\mathcal{A}$ is called* weakly A-locally free in $p \in X$ *if $\mathcal{A}|_p \cong A \otimes \Bbbk(p)$ as $\Bbbk(p)$-algebras, where $\Bbbk(p)$ is the residue field of p.*
- *$\mathcal{A}$ is called* formally A-locally free in $p \in X$ *if $\widehat{\mathcal{A}}_p \cong A \otimes \widehat{\mathcal{O}}_{X,p}$ as $\widehat{\mathcal{O}}_{X,p}$-algebras.*
- *$\mathcal{A}$ is called* étale A-locally free in $p \in X$ *if there exists an étale morphism $f\colon U \to X$ of $\Bbbk$-schemes such that $p \in f(U)$ and $f^*\mathcal{A} \cong A \otimes \mathcal{O}_U$ as sheaves of algebras.*
- *$\mathcal{A}$ is called* Zarsiki A-locally free in $p \in X$ *if there exists an open neighbourhood U of p such that $\mathcal{A}|_U \cong A \otimes \mathcal{O}_U$ as sheaves of algebras.*
- *$\mathcal{A}$ is called* weakly *(resp.* formally, étale, Zariski*)* A-locally free *if it has this property in all $p \in X$.*

Remark 3.7. — Let $p \in X$ and A be a finite-dimensional $\Bbbk$-algebra. Obviously, $\mathcal{A}$ is weakly A-locally free in $p \in X$ if it is formally A-locally free in p. Since open immersions are étale, $\mathcal{A}$ is both formally and étale A-locally free in p if it is Zariski A-locally free in p. Furthermore, if $\Bbbk = \overline{\Bbbk}$ and p is closed, $\mathcal{A}$ is formally A-locally free in p if it is étale A-locally free in p.

Remark 3.8. — Let A be a finite-dimensional $\Bbbk$-algebra. Note that if $\mathcal{A}$ is étale A-locally free in a point $p \in X$ it is already étale A-locally free in an open neighbourhood of p, since étale morphisms are open. Assume we have a set of étale morphisms $f_i\colon U_i \to X$ for $i \in \{1, \dots, n\}$ such that $X = \bigcup_{i=1}^n f_i(U_i)$. Then $f \coloneqq \coprod_{i=1}^n f_i\colon Y \coloneqq \coprod_{i=1}^n U_i \to X$ is surjective and étale. If U_i is affine for all $i \in \{1, \dots, n\}$, Y is also affine. This shows that if $\mathcal{A}$ is étale A-locally free, there exists a surjective étale morphism $f\colon Y \to X$ of $\Bbbk$-schemes such that Y is affine and $f^*\mathcal{A} \cong \mathfrak{g} \otimes \mathcal{O}_Y$. Here we used that X is quasi-compact.

Lemma 3.9. — *Let $\mathcal{A}$ be a sheaf of algebras on a finite-type $\Bbbk$-scheme X and A be a finite-dimensional $\Bbbk$-algebra. Furthermore, let $\pi\colon \overline{X} \coloneqq X \times$* Spec$(\overline{\Bbbk}) \to X$ *be the canonical projection (where we recall that $\overline{\Bbbk}$ denotes the algebraic closure of $\Bbbk$) and write $\overline{A} \coloneqq A \otimes \overline{\Bbbk}$.*

The sheaf $\mathcal{A}$ is étale A-locally free in a point $p \in X$ if and only if $\pi^*\mathcal{A}$ is étale $\overline{A}$-locally free in all $q \in \pi^{-1}(p)$.

Proof.

Step 1: Setup. — The "only if" part follows directly from the fact that the property of being étale is stable under base change; see [18, Proposition 17.3.3 (iii)]. It remains to prove the "if" part. Étale A-local triviality is a local property, so we can assume that $M = \Gamma(X, \mathcal{A})$ is a free R-algebra for $X = \operatorname{Spec}(R)$, where $R := \Bbbk[x_1, \dots, x_m]/I$ for some ideal $I \subseteq \Bbbk[x_1, \dots, x_m]$. Then $\overline{X} \cong \operatorname{Spec}(\overline{R})$ for $\overline{R} := \overline{\Bbbk}[x_1, \dots, x_m]/\overline{\Bbbk}I$ and the natural injective morphism $\iota\colon R \to \overline{R}$ induces $\pi\colon \overline{X} \to X$. By definition, p is a prime ideal of R. Fix $\bar{q} \in \pi^{-1}(p)$, i.e. $\bar{q} \subset \overline{R}$ is a prime ideal such that $\iota^{-1}(\bar{q}) = p$. The étale A-local triviality of $\pi^*\mathcal{A}$ in $\bar{q}$ can now be formulated as: there exists an étale morphism $\overline{f}\colon \overline{R} \to \overline{S}$ of $\overline{\Bbbk}$-algebras, such that $\bar{q} = \overline{f}^{-1}(\bar{r})$ for some prime ideal $\bar{r} \subset S$, and an isomorphism

$$\psi\colon M \otimes_R \overline{S} \cong (M \otimes_R \overline{R}) \otimes_{\overline{R}} \overline{S} \longrightarrow \overline{A} \otimes_{\overline{\Bbbk}} \overline{S} \cong A \otimes \overline{S}$$

of $\overline{S}$-algebras. We may assume that $\overline{S} = \overline{\Bbbk}[x_1, \dots, x_n]/(s_1, \dots, s_k)$ for some $s_1, \dots, s_k \in \overline{\Bbbk}[x_1, \dots, x_n]$. The morphism $\overline{f}$ is completely determined by the images $f_i := \overline{f}(x_i + \overline{\Bbbk}I) \in \overline{S}$ of $x_i + \overline{\Bbbk}I \in \overline{R}$ for $i \in \{1, \dots, m\}$. We can describe ψ by a matrix $\widetilde{\psi} \in \operatorname{Mat}_{d\times d}(\overline{S})$ after choosing a basis of A. Here $d = \dim(A)$.

Step 2: $\overline{\Bbbk}$ can be replaced by some finite field extension $\Bbbk'$ of $\Bbbk$. — Since $\overline{\Bbbk}$ is algebraic over $\Bbbk$, we can choose a finite field extension $\Bbbk'$ of $\Bbbk$ such that $s_1, \dots, s_k \in \Bbbk'[x_1, \dots, x_n]$, $f_1, \dots, f_m, \det(\widetilde{\psi})^{-1} \in S$ and $\widetilde{\psi} \in \operatorname{Mat}_{d\times d}(S)$ for

$$R' := \Bbbk'[x_1, \dots, x_m]/\Bbbk' I \text{ and } S := \Bbbk'[x_1, \dots, x_n]/(s_1, \dots, s_k).$$

Here we used that only finitely many elements of $\overline{\Bbbk}$ appeared in these constructions. The assignment $x_i + \Bbbk' I \mapsto f_i$ for $i \in \{1, \dots, m\}$ defines a $\Bbbk$-algebra morphism $f\colon R' \to S$ making the diagram

$$\begin{array}{ccc} R'\iota' & \xrightarrow{\;f\;} & S \\ {\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle \jmath} \\ \overline{R} & \xrightarrow[\;\overline{f}\;]{} & \overline{S} \end{array} \tag{3.5}$$

commutative, where the vertical maps are the canonical ones. In particular, it holds that $f^{-1}(r) = q$ for $q := \iota'^{,-1}(\bar{q})$ and $r := \jmath^{-1}(\bar{r})$.

Step 3: Concluding the proof. — Since $\overline{f}$ can be identified with $f \otimes_{\Bbbk'} \operatorname{id}_{\overline{\Bbbk}}$, f is étale since $\overline{f}$ is; see [18, Proposition 17.7.1 (ii)]. Furthermore, $R \to R'$ is étale since $\Bbbk \to \Bbbk'$ is finite, the characteristic of $\Bbbk$ is 0 and $R' \cong R \otimes \Bbbk'$. The composition $g\colon R \to S$ of the canonical morphism $\iota''\colon R \to R'$ with f is thus étale and satisfies $g^{-1}(r) = \iota''^{,-1}(f^{-1}(r)) = \iota^{-1}(\overline{f}^{-1}(\bar{r})) = p$, where we used $\iota = \iota'\iota''$ and (3.5). The matrix $\widetilde{\psi}$ defines a morphism $M \otimes_R S \to A \otimes S$ of S-algebras, which is bijective due to $\det(\widetilde{\psi})^{-1} \in S$. This is equivalent to the fact that $\mathcal{A}$ is étale A-locally free in the point p. □

The following result can be seen as an algebro-geometric version of the result on local triviality of Lie algebra bundles in [35].

Theorem 3.10. — *Let $\mathcal{A}$ be a finite locally free sheaf of algebras on a reduced finite-type $\Bbbk$-scheme X and A be a finite-dimensional $\Bbbk$-algebra. Furthermore, let $\pi\colon X \times \operatorname{Spec}(\overline{\Bbbk}) \to X$ be the canonical projection and write $\overline{A} := A \otimes \overline{\Bbbk}$.*

Then $\mathcal{A}$ is étale A-locally free if and only if $\pi^*\mathcal{A}$ is weakly $\overline{A}$-locally free in all closed points $p \in X$.

Proof.

Step 1: Setup. — By Lemma 3.9 we may assume that $\Bbbk = \overline{\Bbbk}$ and so we have to show that $\mathcal{A}$ is étale A-locally free if and only if $\mathcal{A}|_p \cong A$ for all $p \in X$ closed. The "only if" part was already discussed in Remark 3.7. It remains to prove the "if" part. Recall that an algebraic prevariety is a locally ringed space associated to the closed points of a reduced $\Bbbk$-scheme of finite type and that the category of reduced $\Bbbk$-schemes of finite type is equivalent to the category of algebraic prevarieties. We are therefore permitted to work in the latter category.

Étale local triviality is local. Therefore, we can assume that X is an affine variety, i.e. the ringed space associated to the closed points of an reduced affine $\Bbbk$-scheme. Let us identify A with a $\Bbbk$-algebra of the form $(\Bbbk^d, \mu_A)$. The $\Gamma(X, \mathcal{O}_X)$-algebra $\Gamma(X, \mathcal{A})$ can be identified with the $\Gamma(X, \mathcal{O}_X)$-module of all regular maps $X \to \Bbbk^d$ equipped with the multiplication map $\mu_{\mathcal{A}}\colon \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{A}$ defined by some regular map $\theta\colon X \to M = \operatorname{Hom}(\Bbbk^d \otimes \Bbbk^d, \Bbbk^d)$ via $\mu_{\mathcal{A}}(a \otimes b)(p) = \theta(p)(a(p) \otimes b(p))$ for all $a, b\colon X \to \Bbbk^d$ regular and $p \in X$. By definition, $\mathcal{A}|_p = (\Bbbk^d, \theta(p))$ for all $p \in X$. The group $G = \operatorname{GL}(d, \Bbbk)$ acts on M by

$$(3.6) \qquad (L \cdot \vartheta)(v \otimes w) = L^{-1}\vartheta(Lv \otimes Lw) \qquad \forall\, L \in G, \vartheta \in M, v, w \in \Bbbk^d.$$

The orbit $G \cdot \mu_A$ coincides with the set of multiplications on $\Bbbk^d$ determining an algebra structure isomorphic to A. Therefore, $\theta(X) \subseteq G \cdot \mu_A$ by assumption.

Step 2: The canonical map $o\colon G \to G\cdot\mu_A$ is a surjective smooth morphism of algebraic prevarieties. — Consider the stabiliser H of μ_A in G. The canonical map $o\colon G \to G/H \cong G \cdot \mu_A$ defined by $L \mapsto L \cdot \mu_A$ is a faithfully flat morphism of algebraic prevarieties and the induced morphism $G\times_{G\cdot\mu_A} G\to G\times H$ is an isomorphism; see e.g. [40]. Note that the pull-back diagram

$$\begin{array}{ccc} G\times_{G\cdot\mu_A} G\cong G\times H & \longrightarrow & G \\ \big\downarrow & & \big\downarrow \\ G & \xrightarrow[o]{} & G\cdot\mu_A, \end{array} \tag{3.7}$$

combined with the fact that H and hence $G\times H\to G$ is smooth and o is flat, implies that o is smooth; see [18, Proposition 17.7.4].

Step 3: For all $p\in X$ there exists an étale morphism $f\colon Y\to X$ and a morphism $s\colon Y\to G$ such that $p\in f(Y)$ and $os=\theta f$. — Consider the pull-back diagram

$$\begin{array}{ccc} G\times_{G\cdot\mu_A} X & \longrightarrow & G \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle o} \\ X & \xrightarrow[\theta]{} & G\cdot\mu_A, \end{array} \tag{3.8}$$

The morphism g is surjective and smooth since o is; see e.g. [23, Proposition 6.15 (3)]. Let $p\in X$ be an arbitrary point. Using the construction in [18, Corollaire 17.16.3], we see that there exists a locally closed affine subvariety $Y\subseteq G\times_{G\cdot\mu_A} X$ such that $f:=g|_Y$ is étale and $p\in f(Y)$. Let s be the restriction of the canonical projection $G\times_{G\cdot\mu_A} X\to G$ to Y. By construction $os=\theta f$.

Step 4: s induces an isomorphism $\psi\colon f^*\mathcal{A}\to A\otimes\mathcal{O}_Y$. — We can identify $\Gamma(Y,f^*\mathcal{A})$ with the $\Gamma(Y,\mathcal{O}_Y)$-module of all regular maps $Y\to\Bbbk^d$ equipped with the multiplication $\mu_{f^*\mathcal{A}}$ determined by $\mu_{f^*\mathcal{A}}(a\otimes b)(q)=\theta(f(q))(a(q)\otimes b(q))$ for all $a,b\colon Y\to\Bbbk^d$ regular and $q\in Y$. Evaluating $os=\theta f$ in an arbitrary $q\in Y$ results in $s(q)\cdot\mu_A=\theta(f(q))$, hence

$$\begin{aligned} s(q)\theta(f(q))(a(q)\otimes b(q)) &= s(q)(s(q)\cdot\mu_A)(a(q)\otimes b(q)) \\ &= \mu_A(s(q)a(q)\otimes s(q)b(q)) \end{aligned}$$

for all $a,b\colon Y\to\Bbbk^d$ regular and $q\in Y$. This shows that the $\Gamma(Y,\mathcal{O}_Y)$-linear automorphism ψ of $\{a\colon Y\to\Bbbk^d \mid a \text{ is regular}\}$ defined by $\psi(a)(q)=s(q)a(q)$ for all $a\colon Y\to\Bbbk^d$ regular and $q\in Y$, induces an isomorphism $\psi\colon f^*\mathcal{A}\to A\otimes\mathcal{O}_Y$ of sheaves of algebras. □

The following result can be seen as an algebro-geometric analog of [36, Lemma 2.1].

THEOREM 3.11. — *Let $\mathcal{A}$ be a finite locally free sheaf of Lie algebras on a reduced $\Bbbk$-scheme X of finite type and $\mathcal{A}|_p$ be semi-simple for some closed point $p \in X$. Then $\mathcal{A}$ is étale $\mathcal{A}|_p$-locally free in $p \in X$.*

Proof. — Lemma 3.5 and Cartan's criterion for semi-simplicity imply that $\pi^*\mathcal{A}|_q$ is semi-simple for all $q \in \pi^{-1}(p)$ if and only if $\mathcal{A}|_p$ is, where $\pi\colon X \times \operatorname{Spec}(\bar{\Bbbk}) \to X$ denotes the canonical projection. Therefore, using Lemma 3.9, we may assume that $\Bbbk$ is algebraically closed. As in Step 1 in the proof of Theorem 3.10, we can proceed to work in the category of algebraic prevarieties, assume that X is an affine variety and that $\mathcal{A}$ is free of rank d. Let $B \subseteq M = \operatorname{Hom}(\Bbbk^d \otimes \Bbbk^d, \Bbbk^d)$ be the affine subvariety of all of possible Lie brackets on $\Bbbk^d$. Then $\Gamma(X, \mathcal{A})$ can be identified with the $\Gamma(X, \mathcal{O}_X)$-module of all regular maps $Y \to \Bbbk^d$ equipped with the Lie bracket $\mu_{\mathcal{A}}$ defined by a regular map $\theta\colon X \to B$ via $\mu_{\mathcal{A}}(a \otimes b)(q) = \theta(q)(a(q) \otimes b(q))$ for all $a, b\colon X \to \Bbbk^d$ regular and $q \in X$.

The action of $G = \mathrm{GL}(n, \Bbbk)$ on M, introduced in Step 1 of the proof of Theorem 3.10, restricts to an action on B. In particular, the orbit $G \cdot \theta(p)$ coincides with the set of Lie brackets on $\Bbbk^d$ determining an Lie algebra structure isomorphic to $\mathcal{A}|_p = (\Bbbk^d, \theta(p))$. Combining [43, Theorem 7.2] with Whitehead's Lemma and the fact that $\mathcal{A}|_p$ is semi-simple, we see that $G \cdot \theta(p) \subseteq B$ is open and as a consequence $U := \theta^{-1}(G \cdot \theta(p))$ is an open neighbourhood of p. For all $q \in U$, we have $\mathcal{A}|_q \cong (\Bbbk^d, \theta(p)) = \mathcal{A}|_p$, so Theorem 3.10 asserts that $\mathcal{A}|_U$ is étale A-locally free. □

3.3. Sheaves of algebras and lattices

In this subsection we will combine the results of [44] with a version of the geometrization scheme presented in [41, 42] to derive a connection between certain subalgebras of Laurent series algebras and sheaves of algebras on projective curves. The application of this procedure to the subalgebra associated to a formal generalized r-matrix will result in the geometrization of this r-matrix in Section 4.

DEFINITION 3.12. — *For a finite-dimensional $\Bbbk$-algebra A, a subspace $W \subseteq A(\!(z)\!)$ is called* A-lattice *if it is a subalgebra of $A(\!(z)\!)$ satisfying*

$$\dim(A[\![z]\!] \cap W) =: h_0 < \infty \text{ and } \dim(A(\!(z)\!)/(A[\![z]\!] + W)) =: h_1 < \infty.$$

The pair (h_0, h_1) is called index *of W.*

Example 3.13. — Let $r \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$ be a formal generalized r-matrix for some finite-dimensional semi-simple Lie algebra $\mathfrak{g}$ over $\Bbbk$. The Lie algebra $\mathfrak{g}(r)$ is a $\mathfrak{g}$-lattice of index $(0, 0)$.

Definition 3.14. — *For a vector space V over $\Bbbk$, a subset $M \subseteq V(\!(z)\!)$ and an integer k, we put $M_k := M \cap z^{-k}V[\![z]\!]$. An element $a(z) = vz^{-k} + \dots \in M_k \setminus M_{k-1}$ is said to have* order k *and* main part vz^{-k}.

Proposition 3.15. — *Let O be an unital $\Bbbk$-lattice of index (h_0, h_1).*

(1) *O has Krull dimension 1 and $h_0 = 1$, i.e. $O_0 = O \cap \Bbbk[\![z]\!] = \Bbbk$.*
(2) *$\mathrm{gr}(O) := \bigoplus_{k \in \mathbb{N}_0} t^k O_k \subseteq O[t]$ is a graded unital $\Bbbk$-subalgebra and $X := \mathrm{Proj}(\mathrm{gr}(O))$ is an integral projective curve over $\Bbbk$ of arithmetic genus h_1.*
(3) *There exists a $\Bbbk$-rational smooth point p and a $\Bbbk$-algebra isomorphism $c\colon \widehat{\mathcal{O}}_{X,p} \to \Bbbk[\![z]\!]$ inducing an isomorphism $\mathrm{Spec}(O) \to X \setminus \{p\}$. More precisely, the extension of c to an isomorphism $\mathrm{Q}(\widehat{\mathcal{O}}_{X,p}) \to \Bbbk(\!(z)\!)$, which will again be denoted by c, satisfies $c(\Gamma(X \setminus \{p\}, \mathcal{O}_X)) = O$.*

Proof.

Step 1: $O_0 = \Bbbk$, $\mathrm{gr}(O) \subseteq O[t]$ is a graded unital $\Bbbk$-subalgebra and X is integral. — That $\mathrm{gr}(O)$ is a graded $\Bbbk$-subalgebra of $O[t]$ follows directly from $O_k O_\ell \subseteq O_{k+\ell}$ for all $k, \ell \in \mathbb{Z}$. The space $O_0 = O \cap \Bbbk[\![z]\!]$ contains $\Bbbk$ since O is unital. If $f \in O_0 \setminus \Bbbk$ would exist, O_0 would contain the infinite linearly independent set $\{(f(z) - f(0))^n \mid n \in \mathbb{N}\}$, contradicting $\dim(O_0) < \infty$. Finally, X is integral, since $O[t]$, and thus $\mathrm{gr}(O)$, is an integral domain.

Step 2: O has Krull dimension 1, i.e. (1) *is true.* — The condition

$$\dim(\Bbbk(\!(z)\!)/(\Bbbk[\![z]\!] + O)) < \infty$$

implies that for sufficiently large $r \in \mathbb{N}$ there exist elements f and g of O with main parts z^{-r} and z^{-r-1} respectively. $O_0 = \Bbbk$ implies that the canonical projection $O \to \Bbbk[z^{-1}]$ is injective. Consequently, $\Bbbk[f, g] \subseteq O$ is of finite codimension, since for every $\ell_1 \geqslant r$ and $0 \leqslant \ell_2 \leqslant r-1$ the element $f^{\ell_1 - \ell_2} g^{\ell_2}$ has main part $z^{-\ell_1 r - \ell_2}$. Therefore, the Krull dimension of O and the Krull dimension of $\Bbbk[f, g]$ coincide. The latter is one, since, if $h_1, \dots, h_k$ is a basis of $\Bbbk[f, g]_{r(r+1)-1}$, we have

$$\Bbbk[f, g]_{r(r+1)-1} \ni f^{r+1} - g^r = c_1 h_1 + \dots + c_k h_k \tag{3.9}$$

for some $c_1, \dots, c_k \in \Bbbk$, which is a polynomial relation of f and g.

Step 3: Construction of p and c. — By definition of $X = \mathrm{Proj}(\mathrm{gr}(O))$, the homogeneous prime ideal $p := (t) = \bigoplus_{k \in \mathbb{N}_0} t^{k+1} O_k$ generated by $t \in \mathrm{gr}(O)$ is a point of X. Observe that $t^k h$ is an element of the homogeneous elements

S in $\mathrm{gr}(O) \setminus (t)$ if and only if h has order k. This shows

$$\mathcal{O}_{X,p} = \left(S^{-1}\,\mathrm{gr}(O)\right)_0 = \{a/h \mid a, h \in O, a/h \in \Bbbk[\![z]\!]\} = \mathrm{Q}(O) \cap \Bbbk[\![z]\!]. \tag{3.10}$$

Choosing f, g as in Step 2 yields $u \coloneqq f/g \in \mathrm{Q}(O) \cap z\Bbbk[\![z]\!]^\times$. Therefore, $\Bbbk[u] \subseteq \mathrm{Q}(O) \cap \Bbbk[\![z]\!] \subseteq \Bbbk[\![z]\!]$ and $\Bbbk[\![u]\!] = \Bbbk[\![z]\!]$ results in an isomorphism $c\colon \widehat{\mathcal{O}}_{X,p} \to \Bbbk[\![z]\!]$. We conclude that p is $\Bbbk$-rational and smooth.

Step 4: The affine open subset $\mathrm{d}_+(t) \subseteq X$ *is isomorphic to* $\mathrm{Spec}(O)$ *and* $c(\Gamma(\mathrm{d}_+(t), \mathcal{O}_X)) = O$. — Since $\Bbbk[\![z]\!]_{(u)} = \Bbbk(\!(z)\!)$ for u from Step 3, we can see from (3.10) that the rational functions on X can be identified with $\mathrm{Q}(O)$ via the isomorphism $c\colon \mathrm{Q}(\widehat{\mathcal{O}}_{X,p}) \to \Bbbk(\!(z)\!)$. More precisely, we can deduce that $c(\Gamma(\mathrm{d}_+(t^k h), \mathcal{O}_X)) = \mathrm{gr}(O)[(t^k h)^{-1}]_0 \subseteq \mathrm{Q}(O)$ for all $t^k h \in \mathrm{gr}(O)$, i.e. c induces the natural isomorphisms $\Gamma(\mathrm{d}_+(t^k h), \mathcal{O}_X) \cong \mathrm{gr}(O)[(t^k h)^{-1}]_0$. In particular, we see that

$$c(\Gamma(\mathrm{d}_+(t), \mathcal{O}_X)) = \mathrm{gr}(O)[t^{-1}]_0 = O. \tag{3.11}$$

Therefore, c defines an isomorphism $\mathrm{Spec}(O) \to \mathrm{d}_+(t)$.

Step 5: X is an integral projective curve over $\Bbbk$. — Using the Steps 1, 2 and 4, it remains to show that X is a $\Bbbk$-scheme of finite type, since then $\dim(X) = \dim(\mathrm{d}_+(t)) = 1$. Thus, we have to show that $\mathrm{gr}(O)$ is a finitely-generated $\Bbbk$-algebra; see e.g. [23, Lemma 13.9 (2) and Proposition 13.12]. We will prove that each basis B of the finite dimensional space $\bigoplus_{k=0}^{r^2} t^k O_k$, containing $t, t^r f$ and $t^{r+1} g$, generates $\mathrm{gr}(O)$, where f, g and r are as in Step 2. Let us write R for the $\Bbbk$-subalgebra of $\mathrm{gr}(O)$ generated by B. We prove by induction on $k \geqslant r^2$ that $t^k O_k \subseteq R$, which is obvious for $k = r^2$. By induction assumption $t^{k-1} O_{k-1} \subseteq R$, and thus $t^k O_{k-1} \subseteq R$. Let $h \in O$ have main part $a z^{-k}$ and $\ell_1 \geqslant r, 0 \leqslant \ell_2 \leqslant r-1$ be such that $k = \ell_1 r + \ell_2$. Then

$$t^k h - a(t^r f)^{\ell_1 - \ell_2}(t^{r+1} g)^{\ell_2} = t^k h - a t^k f^{\ell_1 - \ell_2} g^{\ell_2} \in t^k O_{k-1} \subseteq R,$$

proving $t^k h \in R$ and this concludes the induction.

Step 6: $\mathrm{d}_+(t) = X \setminus \{p\}$ *and* $\mathrm{H}^1(\mathcal{O}_X) = h_1$. — Since $O \cap \Bbbk[\![z]\!] = \Bbbk$ by Step 1, the open subscheme $D_+(t) \cup \mathrm{d}_+\left(t^{r+1} g\right) = D_+(t) \cup \{p\}$, where g was chosen in Step 2, of X is not affine and hence a proper $\Bbbk$-scheme by [28, Chapter IV, Exercise 1.4]. Therefore, the embedding $\iota(\mathrm{Spec}(O)) \cup \{p\} \to X$ is closed, so $X = \iota(\mathrm{Spec}(O)) \cup \{p\}$, since X is integral. For every coherent sheaf $\mathcal{F}$ on X, we have an exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathcal{F}) \longrightarrow \Gamma(X \setminus \{p\}, \mathcal{F}) \oplus \widehat{\mathcal{F}} \longrightarrow \mathrm{Q}(\widehat{\mathcal{F}}) \longrightarrow \mathrm{H}^1(\mathcal{F}) \longrightarrow 0; \tag{3.12}$$

see e.g. [45, Proposition 3]. Setting $\mathcal{F} = \mathcal{O}_X$ and applying c results in the desired $\mathrm{H}^1(\mathcal{O}_X) = \dim(\Bbbk((z))/(\Bbbk[\![z]\!] + O)) = h_1$. □

Theorem 3.16. — *Let O be an unital $\Bbbk$-lattice, X, p and c be the associated geometric datum from Proposition 3.15 and W be some A-lattice of index (h_0, h_1) satisfying $OW \subseteq W$ for some finite-dimensional $\Bbbk$-algebra A.*

(1) $\mathrm{gr}(W) := \bigoplus_{k \in \mathbb{Z}} t^k W_k \subseteq W[t, t^{-1}]$ *is a graded* $\mathrm{gr}(O)$*-subalgebra and the associated sheaf of algebras $\mathcal{A}$ on X is coherent and torsion-free.*

(2) *There is a natural c-equivariant isomorphism $\zeta\colon \widehat{\mathcal{A}}_p \to A[\![z]\!]$ such that the induced map $\zeta\colon \mathrm{Q}(\widehat{\mathcal{A}}_p) \to A((z))$ satisfies $\zeta(\Gamma(X \setminus \{p\}, \mathcal{A})) = W$.*

(3) $\mathrm{H}^0(\mathcal{A}) = h_0$ *and* $\mathrm{H}^1(\mathcal{A}) = h_1$.

Proof.

Step 1: $\mathrm{gr}(W) \subseteq W[t, t^{-1}]$ *is a graded* $\mathrm{gr}(O)$*-subalgebra and $\mathcal{A}$ is torsion-free.* — The fact that $O_k W_\ell, \mu_A(W_k \otimes W_\ell) \subseteq W_{k+\ell}$ for all $k, \ell \in \mathbb{Z}$ immediately implies that $\mathrm{gr}(W) \subseteq W[t, t^{-1}]$ is a graded $\mathrm{gr}(O)$-subalgebra. Here $\mu_A\colon A((z)) \otimes_{\Bbbk((z))} A((z)) \to A((z))$ denotes the multiplication map of $A((z))$, which can be identified with the $\Bbbk((z))$-linear extension of the multiplication map of A. It is obvious that $\mathcal{A}$ is torsion-free since $\mathrm{gr}(W)$ is.

Step 2: $\mathcal{A}$ is coherent. — Since X is noetherian, we have to prove that $\mathrm{gr}(W)$ is finitely-generated by [28, Proposition 5.11 (c)]. Choose $r \in \mathbb{N}$ such that for all $k \geqslant r$ and for all $v \in A$ there exists an element in W with main part vz^{-k} as well as an element in O with main part z^{-k}. Since $W_0 = W \cap A[\![z]\!]$ is finite-dimensional, W_k is too, for all $k \in \mathbb{Z}$ and $W_{-k} = \{0\}$ for k sufficiently large. Let B be any basis of the finite-dimensional vector space $\bigoplus_{k=-\infty}^{2r} t^k W_k \subseteq \mathrm{gr}(W)$ and M be the $\mathrm{gr}(O)$-submodule of $\mathrm{gr}(W)$ spanned by B. From $\Bbbk = O_0$ we see that $t^k W_k \subseteq M$ for all $k \leqslant 2r$. We show $\mathrm{gr}(W) = M$ through proving $t^k W_k \subseteq M$ by induction on $k \geqslant 2r$, which has already been verified for $k = 2r$. By induction assumption $t^{k-1} W_{k-1} \subseteq M$, which immediately implies that $t^k W_{k-1} \subseteq M$ since $t \in \mathrm{gr}(O)$. Let $a \in W$ have main part vz^{-k}. There exists $b \in W$ with main part vz^{-r} and $h \in O$ with main part $z^{-r-\ell}$ for $\ell = k - 2r$. Therefore, $t^k a - t^{r+\ell} h t^r a \in t^k W_{k-1} \subseteq M$. The observations $t^r b \in M$ and $t^{r+\ell} h \in \mathrm{gr}(O)$ show that $t^k a \in M$, which concludes the induction.

Step 3: Construction of ζ and proof of $\zeta(\Gamma(X \setminus \{p\}, \mathcal{A})) = W$. — The same reasoning as in Step 3 of Proposition 3.15 yields $\mathcal{A}_p \cong \mathrm{Q}(W) \cap A[\![z]\!]$. For each $v \in A$ exists $k \in \mathbb{N}$ such that there is some $a \in W$ with main part vz^{-k} and $h \in O$ with main part z^{-k}. Therefore, $a/h \in \mathrm{Q}(W) \cap A[\![z]\!]$ satisfies $a(0) = v$. This shows that $\Bbbk[\![z]\!](Q(W) \cap A[\![z]\!]) = A[\![z]\!]$. Using c we get ζ as the composition

$$\widehat{\mathcal{A}}_p \cong \mathcal{A}_p \otimes_{\mathcal{O}_{X,p}} \widehat{\mathcal{O}}_{X,p} \cong (\mathrm{Q}(W) \cap A[\![z]\!]) \otimes_{\mathrm{Q}(O) \cap \Bbbk[\![z]\!]} \Bbbk[\![z]\!] \cong A[\![z]\!],$$

where the last isomorphism is given by multiplication. The same arguments as in Step 4 of the proof of Proposition 3.15 imply that $\zeta(\Gamma(\mathrm{d}_+(t), \mathcal{A})) = \mathrm{gr}(W)[t^{-1}]_0 = W$, so $\mathrm{d}_+(t) = X \setminus \{p\}$ concludes the proof.

Step 4: $\mathrm{H}^0(\mathcal{A}) = h_0, \mathrm{H}^1(\mathcal{A}) = h_1$. — Applying ζ to (3.12) for $\mathcal{F} = \mathcal{A}$ results in $\mathrm{H}^0(\mathcal{A}) = h_0$ and $\mathrm{H}^1(\mathcal{A}) = h_1$. □

Definition 3.17. — *We call a quintuple* $((X, \mathcal{A}), (p, c, \zeta))$ geometric A-lattice model *for a finite-dimensional* $\Bbbk$*-algebra* A *if:*

- $\mathcal{A}$ *is a torsion-free coherent sheaf of algebras on an integral projective curve* X *over* $\Bbbk$*;*
- $p \in X$ *is a* $\Bbbk$*-rational smooth point equipped with a* $\Bbbk$*-algebra isomorphism* $c\colon \widehat{\mathcal{O}}_{X,p} \to \Bbbk[\![z]\!]$ *and a* c*-equivariant isomorphism* $\zeta\colon \widehat{\mathcal{A}}_p \to A[\![z]\!]$.

Furthermore, if O *is a unital* $\Bbbk$*-lattice and* W *is some* A*-lattice satisfying* $OW \subseteq W$*, we write* $\mathrm{GD}(O, W) := ((X, \mathcal{A}), (p, c, \zeta))$ *for the geometric* A*-lattice model associated to the pair* (O, W) *by virtue of Theorem 3.16.*

Definition 3.18. — *Let* A *be a finite-dimensional* $\Bbbk$*-algebra. We call two geometric* A*-lattice models* $((X_1, \mathcal{A}_1), (p_1, c_1, \zeta_1))$ *and* $((X_2, \mathcal{A}_2), (p_2, c_2, \zeta_2))$ isomorphic *if there exists an isomorphism* $f\colon X_2 \to X_1$*, mapping* p_2 *to* p_1*, and an isomorphism* $\psi\colon \mathcal{A}_1 \to f_*\mathcal{A}_2$.

Remark 3.19. — The construction of geometric A-lattice models from lattices can be inverted: let A be a finite-dimensional $\Bbbk$-algebra and

$$((X, \mathcal{A}), (p, c, \zeta))$$

be a geometric A-lattice model. Then the exact sequence (3.12) for $\mathcal{F} \in \{\mathcal{O}_X, \mathcal{A}\}$ implies that $O := c(\Gamma(X \setminus \{p\}, \mathcal{O}_X))$ is a $\Bbbk$-lattice and $W := \zeta(\Gamma(X \setminus \{p\}, \mathcal{A}))$ is an A-lattice such that $OW \subseteq W$. In fact, $\mathrm{GD}(O, W) \cong ((X, \mathcal{A}), (p, c, \zeta))$ holds, where "$\cong$" is the isomorphism of geometric A-lattice models from Definition 3.18.

Remark 3.20. — Let O_1 be a unital $\Bbbk$-lattice, W be some A-lattice satisfying $O_1W \subseteq W$, and consider the associated geometric datum $\mathrm{GD}(O_1, W) := ((X_1, \mathcal{A}_1), (p_1, c_1, \zeta_1))$. Now let O_2 be a $\Bbbk$-algebra extension of O_1 such that

$$O_2W \subseteq W \quad \text{and} \quad \mathrm{GD}(O_2, W) := ((X_2, \mathcal{A}_2), (p_2, c_2, \zeta_2)).$$

Then $\dim(O_2/O_1) < \infty$ and $O_1 \to O_2$ induces a finite morphism $\nu\colon X_2 \to X_1$ which maps p_2 to p_1 and is compatible with c_1 and c_2. Moreover, $\mathcal{A}_1 = \nu_*\mathcal{A}_2$ and this equality identifies ζ_1 with ζ_2.

We call a geometric A-lattice model $((X,\mathcal{A}),(p,c,\zeta))$ a *trivial cover* of $((X_1,\mathcal{A}_1),(p_1,c_1,\zeta_1))$ if it is isomorphic to $((X_2,\mathcal{A}_2),(p_2,c_2,\zeta_2))$ for some $\Bbbk$-algebra extension O_2 of O_1. Observe that we may chose $O_2 = \mathrm{Mult}(W) := \{f \in \Bbbk(\!(z)\!) \mid fW \subseteq W\}$.

Definition 3.21. — *Let A be a finite-dimensional $\Bbbk$-algebra. We call two geometric A-lattice models $((X_1,\mathcal{A}_1),(p_1,c_1,\zeta_1))$ and $((X_2,\mathcal{A}_2),(p_2,c_2,\zeta_2))$* equivalent *if there are isomorphic (see Definition 3.18) trivial covers (see Remark 3.20) of $((X_1,\mathcal{A}_1),(p_1,c_1,\zeta_1))$ and $((X_2,\mathcal{A}_2),(p_2,c_2,\zeta_2))$.*

Remark 3.22. — Let A be a finite-dimensional $\Bbbk$-algebra, O be a unital $\Bbbk$-lattice, W be some A-lattice satisfying $OW \subseteq W$, and $\mathrm{GD}(O,W) = ((X,\mathcal{A}),(p,c,\zeta))$.

(1) If A is a Lie algebra, $\mathcal{A}$ is a sheaf of Lie algebras, since the sections of $\mathcal{A}$ can be identified with subalgebras of $A(\!(z)\!)$ using ζ.
(2) $\mathcal{A}$ is formally A-locally free in p, hence weakly A-locally free in p. Therefore, Theorem 3.11 implies that $\mathcal{A}$ is étale A-locally free in p if A is a semi-simple Lie algebra.

The problem of the existence of a $\Bbbk$-lattice stabilizing some A-lattice is considered in [44] for a finite-dimensional simple $\Bbbk$-algebra A in the case that $\Bbbk$ is algebraically closed. However, the methods actually apply to simple Lie algebras over arbitrary fields of characteristic 0 under an additional assumption.

Definition 3.23. — *Let A be a finite-dimensional $\Bbbk$-algebra and C be the centralizer of the subspace of* $\mathrm{End}(A)$ *generated by all left and right multiplication maps of A, i.e. $L(ab) = aL(b) = L(a)b$ for all $L \in C, a, b \in A$. C is called* centroid *of A and A is said to be* central *if $C = \Bbbk\,\mathrm{id}_A$.*

Remark 3.24. — Every finite-dimensional simple $\Bbbk$-algebra is central if $\Bbbk$ is algebraically closed. Indeed, [30, Chapter X, Theorem 1] tells us that the centroid of a finite-dimensional simple $\Bbbk$-algebra is a field and as a consequence a finite field extension of $\Bbbk$.

Theorem 3.25 ([44]). — *Let A be a finite-dimensional, central, simple $\Bbbk$-algebra and W be some A-lattice. The unital $\Bbbk$-algebra*

$$\mathrm{Mult}(W) := \{f \in \Bbbk(\!(z)\!) \mid fW \subseteq W\}$$

is a $\Bbbk$-lattice.

Proof.

Step 1: Setup. — Let us denote by r_a (resp. ℓ_a) the right (resp. left) multiplication by an element $a \in A(\!(z)\!)$ considered as an element in $\mathrm{End}_{\Bbbk(\!(z)\!)}(A(\!(z)\!)) \cong \mathrm{End}(A)(\!(z)\!)$, i.e. $\ell_a(b) = ab = r_b(a)$ for all $a, b \in A(\!(z)\!)$. Note that $r_a, \ell_a \in \mathrm{End}(A)$ for $a \in A$. The fact that A is central combined with [30, Chapter X, Theorem 4.] implies that the subalgebra of $\mathrm{End}(A)$ generated by $\{r_a, \ell_a \mid a \in A\}$ equals $\mathrm{End}(A)$. Let $J \subseteq \mathrm{End}(A)(\!(z)\!)$ be the subalgebra generated by $\{r_a, \ell_a \mid a \in W\}$.

Step 2: For every $L \in \mathrm{End}(A)$ there exists $r(L) \in \mathbb{N}_0$ such that for all $k \in \mathbb{N}_0$ there is an element of J with main part $z^{-k-r(L)}L$. — There exists a non-commutative polynomial $f = f(x_1, \dots, x_q)$ and $a_1, \dots, a_q \in A$ such that $f(m_1, \dots, m_q) = L$, where $m_i \in \{r_{a_i}, \ell_{a_i}\}$ for all $i \in \{1, \dots, q\}$. Let $\{f_i\}_{i=1}^q$ be the unique polynomials defined by the following inductive process: f_q is the sum of all monomials of f depending on x_q and, if $f_{i+1}, \dots, f_q$ are given, f_i is the sum of all monomials of $f - f_{i+1} - \dots - f_q$ which depend on x_i. By construction, f_i depends only on $x_1, \dots, x_i$ and every monomial of f_i contains a factor x_i. Let f_{ij} denote the homogeneous component of f_i of degree j and $g_{ij}(x_1, \dots, x_i; y_i)$ be the polynomial where the left most x_i appearing in every monomial of f_{ij} is changed to y_i. Since W is an A-lattice, we can chose $s \in \mathbb{N}$ such that for all $i \in \{1, \dots, q\}$ and $k \in \mathbb{N}_0$ there exists b_i^k in W with main part $z^{-s-k}a_i$. Let $\widetilde{m}_i^k$ be the left (resp. right) multiplication by b_i^k if m_i is the left (resp. right) multiplication by a_i. By construction $g_{ij}\left(\widetilde{m}_1^0, \dots, \widetilde{m}_i^0; \widetilde{m}_i^k\right)$ has main part $z^{-k-sj}f_{ij}(m_1, \dots, m_i)$. Thus,

$$\sum_{i=1}^{q} \sum_{j=1}^{\deg(f_i)} g_{ij}\left(\widetilde{m}_1^0, \dots, \widetilde{m}_i^0; \widetilde{m}_i^{k+s(\deg(f)-j)}\right) \tag{3.13}$$

has main part $z^{-k-s\deg(f)}L$ and putting $r(L) \coloneqq s\deg(f)$ proves the statement.

Step 3: There exists $r \in \mathbb{N}_0$ such that for all $L \in \mathrm{End}(A)$ and $k \in \mathbb{N}_0$ there is an element of J with main part $z^{-r-k}L$. — Let $\{L_i\}_{i=1}^n$ be a basis of $\mathrm{End}(A)$. Using Step 2, the result follows by choosing

$$r \coloneqq \max\{r(L_i) \mid i \in \{1, \dots, q\}\}.$$

Step 4: $\mathrm{Mult}(W)$ is a $\Bbbk$-lattice. — In [6] the author constructs a non-commutative homogeneous polynomial $P = P(x_1, \dots, x_q)$ in $q \coloneqq 2\dim(A)^2$-variables which is linear in each variable and non-vanishing with values in $\Bbbk\,\mathrm{id}_A$ if evaluated on $\mathrm{End}(A)$. In particular, we may choose $L_1, \dots, L_q \in \mathrm{End}(A)$ such that $P(L_1, \dots, L_q) = \mathrm{id}_A$. Let $r \in \mathbb{N}$ be the integer from

Step 3, i.e. for all $k \in \mathbb{N}_0$ and $i \in \{1, \dots, q\}$ we may chose $\widetilde{L}_i^k \in J$ with main part $z^{-r-k}L_i$. We have

$$P\left(\widetilde{L}_1^0, \dots, \widetilde{L}_{q-1}^0, \widetilde{L}_q^{\ell-rq}\right) = f_\ell \,\mathrm{id}_A \tag{3.14}$$

for all $\ell \geqslant rq$, where $f_\ell \in \Bbbk((z))$ has main part $z^{-\ell}$. Since the left-hand side of (3.14) is an element of J, we can see that $f_\ell \in \mathrm{Mult}(W)$. This concludes the proof. □

3.4. Sheaves of Lie algebras on one-dimensional algebraic groups

Recall that a one-dimensional connected complex algebraic group is, up to isomorphism, either an elliptic curve, $\mathbb{C}$, or $\mathbb{C}^\times$. In this section, we will see that the sheaves of Lie algebras with simple fibers on these curves are completely classified. This will be important in our new proof of the Belavin–Drinfeld trichotomy in Section 4.7.

The classification in the affine cases relies on the notion of torsors, which give a description of isomorphism classes of étale locally trivial sheaves of algebras. Let us briefly recall said notion in the limited scope needed in the following, we refer to e.g. [39, Section III.4] for more details. Let X be a $\Bbbk$-scheme of finite type and G be a group scheme over X. Note that étale coverings of X are without loss of generality surjective étale morphisms $Y \to X$; see Remark 3.8. For a surjective étale morphism $Y \to X$, the set $Z^1(Y/X, G)$ of *1-cocycles* trivialized on Y with values in G consists of morphisms $g\colon Y \times_X Y \to G$ of X-schemes satisfying $\mathrm{pr}_{31}^*(g) = \mathrm{pr}_{32}^*(g)\,\mathrm{pr}_{21}^*(g)$, where

$$\mathrm{pr}_{ij}\colon Y \times_X Y \times_X Y \longrightarrow Y \times_X Y \qquad (y_1, y_2, y_3) \longmapsto (y_i, y_j) \tag{3.15}$$

are the canonical projections for $ij \in \{21, 31, 32\}$. Two 1-cocycles $g, g' \in Z^1(Y/X, G)$ are called *cohomologous*, written in symbols as $g \sim g'$, if there exists a morphism $h\colon Y \to G$ of X-schemes such that $g' = \mathrm{pr}_2^*(h) g \,\mathrm{pr}_1^*(h)^{-1}$, where $\mathrm{pr}_1, \mathrm{pr}_2\colon Y \times_X Y \to Y$ are the canonical projections. We write $\check{\mathrm{H}}^1(Y/X, G) := Z^1(Y/X, G)/\sim$. If we have another surjective étale map $Y' \to X$ factoring over $Y \to X$, there exists a natural induced map $\check{\mathrm{H}}(Y/X, G) \to \check{\mathrm{H}}(Y'/X, G)$. The set of étale G-torsors is given by

$$\check{\mathrm{H}}^1(X_{\text{ét}}, G) := \varinjlim \check{\mathrm{H}}^1(Y/X, G),$$

where the limit is taken over the directed set of surjective étale morphisms $Y \to X$.

Lemma 3.26. — *Let $\Bbbk = \overline{\Bbbk}$, X be a reduced $\Bbbk$-scheme of finite type, and A be a finite-dimensional $\Bbbk$-algebra. The isomorphism classes of étale A-locally free sheaves of algebras are in bijection with the set*

$$\check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{Aut}_{\Bbbk\text{-alg}}(A)_X), \quad \textit{where } \mathrm{Aut}_{\Bbbk\text{-alg}}(A)_X := X \times G$$

for the unique group scheme G over $\mathrm{Spec}(\Bbbk)$ *with closed points* $\mathrm{Aut}_{\Bbbk\text{-alg}}(A)$.

Proof. — Let E be the set of isomorphism classes of étale A-locally free sheaves of algebras.

Step 1: Construction of $E \to \check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{Aut}_{\Bbbk\text{-alg}}(A)_X)$. — Let $\mathcal{A}$ be an étale A-locally free sheaf over X. Under consideration of Remark 3.8, there exists a surjective étale morphism $f\colon Y \to X$ of $\Bbbk$-schemes such that we may choose an isomorphism $f^*\mathcal{A} \cong A \otimes \mathcal{O}_Y$ of sheaves of algebras. Let $\mathrm{pr}_1, \mathrm{pr}_2\colon Y \times_X Y \to Y$ be the canonical projections. The chain of isomorphisms

$$A \otimes \mathcal{O}_{Y \times_X Y} \cong \mathrm{pr}_1^* f^*\mathcal{A} \cong \mathrm{pr}_2^* f^*\mathcal{A} \cong A \otimes \mathcal{O}_{Y \times_X Y}$$

determines a regular map from the algebraic prevariety of closed points of $Y \times_X Y$ to $\mathrm{Aut}_{\Bbbk\text{-alg}}(A)$. This induces an unique morphism $Y \times_X Y \to G$ of $\Bbbk$-schemes, which in turn defines a morphism $g\colon Y \times_X Y \to X \times G$ of X-schemes. It is straight forward to see that $\mathrm{pr}_{31}^*(g) = \mathrm{pr}_{32}^*(g)\,\mathrm{pr}_{21}^*(g)$ for the canonical projections $\mathrm{pr}_{ij}\colon Y \times_X Y \times_X Y \to Y \times_X Y$, where $ij \in \{21, 31, 32\}$, and that a sheaf of algebras isomorphic to $\mathcal{A}$ defines a 1-cocycle cohomologous to g. Thus, we have constructed a map $E \to \check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{Aut}_{\Bbbk\text{-alg}}(A)_X)$.

Step 2: Construction of the inverse of $E \to \check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{Aut}_{\Bbbk\text{-alg}}(A)_X)$. — Let $f\colon Y \to X$ be a surjective étale morphism and $g\colon Y \times_X Y \to X \times G$ be a 1-cocycle. Then g defines an isomorphism $\psi\colon A \otimes \mathcal{O}_{Y \times_X Y} \to A \otimes \mathcal{O}_{Y \times_X Y}$ of sheaves of algebras such that $\mathrm{pr}_{31}^*(\psi) = \mathrm{pr}_{32}^*(\psi)\,\mathrm{pr}_{21}^*(\psi)$ for the canonical projections $\mathrm{pr}_{ij}\colon Y \times_X Y \times_X Y \to Y \times_X Y$, where $ij \in \{21, 31, 32\}$. Therefore, we obtain a quasi-coherent sheaf $\mathcal{A}$ on X equipped with an isomorphism $f^*\mathcal{A} \cong A \otimes \mathcal{O}_Y$ using faithfully flat descent; see e.g. [39, Proposition 2.22]. If X and Y are affine, $\mathcal{A}$ is simply the direct image via $Y \to X$ of the kernel of $\psi\,\mathrm{pr}_1^* - \mathrm{pr}_2^*\colon A \otimes \mathcal{O}_Y \to A \otimes \mathcal{O}_{Y \times Y}$. This can be used to see that $\mathcal{A}$ is a subsheaf of algebras of $A \otimes \mathcal{O}_Y$, since ψ is an automorphism of sheaves of algebras, and that a 1-cocycle cohomologous to g defines a sheaf of algebras isomorphic to $\mathcal{A}$. The resulting map $\check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{Aut}_{\Bbbk\text{-alg}}(A)_X) \to E$ is clearly inverse to $E \to \check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{Aut}_{\Bbbk\text{-alg}}(A)_X)$. □

Theorem 3.27. — *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\Bbbk$, $\Bbbk = \overline{\Bbbk}$, and $\mathcal{A}$ be a weakly $\mathfrak{g}$-locally free sheaf of Lie algebras on a $\Bbbk$-scheme X of finite type.*

(1) *If* $X = \mathrm{Spec}(\Bbbk[u,u^{-1}])$, *there exists* $\sigma \in \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})$ *of order* $m \in \mathbb{N}$ *and a primitive* m^{th} *root of unity* $\varepsilon \in \Bbbk$ *such that* $\mathcal{A}$ *is isomorphic to the sheaf of Lie algebras associated to the twisted loop algebra*

$$\mathfrak{L}(\mathfrak{g},\sigma) \coloneqq \left\{ f(\widetilde{u}) \in \mathfrak{g}[\widetilde{u},\widetilde{u}^{-1}] \,\middle|\, f(\varepsilon\widetilde{u}) = \sigma(f(\widetilde{u})) \right\}$$

on X, *where the module structure of* $\mathfrak{L}(\mathfrak{g},\sigma)$ *is defined by* $u = \widetilde{u}^m$.

(2) *If* $X = \mathrm{Spec}(\Bbbk[u])$, $\mathcal{A}$ *is isomorphic to the sheaf of Lie algebras associated to* $\mathfrak{g}[u]$ *on* X.

Proof.

Step 1: Setup. — In both cases Theorem 3.10 states that $\mathcal{A}$ is automatically étale $\mathfrak{g}$-locally free on X and for this reason, up to isomorphism, determined by an element of $\check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})_X)$ by virtue of Lemma 3.26. The arguments in [46] imply that, since the Lie group of $\mathfrak{g}$ is reductive, there is a canonical injection $\check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})_X) \to \check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{Out}(\mathfrak{g})_X)$, where $\mathrm{Out}(\mathfrak{g})$ is the automorphism group of the Dynkin diagram of $\mathfrak{g}$ and $\mathrm{Out}(\mathfrak{g})_X \coloneqq X \times \mathrm{Out}(\mathfrak{g})$. The case of $X = \mathrm{Spec}(\Bbbk[u,u^{-1}])$ is thereby considered explicitly in [46] while the case of $X = \mathrm{Spec}(\Bbbk[u])$ works analogous. Since $\mathrm{Out}(\mathfrak{g})$ is finite, we have a bijection of

$$\check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{Out}(\mathfrak{g})_X)$$

and the non-abelian continuous cohomology group $\mathrm{H}^1(\pi_1(X,x), \mathrm{Out}(\mathfrak{g}))$, where $x \in X$ is a closed point, $\pi_1(X,x)$ is the associated étale fundamental group and the action of $\pi_1(X,x)$ on $\mathrm{Out}(\mathfrak{g})$ is trivial.

Step 2: Proof of (1). — If $X = \mathrm{Spec}(\Bbbk[u,u^{-1}])$, we can choose $x = (u-1)$ and then it is explained in [46] that $\mathrm{H}^1(\pi_1(X,x), \mathrm{Out}(\mathfrak{g}))$ is in bijection with the conjugacy classes in $\mathrm{Out}(\mathfrak{g})$ and the $\mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})_X$-torsor of $\mathfrak{L}(\mathfrak{g},\sigma)$ is mapped to the conjugacy class of the class of σ^{-1} in $\mathrm{Out}(\mathfrak{g})$. In particular, every $\mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})_X$-torsor is represented by some $\mathfrak{L}(\mathfrak{g},\sigma)$ for an appropriate σ.

Step 3: Proof of (2). — If $X = \mathrm{Spec}(\Bbbk[u])$ and $x = (u)$, the group $\pi_1(X,x)$ is trivial. Therefore, $\check{\mathrm{H}}^1(X_{\text{ét}}, \mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})_X)$ has only one element, consisting of the trivial $\mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})_X$-torsor on X, i.e. the one represented by $\mathfrak{g}[u]$. □

Let us now consider an elliptic curve X over $\Bbbk = \mathbb{C}$ and write $X^{\mathrm{an}} = (X^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}})$ for the locally ringed space obtained by considering this curve as a complex manifold. Then X^{an} is isomorphic to $\mathbb{C}/\Lambda$ for some lattice $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2 \subset \mathbb{C}$ of rank two. Let $\iota\colon X^{\mathrm{an}} \to X$ denote the morphism of locally ringed spaces which identifies points of X^{an} with closed points of X on the level of topological spaces, while $\iota^\flat\colon \mathcal{O}_X \to \iota_*\mathcal{O}_X^{\mathrm{an}}$ recognizes regular functions as holomorphic ones. Then the results of Serre's GAGA [48], tell us that $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}} := \iota^*\mathcal{F}$ defines an equivalence between the category of locally free sheaves on X and the category of sheaves of sections of holomorphic vector bundles over X^{an}. Furthermore, the field of global meromorphic functions on X^{an} coincides with the field of rational functions of X; see e.g. [25, Section 3.1]. Consequently, the pull-back ι^* identifies $\Gamma(U,\mathcal{F})$, for $U \subseteq X$ open, with the global meromorphic sections of $\mathcal{F}^{\mathrm{an}}$ that are holomorphic on $\iota^{-1}(U)$.

Let $\pi\colon \mathbb{C} \to \mathbb{C}/\Lambda \cong X^{\mathrm{an}}$ be the canonical map, $\mathcal{A}$ be a locally free sheaf of rank d on X and $E \to X^{\mathrm{an}}$ be the vector bundle with sheaf of holomorphic sections $\mathcal{A}^{\mathrm{an}}$. In e.g. [29] it is explained that, since π^*E is trivial, E is determined by some holomorphic map $\phi\colon \Lambda \times \mathbb{C} \to \mathrm{GL}(d,\mathbb{C})$ satisfying

$$\Phi(\lambda+\lambda',z) = \Phi(\lambda,z+\lambda')\Phi(\lambda',z) \qquad \lambda,\lambda' \in \Lambda, z \in \mathbb{C}, \tag{3.16}$$

called *factor of automorphy*, in the sense that

$$E = \mathbb{C} \times \mathbb{C}^d/\sim \qquad (z,a) \sim (z+\lambda,\Phi(\lambda,z)a), \quad \forall\, \lambda \in \Lambda. \tag{3.17}$$

Assume that $\mathcal{A}$ is an étale $\mathfrak{g}$-locally free sheaf of Lie algebras for some finite-dimensional complex Lie algebra $\mathfrak{g}$. Then it is easy to see that E is a holomorphic fiber bundle with fiber $\mathfrak{g}$ and structure group $\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$. Therefore, [24, Satz 6] implies that $\pi^*E \cong \mathbb{C} \times \mathfrak{g}$ as holomorphic fiber bundles. This implies that Φ takes values in $\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$.

Theorem 3.28. — *Let $\mathcal{A}$ be an acyclic (i.e. $\mathrm{H}^1(\mathcal{A}) = 0$) sheaf of Lie algebras on X, which is weakly $\mathfrak{g}$-locally free in all closed points of X for some simple, finite-dimensional, complex Lie algebra $\mathfrak{g}$. There exists $n \in \mathbb{N}_0$ and $0 < m < n$ such that $\gcd(n,m) = 1$, $\mathfrak{g} \cong \mathfrak{sl}(n,\mathbb{C})$ and $\mathcal{A}^{\mathrm{an}}$ is isomorphic to the sheaf of holomorphic sections of $\mathbb{C} \times \mathfrak{sl}(n,\mathbb{C})/\sim$, where for $\varepsilon := \exp(2\pi i m/n)$ and*

$$T_1 := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \varepsilon & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varepsilon^{n-1} \end{pmatrix}, \qquad T_2 := \begin{pmatrix} 0 & \dots & 0 & 1 \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}.$$

the relation $\sim$ is defined by $(z,a) \sim (z+\lambda_1, T_1 a T_1^{-1}) \sim (z+\lambda_2, T_2 a T_2^{-1})$.

Proof.

Step 1: $\mathrm{H}^0(\mathcal{A}) = 0 = \mathrm{H}^1(\mathcal{A})$. — The morphism $\mathcal{A} \to \mathcal{A}^*$ induced by the Killing form of $\mathcal{A}$ is an isomorphism, since $\mathcal{A}|_p \cong \mathfrak{g}$ is simple for all $p \in X$ closed; see Lemma 3.5. Therefore, using Serre duality, we see that $\mathrm{H}^0(\mathcal{A}) = \mathrm{H}^1(\mathcal{A}^*) = \mathrm{H}^1(\mathcal{A}) = 0$.

Step 2: Description of $\Gamma(X \setminus \{p\}, \mathcal{A})$, *where* p *is the image of* $\overline{0}$ *under* $\mathbb{C}/\Lambda \to X^{\mathrm{an}} \to X$. — Theorem 3.10 states that $\mathcal{A}$ is étale $\mathfrak{g}$-locally trivial. As argued above, this implies that the holomorphic vector bundle $E \to X^{\mathrm{an}}$ with holomorphic sheaf of sections $\mathcal{A}^{\mathrm{an}}$ is determined by a factor of automorphy $\Phi\colon \mathbb{C} \times \Lambda \to \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$. In particular, $\Gamma(X \setminus \{p\}, \mathcal{A})$ can be identified with the algebra of meromorphic functions $a\colon \mathbb{C} \to \mathfrak{g}$ holomorphic on $\mathbb{C} \setminus \Lambda$ and satisfying $a(z+\lambda) = \Phi(\lambda, z)a(z)$ for all $\lambda \in \Lambda, z \in \mathbb{C} \setminus \Lambda$.

Step 3: Φ *is locally constant up to isomorphism.* — The Taylor series of elements of $\mathcal{A}^{\mathrm{an}}$ with respect to a chosen holomorphic coordinate z on $\mathbb{C}$ combined with the canonical isomorphism $\widehat{\mathcal{A}}_p \cong \widehat{\mathcal{A}}_p^{\mathrm{an}}$ results in an isomorphism $\zeta\colon \widehat{\mathcal{A}}_p \to \mathfrak{g}[\![z]\!]$. Using Step 1 and (3.12) yields

$$\mathfrak{g}(\!(z)\!) = \mathfrak{g}[\![z]\!] \oplus \zeta(\Gamma(X \setminus \{p\}, \mathcal{A})),$$

where we note that ζ identifies sections of $\mathcal{A}$, viewed as meromorphic functions $\mathbb{C} \to \mathfrak{g}$ with their Laurent series in $z = 0$. Therefore, Proposition 2.14 implies that there exists a normalized formal generalized r-matrix r such that $\mathfrak{g}(r) = \zeta(\Gamma(X \setminus \{p\}, \mathcal{A}))$. We can chose a global 1-form η on X such that $\pi^*\iota^*\eta = \mathrm{d}z$ as a holomorphic 1-form on $\mathbb{C}$. The residue theorem implies that

$$\mathrm{res}_0\, \kappa(\zeta(a), \zeta(b))\mathrm{d}z = \mathrm{res}_p\, K(a,b)\eta = 0$$

for all $a, b \in \Gamma(X \setminus \{p\}, \mathcal{A})$, where K is the Killing form of $\mathcal{A}$. Therefore, $\mathfrak{g}(r)^\perp = \mathfrak{g}(r)$ and Proposition 2.21 forces r to be skew-symmetric. Combining Lemma 4.5 and Proposition 2.29, we may assume that $\mathfrak{g}(r)$, and therefore $\zeta(\Gamma(X \setminus \{p\}, \mathcal{A}))$, is closed under the derivation with respect to z, after probably replacing $\mathcal{A}$ with an isomorphic sheaf of Lie algebras. In particular, we have

$$\Phi(\lambda, z)\frac{\mathrm{d}a}{\mathrm{d}z}(z) = \frac{\mathrm{d}a}{\mathrm{d}z}(z+\lambda) = \frac{\partial \Phi}{\partial z}(\lambda, z)a(z) + \Phi(\lambda, z)\frac{\mathrm{d}a}{\mathrm{d}z}(z). \tag{3.18}$$

for every $a \in \Gamma(X \setminus \{p\}, \mathcal{A})$. Therefore, $\frac{\partial \Phi}{\partial z}(\lambda, z)a(z) = 0$ for all $z \in \mathbb{C} \setminus \Lambda$, $a \in \Gamma(X \setminus \{p\}, \mathcal{A})$. Since $\zeta(\Gamma(X \setminus \{p\}, \mathcal{A})) \otimes \mathbb{C}(\!(z)\!) = \mathfrak{g}(\!(z)\!)$, we see that $\frac{\partial \Phi}{\partial z}(\lambda, z) = 0$, and thus $\Phi_\lambda \coloneqq \Phi(\lambda, z) \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ is independent of z.

Step 4: $P \coloneqq \{\Phi_\lambda\}_{\lambda \in \Lambda}$ *is a finite abelian group.* — Equation (3.16) gives $\Phi_\lambda \Phi_{\lambda'} = \Phi_{\lambda + \lambda'}$ for all $\lambda, \lambda' \in \Lambda$, so Φ is a commutative subgroup of $\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ generated by $\Phi_1 \coloneqq \Phi_{\lambda_1}, \Phi_2 \coloneqq \Phi_{\lambda_2}$. A non-zero element in $\mathfrak{g}$ which is fixed by all elements in P would define a global section of $\mathcal{A}$. Hence such an element does not exist by Step 1. Assume that P has infinite order and let $\mathfrak{s}$ be the Lie algebra of the smallest algebraic subgroup S of $\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ containing P. Since P is infinite $\mathfrak{s}$ can be identified with a non-zero subalgebra of $\mathfrak{g}$ and since P is abelian and dense (with respect to the

Zariski topology) in S, it can be shown that S is abelian. Therefore, the action of S on $\mathfrak{s}$ is trivial and each non-zero element of $\mathfrak{s}$ is fixed by all elements in P. This is a contradiction. We can conclude that P has finite order.

Step 5: Concluding the proof. — The commuting automorphisms Φ_1 and Φ_2 have finite order and no non-zero common fixed vector. Thus, [10, Theorem 9.3] implies that $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(n,\mathbb{C})$ via an isomorphism which identifies Φ_1 and Φ_2 with the conjugations by T_1 and T_2 respectively, for an appropriate choice of m. □

Remark 3.29. — For our purposes, it is actually sufficient that $\mathcal{A}^{\mathrm{an}}$ is isomorphic to the sheaf of sections of $\mathbb{C}\times\mathfrak{g}/\sim$, where

$$(z,a)\sim(z+\lambda_1,\Phi_1 a)\sim(z+\lambda_2,\Phi_2 a),$$

for some $\Phi_1,\Phi_2\in\operatorname{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ of finite order. Therefore, we could adjust the statement of Theorem 3.28 accordingly in order to drop the reference to [10] in Step 5 of its proof and stay self-contained. Nevertheless, we chose this presentation to show a conclusive result to the given classification problem.

4. Algebraic geometry of formal generalized r-matrices

Let $\mathfrak{g}$ be a finite-dimensional, central, simple Lie algebra over a field $\Bbbk$ of characteristic 0.

4.1. Geometrization of generalized r-matrices

Let $r\in(\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ be a formal generalized r-matrix. The algebra $\mathfrak{g}(r)$ is a $\mathfrak{g}$-lattice of index $(0,0)$ by Proposition 2.14, so Theorem 3.25 implies that every unital subalgebra O of

$$\operatorname{Mult}(\mathfrak{g}(r))=\{f\in\Bbbk((z))\mid f\mathfrak{g}(r)\subseteq\mathfrak{g}(r)\}$$

of finite codimension is a $\Bbbk$-lattice satisfying $O\mathfrak{g}(r)\subseteq\mathfrak{g}(r)$. Therefore, Theorem 3.16 provides a geometric datum $\operatorname{GD}(O,\mathfrak{g}(r))=((X,\mathcal{A}),(p,c,\zeta))$, where:

- X is an integral projective curve over $\Bbbk$ of arithmetic genus
  $$\dim(\Bbbk((z))/(\Bbbk[\![z]\!]+O))$$
  that comes equipped with a $\Bbbk$-algebra isomorphism $c\colon\widehat{\mathcal{O}}_{X,p}\to\Bbbk[\![z]\!]$ at a $\Bbbk$-rational smooth point $p\in X$, which induces an isomorphism $\operatorname{Spec}(O)\to X\setminus\{p\}$;

- $\mathcal{A}$ is a coherent sheaf of Lie algebras on X which satisfies $\mathrm{H}^0(\mathcal{A}) = 0 = \mathrm{H}^1(\mathcal{A})$, is étale $\mathfrak{g}$-locally free in p (see Remark 3.22) and admits a natural c-equivariant Lie algebra isomorphism $\zeta\colon \widehat{\mathcal{A}}_p \to \mathfrak{g}[\![z]\!]$ such that $\zeta(\Gamma(X \setminus \{p\}, \mathcal{A})) = \mathfrak{g}(r)$.

Remark 4.1. — Let us note that, if

$$P \subseteq \mathrm{Mult}(\mathfrak{g}(r)) = \{f \in \Bbbk(\!(z)\!) \mid f\mathfrak{g}(r) \subseteq \mathfrak{g}(r)\}$$

is another unital subalgebra of finite codimension, $\mathrm{GD}(P, \mathfrak{g}(r))$ is equivalent to $((X, \mathcal{A}), (p, c, \zeta))$ in the sense of Definition 3.21.

Definition 4.2. — *Let SolGCYBE be the set of formal generalized r-matrices in $(\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$ and GeomGCYBE be the set of geometric $\mathfrak{g}$-lattice models $((X, \mathcal{A}), (p, c, \zeta))$ satisfying $\mathrm{H}^0(\mathcal{A}) = 0 = \mathrm{H}^1(\mathcal{A})$. Then Remark 4.1 states that $r \mapsto \mathrm{GD}(O, \mathfrak{g}(r))$, where $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$ is an arbitrary unital subalgebra of finite codimension, defines a map*

$$\mathrm{SolGCYBE} \longrightarrow \mathrm{GeomGCYBE}\,/\sim,$$

where "$\sim$" is the equivalence from Definition 3.21. We call this map geometrization *of formal generalized r-matrices. In particular, we say that $((X, \mathcal{A}), (p, c, \zeta))$ is a geometrization of a formal generalized r-matrix r, if $((X, \mathcal{A}), (p, c, \zeta)) = \mathrm{GD}(O, \mathfrak{g}(r))$, where $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$ is any unital subalgebra of finite codimension*

Example 4.3. — Let $((X, \mathcal{A}), (p, c, \zeta))$ be a geometrization of a formal generalized r-matrix $r \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$. If, up to equivalence, $X \cong \mathbb{P}^1_{\Bbbk}$ holds, then $O := \mathrm{Mult}(\mathfrak{g}(r))$ satisfies $\Bbbk(\!(z)\!) = \Bbbk[\![z]\!] + O$. This and $O \cap \Bbbk[\![z]\!] = \Bbbk$ can be used to deduce that $O = \Bbbk[u^{-1}]$ for an arbitrary $u \in z\Bbbk[\![z]\!]^{\times}$ such that $u^{-1} \in O$. Let $w \in z\Bbbk[\![z]\!]^{\times}$ be the compositional inverse of u, i.e. $w(u(z)) = z$. Then $\widetilde{r}(x, y) := r(w(x), w(y))$ satisfies $\mathrm{Mult}(\mathfrak{g}(\widetilde{r})) = \Bbbk[z^{-1}]$. In other words, $\mathfrak{g}(\widetilde{r})$ is homogeneous in the sense of Example 2.19. In particular, the formal generalized r-matrices r for which $X \cong \mathbb{P}^1_{\Bbbk}$ holds up to equivalence of geometric $\mathfrak{g}$-lattice data are exactly those equivalent to the generalized r-matrices described in Example 2.19.

Remark 4.4. — Let $\Bbbk'$ be a field extension of $\Bbbk$ and let $r_{\Bbbk'} \in (\mathfrak{g}_{\Bbbk'} \otimes_{\Bbbk'} \mathfrak{g}_{\Bbbk'})(\!(x)\!)[\![y]\!]$ be the generalized r-matrix obtained by extension of scalars from a generalized r-matrix $r \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$; see Remark 2.9. Note that $\mathfrak{g}_{\Bbbk'} := \mathfrak{g} \otimes_{\Bbbk} \Bbbk'$ is a central simple Lie algebra over $\Bbbk'$ by [30, Chapter X, Theorem 3]. If $O_{\Bbbk'}$ denotes the image of $O \otimes \Bbbk'$ under the multiplication map $\Bbbk(\!(z)\!) \otimes \Bbbk' \to \Bbbk'(\!(z)\!)$, the geometric datum $\mathrm{GD}(O_{\Bbbk'}, \mathfrak{g}(r_{\Bbbk'})) = ((X_{\Bbbk'}, \mathcal{A}_{\Bbbk'}), (p_{\Bbbk'}, c_{\Bbbk'}, \zeta_{\Bbbk'}))$ satisfies the following compatibility conditions:

- $O_{\Bbbk'} \cong O \otimes \Bbbk'$ induces an isomorphism $X_{\Bbbk'} \cong X \times \mathrm{Spec}(\Bbbk')$ such that the canonical map $O \to O_{\Bbbk'}$ is compatible with the canonical morphism $\pi\colon X_{\Bbbk'} \to X$. Furthermore, $\pi(p_{\Bbbk'}) = p$ and the following diagram commutes

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,p} & \xrightarrow{c} & \Bbbk[\![z]\!] \\ {\scriptstyle \widehat{\pi}^\sharp_p}\downarrow & & \downarrow \\ \widehat{\mathcal{O}}_{X_{\Bbbk'},p_{\Bbbk'}} & \xrightarrow[c_{\Bbbk'}]{} & \Bbbk'[\![z]\!]. \end{array}$$

- The multiplication map $\mathfrak{g}(r)\otimes\Bbbk' \cong \mathfrak{g}_{\Bbbk'}(r_{\Bbbk'})$ induces an isomorphism $\pi^*\mathcal{A} \cong \mathcal{A}_{\Bbbk'}$ of sheaves of Lie algebras such that the following diagram commutes

$$\begin{array}{ccc} \widehat{\mathcal{A}}_p & \xrightarrow{\zeta} & \mathfrak{g}[\![z]\!] \\ \downarrow & & \downarrow \\ \widehat{\mathcal{A}}_{\Bbbk',p_{\Bbbk'}} & \xrightarrow[\zeta_{\Bbbk'}]{} & \mathfrak{g}_{\Bbbk'}[\![z]\!]. \end{array}$$

In particular, if $\Bbbk \to \Bbbk'$ is Galois with Galois group G, then G acts on $X_{\Bbbk'}$ by automorphisms of X-schemes and on $\mathcal{A}_{\Bbbk'}$ by $\mathcal{O}_X$-linear automorphisms of sheaves of Lie algebras in such a way that X and $\mathcal{A}$ are the respective fixed objects.

LEMMA 4.5. — *For $i \in \{1,2\}$, let $r_i \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$ be a generalized r-matrix with a geometrization $((X_i,\mathcal{A}_i),(p_i,c_i,\zeta_i))$. The series r_1 and r_2 are equivalent in the sense of Definition 2.11 if and only if $((X_1,\mathcal{A}_1),(p_1,c_1,\zeta_1))$ and $((X_2,\mathcal{A}_2),(p_2,c_2,\zeta_2))$ are equivalent in the sense of Definition 3.21.*

Proof.
"$\Longrightarrow$". — Let (λ,w,φ) be an equivalence of r_1 to r_2. We have to show that there are isomorphic trivial covers of $((X_1,\mathcal{A}_1),(p_1,c_1,\zeta_1))$ and $((X_2,\mathcal{A}_2),(p_2,c_2,\zeta_2))$ in the sense of Remark 3.20. Therefore, we can consider $O_1 = \mathrm{Mult}(\mathfrak{g}(r_1))$ and $O_2 = \mathrm{Mult}(\mathfrak{g}(r_2))$ and show that

$$\begin{aligned} &\mathrm{GD}(O_1,\mathfrak{g}(r_1)) = ((X_1,\mathcal{A}_1),(p_1,c_1,\zeta_1)) \text{ and} \\ &\mathrm{GD}(O_2,\mathfrak{g}(r_2)) = ((X_2,\mathcal{A}_2),(p_2,c_2,\zeta_2)) \end{aligned} \tag{4.1}$$

are isomorphic.

The assignment $p(z) \mapsto p_w(z) := p(w(z))$ defines an isomorphism $O_1 \to O_2$ resulting in a graded isomorphism

$$O_1[t] \supseteq \mathrm{gr}(O_1) \xrightarrow{f^\flat} \mathrm{gr}(O_2) \subseteq O_2[t]$$

which maps t to t. We can see from e.g. [23, Section 13.2] that this induces an isomorphism $f\colon X_2 \to X_1$. This isomorphism is seen to satisfy $f(p_2) = p_1$ after recalling the construction of p_1 and p_2 in Step 3 of the proof of Proposition 3.15. Lemma 2.17 provides an isomorphism $\mathfrak{g}(r_1) \to \mathfrak{g}(r_2)$ defined by $a(z) \mapsto \varphi(z)a(w(z))$. This induces a graded $f^\flat$-equivariant isomorphism $\mathrm{gr}(\mathfrak{g}(r_1)) \to \mathrm{gr}(\mathfrak{g}(r_2))$. The process of associating a quasi-coherent sheaf to a graded module is functorial (see e.g. [23, Section 13.4]) and the sheaf associated to $\mathrm{gr}(\mathfrak{g}(r_2))$ equipped with the $\mathrm{gr}(O_1)$-module structure induced by $f^\flat$ is exactly $f_*\mathcal{A}_2$. Thus, we obtain an isomorphism $\psi\colon \mathcal{A}_1 \to f_*\mathcal{A}_2$.

"$\Longleftarrow$". — After replacing O_1 and O_2 by some $\Bbbk$-algebra extensions, we may assume that the geometric A-lattice models $((X_1,\mathcal{A}_1),(p_1,c_1,\zeta_1))$ and $((X_2,\mathcal{A}_2),(p_2,c_2,\zeta_2))$ are isomorphic. In particular, there exists an isomorphism $f\colon X_2 \to X_1$, mapping p_2 to p_1, and an isomorphism $\psi\colon \mathcal{A}_1 \to f_*\mathcal{A}_2$.

Let w be the image of z under

$$\Bbbk[\![z]\!] \xrightarrow{c_1^{-1}} \widehat{\mathcal{O}}_{X_1,p_1} \xrightarrow{\widehat{f_p^\sharp}} \widehat{\mathcal{O}}_{X_2,p_2} \xrightarrow{c_2} \Bbbk[\![z]\!].$$

Then the $\Bbbk$-linear isomorphism

$$\mathfrak{g}[\![z]\!] \xrightarrow{\zeta_1^{-1}} \widehat{\mathcal{A}}_{1,p_1} \xrightarrow{\widehat{\psi}_{p_1}} \widehat{\mathcal{A}}_{2,p_2} \xrightarrow{\zeta_2} \mathfrak{g}[\![z]\!],$$

where we implicitly used $\left(\widehat{f_*\mathcal{A}_2}\right)_{p_1} \cong \widehat{\mathcal{A}}_{2,p_2}$, takes the form $a(z) \mapsto \varphi(z)\, a(w(z))$ for some $\varphi \in \mathrm{Aut}_{\Bbbk[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!])$. Lemma 2.17 and the calculation

$$\zeta_2\widehat{\psi}_{p_1}\zeta_1^{-1}(\mathfrak{g}(r_1)) = \zeta_2\widehat{\psi}_{p_1}(\Gamma(X_1 \setminus \{p_1\},\mathcal{A}_1)) = \zeta_2(\Gamma(X_2 \setminus \{p_2\},\mathcal{A}_2)) = \mathfrak{g}(r_2)$$

verify that r_1 is equivalent to r_2. □

Remark 4.6. — A special case of Lemma 4.5 was used in [3] to classify twists of the standard bialgebra structure on twisted loop algebras by reducing it to the classification of trigonometric r-matrices given in [9].

Lemma 4.7. — *Let $r \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$ be a formal generalized r-matrix and $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$ be a unital subalgebra of finite codimension. Furthermore, let $\widetilde{O}$ be the integral closure of O and put $\widetilde{\mathfrak{g}(r)} := \widetilde{O}\mathfrak{g}(r)$. Then $\widetilde{O}$ is naturally a $\Bbbk$-lattice and $\widetilde{\mathfrak{g}(r)}$ is a $\mathfrak{g}$-lattice. Consider $\mathrm{GD}(O,\mathfrak{g}(r)) = ((X,\mathcal{A}),(p,c,\zeta))$ and $\mathrm{GD}(\widetilde{O},\widetilde{\mathfrak{g}(r)}) = ((\widetilde{X},\widetilde{\mathcal{A}}),(\widetilde{p},\widetilde{c},\widetilde{\zeta}))$.*

(1) *The canonical map $O \subseteq \widetilde{O}$ induces a morphism $\nu\colon \widetilde{X} \to X$ such that $\nu(\widetilde{p}) = p$ and ν is the normalization of X.*

(2) *The canonical inclusion $\mathfrak{g}(r) \subseteq \widetilde{\mathfrak{g}(r)}$ induces an injective morphism $\iota\colon \mathcal{A} \to \nu_*\widetilde{\mathcal{A}}$ of sheaves of Lie algebras having a torsion cokernel.*

(3) *The morphisms ν and ι induce isomorphisms $\mathcal{O}_{X,p} \cong \mathcal{O}_{\widetilde{X},\widetilde{p}}$ and $\mathcal{A}_p \cong \widetilde{\mathcal{A}}_{\widetilde{p}}$ identifying $\widetilde{c}$ and $\widetilde{\zeta}$ with c and ζ respectively.*

Proof. — We have $\widetilde{O} \subseteq \mathrm{Q}(\widetilde{O}) = \mathrm{Q}(O) \subseteq \Bbbk((z))$ is a unital subalgebra and it is well-known that $O \to \widetilde{O}$ is finite. Thus, $\widetilde{O}$ is a $\Bbbk$-lattice and $\widetilde{\mathfrak{g}(r)}$ is a $\mathfrak{g}$-lattice. Let $\mathrm{GD}(\widetilde{O}, \widetilde{\mathfrak{g}(r)}) = ((\widetilde{X}, \widetilde{\mathcal{A}}), (\widetilde{p}, \widetilde{c}, \widetilde{\zeta}))$. Then $\mathcal{O}_{X,p} \cong \mathrm{Q}(O) \cap \Bbbk[\![z]\!] = \mathrm{Q}(\widetilde{O}) \cap \Bbbk[\![z]\!] \cong \mathcal{O}_{\widetilde{X},\widetilde{p}}$ is compatible with $O \to \widetilde{O}$, resulting in the morphism $\nu\colon \widetilde{X} \to X$ mapping $\widetilde{p}$ to p. The morphism ν is finite and $\widetilde{X}$ is smooth, so $\nu\colon \widetilde{X} \to X$ is the normalization of X. Now $\mathfrak{g}(r) \to \widetilde{\mathfrak{g}(r)}$ induces an injective graded morphism $\mathrm{gr}(\mathfrak{g}(r)) \to \mathrm{gr}\left(\widetilde{\mathfrak{g}(r)}\right)$ which results in the injective morphism $\iota\colon \mathcal{A} \to \nu_*\widetilde{\mathcal{A}}$. Clearly, ι is an isomorphism locally around p, thus we see that the cokernel of ι is a torsion sheaf. The canonical maps $\nu_p^\sharp\colon \mathcal{O}_{X,p} \to \mathcal{O}_{\widetilde{X},\widetilde{p}}$ and $\iota_p\colon \mathcal{A}_p \to \left(\nu_*\mathcal{A}\right)_{\widetilde{p}} \cong \widetilde{\mathcal{A}}_{\widetilde{p}}$ are isomorphisms and are easily seen to identify $\widetilde{c}$ and $\widetilde{\zeta}$ with c and ζ. ☐

Proposition 4.8. — *Using the same notation as in Lemma 4.7, the following results are true.*

(1) *The genus $\widetilde{g} := \mathrm{H}^1(\mathcal{O}_{\widetilde{X}})$ of $\widetilde{X}$ is at most one.*
(2) *Let $\Bbbk((z))$ be equipped with the $\Bbbk$-bilinear form defined by $(f,g) \mapsto \mathrm{res}_0\, fg\mathrm{d}z$. Then*

$$\widetilde{O}^\perp \mathfrak{g}(r) \subseteq \widetilde{O}^\perp \widetilde{\mathfrak{g}(r)} \subseteq \widetilde{\mathfrak{g}(r)}^{\perp} \subseteq \mathfrak{g}(r)^\perp.$$

(3) *If $\widetilde{g} = 1$ and $O = \mathrm{Mult}(\mathfrak{g}(r))$, we have: $\mathfrak{g}(r) = \widetilde{\mathfrak{g}(r)}$, $X = \widetilde{X}$ and the Killing form of $\widetilde{\mathcal{A}}$ (see Definition 3.4) is perfect. Furthermore, there exists $\lambda \in \Bbbk[\![z]\!]$ and $w \in z\Bbbk[\![z]\!]^\times$ such that $\widetilde{r}(x,y) = \lambda(y) r(w(x), w(y))$ is normalized and skew-symmetric. In particular, r is equivalent to the normalized formal r-matrix $\widetilde{r}$.*

Proof.

Step 1: $\mathrm{H}^1(\widetilde{\mathcal{A}}) = 0 = \mathrm{H}^1(\widetilde{\mathcal{A}}^*)$. — Using $\mathrm{H}^1(\mathrm{Cok}(\iota)) = 0 = \mathrm{H}^1(\mathcal{A})$ in the long exact sequence in cohomology of

$$0 \longrightarrow \mathcal{A} \xrightarrow{\iota} \nu_*\widetilde{\mathcal{A}} \longrightarrow \mathrm{Cok}(\iota) \longrightarrow 0, \tag{4.2}$$

implies that $\mathrm{H}^1(\widetilde{\mathcal{A}}) = 0$. Here we used ι from Lemma 4.7 and the fact that $\mathrm{Cok}(\iota)$ is a torsion sheaf. Let $K^{\mathrm{a}}\colon \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}^*$ be the canonical $\mathcal{O}_{\widetilde{X}}$-linear morphism induced by the Killing form K of $\widetilde{\mathcal{A}}$. Lemma 3.5 implies that $K|_p$ is the Killing form of $\widetilde{\mathcal{A}}|_p \cong \mathcal{A}|_p \cong \mathfrak{g}$. In particular, $K|_p$ is non-degenerate, so $K^{\mathrm{a}}|_p$ is an isomorphism. Therefore, $\mathrm{Ker}(K^{\mathrm{a}})$ and $\mathrm{Cok}(K^{\mathrm{a}})$ are torsion sheaves. Since $\widetilde{\mathcal{A}}$ is locally free, this forces $\mathrm{Ker}(K^{\mathrm{a}})$ to vanish.

Using $\mathrm{H}^1(\mathrm{Cok}(K^{\mathrm{a}})) = 0 = \mathrm{H}^1(\widetilde{\mathcal{A}})$ in the long exact sequence in cohomology of

$$0 \longrightarrow \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{A}}^* \longrightarrow \mathrm{Cok}(K^{\mathrm{a}}) \longrightarrow 0 \tag{4.3}$$

results in $\mathrm{H}^1(\widetilde{\mathcal{A}}^*) = 0$.

Step 2: $\widetilde{g} \leqslant 1$, i.e. (1) *holds.* — The Riemann–Roch theorem for $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{A}}^*$ (e.g. in the version of [38, Chapter 7, Exercise 3.3]) reads

$$0 \leqslant \mathrm{H}^0(\widetilde{\mathcal{A}}) - \mathrm{H}^1(\widetilde{\mathcal{A}}) = \deg(\det(\widetilde{\mathcal{A}})) + (1-\widetilde{g})\,\mathrm{rank}(\widetilde{\mathcal{A}})$$
$$0 \leqslant \mathrm{H}^0(\widetilde{\mathcal{A}}^*) - \mathrm{H}^1(\widetilde{\mathcal{A}}^*) = -\deg(\det(\widetilde{\mathcal{A}})) + (1-\widetilde{g})\,\mathrm{rank}(\widetilde{\mathcal{A}}),$$

where we used that

$$\det(\widetilde{\mathcal{A}}^*) = \det(\widetilde{\mathcal{A}})^* \quad \text{implies} \quad \deg(\det(\widetilde{\mathcal{A}}^*)) = -\deg(\det(\widetilde{\mathcal{A}})).$$

We conclude $\widetilde{g} \leqslant 1$.

Step 3: The statement in (2) *holds.* — The Killing form K of $\widetilde{\mathcal{A}}$ induces a commutative diagram

$$\begin{array}{ccc} \Gamma(U,\widetilde{\mathcal{A}}) \times \Gamma(U,\widetilde{\mathcal{A}}) & \xrightarrow{K_U} & \Gamma(U,\mathcal{O}_{\widetilde{X}}) \\ {\scriptstyle \widetilde{\zeta}\times\widetilde{\zeta}}\downarrow & & \downarrow{\scriptstyle \widetilde{c}} \\ \mathfrak{g}((z)) \times \mathfrak{g}((z)) & \xrightarrow[\kappa]{} & \Bbbk((z)) \end{array} \tag{4.4}$$

for all affine open $U \subseteq X$ such that $\widetilde{\mathcal{A}}|_U$ is free. Consequently, (4.4) is commutative for all $U \subseteq X$ open by a gluing argument. In particular, $\kappa(a,b) \in \widetilde{c}(\Gamma(\widetilde{X} \setminus \{\widetilde{p}\}, \mathcal{O}_{\widetilde{X}})) = \widetilde{O}$ for all $a, b \in \widetilde{\mathfrak{g}(r)}$. Hence, we see that

$$\kappa_0(fa,b) = \mathrm{res}_0\, \kappa(fa,b)\mathrm{d}z = \mathrm{res}_0\, f\kappa(a,b)\mathrm{d}z = 0$$

for all $f \in \widetilde{O}^\perp$. Therefore, $fa \in \widetilde{\mathfrak{g}(r)}^\perp$ and we can complete the chain of inclusions by observing that $\mathfrak{g}(r) \subseteq \widetilde{\mathfrak{g}(r)}$ implies $\widetilde{\mathfrak{g}(r)}^\perp \subseteq \mathfrak{g}(r)$.

Step 4: If $\widetilde{g} = 1$, we have $\mathrm{Mult}(\mathfrak{g}(r)) = \widetilde{O}$ *and* $\mathrm{GD}(\widetilde{O}, \mathfrak{g}(r)) = ((\widetilde{X}, \widetilde{\mathcal{A}}), (\widetilde{p}, \widetilde{c}, \widetilde{\zeta}))$. — Let $\Omega^1_{\widetilde{X}}$ be the sheaf of regular 1-forms on $\widetilde{X}$. We have $\mathrm{H}^1(\Omega^1_{\widetilde{X}}) = \Bbbk \cdot \eta$ for some global 1-form η on $\widetilde{X}$ and this choice defines an isomorphism $\Omega^1_{\widetilde{X}} \cong \mathcal{O}_{\widetilde{X}}$. Serre duality (e.g. in the version [38, Chapter 6, Remark 4.20 and Theorem 4.32]) and Step 1 provide that $0 = \mathrm{H}^1(\widetilde{\mathcal{A}}^*) = \mathrm{H}^0(\widetilde{\mathcal{A}})$. Combined with the fact that $\mathrm{H}^1(\mathcal{A}) = 0$ and $\mathrm{Cok}(\iota)$ is torsion, the long exact sequence of (4.2) in cohomology implies that $\iota\colon \mathcal{A} \to \nu_*\widetilde{\mathcal{A}}$ is an isomorphism. In particular, $\widetilde{\mathfrak{g}(r)} = \mathfrak{g}(r)$, $\mathrm{Mult}(\mathfrak{g}(r)) = \widetilde{O}$ and $X = \widetilde{X}$ in case that $O = \mathrm{Mult}(\mathfrak{g}(r))$.

Step 5: Concluding the proof of (3). — Let $\mathrm{d}u(z) = u'(z)\mathrm{d}z = c^*(\eta)$ for some global 1-form η on X (see Remark 4.9 below) and $a, b \in \Gamma(X \setminus \{p\}, \mathcal{A})$. If $w \in z\Bbbk[\![z]\!]^\times$ is the series uniquely determined by $u(w(z)) = z$ and $\widetilde{a} \coloneqq \zeta(a), \widetilde{b} \coloneqq \zeta(b)$, we may calculate

$$\begin{aligned}\kappa_0\left(\widetilde{a}(w(z)), \widetilde{b}(w(z))\right) &= \operatorname{res}_0 \kappa\left(\widetilde{a}(w(z)), \widetilde{b}(w(z))\right)\mathrm{d}z \\ &= \operatorname{res}_0 \kappa\left(\widetilde{a}(z), \widetilde{b}(z)\right)\mathrm{d}u(z) = \operatorname{res}_p K(a,b)\eta = 0.\end{aligned}$$

Here the second to last equality uses (4.4) and Remark 4.9 below while the last equality is due to the residue theorem [55, Corollary of Theorem 3] under consideration of $K(a,b)\eta \in \Gamma(X \setminus \{p\}, \Omega_X)$. Thus, the image W of $\mathfrak{g}(r)$ under $a(z) \mapsto a(w(z))$ satisfies $W^\perp = W$. Lemma 2.17 states that $W = \mathfrak{g}(\widetilde{r})$ for $\widetilde{r}(x,y) = \lambda(y)r(w(x), w(y))$, where $\lambda \in \Bbbk[\![z]\!]$ is arbitrary. This shows that $\widetilde{r}$ is a formal r-matrix if we chose λ in such a way that $\widetilde{r}$ is normalized; see Lemma 2.24. That K is indeed perfect follows from $\mathrm{Cok}(K^{\mathrm{a}}) = 0$, which is a consequence of using $\mathrm{H}^1(\widetilde{\mathcal{A}}) = \mathrm{H}^0(\widetilde{\mathcal{A}}^*) = 0$ in the long exact sequence of (4.3) in cohomology. Here we used Serre duality again. □

Remark 4.9. — Let $((X, \mathcal{A}), (p, c, \zeta))$ be a geometric $\mathfrak{g}$-lattice model and ω_X be the dualizing sheaf of X. Since p is smooth, $\omega_{X,p}$ can be identified with the Kähler differentials $\Omega^1_{\mathcal{O}_{X,p}/\Bbbk}$. Therefore, using e.g. [37, Corollary 12.5 and Example 12.7], we obtain a c-equivariant isomorphism

$$c^* : \widehat{\omega}_{X,p} \longrightarrow \Bbbk[\![z]\!]\mathrm{d}z.$$

More precisely, the differential $\mathcal{O}_{X,p} \to \Omega^1_{\mathcal{O}_{X,p}/\Bbbk} \cong \omega_{X,p}$ induces a continuous differential $\mathrm{d}\colon \widehat{\mathcal{O}}_{X,p} \to \widehat{\omega}_{X,p}$ whose image generates $\widehat{\omega}_{X,p}$ and $c^*(\mathrm{d}f) = c(f)'\mathrm{d}z$. The isomorphism c^* respects residues by [55, Theorem 2], i.e. $\operatorname{res}_p \eta = \operatorname{res}_0 c^*\eta$ for all $\eta \in \mathrm{Q}(\widehat{\omega}_{X,p})$. Here we extended c^* to a map $\mathrm{Q}(\widehat{\omega}_{X,p}) \to \Bbbk(\!(z)\!)\mathrm{d}z$.

4.2. Geometric trichotomy of formal r-matrices

Recall that $\mathfrak{g}$ is a finite-dimensional, central, simple Lie algebra over $\Bbbk$.

THEOREM 4.10. — *Let $r \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$ be a normalized formal r-matrix. There exists a unital subalgebra*

$$O \subseteq \mathrm{Mult}(\mathfrak{g}(r)) = \{f \in \Bbbk(\!(z)\!) \mid f\mathfrak{g}(r) \subseteq \mathfrak{g}(r)\}$$

of finite codimension such that the associated geometric datum $\mathrm{GD}(O, \mathfrak{g}(r)) = ((X, \mathcal{A}), (p, c, \zeta))$ *has the following properties:*

(1) *X has arithmetic genus one.*
(2) *Let $\breve{X}$ be the smooth locus of X. The locally free sheaf $\mathcal{A}|_{\breve{X}}$ of Lie algebras is étale $\mathfrak{g}$-locally free and its Killing form extends uniquely to an invariant perfect pairing $\mathcal{A} \times \mathcal{A} \to \mathcal{O}_X$.*
(3) *There exists a global generator η of the dualizing sheaf ω_X of X such that $c^*(\widehat{\eta}_p) = \mathrm{d}z$ in the notation of Remark 4.9.*

Proof.

Step 1: Setup. — Let $\widetilde{O}$ be the integral closure of $\mathrm{Mult}(\mathfrak{g}(r))$ and $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$ be an, for the moment arbitrary, unital subalgebra of finite codimension. Note that $\widetilde{O}$ is also the integral closure of O and the genus of the normalization of X is $\widetilde{g} \coloneqq \dim(\Bbbk((z))/(\Bbbk[\![z]\!] + \widetilde{O}))$. Proposition 4.8 implies that $\widetilde{g} \in \{0,1\}$.

Step 2: It is possible to chose O such that X has arithmetic genus one. For $\widetilde{g} = 1$ this is already proven in Proposition 4.8, so we may assume $\widetilde{g} = 0$, i.e. $\Bbbk[\![z]\!] + \widetilde{O} = \Bbbk((z))$. Since $\widetilde{O} \cap \Bbbk[\![z]\!] = \Bbbk$, the canonical projection $O \to \Bbbk[z^{-1}]$ is injective and as a consequence $\widetilde{O} = \Bbbk[u]$ for an arbitrary $u \in z^{-1}\Bbbk[\![z]\!]^\times \cap \widetilde{O}$. Proposition 4.8 (2) states that

$$\widetilde{O}^\perp \mathfrak{g}(r) \subseteq \mathfrak{g}(r)^\perp = \mathfrak{g}(r), \tag{4.5}$$

where $\Bbbk((z))$ is equipped with the bilinear form $(f,g) \mapsto \mathrm{res}_0\, fg\mathrm{d}z$ and the last equality follows from Proposition 2.21 and Lemma 2.24. In particular, $\widetilde{O}^\perp \subseteq \mathrm{Mult}(\mathfrak{g}(r)) \subseteq \widetilde{O}$. Since for all $k \in \mathbb{N}_0$

$$\mathrm{res}_0\, u^k u' \mathrm{d}z = \mathrm{res}_0 \frac{1}{k+1}\left(u^{k+1}\right)' \mathrm{d}z = 0 \tag{4.6}$$

holds, where $(\cdot)'$ denotes the formal derivative with respect to z, we see that $u'\widetilde{O} \subseteq \widetilde{O}^\perp \subseteq \widetilde{O}$. Thus, (4.5) implies that $\Bbbk + u'\widetilde{O} \subseteq \mathrm{Mult}(\mathfrak{g}(r))$. The fact that $u' \in \widetilde{O} = \Bbbk[u]$ has order two can be used to see that $\Bbbk + u'\widetilde{O} = \Bbbk[u', u'u]$ is an unital $\Bbbk$-subalgebra of $\mathrm{Mult}(\mathfrak{g}(r))$ such that $\dim(\Bbbk((z))/(\Bbbk[\![z]\!] + \Bbbk[u', u'u])) = 1$. Choosing $O = \Bbbk[u', u'u]$ implies that X has arithmetic genus 1.

Step 3: Choosing O as in Step 2, the Killing form of $\mathcal{A}|_{\breve{X}}$ extends to a paring $\mathcal{A} \times \mathcal{A} \to \mathcal{O}_X$. — We have $\widetilde{X} = X = \breve{X}$ for $\widetilde{g} = 1$ by virtue of Proposition 4.8, so we may assume $\widetilde{g} = 0$, i.e. $\widetilde{O} = \Bbbk[u]$ and $O = \Bbbk[u', u'u]$. Since $X = \breve{X} \cup (X \setminus \{p\})$, we have to define the paring $\mathcal{A} \times \mathcal{A} \to \mathcal{O}_X$ on the affine open set $X \setminus \{p\}$ and show that it is compatible with the Killing form of $\mathcal{A}|_{\breve{X}}$. The diagram (4.4) implies that $\kappa\colon \mathfrak{g}((z)) \times \mathfrak{g}((z)) \to \Bbbk((z))$ restricts to a mapping $\widetilde{\mathfrak{g}(r)} \times \widetilde{\mathfrak{g}(r)} \to \widetilde{O}$ and it suffices to show that this pairing restricts further to $\mathfrak{g}(r) \times \mathfrak{g}(r) \to O$. Observe that $\mathfrak{g}(r)^\perp = \mathfrak{g}(r)$ implies that κ restricts to a bilinear map

$$\mathfrak{g}(r) \times \mathfrak{g}(r) \longrightarrow P := \{f \in \widetilde{O} \mid \operatorname{res}_0 f\mathrm{d}z = 0\}. \tag{4.7}$$

We have $O \subseteq \widetilde{O}^\perp \subseteq P$, since $1 \in \widetilde{O}$. The codimension of P in $\widetilde{O}$ is one, i.e. $\widetilde{O}/P$ is spanned by $u + P$. The same is true for $\widetilde{O}/O$ proving $P = O$ and concluding the proof.

Step 4: The pairing $\mathcal{A} \times \mathcal{A} \to \mathcal{O}_X$ constructed in Step 3 is perfect. — Similarly as in Step 1 of the proof of Proposition 4.8, the morphism $\mathcal{A} \to \mathcal{A}^*$ induced by $\mathcal{A} \times \mathcal{A} \to \mathcal{O}_X$ is injective with a torsion cokernel $\mathcal{C}$. The long exact sequence in cohomology induced by

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}^* \longrightarrow \mathcal{C} \longrightarrow 0$$

combined with $\mathrm{H}^0(\mathcal{A}) = 0 = \mathrm{H}^1(\mathcal{A})$ implies $\mathrm{H}^0(\mathcal{C}) = \mathrm{H}^0(\mathcal{A}^*) = \mathrm{H}^1(\mathcal{A}) = 0$. Here we used the Serre duality (in e.g. the version [38, Chapter 6, Remark 4.20 and Theorem 4.32]), where we note that $\mathrm{H}^1(\mathcal{O}_X) = 1$ and the fact that X is locally a complete intersection (see e.g. Remark 4.11 below) imply that the dualizing sheaf of X is trivial. We conclude that $\mathcal{C} = 0$. Thus, $\mathcal{A} \to \mathcal{A}^*$ is an isomorphism. This is equivalent to saying that $\mathcal{A} \times \mathcal{A} \to \mathcal{O}_X$ is perfect.

Step 5: Reduction of the remaining statements to the case that $\Bbbk$ is algebraically closed. — It remains to prove that $\mathcal{A}|_{\breve{X}}$ is étale $\mathfrak{g}$-locally free and that there exists $\eta \in \Gamma(X, \omega_X)$ such that $c^*(\widehat{\eta}_p) = \mathrm{d}z$. The latter statement is equivalent to $c^*(\widehat{\eta}_p) \in \Bbbk\mathrm{d}z$ for all $\eta \in \Gamma(X, \omega_X)$, since ω_X is trivial. Let

$$\mathrm{GD}\left(O_{\bar{\Bbbk}}, \mathfrak{g}(r_{\bar{\Bbbk}})\right) = \left((X_{\bar{\Bbbk}}, \mathcal{A}_{\bar{\Bbbk}}), (p_{\bar{\Bbbk}}, c_{\bar{\Bbbk}}, \zeta_{\bar{\Bbbk}})\right) \quad \text{and} \quad \pi\colon X_{\bar{\Bbbk}} \longrightarrow X \tag{4.8}$$

be as in Remark 4.4, where we recall that $\bar{\Bbbk}$ is the algebraic closure of $\Bbbk$. Note that $r_{\bar{\Bbbk}}$ is a normalized formal r-matrix, π factors as $X_{\bar{\Bbbk}} \cong X \times \operatorname{Spec}(\bar{\Bbbk}) \to X$, where the second map is the canonical projection, and $\mathcal{A}_{\bar{\Bbbk}} \cong \pi^*\mathcal{A}$. Lemma 3.9 states that $\mathcal{A}|_{\breve{X}}$ is étale $\mathfrak{g}$-locally free if $\mathcal{A}_{\bar{\Bbbk}}|_{\breve{X}_{\bar{\Bbbk}}}$ is étale $\mathfrak{g}\otimes\bar{\Bbbk}$-locally free, where $\pi^{-1}(\breve{X}) = \breve{X}_{\bar{\Bbbk}}$ is the smooth locus of $X_{\bar{\Bbbk}}$. Furthermore, $\pi^*\omega_X \cong \omega_{X_{\bar{\Bbbk}}}$ is the dualizing sheaf of $X_{\bar{\Bbbk}}$ (see e.g. [16, Theorem 3.6.1]) and the image of $c^*(\widehat{\eta}_p)$ under $\Bbbk[\![z]\!]\mathrm{d}z \to \bar{\Bbbk}[\![z]\!]\mathrm{d}z$ equals $c_{\bar{\Bbbk}}^*\big(\widehat{\pi^*\eta}\big)_{p_{\bar{\Bbbk}}}$, hence $c^*(\widehat{\eta}_p) \in \Bbbk\mathrm{d}z$ if and only if $c_{\bar{\Bbbk}}^*\big(\widehat{\pi^*\eta}\big)_{p_{\bar{\Bbbk}}} \in \bar{\Bbbk}\mathrm{d}z$. Summarized, we may assume that $\Bbbk = \bar{\Bbbk}$.

Step 6: $\mathcal{A}|_{\breve{X}}$ is étale $\mathfrak{g}$-locally free. — By Step 5 we may assume that $\Bbbk$ is algebraically closed. Using Step 4 and Lemma 3.5, we see that $\mathcal{A}|_q$ is semi-simple for all closed $q \in \breve{X}$. Therefore, $\mathcal{A}$ is étale $\mathcal{A}|_q$-locally free in each closed point $q \in \check{X}$. Consequently, $\mathcal{A}$ is weakly $\mathcal{A}|_q$-locally free in all closed points in some open neighbourhood of q, since étale maps are open; see also Remark 3.7. This forces $\mathcal{A}|_q \cong \mathcal{A}|_p \cong \mathfrak{g}$ for all $q \in \check{X}$ closed, since $\check{X}$ is connected. Theorem 3.10 states that $\mathcal{A}|_{\check{X}}$ is étale $\mathfrak{g}$-locally free.

Step 7: There exists $\eta \in \Gamma(X, \omega_X)$ such that $c^*(\widehat{\eta}_p) = \mathrm{d}z$. — By Step 5 we may assume that $\Bbbk$ is algebraically closed. Let η be a non-zero global section of the dualizing sheaf ω_X of X and $c^*(\widehat{\eta}_p) = \mathrm{d}w(z) = w'(z)\mathrm{d}z$ for some $w(z) \in z\Bbbk[\![z]\!]$. The dualizing sheaf can be identified with the sheaf of Rosenlicht regular 1-forms, i.e. η is a rational 1-form on $\widetilde{X}$, where we recall that $\nu \colon \widetilde{X} \to X$ is the normalization of X, which is regular on $\nu^{-1}(\check{X})$ and satisfies

$$\sum_{q \in \nu^{-1}(s)} \operatorname{res}_q f\eta = 0 \qquad \forall\, s \in X \text{ singular and closed, } \ f \in \mathcal{O}_{X,s};$$

see e.g. [16, Theorem 5.2.3]. The residue theorem on $\widetilde{X}$ implies that $\operatorname{res}_p f\eta = 0$ for all $f \in \Gamma(X \setminus \{p\}, \mathcal{O}_X)$. Combining this with Remark 4.9 and using diagram (4.4) results in

$$\operatorname{res}_0 \kappa(\zeta(a), \zeta(b)) w' \mathrm{d}z = \operatorname{res}_p K(a,b)\eta = 0$$

for all $a, b \in \Gamma(X \setminus \{p\}, \mathcal{A})$. This implies that $w'\mathfrak{g}(r) \subseteq \mathfrak{g}(r)^\perp = \mathfrak{g}(r)$. In other words, we obtain $w' \in \operatorname{Mult}(\mathfrak{g}(r)) \cap \Bbbk[\![z]\!]^\times = \Bbbk^\times$. We can conclude the proof by replacing η with $(w')^{-1}\eta \in \mathrm{H}^0(X, \omega_X)$. □

Remark 4.11. — Every integral projective curve X over $\Bbbk$ of arithmetic genus one with a $\Bbbk$-rational smooth point p is a plane cubic curve, i.e. determined by one cubic equation. This can be seen for example from (3.12): fix an isomorphism $c \colon \widehat{\mathcal{O}}_{X,p} \to \Bbbk[\![z]\!]$ write $O := c(\Gamma(X \setminus \{p\}, \mathcal{O}_X))$ and note that the codimension of $\Bbbk[\![z]\!] + O$ in $\Bbbk(\!(z)\!)$ is one. Then $O \cap \Bbbk[\![z]\!] = c(\Gamma(X, \mathcal{O}_X)) = \Bbbk$ implies that $O = \Bbbk[f,g]$, where f has order 2 and g has order 3. After probably adjusting f and g, we get $g^2 = f^3 + af + b$ for some $a, b \in \Bbbk$. This is a minimal polynomial relation between f and g, i.e.

$$O \cong \Bbbk[x,y] / \left(y^2 - x^3 - ax - b\right).$$

The curve X is smooth if and only if $4a^3 + 27b^2 \neq 0$, in which case it is elliptic, and has a unique nodal (resp. cuspidal) singularity if $4a^3 = -27b^2 \neq 0$ (resp. $a = b = 0$). In the singular cases X is rational, i.e. X has the normalization $\nu \colon \mathbb{P}^1_{\Bbbk} \to X$.

Definition 4.12. — *Let SolCYBE be the set of normalized formal r-matrices in $(\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$ and GeomCYBE be the set of geometric $\mathfrak{g}$-lattice models $((X,\mathcal{A}),(p,c,\zeta))$ satisfying*

- $\mathrm{H}^0(\mathcal{A}) = 0 = \mathrm{H}^1(\mathcal{A})$;
- *X is an irreducible cubic plane curve;*
- *The restriction $\mathcal{A}|_{\breve{X}}$ is étale $\mathfrak{g}$-locally free and its Killing form extends to an invariant perfect pairing $\mathcal{A}\times\mathcal{A}\to\mathcal{O}_X$. Here, $\breve{X}\subseteq X$ is the set of smooth points.*

Then the map

$$\mathrm{SolCYBE} \longrightarrow \mathrm{GeomCYBE}$$

defined by $r\mapsto \mathrm{GD}(O,\mathfrak{g}(r))$, where $O\subseteq \mathrm{Mult}(\mathfrak{g}(r))$ is the subalgebra from Theorem 4.10, is called geometrization *of normalized formal r-matrices.*

Remark 4.13. — Recall that two normalized formal r-matrices

$$r_1, r_2 \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$$

are equivalent if and only if there exists $\lambda\in\Bbbk^\times$ and $\varphi\in\mathrm{Aut}_{\Bbbk[\![z]\!]\text{-alg}}(\mathfrak{g}[\![z]\!])$ such that

$$r_2(x,y) = \lambda(\varphi(x)\otimes\varphi(y))r_1(\lambda x,\lambda y).$$

Therefore, it is easy to see that the geometric data $((X_1,\mathcal{A}_1),(p_1,c_1,\zeta_1))$ and $((X_2,\mathcal{A}_2),(p_2,c_2,\zeta_2))$ associated to r_1 and r_2 in Theorem 4.10 are isomorphic if and only if r_1 and r_2 are equivalent.

4.3. The geometric r-matrix

Recall that $\mathfrak{g}$ is a finite-dimensional, central, simple Lie algebra over the field $\Bbbk$ of characteristic 0. Fix a generalized r-matrix $r\in(\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$ and let $((X,\mathcal{A}),(p,c,\zeta))$ be a geometrization of r in the sense of Definition 4.2.

The sheaf $\mathcal{A}$ is étale $\mathfrak{g}$-locally free in p by Theorem 3.11. Since étale morphisms are open, we can chose a smooth open neighbourhood C of p such that $\mathcal{A}|_C$ is étale $\mathfrak{g}$-locally free. After probably shrinking C, there exists a non-vanishing 1-form η on C. We have obtained a geometric datum $((X,\mathcal{A}),(C,\eta))$ consisting of

- an integral projective curve X over $\Bbbk$, a non-empty smooth open subscheme $C\subseteq X$ and a non-vanishing 1-form η on C as well as
- a coherent sheaf of Lie algebras $\mathcal{A}$ on X such that $\mathrm{H}^0(\mathcal{A}) = 0 = \mathrm{H}^1(\mathcal{A})$ and $\mathcal{A}|_C$ is étale $\mathfrak{g}$-locally free.

Remark 4.14. — For $\Bbbk = \bar{\Bbbk}$ the geometric datum $((X, \mathcal{A}), (C, \eta))$ satisfies the axioms used in [11] to construct a geometric analog of a generalized r-matrix called *geometric r-matrix*. Indeed, Theorem 3.10 implies that $\mathcal{A}|_C$ is étale $\mathfrak{g}$-locally free if and only if it is weakly $\mathfrak{g}$-locally free for $\Bbbk = \bar{\Bbbk}$. Therefore, the above conditions can be seen as an appropriate generalization of the axioms used in [11] if one works over a not necessarily algebraically closed ground field. In the following we recall the construction of the geometric r-matrix and observe that it works in our generalized setting.

Remark 4.15. — If r is normalized and skew-symmetric and X, η are chosen as in Theorem 4.10, we may assume that C is the smooth locus of X. If $\Bbbk = \bar{\Bbbk}$, we can see that $((X, \mathcal{A}), (C, \eta))$ satisfies the geometric axiomatization of skew-symmetry given in [11, Theorem 4.3], where the third condition in said theorem can be seen as a consequence of the fact that the Killing form of $\mathcal{A}|_C$ extends to a pairing $\mathcal{A} \times \mathcal{A} \to \mathcal{O}_X$ by Theorem 4.10 (2); see [21, Theorem 1.2 (2)].

Let Δ denote the image of the diagonal embedding $\delta \colon C \to X \times C$. The choice of non-vanishing 1-form η induces the so-called *diagonal residue sequence*

$$0 \longrightarrow \mathcal{O}_{X\times C} \longrightarrow \mathcal{O}_{X\times C}(\Delta) \xrightarrow{\operatorname{res}^\eta_\Delta} \delta_*\mathcal{O}_C \longrightarrow 0. \tag{4.9}$$

The map $\operatorname{res}^\eta_\Delta$ is thereby determined as follows: for a closed point $q \in C$ with local parameter u defined on an affine open subset U of C, the sheaves Ω^1_C and $\mathcal{O}_{X\times C}(-\Delta)$ are locally generated by $\mathrm{d}u$ and

$$u - v \coloneqq u \otimes 1 - 1 \otimes u \in \Gamma(U, \mathcal{O}_X) \otimes \Gamma(U, \mathcal{O}_X) \cong \Gamma(U \times U, \mathcal{O}_{X\times X})$$

around q and (q,q) respectively; $\operatorname{res}^\eta_\Delta$ maps $(u-v)^{-1}$ to μ, where μ is defined by $\eta_q = \mu \mathrm{d}u$. Tensoring (4.9) with $\mathcal{A}\boxtimes\mathcal{A}|_C \coloneqq \operatorname{pr}_1^*\mathcal{A} \otimes_{\mathcal{O}_{X\times C}} \operatorname{pr}_2^*\mathcal{A}|_C$,, where $X \xleftarrow{\operatorname{pr}_1} X \times C \xrightarrow{\operatorname{pr}_2} C$ are the canonical projections, gives rise to a short exact sequence

$$0 \longrightarrow \mathcal{A} \boxtimes \mathcal{A}|_C \longrightarrow \mathcal{A} \boxtimes \mathcal{A}|_C(\Delta) \longrightarrow \delta_*(\mathcal{A}|_C \otimes_{\mathcal{O}_C} \mathcal{A}|_C) \longrightarrow 0. \tag{4.10}$$

The Künneth formula implies that

$$\begin{aligned} &\mathrm{H}^0(\mathcal{A} \boxtimes \mathcal{A}|_C) = \mathrm{H}^0(\mathcal{A}) \otimes \mathrm{H}^0(\mathcal{A}|_C) = 0 \text{ and} \\ &\mathrm{H}^1(\mathcal{A} \boxtimes \mathcal{A}|_C) = \mathrm{H}^1(\mathcal{A}) \otimes \mathrm{H}^0(\mathcal{A}|_C) \oplus \mathrm{H}^0(\mathcal{A}) \otimes \mathrm{H}^1(\mathcal{A}|_C) = 0, \end{aligned} \tag{4.11}$$

where we used $\mathrm{H}^0(\mathcal{A}) = 0 = \mathrm{H}^1(\mathcal{A})$. The long exact sequence in cohomology induced by (4.10) yields an isomorphism $R \colon \mathrm{H}^0(\mathcal{A} \boxtimes \mathcal{A}|_C(\Delta)) \to \mathrm{H}^0(\mathcal{A}|_C \otimes \mathcal{A}|_C)$.

Remark 3.8 states that there exists a surjective étale morphism $f:Y\to C$ such that $f^*\mathcal{A}|_C\cong\mathfrak{g}\otimes\mathcal{O}_Y$, while Lemma 3.5 asserts that the inverse image f^*K of the Killing form K of $\mathcal{A}|_C$ can be identified with the Killing form of $\mathfrak{g}\otimes\mathcal{O}_Y$. The pairing f^*K is prefect due to the simplicity of $\mathfrak{g}$. Thus, the surjectivity of f implies that K is also perfect. This implies that the morphism

$$\widetilde{K}\colon \mathcal{A}|_C\otimes_{\mathcal{O}_C}\mathcal{A}|_C\longrightarrow \mathcal{E}nd_{\mathcal{O}_C}(\mathcal{A}|_C),$$

defined by $a\otimes b\mapsto K_U(b,-)a$ for all affine open $U\subseteq C$ and $a,b\in\Gamma(U,\mathcal{A})$, is an isomorphism. Summarized, we obtain an isomorphism $\Phi:=\widetilde{K}R\colon \mathrm{H}^0(\mathcal{A}\boxtimes\mathcal{A}|_C(\Delta))\to\mathrm{End}_{\mathcal{O}_C}(\mathcal{A}|_C)$.

Definition 4.16. — *The section* $\rho:=\Phi^{-1}(\mathrm{id}_{\mathcal{A}|_C})\in\mathrm{H}^0(\mathcal{A}\boxtimes\mathcal{A}|_C(\Delta))$ *is called* geometric r-matrix *of* $((X,\mathcal{A}),(C,\eta))$.

Let us begin the discussion of the geometric r-matrix by observing that geometric r-matrices, constructed from equivalent formal generalized r-matrices in the way presented above, are equivalent in an geometric sense.

Lemma 4.17. — *For* $i\in\{1,2\}$*, let* $r_i\in(\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$ *be a generalized r-matrix and denote the associated geometric datum by*

$$\mathrm{GD}(\mathrm{Mult}(\mathfrak{g}(r_i)),\mathfrak{g}(r_i))=((X_i,C_i),(p_i,c_i,\zeta_i)).$$

Assume that r_1 *and* r_2 *are equivalent and* $f\colon X_2\to X_1$ *and* $\phi\colon\mathcal{A}_1\to f_*\mathcal{A}_2$ *are the isomorphisms provided by Lemma 4.5. We can chose a geometric datum* $((X_i,\mathcal{A}_i),(C_i,\eta_i))$ *for* $i\in\{1,2\}$ *as in the beginning of this section in such a way that* $f^{-1}(C_1)=C_2$ *and* $f^*\eta_1=\eta_2$*. In this case, the associated geometric r-matrices* ρ_1,ρ_2 *satisfy* $\rho_2=(f^*(\phi)\boxtimes f^*(\phi))(f\times f)^*\rho_1$.

Proof. — It is obvious that we may chose the geometric datum in such a way that $f^{-1}(C_1)=C_2$ and $f^*\eta_1=\eta_2$. Consider the commutative diagram (4.12)

$$\begin{array}{ccccc} f^*\mathcal{A}_1\boxtimes f^*\mathcal{A}_1|_C & \hookrightarrow & f^*\mathcal{A}_1\boxtimes f^*\mathcal{A}_1|_C(\Delta_2) & \twoheadrightarrow & \delta_{2,*}\,\mathcal{E}nd_{\mathcal{O}_{C_2}}(f^*\mathcal{A}_1|_{C_1}) \\ \downarrow{\scriptstyle f^*(\phi)\boxtimes f^*(\phi)} & & \downarrow{\scriptstyle f^*(\phi)\boxtimes f^*(\phi)} & & \downarrow{\scriptstyle f^*(\phi)-f^*(\phi)^{-1}} \\ \mathcal{A}_2\boxtimes\mathcal{A}_2|_C & \hookrightarrow & \mathcal{A}_2\boxtimes\mathcal{A}_2|_C(\Delta_2) & \twoheadrightarrow & \delta_{2,*}\,\mathcal{E}nd_{\mathcal{O}_{C_2}}(\mathcal{A}_2|_{C_2}), \end{array}$$

where the upper row is $(f\times f)^*$ of (4.10) for $\mathcal{A}=\mathcal{A}_1$, the lower row is (4.10) for $\mathcal{A}=\mathcal{A}_2$, Δ_2 is the image of the diagonal embedding $\delta_2\colon C_2\to X_2\times C_2$ and the isomorphisms $\mathcal{A}_i|_{C_i}\otimes\mathcal{A}_i|_{C_i}\to\mathcal{E}nd_{\mathcal{O}_{C_i}}(\mathcal{A}_i|_{C_i})$ for $i\in\{1,2\}$ were used. Here, the commutativity of the right square follows form the fact that for a free Lie algebra $\mathfrak{L}$ of finite rank over an unital commutative ring R

with Killing form $\kappa_{\mathfrak{L}}$, the adjoint of $\psi \in \mathrm{Aut}_{R\text{-alg}}(\mathfrak{L})$ with respect to $\kappa_{\mathfrak{L}}$ coincides with ψ^{-1}, so

$$\widetilde{\kappa}_{\mathfrak{L}}(\psi(a) \otimes \psi(b)) = \psi\widetilde{\kappa}_{\mathfrak{L}}(a \otimes b)\psi^{-1} \tag{4.13}$$

holds for $\widetilde{\kappa}_{\mathfrak{L}}(a \otimes b) := \kappa_{\mathfrak{L}}(b, \cdot)a \in \mathrm{End}_R(\mathfrak{L})$ and $a, b \in \mathfrak{L}$.

It is straight forward to show that the inverse image along $f \times f$ of the diagonal residue sequence (4.9) for $X = X_1$ and $\eta = \eta_1$ coincides with the diagonal residue sequence for $X = X_2$ and $\eta = f^*\eta_1 = \eta_2$. This implies that $(f \times f)^*\rho_1$ is mapped to $\mathrm{id}_{f^*\mathcal{A}_1|_{C_1}}$ under $\mathrm{H}^0(f^*\mathcal{A}_1 \boxtimes f^*\mathcal{A}_1|_{C_1}(\Delta_2)) \to \mathrm{End}_{\mathcal{O}_{C_2}}(f^*\mathcal{A}_1|_{C_1})$. Thus, the commutativity of (4.12) implies that

$$(f^*(\phi) \boxtimes f^*(\phi))(f \times f)^*\rho_1 \longmapsto \mathrm{id}_{\mathcal{A}_2|_{C_2}} \tag{4.14}$$

under $\mathrm{H}^0(\mathcal{A}_2 \boxtimes \mathcal{A}_2|_{C_2}(\Delta_2)) \to \mathrm{End}_{\mathcal{O}_{C_2}}(\mathcal{A}_2|_{C_2})$, so

$$\rho_2 = (f^*(\phi) \boxtimes f^*(\phi))(f \times f)^*\rho_1$$

holds. □

The geometric r-matrix has a simple pole along the diagonal with predetermined residue. This can be used to derive a local standard form in the vein of the standard form of formal generalized r-matrices.

Lemma 4.18. — *There exists an affine open neighbourhood $U \subseteq C$ of p admitting a local parameter $u \in \Gamma(U, \mathcal{O}_X)$ of p such that* $\mathrm{d}u$ *generates $\Gamma(U, \Omega^1_C)$ and $u - v$ generates $\Gamma(U \times U, \mathcal{O}_{X\times C}(-\Delta))$. Let $\mu \in \Gamma(U, \mathcal{O}_X)$ be defined by $\eta = \mu^{-1}\mathrm{d}u$ and fix an arbitrary preimage χ of $\mathrm{id}_{\mathcal{A}|_U}$ under the surjective map*

$$\Gamma(U \times U, \mathcal{A} \boxtimes \mathcal{A}|_C) \longrightarrow \Gamma(U, \mathcal{A}|_C \otimes_{\mathcal{O}_C} \mathcal{A}|_C) \xrightarrow{\widetilde{K}_U} \mathrm{End}_{\mathcal{O}_U}(\mathcal{A}|_U). \tag{4.15}$$

There exists $t \in \Gamma(U \times U, \mathcal{A} \boxtimes \mathcal{A})$ such that $\rho|_{U\times U} = \frac{1\otimes\mu}{u-v}\chi + t$.

Proof. — Let $u \in \Gamma(V, \mathcal{O}_X)$ be any local parameter of p, where $V \subseteq C$ is an affine open neighbourhood of p. Chose U to be some affine open subset of the intersection of

- the projection of the open set $\{(q, q') \in V \times V \mid (u - v)(q, q') \neq 0\}$ to the first component with
- an open neighbourhood of p where $\mathrm{d}u$ generates Ω^1_C.

It is straightforward to show that $\mathrm{d}u$ generates $\Gamma(U, \Omega^1_C)$ and $u-v$ generates $\Gamma(U \times U, \mathcal{O}_{X\times C}(-\Delta))$. By construction, $\mathrm{res}^\eta_\Delta \frac{1\otimes\mu}{u-v} = 1$. Thus, we can see that $\rho|_{U\times U}$ and $\frac{1\otimes\mu}{u-v}\chi$ map to $\mathrm{id}_{\mathcal{A}|_U}$ under

$$\Gamma(U \times U, \mathcal{A} \boxtimes \mathcal{A}|_C(\Delta)) \longrightarrow \Gamma(U, \mathcal{A}|_C \otimes \mathcal{A}|_C) \longrightarrow \mathrm{End}(\mathcal{A}|_U). \tag{4.16}$$

Applying the exact functor $\Gamma(U\times U,\cdot)$ to (4.10) implies that $\rho|_{U\times U}-\frac{1\otimes\chi}{u-v}\chi$ is an element of $\Gamma(U\times U,\mathcal{A}\boxtimes\mathcal{A})$, concluding the proof. □

Recall that c induces an isomorphism $\mathrm{Q}(\widehat{\mathcal{O}}_{X,p})\to\Bbbk((z))$ which is again denoted by c and for all open $U\subseteq X$ we have a natural inclusion $\Gamma(U,\mathcal{O}_X)\subseteq\mathrm{Q}(\widehat{\mathcal{O}}_{X,p})$.

Definition 4.19. — *Let $\mathfrak{X}$ be the locally ringed space with underlying topological space X and sheaf of rings $\mathcal{O}_{\mathfrak{X}}$ defined by $\Gamma(U,\mathcal{O}_{\mathfrak{X}}):=c(\Gamma(U,\mathcal{O}_X))[\![y]\!]$ for all $U\subseteq X$ open. Furthermore, let $\jmath=(\jmath,\jmath^\flat)\colon\mathfrak{X}\to X\times X$ be the morphism of ringed spaces defined as follows: $\jmath$ is the composition $X\to X\times\{p\}\to X\times X$ and for an affine open $U\subseteq X$ and an affine open neighbourhood $V\subseteq X$ of p the morphism $\jmath^\flat\colon\Gamma(U\times V,\mathcal{O}_{X\times X})\to\Gamma(U,\mathcal{O}_{\mathfrak{X}})$ is the projective limit over k of the morphisms*

$$\Gamma(U\times V,\mathcal{O}_{X\times X})\cong\Gamma(U,\mathcal{O}_X)\otimes\Gamma(V,\mathcal{O}_X)\xrightarrow{c\otimes\bar{c}_k}c(\Gamma(U,\mathcal{O}_X))\otimes\Bbbk[y]/(y^k),$$

where $\bar{c}_k$ is the composition of c with the canonical map $\Bbbk[\![z]\!]\to\Bbbk[z]/(z^k)$.

Remark 4.20. — The locally ringed space $\mathfrak{X}$ coincides, up to application of c and the identification $X\cong X\times\{p\}$, with the formal scheme obtained by completing $X\times X$ along $X\times\{p\}$. In particular, $\mathfrak{X}$ is indeed a locally ringed space. Using the Künneth formula and ζ, it is possible to identify $\jmath^*(\mathcal{A}\boxtimes\mathcal{A})$ with the sheaf of Lie algebras on $\mathfrak{X}$ defined by $U\mapsto(\zeta(\Gamma(U,\mathcal{A}))\otimes\mathfrak{g})[\![y]\!]$ for all $U\subseteq X$ open. The canonical morphism $\jmath^*\colon\mathcal{A}\boxtimes\mathcal{A}\to\jmath_*\jmath^*(\mathcal{A}\boxtimes\mathcal{A})$ is injective, since, up to isomorphism, it amounts to the completion of a coherent sheaf at a sheaf of ideals. In particular, this construction yields an injective map

$$\jmath^*\colon\Gamma(X\times C\setminus\Delta,\mathcal{A}\boxtimes\mathcal{A})\longrightarrow\Gamma(X\setminus\{p\},\jmath^*(\mathcal{A}\boxtimes\mathcal{A}))=(\mathfrak{g}(r)\otimes\mathfrak{g})[\![y]\!],\tag{4.17}$$

where $\jmath^{-1}(X\times C\setminus\Delta)=X\setminus\{p\}$ and $\zeta(\Gamma(X\setminus\{p\},\mathcal{A}))=\mathfrak{g}(r)$ was used.

Definition 4.21. — *We call the image of a local section s of $\mathcal{A}\boxtimes\mathcal{A}$ under $\jmath^*$* formal trivialization *of s at (p,p) with respect to (c,ζ).*

The following statement can be seen as a generalization of [11, Theorem 6.4].

Theorem 4.22. — *Let $\mathfrak{g}$ be a finite-dimensional, central, simple Lie algebra over a field $\Bbbk$ of characteristc 0, $r\in(\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ be a formal generalized r-matrix, and $((X,\mathcal{A}),(p,c,\zeta))$ be a geometrization of r. Chose a smooth open neighbourhood C of p such that $\mathcal{A}|_C$ is étale $\mathfrak{g}$-locally free and let η be a non-vanishing 1-form on C. Finally, let ρ be the geometric r-matrix of $((X,\mathcal{A}),(C,\eta))$. The following results are true:*

(1) *The formal trivialization* $\jmath^*\rho$ *(see* (4.17)*) of* $\rho \in \mathrm{H}^0(\mathcal{A} \boxtimes \mathcal{A}|_C(\Delta)) \subseteq \Gamma(X \times C \setminus \Delta, \mathcal{A} \boxtimes \mathcal{A})$ *at* (p,p) *with respect to* (c,ζ) *is equal to* $\lambda(y)r(x,y)$. *Here,* $\lambda := (\lambda_1\lambda_2)^{-1}$, *where* λ_1 *and* λ_2 *are determined by* $c^*(\widehat{\eta}_p) = \lambda_1(z)dz$ *and* $r(x,y) - \lambda_2(y)r_{\mathrm{Yang}}(x,y) \in (\mathfrak{g} \otimes \mathfrak{g})[\![x,y]\!]$ *respectively;*

(2) *If* r *is skew-symmetric and normalized and* X, η *are chosen as in Theorem 4.10, the formal trivialization of* ρ *at* (p,p) *with respect to* (c,ζ) *is* r.

Proof.

Step 1: Setup. — Choose U, u, μ, χ and t as in Lemma 4.18, i.e.

$$\rho|_{U\times U} = \frac{1 \otimes \mu}{u - v}\chi + t. \tag{4.18}$$

Let us write $\widetilde{\mu} := c(\mu), \widetilde{u} := c(u) \in \Bbbk[\![z]\!]$. Then

$$\jmath^\flat(1 \otimes \mu) = \widetilde{\mu}(y), \jmath^\flat(u - v) = \widetilde{u}(x) - \widetilde{u}(y) \in \Bbbk[\![x,y]\!]. \tag{4.19}$$

Similarly, let $\widetilde{t} := \jmath^*(t), \widetilde{\chi} := \jmath^*(\chi) \in (\mathfrak{g} \otimes \mathfrak{g})[\![x,y]\!]$.

Step 2: The image of ρ *under* (4.17) *is in standard form.* — The diagram
(4.20)

$$\begin{array}{ccccc} \Gamma(X \times C \setminus \Delta, \mathcal{A} \boxtimes \mathcal{A}) & \xrightarrow{\jmath^*} & (\mathfrak{g}(r) \otimes \mathfrak{g})[\![y]\!] & \longrightarrow & (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!] \\ \downarrow & & & \nearrow & \\ \Gamma(U \times U \setminus \Delta, \mathcal{A} \boxtimes \mathcal{A}) & \xrightarrow[\jmath^*]{} & (\zeta(\Gamma(U \setminus \{p\}, \mathcal{A})) \otimes \mathfrak{g})[\![y]\!] & & \end{array}$$

commutes. Therefore, using the notations from Step 1 and (4.18), the image of ρ under (4.17) is of the form

$$\widetilde{r}(x,y) = \frac{\widetilde{\mu}(y)}{\widetilde{u}(x) - \widetilde{u}(y)}\widetilde{\chi}(x,y) + \widetilde{t}(x,y).$$

Lemma 2.3 can be used to see that $(\widetilde{u}(x) - \widetilde{u}(y))^{-1} - (\widetilde{u}'(y)(x-y))^{-1} \in \Bbbk[\![x,y]\!]$ and $\widetilde{\chi}(x,y) - \gamma \in (x-y)(\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$. Moreover, $\eta_p = \mu^{-1}\mathrm{d}u$ implies that $\lambda_1 = \widetilde{u}'/\widetilde{\mu}$. Summarized, we obtain

$$\widetilde{r}(x,y) = \frac{\gamma}{\lambda_1(y)(x-y)} + s(x,y) \tag{4.21}$$

for some $s \in (\mathfrak{g} \otimes \mathfrak{g})[\![x,y]\!]$.

Step 3: Concluding the proof. — The image $\widetilde{r}$ of ρ under (4.17) is by definition in $(\mathfrak{g}(r) \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$. But so is $\lambda(y)r(x,y) \in \lambda_1(y)^{-1}r_{\mathrm{Yang}}(x,y) + (\mathfrak{g} \otimes \mathfrak{g})[\![x,y]\!]$ and

$$\widetilde{r}(x,y) - \lambda(y)r(x,y) \in (\mathfrak{g}(r) \otimes \mathfrak{g})[\![y]\!] \cap (\mathfrak{g} \otimes \mathfrak{g})[\![x,y]\!] = \{0\},$$

concludes the proof of Theorem 4.22. □

We conclude the section with discussing how the property of solving the CYBE resp. GCYBE can be formulated more globally using the geometric r-matrix. For this purpose we need to sheafify Notation 2.5. In the following we assume that C is affine.

Notation 4.23. — Let $\mathcal{U}$ be the quasi-coherent sheaf on the affine scheme C associated to the universal enveloping sheaf of $\mathrm{H}^0(\mathcal{A}|_C)$ as $\mathrm{H}^0(\mathcal{O}_C)$-Lie algebra and $\iota\colon \mathrm{H}^0(\mathcal{A}|_C) \to \mathrm{H}^0(\mathcal{U})$ be the canonical map. For $ij \in \{12, 13, 23\}$, let $\pi_{ij}\colon C\times C\times C \to C\times C$ denote the natural projections defined through $(x_1, x_2, x_3) \mapsto (x_i, x_j)$ and note that there are natural maps $(\cdot)^{ij}\colon \mathcal{A}|_C \boxtimes \mathcal{A}|_C \to \pi_{ij,*}(\mathcal{U}\boxtimes\mathcal{U}\boxtimes\mathcal{U})$ defined, under consideration of the Künneth formulas

$$\mathrm{H}^0(\mathcal{A}|_C \boxtimes \mathcal{A}|_C) \cong \mathrm{H}^0(\mathcal{A}|_C) \otimes \mathrm{H}^0(\mathcal{A}|_C)$$
$$\mathrm{H}^0(\mathcal{U}\boxtimes\mathcal{U}\boxtimes\mathcal{U}) \cong \mathrm{H}^0(\mathcal{U}) \otimes \mathrm{H}^0(\mathcal{U}) \otimes \mathrm{H}^0(\mathcal{U}),$$

by $t^{12} = \iota(a)\otimes\iota(b)\otimes 1, t^{13} = \iota(a)\otimes 1\otimes\iota(b)$ and $t^{23} = 1\otimes\iota(a)\otimes\iota(b)$ for $t = a\otimes b \in \mathrm{H}^0(\mathcal{A}|_C)\otimes\mathrm{H}^0(\mathcal{A}|_C)$. Furthermore, if $\sigma\colon C\times C \to C\times C$ denotes the map $(x,y)\mapsto(y,x)$, let $\overline{(\cdot)}\colon \mathcal{A}|_C \boxtimes \mathcal{A}|_C \to \sigma_*(\mathcal{A}|_C \boxtimes \mathcal{A}|_C)$ be the morphism defined on global sections by the σ-equivariant automorphism $a\otimes b \mapsto -b\otimes a$.

Remark 4.24. — It can be shown that ι is injective; see [21, Lemma 1.6].

The following result can be proven as [11, Theorem 3.11 & Theorem 4.3]. However, we will sketch a possible self-contained proof.

THEOREM 4.25. — *Using the same assumptions as in Theorem 4.22, the geometric r-matrix ρ, treated as an element of $\Gamma(C\times C\setminus\Delta, \mathcal{A}\boxtimes\mathcal{A})$, solves the* geometric GCYBE

$$\left[\rho^{12}, \rho^{13}\right] + \left[\rho^{12}, \rho^{23}\right] + \left[\rho^{13}, \bar{\rho}^{23}\right] = 0. \tag{4.22}$$

Here, the commutators on the left-hand side are understood in $\Gamma(C\times C\times C\setminus\Sigma, \mathcal{U}\boxtimes\mathcal{U}\boxtimes\mathcal{U})$ for

$$\Sigma = \{(x_1,x_2,x_3)\in C\times C\times C \mid x_i \neq x_j, i\neq j\}.$$

Furthermore, if r is normalized and skew-symmetric and we use the same assumptions as in Theorem 4.22(2), we have $\bar{\rho} = \rho$.

Sketch of proof. — Similar calculations as in Remark 2.8 show that the left-hand side of the geometric GCYBE (4.22) is actually contained in $\Gamma(C\times C\times C\setminus\Sigma, \mathcal{A}\boxtimes\mathcal{A}\boxtimes\mathcal{A})$. It is possible to construct an injective morphism

$$\Gamma(C\times C\times C\setminus\Sigma, \mathcal{A}\boxtimes\mathcal{A}\boxtimes\mathcal{A}) \longrightarrow (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})(\!(x_1)\!)(\!(x_2)\!)[\![x_3]\!], \tag{4.23}$$

in the same vein as (4.17), where the only additional observation necessary in this construction is that $\Gamma(C \times C \setminus \Delta, \mathcal{A} \boxtimes \mathcal{A}) \to (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ extends to a morphism

$$\Gamma((C \setminus \{p\}) \times (C \setminus \{p\}) \setminus \Delta, \mathcal{A} \boxtimes \mathcal{A}) \longrightarrow (\mathfrak{g} \otimes \mathfrak{g})((x))((y)). \tag{4.24}$$

Using Theorem 4.22, it can be shown that the map (4.23) sends the left hand side of (4.22) to $\mathrm{GCYB}(\widetilde{r})$, where $\widetilde{r}(x,y) = \lambda(y)r(x,y)$ for some $\lambda \in \Bbbk[\![z]\!]^\times$. This is 0 since r is a formal generalized r-matrix. The injectivity of (4.23) implies that ρ solves (4.22). Furthermore, it can be shown that $\bar{\rho}$ is mapped to $\bar{r}$ via the injective map $\Gamma(C \times C \setminus \Delta, \mathcal{A} \boxtimes \mathcal{A}) \to (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$. Therefore $r = \bar{r}$ implies $\rho = \bar{\rho}$. □

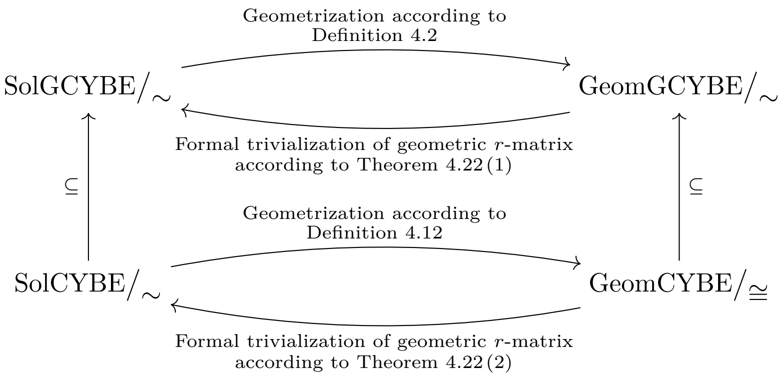
4.4. Summary of the algebraic geometry of formal generalized r-matrices

As always, $\mathfrak{g}$ denotes a central, simple, finite-dimensional Lie algebra over the field $\Bbbk$ of characteristic 0. Recall that SolGCYBE (resp. SolCYBE) is the set of formal generalized r-matrices (resp. normalized r-matrices) over $\mathfrak{g}$. Furthermore, GeomGCYBE is the set of geometric $\mathfrak{g}$-lattice models $((X, \mathcal{A}), (p, c, \zeta))$ such that $\mathrm{H}^0(\mathcal{A}) = 0 = \mathrm{H}^1(\mathcal{A})$ and GeomCYBE $\subseteq$ GeomGCYBE is the subset of those lattice models where additionally:

- X is an irreducible cubic plane curve;
- The restriction $\mathcal{A}|_{\breve{X}}$ is étale $\mathfrak{g}$-locally free and its Killing form extends to an invariant perfect pairing $\mathcal{A} \times \mathcal{A} \to \mathcal{O}_X$. Here, $\breve{X} \subseteq X$ is the set of smooth points.

The choice of (C, η) in Theorem 4.22 is irrelevant up to equivalence of formal generalized r-matrices. Furthermore, Proposition 2.14 and Remark 3.19 imply that all geometric $\mathfrak{g}$-lattice models $((X, \mathcal{A}), (p, c, \zeta))$, satisfying $\mathrm{H}^0(\mathcal{A}) = 0 = \mathrm{H}^1(\mathcal{A})$, are isomorphic to $\mathrm{GD}(O, \mathfrak{g}(r))$ for an appropriate formal generalized r-matrix r and a unital subalgebra $O \subseteq \mathrm{Mult}(\mathfrak{g}(r))$. Therefore, the trivialization of the geometric r-matrix in Theorem 4.22 defines a map GeomGCYBE $\to$ SolGCYBE$/\sim$. Furthermore, Theorem 4.22 (2) states that this map restricts to GeomCYBE $\to$ SolCYBE$/\sim$, if one chooses

$\eta \in \mathrm{H}^0(\omega_X) \cong \Bbbk$. Therefore, we obtain the following commutative diagram:



Here, the horizontal maps are mutually inverse bijections. Furthermore, "$\sim$" in the left column denotes the equivalence of formal generalized r-matrices from Definition 2.11. Similarly, "$\sim$" (resp. "$\cong$") in the right column denotes the equivalence (resp. isomorphism) of geometric $\mathfrak{g}$-lattice models from Definition 3.21 (resp. Definition 3.18). The fact that the equivalence classes are mapped onto each other in the way depicted follows from Lemma 4.5 and Remark 4.13.

The reminder of this section consists of structural consequences for generalized r-matrices coming from the above commutative diagram.

4.5. Algebraization of formal generalized r-matrices

Recall that $\mathfrak{g}$ is a finite-dimensional, central, simple Lie algebra over a field $\Bbbk$ of characteristic 0. Let $r \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$ be a generalized r-matrix and $((X, \mathcal{A}), (p, c, \zeta))$ be a geometrization of r.

Chose a smooth open neighbourhood C of p such that $\mathcal{A}|_C$ is étale $\mathfrak{g}$-locally free and there exists a non-vanishing 1-form η on C. Let $\rho \in \Gamma(X \times C \setminus \Delta, \mathcal{A} \boxtimes \mathcal{A})$ be the geometric r-matrix of $((X, \mathcal{A}), (C, \eta))$. If we fix an étale morphism $f\colon Y \to X$ such that $p \in f(Y) \subseteq C$ and there exists an isomorphism $\psi\colon f^*\mathcal{A} \to \mathfrak{g} \otimes \mathcal{O}_Y$ of sheaves of Lie algebras, the morphism $\psi \boxtimes \psi$ defines an isomorphism $(f \times f)^*(\mathcal{A} \boxtimes \mathcal{A}) \cong (\mathfrak{g} \otimes \mathfrak{g}) \otimes \mathcal{O}_{Y \times Y}$. Therefore, we can consider

$$\varrho := (\psi \boxtimes \psi)(f \times f)^*\rho \in (\mathfrak{g} \otimes \mathfrak{g}) \otimes \Gamma\left(Y \times Y \setminus \Delta_f, \mathcal{O}_{Y \times Y}\right), \tag{4.25}$$

where $\Delta_f = (f \times f)^{-1}(\Delta) = \{(x, y) \in Y \times Y \mid f(x) = f(y)\}$. Observe that for $\Bbbk = \overline{\Bbbk}$ this section can be identified with a rational map $\varrho\colon Y(\Bbbk) \times Y(\Bbbk) \to \mathfrak{g} \otimes \mathfrak{g}$, where $Y(\Bbbk)$ is the algebraic variety of closed points of Y.

Let $q \in f^{-1}(p)$ be a closed point. Using e.g. [17, Theorem 19.6.4] it is possible to chose a $\Bbbk$-algebra isomorphism $c' \colon \widehat{\mathcal{O}}_{Y,q} \to \Bbbk(q)[\![z]\!]$, where $\Bbbk(q)$ is the residue field of q. Similar to the construction of $\mathfrak{X}$ in Notation 4.19, we have a locally ringed space $\mathfrak{Y}$ equipped with a morphism of locally ringed spaces $\jmath' \colon \mathfrak{Y} \to Y \times Y$ whose topological space coincides with the one of Y and for all $U \subseteq Y$ open $\Gamma(U, \mathcal{O}_{\mathfrak{Y}}) = c'(\Gamma(U, \mathcal{O}_Y))[\![y]\!]$, where c' is extended to an isomorphism $\mathrm{Q}(\widehat{\mathcal{O}}_{Y,q}) \to \Bbbk(q)(\!(z)\!)$. Clearly

$$\begin{aligned} (4.26) \quad &\Gamma\big(V, \jmath'^{,*}((\mathfrak{g} \otimes \mathfrak{g}) \otimes \mathcal{O}_{Y \times Y})\big) \\ &\qquad = (\mathfrak{g} \otimes \mathfrak{g}) \otimes c'(\Gamma(U, \mathcal{O}_Y))[\![y]\!] \subseteq \big(\mathfrak{g}_{\Bbbk(q)} \otimes_{\Bbbk(q)} \mathfrak{g}_{\Bbbk(q)}\big)(\!(x)\!)[\![y]\!] \end{aligned}$$

for $V \subseteq Y$ open such that $\jmath'^{,-1}(V) = U$.

Definition 4.26. — *Let us call the image of a local section s of $(\mathfrak{g} \otimes \mathfrak{g}) \otimes \mathcal{O}_{Y \times Y}$ under $\jmath'^{,*}$* formal trivialization *of s at (q,q) with respect to c'.*

Theorem 4.27. — *Let $r_{\Bbbk(q)} \in (\mathfrak{g}_{\Bbbk(q)} \otimes_{\Bbbk(q)} \mathfrak{g}_{\Bbbk(q)})(\!(x)\!)[\![y]\!]$ be the formal generalized r-matrix constructed from r in Remark 2.9. Then the formal trivialization $\jmath'^{,*}\varrho$ of ϱ is equivalent to $r_{\Bbbk(q)}$, where ϱ is defined in (4.25). In particular, if $\Bbbk = \overline{\Bbbk}$, r is equivalent to the trivialization of a rational map $\varrho \colon Y(\Bbbk) \times Y(\Bbbk) \to \mathfrak{g} \otimes \mathfrak{g}$ in (q,q).*

Proof. — Using Theorem 4.22 and Remark 2.9 we see that the image of ρ under

$$\Gamma(X \times C \setminus \Delta, \mathcal{A} \boxtimes \mathcal{A}) \xrightarrow{\jmath^*} (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!] \longrightarrow \big(\mathfrak{g}_{\Bbbk(q)} \otimes_{\Bbbk(q)} \mathfrak{g}_{\Bbbk(q)}\big)(\!(x)\!)[\![y]\!]$$

is equivalent to $r_{\Bbbk(q)}$ by a rescaling $\lambda \in \Bbbk[\![z]\!]^\times \subseteq \Bbbk(q)[\![z]\!]^\times$. The image of z under

$$\Bbbk[\![z]\!] \xrightarrow{c^{-1}} \widehat{\mathcal{O}}_{X,p} \xrightarrow{\widehat{f}_p^\sharp} \widehat{\mathcal{O}}_{Y,q} \xrightarrow{c'} \Bbbk(q)[\![z]\!]$$

defines a coordinate transform $w \in z\Bbbk(q)[\![z]\!]^\times$. Similarly, the chain of maps

$$\mathfrak{g}[\![z]\!] \xrightarrow{\zeta^{-1}} \widehat{\mathcal{A}}_p \longrightarrow \widehat{f^*\mathcal{A}}_q \xrightarrow{\widehat{\psi}_q} \mathfrak{g} \otimes \widehat{\mathcal{O}}_{Y,q} \xrightarrow{\mathrm{id}_\mathfrak{g} \otimes c'} \mathfrak{g}_{\Bbbk(q)}[\![z]\!],$$

where the middle arrow is the composition of $\widehat{f}_p^*$ with the completion of the canonical map $(f_* f^* \mathcal{A})_p \to f^*\mathcal{A}_q$, induces a gauge transformation

$\varphi \in \mathrm{Aut}_{\Bbbk(q)\text{-alg}}(\mathfrak{g}_{\Bbbk(q)}[\![z]\!])$. It is straight forward to show from the constructions of $\mathfrak{X}$ and $\mathfrak{Y}$ that the diagram

$$\begin{array}{ccc}
\Gamma(X \times C \setminus \Delta, \mathcal{A} \boxtimes \mathcal{A}) & \xrightarrow{\ \jmath^*\ } & (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!] \\
\Big\downarrow{\scriptstyle (f\times f)^*} & & \Big\downarrow \\
 & & \left(\mathfrak{g}_{\Bbbk(q)} \otimes_{\Bbbk(q)} \mathfrak{g}_{\Bbbk(q)}\right)(\!(x)\!)[\![y]\!] \\
 & & \Big\uparrow{\scriptstyle \jmath',*} \\
\Gamma(Y \times Y, f^*\mathcal{A} \boxtimes f^*\mathcal{A}) & \xrightarrow[\psi\boxtimes\psi]{} & (\mathfrak{g} \otimes \mathfrak{g}) \otimes \Gamma(Y \times Y \setminus \Delta_f, \mathcal{O}_{Y\times Y})
\end{array}$$

commutes, where the arrow in the upper right is defined by $s(x,y) \mapsto (\varphi(x) \otimes \varphi(y))s(w(x), w(y))$ for all $s \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$. In particular, this implies that the series $\jmath'^{,*}\varrho$ is equivalent to $r_{\Bbbk(q)}$ via the equivalence (λ, w, φ). If $\Bbbk = \overline{\Bbbk}$, we can apply a coordinate transformation to assume that c' is induced by a local parameter of q. This proves the second part of the statement. $\square$

Passing to the smooth completion Z of the normalization of Y to a projective curve, we can consider ϱ as a rational section of $(\mathfrak{g} \otimes \mathfrak{g}) \otimes \mathcal{O}_{Z\times Z}$, where now Z is an integral smooth projective curve over $\Bbbk$. In particular, if $\Bbbk = \mathbb{C}$, we obtain the following result.

Corollary 4.28. — *Let $\mathfrak{g}$ be a finite-dimensional, simple, complex Lie algebra and $r \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$ be a formal generalized r-matrix. Then r is equivalent to a formal trivialization of a rational map $R \otimes R \to \mathfrak{g} \otimes \mathfrak{g}$ for some compact Riemann surface R.*

Remark 4.29. — The section ϱ from (4.25) is the solution of yet another version of the GCYBE. For $ij \in \{12, 13, 23\}$, let $\pi'_{ij} \colon Y \times Y \times Y \to Y \times Y$ be the canonical projections $(y_1, y_2, y_3) \to (y_i, y_j)$ and ϱ^{ij} be the image of ϱ under the diagonal arrow in

$$\begin{array}{ccc}
(\mathfrak{g} \otimes \mathfrak{g}) \otimes \mathcal{O}_{Y\times Y} & \xrightarrow{\ \pi'^{,*}_{ij}\ } & (\mathfrak{g} \otimes \mathfrak{g}) \otimes \pi_{ij,*}\mathcal{O}_{Y\times Y\times Y} \\
 & \searrow & \Big\downarrow{\scriptstyle (\cdot)^{ij}\otimes \mathrm{id}} \\
 & & \left(\mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})\right) \otimes \pi_{ij,*}\mathcal{O}_{Y\times Y\times Y},
\end{array}$$

where Notation 2.5 was used. Furthermore, let $\bar{\varrho} = -\tau'\sigma'^{,*}\varrho$, where $\sigma' \colon Y \times Y \to Y \times Y$ is given by $\sigma'(x,y) = (y,x)$ and $\tau' \colon (\mathfrak{g} \otimes \mathfrak{g}) \otimes \mathcal{O}_{Y\times Y} \to (\mathfrak{g} \otimes \mathfrak{g}) \otimes \mathcal{O}_{Y\times Y}$ is the $\mathcal{O}_{Y\times Y}$-linear extension of $a \otimes b \mapsto b \otimes a$ for $a, b \in \mathfrak{g}$. Then we have

$$\left[\varrho^{12}, \varrho^{13}\right] + \left[\varrho^{12}, \varrho^{23}\right] + \left[\varrho^{13}, \bar{\varrho}^{23}\right] = 0 \tag{4.27}$$

and if r is skew-symmetric, the identity $\varrho = \bar{\varrho}$ holds. This can be proven exactly as was sketched for Theorem 4.25 using Theorem 4.27.

Example 4.30. — Let $\mathfrak{g} = \mathfrak{so}(n, \Bbbk)$ and $X_{ij} = E_{ij} - E_{ji}$, where $E_{ij} = (\delta_{ik}\delta_{j\ell})_{k,\ell=1}^n$. In [49] the author examines

$$r(x,y) = \frac{1}{x-y} \sum_{1 \leqslant i < j \leqslant n} \frac{\alpha_i(x)\alpha_j(x)}{\alpha_i(y)\alpha_j(y)} X_{ij} \otimes X_{ji},$$

where $\alpha_i(z) := (1-a_iz)^{1/2} = \sum_{k=0}^{\infty}(-1)^k\binom{1/2}{k}(a_iz)^k \in \Bbbk[\![z]\!]$ for some non-zero $a_i \in \Bbbk$, $i \in \{1,\dots,n\}$, in the case that $\Bbbk = \mathbb{C}$ and considered as a meromorphic function. If we write $A = \operatorname{diag}(a_1,\dots,a_n)$, it is easy to see that $\mathfrak{g}(r)$ is the image of $z^{-1}\mathfrak{g}[z^{-1}]$ under

$$a(z) \longmapsto (1-Az)^{1/2}a(z)(1-Az)^{1/2}.$$

This is a homogeneous subalgebra of $\mathfrak{g}(\!(z)\!)$ the sense of Example 2.19. Therefore, r is a non-skew-symmetric normalized generalized r-matrix by Proposition 2.14. The associated sheaf of Lie algebras $\mathcal{A}$ on $X = \mathbb{P}^1_{\Bbbk}$ is obtained by gluing $\mathfrak{g}(r)$ and $\Gamma(U,\mathcal{A}) := (1-Az)^{1/2}\mathfrak{g}[z](1-Az)^{1/2}$, where $\operatorname{Spec}(\Bbbk[z]) \cong U := \mathbb{P}^1_{\Bbbk} \setminus \{\infty\} \subset \mathbb{P}^1_{\Bbbk}$, along $\operatorname{Spec}(\Bbbk[z,z^{-1}])$. Note that $p = (z) \in U$.

Consider the canonical map

$$\Bbbk[z] \longrightarrow R := \Bbbk[z,u_1,\dots,u_n]/\left(u_1^2-(1-a_1z),\dots,u_n^2-(1-a_nz)\right).$$

Observe that $u_i \mapsto \alpha_i$ defines an injective map $R \to \Bbbk[\![z]\!]$ which identifies R with the subalgebra of $\Bbbk[\![z]\!]$ generated by $\{\alpha_1,\dots,\alpha_n\}$. Let $Y = \operatorname{Spec}(S)$, where $S := R_u$ for $u := u_1 \cdot \ldots \cdot u_n$, and $f\colon Y \to U$ be the map induced by $\Bbbk[z] \to R \to R_u = S$. Then f is étale by [39, Corollary 3.16] and maps the $\Bbbk$-rational point $\overline{(z,u_1-1,\dots,u_n-1)}_u \in Y$ to $p = (z)$. Furthermore, the chain of maps

$$\Gamma(Y, f^*\mathcal{A}) \cong \Gamma(U,\mathcal{A}) \otimes_{\Bbbk[z]} S \longrightarrow \mathfrak{g} \otimes S \cong \Gamma(Y, \mathfrak{g} \otimes \mathcal{O}_Y),$$

where the middle arrow is defined by $\alpha_i\alpha_jX_{ij} \otimes 1 \mapsto X_{ij} \otimes u_iu_j$, is an isomorphism of Lie algebras over S. Here we used that $\{\alpha_i\alpha_jX_{ij}\}_{i\neq j}$ forms a $\Bbbk[z]$-basis of $\Gamma(U,\mathcal{A})$ and $u_iu_j \in S$ is a unit for all $i,j \in \{1,\dots,n\}$. Thus, $f^*\mathcal{A} \cong \mathfrak{g} \otimes \mathcal{O}_Y$ and r can be obtained from

$$\sum_{1 \leqslant i < j \leqslant n} (X_{ij} \otimes X_{ji}) \otimes \frac{u_iu_j \otimes (u_iu_j)^{-1}}{z \otimes 1 - 1 \otimes z} \in (\mathfrak{g} \otimes \mathfrak{g}) \otimes (S \otimes S)\left[\frac{1}{z \otimes 1 - 1 \otimes z}\right]$$

in the way described in this section.

4.6. The Belavin–Drinfeld trichotomy of complex r-matrices

We are now able to proof the following version of the Belavin–Drinfeld trichotomy [9, Theorem 1.1] in two spectral parameters. The proof presented in Section 4.7 below completely relies on the algebro-geometric methods established in this paper and is hence independent of [9].

The following notation is convinient in order to state the desired result.

Definition 4.31. — *For a meromorphic map $s\colon U\times U\to\mathfrak{g}\otimes\mathfrak{g}$ such that $z\mapsto s(z,0)$ is meromorphic, let $\Theta s\in(\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ denote the series*

$$\Theta s := \sum_{k=0}^{\infty}\frac{1}{k!}\frac{\partial^k s}{\partial y^k}(x,0)y^k, \tag{4.28}$$

i.e. Θs is the Taylor expansion of s in the second variable in 0.

Theorem 4.32. — *Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra, $r\in(\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ be a normalized formal r-matrix, and X be the irreducible cubic plane curve associated to r in Theorem 4.10 (see also Remark 4.11). The following results are true.*

(1) *X is elliptic if and only if r is gauge equivalent to $\Theta\varrho$ for a meromorphic map $\varrho\colon\mathbb{C}\times\mathbb{C}\to\mathfrak{g}\otimes\mathfrak{g}$ satisfying $\varrho(x+\lambda,y+\lambda')=\varrho(x,y)$ for all λ and λ' in some rank-two lattice in $\mathbb{C}$. Here, Θ was defined in Definition 4.31.*

(2) *X is nodal if and only if r is gauge equivalent to $\Theta\varrho$, where*

$$\begin{aligned}\varrho(x,y)\\ =\frac{1}{\exp(x-y)-1}\sum_{k=0}^{m-1}\exp\left(\frac{k(x-y)}{m}\right)\gamma_k+t\left(\exp\left(\frac{x}{m}\right),\exp\left(\frac{y}{m}\right)\right)\end{aligned} \tag{4.29}$$

for some $\sigma\in\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ of order m and $t\in\mathfrak{L}(\mathfrak{g},\sigma)\otimes\mathfrak{L}(\mathfrak{g},\sigma)$. Here, $\mathfrak{L}(\mathfrak{g},\sigma)$ is defined in Theorem 3.27 and $\gamma=\sum_{k=0}^{m-1}\gamma_k$ is the unique decomposition such that $(\sigma\otimes 1)\gamma_k=\exp(2\pi ik/m)\gamma_k$.

(3) *X is cuspidal if and only if r is gauge equivalent to $\Theta\varrho$, where $\varrho(x,y)=\frac{\gamma}{x-y}+t(x,y)$ for some $t\in(\mathfrak{g}\otimes\mathfrak{g})[x,y]$.*

Remark 4.33. — The sheaves of Lie algebras coming from complex r-matrices have been studied in a case by case fashion using the Belavin–Drinfeld trichotomy [9, Theorem 1.1]. In the elliptic case it is shown in [12] that these sheaves, which are described in Theorem 3.28, are all of the form $\mathrm{Ker}(\mathrm{Tr}_{\mathcal{F}}\colon\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F})\to\mathcal{O}_X)$ for some simple vector bundle $\mathcal{F}$ on a complex elliptic curve X. In the rational case the corresponding sheaves on the cuspidal curve were constructed in [11], using the structure theory of rational r-matrices from [53, 54].

The sheaves of Lie algebras on the nodal curve corresponding to trigonometric r-matrices were recently constructed independently in [2, 47], completing the geometrization of non-degenerate solutions of the CYBE. In [47], the author calculates the set of multipliers of the subalgebras associated to trigonometric r-matrices using the classification of said r-matrices from [9] and then applies the geometrization procedure presented in Subsection 3.3. The construction in [2] uses a different approach based on twisting the standard Lie bialgebra structure of loop algebras and a result in [31], which can be seen as an analog of the theory of maximal orders from [53, 54] for subalgebras of Kac–Moody algebras.

The derivation of Theorem 4.32 simultaneously yields a new proof of the Belavin–Drinfeld trichotomy and an alternative way to execute the geometrization program for non-degenerate solutions of the CYBE.

4.7. Proof of the Belavin–Drinfeld trichotomy 4.32.

Let $((X,\mathcal{A}),(p,c,\zeta))$ as well as $\eta \in \mathrm{H}^0(\omega_X)$ be the geometric datum associated to r in Theorem 4.10. Furthermore, let $X^{\mathrm{an}} = (X^{\mathrm{an}},\mathcal{O}_X^{\mathrm{an}})$ be the complex analytic space associated to X, $\iota\colon X^{\mathrm{an}} \to X$ be the canonical morphism of locally ringed spaces and write $\breve{X} \subseteq X$, $\iota^{-1}(\breve{X}) = \breve{X}^{\mathrm{an}}$ for the respective smooth loci. If $X = \breve{X}$ is smooth, it is a complex elliptic curve, so there exists a lattice $\Lambda \subseteq \mathbb{C}$ of rank two as well as a biholomorphic map $\widetilde{\nu}\colon \mathbb{C}/\Lambda \to X^{\mathrm{an}}$ such that $\widetilde{\nu}(\Lambda) = p$. Otherwise, X has a unique singular closed point s and the normalization $\nu\colon \mathbb{P}^1_{\mathbb{C}} \to X$, where one of the following cases occurs:

- s is nodal and we can choose coordinates $(u_0 : u_1)$ on $\mathbb{P}^1_{\mathbb{C}}$ such that $\nu^{-1}(s) = \{(1:0),(0:1)\}$ and $\nu(1:1) = p$. In particular, ν restricts to an isomorphism $\mathrm{Spec}(\mathbb{C}[u,u^{-1}]) \to \breve{X}$ for $u = u_1/u_0$ and induces a biholomorphic map $\widetilde{\nu}\colon \mathbb{C}^\times \to \breve{X}^{\mathrm{an}}$.
- s is a cuspidal and we can choose coordinates $(u_0 : u_1)$ on $\mathbb{P}^1_{\mathbb{C}}$ such that $\nu^{-1}(s) = \{(0:1)\}$ and $\nu(1:0) = p$. In particular, ν restricts to an isomorphism $\mathrm{Spec}(\mathbb{C}[u]) \to \breve{X}$ for $u = u_1/u_0$ and induces a biholomorphic map $\widetilde{\nu}\colon \mathbb{C} \to \breve{X}^{\mathrm{an}}$.

Summarized, we have a holomorphic covering $\widetilde{\pi}\colon \mathbb{C} = (\mathbb{C}, \mathcal{O}^{\mathrm{an}}_{\mathbb{C}}) \to \breve{X}^{\mathrm{an}}$ satisfying $\iota\widetilde{\pi}(0) = p$, defined by

$$\widetilde{\pi}(\widetilde{z}) = \begin{cases} \widetilde{\nu}(\widetilde{z}+\Lambda) & \text{if } X \text{ is elliptic} \\ \widetilde{\nu}\,(\exp(\widetilde{z})) & \text{if } X \text{ is nodal} \\ \widetilde{\nu}(\widetilde{z}) & \text{if } X \text{ is cuspidal,} \end{cases} \tag{4.30}$$

in some holomorphic coordinates $\widetilde{z}$ on $\mathbb{C}$. The invertible sheaf $\Omega^{\mathrm{an}}_{\breve{X}} = \iota^*\Omega_{\breve{X}}$ can be identified with the sheaf of holomorphic 1-forms on $\breve{X}^{\mathrm{an}}$, so $\iota^*\eta$ can be viewed as holomorphic 1-form. Let us write $\pi := \iota\widetilde{\pi}$. We can assume that $\pi^*\eta = \mathrm{d}\widetilde{z}$. Indeed, it is well-known that there exists a $\lambda \in \Bbbk^\times$ such that $\pi^*\eta = \lambda\mathrm{d}\widetilde{z}$ if X is elliptic and

$$\nu^*(\eta) = \begin{cases} \lambda\mathrm{d}u/u & \text{if } X \text{ is nodal} \\ \lambda\mathrm{d}u & \text{if } X \text{ is cuspidal.} \end{cases} \tag{4.31}$$

We can achieve that $\lambda = 1$ by replacing r with $\lambda r(\lambda x, \lambda y)$.

Lemma 4.34. — *Let $\theta\colon \widehat{\mathcal{O}}^{\mathrm{an}}_{\mathbb{C},0} \to \mathbb{C}[\![\widetilde{z}]\!]$ be the isomorphism defined by the Taylor series in 0. Then the diagram*

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,p} & \xrightarrow{\widehat{\pi}_0^\sharp} & \widehat{\mathcal{O}}^{\mathrm{an}}_{\mathbb{C},0} \\ \downarrow c & & \downarrow \theta \\ \mathbb{C}[\![z]\!] & \xrightarrow[z\mapsto\widetilde{z}]{} & \mathbb{C}[\![\widetilde{z}]\!] \end{array} \tag{4.32}$$

commutes. Furthermore, any series in $\mathbb{C}(\!(z)\!)$ representing a rational function on X coincides with the Laurent series of a meromorphic function on $\mathbb{C}$ in 0. This meromorphic function is elliptic (resp. a rational function of exponentials, resp. rational) if and only if X is elliptic (resp. nodal, resp. cuspidal).

Proof. — Similar to Remark 4.9, the canonical derivations $\mathcal{O}_{X,p} \to \omega_{X,p}$ and $\mathcal{O}^{\mathrm{an}}_{\mathbb{C},0} \to \Omega^{\mathrm{an}}_{\mathbb{C},0}$ induce continuous derivations $\widehat{\mathcal{O}}_{X,p} \to \widehat{\omega}_{X,p}$ and $\widehat{\mathcal{O}}^{\mathrm{an}}_{\mathbb{C},0} \to \widehat{\Omega}^{\mathrm{an}}_{\mathbb{C},0}$ whose images generate the respective modules. These derivations will both be denoted by d, since it will be clear from the context which one is in use. The completion $\widehat{\omega}_{X,p} \to \widehat{\Omega}^{\mathrm{an}}_{\mathbb{C},0}$ of

$$\omega_{X,p} \xrightarrow{\pi_p^*} (\pi_*\pi^*\Omega^{\mathrm{an}}_{\mathbb{C}})_p \longrightarrow \Omega^{\mathrm{an}}_{\mathbb{C},0} \tag{4.33}$$

is described by $\mathrm{d}f \mapsto \mathrm{d}\widehat{\pi}_0^\sharp(f)$ for all $f \in \widehat{\mathcal{O}}_{X,p}$. The identity $c^*(\widehat{\eta}_p) = \mathrm{d}z$ implies that $\widehat{\eta}_p = \mathrm{d}c^{-1}(z)$ (see Remark 4.9) and $\pi^*\eta = \mathrm{d}\widetilde{z}$ implies that

$\mathrm{d}\widehat{\pi}_0^\sharp(c^{-1}(z)) = \mathrm{d}\widetilde{z}$. This yields $\widehat{\pi}_0^\sharp(c^{-1}(z)) = \widetilde{z}$, i.e. (4.32) is commutative, since $\theta(\widetilde{z}) = \widetilde{z}$.

Let f be a rational function of X. Then $\pi^\flat(f)(\widetilde{z}) = f(\pi(\widetilde{z}))$ is a meromorphic function on $\mathbb{C}$ and its Laurent series in 0 coincides with its image of the extension of θ to the respective quotient fields. The commutativity of (4.32) implies that this Laurent series evaluated in z coincides with $c(f) \in \mathbb{C}(\!(z)\!)$. Looking at (4.30), we can see that $f(\pi(\widetilde{z}))$ is elliptic if and only if X is elliptic, a rational function of exponentials if and only if X is nodal and a rational function if and only if X is cuspidal. Here we used $f\iota\widetilde{\nu} = f\nu\iota$ and the fact that $f\nu$ is a rational function on $\mathbb{P}^1_{\mathbb{C}}$ if X is singular, i.e. simply a quotient of two polynomials. □

Lemma 4.35. — *The "if" holds in all three parts of Theorem 4.32.*

Proof. — Assume that ϱ is of the given form and let $\widetilde{r} := \Theta\varrho \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$. By Lemma 2.16, $\mathfrak{g}(\widetilde{r})$ is generated by $\{(1 \otimes \alpha)\varrho(z,0) \mid \alpha \in \mathfrak{g}^*\}$, so it can be identified with a subalgebra of meromorphic maps $\mathbb{C} \to \mathfrak{g}$ which are elliptic in case (1), rational functions of $\exp(z/m)$ in case (2) and rational in case (3). Since r and $\widetilde{r}$ are gauge equivalent, it holds that

$$\mathrm{Mult}(\mathfrak{g}(r)) = \mathrm{Mult}(\mathfrak{g}(\widetilde{r})) = \{f \in \mathfrak{g}(\!(z)\!) \mid f\mathfrak{g}(\widetilde{r}) \subseteq \mathfrak{g}(\widetilde{r})\}\,.$$

We have seen in Step 3 of the proof of Theorem 3.25 that there exists a subalgebra $O \subseteq \mathrm{Mult}(\mathfrak{g}(\widetilde{r}))$ of finite codimension with the property: for every $f \in O$ exists a non-commutative polynomial $P = P(x_1, \dots, x_q)$ and elements $a_1, \dots, a_q \in \mathfrak{g}(\widetilde{r})$ satisfying $f\,\mathrm{id}_{\mathfrak{g}} = P(\mathrm{add}(a_1), \dots, \mathrm{add}(a_q))$. In particular, O consists of elliptic functions in case (1), rational functions of exponentials in case (2) and rational functions in case (3). Since the quotient field of O coincides with the rational functions on X, this observation combined with Lemma 4.34 proves all "if" directions. □

Lemma 4.36. — *Let ρ be the geometric r-matrix of $((X,\mathcal{A}),(C,\eta))$. There exists an isomorphism $\psi\colon \pi^*\mathcal{A} \to \mathfrak{g} \otimes \mathcal{O}_{\mathbb{C}}$ such that the meromorphic map $\varrho\colon \mathbb{C} \times \mathbb{C} \to \mathfrak{g} \otimes \mathfrak{g}$ defined by*

$$\varrho = (\psi \boxtimes \psi)(\pi \times \pi)^* \rho|_{C \times C} \tag{4.34}$$

has the following property depending on X:

1. *If X is elliptic, $\varrho(x+\lambda, y+\lambda') = \varrho(x,y)$ for some $n \in \mathbb{N}$ and all $\lambda, \lambda' \in n\Lambda$.*
2. *If X is nodal, ϱ is of the form (4.29) for some $t \in \mathfrak{L}(\mathfrak{g},\sigma) \otimes \mathfrak{L}(\mathfrak{g},\sigma)$, where σ is some automorphism of $\mathfrak{g}$ of order m and $\gamma = \sum_{k=0}^{m-1} \gamma_k$ is the unique decomposition such that $(\sigma \otimes 1)\gamma_k = \exp(2\pi i k/m)\gamma_k$.*
3. *If X is cuspidal, $\varrho(x,y) = \frac{\gamma}{x-y} + t(x,y)$ for some $t \in (\mathfrak{g} \otimes \mathfrak{g})[x,y]$.*

Proof.

Construction of ϱ in (1). — Since $\mathcal{A}$ is weakly $\mathfrak{g}$-locally free, Theorem 3.28 provides an isomorphism $\psi_1 \colon \widetilde{\nu}^* \iota^* \mathcal{A} \to \mathcal{S}$, where $\mathcal{S}$ is the sheaf on $\mathbb{C}/\Lambda$ of holomorphic sections of

$$\mathbb{C} \times \mathfrak{sl}(n,\mathbb{C})/\sim \qquad (z,a) \sim \left(z+\lambda_1, T_1 a T_1^{-1}\right) \sim \left(z+\lambda_2, T_2 a T_2^{-1}\right),$$

for T_1 and T_2 as defined in Theorem 3.28. Let $\psi_2 \colon \operatorname{pr}^* \mathcal{S} \to \mathfrak{g} \otimes \mathcal{O}_{\mathbb{C}}^{\mathrm{an}}$ denote the canonical isomorphism, where $\operatorname{pr} \colon \mathbb{C} \to \mathbb{C}/\Lambda$ is the canonical projection. Then $\psi_2 \operatorname{pr}^* a \colon \mathbb{C} \to \mathfrak{g}$ is an $n\Lambda$-periodic meromorphic function for any rational section a of $\mathcal{S}$, since $T_1^n = T_2^n = \mathrm{id}_{\mathfrak{g}}$. Therefore,

$$\varrho \coloneqq (\psi \boxtimes \psi)(\pi \times \pi)^* \rho \colon \mathbb{C} \times \mathbb{C} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$$

is a meromorphic map satisfying the desired periodicity, where $\psi \coloneqq \psi_2(\operatorname{pr}^* \psi_1)$ and $\pi = \iota \widetilde{\nu} \operatorname{pr}$ was used.

Construction of ϱ in (2). — Since $\mathcal{A}|_{\breve{X}}$ is weakly $\mathfrak{g}$-locally free, Theorem 3.27 (1) provides an isomorphism $\psi_1 \colon \nu^*(\mathcal{A}|_{\breve{X}}) \to \mathcal{L}$, where $\mathcal{L}$ is the sheaf associated to $\mathfrak{L}(\mathfrak{g},\sigma) \subseteq \mathfrak{g}[\widetilde{u},\widetilde{u}^{-1}]$ on $\operatorname{Spec}(\mathbb{C}[u,u^{-1}]) = \nu^{-1}(\breve{X})$, for some $\sigma \in \operatorname{Aut}_{\mathbb{C}\text{-alg}}(\mathfrak{g})$ of finite order m. Using Lemma 4.18 for $U = \breve{X}$, the local parameter $\nu^{\flat,-1}(u-1)$ of p, and $\chi = (\psi_1 \boxtimes \psi_1)^{-1}(\nu^{-1} \times \nu^{-1})^* \widetilde{\chi}$, where

$$\begin{aligned}(4.35) \quad \widetilde{\chi} \coloneqq \sum_{j=0}^{m-1} (\widetilde{u}/\widetilde{v})^j \gamma_j \in \mathfrak{L}(\mathfrak{g},\sigma) \otimes \mathfrak{L}(\mathfrak{g},\sigma) &\subseteq \mathfrak{g}\left[\widetilde{u},\widetilde{u}^{-1}\right] \otimes \mathfrak{g}\left[\widetilde{u},\widetilde{u}^{-1}\right] \\ &\cong (\mathfrak{g} \otimes \mathfrak{g})\left[\widetilde{u},\widetilde{u}^{-1},\widetilde{v},\widetilde{v}^{-1}\right],\end{aligned}$$

results in

$$(4.36) \qquad \widetilde{\rho} \coloneqq (\psi_1 \boxtimes \psi_1)(\nu \times \nu)^* \rho|_{\breve{X} \times \breve{X}} = \frac{v\widetilde{\chi}}{u-v} + t,$$

for some $t \in \mathfrak{L}(\mathfrak{g},\sigma) \otimes \mathfrak{L}(\mathfrak{g},\sigma)$. Here we used $\widetilde{u}^m = u$ and $\eta = \mathrm{d}u/u$. Moreover, it was used that $(\sigma \otimes \sigma)\gamma = \gamma$ gives $\gamma = \sum_{j=0}^{m-1} \gamma_j$ and this implies that χ is indeed a preimage of $\mathrm{id}_{\mathcal{A}|_U}$ under (4.15). The mapping $a(\widetilde{u}) \mapsto a(\exp(z/m)))$ induces an isomorphism $\psi_2 \colon \exp^* \iota^* \mathcal{L} \to \mathfrak{g} \otimes \mathcal{O}_{\mathbb{C}}^{\mathrm{an}}$ such that

$$\begin{aligned}(4.37) \quad \varrho &\coloneqq (\psi_2 \boxtimes \psi_2)(\iota \exp \times \iota \exp)^* \widetilde{\rho}|_{\breve{X} \times \breve{X}} \\ &= (\psi \boxtimes \psi)(\pi \times \pi)^* \rho|_{\breve{X} \times \breve{X}} \colon \mathbb{C} \times \mathbb{C} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}\end{aligned}$$

is of the form (4.29), where $\psi \coloneqq \psi_2\left(\exp^* \iota^* \psi_1\right)$ and $\nu \iota \exp = \iota \widetilde{\nu} \exp = \pi$ was used.

Construction of ϱ in (3). — Since $\mathcal{A}|_{\breve{X}}$‘ is weakly $\mathfrak{g}$-locally free, Theorem 3.27 (2) provides an isomorphism $\psi\colon \nu^*(\mathcal{A}|_{\breve{X}}) \to \mathfrak{g}\otimes\mathcal{O}_{\operatorname{Spec}(\mathbb{C}[u])}$. Using Lemma 4.18 for $U = \breve{X}$, the local parameter $\nu^{\flat,-1}(u)$ of p, and

$$\chi = (\psi\boxtimes\psi)^{-1}\left(\nu^{-1}\times\nu^{-1}\right)^*\gamma,$$

where $\gamma\in\mathfrak{g}\otimes\mathfrak{g}$ is considered as a global section

$$(\mathfrak{g}\otimes\mathcal{O}_{\operatorname{Spec}(\mathbb{C}[u])})\boxtimes(\mathfrak{g}\otimes\mathcal{O}_{\operatorname{Spec}(\mathbb{C}[u])})\cong(\mathfrak{g}\otimes\mathfrak{g})\otimes\mathcal{O}_{\operatorname{Spec}(\mathbb{C}[u,v])},$$

results in the desired

$$\varrho := (\psi\boxtimes\psi)(\nu\times\nu)^*\rho|_{\breve{X}\times\breve{X}} = \frac{\gamma}{u-v}+t, \tag{4.38}$$

for some $t\in(\mathfrak{g}\otimes\mathfrak{g})[u,v]$. Here we used $\eta = \mathrm{d}u$ and the fact that χ is obviously a preimage of $\mathrm{id}_{\mathcal{A}|_U}$ under (4.15). □

Lemma 4.37. — *The series $\Theta\varrho$ is gauge equivalent to r, where Θ was defined in Definition 4.31.*

Proof. — In all three cases, $\varrho = (\psi\times\psi)(\pi\times\pi)^*\rho|_{\breve{X}\times\breve{X}}$ for an isomorphism $\psi\colon\pi^*\mathcal{A}\to\mathfrak{g}\otimes\mathcal{O}_{\mathbb{C}}$. Using Lemma 4.34, we can see that the composition

$$\mathfrak{g}[\![z]\!]\xrightarrow{\zeta^{-1}}\widehat{\mathcal{A}}_p\longrightarrow\widehat{\pi^*\mathcal{A}_0}\xrightarrow{\widehat{\psi}_0}\mathfrak{g}\otimes\widehat{\mathcal{O}}^{\mathrm{an}}_{\mathbb{C},0}\xrightarrow{\mathrm{id}_{\mathfrak{g}}\otimes\theta}\mathfrak{g}[\![z]\!] \tag{4.39}$$

defines a $\mathbb{C}[\![z]\!]$-linear Lie algebra automorphism φ of $\mathfrak{g}[\![z]\!]$. Here, the second arrow is the isomorphism obtained by completing $\mathcal{A}_p\to(\pi_*\pi^*\mathcal{A})_p\to\mathcal{A}_p\otimes_{\mathcal{O}_{X,p}}\mathcal{O}^{\mathrm{an}}_{\mathbb{C},0}\cong(\pi^*\mathcal{A})_0$ and θ is the map defined by the Taylor expansion in 0. It is straight forward to show that the diagram

$$\begin{array}{ccc}
\Gamma\left(\breve{X}\times\breve{X}\setminus\Delta,\mathcal{A}\boxtimes\mathcal{A}\right) & \xrightarrow{\jmath^*} & (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!] \\
 & & \big\downarrow{\scriptstyle\varphi(x)\otimes\varphi(y)} \\
{\scriptstyle(\pi\times\pi)^*}\big\downarrow & & (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!] \\
 & & \big\uparrow{\scriptstyle\Theta} \\
\Gamma(\mathbb{C}\times\mathbb{C}\setminus\Delta_\pi,\pi^*\mathcal{A}\boxtimes\pi^*\mathcal{A}) & \xrightarrow[\psi\boxtimes\psi]{} & (\mathfrak{g}\otimes\mathfrak{g})\otimes\Gamma(\mathbb{C}\times\mathbb{C}\setminus\Delta_\pi,\mathcal{O}^{\mathrm{an}}_{\mathbb{C}\times\mathbb{C}})
\end{array} \tag{4.40}$$

commutes, where $\Delta_\pi := \{(x,y)\in\mathbb{C}\times\mathbb{C}\mid\pi(x)=\pi(y)\}$ and $\jmath$ was defined in Notation 4.19. The upper row maps $\rho|_{\breve{X}\times\breve{X}}$ to $(\varphi(x)\otimes\varphi(y))r(x,y)$ by virtue of Theorem 4.22, so we can conclude that $(\varphi(x)\otimes\varphi(y))r(x,y) = \Theta\varrho(x,y)$. □

Appendix A. Real and complex analytic generalized r-matrices

In this subsection we define the classic complex analytic notion of generalized r-matrices and relate these to formal generalized r-matrices over $\mathbb{C}$. It is also possible to consider a real analytic context. Both cases work rather parallel and are developed simultaneously in the following. For this purpose, let $\Bbbk \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\Bbbk$.

Definition A.1. — *For two real finite-dimensional vector spaces V, W and $U \subseteq V$ open, a* (real) meromorphic map $U \to W$ *is a map defined on a dense open subset of U with values in W which is the restriction of a meromorphic map (in the usual complex analytic sense) $\widetilde{U} \to W \otimes_{\mathbb{R}} \mathbb{C}$ to U, where $\widetilde{U} \subseteq V \otimes_{\mathbb{R}} \mathbb{C}$ is an open neighbourhood of U.*

Remark A.2. — This definition could be done without reference to complex numbers by considering maps defined on a dense subset of U with values in V which are locally quotients of analytic maps.

Definition A.3. — *Let $U \subset \Bbbk$ be a connected open subset and $r \colon U \times U \to \mathfrak{g} \otimes \mathfrak{g}$ be a meromorphic map. Then r is called a* generalized r-matrix *if it solves the* generalized classical Yang–Baxter equation *(GCYBE)*

$$\begin{aligned}(A.1)\quad &\left[r^{12}(x_1, x_2), r^{13}(x_1, x_3)\right] + \left[r^{12}(x_1, x_2), r^{23}(x_2, x_3)\right] \\ &\qquad + \left[r^{32}(x_3, x_2), r^{13}(x_1, x_3)\right] = 0.\end{aligned}$$

and r-matrix *if it solves the* classical Yang–Baxter equation *(CYBE)*

$$\begin{aligned}(A.2)\quad &\left[r^{12}(x_1, x_2), r^{13}(x_1, x_3)\right] + \left[r^{12}(x_1, x_2), r^{23}(x_2, x_3)\right] \\ &\qquad + \left[r^{13}(x_1, x_3), r^{23}(x_2, x_3)\right] = 0.\end{aligned}$$

for all $x_1, x_2, x_3 \in \Bbbk$ where these equations are defined respectively. Here, Notation 2.5 as well as $r^{ij}(x_i, x_j) = r(x_i, x_j)^{ij}, r^{32}(x_3, x_2) = \tau(r(x_2, x_3))^{23}$ was used and the brackets on the left hand side of both equations are understood as the usual commutators in $\mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$. If we want to emphasize that $\Bbbk = \mathbb{C}$ (resp. $\Bbbk = \mathbb{R}$) we call r complex *(resp.* real*) and we say that r is* analytic *if we want to emphasize that we are not in the formalism of Section 2.*

Definition A.4. — *A meromorphic map $r \colon U \times U \to \mathfrak{g} \otimes \mathfrak{g}$, where $U \subseteq \Bbbk$ is a connected open subset, is called*

- non-degenerate *if for some (x_0, y_0) in the domain of r the linear map $\mathfrak{g}^* \to \mathfrak{g}$ defined by $\alpha \mapsto (1 \otimes \alpha) r(x_0, y_0)$ is bijective and*

- skew-symmetric *if* $r(x,y) = -\tau(r(x,y))$, *where* $\tau(a \otimes b) = b \otimes a$ *for* $a, b \in \mathfrak{g}$.

Remark A.5. — If $r\colon U \times U \to \mathfrak{g} \otimes \mathfrak{g}$ is a non-degenerate meromorphic map, where $U \subseteq \Bbbk$ is a connected open subset, the map $\mathfrak{g}^* \to \mathfrak{g}$ defined by $\alpha \mapsto (1 \otimes \alpha)r(x,y)$ is bijective for all (x,y) in a dense open subset of the domain of definition of r.

Assume that $\mathfrak{g}$ is semi-simple with Killing form κ and Casimir element $\gamma \in \mathfrak{g} \otimes \mathfrak{g}$. Then $\widetilde{\kappa}(\gamma) = \mathrm{id}_{\mathfrak{g}}$, where $\widetilde{\kappa}\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is the isomorphism defined by $a \otimes b \mapsto \kappa(b, \cdot)a$. Clearly, the linear map $\mathfrak{g}^* \to \mathfrak{g}$ defined by a tensor $t \in \mathfrak{g} \otimes \mathfrak{g}$ is bijective if and only if $\det(\widetilde{\kappa}(t)) \neq 0$. Consider a meromorphic map r of the form

$$r(x,y) = \frac{\lambda(y)\gamma}{x-y} + r_0(x,y), \tag{A.3}$$

where $\lambda\colon U \to \Bbbk^\times$ and $r_0\colon U \times U \to \mathfrak{g} \otimes \mathfrak{g}$ are analytic for some connected open $U \subseteq \Bbbk$. Then $(x-y)\widetilde{\kappa}(r(x,y))$ equals $\lambda(y)\,\mathrm{id}_{\mathfrak{g}}$ for $x = y$. Thus, its determinant is non-vanishing in an open neighbourhood of

$$\{(x,y) \in U \times U \mid x = y\}.$$

In particular, r is seen to be non-degenerate. It turns out that every non-degenerate (generalized) r-matrix is locally of the form (A.3) if $\mathfrak{g}$ is central simple. The proof in the case of a solution of the CYBE is mentioned in [8] and the GCYBE case is proven similar. However, since the details are omitted in [8] and the real case is not discussed, we will present a proof using the coordinate-free methods from [34].

Proposition A.6. — *Let $\mathfrak{g}$ be a finite-dimensional central simple Lie algebra and $r\colon U \times U \to \mathfrak{g} \otimes \mathfrak{g}$ be a non-degenerate generalized r-matrix (resp. r-matrix), where $U \subseteq \Bbbk$ is connected and open. After probably shrinking U and shifting the origin, $0 \in U$ and there exists analytic maps $\lambda\colon U \to \Bbbk^\times$ and $r_0\colon U \times U \to \mathfrak{g} \otimes \mathfrak{g}$ such that*

$$r(x,y) = \frac{\lambda(y)}{x-y}\gamma + r_0(x,y). \tag{A.4}$$

In particular, Θr is a formal generalized r-matrix (resp. formal r-matrix) for Θ defined in Definition 4.31.

Proof.

Step 1: Translating the CYBE and GCYBE into an endomorphism language. — Consider the isomorphisms $\widetilde{\kappa}\colon \mathfrak{g}\otimes\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ and $\widetilde{\kappa}_3\colon \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \to \mathrm{Hom}(\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g})$ defined by

$$\widetilde{\kappa}(a_1\otimes a_2)(b_1) = \kappa(a_2,b_1)a_1$$
$$\widetilde{\kappa}_3(a_1\otimes a_2\otimes a_3)(b_1\otimes b_2) = \kappa(b_2,a_3)\kappa(b_1,a_2)a_1$$

for all $a_1,a_2,a_3,b_1,b_2 \in \mathfrak{g}$. Assume first that r is a generalized r-matrix. Then, applying $\widetilde{\kappa}_3$ to the GCYBE (A.1) and evaluating in an arbitrary $t_1\otimes t_2 \in \mathfrak{g}\otimes\mathfrak{g}$ yields

$$\begin{aligned}(\text{A.5})\quad &[\widetilde{\kappa}(r(x_1,x_2))t_1,\widetilde{\kappa}(r(x_1,x_3))t_2]\\ &\quad= \widetilde{\kappa}(r(x_1,x_2))[t_1,\widetilde{\kappa}(r(x_2,x_3))t_2] + \widetilde{\kappa}(r(x_1,x_3))[\widetilde{\kappa}(r(x_3,x_2))t_1,t_2],\end{aligned}$$

where it was used that e.g. for any $a,b,c,d\in\mathfrak{g}$ we have:

$$\begin{aligned}\widetilde{\kappa}_3\left([(a\otimes b)^{32},(c\otimes d)^{13}]\right)(t_1\otimes t_2) &= \widetilde{\kappa}_3(c\otimes b\otimes[a,d])(t_1\otimes t_2)\\ &= \kappa([a,d],t_2)\kappa(b,t_1)c = \kappa([\kappa(b,t_1)a,d],t_2)c\\ &= -\kappa([\kappa(b,t_1)a,t_2],d)c = -\widetilde{\kappa}(c\otimes d)[\widetilde{\kappa}(a\otimes b)t_1,t_2].\end{aligned}$$

The other identities used can be derived similarly; see [34, Proposition 2.14]. If r is a solution of the CYBE (A.2), we find

$$\begin{aligned}(\text{A.6})\quad &[\widetilde{\kappa}(r(x_1,x_2))t_1,\widetilde{\kappa}(r(x_1,x_3))t_2]\\ &\quad= \widetilde{\kappa}(r(x_1,x_2))[t_1,\widetilde{\kappa}(r(x_2,x_3))t_2] - \widetilde{\kappa}(r(x_1,x_3))[\widetilde{\kappa}(r(x_2,x_3))^*t_1,t_2],\end{aligned}$$

by applying $\widetilde{\kappa}_3$. Here $(\cdot)^*$ denotes the adjoint with respect to κ.

Step 2: r has poles along the diagonal. — By Remark A.5 (1) we can chose a point (u_0,v_0) in the domain of r such that $\widetilde{\kappa}(r(u_0,v_0))$ is an isomorphism and $u\mapsto T_u := \widetilde{\kappa}(r(u,v_0))$ is analytic along the line connecting u_0 and v_0 but excluding v_0. We prove by contradiction that r has a pole at (v_0,v_0). Assume that r is analytic in (v_0,v_0), i.e. $u\mapsto T_u$ is analytic in v_0. The equations (A.5) and (A.6) reduce to

$$(\text{A.7})\qquad [T_ut_1,T_ut_2] = T_u([t_1,T_{v_0}t_2]+[T_{v_0}t_1,t_2])$$
$$(\text{A.8})\qquad [T_ut_1,T_ut_2] = T_u([t_1,T_{v_0}t_2]-[T_{v_0}^*t_1,t_2]).$$

respectively by setting $x_1 = u, x_2,x_3 = v_0$. Applying $\psi_u := T_u\circ T_{u_0}^{-1}$ to (A.7) and (A.8) evaluated at $u=u_0$ results in

$$(\text{A.9})\qquad \psi_u[T_{u_0}t_1,T_{u_0}t_2] = T_u([t_1,T_{v_0}t_2]+[T_{v_0}t_1,t_2]),$$
$$(\text{A.10})\qquad \psi_u[T_{u_0}t_1,T_{u_0}t_2] = T_u([t_1,T_{v_0}t_2]-[T_{v_0}^*t_1,t_2]).$$

Comparing these equations with (A.7) and (A.8) evaluated at $u = u_0$ and using the fact that T_{u_0} is bijective, we see that ψ_u is a Lie algebra homomorphism in both cases. Therefore, the fact that ψ_u is orthogonal with respect to κ if it is invertible, implies that $\det(\psi_u) \in \{0, \pm 1\}$; see e.g. [34, Lemma 2.3.] for details. A continuity argument and $\psi_{u_0} = \mathrm{id}_\mathfrak{g}$ forces ψ_{v_0} and consequently $T_{v_0} = \psi_{v_0} \circ T_{u_0}$ to be an isomorphism.

Setting $u = v_0$ in equation (A.7), we see that $T_{v_0}^{-1}$ is an invertible derivation of $\mathfrak{g}$, contradicting the simplicity of $\mathfrak{g}$. Setting $u = v_0$ in equation (A.8) leads to the same contradiction, considering the fact that

$$\det(T_{v_0}) \neq 0 \implies T_{v_0}^* = -T_{v_0}. \tag{A.11}$$

The proof of (A.11) can be found in [34, Lemma 3.2 and Lemma 3.4] for $\Bbbk = \mathbb{C}$ and uses Schur's Lemma as well as the fact that every automorphism of $\mathfrak{g}$ has a fixed vector. Since $\mathfrak{g}$ is assumed to be central, Schur's Lemma applies for $\Bbbk = \mathbb{R}$ and an automorphism of $\mathfrak{g}$ without fixed vector defines one on the simple complex Lie algebra $\mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ by extension of scalars. Thus, the proof in [34, Lemma 3.2 and Lemma 3.4] also applies to the case $\Bbbk = \mathbb{R}$. Summarized, the assumption that r is a solution of either the CYBE or GCYBE without a pole along the diagonal leads to a contradiction. We have shown that r has a pole along the diagonal.

Step 3: After shrinking U, $r(x,y) = \frac{\lambda(y)\gamma}{(x-y)^k} + \frac{f(x,y)}{(x-y)^{k-1}}$. — Using Lemma 2.3 for $V = \mathfrak{g} \otimes \mathfrak{g}$ as well as $\bigcap_{k=0}^\infty (x-y)^k V[\![x,y]\!] = \{0\}$, we can find a $k \in \mathbb{N}_0$ and shrink U in such a way that $s(x,y) = (x-y)^k r(x,y)$ is analytic on $U \times U$ and $h(z) := s(z,z)$ is an analytic function on U which is not identically 0. After probably shrinking U further, we may assume that h is non-vanishing and $s(x,y) - h(y) = (x-y)f(x,y)$ for an analytic function $f \colon U \times U \to \mathfrak{g} \otimes \mathfrak{g}$. Multiplying either (A.7) or (A.8) with $(x_1 - x_2)^k$ and setting $x_1 = x_2$ results in

$$[\widetilde{\kappa}(h(x_2))t_1, \widetilde{\kappa}(r(x_2,x_3))t_2] = \widetilde{\kappa}(h(x_2))[t_1, \widetilde{\kappa}(r(x_2,x_3))t_2] \tag{A.12}$$

in both cases. Choosing x_3 in such a way that $\widetilde{\kappa}(r(x_2,x_3))$ is an isomorphism, we see that $\widetilde{\kappa}(h(x_2))$ is an equivariant endomorphism of $\mathfrak{g}$ with respect to the adjoint representation. In other words, $\kappa(h(x_2))$ is in the centroid of $\mathfrak{g}$ (see Definition 3.23). Hence, $\widetilde{\kappa}(h(x_2)) = \lambda(x_2)\,\mathrm{id}_\mathfrak{g}$ since $\mathfrak{g}$ is central, where $\lambda \colon U \to \Bbbk^\times$ is an analytic function. This implies that $h(x_2) = \lambda(x_2)\gamma$. Summarized, we obtain

$$r(x,y) = \frac{\lambda(y)\gamma}{(x-y)^k} + \frac{f(x,y)}{(x-y)^{k-1}}.$$

Step 4: $k=1$. — Assume that $k>1$. Then

$$(x_1-x_2)^{k-1}\left[r^{32}(x_3,x_2),r^{13}(x_1,x_3)\right]$$

and

$$(x_1-x_2)^{k-1}\left[r^{13}(x_1,x_3),r^{23}(x_2,x_3)\right]$$

vanish for $x_1=x_2$. Therefore, multiplying the CYBE or the GCYBE with $(x_1-x_2)^{k-1}$, using

$$\left[\gamma^{12},r(x_2,y_3)^{23}\right]=-\left[\gamma^{12},r(x_2,y_3)^{13}\right]$$

and taking the limit $x_1\to x_2$ results in

$$\begin{aligned}0&=\left[h(x_2)^{12},\partial_{x_2}r(x_2,x_3)^{13}\right]+\left[f(x_2,x_2)^{12},r(x_2,x_3)^{13}+r(x_2,x_3)^{23}\right]\\&=\left[h(x_2)^{12},(x_2-x_3)^{-k}\partial_{x_2}s(x_2,x_3)^{13}-k(x_2-x_3)^{-k-1}s(x_2,x_3)^{13}\right]\\&\quad+\left[f(x_2,x_2)^{12},r(x_2,x_3)^{13}+r(x_2,x_3)^{13}\right],\end{aligned}$$

where h,s and f are defined in Step 3 and the limit definition of ∂_{x_2} was used. Multiplying this with $(x_2-x_3)^{k+1}$ and taking the limit $x_1\to x_3$ under consideration of $h(z)=\lambda(z)\gamma$ yields $-k\lambda(x_3)^2[\gamma^{12},\gamma^{13}]=0$. This contradicts the fact that $[\gamma^{12},\gamma^{13}]\neq 0$. Indeed, $[\gamma^{12},\gamma^{13}]$ maps to γ with respect to the linear map defined by $a\otimes b\otimes c\mapsto[b,a]\otimes c$; see (2.23). Therefore, the assumption $k>1$ leads to a contradiction and we can conclude that $k=1$. □

Remark A.7. — A formal generalized r-matrix associated to a non-degenerate analytic generalized r-matrix r is skew-symmetric if and only if r is skew-symmetric. In particular, Proposition 2.21 gives a new proof of the fact that non-degenerate analytic generalized r-matrices are exactly the non-degenerate analytic r-matrices.

Definition A.8. — *A meromorphic map $s\colon U\times U\to\mathfrak{g}\otimes\mathfrak{g}$ is said to be in* standard form *if $U\subseteq\Bbbk$ is a connected open neighbourhood of $0\in\Bbbk$ such that*

$$s(x,y)=\frac{\lambda(y)\gamma}{x-y}+s_0(x,y)\tag{A.13}$$

for some analytic maps $\lambda\colon U\to\Bbbk$ and $s_0\colon U\times U\to\mathfrak{g}\otimes\mathfrak{g}$

Remark A.9. — Proposition A.6 allows us to restrict our study of non-degenerate generalized r-matrices to those in standard form.

Definition A.10. — *Let* $r\colon U\times U\to\mathfrak{g}\otimes\mathfrak{g}$, $\widetilde{r}\colon V\times V\to\mathfrak{g}\otimes\mathfrak{g}$ *be meromorphic maps in standard form. Then* $\widetilde{r}$ *is called* analytically equivalent *to* r *if*

$$\widetilde{r}(x,y)=\lambda(y)(\varphi(x)\otimes\varphi(y))r(w(x),w(y)),$$

where the triple (λ,w,φ) *is called an* analytic equivalence *and consists of*

- *an analytic embedding* $w\colon W\to U$, *for some connected open neighbourhood* $W\subseteq V$ *of* 0 *satisfying* $w(0)=0$, *called* coordinate transformation,
- *a non-zero analytic* $\lambda\colon W\to\Bbbk$ *called* rescaling *and*
- *an analytic* $\varphi\colon W\to\mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})$ *called* gauge transformation*.*

Remark A.11. — A similar result to Lemma 2.12 holds for analytic equivalences. More precisely, analytic equivalences preserve non-degeneracy and the property of solving the GCYBE (A.1). Furthermore, analytic equivalences with constant rescaling part preserve skew-symmetry and the property of solving the CYBE (A.2). The proof of these statements uses a reduction to the complex case and the identity theorem (see e.g. [27, Chapter I.A, Theorem 6]), under consideration that the domain of definition of a complex meromorphic function on a connected open set is connected.

Remark A.12. — Using Proposition A.6 and an appropriate rescaling, we can see that non-degenerate analytic generalized r-matrices are in normalized standard form up to equivalence, i.e. in the form (A.4) with $\lambda=1$.

Clearly, Θr is a formal generalized r-matrix for every solution of the GCYBE r in standard form, where Θ was defined in Definition 4.31. The following result shows that Θ essentially identifies the formal and analytic setting in this way.

Proposition A.13. — *Every formal generalized* r*-matrix over* $\mathbb{C}$ *(resp.* $\mathbb{R}$*) is of the form* Θr *for a complex (resp. real) analytic generalized* r*-matrix in standard form. Furthermore, two complex (resp. real) analytic generalized* r*-matrices* r_1 *and* r_2 *in standard form are analytically equivalent if and only if* Θr_1 *and* Θr_2 *are equivalent. In other words,* Θ *defines a bijection between equivalence classes of formal generalized* r*-matrices over* $\mathbb{C}$ *(resp.* $\mathbb{R}$*) and analytic equivalence classes of complex (resp. real) analytic generalized* r*-matrices in standard form.*

Proof.

Step 1: Setup. — Let $i\in\{1,2\}$ and $r_i(x,y)\in(\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]$ be a formal generalized r-matrix and

$$\mathrm{GD}(\mathrm{Mult}(\mathfrak{g}(r_i)),\mathfrak{g}(r_i)):=((X_i,\mathcal{A}_i),(p_i,c_i,\zeta_i))$$

be the associated geometric datum. Pick a smooth open neighbourhood C_i of p_i such that $\mathcal{A}_i|_{C_i}$ is étale $\mathfrak{g}$-locally free and there exists a non-vanishing 1-form η_i on C_i. The geometric r-matrix defined by $((X_i, \mathcal{A}_i), (C_i, \eta_i))$ will be denoted by ρ_i.

Step 2: r_i is equivalent to a Taylor series of an analytic generalized r-matrix ϱ_i. — Let $C_i^{\mathrm{an}} = (C_i^{\mathrm{an}}, \mathcal{O}_{C_i}^{\mathrm{an}})$ be the analytic manifold defined by the $\Bbbk$-rational points of C_i. Note that C_i is 1-dimensional, where in the real case this may be seen through the implicit function theorem and the fact that $p_i \in C_i^{\mathrm{an}}$. Let $U_i \to C_i^{\mathrm{an}}$ be an analytic parameterization around p_i, where $U_i = (U_i, \mathcal{O}_{U_i}^{\mathrm{an}})$ is the locally ringed space associated to an open disc if $\Bbbk = \mathbb{C}$ (resp. an open interval if $\Bbbk = \mathbb{R}$) around the origin. We write $\iota_i \colon U_i \to C_i^{\mathrm{an}} \to X_i$ for the resulting morphism of locally ringed spaces such that $\iota_i(0) = p_i$. The sheaf of Lie algebras $\iota_i^* \mathcal{A}_i$ can be identified with an analytic fiber bundle on U_i with fiber $\mathfrak{g}$ and structure group $\mathrm{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})$. Indeed, for $\Bbbk = \mathbb{C}$ this follows from Theorem 3.10 and the observation that étale $\mathfrak{g}$-local triviality implies local triviality in the complex topology, while for $\Bbbk = \mathbb{R}$ this is due to [36, Lemma 2.1]. These fiber bundles are always trivial since U_i is contractible; see [24, Satz 6] for the complex and [26, Chapter VIII,Propositions 1.10 & 1.19] for the real case. Thus, there exists an isomorphism $\psi_i \colon \iota_i^* \mathcal{A}_i \to \mathfrak{g} \otimes \mathcal{O}_{U_i}^{\mathrm{an}}$ of sheaves of Lie algebras and $\psi_i \boxtimes \psi_i$ defines an isomorphism $\iota_i^* \mathcal{A}_i \boxtimes \iota_i^* \mathcal{A}_i \to (\mathfrak{g} \otimes \mathfrak{g}) \otimes \mathcal{O}_{U_i \times U_i}^{\mathrm{an}}$. Consider the meromorphic map

$$\varrho_i \coloneqq (\psi_i \boxtimes \psi_i)(\iota_i \times \iota_i)^* \rho_i \colon U_i \times U_i \longrightarrow \mathfrak{g} \otimes \mathfrak{g}.$$

The Taylor series of ϱ_i in the second variable in the preimage of p_i under ι_i is equivalent to r_i. This can be deduced with an argument similar to the proof of Theorem 4.27.

Step 3: An equivalence between r_1 and r_2 defines an analytical equivalence between ϱ_1 and ϱ_2. — The equivalence between r_1 and r_2 defines isomorphisms $f \colon X_2 \to X_1$ and $\phi \colon \mathcal{A}_1 \to f_* \mathcal{A}_2$ such that, after probably adjusting C_1, C_2, η_1 and η_2, we have $f(p_2) = p_1$, $f^{-1}(C_1) = C_2$, $f^* \eta_1 = \eta_2$ and

$$(f^*(\phi) \boxtimes f^*(\phi))(f \times f)^* \rho_1 = \rho_2; \tag{A.14}$$

see Lemma 4.17. Application of $(\psi_2 \boxtimes \psi_2)(\iota_2 \times \iota_2)^*$ results in

$$\big(\psi_2\big((f\iota_2)^* \phi\big) \boxtimes \psi_2\big((f\iota_2)^* \phi\big)\big)(f\iota_2 \times f\iota_2)^* \rho_1 = \varrho_2. \tag{A.15}$$

After probably shrinking U_2, there exists an analytic embedding $w\colon U_2 \to U_1$ such that $\iota_1 w = f\iota_2$, since f is an isomorphism and maps p_2 to p_1. In particular, $w(0)=0$. We can rewrite (A.15) as $(\varphi(x)\otimes\varphi(y))\varrho_1(w(x),w(y))=\varrho_2(x,y)$, where $\varphi\colon U_2 \to \operatorname{Aut}_{\Bbbk\text{-alg}}(\mathfrak{g})$ is the analytic map induced by the chain

$$\text{(A.16)}\quad \mathfrak{g}\otimes\mathcal{O}_{U_2} \xrightarrow{w^*\left(\psi_1^{-1}\right)} w^*\iota_1^*\mathcal{A}_1 = (f\iota_2)^*\mathcal{A}_1 \xrightarrow{(f\iota_2)^*\phi} \iota_2^*\mathcal{A}_2 \xrightarrow{\psi_2} \mathfrak{g}\otimes\mathcal{O}_{U_2}$$

of isomorphisms of sheaves of Lie algebras. □

Remark A.14. — Let us point out that for a meromorphic map s in standard form, Θs is skew-symmetric if and only if s is. Therefore, Proposition A.13 induces a bijection between equivalence classes of formal r-matrices and analytic r-matrices as well.

Appendix B. Notation and conventions

Throughout this document $\Bbbk$ denotes a field of characteristic 0 with algebraic closure $\bar{\Bbbk}$. Furthermore, $\mathfrak{g}$ is a finite-dimensional Lie algebra over $\Bbbk$ which is semi-simple in Section 2 and central simple in Section 4. By convention the set of natural numbers $\mathbb{N}$ excludes 0 and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

B.1. Algebra.

For a unital commutative ring R and R-modules M, N, the space of R-linear maps $M \to N$ (resp. $M \to M$) is denoted by $\operatorname{Hom}_R(M,N)$ (resp. $\operatorname{End}_R(M)$), while the tensor product of M and N is written as $M\otimes_R N$. For $R=\Bbbk$ the indices are omitted. The invertible elements of R are denoted by $R^\times$, and $M^* := \operatorname{Hom}_R(M,R)$ is the dual module of M. If R is a domain, $\operatorname{Q}(R) := (R\setminus\{0\})^{-1}R$ denotes its quotient field and we write $\operatorname{Q}(M) := M\otimes_R \operatorname{Q}(R)$. Let $f\colon R \to R'$ be a morphism of unital commutative rings and M' be an R'-module. We say that a map $g\colon M \to M'$ is f-equivariant if it is a group homomorphism satisfying $g(rm) = f(r)g(m)$ for all $r\in R, m\in M$.

In this text, an R-algebra A satisfies no additional assumptions, i.e. $A = (A,\mu_A)$ consists of an R-module A equipped with a multiplication map $\mu_A\colon A\otimes_R A \to A$. In particular, a Lie algebra over R is an R-algebra. The group of invertible R-algebra endomorphisms of A, i.e. invertible R-linear maps $f\colon A \to A$ satisfying $f\mu_A = \mu_A(f\otimes f)$, will be denoted by

$\mathrm{Aut}_{R\text{-alg}}(A)$. We note that $\oplus$ will always denote the direct sum of modules and not of algebras. If A is a Lie algebra with adjoint representation $\mathrm{add}\colon A \to \mathrm{End}_R(A)$, we write $[a \otimes 1, t] := (\mathrm{add}(a) \otimes \mathrm{id}_A)t$ for all $a \in A, t \in A \otimes_R A$ and define $[1 \otimes a, t]$ and $[a \otimes 1 + 1 \otimes a, t]$ in a similar fashion.

B.2. Algebraic geometry.

Let $X = (X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules. For a morphism $f\colon X \to Y = (Y, \mathcal{O}_Y)$ of ringed spaces, we denote the additional structure morphism by $f^\flat\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ and write $f^\sharp\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ for the induced morphism. The set of $\mathcal{O}_X$-module homomorphisms $\mathcal{F} \to \mathcal{G}$ (resp. $\mathcal{F} \to \mathcal{F}$) is denoted by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ (resp. $\mathrm{End}_{\mathcal{O}_X}(\mathcal{F})$) while its sheaf counterpart is denoted by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ (resp. $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F})$). In particular, we write $\mathcal{F}^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. The tensor product of $\mathcal{F}$ and $\mathcal{G}$ is written as $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$. By abuse of notation, we write $f^* = f^*_{\mathcal{F}}\colon \mathcal{F} \to f_*f^*\mathcal{F}$ for the canonical morphism.

Assume that X and Y are S-schemes. The fiber product of X and Y over S is denoted by $X \times_S Y$ and $\mathcal{F}|_p$ is the fiber of $\mathcal{F}$ in a point $p \in X$. If $S = \mathrm{Spec}(\Bbbk)$, the index S is omitted and $\mathrm{H}^n(\mathcal{F})$ denotes the n^{th} global cohomology group of $\mathcal{F}$, while $\mathrm{H}^n(\mathcal{F})$ denotes its dimension over $\Bbbk$, if said space is finite-dimensional. In particular, $\mathrm{H}^0(\mathcal{F}) = \Gamma(X, \mathcal{F})$ is the space of global sections of $\mathcal{F}$.

B.3. Formal series

For a module M over a unital commutative ring R, the module of formal power series in the formal variable z with coefficients in M is denoted by $M[\![z]\!]$. The module $R[\![z]\!]$ is again a unital commutative ring and $M[\![z]\!]$ is a $R[\![z]\!]$-module. Then $M(\!(z)\!) := M[\![z]\!][z^{-1}]$ is the module of formal Laurent series. We note that if M is an R-algebra the module $M[\![z]\!]$ (resp. $M(\!(z)\!)$) is naturally an $R[\![z]\!]$-algebra (resp. $R(\!(z)\!)$-algebra). Elements p in $M(\!(z)\!)$ (resp. $M(\!(x_1)\!)\dots(\!(x_k)\!)$) will sometimes be denoted with the formal variable (resp. variables) for convenience, i.e. $p = p(z)$ (resp. $p = p\ (x_1, \dots, x_k)$). Moreover, a generic element $p \in M(\!(z)\!)$ is written $p(z) = \sum_{k \gg -\infty} p_k z^k$ and $p'(z) = \sum_{k \gg -\infty} kp_k z^{k-1}$ denotes the formal derivative of p. Finally, if $p(z) \in mz^{-k} + z^{-k+1}M[\![z]\!]$, it is said to be of order k with main part mz^{-k}.

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