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TOWARDS A DERIVATION OF CLASSICAL ELECTRODYNAMICS OF CHARGES AND FIELDS FROM QED

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ABSTRACT. — The purpose of this article is twofold:

- (1) On one hand, we rigorously derive the Newton–Maxwell equation in the Coulomb gauge from first principles of quantum electrodynamics in agreement with the formal Bohr correspondence principle of quantum mechanics.
- (2) On the other hand, we establish the global well-posedness of the Newton–Maxwell system on energy-spaces under weak assumptions on the charge distribution.

Both results improve the state of the art, and are obtained by incorporating semiclassical and measure theoretical techniques. One of the novelties is the use of quantum propagation properties in order to build global solutions of the Newton–Maxwell equation.

RÉSUMÉ. — L’objectif de cet article est double :

- (1) D’une part, la dérivation rigoureuse et cohérente avec le principe de correspondance de Bohr de l’équation de Newton–Maxwell dans la jauge de Coulomb à partir de l’électrodynamique quantique.
- (2) D’autre part, la preuve du caractère bien posé du système de Newton–Maxwell sur des espaces d’énergies adaptés avec des hypothèses faibles sur la distribution des charges.

Les deux résultats améliorent l’état de l’art et sont obtenus en incorporant des techniques semi-classiques et de la théorie des mesures. L’une des nouveautés est l’utilisation des propriétés de propagation quantique afin de construire des solutions globales de l’équation de Newton–Maxwell.

Keywords: Pauli–Fierz, Newton–Maxwell system, semiclassical analysis, Wigner measures, Liouville equation, generalized flow.

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1. Introduction

The dynamics of charged particles in interaction with the electromagnetic field is a topic that has been at the heart of physical and mathematical investigations for the most part of the last two centuries, and it is still of great interest for a large community of scientists, [43]. In the current state of the art, we are still far from a complete understanding of such dynamics, especially from a mathematical perspective: the interaction carries singularities of various nature both at the classical and quantum levels, that are quite difficult to deal with, see e.g., [47, 61]. It is well-known that, from a classical standpoint, an atomic system is not stable: according to Larmor's formula electrons would lose energy by radiation when accelerated by the nucleus' electrostatic force, thus collapsing! In addition, the charge distribution of a particle cannot be concentrated at a single point. Otherwise, it would yield an infinite electrostatic energy caused by the self-interaction of the particle with its own field; and consequently leading to known pathological behaviors of the Abraham–Lorentz radiation reaction force. On the other hand, if one passes to the quantum theory that was indeed developed in good part to address the issues raised by classical electrodynamics, a stable atomic model can be defined. However, the relativistic nature of the quantum electromagnetic field introduces new divergences, both infrared and ultraviolet, that require a so-called renormalization procedure to be dealt with. Unfortunately, renormalization for quantum electrodynamics, both with relativistic lepton fields and nonrelativistic point charges, has not yet been put on firm mathematical grounds, see [59] for a more detailed discussion. In the absence of such Lorentz covariant fundamental theory, one can alternatively state that the classical theory of electrodynamics is not appropriate for describing the behavior of an electron at a distance less than its Compton wavelength; and thus justifying the introduction of extended non-point charge distributions.

The aim of this paper is to study the mathematical interplay between the quantum and classical features of electrodynamics, in the case of extended and nonrelativistic charges in interaction with the electromagnetic field. Physically, we are thinking about molecules, ions or atoms, possibly different among each other, interacting both among themselves, with a pair potential, and with the electromagnetic field. From a classical standpoint, the dynamics is governed by the Newton–Maxwell system of equations, a coupled PDE-ODE nonlinear system, while at the quantum level the linear evolution is generated by the so-called Pauli–Fierz Hamiltonian, [14].

We rigorously derive the classical dynamics starting from the quantum dynamics in the limit $\hbar \rightarrow 0$. Here \hbar is, as usual in semiclassical analysis, a dimensionless parameter measuring how strong the quantum effects are in the system. Thus, verifying the validity of Bohr’s correspondence principle. In addition, we prove global well-posedness of the classic Newton–Maxwell system for “rough” charge distributions.

Bohr’s correspondence principle states that in the limit of large quantum numbers, i.e., at scales where the noncommutative nature of observables becomes irrelevant, the physical description of a quantum system should converge to the one dictated by its classical analogue. While such commonly accepted physical principle is nowadays mathematically understood to a good extent for particle systems, less is known for field theories. For the electrodynamics of extended charges, there are partial results available, limited only to coherent initial quantum states of minimal uncertainty, with no initial correlation between the particle’s and field’s subsystems [45].

We extend such analysis in two directions, in Theorem 1.4: On one hand, we relax the assumptions on the charge distributions of the particles, and on the other hand, we prove the correspondence principle for a very general class of quantum states, including highly correlated and incoherent ones. Moreover, our approach is different from the one in [45] and uses the more general concept of Wigner measures. This allows us to formulate an essentially complete mathematical picture of the correspondence principle for such fundamental interaction.

A second element of novelty is that we use the semiclassical features of quantum dynamics *to prove results on the classical dynamics* in a way that was not, up to our knowledge, explored before in QFT. In fact, we prove by infinite-dimensional semiclassical analysis and measure theoretical techniques, global well-posedness for the Newton–Maxwell system, under weak regularity assumptions on the particles’ charge distributions, see Theorem B.8 and Theorem 1.3. The regularity of the charge distribution affects the fixed-point estimates for such nonlinear evolution system, and it plays a crucial role for the ODE part of the system: below a certain regularity threshold the vector field of interaction is not Lipschitz, and thus it is not possible to close a standard fixed point argument as in [28]. While on the PDE part a lack of regularity may be compensated by finer spacetime estimates, for the ODE part there is less space of maneuver, see Section 2.2 for a discussion. Instead, we take a “quantum detour”, exploiting the well-defined quantum dynamics to define the classical one; we find this approach very natural from a physical perspective as classical theories are only an

approximation of quantum ones, even if mathematically it could seem at first sight surprising, it reflects the regularizing character of the quantum dynamics. We also prove some new interesting regularity propagation estimates, inherited from the corresponding quantum ones, see Proposition 5.7. Moreover, we conjecture that the assumptions (A_1) – $(A_2^{(1)})$ used in this article are thresholds for the Hadamard stability of the Newton–Maxwell equation in the space X^1 defined in (1.8).

One of the main tools in our investigation is *Wigner measures*. This mathematical object is of course well-known in homogenization and in semiclassical analysis over finite dimensional spaces, see for instance [1, 21, 34, 50, 60]. In recent years, these measures have been extended on the one hand to many-body theory in the mean-field scaling [8, 10], and on the other to interacting quantum fields [4, 5]. Our analysis here expands the previous investigations to systems of quantum particles coupled to quantized fields. In addition, we provide here several improvements of those techniques summarized in Appendices B and C.

Let us conclude this general introduction by providing some references. In addition to [28] there are quite some works devoted to the Newton–Maxwell system of equations [11, 15, 16, 18, 19, 40, 41, 42, 44, 46]. Whereas the Pauli–Fierz dynamics and nonrelativistic quantum electrodynamics have attracted a great interest from the mathematical physics community, see, e.g., [12, 13, 17, 20, 27, 31, 32, 33, 35, 36, 53]. Apart from the already mentioned paper [45], let us mention some other works where the classical limit of the Pauli–Fierz Hamiltonian has been studied in a different scaling, and with a different physical interpretation [24, 25, 26, 48, 49].

In the rest of this introduction we will define some notations, present the mathematical setting of the problem and conclude with the statement of our main results, Theorems 1.3 and 1.4.

In Section 2, we focus on the classical Newton–Maxwell system and its uniqueness properties.

Section 3 is devoted to the quantum system, in particular we provide some new uniform estimates for the Pauli–Fierz Hamiltonian that are crucial to our analysis.

In Section 4, we study the classical limit $\hbar \rightarrow 0$ of the quantum dynamics and derive the characteristic equation satisfied by its Wigner measures.

Finally, Section 5 is devoted to the proof of Theorems 1.3 and 1.4.

Appendix A is devoted to introducing fundamental tools for establishing uniform bounds related to the annihilation and creation operators.

Appendix B contains abstract results — of independent interest — concerning:

- (1) Equivalence between characteristic and Liouville equations.
- (2) Probabilistic representation of measure-valued solutions of Liouville equations.
- (3) Construction of generalized global flow for abstract initial value problems.

The measure theoretical techniques presented in Appendix B are crucial in the proof of our main results. It puts in a conclusive form several ideas that appeared previously in [6, 7, 57].

Appendix C establishes convergence results towards Wigner measures under optimal regularity assumptions on quantum states.

Notations and main results. — In this paragraph we first introduce the precise mathematical formulation of the quantum and classical models of electrodynamics that we shall consider and then state our main results. Recall that classical electrodynamics is governed by the Newton–Maxwell equation while quantum electrodynamics is described by the Pauli–Fierz Hamiltonian. Our main contributions are summarized in Theorem 1.3 where existence of unique global solutions of the Newton–Maxwell equation is established and in Theorem 1.4 where the above solutions are derived as the classical limit $\hbar \rightarrow 0$ of quantum Pauli–Fierz dynamics.

1.1. Newton–Maxwell equation

Consider n extended classical particles in the configuration space \mathbb{R}^d , with dimension $d \geq 3$, interacting with an electromagnetic field. The i^{th} particle has an assigned mass $m_i > 0$ and a charge distribution⁽¹⁾

$$\varphi_i : \mathbb{R}^d \longrightarrow \mathbb{R}$$

and their dynamics is completely characterized by their momenta $p_i = (p_i^\nu)_{\nu=1,\dots,d} \in \mathbb{R}^d$, and positions $q_i = (q_i^\nu)_{\nu=1,\dots,d} \in \mathbb{R}^d$. On the other hand, it is convenient for our purposes to describe the electromagnetic field in the Coulomb gauge by a complex vector field

$$\alpha = (\alpha_\lambda)_{\lambda=1,\dots,d-1} : \mathbb{R}^d \longrightarrow \mathbb{C}^{d-1}.$$

⁽¹⁾The total charge of the i^{th} particle is given, up to a dimensional constant, by $\int_{\mathbb{R}^d} \varphi_i(x) dx \in \mathbb{R}$. Therefore, our discussion includes as well the electrodynamics of charges that are globally neutral, but with a nontrivial charge distribution, like for example a water molecule.

Thus the Hamiltonian of the Newton–Maxwell system takes the form

$$(1.1) \quad H(p, q, \alpha) = \sum_{i=1}^n \frac{1}{2m_i} (p_i - A_i(q_i, \alpha))^2 + V(q) + \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \bar{\alpha}_\lambda(k) |k| \alpha_\lambda(k) dk.$$

Here

$$p = (p_1, \dots, p_n) \in \mathbb{R}^{dn}, \quad q = (q_1, \dots, q_n) \in \mathbb{R}^{dn}$$

and V is the smeared d -dimensional Coulomb potential

$$(1.2) \quad V(q) = \sum_{1 \leq i < j \leq n} w_{ij} (q_i - q_j),$$

such that

$$(1.3) \quad w_{ij} = g \varphi_i * \frac{1}{\omega^{d-2}} * \varphi_j,$$

where $g > 0$ is a suitable (dimensional) constant. Here we set

$$(1.4) \quad \omega(k) = |k|, \quad k \in \mathbb{R}^d.$$

In this paper we will require minimal mathematical assumptions on the potential V that therefore could be different from the electrostatic potential (1.2) above, and on the Fourier transform of the φ_i . See hypotheses (A_0) – (A_1) and $(A_2^{(\sigma)})$ below. The electromagnetic vector potential $A_i(q_i, \alpha) \in \mathbb{R}^d$ smeared by the charge distribution φ_i is defined by

$$(1.5) \quad A_i(q_i, \alpha) = \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \frac{\epsilon_\lambda(k)}{\sqrt{2|k|}} (\bar{\chi}_i(k) \alpha_\lambda(k) e^{2\pi i k \cdot q_i} + \chi_i(k) \bar{\alpha}_\lambda(k) e^{-2\pi i k \cdot q_i}) dk$$

where

$$\left\{ \epsilon_\lambda(k), \frac{k}{|k|} \right\}_{\lambda=1, \dots, d-1}$$

is an O.N.B of \mathbb{R}^d and

$$(1.6) \quad \chi_i(k) = \mathcal{F}^{-1}[\varphi_i](k) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot k} \varphi_i(x) dx.$$

Let us recall that the scalar potential is not a dynamical variable in the Coulomb gauge. Moreover, we denote the components of the vector potential by

$$A_i(q_i, \alpha) = (A_i^\nu(q_i, \alpha))_{\nu=1, \dots, d}.$$

The Newton–Maxwell equations of motion read as:

$$(1.7a) \quad \partial_t p_i = \frac{1}{m_i} (p_i - A_i(q_i, \alpha)) \cdot \nabla_{q_i} A_i(q_i, \alpha) - \nabla_{q_i} V(q),$$

$$(1.7b) \quad \partial_t q_i = \frac{1}{m_i} (p_i - A_i(q_i, \alpha)),$$

$$(1.7c) \quad i\partial_t \alpha_\lambda(k) = |k| \alpha_\lambda(k) - \sum_{i=1}^n \frac{1}{m_i} \frac{\chi_i(k)}{\sqrt{2|k|}} (p_i - A_i(q_i, \alpha)) \cdot \epsilon_\lambda(k) e^{-2\pi i k \cdot q_i}.$$

Consider respectively the following weighted L^2 spaces $\dot{\mathfrak{H}}^\sigma$ and \mathfrak{H}^σ with the norms

$$\|\alpha\|_{\dot{\mathfrak{H}}^\sigma} = \left(\sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} |k|^{2\sigma} |\alpha_\lambda(k)|^2 dk \right)^{1/2},$$

$$\|\alpha\|_{\mathfrak{H}^\sigma} = \left(\sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} (1 + |k|^2)^\sigma |\alpha_\lambda(k)|^2 dk \right)^{1/2},$$

and define the functional inner product spaces $\dot{X}^\sigma = \mathbb{R}^{dn} \times \mathbb{R}^{dn} \times \dot{\mathfrak{H}}^\sigma$ and $X^\sigma = \mathbb{R}^{dn} \times \mathbb{R}^{dn} \times \mathfrak{H}^\sigma$ endowed respectively with the norms

$$(1.8) \quad \|u\|_{\dot{X}^\sigma}^2 = \sum_{i=1}^n (|p_i|^2 + |q_i|^2) + \|\alpha\|_{\dot{\mathfrak{H}}^\sigma}^2,$$

$$\|u\|_{X^\sigma}^2 = \sum_{i=1}^n (|p_i|^2 + |q_i|^2) + \|\alpha\|_{\mathfrak{H}^\sigma}^2,$$

with $u = (p, q, \alpha)$ in \dot{X}^σ or X^σ . The spaces X^σ with the above inner product are always complete, and thus Hilbert, while \dot{X}^σ are complete only for $\sigma < \frac{d}{2}$. Let us remark that with the above notation, the p_i and q_i are d -dimensional vectors. Throughout the article, we will use indifferently the canonical identifications,

$$\dot{X}^\sigma \equiv \mathbb{C}^{dn} \times \dot{\mathfrak{H}}^\sigma,$$

$$X^\sigma \equiv \mathbb{C}^{dn} \times \mathfrak{H}^\sigma,$$

implemented by the complex structure

$$z \equiv q + ip,$$

where i denotes the imaginary unit. Note that $\mathfrak{H}^0 = L^2(\mathbb{R}^d, \mathbb{C}^{d-1})$ and $X^0 = \dot{X}^0 = \mathbb{C}^{dn} \times \mathfrak{H}^0$.

1.2. Pauli–Fierz Hamiltonian

The formal quantization of the above Newton–Maxwell system gives the so-called Pauli–Fierz model of non-relativistic quantum electrodynamics that we recall below. The Hilbert space of the quantized particles-field system is

$$\mathcal{H} = L^2(\mathbb{R}_x^{dn}) \otimes \Gamma_s(\mathfrak{H}^0),$$

where $\Gamma_s(\mathfrak{H}^0)$ denotes the symmetric Fock space over \mathfrak{H}^0 . Let $\widehat{p} = (\widehat{p}_1, \dots, \widehat{p}_n)$ and $\widehat{q} = (\widehat{q}_1, \dots, \widehat{q}_n)$ denote the particles momenta and position operators,

$$\widehat{p}_i = -i\hbar \nabla_{x_i}, \quad \widehat{q}_i = x_i.$$

The \hbar -scaled annihilation-creation operators for the field are defined for any $f = (f_1, \dots, f_{d-1}) \in \mathfrak{H}^0$ as

$$(1.9) \quad \widehat{a}(f) = \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \bar{f}_\lambda(k) \widehat{a}_\lambda(k) dk, \quad \widehat{a}^*(f) = \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} f_\lambda(k) \widehat{a}_\lambda^*(k) dk,$$

with $\widehat{a}_\lambda(k)$ and $\widehat{a}_\lambda^*(k)$ are the annihilation-creation operator-valued distributions satisfying the \hbar -scaled canonical commutation relations:

$$(1.10) \quad [\widehat{a}_\lambda(k), \widehat{a}_{\lambda'}^*(k')] = \hbar \delta(k - k') \delta_{\lambda, \lambda'}.$$

With these notations the field Hamiltonian is given by

$$(1.11) \quad \widehat{H}_f = \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} |k| \widehat{a}_\lambda^*(k) \widehat{a}_\lambda(k) dk,$$

and the smeared quantum electromagnetic vector potential is defined as

$$A_i(\widehat{q}_i, \widehat{a}) = \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \frac{\epsilon_\lambda(k)}{\sqrt{2|k|}} (\bar{\chi}_i(k) \widehat{a}_\lambda(k) e^{2\pi i k \cdot \widehat{q}_i} + \chi_i(k) \widehat{a}_\lambda^*(k) e^{-2\pi i k \cdot \widehat{q}_i}) dk.$$

Here, V and χ_i are respectively the same potential and form factor as for the Newton–Maxwell system. In particular, χ_i is the Fourier transform of the i^{th} particle's charge distribution φ_i and $(\epsilon_\lambda(k))_{\lambda=1, \dots, d-1}$ are the above defined polarization vectors satisfying, for almost all $k \in \mathbb{R}^d$ and for all $\lambda, \lambda' \in \{1, \dots, d-1\}$, the identities:

$$(1.12) \quad k \cdot \epsilon_\lambda(k) = 0 \quad \text{and} \quad \epsilon_\lambda(k) \cdot \epsilon_{\lambda'}(k) = \delta_{\lambda, \lambda'}.$$

The Pauli–Fierz Hamiltonian of n -particles interacting with the quantized electromagnetic field takes then the form

$$(1.13) \quad \widehat{H}_\hbar = \sum_{i=1}^n \frac{1}{2m_i} (\widehat{p}_i - A_i(\widehat{q}_i, \widehat{a}))^2 + V(\widehat{q}) + \widehat{H}_f.$$

It is known that the above Pauli–Fierz Hamiltonian is self-adjoint under convenient assumptions on the form factors χ_i and the potential V (see Section 3 for more details). For convenience, let us also define the non-interacting Hamiltonian,

$$\widehat{H}_\hbar^0 = \sum_{i=1}^n \frac{1}{2m_i} \widehat{p}_i^2 + \widehat{H}_f,$$

the \hbar -scaled number operator

$$(1.14) \quad \widehat{N}_\hbar = \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \widehat{a}_\lambda^*(k) \widehat{a}_\lambda(k) dk,$$

and the weighted modulus momentum operator defined for $\sigma \geq 0$ as

$$(1.15) \quad d\Gamma(|k|^{2\sigma}) = \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} |k|^{2\sigma} \widehat{a}_\lambda^*(k) \widehat{a}_\lambda(k) dk.$$

1.3. Wigner measures

In order to establish the Bohr correspondence principle between quantum and classical electrodynamics it is useful to use the very general concept of Wigner (or semiclassical) measures, and not to restrict to coherent states only. The semiclassical techniques of Wigner measures are a well-known tool that efficiently relates the quantum and classical states while taking the effective limit $\hbar \rightarrow 0$. In particular, Wigner measures have been extensively studied in finite dimensions (see, e.g., [50] and references therein contained). Moreover, such concept has been extended to infinite-dimensional phase spaces in [8] and applied to some fundamental examples of many-body and quantum field theories in [4, 5, 9, 10].

In the sequel, we denote by $\mathfrak{P}(X^\sigma)$ the set of all Borel probability measures over X^σ and recall that a density matrix on \mathcal{H} is a normalized non-negative trace class operator on \mathcal{H} .

DEFINITION 1.1 (Wigner measures). — *A Borel probability measure $\mu \in \mathfrak{P}(X^0)$ is a Wigner measure of a family of density matrices $(\varrho_\hbar)_{\hbar \in (0,1)}$ on the Hilbert space \mathcal{H} if and only if there exists a countable subset $\mathcal{E} \subset (0,1)$ with $0 \in \overline{\mathcal{E}}$ (the closure of \mathcal{E}) such that for any $\xi = (p, q, \alpha) \in X^0$:*

$$(1.16) \quad \lim_{\substack{\hbar \rightarrow 0 \\ \hbar \in \mathcal{E}}} \text{Tr} [\varrho_\hbar \mathcal{W}_\hbar(2\pi q, -2\pi p, \sqrt{2\pi}\alpha)] = \int_{X^0} e^{2\pi i \Re \langle \xi, u \rangle_{X^0}} d\mu(u).$$

Here, $\mathcal{W}_\hbar(\cdot)$ depends on the parameter \hbar and denotes the Weyl–Heisenberg operator defined according to (4.2), (4.4) and (4.6).

DEFINITION 1.2 (Wigner measures set). — *The above definition extends to any family of density matrices $(\varrho_{\hbar})_{\hbar \in \mathcal{J}}$ indexed by an arbitrary subset \mathcal{J} of the interval $(0, 1)$ such that $0 \in \overline{\mathcal{J}}$ (the closure of \mathcal{J}). In such case, we simply denote the collection of all the Wigner measures of $(\varrho_{\hbar})_{\hbar \in \mathcal{J}}$ by*

$$(1.17) \quad \mathcal{M}(\varrho_{\hbar}, \hbar \in \mathcal{J}).$$

Note that in general the set of Wigner measures $\mathcal{M}(\varrho_{\hbar}, \hbar \in (0, 1))$ is not empty if a mild assumption on the density matrices is assumed (see Proposition 4.1). Existence of Wigner measures is a consequence of Weyl commutation relations, Prokhorov's theorem and Arzelà-Ascoli's theorem (see [8, Theorem 6.2]). Moreover, by extracting subsequences one can always choose a family of density matrices with a single Wigner measure, i.e., there exists \mathcal{J} such that (1.17) is a singleton.

Now, the convergence of quantum electrodynamics towards classical electrodynamics when $\hbar \rightarrow 0$ can be reformulated as the formal commutative diagram:

$$\begin{array}{ccc} \varrho_{\hbar} & \xrightarrow{e^{-\frac{i}{\hbar}t\hat{H}_{\hbar}}(\cdot)e^{\frac{i}{\hbar}t\hat{H}_{\hbar}}} & \varrho_{\hbar}(t) \\ \downarrow \hbar \rightarrow 0 & & \downarrow \hbar \rightarrow 0 \\ \mu_0 & \xrightarrow{(\Phi_t)_\#(\cdot)} & \mu_t \end{array}$$

Indeed, consider a family of density matrices $(\varrho_{\hbar})_{\hbar \in (0,1)}$ at time $t = 0$; then its time evolution satisfies

$$\varrho_{\hbar}(t) = e^{-\frac{i}{\hbar}t\hat{H}_{\hbar}} \varrho_{\hbar} e^{\frac{i}{\hbar}t\hat{H}_{\hbar}}.$$

In order to link the quantum and classical dynamics as $\hbar \rightarrow 0$, it is enough to show that if the family of states $(\varrho_{\hbar})_{\hbar \in (0,1)}$ admits a single Wigner measure μ_0 , i.e.,

$$\mathcal{M}(\varrho_{\hbar}, \hbar \in (0, 1)) = \{\mu_0\},$$

then the family $(\varrho_{\hbar}(t))_{\hbar \in (0,1)}$ admits a unique Wigner measure at any time $t \in \mathbb{R}$ given by the push-forward measure

$$(1.18) \quad \mu_t = (\Phi_t)_\# \mu_0,$$

with Φ_t the flow that solves the classical Newton–Maxwell equation. In other words, the right-hand side of (1.18) denotes the image measure or

pushforward measure defined for all Borel sets B of X^σ as

$$(\Phi_t)_\# \mu_0(B) = \mu_0((\Phi_t)^{-1}(B)).$$

It is worth noting that Bohr's correspondence principle scheme was successfully proved to be true for the Nelson model with Yukawa interaction, in a different scaling (in which the particles are many but remain quantum), both with and without ultraviolet cutoffs in [4] and [5], respectively.

1.4. Main results

The two ingredients to be chosen in the quantum and classical systems (1.1)–(1.13) are the potential V and the form factors χ_i . Although V and χ_i are physically related through the charge distributions φ_i , we prefer to consider a more general context by picking the following hypotheses, for all $i = 1, \dots, n$:

$$\begin{aligned} (A_0) \quad & V \in \mathcal{C}_b^2(\mathbb{R}^{dn}, \mathbb{R}), \\ (A_1) \quad & \omega^{-1} \chi_i \in L^2(\mathbb{R}^d), \\ (A_2^{(\sigma)}) \quad & \omega^{\frac{3}{2}-\sigma} \chi_i \in L^2(\mathbb{R}^d). \end{aligned}$$

The first condition means that V is a bounded \mathcal{C}^2 function with its first and second derivatives bounded. In our analysis the relevant range for the parameter σ in the assumption $(A_2^{(\sigma)})$ is the interval $[\frac{1}{2}, 1]$. Notice that for $\sigma, \sigma' \in [\frac{1}{2}, 1]$ such that $\sigma \geq \sigma'$, one has

$$(A_2^{(\sigma')}) \longrightarrow (A_2^{(\sigma)}).$$

In particular, $(A_2^{(1)})$ holds true if we assume $(A_2^{(\sigma)})$ for some $\sigma \in [\frac{1}{2}, 1]$. Moreover, by interpolation if χ_i satisfies (A_1) and $(A_2^{(\sigma)})$, then for all $-1 \leq \lambda \leq \frac{3}{2} - \sigma$,

$$\omega^\lambda \chi_i \in L^2(\mathbb{R}^d).$$

Our first result concerns the flow of the Newton–Maxwell system of equations, and reads as follows.

THEOREM 1.3 (Generalized global flow). — *Let $\sigma \in [\frac{1}{2}, 1]$ and assume that (A_0) , (A_1) and $(A_2^{(\sigma)})$ are satisfied. Then for any initial condition $u_0 \in X^\sigma$ there exists a unique global strong solution $u(\cdot) \in \mathcal{C}(\mathbb{R}, X^\sigma) \cap$*

$\mathcal{C}^1(\mathbb{R}, X^{\sigma-1})$ of the Newton–Maxwell equation (1.7). Moreover, the generalized global flow

$$(1.19) \quad \begin{aligned} \Phi_t : X^\sigma &\longrightarrow X^\sigma \\ u_0 &\longmapsto u(t) \end{aligned}$$

is Borel measurable.

The above theorem is a consequence of a uniqueness property for the Newton–Maxwell solutions, proved in Proposition 2.6, combined with the existence of global solutions derived from the Pauli–Fierz quantum dynamics by means of the general theory in Appendix B. Its proof, provided in Section 5, is quite different from standard nonlinear fixed-point arguments which require the charge distribution to be more regular. We also prove, by the same method, further quantitative propagation estimates satisfied by the Newton–Maxwell flow in Proposition 5.7.

Let now $(\varrho_{\hbar})_{\hbar \in (0,1)}$ be a family of density matrices on the total Hilbert space \mathcal{H} of the particles-field quantum system, i.e., ϱ_{\hbar} is a normalized non-negative trace class operator on \mathcal{H} . The main assumptions on the family of states $(\varrho_{\hbar})_{\hbar \in (0,1)}$ are:

$$\begin{aligned} (S_0^{(\delta)}) \quad & \exists C_0 > 0, \forall \hbar \in (0, 1), \quad \text{Tr}[\varrho_{\hbar}(\hat{H}_{\hbar}^0 + 1)^{\delta}] \leq C_0, \\ (S_1^{(\delta)}) \quad & \exists C_1 > 0, \forall \hbar \in (0, 1), \quad \text{Tr}[\varrho_{\hbar}(\hat{N}_{\hbar} + \hat{q}^2 + 1)^{\delta}] \leq C_1, \\ (S_2^{(\sigma)}) \quad & \exists C_2 > 0, \forall \hbar \in (0, 1), \quad \text{Tr}[\varrho_{\hbar} d\Gamma(|k|^{2\sigma})] \leq C_2. \end{aligned}$$

Under assumptions, we can prove the Bohr correspondence principle.

THEOREM 1.4 (Classical limit). — *Let $\delta \in (0, 1]$, $\sigma \in [\frac{1}{2}, 1]$ and we assume that (A_0) , (A_1) and $(A_2^{(\sigma)})$ are satisfied. Let $(\varrho_{\hbar})_{\hbar \in (0,1)}$ be a family of density matrices on \mathcal{H} satisfying the assumptions $(S_0^{(\delta)})$, $(S_1^{(\delta)})$ and $(S_2^{(\sigma)})$. Assume that for some probability measure $\mu_0 \in \mathfrak{P}(X^0)$*

$$(1.20) \quad \mathcal{M}(\varrho_{\hbar}, \hbar \in (0, 1)) = \{\mu_0\}.$$

Then for all times $t \in \mathbb{R}$,

$$\mathcal{M}(e^{-i\frac{t}{\hbar}\hat{H}_{\hbar}} \varrho_{\hbar} e^{i\frac{t}{\hbar}\hat{H}_{\hbar}}, \hbar \in (0, 1)) = \{\mu_t\},$$

with $\mu_t \in \mathfrak{P}(X^0)$ satisfying the following statements:

- (1) μ_t is a Borel probability measure on X^σ ,
- (2) μ_t is the push-forward measure of μ_0 by the generalized flow Φ_t of the Newton–Maxwell equation constructed in Theorem 1.3, i.e., for all Borel subsets B of X^σ , and all $t \in \mathbb{R}$,

$$(1.21) \quad \mu_t(B) = (\Phi_t)_\# \mu_0(B) = \mu_0((\Phi_t)^{-1}(B)).$$

Moreover, in the general case where (1.20) is not assumed the statement below holds true for all times,

$$(1.22) \quad \mathcal{M}(e^{-i\frac{t}{\hbar}\hat{H}_{\hbar}}\varrho_{\hbar}e^{i\frac{t}{\hbar}\hat{H}_{\hbar}}, \hbar \in (0, 1)) = \{(\Phi_t)_{\#}\mu_0, \mu_0 \in \mathcal{M}(\varrho_{\hbar}, \hbar \in (0, 1))\}.$$

Remark 1.5.

- (1) The case $\sigma = 1$ corresponds to the weakest requirement for the form factors χ_i while for $\sigma = \frac{1}{2}$ the assumptions on the quantum states $(S_2^{(\sigma)})$ are the weakest.
- (2) The assumption (1.20) means that for any sequence $(\hbar_n)_{n \in \mathbb{N}}$ with $\hbar_n \rightarrow 0$ and any $\xi \in X^0$, the equality (1.16) holds true with the unique measure μ_0 on the right-hand side.
- (3) The statement (1.22) can be complemented by the fact that, given $\mu_0 \in \mathcal{M}(\varrho_{\hbar}, \hbar \in (0, 1))$ and the associated family $(\varrho_{\hbar})_{\hbar \in \mathcal{E}}$ that converges to it, then $(\varrho_{\hbar}(t))_{\hbar \in \mathcal{E}}$ converges to $(\Phi_t)_{\#}\mu_0$ for all $t \in \mathbb{R}$. Convergence here is according to (1.16) and Definition 1.1.
- (4) Most of the analysis in this article is devoted to the case $\delta = 1$ and (1.20). The proof of Theorem 1.4 in the general case is given only at the end of Section 5 (Proof of Theorem 1.4 in Section 5).
- (5) Examples of density matrices satisfying the above hypothesis $(S_0^{(\delta)})$ – $(S_2^{(\sigma)})$ include coherent states, product states or superpositions. See Section 4 and [8].

2. Newton–Maxwell equation

2.1. Uniqueness of solutions to the Newton–Maxwell equation

In this section, we prove a uniqueness result for solutions of the Newton–Maxwell equation (1.7) on the spaces X^{σ} with $\sigma \in [\frac{1}{2}, 1]$. Note that from now on we will refer only to V and χ_i , $i = 1, \dots, n$, satisfying the hypothesis (A_0) , (A_1) and $(A_2^{(\sigma)})$, and no more to the charge distributions φ_i s. In particular, the following lemma justifies the assumption (A_0) on the potential V by showing it to be true for the electrostatic potentials between the charges.

LEMMA 2.1. — *Let $\sigma \in [\frac{1}{2}, 1]$ and assume that (A_1) and $(A_2^{(\sigma)})$ are satisfied. Then the potentials w_{ij} , $i < j = 1, \dots, n$, defined by (1.3), belong to $\mathcal{C}_b^2(\mathbb{R}^d)$.*

Proof. — First, the expression (1.3) is well defined as a tempered distribution. Hence

$$w_{ij} = \mathcal{F}[\mathcal{F}^{-1}[w_{ij}]] = g \mathcal{F}\left[\frac{\mathcal{F}^{-1}[\varphi_i]\mathcal{F}^{-1}[\varphi_j]}{\omega^2}\right] = g \mathcal{F}\left[\frac{\chi_i \chi_j}{\omega^2}\right],$$

where \mathcal{F} denotes the Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$, and \mathcal{F}^{-1} its inverse (see also (1.6)). Now, the assumption (A_1) yields that $w_{ij} \in \mathcal{C}_\infty(\mathbb{R}^d)$ as the Fourier transform of an L^1 function. Repeating the same argument for the derivatives one proves the result. \square

Let us now prove some preliminary lemmas, concerning the electromagnetic vector potential and its derivative.

LEMMA 2.2. — *Let $\sigma \in [\frac{1}{2}, 1]$ and assume that (A_1) and $(A_2^{(\sigma)})$ are satisfied. Then*

$$(1) \quad \forall (q, \alpha) \in \mathbb{R}^{dn} \times \mathfrak{H}^0, \forall i \in \{1, \dots, n\},$$

$$|A_i(q, \alpha)| \leq \sqrt{2(d-1)} \left\| \frac{\chi_i}{\sqrt{\omega}} \right\|_{L^2} \|\alpha\|_{\mathfrak{H}^0},$$

$$(2) \quad \forall (q, \alpha) \in \mathbb{R}^{dn} \times \dot{\mathfrak{H}}^{1/2}, \forall i \in \{1, \dots, n\},$$

$$|A_i(q, \alpha)| \leq \sqrt{2(d-1)} \left\| \frac{\chi_i}{\omega} \right\|_{L^2} \|\alpha\|_{\dot{\mathfrak{H}}^{1/2}}.$$

Proof. — By the Cauchy–Schwarz inequalities with respect to both λ and k :

$$\begin{aligned} |A_i(q, \alpha)| &\leq \left| \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \frac{1}{\sqrt{2|k|}} (\bar{\chi}_i(k) \epsilon_\lambda(k) \alpha_\lambda(k) e^{2\pi i k \cdot q} + \text{h.c.}) dk \right| \\ &\leq \begin{cases} 2 \left(\sum_{\lambda=1}^{d-1} \left\| \frac{\chi_i}{\sqrt{2\omega}} \right\|_{L^2}^2 \right)^{\frac{1}{2}} \|\alpha\|_{\mathfrak{H}^0} \\ \text{or} \\ 2 \left(\sum_{\lambda=1}^{d-1} \left\| \frac{\chi_i}{\sqrt{2\omega}} \right\|_{L^2}^2 \right)^{\frac{1}{2}} \|\alpha\|_{\dot{\mathfrak{H}}^{1/2}}. \end{cases} \end{aligned} \quad \square$$

LEMMA 2.3. — *Let $\sigma \in [\frac{1}{2}, 1]$ and assume that (A_1) and $(A_2^{(\sigma)})$ are satisfied. Then for all $\nu = 1, \dots, d$:*

$$(1) \quad \forall (q, \alpha) \in \mathbb{R}^{dn} \times \mathfrak{H}^0, \forall i \in \{1, \dots, n\},$$

$$|\nabla_q A_i^\nu(q, \alpha)| \leq \sqrt{2(d-1)} \|\omega^{1/2} \chi_i\|_{L^2} \|\alpha\|_{\mathfrak{H}^0},$$

$$(2) \quad \forall (q, \alpha) \in \mathbb{R}^{dn} \times \mathfrak{H}^{1/2}, \forall i \in \{1, \dots, n\},$$

$$|\nabla_q A_i^\nu(q, \alpha)| \leq \sqrt{2(d-1)} \|\chi_i\|_{L^2} \|\alpha\|_{\mathfrak{H}^{1/2}}.$$

Proof. — By the Cauchy–Schwarz inequality:

$$\begin{aligned} |\nabla_q A_i^\nu(q, \alpha)| &\leq \left| \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \frac{k}{\sqrt{2|k|}} (\bar{\chi}_i(k) \epsilon_\lambda^\nu(k) \alpha_\lambda(k) e^{2\pi i k \cdot q} + \text{h.c.}) dk \right| \\ &\leq \begin{cases} 2 \left(\sum_{\lambda=1}^{d-1} \frac{1}{2} \|\omega^{1/2} \chi_i\|_{L^2}^2 \right)^{\frac{1}{2}} \|\alpha\|_{\mathfrak{H}^0} \\ \text{or} \\ 2 \left(\sum_{\lambda=1}^{d-1} \frac{1}{2} \|\chi_i\|_{L^2}^2 \right)^{\frac{1}{2}} \|\alpha\|_{\mathfrak{H}^{1/2}}. \end{cases} \end{aligned} \quad \square$$

LEMMA 2.4. — Let $\sigma \in [\frac{1}{2}, 1]$ and assume that (A_1) and $(A_2^{(\sigma)})$ are satisfied. Then for all $q_1, q_2 \in \mathbb{R}^{dn}$ and $\nu = 1, \dots, d$:

$$(1) \quad \forall \alpha_1, \alpha_2 \in \mathfrak{H}^0, \forall i \in \{1, \dots, n\},$$

$$\begin{aligned} |A_i(q_1, \alpha_1) - A_i(q_2, \alpha_2)| \\ \lesssim \left\| \frac{\chi_i}{\sqrt{\omega}} \right\|_{L^2} \|\alpha_1 - \alpha_2\|_{\mathfrak{H}^0} + |q_1 - q_2| \|\sqrt{\omega} \chi_i\|_{L^2} \|\alpha_2\|_{\mathfrak{H}^0}. \end{aligned}$$

$$(2) \quad \forall \alpha_1, \alpha_2 \in \mathfrak{H}^\sigma, \forall i \in \{1, \dots, n\},$$

$$\begin{aligned} |\nabla_{q_1} A_i^\nu(q_1, \alpha_1) - \nabla_{q_2} A_i^\nu(q_2, \alpha_2)| \\ \lesssim \|\sqrt{\omega} \chi_i\|_{L^2} \|\alpha_1 - \alpha_2\|_{\mathfrak{H}^0} + |q_1 - q_2| \|\omega^{\frac{3}{2}-\sigma} \chi_i\|_{L^2} \|\alpha_2\|_{\mathfrak{H}^\sigma}. \end{aligned}$$

Proof. — Estimating the left-hand side of (1), the Cauchy–Schwarz inequality yields:

$$\begin{aligned} |A_i(q_1, \alpha_1) - A_i(q_2, \alpha_2)| \\ \lesssim \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \left| \frac{\chi_i(k)}{\sqrt{|k|}} \right| (|\alpha_{1,\lambda}(k) - \alpha_{2,\lambda}(k)| + |(e^{2\pi i k \cdot q_1} - e^{2\pi i k \cdot q_2}) \alpha_{2,\lambda}(k)|) dk \\ \lesssim \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \left| \frac{\chi_i(k)}{\sqrt{|k|}} \right| (|\alpha_{1,\lambda}(k) - \alpha_{2,\lambda}(k)| + |k| |q_1 - q_2| |\alpha_{2,\lambda}(k)|) dk \\ \lesssim \left\| \frac{\chi_i}{\sqrt{\omega}} \right\|_{\mathfrak{H}^0} \|\alpha_1 - \alpha_2\|_{\mathfrak{H}^0} + |q_1 - q_2| \|\sqrt{\omega} \chi_i\|_{L^2} \|\alpha_2\|_{\mathfrak{H}^0}. \end{aligned}$$

A similar argument gives

$$\begin{aligned}
& \left| \nabla_{q_1} A_i^\nu(q_1, \alpha_1) - \nabla_{q_2} A_i^\nu(q_2, \alpha_2) \right| \\
& \lesssim \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \sqrt{|k|} |\chi_i(k)| (|\alpha_{1,\lambda}(k) - \alpha_{2,\lambda}(k)| + |k| |q_1 - q_2| |\alpha_{2,\lambda}(k)|) dk \\
& \lesssim \left\| \sqrt{\omega} \chi_i \right\|_{\dot{S}^0} \|\alpha_1 - \alpha_2\|_{\dot{S}^0} + |q_1 - q_2| \left\| \omega^{\frac{3}{2}-\sigma} \chi_i \right\|_{L^2} \|\alpha_2\|_{\dot{S}^\sigma}. \quad \square
\end{aligned}$$

The Newton–Maxwell system (1.7) is a semi-linear PDE of the form

$$(2.1) \quad \begin{cases} \frac{d}{dt} u(t) = \mathcal{L}(u(t)) + F(u(t)), \\ u(0) = u_0, \end{cases}$$

where $u_0 \in X^\sigma$, $t \rightarrow u(t) = (p(t), q(t), \alpha(t))$ is a solution,

$$\mathcal{L}(u) = (0, 0, -i|k|\alpha)$$

is a linear operator defined for all $u = (p, q, \alpha) \in \dot{X}^1$, and F is nonlinear and given by

$$(2.2a) \quad F(u)_{p_i} = \frac{1}{m_i} (p_i - A_i(q_i, \alpha)) \cdot \nabla_{q_i} A_i(q_i, \alpha) - \nabla_{q_i} V(q),$$

$$(2.2b) \quad F(u)_{q_i} = \frac{1}{m_i} (p_i - A_i(q_i, \alpha)),$$

$$(2.2c) \quad F(u)_{\alpha_\lambda}(k) = i \sum_{i=1}^n \frac{1}{m_i} \frac{\chi_i(k)}{\sqrt{2|k|}} (p_i - A_i(q_i, \alpha)) \cdot \epsilon_\lambda(k) e^{-2\pi i k \cdot q_i}.$$

Here the notations $(\cdot)_{p_i}$, $(\cdot)_{q_i}$ and $(\cdot)_{\alpha_\lambda}$ refer to the components p_i , q_i and α_λ respectively. We note also that F contains a linear part in (2.2b).

For convenience, given a finite collection of functions $\{f_i, \dots, f_n\}$, all belonging to the same Banach space X , we adopt the following notation:

$$\|f\|_X = \max_{i \in \{1, \dots, n\}} \|f_i\|_X.$$

The vector field F representing the Newton–Maxwell nonlinearity has the following property.

PROPOSITION 2.5. — *Let $\sigma \in [\frac{1}{2}, 1]$ and assume that (A_0) , (A_1) and $(A_2^{(\sigma)})$ are satisfied. Then the nonlinearity $F : X^\sigma \rightarrow X^\sigma$ is a continuous vector field which is bounded on bounded set.*

Proof. — One verifies, thanks to Lemmas 2.2 and 2.3 that F maps $X^{1/2}$ into X^1 and is bounded on bounded sets, with the estimates below holding true:

$$(2.3a) \quad F(u)_{p_i} \lesssim (|p| + \|\alpha\|_{\mathfrak{H}^0}) \left\| \frac{\chi_i}{\sqrt{\omega}} \right\|_{L^2} \|\sqrt{\omega} \chi_i\|_{L^2} \|\alpha\|_{\mathfrak{H}^0} + \sup_q |\nabla_q V(q)|,$$

$$(2.3b) \quad F(u)_{q_i} \lesssim |p| + \left\| \frac{\chi_i}{\sqrt{\omega}} \right\|_{L^2} \|\alpha\|_{\mathfrak{H}^0},$$

$$(2.3c) \quad \|F(u)_{\alpha_\lambda}\|_{\mathfrak{H}^1} \lesssim \|\sqrt{\omega} \chi\|_{L^2} \left(|p| + \left\| \frac{\chi}{\sqrt{\omega}} \right\|_{L^2} \|\alpha\|_{\mathfrak{H}^0} \right),$$

$$(2.3d) \quad \|F(u)_{\alpha_\lambda}\|_{\mathfrak{H}^0} \lesssim \left\| \frac{\chi}{\sqrt{\omega}} \right\|_{L^2} \left(|p| + \left\| \frac{\chi}{\sqrt{\omega}} \right\|_{\mathfrak{H}^0} \|\alpha\|_{\mathfrak{H}^0} \right).$$

In order to prove continuity of F for $\sigma \in [\frac{1}{2}, 1]$, it is enough to show the continuity of the following three maps over $\mathbb{R}^{dn} \times \mathfrak{H}^\sigma$, for any $i = 1, \dots, n$:

- (1) $(q, \alpha) \rightarrow A_i^\nu(q, \alpha) \in \mathbb{R}$
- (2) $(q, \alpha) \rightarrow \nabla_q A_i^\nu(q, \alpha) \in \mathbb{R}$
- (3) $(q, \alpha) \rightarrow \frac{\chi_i(k)}{\sqrt{|k|}} e^{-2\pi i k \cdot q} \in \mathfrak{H}^\sigma$.

Indeed, continuity of the maps (1) and (2) is a consequence of Lemma 2.4 while the map (3) is continuous thanks to the dominated convergence theorem and to the bounds,

$$\begin{aligned} & \left\| \frac{\chi_i(k)}{\sqrt{|k|}} e^{-2\pi i k \cdot q_1} - \frac{\chi_i(k)}{\sqrt{|k|}} e^{-2\pi i k \cdot q_2} \right\|_{L^2}^2 \\ & \leq \int_{\mathbb{R}^d} \left(\left| \frac{\chi_i(k)}{\sqrt{|k|}} \right| e^{-2\pi i k \cdot q_1} - e^{-2\pi i k \cdot q_2} \right)^2 dk, \end{aligned}$$

$$\begin{aligned} & \left\| \frac{\chi_i(k)}{\sqrt{|k|}} e^{-2\pi i k \cdot q_1} - \frac{\chi_i(k)}{\sqrt{|k|}} e^{-2\pi i k \cdot q_2} \right\|_{\mathfrak{H}^\sigma}^2 \\ & \leq \int_{\mathbb{R}^d} \left(|k|^{\sigma-1/2} |\chi_i(k)| \left| e^{-2\pi i k \cdot q_1} - e^{-2\pi i k \cdot q_2} \right| \right)^2 dk, \end{aligned}$$

since $\sigma - \frac{1}{2} \leq \frac{3}{2} - \sigma$ for $\sigma \in [\frac{1}{2}, 1]$. \square

Let I be an open interval containing the origin. We are interested in *strong* solutions u of the Newton–Maxwell equation (1.7) or equivalently (2.1) such that

$$u \in \mathcal{C}(I, X^\sigma) \cap \mathcal{C}^1(I, X^{\sigma-1}),$$

with $\sigma \in [\frac{1}{2}, 1]$ and (2.1) is satisfied for all $t \in I$. In particular, these solutions verify the following Duhamel formula for all $t \in I$,

$$(2.4) \quad u(t) = \mathcal{D}_0(t)u(0) + \int_0^t \mathcal{D}_0(t-s)F(u(s)) \, ds$$

where $\mathcal{D}_0(\cdot)$ is the free field flow defined as

$$(2.5) \quad \mathcal{D}_0(t)(p, q, \alpha) = (p, q, e^{-it\omega} \alpha).$$

PROPOSITION 2.6. — *Let $\sigma \in [\frac{1}{2}, 1]$ and assume that (A_0) , (A_1) and $(A_2^{(\sigma)})$ are satisfied. Consider an open interval I containing the origin and let $u_1, u_2 \in \mathcal{C}(I, X^\sigma)$ be two strong solutions of the Newton–Maxwell equation (2.1) such that*

$$u_1(0) = u_2(0).$$

Then $u_1(t) = u_2(t)$ for all $t \in I$.

Proof. — For any compact interval $J \subset I$, note that

$$M = \max_{i=1,2} \left(\max_J \|u_i(t)\|_{X^\sigma} \right) < \infty.$$

Using the Duhamel formula (2.4), one finds that for all $t \geq 0$,

$$\|u_1(t) - u_2(t)\|_{X^0} \leq \int_0^t \|F(u_1(s)) - F(u_2(s))\|_{X^0} \, ds.$$

The following estimate holds true (we drop the s -dependence in the right-hand side):

$$\begin{aligned} & \left| (F(u_1(s)) - F(u_2(s)))_{p_i} \right| \\ & \lesssim \sum_{\nu=1}^d \left\{ (|p_{1,i}^\nu - p_{2,i}^\nu| + |A_i^\nu(q_{1,i}, \alpha_1) - A_i^\nu(q_{2,i}, \alpha_2)|) |\nabla_{q_{2,i}} A_i^\nu(q_{2,i}, \alpha_2)| \right. \\ & \quad \left. + (|p_{2,i}^\nu| + |A_i^\nu(q_{2,i}, \alpha_2)|) |\nabla_{q_{1,i}} A_i^\nu(q_{1,i}, \alpha_1) - \nabla_{q_{1,i}} A_i^\nu(q_{1,i}, \alpha_1)| \right\} \\ & \quad + |\nabla_{q_{1,i}} V(q_1) - \nabla_{q_{2,i}} V(q_2)| \\ & \lesssim |p_1(s) - p_2(s)| + |q_1(s) - q_2(s)| + \|\alpha_1(s) - \alpha_2(s)\|_{\mathfrak{H}^0}, \end{aligned}$$

by Lemmas 2.2-2.4. Similarly,

$$\begin{aligned} & \left| (F(u_1(s)) - F(u_2(s)))_{q_i} \right| \\ & \lesssim \sum_{\nu=1}^d \left\{ |p_{1,i}^\nu - p_{2,i}^\nu| + |A_i^\nu(q_{1,i}, \alpha_1) - A_i^\nu(q_{2,i}, \alpha_2)| \right\} \\ & \lesssim |p_1(s) - p_2(s)| + |q_1(s) - q_2(s)| + \|\alpha_1(s) - \alpha_2(s)\|_{\mathfrak{H}^0} \end{aligned}$$

and

$$\begin{aligned}
& \left\| (F(u_1(s)) - F(u_2(s)))_{\alpha_\lambda} \right\|_{\mathfrak{H}^0} \\
& \lesssim \sum_{i=1}^n \left\{ |p_{1,i} - p_{2,i}| + |A_i(q_{1,i}, \alpha_1) - A_i(q_{2,i}, \alpha_2)| + |q_{1,i} - q_{2,i}| \right\} \\
& \lesssim |p_1(s) - p_2(s)| + |q_1(s) - q_2(s)| + \|\alpha_1(s) - \alpha_2(s)\|_{\mathfrak{H}^0}.
\end{aligned}$$

Hence

$$\begin{aligned}
& \|u_1(t) - u_2(t)\|_{X^0} \\
& \lesssim \int_0^t |p_1(s) - p_2(s)| + |q_1(s) - q_2(s)| + \|\alpha_1(s) - \alpha_2(s)\|_{\mathfrak{H}^0} \, ds \\
& \leq C \int_0^t \|u_1(s) - u_2(s)\|_{X^0} \, ds,
\end{aligned}$$

and so Grönwall's inequality yields the result. \square

2.2. Local well-posedness

We indicate here the main difficulty in proving the local well-posedness in X^1 of the Newton–Maxwell equation by means of a standard fixed-point argument. Take $(p_0, q_0, \alpha_0) \in X^1$ an initial data, $T > 0$ a given time and R a positive real number. Consider the natural complete metric space

$$E = \left\{ u \in \mathcal{C}([0, T], X^1) : \sup_{s \in [0, T]} \|u(s) - (p_0, q_0, e^{-is\omega} \alpha_0)\|_{X^1} \leq R \right\}.$$

So, the question is to prove the existence of a unique fixed-point for the mapping

$$\begin{aligned}
\tau : E & \longrightarrow E \\
u & \longmapsto \tau(u),
\end{aligned}$$

defined by

$$\tau(u)(t) = (p_0, q_0, e^{-it\omega} \alpha_0) + \int_0^t (F(u(s))_p, F(u(s))_q, e^{-i(t-s)\omega} F(u(s))_\alpha) \, ds,$$

where F is the vector field given in (2.2). Firstly, one has to choose $T > 0$ small enough such that τ maps E into itself. Secondly, to prove the

contraction property for τ it is sufficient to establish the following inequality for all $u_1, u_2 \in E$,

$$(2.6) \quad \int_0^T \|F(u_1(s)) - F(u_2(s))\|_{X^1} ds \leq TC(R) \sup_{s \in [0, T]} \|u_1(s) - u_2(s)\|_{X^1},$$

with $C(R)$ is a constant depending essentially on R and T has to be chosen small enough such that $TC(R) < 1$. Now, the estimate (2.6) requires essentially the bounds

$$\left\| \frac{\chi_j}{\sqrt{2\omega}} (e^{-2\pi i k \cdot q_{j,1}(s)} - e^{-2\pi i k \cdot q_{j,2}(s)}) \right\|_{\dot{J}^1} \lesssim \sup_{s \in [0, T]} |q_{j,1}(s) - q_{j,2}(s)|$$

to be true while we only have $\omega^{1/2}\chi_j \in L^2$. Hence the argument can not be closed in this way and probably more refined analysis is needed. In particular taking advantage of the time oscillation on front of the field component could be helpful; but even in this case it is unclear to us how to close the estimates.

Finally, examining the vector field F in (2.2), one notices that a small variation on the positions of the particles leads to a large variation of the fields in the L^2 -norm whenever we assume only $\omega^{1/2-\varepsilon}\chi_j \in L^2$ for an arbitrary small $\varepsilon > 0$. In some sense, this indicates that the assumptions (A_1) – $(A_2^{(1)})$ are thresholds for the Hadamard stability of the Newton–Maxwell equation. We believe that these questions of fixed-point and stability are interesting in themselves and relevant for classical electrodynamics and we hope that our work will stimulate some interest on them, see [54] and [56].

3. Pauli–Fierz Hamiltonians

In this section, we review some features of the Pauli–Fierz Hamiltonian and prove uniform estimates that are useful in the analysis of the classical limit.

3.1. Self-adjointness and uniform estimates

Recall that such Hamiltonian describes a system of many quantum d -dimensional extended charges ($d \geq 3$) interacting with the quantized electromagnetic field in the Coulomb gauge, see e.g. [59]. Moreover, it is known

that the Pauli–Fierz operator \hat{H}_{\hbar} in (1.13) is essentially self-adjoint under suitable regularity assumptions on the charge distributions. For various approaches to the study of self-adjointness for Pauli–Fierz-type Hamiltonians, see for instance [29, 37, 38, 39, 51].

THEOREM 3.1 ([39]). — Assume (A_0) , (A_1) and $(A_2^{(1)})$. Then \hat{H}_{\hbar} is self-adjoint on $D(\hat{H}_{\hbar}^0) = D(\hat{p}^2) \cap D(\hat{H}_{\text{f}})$.

We recall some useful inequalities on \hat{a}^{\sharp} and \hat{H}_{f} . Suppose that

$$\|f_j/\sqrt{\omega}\|_{L^2} < \infty, \quad j = 1, \dots, m.$$

Let $\psi \in D(\hat{H}_{\text{f}}^{m/2})$. Then

$$\psi \in D\left(\prod_{j=1}^m \hat{a}(f_j)\right) \cap D\left(\prod_{j=1}^m \hat{a}^*(f_j)\right)$$

and there exists a constant $c > 0$ depending only on $\|f_j/\sqrt{\omega}\|_{L^2}$ such that

$$(3.1) \quad \left\| \prod_{j=1}^m \hat{a}(f_j) \psi \right\| \leq c \|\hat{H}_{\text{f}}^{m/2} \psi\|,$$

$$(3.2) \quad \left\| \prod_{j=1}^m \hat{a}^*(f_j) \psi \right\| \leq c \|(\hat{H}_{\text{f}} + 1)^{m/2} \psi\|.$$

LEMMA 3.2. — Assume (A_0) , (A_1) and $(A_2^{(1)})$. Then there exist three constants $c, C, \gamma > 0$ independent of \hbar such that for all $\hbar \in (0, 1)$ and for all $\psi \in D(\hat{H}_{\hbar}^0)$,

$$(3.3) \quad c \|(\hat{H}_{\hbar} + \gamma) \psi\| \leq \|(\hat{H}_{\hbar}^0 + 1) \psi\| \leq C \|(\hat{H}_{\hbar} + \gamma) \psi\|.$$

Proof. — We set $A_j = A_j(\hat{q}_j, \hat{a})$, $p_j = \hat{p}_j$, $H^0 = \hat{H}_{\hbar}^0$, $H_{\text{f}} = \hat{H}_{\text{f}}$, and $H = \hat{H}_{\hbar}$ for simplicity, and we assume that ψ is sufficiently smooth. By Theorem 3.1 and the closed graph theorem we obtain (3.3), but it is not clear how c and C depend on \hbar . Hence we shall prove the lemma from scratch. In the proof below constants c_j are independent of $\hbar \in (0, 1]$ and depend at most on the L^2 -norms of χ_i/ω , $\chi_i/\sqrt{\omega}$, χ_i and $\sqrt{\omega}\chi_i$. We prove (3.3) for the case of $V = 0$ in what follows. In the case of $V \neq 0$, (3.3) can be also derived straightforwardly from (3.3) with $V = 0$, since V is bounded.

First we show the second inequality of (3.3). We estimate both $\|H\psi\|^2$ and $\sum_{j=1}^n \left\| \frac{1}{2m_j} p_j^2 \psi \right\|^2$. Using that A_j is divergence free due to the Coulomb gauge, we have the identity

$$(p_j - A_j)^2 \psi = p_j^2 \psi - 2A_j(p_j - A_j)\psi + A_j^2 \psi,$$

and thus

$$\begin{aligned} p_j^2 \psi &= (p_j - A_j)^2 \psi + 2A_j(p_j - A_j)\psi - A_j^2 \psi, \\ \|p_j^2 \psi\|^2 &\leq c_1(\|(p_j - A_j)^2 \psi\|^2 + \|A_j(p_j - A_j)\psi\|^2 + \|A_j^2 \psi\|^2). \end{aligned}$$

We obtain that

$$\begin{aligned} \|A_j(p_j - A_j)\psi\|^2 &\leq c_2 \langle (p_j - A_j)\psi, (H_f + 1)(p_j - A_j)\psi \rangle \\ &= c_2 \langle (p_j - A_j)^2 \psi, (H_f + 1)\psi \rangle + c_2 \langle (p_j - A_j)\psi, -[H_f, A_j]\psi \rangle. \end{aligned}$$

Note that $|\langle (p_j - A_j)\psi, -[H_f, A_j]\psi \rangle| \leq c_3 \|(p_j - A_j)\psi\| \|(H_f + 1)^{1/2} \psi\|$. Then

$$\|A_j(p_j - A_j)\psi\|^2 \leq c_4(\|(p_j - A_j)^2 \psi\|^2 + \|(H_f + 1)\psi\|^2 + \|\psi\|^2).$$

Since from (3.1)–(3.2),

$$\|A_j^2 \psi\| \leq c \|(H_f + 1)\psi\|$$

follows, we conclude that

$$(3.4) \quad \sum_{j=1}^n \left\| \frac{1}{2m_j} p_j^2 \psi \right\|^2 \leq c_5 \left(\sum_{j=1}^n \left\| \frac{1}{2m_j} (p_j - A_j)^2 \psi \right\|^2 + \|(H_f + 1)\psi\|^2 + \|\psi\|^2 \right).$$

Next we estimate $\|H\psi\|^2$. We have

$$\begin{aligned} (3.5) \quad \|H\psi\|^2 &= \left\| \sum_{j=1}^n \frac{1}{2m_j} (p_j - A_j)^2 \psi + H_f \psi \right\|^2 \\ &= \left\| \sum_{j=1}^n \frac{1}{2m_j} (p_j - A_j)^2 \psi \right\|^2 + \|H_f \psi\|^2 \\ &\quad + 2\Re \left\langle \sum_{j=1}^n \frac{1}{2m_j} (p_j - A_j)^2 \psi, H_f \psi \right\rangle. \end{aligned}$$

Since we observe that commutation relations

$$[p_j - A_j, p_i - A_i]\psi = 0$$

for any i and j , we have

$$\begin{aligned}
& \left\| \sum_{j=1}^n \frac{1}{2m_j} (p_j - A_j)^2 \psi \right\|^2 \\
&= \sum_{j=1}^n \left\| \frac{1}{2m_j} (p_j - A_j)^2 \psi \right\|^2 + \sum_{i \neq j} \frac{1}{2m_j} \frac{1}{2m_i} \langle (p_j - A_j)^2 \psi, (p_i - A_i)^2 \psi \rangle \\
&\geq \sum_{j=1}^n \left\| \frac{1}{2m_j} (p_j - A_j)^2 \psi \right\|^2.
\end{aligned}$$

For arbitrary $\epsilon > 0$ and $\delta > 0$ we see that

$$\begin{aligned}
& \sum_{j=1}^n \frac{1}{2m_j} \|(p_j - A_j)\psi\| \|(H_f + 1)^{1/2}\psi\| \\
&\leq \epsilon \sum_{j=1}^n \frac{1}{2m_j} \|(p_j - A_j)\psi\|^2 + \frac{1}{4\epsilon} \sum_{j=1}^n \frac{1}{2m_j} \|(H_f + 1)^{1/2}\psi\|^2 \\
&\leq \epsilon \sum_{j=1}^n \frac{1}{2m_j} \|(p_j - A_j)\psi\|^2 + \frac{1}{4\epsilon} \sum_{j=1}^n \frac{1}{2m_j} \left(\delta \|H_f \psi\|^2 + \frac{1}{4\delta} \|\psi\|^2 + \|\psi\|^2 \right) \\
&\leq \epsilon \sum_{j=1}^n \frac{1}{2m_j} \|(p_j - A_j)\psi\|^2 + \frac{1}{4\epsilon} \sum_{j=1}^n \frac{1}{2m_j} \delta \|H_f \psi\|^2 \\
&\quad + \frac{1}{4\epsilon} \sum_{j=1}^n \frac{1}{2m_j} \left(\frac{1}{4\delta} + 1 \right) \|\psi\|^2
\end{aligned}$$

Taking $\delta = 4\epsilon^2 / \sum_{j=1}^n \frac{1}{2m_j}$, we then have

$$\begin{aligned}
& \sum_{j=1}^n \frac{1}{2m_j} \|(p_j - A_j)\psi\| \|(H_f + 1)^{1/2}\psi\| \\
&\leq \epsilon \sum_{j=1}^n \frac{1}{2m_j} \|(p_j - A_j)\psi\|^2 + \epsilon \|H_f \psi\|^2 \\
&\quad + \frac{1}{4\epsilon} \sum_{j=1}^n \frac{1}{2m_j} \left(\frac{1}{16\epsilon^2} \sum_{j=1}^n \frac{1}{2m_j} + 1 \right) \|\psi\|^2.
\end{aligned}$$

Hence for arbitrary small $\epsilon > 0$,

$$\begin{aligned}
 (3.6) \quad & \Re \left\langle \sum_{j=1}^n \frac{1}{2m_j} (p_j - A_j)^2 \psi, H_f \psi \right\rangle \\
 &= \sum_{j=1}^n \frac{1}{2m_j} \langle (p_j - A_j) \psi, H_f (p_j - A_j) \psi \rangle \\
 &\quad + \sum_{j=1}^n \frac{1}{2m_j} \Re \langle (p_j - A_j) \psi, [H_f, -A_j] \psi \rangle \\
 &\geq -c_6 \sum_{j=1}^n \frac{1}{2m_j} \| (p_j - A_j) \psi \| \| (H_f + 1)^{1/2} \psi \| \\
 &\geq -\epsilon \sum_{j=1}^n \frac{1}{2m_j} \| (p_j - A_j)^2 \psi \|^2 - \epsilon \| H_f \psi \|^2 - c_7 \| \psi \|^2.
 \end{aligned}$$

Here

$$c_7 = \frac{c_6}{4\epsilon} \sum_{j=1}^n \frac{1}{2m_j} \left(\frac{c_6^2}{16\epsilon^2} \sum_{j=1}^n \frac{1}{2m_j} + 1 \right).$$

By (3.5) and (3.6), there exists $c_8 > 0$ such that

$$(3.7) \quad \| H \psi \|^2 + c_9 \| \psi \|^2 \geq c_8 \left(\sum_{j=1}^n \left\| \frac{1}{2m_j} (p_j - A_j)^2 \psi \right\|^2 + \| H_f \psi \|^2 \right).$$

Then combining (3.4) and (3.7) yield

$$\begin{aligned}
 \| (H^0 + 1) \psi \|^2 &\leq 2 \left\| \sum_{j=1}^n \frac{1}{2m_j} p_j^2 \psi \right\|^2 + 2 \| H_f \psi \|^2 \\
 &\leq c_{10} \left(\sum_{j=1}^n \left\| \frac{1}{2m_j} p_j^2 \psi \right\|^2 + \| H_f \psi \|^2 \right) \\
 &\leq c_{11} \left(\sum_{j=1}^n \left\| \frac{1}{2m_j} (p_j - A_j)^2 \psi \right\|^2 + \| H_f \psi \|^2 + \| \psi \|^2 \right) \\
 &\leq c_{12} (\| H \psi \|^2 + \| \psi \|^2) \leq c_{13} \| (H + 1) \psi \|^2.
 \end{aligned}$$

It finally follows that

$$(3.8) \quad \| (H^0 + 1) \psi \| \leq c_{13} \| (H + 1) \psi \|.$$

Secondly we show the first inequality of (3.3), it is however easier than the second inequality. We have

$$\|H\psi\| \leq \|H^0\psi\| + \left\| -\sum_{j=1}^n \frac{1}{m_j} A_j p_j \psi + \sum_{j=1}^n \frac{1}{2m_j} A_j^2 \psi \right\|$$

and

$$\left\| -\sum_{j=1}^n \frac{1}{m_j} A_j p_j \psi + \sum_{j=1}^n \frac{1}{2m_j} A_j^2 \psi \right\| \leq c_{14} \left(\left\| \sum_{j=1}^n \frac{1}{2m_j} p_j^2 \psi \right\| + \|H_f \psi\| + \|\psi\| \right).$$

Then it follows that

$$(3.9) \quad \|(H+1)\psi\| \leq c_{15} \|(H^0+1)\psi\|.$$

By (3.8) and (3.9) we see that

$$\frac{1}{c_{15}} \|(H+1)\psi\| \leq \|(H^0+1)\psi\| \leq c_{14} \|(H+1)\psi\|$$

and the proof is complete for $V = 0$. \square

From Lemma 3.2 we can immediately obtain a form version of inequalities. Suppose that $\hat{H}_\hbar + a > 0$ and $\hat{H}_\hbar^0 + b > 0$ for some $a, b \in \mathbb{R}$.

COROLLARY 3.3. — Assume (A_0) , (A_1) and $(A_2^{(1)})$. Then there exist $c, C > 0$ independent of \hbar such that for all $\hbar \in (0, 1)$ and for all $\psi \in D(\hat{H}_\hbar^0)$,

$$(3.10) \quad c\langle\psi, (\hat{H}_\hbar + a)\psi\rangle \leq \langle\psi, (\hat{H}_\hbar^0 + b)\psi\rangle \leq C\langle\psi, (\hat{H}_\hbar + a)\psi\rangle.$$

In particular

$$(3.11) \quad c\|(\hat{H}_\hbar + a)^{1/2}\psi\| \leq \|(\hat{H}_\hbar^0 + b)^{1/2}\psi\| \leq C\|(\hat{H}_\hbar + a)^{1/2}\psi\|$$

for $\psi \in D((\hat{H}_\hbar^0)^{1/2})$.

Proof. — The proof for (3.10) is similar but even simpler than (3.3). To show (3.11) we take an approximation. (3.11) is true for $\psi \in D(\hat{H}_\hbar^0)$. Let $\psi \in D((\hat{H}_\hbar^0)^{1/2})$. There exists a sequence $\psi_n \in D(\hat{H}_\hbar^0)$ such that $\psi_n \rightarrow \psi$ and $(\hat{H}_\hbar^0)^{1/2}\psi_n \rightarrow (\hat{H}_\hbar^0)^{1/2}\psi$ as $n \rightarrow \infty$. Hence $\hat{H}_\hbar^{1/2}\psi_n$ is a Cauchy sequence. By the closedness of $\hat{H}_\hbar^{1/2}$ we see that $\psi \in D(\hat{H}_\hbar^{1/2})$ and the inequality of the left-hand side of (3.11) follows. The inequality of the right-hand side of (3.11) can be similarly proven. \square

3.2. Regularity properties

In order to understand the behavior of the Pauli–Fierz dynamics as $\hbar \rightarrow 0$, it is convenient to control the expectation of some suitable quantum observables *uniformly in* $\hbar \in (0, 1)$. Namely, we consider the number operator (1.14), the modulus momentum operator (1.15) and the squared position operator \widehat{q}^2 , and establish the following regularity propagation estimates for the Pauli–Fierz dynamics. These estimates prove to be very useful in Section 4.

3.2.1. Number estimates

The aim of the present subsection is to study the propagation in time of the photon number operator estimates on a suitable dense domain. The commutation relations of \widehat{N}_\hbar with $\widehat{a}_\lambda(f)$ and $\widehat{a}_\lambda^*(f)$ have a simple form:

$$\begin{aligned} [\widehat{N}_\hbar, \widehat{a}_\lambda(f)] &= -\hbar \widehat{a}_\lambda(f), \\ [\widehat{N}_\hbar, \widehat{a}_\lambda^*(f)] &= \hbar \widehat{a}_\lambda^*(f). \end{aligned}$$

We shall characterize a dense domain $\mathcal{D} \subset D(\widehat{N}_\hbar)$ such that for all $\psi \in \mathcal{D}$ and for all $t \in \mathbb{R}$,

$$\psi(t) = \widehat{U}_\hbar(t)\psi = e^{-i\frac{t}{\hbar}\widehat{H}_\hbar} \psi \in D(\widehat{N}_\hbar).$$

To do that, it is technically convenient to introduce a bounded and positive, approximation of \widehat{N}_\hbar that strongly converges monotonically to it:

$$\widehat{N}_\hbar(\delta) = \widehat{N}_\hbar e^{-\delta \widehat{N}_\hbar}, \quad 0 < \delta < 1.$$

It is straightforward to verify that on their dense common domain of definition, \widehat{H}_\hbar^0 and $\widehat{N}_\hbar(\delta)$ commute for all $\delta \geq 0$:

$$[\widehat{H}_\hbar^0, \widehat{N}_\hbar(\delta)] = 0.$$

Let $\psi \in D(\widehat{H}_\hbar^0)$, and define

$$M(t) = \|\widehat{N}_\hbar(\delta)\psi(t)\|^2.$$

$M(t)$ is finite for every $t \in \mathbb{R}$, since $\widehat{N}_\hbar(\delta)$ is a bounded operator. In addition, the map $t \mapsto M(t)$ is differentiable, with

$$(3.12) \quad \dot{M}(t) = \frac{i}{\hbar} (\langle \widehat{N}_\hbar(\delta) \widehat{H}_\hbar \psi(t), \widehat{N}_\hbar(\delta) \psi(t) \rangle - \langle \widehat{N}_\hbar(\delta) \psi(t), \widehat{N}_\hbar(\delta) \widehat{H}_\hbar \psi(t) \rangle).$$

In light of (3.12), let us introduce the densely defined sesquilinear form $q_{\delta, \hbar} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ defined by

$$q_{\delta, \hbar}(\phi, \psi) = \frac{i}{\hbar} (\langle \widehat{N}_{\hbar}(\delta) \widehat{H}_{\hbar} \phi, \widehat{N}_{\hbar}(\delta) \psi \rangle - \langle \widehat{N}_{\hbar}(\delta) \phi, \widehat{N}_{\hbar}(\delta) \widehat{H}_{\hbar} \psi \rangle),$$

for any $\phi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}^{dn}) \otimes \Gamma_{\text{fin}}$, where Γ_{fin} is the subspace of the Fock space $\Gamma_s(\mathfrak{H}^0)$ composed of vectors with finitely many nonvanishing components. Let us omit the explicit dependence on δ and \hbar of the sesquilinear form, i.e., $q \equiv q_{\delta, \hbar}$. It is possible to rewrite q by using the commutator between $\widehat{N}_{\hbar}(\delta)$ and \widehat{H}_{\hbar} on $\mathcal{C}_0^\infty(\mathbb{R}^{dn}) \otimes \Gamma_{\text{fin}}$:

$$q(\phi, \psi) = \frac{i}{\hbar} (\langle [\widehat{N}_{\hbar}(\delta), \widehat{H}_{\hbar}] \phi, \widehat{N}_{\hbar}(\delta) \psi \rangle - \langle \widehat{N}_{\hbar}(\delta) \phi, [\widehat{N}_{\hbar}(\delta), \widehat{H}_{\hbar}] \psi \rangle).$$

LEMMA 3.4. — Assume (A_0) , (A_1) and $(A_2^{(1)})$. Then there exists a constant $C > 0$ such that for all $\hbar \in (0, 1)$ and $\phi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}^{dn}) \otimes \Gamma_{\text{fin}}$:

$$\|q(\phi, \psi)\| \leq C (\|(\widehat{H}_{\hbar}^0 + 1)\phi\| \|\widehat{N}_{\hbar}(\delta)\psi\| + \|\widehat{N}_{\hbar}(\delta)\phi\| \|(\widehat{H}_{\hbar}^0 + 1)\psi\|).$$

Moreover, there exists a constant $C' > 0$ such that

$$\|q(\phi, \psi)\| \leq C' (\|(\widehat{H}_{\hbar} + 1)\phi\| \|\widehat{N}_{\hbar}(\delta)\psi\| + \|\widehat{N}_{\hbar}(\delta)\phi\| \|(\widehat{H}_{\hbar} + 1)\psi\|).$$

Proof. — In this proof C denotes a constant independent of \hbar and δ . The commutator can be computed explicitly as follows

$$[\widehat{N}_{\hbar}(\delta), \widehat{H}_{\hbar}] = \frac{1}{2} \sum_{j=1}^n \sum_{\mu=1}^d ([\widehat{N}_{\hbar}(\delta), D_{\mu}^j] D_{\mu}^j + D_{\mu}^j [\widehat{N}_{\hbar}(\delta), D_{\mu}^j]),$$

where $D^j = \widehat{p}_j - A_j$ is the covariant derivative. In addition,

$$[\widehat{N}_{\hbar}(\delta), D_{\mu}^j] = -i(\hbar \Pi_{j, \mu} + \widehat{N}_{\hbar} \Pi_{j, \mu}(\delta)) e^{-\delta \widehat{N}_{\hbar}},$$

where Π_j is the conjugate momentum of A_j , i.e.,

$$\Pi_j = i \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} (v_{\lambda}(k, \widehat{q}_j) \widehat{a}_{\lambda}^*(k) - \bar{v}_{\lambda}(k, \widehat{q}_j) \widehat{a}_{\lambda}(k)) dk,$$

with

$$v_{\lambda}(k, \widehat{q}_j) = \frac{\epsilon_{\lambda}(k)}{\sqrt{2|k|}} \chi_j(k) e^{2\pi i k \cdot \widehat{q}_j}$$

and

$$(3.13) \quad \Pi_j(\delta) = i \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} (v_{\lambda}(k, \widehat{q}_j) (e^{-\hbar \delta} - 1) \widehat{a}_{\lambda}^*(k) - \bar{v}_{\lambda}(k, \widehat{q}_j) (e^{\hbar \delta} - 1) \widehat{a}_{\lambda}(k)) dk.$$

The above formula (3.13) follows from the commutation relations

$$\begin{aligned} [e^{-\delta \hat{N}_h}, \hat{a}_\lambda^*(f)] &= \hat{a}_\lambda^*((e^{-\hbar\delta} - 1)f) e^{-\delta \hat{N}_h}, \\ [e^{-\delta \hat{N}_h}, \hat{a}_\lambda(f)] &= \hat{a}_\lambda((e^{\hbar\delta} - 1)f) e^{-\delta \hat{N}_h}. \end{aligned}$$

The operators $\Pi_j(\delta)$ are not symmetric, but closed on a suitable domain. Therefore,

$$\begin{aligned} & \frac{i}{\hbar} [\hat{N}_h(\delta), \hat{H}_h] \\ &= \frac{1}{2} \sum_{j=1}^n \left((\Pi_j + \hat{N}_1 \Pi_j(\delta)) e^{-\delta \hat{N}_h} \cdot D^j + D^j \cdot (\Pi_j + \hat{N}_1 \Pi_j(\delta)) e^{-\delta \hat{N}_h} \right) \\ &= \frac{1}{2} \sum_{j=1}^n (I_1^j + I_2^j + I_3^j + I_4^j), \end{aligned}$$

where \hat{N}_1 is the standard \hbar -independent number operator, and

$$\begin{aligned} I_1^j &= \Pi_j e^{-\delta \hat{N}_h} \cdot D^j, & I_2^j &= \hat{N}_1 \Pi_j(\delta) e^{-\delta \hat{N}_h} \cdot D^j, \\ I_3^j &= D^j \cdot \Pi_j e^{-\delta \hat{N}_h}, & I_4^j &= D^j \cdot \hat{N}_1 \Pi_j(\delta) e^{-\delta \hat{N}_h}. \end{aligned}$$

The terms I_1^j – I_4^j are estimated separately.

$$I_1^j = \underbrace{e^{-\delta \hat{N}_h} \Pi_j \cdot D^j}_{I_1^{(1)}} + \underbrace{A_j(\delta) e^{-\delta \hat{N}_h} \cdot D^j}_{I_1^{(2)}},$$

where $A_j(\delta) = -\Pi_j(\delta)$. We note that

$$\|I_1^{(1)} \psi\| \leq \|\Pi_j \cdot D^j \psi\|.$$

Now observe that

$$\begin{aligned} \|\Pi_j \cdot \hat{p} \psi\| &\leq C \|(\hat{H}_h^0 + 1) \psi\|, \\ \|\Pi_j \cdot A_j \psi\| &\leq C \|(\hat{H}_h^0 + 1) \psi\|. \end{aligned}$$

Combining the above estimates, we get

$$\|I_1^{(1)} \psi\| \leq C \|(\hat{H}_h^0 + 1) \psi\|,$$

where C does not depend on δ and \hbar . To complete the estimate for I_1^j , we estimate $I_1^{(2)}$. Let $(E_\lambda)_{\lambda \in \mathbb{R}^+}$ be the spectral measure of a positive self-adjoint operator T evaluated on ψ . Then we see that for all $\alpha \in \frac{1}{2}\mathbb{N}$,

$$\|T^\alpha e^{-tT} \psi\|^2 = \int_0^\infty \lambda^{2\alpha} e^{-2t\lambda} dE_\lambda,$$

and note that $\frac{(2t\lambda)^{2\alpha}}{(2\alpha)!} e^{-2t\lambda} \leq 1$ for all $\lambda \in \mathbb{R}_+$. Therefore

$$\|T^\alpha e^{-tN}\| \leq \frac{\sqrt{(2\alpha)!}}{(2t)^\alpha}$$

follows for any $t > 0$. In particular we obtain that

$$(3.14) \quad \|\hat{N}_1^\alpha e^{-\hbar\delta\hat{N}_1}\| \leq \frac{\sqrt{(2\alpha)!}}{(2\hbar\delta)^\alpha},$$

$$(3.15) \quad \|(\hat{N}_1 + 1)^\alpha e^{-\hbar\delta(\hat{N}_1+1)}\| \leq \frac{\sqrt{(2\alpha)!}}{(2\hbar\delta)^\alpha}.$$

As a result we have

$$\begin{aligned} \|A_j(\delta) e^{-\delta\hat{N}_\hbar} \cdot D^j \psi\| &\leq 2\|\chi_j/\sqrt{\omega}\| e^{\hbar\delta} \sqrt{\hbar} (|e^{\hbar\delta} - 1| + |e^{-\hbar\delta} - 1|) \\ &\quad \times \|(\hat{N}_1 + 1)^{1/2} e^{-\hbar\delta(\hat{N}_1+1)}\| \sum_{\mu=1}^d \|D_\mu^j \psi\| \\ &\leq \sqrt{2}\|\chi_j/\sqrt{\omega}\| e^{\hbar\delta} \sqrt{\hbar} \frac{|e^{\hbar\delta} - 1| + |e^{-\hbar\delta} - 1|}{\sqrt{\hbar\delta}} \sum_{\mu=1}^d \|D_\mu^j \psi\|. \end{aligned}$$

Note that

$$(3.16) \quad \sup_{\xi \in [0,1]} \frac{|e^{\pm\xi} - 1|}{\xi} \leq e - 1.$$

Since $\sum_{\mu=1}^d \|D_\mu^j \psi\| \leq c\|(\hat{H}_\hbar^0 + 1)\psi\|$ with some c , we see that

$$\|A_j(\delta) e^{-\delta\hat{N}_\hbar} \cdot D^j \psi\| \leq C_1(\hbar, \delta) \|(\hat{H}_\hbar^0 + 1)\psi\|.$$

Here

$$\begin{aligned} C_1(\hbar, \delta) &= \hbar\sqrt{\delta}c\sqrt{2}\|\chi_j/\sqrt{\omega}\| e^{\hbar\delta} \frac{|e^{\hbar\delta} - 1| + |e^{-\hbar\delta} - 1|}{\hbar\delta} \\ &\leq 2\sqrt{2}ce(e-1)\|\chi_j/\sqrt{\omega}\|. \end{aligned}$$

Therefore,

$$\|I_1^j \psi\| \leq C\|(\hat{H}_\hbar^0 + 1)\psi\|.$$

Then, we estimate I_2^j ; for this term we need the bounds (3.14) both with $\alpha = 1$ and $\alpha = \frac{3}{2}$. Again, let us split it in two parts:

$$I_2^j = \hat{N}_1 \Pi_j(\delta) e^{-\delta\hat{N}_\hbar} \cdot D^j = \underbrace{\hat{N}_1 e^{-\delta\hat{N}_\hbar} \Pi_j(\delta) \cdot D^j}_{I_2^{(1)}} + \underbrace{\hat{N}_1 A_j^{(2)}(\delta) e^{-\delta\hat{N}_\hbar} \cdot D^j}_{I_2^{(2)}},$$

where

$$A_j^{(2)}(\delta) = i \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \left(v_\lambda(k, \hat{q}_j)(1 - e^{-\hbar\delta})^2 \hat{a}_\lambda^*(k) - \bar{v}_\lambda(k, \hat{q}_j)(1 - e^{-\hbar\delta})^2 \hat{a}_\lambda(k) \right) dk.$$

Similarly to $I_1^{(2)}$, we obtain

$$\|I_2^{(1)}\psi\| \leq C_2(\hbar, \delta) \|(\hat{H}_\hbar^0 + 1)\psi\|.$$

Here

$$\begin{aligned} C_2(\hbar, \delta) &= 3\sqrt{2}d(\|\chi_j/\sqrt{\omega}\| + \|\chi_j/\omega\|) \frac{|e^{\hbar\delta} - 1| + |e^{-\hbar\delta} - 1|}{\hbar\delta} \\ &\leq 3\sqrt{2}(e - 1)d(\|\chi_j/\sqrt{\omega}\| + \|\chi_j/\omega\|). \end{aligned}$$

For $I_2^{(2)}$ one gets

$$\|I_2^{(2)}\psi\| \leq C_3(\hbar, \delta) \|(\hat{H}_\hbar^0 + 1)\psi\|.$$

Here

$$\begin{aligned} C_3(\hbar, \delta) &= \sqrt{3}d\|\chi_j/\sqrt{\omega}\| e^{\hbar\delta} \hbar\sqrt{\delta} \frac{(e^{\hbar\delta} - 1)^2 + (e^{-\hbar\delta} - 1)^2}{(\hbar\delta)^2} \\ &\leq \sqrt{32}e(e - 1)^2 d\|\chi_j/\sqrt{\omega}\|. \end{aligned}$$

Summing up, we have

$$\|I_2^j\psi\| \leq C \|(\hat{H}_\hbar^0 + 1)\psi\|.$$

The estimate of I_3^j is easy, yielding

$$\|I_3^j\psi\| \leq C \|(\hat{H}_\hbar^0 + 1) e^{-\delta\hat{N}_\hbar} \psi\| \leq C \|(\hat{H}_\hbar^0 + 1)\psi\|.$$

Finally, we estimate

$$I_4^j = D^j \cdot \hat{N}_1 \Pi_j(\delta) e^{-\delta\hat{N}_\hbar} = \underbrace{D^j \cdot \Pi_j(\delta) \hat{N}_1 e^{-\delta\hat{N}_\hbar}}_{I_4^{(1)}} + \underbrace{D^j \cdot A_j(\delta) e^{-\delta\hat{N}_\hbar}}_{I_4^{(2)}}.$$

Hence

$$\begin{aligned} \|I_4^{(1)}\psi\| &\leq 6(\|\chi_j/\sqrt{\omega}\| + \|\chi_j/\omega\|) \\ &\quad \times (|e^{\hbar\delta} - 1| + |e^{-\hbar\delta} - 1|) \|(\hat{H}_\hbar^0 + 1) N_1 e^{-\delta\hat{N}_\hbar} \psi\| \\ &\leq C_4(\hbar, \delta) \|(\hat{H}_\hbar^0 + 1)\psi\|. \end{aligned}$$

Here

$$\begin{aligned} C_4(\hbar, \delta) &= 3\sqrt{2}(\|\chi_j/\sqrt{\omega}\| + \|\chi_j/\omega\|) \frac{|e^{\hbar\delta} - 1| + |e^{-\hbar\delta} - 1|}{\hbar\delta} \\ &\leq 6\sqrt{2}(e - 1)(\|\chi_j/\sqrt{\omega}\| + \|\chi_j/\omega\|). \end{aligned}$$

Estimating $I_4^{(2)}$ is easier, and one gets

$$\|I_4^{(2)}\psi\| \leq C_5(\hbar, \delta) \|(\widehat{H}_\hbar^0 + 1)\psi\|$$

Here

$$\begin{aligned} C_5(\hbar, \delta) &= 6(\|\chi_j/\sqrt{\omega}\| + \|\chi_j/\omega\|)(|e^{\hbar\delta} - 1| + |e^{-\hbar\delta} - 1|) \\ &\leq 12(e - 1)(\|\chi_j/\sqrt{\omega}\| + \|\chi_j/\omega\|). \end{aligned}$$

It yields that

$$\|I_4^j\psi\| \leq C\|(\widehat{H}_\hbar^0 + 1)\psi\|.$$

Putting the results above together, we prove the lemma. \square

With the aid of Lemma 3.4, we prove the following bound on $\dot{M}(t)$.

LEMMA 3.5. — Assume (A_0) , (A_1) and $(A_2^{(1)})$. Then there exists a constant $C > 0$ independent of δ and \hbar such that for any $t \in \mathbb{R}$,

$$\dot{M}(t) \leq C(\|(\widehat{H}_\hbar + 1)\psi\|^2 + M(t)).$$

Proof. — By Lemma 3.4, the domain of q can be uniquely extended to $D(\widehat{H}_\hbar) \times D(\widehat{H}_\hbar)$. In particular, we have

$$\begin{aligned} |\dot{M}(t)| &= |q(\psi(t), \psi(t))| \leq C\|(\widehat{H}_\hbar + 1)\psi\| \|\widehat{N}_\hbar(\delta)\psi(t)\| \\ &\leq C(\|(\widehat{H}_\hbar + 1)\psi\|^2 + \|\widehat{N}_\hbar(\delta)\psi(t)\|^2). \end{aligned}$$

Then the lemma is proven. \square

Grönwall's inequality then yields the sought result of propagation.

PROPOSITION 3.6. — Assume (A_0) , (A_1) and $(A_2^{(1)})$. In addition, let $\psi \in D(\widehat{p}^2) \cap D(\widehat{H}_\hbar) \cap D(\widehat{N}_\hbar)$. Then $e^{-i\frac{t}{\hbar}\widehat{H}_\hbar}\psi \in D(\widehat{N}_\hbar)$.

Proof. — By Lemma 3.5, we have

$$M(t) = \int_0^t \dot{M}(s) ds + M(0) \leq C \int_0^t (M(s) + \theta) ds + M(0),$$

where $\theta = \|(\widehat{H}_\hbar + 1)\psi\|^2$. Let $M(t) + \theta = N(t)$. Then

$$M(t) \leq N(t) \leq C \int_0^t N(s) ds + N(0).$$

Then Grönwall's inequality yields

$$M(t) \leq N(t) \leq (\|\widehat{N}_\hbar(\delta)\psi\|^2 + \|(\widehat{H}_\hbar + 1)\psi\|^2) e^{C|t|},$$

where C does not depend on \hbar and δ . For all $\psi \in D(\hat{N}_{\hbar})$,

$$\begin{aligned} \lim_{\delta \downarrow 0} \|\hat{N}_{\hbar}(\delta)\psi(t)\|^2 &= \lim_{\delta \downarrow 0} \int_0^\infty \lambda^2 e^{-2\delta\lambda} dE_\lambda \\ &= \lim_{\delta \downarrow 0} (\|\hat{N}_{\hbar}(\delta)\psi\|^2 + \|(\hat{H}_{\hbar} + 1)\psi\|^2) e^{C|t|} \\ &= (\|\hat{N}_{\hbar}\psi\|^2 + \|(\hat{H}_{\hbar} + 1)\psi\|^2) e^{C|t|}. \end{aligned}$$

Hence by the monotone convergence theorem, it follows that $\psi(t) \in D(\hat{N}_{\hbar})$, and

$$\|\hat{N}_{\hbar}\psi(t)\|^2 \leq (\|\hat{N}_{\hbar}\psi\|^2 + \|(\hat{H}_{\hbar} + 1)\psi\|^2) e^{C|t|}. \quad \square$$

It is immediate to see the corollary below.

COROLLARY 3.7. — Assume (A_0) , (A_1) and $(A_2^{(1)})$. Then there exist two constants $K, C > 0$ such that: for all $\hbar \in (0, 1)$, for all $\psi \in D(\hat{p}^2) \cap D(\hat{H}_{\text{f}}) \cap D(\hat{N}_{\hbar})$, and for all $t \in \mathbb{R}$,

$$\|\hat{N}_{\hbar}\psi(t)\| \leq K(\|\hat{N}_{\hbar}\psi\| + \|(\hat{H}_{\hbar}^0 + 1)\psi\|) e^{C|t|}.$$

Proof. — It follows from Lemma 3.2 and Proposition 3.6. \square

3.2.2. Momentum estimates

The commutation relations of $\hat{N}_{\hbar}^{(\sigma)} = d\Gamma(|k|^{2\sigma})$ with $\hat{a}_\lambda(f)$ and $\hat{a}_\lambda^*(f)$ are

$$\begin{aligned} [\hat{N}_{\hbar}^{(\sigma)}, \hat{a}_\lambda(f)] &= -\hbar \hat{a}_\lambda(|k|^{2\sigma} f), \\ [\hat{N}_{\hbar}^{(\sigma)}, \hat{a}_\lambda^*(f)] &= \hbar \hat{a}_\lambda^*(|k|^{2\sigma} f). \end{aligned}$$

In a similar way to $\hat{N}_{\hbar}(\delta)$ we introduce

$$\hat{N}_{\hbar}^{(\sigma)}(\delta) = \hat{N}_{\hbar}^{(\sigma)} e^{-\delta \hat{N}_{\hbar}^{(\sigma)}}, \sigma \geq 0, \quad 0 < \delta < 1.$$

Let $\psi \in D(\hat{H}_{\hbar}^0)$, and define

$$M^\sigma(t) = \left\langle \psi(t), \hat{N}_{\hbar}^{(\sigma)}(\delta) \psi(t) \right\rangle.$$

The map $t \mapsto M^\sigma(t)$ is differentiable and

$$(3.17) \quad \dot{M}^\sigma(t) = \frac{i}{\hbar} \left\langle \psi(t), [\hat{H}_{\hbar}, \hat{N}_{\hbar}^{(\sigma)}(\delta)] \psi(t) \right\rangle.$$

We set

$$q_{\delta, \hbar}^\sigma(\phi, \psi) = \frac{i}{\hbar} \left\langle \phi, [\hat{H}_{\hbar}, \hat{N}_{\hbar}^{(\sigma)}(\delta)] \psi \right\rangle$$

for any $\phi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}^{dn}) \otimes \Gamma_{\text{fin}}$. Let $q \equiv q_{\delta, \hbar}^\sigma$. Note that $3/2 - \sigma \geq \sigma - 1/2$ if and only if $\sigma \leq 1$. In what follows we assume that $\hat{H}_{\hbar} + a > 0$ and $\hat{H}_{\hbar}^0 + b > 0$. Here a and b are introduced in Corollary 3.3.

LEMMA 3.8. — Assume (A_0) , (A_1) and $(A_2^{(1)})$. For $\sigma \in [0, 1]$ there exists $C > 0$ such that for all $\hbar \in (0, 1)$ and $\phi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}^{dn}) \otimes \Gamma_{\text{fin}}$:

$$|\mathbf{q}^\sigma(\phi, \psi)| \leq C \|\omega^{\sigma-1/2} \chi\| (\|(\widehat{H}_\hbar + 1)^{1/2} \phi\| \|(\widehat{N}_\hbar^{(\sigma)})^{1/2} \psi\| + \|(\widehat{H}_\hbar + 1)^{1/2} \psi\| \|(\widehat{N}_\hbar^{(\sigma)})^{1/2} \phi\|)$$

Proof. — The commutator can be computed explicitly as follows

$$[\widehat{N}_\hbar^{(\sigma)}(\delta), \widehat{H}_\hbar] = \frac{1}{2} \sum_{j=1}^n \sum_{\mu=1}^d ([\widehat{N}_\hbar^{(\sigma)}(\delta), D_\mu^j] D_\mu^j + D_\mu^j [\widehat{N}_\hbar^{(\sigma)}(\delta), D_\mu^j]).$$

Note that

$$\begin{aligned} [e^{-\delta \widehat{N}_\hbar^{(\sigma)}}, \widehat{a}_\lambda(f)] &= e^{-\delta \widehat{N}_\hbar^{(\sigma)}} a_\lambda(\xi f), \\ [e^{-\delta \widehat{N}_\hbar^{(\sigma)}}, \widehat{a}_\lambda^*(f)] &= \widehat{a}_\lambda^*(-\xi f) e^{-\delta \widehat{N}_\hbar^{(\sigma)}}, \end{aligned}$$

where $\xi = 1 - e^{-\delta \hbar |k|^{2\sigma}}$. We have

$$\begin{aligned} &[\widehat{N}_\hbar^{(\sigma)}(\delta), D_\mu^j] \\ &= -i\hbar \left(\Pi_{j,\mu}^\sigma e^{-\delta \widehat{N}_\hbar^{(\sigma)}} + \widehat{N}_\hbar^{(\sigma)} e^{-\delta \widehat{N}_\hbar^{(\sigma)}} \widehat{a}_{j,\mu}(\xi) + \widehat{N}_\hbar^{(\sigma)} \widehat{a}_{j,\mu}^*(-\xi) e^{-\delta \widehat{N}_\hbar^{(\sigma)}} \right) \\ &= -i\hbar (X + Y + Z), \end{aligned}$$

where

$$\begin{aligned} X &= \Pi_{j,\mu}^\sigma e^{-\delta \widehat{N}_\hbar^{(\sigma)}}, \\ Y &= \widehat{N}_\hbar^{(\sigma)} e^{-\delta \widehat{N}_\hbar^{(\sigma)}} \widehat{a}_{j,\mu}(\xi), \\ Z &= \widehat{N}_\hbar^{(\sigma)} \widehat{a}_{j,\mu}^*(-\xi) e^{-\delta \widehat{N}_\hbar^{(\sigma)}}, \end{aligned}$$

Π_j^σ is given by

$$\Pi_j^\sigma = i \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} |k|^{2\sigma-\frac{1}{2}} \epsilon_\lambda(k) (\chi_j(k) e^{-2\pi i k \cdot \hat{q}_j} a_\lambda^*(k) - \bar{\chi}_j(k) e^{2\pi i k \cdot \hat{q}_j} a_\lambda(k)) dk,$$

and

$$\begin{aligned} \widehat{a}_j(\xi) &= i \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} |k|^{-\frac{1}{2}} \epsilon_\lambda(k) \chi_j(k) (1 - e^{-\hbar \delta |k|^{2\sigma}}) e^{2\pi i k \cdot \hat{q}_j} \widehat{a}_\lambda(k) dk, \\ \widehat{a}_j^*(-\xi) &= i \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} |k|^{-\frac{1}{2}} \epsilon_\lambda(k) \chi_j(k) (e^{-\hbar \delta |k|^{2\sigma}} - 1) e^{-2\pi i k \cdot \hat{q}_j} \widehat{a}_\lambda^*(k) dk. \end{aligned}$$

We use the same trick for Y and Z as we used in the estimate on $\widehat{N}_{\hbar} e^{-\delta \widehat{N}_{\hbar}}$. We have

$$\widehat{N}_{\hbar}^{(\sigma)} e^{-\delta \widehat{N}_{\hbar}^{(\sigma)}} \widehat{a}_{j,\mu}(\xi) = \delta \widehat{N}_{\hbar}^{(\sigma)} e^{-\delta \widehat{N}_{\hbar}^{(\sigma)}} \widehat{a}_{j,\mu}(\xi/\delta),$$

where

$$\sup_{0 < \delta < 1} \|\delta \widehat{N}_{\hbar}^{(\sigma)} e^{-\delta \widehat{N}_{\hbar}^{(\sigma)}}\| \leq 1/e,$$

$$(1 - e^{-\hbar \delta |k|^{2\sigma}})/\delta \leq \hbar |k|^{2\sigma}.$$

Let

$$q^{\sigma}(\phi, \psi) = \frac{1}{2} \sum_{j=1}^n \sum_{\mu=1}^d \{ \langle (X + Y + Z)\phi, D_{\mu}^j \psi \rangle + \langle D_{\mu}^j \phi, (X + Y + Z)\psi \rangle \}$$

Note that we have already seen that

$$\sum_{j=1}^n \sum_{\mu=1}^d \|D_{\mu}^j \psi\| \leq C \|(\widehat{H}_{\hbar}^0 + b)^{1/2} \psi\| \leq C \|(\widehat{H}_{\hbar} + a)^{1/2} \psi\|.$$

It is straightforward to see that

$$\begin{aligned} & \frac{1}{2} \sum_{j=1}^n \sum_{\mu=1}^d \{ \langle Y\phi, D_{\mu}^j \psi \rangle + \langle D_{\mu}^j \phi, Y\psi \rangle \} \\ & \leq C \|\omega^{2\sigma-1} \chi_j\| (\|(\widehat{N}_{\hbar}^{(\sigma)})^{1/2} \phi\| \|(\widehat{H}_{\hbar} + a)\psi\| + \|(\widehat{N}_{\hbar}^{(\sigma)})^{1/2} \psi\| \|(\widehat{H}_{\hbar} + a)\phi\|) \end{aligned}$$

We estimate Z :

$$\begin{aligned} & \frac{1}{2} \sum_{j=1}^n \sum_{\mu=1}^d \{ \langle Z\phi, D_{\mu}^j \psi \rangle + \langle D_{\mu}^j \phi, Z\psi \rangle \} \\ & = \langle \widehat{a}_{j\mu}^*(-\xi) \widehat{N}_{\hbar}^{(\sigma)} e^{-\delta \widehat{N}_{\hbar}^{(\sigma)}} \phi, D_{\mu}^j \psi \rangle + \langle D_{\mu}^j \phi, \widehat{a}_{j\mu}^*(-\xi) \widehat{N}_{\hbar}^{(\sigma)} e^{-\delta \widehat{N}_{\hbar}^{(\sigma)}} \psi \rangle \\ & \quad + \langle \widehat{a}_{j\mu}^*(-\omega^{2\sigma} \xi) e^{-\delta \widehat{N}_{\hbar}^{(\sigma)}} \phi, D_{\mu}^j \psi \rangle + \langle D_{\mu}^j \phi, \widehat{a}_{j\mu}^*(-\omega^{2\sigma} \xi) e^{-\delta \widehat{N}_{\hbar}^{(\sigma)}} \psi \rangle. \end{aligned}$$

Taking the Coulomb gauge condition we obtain that

$$\sum_{\mu=1}^d \widehat{a}_{j\mu}(-\xi) D_{\mu}^j = \sum_{\mu=1}^d D_{\mu}^j \widehat{a}_{j\mu}(-\xi)$$

and

$$\begin{aligned}
& \left| \frac{1}{2} \sum_{j=1}^n \sum_{\mu=1}^d \langle \widehat{N}_h^{(\sigma)} e^{-\delta \widehat{N}_h^{(\sigma)}} \phi, \widehat{a}_{j\mu}(-\xi) D_\mu^j \psi \rangle + \langle \widehat{a}_{j\mu}(-\xi) D_\mu^j \phi, \widehat{N}_h^{(\sigma)} e^{-\delta \widehat{N}_h^{(\sigma)}} \psi \rangle \right| \\
&= \left| \frac{1}{2} \sum_{j=1}^n \sum_{\mu=1}^d \left\langle D_\mu^j \delta \widehat{N}_h^{(\sigma)} e^{-\delta \widehat{N}_h^{(\sigma)}} \phi, \widehat{a}_{j\mu}(-\xi/\delta) \psi \right\rangle \right. \\
&\quad \left. + \left\langle \widehat{a}_{j\mu}(-\xi/\delta) \phi, D_\mu^j \delta \widehat{N}_h^{(\sigma)} e^{-\delta \widehat{N}_h^{(\sigma)}} \psi \right\rangle \right| \\
&\leq C \|\omega^{\sigma-1/2} \chi_j\| \left(\|(\widehat{H}_h^0 + b)^{1/2} \delta \widehat{N}_h^{(\sigma)} e^{-\delta \widehat{N}_h^{(\sigma)}} \phi\| \|(\widehat{N}_h^{(\sigma)})^{1/2} \psi\| \right. \\
&\quad \left. + \|(\widehat{H}_h^0 + b)^{1/2} \delta \widehat{N}_h^{(\sigma)} e^{-\delta \widehat{N}_h^{(\sigma)}} \psi\| \|(\widehat{N}_h^{(\sigma)})^{1/2} \phi\| \right) \\
&\leq C \|\omega^{\sigma-1/2} \chi_j\| \left(\|(\widehat{H}_h^0 + b)^{1/2} \phi\| \|(\widehat{N}_h^{(\sigma)})^{1/2} \psi\| \right. \\
&\quad \left. + \|(\widehat{H}_h^0 + b)^{1/2} \phi\| \|(\widehat{N}_h^{(\sigma)})^{1/2} \psi\| \right).
\end{aligned}$$

Similarly

$$\begin{aligned}
& \frac{1}{2} \sum_{j=1}^n \sum_{\mu=1}^d \langle e^{-\delta \widehat{N}_h^{(\sigma)}} \phi, \widehat{a}_{j\mu}(-\omega^{2\sigma} \xi) D_\mu^j \psi \rangle + \langle \widehat{a}_{j\mu}(-\omega^{2\sigma} \xi) D_\mu^j \phi, e^{-\delta \widehat{N}_h^{(\sigma)}} \psi \rangle \\
&= \frac{1}{2} \sum_{j=1}^n \sum_{\mu=1}^d \langle D_\mu^j e^{-\delta \widehat{N}_h^{(\sigma)}} \phi, \widehat{a}_{j\mu}(-\omega^{2\sigma} \xi) \psi \rangle + \langle \widehat{a}_{j\mu}(-\omega^{2\sigma} \xi) \phi, D_\mu^j e^{-\delta \widehat{N}_h^{(\sigma)}} \psi \rangle \\
&\leq C \|\omega^{\sigma-1/2} \chi_j\| \left(\|(\widehat{H}_h^0 + b)^{1/2} e^{-\delta \widehat{N}_h^{(\sigma)}} \phi\| \|(\widehat{N}_h^{(\sigma)})^{1/2} \psi\| \right. \\
&\quad \left. + \|(\widehat{H}_h^0 + b)^{1/2} e^{-\delta \widehat{N}_h^{(\sigma)}} \psi\| \|(\widehat{N}_h^{(\sigma)})^{1/2} \phi\| \right) \\
&\leq C \|\omega^{\sigma-1/2} \chi_j\| \left(\|(\widehat{H}_h^0 + b)^{1/2} \phi\| \|(\widehat{N}_h^{(\sigma)})^{1/2} \psi\| \right. \\
&\quad \left. + \|(\widehat{H}_h^0 + b)^{1/2} \phi\| \|(\widehat{N}_h^{(\sigma)})^{1/2} \psi\| \right).
\end{aligned}$$

Finally we estimate X . The result is

$$\begin{aligned}
& \frac{1}{2} \sum_{j=1}^n \sum_{\mu=1}^d \{ \langle X \phi, D_\mu^j \psi \rangle + \langle D_\mu^j \phi, X \psi \rangle \} \\
&\leq C \|\omega^{\sigma-1/2} \chi_j\| \left(\|(\widehat{H}_h^0 + b)^{1/2} \phi\| \|(\widehat{N}_h^{(\sigma)})^{1/2} \psi\| \right. \\
&\quad \left. + \|(\widehat{H}_h^0 + b)^{1/2} \phi\| \|(\widehat{N}_h^{(\sigma)})^{1/2} \psi\| \right).
\end{aligned}$$

Then

$$|q^\sigma(\phi, \psi)| \leq C \|\omega^{\sigma-1/2} \chi_j\| (\|(\hat{H}_h + a)^{1/2} \phi\| \|(\hat{N}_h^{(\sigma)})^{1/2} \psi\| + \|(\hat{H}_h + a)^{1/2} \psi\| \|(\hat{N}_h^{(\sigma)})^{1/2} \phi\|).$$

Here C is independent of h and δ . \square

With the aid of Lemma 3.8, we prove the following bound on $\dot{M}^\sigma(t)$.

LEMMA 3.9. — Assume (A_0) , (A_1) and $(A_2^{(1)})$. Then for $\sigma \in [0, 1]$ there exists a positive constant C independent of h and δ such that for any $t \in \mathbb{R}$,

$$\dot{M}^\sigma(t) \leq C \langle \psi, (\hat{H}_h + a) \psi \rangle + M^\sigma(t).$$

Proof. — From Lemma 3.8, the lemma follows. \square

Let $Q(T)$ be the form domain of T . Grönwall's inequality then yields the sought result of propagation.

PROPOSITION 3.10. — Assume (A_0) , (A_1) and $(A_2^{(1)})$. Let $\sigma \in [0, 1]$ and $\psi \in Q(\hat{H}_h^0) \cap Q(\hat{N}_h^{(\sigma)})$, then $e^{-i\frac{t}{h}\hat{H}_h} \psi \in Q(\hat{N}_h^{(\sigma)})$.

Proof. — The proof is similar to that of Proposition 3.6. Let $\psi \in D(\hat{H}_h^0)$. By Lemma 3.9, we have

$$M^\sigma(t) = \int_0^t \dot{M}^\sigma(s) ds + M^\sigma(0) \leq C \int_0^t (M^\sigma(s) + \theta) ds + M^\sigma(0),$$

where $\theta = \|(\hat{H}_h + a)^{1/2} \psi\|^2$. Then Grönwall's inequality yields

$$M^\sigma(t) \leq (\|(\hat{N}_h^{(\sigma)}(\delta))^{1/2} \psi\|^2 + \|(\hat{H}_h + a)^{1/2} \psi\|^2) e^{C|t|},$$

where C does not depend on h and δ . This inequality can be extended to $\psi \in Q(\hat{H}_h) \cap Q(\hat{N}_h^{(\sigma)})$. Furthermore for all $\psi \in Q(\hat{H}_h) \cap Q(\hat{N}_h^{(\sigma)})$,

$$\begin{aligned} \lim_{\delta \downarrow 0} \|(\hat{N}_h^{(\sigma)}(\delta))^{1/2} \psi(t)\|^2 &= \lim_{\delta \downarrow 0} \int_0^\infty \lambda e^{-\delta \lambda} dE_\lambda \\ &= \lim_{\delta \downarrow 0} (\|(\hat{N}_h^{(\sigma)}(\delta))^{1/2} \psi\|^2 + \|(\hat{H}_h + a)^{1/2} \psi\|^2) e^{C|t|} \\ &= (\|(\hat{N}_h^{(\sigma)})^{1/2} \psi\|^2 + \|(\hat{H}_h + a)^{1/2} \psi\|^2) e^{C|t|}. \end{aligned}$$

Hence by the monotone convergence theorem, it follows that $\psi(t) \in Q(\hat{N}_h^\sigma)$, and

$$\|(\hat{N}_h^{(\sigma)})^{1/2} \psi(t)\|^2 \leq (\|(\hat{N}_h^{(\sigma)})^{1/2} \psi\|^2 + \|(\hat{H}_h + a)^{1/2} \psi\|^2) e^{C|t|}.$$

Then the proposition follows. \square

From the proof of Proposition 3.10 we obtain the corollary below.

COROLLARY 3.11. — Assume (A_0) , (A_1) and $(A_2^{(1)})$. For $\sigma \in [0, 1]$ there exist two constants $K, C > 0$ such that for all $\hbar \in (0, 1)$, all $\psi \in Q(\hat{H}_\hbar^0) \cap Q(\hat{N}_\hbar^{(\sigma)})$, and all $t \in \mathbb{R}$,

$$\|(\hat{N}_\hbar^{(\sigma)})^{1/2} \psi(t)\| \leq K (\|(\hat{N}_\hbar^{(\sigma)})^{1/2} \psi\| + \|(\hat{H}_\hbar^0 + 1)^{1/2} \psi\|) e^{C|t|}.$$

3.2.3. Particles position estimates

We end this section by proving localization estimates for the particles along the Pauli–Fierz quantum dynamics. In fact, we show below that if an initial state is localized in position then for all times it will stay localized uniformly with respect to $\hbar \in (0, 1)$.

LEMMA 3.12. — Assume (A_0) , (A_1) and $(A_2^{(1)})$. Then there exist constants $K, C > 0$ such that for all $\Psi \in Q(\hat{H}_\hbar^0) \cap Q(\hat{q}^2)$, all $t \in \mathbb{R}$ and all $\hbar \in (0, 1)$:

- (1) $\langle e^{-i\frac{t}{\hbar}\hat{H}_\hbar^0} \Psi, \hat{q}^2 e^{-i\frac{t}{\hbar}\hat{H}_\hbar^0} \Psi \rangle \leq K \langle \Psi, (\hat{H}_\hbar^0 + \hat{q}^2 + 1) \Psi \rangle e^{C|t|},$
- (2) $\langle e^{-i\frac{t}{\hbar}\hat{H}_\hbar} \Psi, \hat{q}^2 e^{-i\frac{t}{\hbar}\hat{H}_\hbar} \Psi \rangle \leq K \langle \Psi, (\hat{H}_\hbar^0 + \hat{q}^2 + 1) \Psi \rangle e^{C|t|}.$

Proof. — The argument is based on Grönwall’s inequality and is similar to the one employed in Subsections 3.2.1 and 3.2.3. So, we just indicate here the key point. In particular, one can use the work of Faris and Lavine [30, Theorem 2] (see also [3, Theorem B.1]). So, it is enough to estimate as quadratic forms on $Q(\hat{H}_\hbar^0) \cap Q(\hat{q}^2)$ the following commutators,

$$(3.18) \quad T_1 = i[\hat{p}_j^2, \hat{q}_j^2] = 2\hbar \hat{p}_j \cdot \hat{q}_j,$$

$$(3.19) \quad T_2 = i[\hat{p}_j \cdot A_j(\hat{q}_j, \hat{a}), \hat{q}_j^2] = 2\hbar \hat{q}_j \cdot A_j(\hat{q}_j, \hat{a}).$$

Since the right-hand sides in (3.18) and (3.19) are bounded respectively by

$$(3.20) \quad \hbar^{-1} T_1 \leq \hat{p}_j^2 + \hat{q}_j^2, \quad \hbar^{-1} T_2 \leq \hat{q}_j^2 + A_j(\hat{q}_j, \hat{a})^2,$$

then by Corollaries 3.3 and A.5,

$$\begin{aligned} N(t) &= \langle e^{-i\frac{t}{\hbar}\hat{H}_\hbar} \Psi, \hat{q}^2 e^{-i\frac{t}{\hbar}\hat{H}_\hbar} \Psi \rangle \\ &\leq N(0) + t K_0 \langle \Psi, (\hat{H}_\hbar^0 + 1) \Psi \rangle + C_0 \int_0^t N(s) ds \\ &\leq K \langle \Psi, (\hat{H}_\hbar^0 + \hat{q}^2 + 1) \Psi \rangle e^{Ct} \end{aligned}$$

for all $t \geq 0$ and some constants K_0, C_0, K, C independent of time and \hbar . Note that the case $t \leq 0$ follows by a simple change of variable. This

proves (2). On the other hand a similar inequality

$$\begin{aligned} M(t) &= \langle e^{-i\frac{t}{\hbar}\hat{H}_h^0} \Psi, \hat{q}^2 e^{-i\frac{t}{\hbar}\hat{H}_h^0} \Psi \rangle \\ &\leq M(0) + tK_0 \langle \Psi, (\hat{H}_h^0 + 1)\Psi \rangle + C_0 \int_0^t M(s) ds \\ &\leq K \langle \Psi, (\hat{H}_h^0 + \hat{q}^2 + 1)\Psi \rangle e^{Ct} \end{aligned}$$

holds true for some possibly different constants. Thus we prove (1). \square

4. Derivation of the characteristic equation

Our approach for the derivation of classical electrodynamics from quantum electrodynamics is based on the analysis of time-dependent Wigner measures $(\mu_t)_{t \in \mathbb{R}}$ of the quantum evolved states $(\varrho_h(t))_{h \in (0,1)}$ given in Theorem 1.4. Due to the complexity generated by the particles-field interactions, it is not possible to compute directly or explicitly these Wigner measures. Therefore, it makes more sense to write a dynamical equation that characterizes the time course of such probabilities. In particular, one obtains such relation if one differentiates with respect to time the characteristic or generating functions of these Wigner measures $(\mu_t)_{t \in \mathbb{R}}$. Such equation is a crucial step in our analysis; it will be given in Proposition 4.19 and named *characteristic equation*.

4.1. Preliminaries

4.1.1. Weyl–Heisenberg operators

The Weyl–Heisenberg translation operators on the particle variables are defined as the unitary operators on $L^2(\mathbb{R}_x^{dn}, \mathbb{C})$,

$$(4.1) \quad \hat{T}(p, q) = e^{i(p \cdot \hat{q} - q \cdot \hat{p})}, \quad \forall (p, q) \in \mathbb{R}^{dn} \times \mathbb{R}^{dn}.$$

The space $\mathbb{R}^{dn} \times \mathbb{R}^{dn} \equiv \mathbb{C}^{2dn}$ is endowed with a complex structure

$$z = z(p, q) = q + ip, \quad \forall (p, q) \in \mathbb{R}^{dn} \times \mathbb{R}^{dn},$$

and a canonical symplectic form

$$\Im \langle z, z' \rangle = q \cdot p' - p \cdot q', \quad \forall (p, q), (p', q') \in \mathbb{R}^{2dn}.$$

Hence Weyl–Heisenberg operators can also be labelled by the phase-space points $z \in \mathbb{C}^{dn}$,

$$(4.2) \quad \widehat{T}(z) = \widehat{T}(p, q) = e^{i\Im\langle \hat{q} + i\hat{p}, z \rangle},$$

satisfying the particles *Weyl commutation relations*:

$$\begin{aligned} \widehat{T}(z)\widehat{T}(z') &= e^{-i\frac{\hbar}{2}\Im\langle z, z' \rangle} \widehat{T}(z + z'), \\ \widehat{T}(z)^* &= \widehat{T}(z)^{-1} = \widehat{T}(-z). \end{aligned}$$

Moreover, the commutations rules below hold true for any $z = q + ip \in \mathbb{C}^{dn}$,

$$(4.3) \quad \begin{aligned} \widehat{T}(z)\widehat{q}\widehat{T}(z)^* &= \widehat{q} - \hbar q, \\ \widehat{T}(z)\widehat{p}\widehat{T}(z)^* &= \widehat{p} - \hbar p. \end{aligned}$$

Similarly, one defines also the unitary Weyl operators on the Fock space $\Gamma_s(\mathfrak{H}^0)$ for any $f \in \mathfrak{H}^0$ according to the formula,

$$(4.4) \quad W(f) = e^{\frac{i}{\sqrt{2}}(\hat{a}(f) + \hat{a}^*(f))}.$$

With these notations the following *Weyl commutations relations* for the field are satisfied,

$$W(f)W(g) = e^{-i\frac{\hbar}{2}\Im\langle f, g \rangle_{\mathfrak{H}^0}} W(f + g), \quad \forall f, g \in \mathfrak{H}^0,$$

where $\Im\langle \cdot, \cdot \rangle_{\mathfrak{H}^0}$ denotes the imaginary part of the scalar product on \mathfrak{H}^0 . Since the total phase-space of the particles-field system is

$$(4.5) \quad X^0 = \mathbb{R}^{2dn} \oplus \mathfrak{H}^0 \equiv \mathbb{C}^{dn} \oplus \mathfrak{H}^0,$$

it follows that the mapping

$$(4.6) \quad (z, \alpha) \in X^0 \longmapsto \mathcal{W}_{\hbar}(z, \alpha) = \widehat{T}(z) \otimes W(\alpha),$$

defines a strongly continuous irreducible representation of the Weyl commutation relations over the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^{dn}) \otimes \Gamma_s(\mathfrak{H}^0).$$

Note that we will use indifferently the notations $\mathcal{W}_{\hbar}(z, \alpha)$, $\mathcal{W}_{\hbar}(p, q, \alpha)$ or $\mathcal{W}_{\hbar}(\xi)$ when $z \equiv (p, q)$ and $\xi = (z, \alpha)$.

4.1.2. Coherent states

Consider the normalized Gaussian function

$$\varphi_0(x) = (\pi\hbar)^{-\frac{dn}{4}} e^{-x^2/2\hbar} \in L^2(\mathbb{R}^{dn}),$$

then the coherent vector, centered on $z_0 \in \mathbb{C}^{dn}$, is defined as

$$(4.7) \quad \varphi_{z_0} = \hat{T}\left(\frac{\sqrt{2}}{i\hbar}z_0\right)\varphi_0.$$

Similarly, for any $\alpha \in \mathfrak{H}^0$ one defines coherent vectors in the Fock spaces as

$$(4.8) \quad W\left(\frac{\sqrt{2}}{i\hbar}\alpha\right)\Omega,$$

where Ω is the vacuum vector in $\Gamma_s(\mathfrak{H}^0)$. Both vectors (4.7) and (4.8) are normalized. For any $u_0 = (p_0, q_0, \alpha_0) \in X^0$ such that $z_0 \equiv (p_0, q_0)$, the one rank orthogonal projection

$$(4.9) \quad \mathcal{C}_{\hbar}(u_0) = \left| \varphi_{z_0} \otimes W\left(\frac{\sqrt{2}}{i\hbar}\alpha_0\right)\Omega \right\rangle \left\langle \varphi_{z_0} \otimes W\left(\frac{\sqrt{2}}{i\hbar}\alpha_0\right)\Omega \right|$$

defines a \hbar -scaled family of density matrices over \mathcal{H} called the coherent states.

4.1.3. Examples of Wigner measures

The following result proved in [8, Theorem 6.2] shows that under mild assumptions any family of quantum states admits at least one Wigner measure.

PROPOSITION 4.1. — *Any family of density matrices $(\varrho_{\hbar})_{\hbar \in (0,1)}$ on \mathcal{H} satisfying the condition:*

$$(4.10) \quad \exists \delta > 0, C_{\delta} > 0 : \forall \hbar \in (0, 1), \quad \text{Tr}[\varrho_{\hbar}(\hat{p}^2 + \hat{q}^2 + \hat{N}_{\hbar})^{\delta}] < C_{\delta},$$

has a non trivial set of Wigner measures, i.e., $\mathcal{M}(\varrho_{\hbar}, \hbar \in (0, 1)) \neq \emptyset$. Moreover, any $\mu \in \mathcal{M}(\varrho_{\hbar}, \hbar \in (0, 1))$ satisfies

$$\int_{X^0} \|u\|_{X^0}^{2\delta} d\mu(u) < \infty.$$

The above result is a consequence of Weyl commutation relations, Arzelà-Ascoli's type theorem, Prokhorov's theorem and pseudo-differential calculus. For further details, we refer the reader to [8, Theorem 6.2].

On the other hand, the set of Wigner measures for a family of coherent states is remarkably simple and it is given below.

LEMMA 4.2. — *Consider for any $u_0 = (p_0, q_0, \alpha_0) \in X^0$ the family of coherent states $(\mathcal{C}_{\hbar}(u_0))_{\hbar \in (0,1)}$ defined in (4.9). Then*

$$(4.11) \quad \mathcal{M}(\mathcal{C}_{\hbar}(u_0), \hbar \in (0, 1)) = \{\delta_{u_0}\},$$

where δ_{u_0} is the Dirac measure centered on u_0 . Furthermore, if $u_0 = (p_0, q_0, \alpha_0) \in X^\sigma$ for some $\sigma \in [\frac{1}{2}, 1]$, then the family $(\mathcal{C}_\hbar(u_0))_{\hbar \in (0,1)}$ satisfies the assumptions $(S_0^{(1)})$, $(S_1^{(1)})$ and $(S_2^{(\sigma)})$.

Proof. — According to Definition 1.1, in order to determine the set of Wigner measures of the family of states $(\mathcal{C}_\hbar(u_0))_{\hbar \in (0,1)}$ it is enough to explicitly compute the limit

$$\lim_{\hbar \rightarrow 0} \text{Tr} [\mathcal{C}_\hbar(u_0) \mathcal{W}_\hbar(2\pi q, -2\pi p, \sqrt{2\pi}\alpha)]$$

for any $\xi = (2\pi q, -2\pi p, \sqrt{2\pi}\alpha) \in X^0$. Indeed, using Weyl commutation relations one shows

$$(4.12) \quad \lim_{\hbar \rightarrow 0} \text{Tr} [\mathcal{C}_\hbar(u_0) \mathcal{W}_\hbar(\xi)] = e^{2\pi i \Re \langle \xi, u_0 \rangle_{X^0}}$$

$$(4.13) \quad = \mathcal{F}^{-1}[\delta_{u_0}](\xi).$$

Furthermore, if $u_0 = (p_0, q_0, \alpha_0) \in X^\sigma$ then one can verify that

$$(4.14) \quad \begin{aligned} \text{Tr} [\mathcal{C}_\hbar(u_0) d\Gamma(|k|)] &= \langle \alpha_0, |k| \alpha_0 \rangle_{\mathfrak{H}^0} = \|\alpha_0\|_{\mathfrak{H}^{1/2}}^2, \\ \text{Tr} [\mathcal{C}_\hbar(u_0) d\Gamma(|k|^{2\sigma})] &= \langle \alpha_0, |k|^{2\sigma} \alpha_0 \rangle_{\mathfrak{H}^0} = \|\alpha_0\|_{\mathfrak{H}^\sigma}^2, \\ \text{Tr} [\mathcal{C}_\hbar(u_0) \hat{N}_\hbar] &= \|\alpha_0\|_{\mathfrak{H}^0}^2. \end{aligned}$$

Such identities can be proved using canonical commutation relations in Fock spaces (See for instance [8, Propositions 2.7 and 2.10]). Similarly, using (4.3) one also proves that

$$(4.15) \quad \text{Tr} [\mathcal{C}_\hbar(u_0) \hat{q}^2] = \langle \varphi_0, (\hat{q} - i\sqrt{2}q_0)^2 \varphi_0 \rangle = \langle \varphi_0, \hat{q}^2 \varphi_0 \rangle - 2q_0^2,$$

$$(4.16) \quad \text{Tr} [\mathcal{C}_\hbar(u_0) \hat{p}^2] = \langle \varphi_0, (\hat{p} - i\sqrt{2}p_0)^2 \varphi_0 \rangle = \langle \varphi_0, \hat{p}^2 \varphi_0 \rangle - 2p_0^2,$$

for φ_0 being the Gaussian function in (4.7) and $z_0 = q_0 + ip_0 \in \mathbb{C}^{dn}$. Since $u_0 = (p_0, q_0, \alpha_0) \in X^\sigma$ for some $\sigma \in [\frac{1}{2}, 1]$, one notices that all the quantities in (4.14), (4.15) and (4.16) are bounded uniformly with respect to $\hbar \in (0, 1)$. \square

A simple observation shows that any Borel probability measure μ_0 over X^0 is a Wigner measure of at least one given family of quantum states.

LEMMA 4.3. — Consider for any $\mu_0 \in \mathfrak{P}(X^0)$ the family of density matrices

$$(4.17) \quad \varrho_\hbar = \int_{X^0} \mathcal{C}_\hbar(u) d\mu_0(u),$$

where $\mathcal{C}_\hbar(u)$ is the coherent state defined in (4.9) and centered on $u = (p, q, \alpha) \in X^0$. Then

$$(4.18) \quad \mathcal{M}(\varrho_\hbar, \hbar \in (0, 1)) = \{\mu_0\}.$$

Furthermore, suppose that

$$(4.19) \quad \int_{X^0} \|u\|_{X^\sigma}^2 d\mu_0 < \infty,$$

then the family of density matrices $(\varrho_{\hbar})_{\hbar \in (0,1)}$ satisfies the assumptions $(S_0^{(1)})$, $(S_1^{(1)})$ and $(S_2^{(\sigma)})$.

Proof. — Using (4.12) and (4.17),

$$(4.20) \quad \lim_{\hbar \rightarrow 0} \operatorname{Tr} [\varrho_{\hbar} \mathcal{W}_{\hbar}(2\pi q, -2\pi p, \sqrt{2\pi}\alpha)] = \int_{X^0} e^{2\pi i \operatorname{Re} \langle \xi, u \rangle_{X^0}} d\mu_0(u) \\ = \mathcal{F}^{-1}[\mu_0](\xi),$$

for all $\xi = (p, q, \alpha) \in X^0$. Hence according to Definition 1.1 the first statement (4.18) is proved since the right-hand side of (4.20) uniquely determines the probability measure μ_0 . The second statement is a consequence of (4.14)–(4.16) and the assumption (4.19). \square

Finally, we note that the set of Wigner measures resulting from a family of quantum states is not a singleton in general. However, for any family $(\varrho_{\hbar})_{\hbar \in \mathcal{J}}$ it is always possible to find a subset $\mathcal{J}_0 \subset \mathcal{J}$, $0 \in \overline{\mathcal{J}}$, such that $\mathcal{M}(\varrho_{\hbar}, \hbar \in \mathcal{J}_0)$ is a singleton.

4.2. Duhamel's formula

4.2.1. Weyl–Heisenberg operator estimates

Here we highlight the fact that the Weyl–Heisenberg operators keep the form domain of the Pauli–Fierz Hamiltonian \widehat{H}_{\hbar} invariant. These properties will be useful to establish a Duhamel formula for quantum evolved states.

LEMMA 4.4. — *For any $\alpha \in \mathfrak{H}^0$ there exists a constant $C > 0$ such that for any $\hbar \in (0, 1)$ and any $\Psi \in D(\widehat{N}_{\hbar})$,*

$$(4.21) \quad \|(\widehat{N}_{\hbar})^{1/2} W(\alpha) \Psi\|_{\Gamma_s(\mathfrak{H}^0)} \leq C \|(\widehat{N}_{\hbar} + 1)^{1/2} \Psi\|_{\Gamma_s(\mathfrak{H}^0)},$$

$$(4.22) \quad \|\widehat{N}_{\hbar} W(\alpha) \Psi\|_{\Gamma_s(\mathfrak{H}^0)} \leq C \|(\widehat{N}_{\hbar} + 1) \Psi\|_{\Gamma_s(\mathfrak{H}^0)}.$$

Proof. — The above estimate is a consequence of the commutator expansion formula,

$$W(\alpha)^* \widehat{N}_{\hbar} W(\alpha) = \widehat{N}_{\hbar} + \frac{i\hbar}{\sqrt{2}} \widehat{a}^*(\alpha) - \frac{i\hbar}{\sqrt{2}} \widehat{a}(\alpha) + \frac{\hbar^2}{2} \|\alpha\|_{\mathfrak{H}^0},$$

and \hbar -scaled standard number estimates (see, e.g., [8, Lemma 2.5]). \square

LEMMA 4.5. — For any $\alpha \in \mathfrak{H}^{1/2}$ there exists a constant $C > 0$ such that for any $\hbar \in (0, 1)$ and any $\Psi \in D(\hat{H}_\hbar^0)$,

$$(4.23) \quad \|(\hat{H}_\hbar^0)^{1/2} W(\alpha) \Psi\|_{\mathcal{H}} \leq C \|(\hat{H}_\hbar^0 + 1)^{1/2} \Psi\|_{\mathcal{H}}.$$

Proof. — It is a simple consequence of the fact that

$$(\hat{H}_\hbar^0 + 1)^{-1/2} [\hat{a}^\sharp(\alpha), \hat{H}_\hbar^0] (\hat{H}_\hbar^0 + 1)^{-1/2},$$

is bounded uniformly with respect to $\hbar \in (0, 1)$, where a^\sharp stands for a creation or an annihilation operator. \square

Similar estimates also hold for the Weyl–Heisenberg operators \hat{T} .

LEMMA 4.6. — For any $z \in \mathbb{C}^{dn}$ there exists $C > 0$ such that for any $\hbar \in (0, 1)$ and any $\Psi \in D((\hat{p}^2 + \hat{q}^2)^{1/2})$,

$$(4.24) \quad \|(\hat{p}^2 + \hat{q}^2)^{1/2} \hat{T}(z) \Psi\|_{L^2(\mathbb{R}^{dn})} \leq C \|(\hat{p}^2 + \hat{q}^2 + 1)^{1/2} \Psi\|_{L^2(\mathbb{R}^{dn})}.$$

Proof. — In order to prove the above estimate, it is enough to evaluate the commutator between the generator of the Weyl–Heisenberg operator \hat{T} and the harmonic oscillator $\hat{p}^2 + \hat{q}^2$. An explicit calculation yields

$$i[\hat{p}^2 + \hat{q}^2, p \cdot \hat{q} - q \cdot \hat{p}] = 2\hbar(q \cdot \hat{q} + p \cdot \hat{p}).$$

Moreover, the right-hand side is bounded uniformly with respect to $\hbar \in (0, 1)$ by $\hat{p}^2 + \hat{q}^2$ in the quadratic form sense. \square

4.2.2. Duhamel formula

The argument behind the Duhamel’s formula in Proposition 4.7 is quite general. To highlight the main ingredients in this step, we list the assumptions at our disposal. For $\hbar \in (0, 1)$:

- (1) $\xi \in X^0 \rightarrow \mathcal{W}_\hbar(\xi)$ is a strongly continuous representation of the Weyl commutation relations on the Hilbert space \mathcal{H} .
- (2) $(\hat{H}_\hbar, \hat{H}_\hbar^0)$ is a couple of self-adjoint operators on \mathcal{H} such that \hat{H}_\hbar^0 is non-negative and the form domain $Q(\hat{H}_\hbar) = D(|\hat{H}_\hbar|^{1/2})$ coincides with that of \hat{H}_\hbar^0 .
- (3) The form domain $Q(\hat{H}_\hbar^0)$ is invariant with respect to the Weyl–Heisenberg operators $\mathcal{W}_\hbar(\xi)$ for all ξ in a dense subset $X^{1/2}$ of the phase-space X^0 .

Let $(\varrho_\hbar)_{\hbar \in (0, 1)}$ be a family of density matrices on \mathcal{H} and define:

$$(4.25) \quad \varrho_\hbar(t) = e^{-i \frac{t}{\hbar} \hat{H}_\hbar} \varrho_\hbar e^{i \frac{t}{\hbar} \hat{H}_\hbar} \quad \text{and} \quad \tilde{\varrho}_\hbar(t) = e^{i \frac{t}{\hbar} \hat{H}_\hbar^0} \varrho_\hbar(t) e^{-i \frac{t}{\hbar} \hat{H}_\hbar^0}.$$

Recall the hypothesis $(S_0^{(1)})$ which specifies the regularity condition that we shall require for the initial quantum states $(\varrho_{\hbar})_{\hbar \in (0,1)}$:

$$\exists C_0 > 0, \forall \hbar \in (0, 1), \quad \text{Tr}[\varrho_{\hbar}(\hat{H}_{\hbar}^0 + 1)] \leq C_0.$$

The uniform bound in the trace means precisely that the operators

$$(4.26) \quad (\hat{H}_{\hbar}^0 + 1)^{1/2} \varrho_{\hbar} (\hat{H}_{\hbar}^0 + 1)^{1/2} \in \mathcal{L}^1(\mathcal{H}),$$

have a finite trace norm, uniformly in $\hbar \in (0, 1)$.

PROPOSITION 4.7. — Assume (A_0) , (A_1) and $(A_2^{(1)})$. Let $(\varrho_{\hbar})_{\hbar \in (0,1)}$ be a family of density matrices satisfying $(S_0^{(1)})$. Then for all $\xi \in X^{1/2}$, all $\hbar \in (0, 1)$ and all $t, t_0 \in \mathbb{R}$,

$$(4.27) \quad \begin{aligned} & \text{Tr}[\mathcal{W}_{\hbar}(\xi) \tilde{\varrho}_{\hbar}(t)] \\ &= \text{Tr}[\mathcal{W}_{\hbar}(\xi) \tilde{\varrho}_{\hbar}(t_0)] - \frac{i}{\hbar} \int_{t_0}^t \text{Tr} \left([\mathcal{W}_{\hbar}(\xi), \hat{H}_I(s)] \tilde{\varrho}_{\hbar}(s) \right) ds, \end{aligned}$$

with

$$(4.28) \quad \hat{H}_I(s) = e^{i \frac{s}{\hbar} \hat{H}_{\hbar}^0} (\hat{H}_{\hbar} - \hat{H}_{\hbar}^0) e^{-i \frac{s}{\hbar} \hat{H}_{\hbar}^0},$$

where the commutator in the right-hand side is interpreted as a bounded quadratic form on $Q(\hat{H}_{\hbar}^0)$.

Proof. — It follows using the Weyl–Heisenberg estimates of Lemma 4.5, Corollary 3.3 and the fundamental theorem of calculus for the function $\mathbb{R} \ni t \mapsto \text{Tr}[\mathcal{W}_{\hbar}(\xi) \tilde{\varrho}_{\hbar}(t)]$ which is of class \mathcal{C}^1 in our case. \square

4.3. Commutator expansion

The main part to be analyzed in the Duhamel formula (4.27) is the commutator $[\mathcal{W}_{\hbar}(\xi), \hat{H}_I(s)]$. In particular, we wish to expand it with respect to the parameter \hbar . A simple computation gives

$$\begin{aligned} \hat{H}_{\hbar} - \hat{H}_{\hbar}^0 &= \sum_{i=1}^n \frac{1}{2m_i} (\hat{p}_i - A_i(\hat{q}_i, \hat{a}))^2 - \frac{1}{2m_i} \hat{p}_i^2 + V(\hat{q}) \\ &= \sum_{i=1}^n \frac{1}{2m_i} \left(A_i(\hat{q}_i, \hat{a})^2 - \hat{p}_i \cdot A_i(\hat{q}_i, \hat{a}) - A_i(\hat{q}_i, \hat{a}) \cdot \hat{p}_i \right) + V(\hat{q}) \\ &= \sum_{i=1}^n \frac{1}{2m_i} \left(A_i(\hat{q}_i, \hat{a})^2 - 2\hat{p}_i \cdot A_i(\hat{q}_i, \hat{a}) \right) + V(\hat{q}). \end{aligned}$$

The last equality is a consequence of the Coulomb gauge. Moreover, one has

$$A_i^\nu(\widehat{q}_i, \widehat{a})^2 = \widehat{a}^*(r_i^\nu)^2 + \widehat{a}(r_i^\nu)^2 + 2\widehat{a}^*(r_i^\nu)\widehat{a}(r_i^\nu) + \hbar \|r_i^\nu\|_{\mathfrak{H}^0}^2,$$

with the function r_i^ν given by

$$(4.29) \quad r_i^\nu = \left(\frac{\chi_i(k)}{\sqrt{2|k|}} e^{-2\pi i k \cdot \widehat{q}_i} \epsilon_\lambda^\nu(k) \right)_{\lambda=1, \dots, d-1} \in L^\infty(\mathbb{R}_{x_i}^d; \mathfrak{H}^0),$$

and

$$\begin{aligned} \widehat{a}^*(r_i^\nu) &= \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \frac{\chi_i(k)}{\sqrt{2|k|}} e^{-2\pi i k \cdot \widehat{q}_i} \epsilon_\lambda^\nu(k) a_\lambda^*(k) dk, \\ \widehat{a}(r_i^\nu) &= \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \frac{\overline{\chi_i(k)}}{\sqrt{2|k|}} e^{2\pi i k \cdot \widehat{q}_i} \epsilon_\lambda^\nu(k) a_\lambda(k) dk. \end{aligned}$$

Using the above notations, one obtains

$$(4.30) \quad \begin{aligned} \widehat{H}_I(0) &= \sum_{i=1}^n \frac{1}{2m_i} \sum_{\nu=1}^d (\widehat{a}^*(r_i^\nu)^2 + \widehat{a}(r_i^\nu)^2 + 2\widehat{a}^*(r_i^\nu)\widehat{a}(r_i^\nu) + \hbar \|r_i^\nu\|_{\mathfrak{H}^0}^2) \\ &\quad - \sum_{i=1}^n \frac{1}{m_i} \sum_{\nu=1}^d \widehat{p}_i^\nu \cdot (\widehat{a}^*(r_i^\nu) + \widehat{a}(r_i^\nu)) + V(\widehat{q}). \end{aligned}$$

In order to use the estimates of Subsection 3.2, it is more convenient to put the quantity $\widehat{H}_I(s)$ defined in (4.28) in the following form.

LEMMA 4.8. — *For any $s \in \mathbb{R}$ the time-evolved interaction term $\widehat{H}_I(s)$ takes the form,*

$$\begin{aligned} \widehat{H}_I(s) &= \sum_{i=1}^n \frac{1}{2m_i} \sum_{\nu=1}^d (a^*(r_i^\nu(s))^2 + a(r_i^\nu(s))^2 + 2a^*(r_i^\nu(s))a(r_i^\nu(s)) + \hbar \|r_i^\nu\|_{\mathfrak{H}^0}^2) \\ &\quad - \sum_{i=1}^n \frac{1}{m_i} \sum_{\nu=1}^d a^*(r_i^\nu(s)) \cdot \widehat{p}_i^\nu + \widehat{p}_i^\nu \cdot a(r_i^\nu(s)) \\ &\quad + V\left(\widehat{q}_1 + s \frac{\widehat{p}_1}{m_1}, \dots, \widehat{q}_n + s \frac{\widehat{p}_n}{m_n}\right), \end{aligned}$$

where

$$(4.31) \quad r_i^\nu(s) = \left(\frac{\chi_i(k)}{\sqrt{2|k|}} \epsilon_\lambda^\nu(k) e^{-2\pi i k \cdot (\widehat{q}_i + s \frac{\widehat{p}_i}{m_i}) + i s |k|} \right)_{\lambda=1, \dots, d-1}.$$

Furthermore, $\widehat{H}_I(s)$ is well defined as a quadratic form on $Q(\widehat{H}_h^0)$ which is relatively \widehat{H}_h^0 -form bounded.

Proof. — Recall that $\widehat{H}_I(s) = e^{i\frac{s}{\hbar}\widehat{H}_h^0} \widehat{H}_I(0) e^{-i\frac{s}{\hbar}\widehat{H}_h^0}$. So, the expression of $\widehat{H}_I(s)$ is a consequence of (4.30) and the commutation relations,

$$\begin{aligned} e^{i\frac{s}{\hbar}\widehat{H}_f} a^\sharp(f) e^{-i\frac{s}{\hbar}\widehat{H}_f} &= a^\sharp(e^{is\omega} f), \\ e^{i\frac{s}{\hbar}\frac{\widehat{p}_i^2}{2m_i}} \widehat{q}_i^\nu e^{-i\frac{s}{\hbar}\frac{\widehat{p}_i^2}{2m_i}} &= \widehat{q}_i^\nu + s\frac{\widehat{p}_i^\nu}{m_i}, \quad \nu = 1, \dots, d. \end{aligned}$$

Additionally, the Coulomb gauge (1.12) condition is used to show that

$$\sum_{\nu=1}^d \widehat{p}_i^\nu \cdot \widehat{a}^*(r_i^\nu(s)) = \sum_{\nu=1}^d a^*(r_i^\nu(s)) \cdot \widehat{p}_i^\nu.$$

Finally, thanks to Corollary A.5 one verifies that $\widehat{H}_I(s)$ is a \widehat{H}_h^0 -form bounded quadratic form. \square

As mentioned before, our main purpose here is to establish a semiclassical expansion of the commutator

$$(4.32) \quad [\mathcal{W}_h(\xi), \widehat{H}_I(s)] = \left(\mathcal{W}_h(\xi) \widehat{H}_I(s) \mathcal{W}_h(\xi)^* - \widehat{H}_I(s) \right) \mathcal{W}_h(\xi).$$

LEMMA 4.9. — *For any $s \in \mathbb{R}$ and any $\xi = (p, q, \alpha) \in X^{1/2}$ the following identity holds true in the sense of quadratic forms in $Q(\widehat{H}_h^0)$,*

$$\begin{aligned} & \mathcal{W}_h(\xi) \widehat{H}_I(s) \mathcal{W}_h(\xi)^* \\ &= \sum_{i=1}^n \frac{1}{2m_i} \sum_{\nu=1}^d \left\{ \left(a^*(\widetilde{r}_i^\nu(s)) + \frac{i\hbar}{\sqrt{2}} \langle \alpha, \widetilde{r}_i^\nu(s) \rangle \right)^2 + \left(a(\widetilde{r}_i^\nu(s)) - \frac{i\hbar}{\sqrt{2}} \langle \widetilde{r}_i^\nu(s), \alpha \rangle \right)^2 \right. \\ & \quad \left. + 2 \left(a^*(\widetilde{r}_i^\nu(s)) + \frac{i\hbar}{\sqrt{2}} \langle \alpha, \widetilde{r}_i^\nu(s) \rangle \right) \left(a(\widetilde{r}_i^\nu(s)) - \frac{i\hbar}{\sqrt{2}} \langle \widetilde{r}_i^\nu(s), \alpha \rangle \right) + \hbar \|r_i^\nu\|_{L^2}^2 \right\} \\ & \quad - \sum_{i=1}^n \frac{1}{m_i} \sum_{\nu=1}^d \left(a^*(\widetilde{r}_i^\nu(s)) + \frac{i\hbar}{\sqrt{2}} \langle \alpha, \widetilde{r}_i^\nu(s) \rangle \right) (\widehat{p}_i^\nu - \hbar p_i^\nu) \\ & \quad - \sum_{i=1}^n \frac{1}{m_i} \sum_{\nu=1}^d (\widehat{p}_i^\nu - \hbar p_i^\nu) \left(a(\widetilde{r}_i^\nu(s)) - \frac{i\hbar}{\sqrt{2}} \langle \widetilde{r}_i^\nu(s), \alpha \rangle \right) \\ & \quad + V \left(\widehat{q}_1 + s\frac{\widehat{p}_1}{m_1} - \hbar \left(q_1 + s\frac{p_1}{m_1} \right), \dots, \widehat{q}_n + s\frac{\widehat{p}_n}{m_n} - \hbar \left(q_n + s\frac{p_n}{m_n} \right) \right). \end{aligned}$$

where

$$\widetilde{r}_i^\nu(s) = r_i^\nu(s) e^{2\pi i \hbar k \cdot (q_i + s\frac{p_i}{m_i})}.$$

Proof. — The stated identity is a consequence of the commutation relations

$$\begin{aligned} W(\alpha)a^*(f)W(\alpha)^* &= a^*(f) + i\frac{\hbar}{\sqrt{2}}\langle\alpha, f\rangle_{\mathfrak{H}^0}, \\ W(\alpha)a(f)W(\alpha)^* &= a(f) - i\frac{\hbar}{\sqrt{2}}\langle f, \alpha\rangle_{\mathfrak{H}^0}, \end{aligned}$$

and

$$\begin{aligned} \widehat{T}(z)\widehat{p}_i^\nu\widehat{T}(z)^* &= \widehat{p}_i^\nu - \hbar p_i^\nu, \\ \widehat{T}(z)\widehat{q}_i^\nu\widehat{T}(z)^* &= \widehat{q}_i^\nu - \hbar q_i^\nu, \end{aligned}$$

where $z \equiv (p, q) \in \mathbb{R}^{2dn}$. In particular, we have

$$\widehat{T}(z)r_i^\nu(s)\widehat{T}(z)^* = \widetilde{r}_i^\nu(s) \in \mathcal{L}(L^2(\mathbb{R}_{x_i}^d); L^2(\mathbb{R}_{x_i}^d) \otimes \mathfrak{H}^0). \quad \square$$

LEMMA 4.10. — *For any $s \in \mathbb{R}$ and any $\xi = (p, q, \alpha) \in X^{1/2}$ the following semiclassical commutator expansion holds true,*

$$(4.33) \quad \frac{1}{\hbar}[\mathcal{W}_\hbar(\xi), \widehat{H}_I(s)] = (M(s, \hbar, \xi) + \hbar R(s, \hbar, \xi))\mathcal{W}_\hbar(\xi)$$

as a quadratic form in $Q(\widehat{H}_\hbar^0)$ with the principal part M given by

$$\begin{aligned} (4.34) \quad M(s, \hbar, \xi) &= \sum_{i=1}^n \frac{1}{2m_i} \left[\frac{i}{\sqrt{2}} \sum_{\nu=1}^d \langle \alpha, r_i^\nu(s) \rangle a^*(r_i^\nu(s)) - \langle r_i^\nu(s), \alpha \rangle a(r_i^\nu(s)) \right] + \text{h.c.} \\ &\quad - \sum_{i=1}^n \frac{1}{m_i} \left[\frac{i}{\sqrt{2}} \sum_{\nu=1}^d a^*(r_i^\nu(s)) \langle r_i^\nu(s), \alpha \rangle \right] + \text{h.c.} \\ &\quad - \sum_{i=1}^n \frac{1}{m_i} \left[\frac{i}{\sqrt{2}} \sum_{\nu=1}^d \langle \alpha, r_i^\nu(s) \rangle \widehat{p}_i^\nu - \sum_{\nu=1}^d a^*(r_i^\nu(s)) p_i^\nu \right] + \text{h.c.} \\ &\quad - \nabla V \left(\widehat{q}_1 + s \frac{\widehat{p}_1}{m_1}, \dots, \widehat{q}_n + s \frac{\widehat{p}_n}{m_n} \right) \cdot \left(q_1 + s \frac{p_1}{m_1}, \dots, q_n + s \frac{p_n}{m_n} \right), \end{aligned}$$

and there exists a constant $c > 0$ so that for all $\hbar \in (0, 1)$, $s \in \mathbb{R}$ and $\xi \in X^{1/2}$,

$$(4.35) \quad \|(\widehat{H}_\hbar^0 + 1)^{-1/2} M(s, \hbar, \xi) (\widehat{H}_\hbar^0 + 1)^{-1/2}\|_{\mathcal{L}(\mathcal{H})} \leq c(1 + |s|)\|\xi\|_{X^0},$$

$$(4.36) \quad \|(\widehat{H}_\hbar^0 + 1)^{-1/2} R(s, \hbar, \xi) (\widehat{H}_\hbar^0 + 1)^{-1/2}\|_{\mathcal{L}(\mathcal{H})} \leq c(1 + |s|)^2(1 + \|\xi\|_{X^0})^2.$$

Proof. — The commutator expansion (4.33) is proved by using Lemma 4.9, the identity (4.32) and the Taylor expansion for \tilde{r}_i^ν and for the potential V with respect to \hbar , i.e.,

$$(4.37) \quad \tilde{r}_i^\nu(s) = r_i^\nu(s) \left(1 + \hbar \frac{e^{2\pi i \hbar k \cdot (q_i + s \frac{p_i}{m_i})} - 1}{\hbar} \right) = r_i^\nu(s) + \hbar g_i^\nu(\hbar, s),$$

and

$$(4.38) \quad V(\hat{q} + s\hat{\mathbf{p}} - \hbar(q + s\mathbf{p})) = V(\hat{q} + s\hat{\mathbf{p}}) - \hbar \nabla V(\hat{q} + s\hat{\mathbf{p}})(q + s\mathbf{p}) + \hbar^2 \theta(\hbar, s),$$

where $g_i^\nu(\hbar, s)$ and $\theta(\hbar, s)$ are the reminder terms, $\hat{\mathbf{p}} = \left(\frac{\hat{p}_i}{m_i} \right)_{i=1, \dots, n}$ and $\mathbf{p} = \left(\frac{p_i}{m_i} \right)_{i=1, \dots, n}$. In particular, using the elementary estimate $|e^{iy} - 1| \leq \sqrt{2}|y|$ for all $y \in \mathbb{R}$, one shows that g_i^ν satisfy the following uniform bounds with respect to $\hbar \in (0, 1)$ and $s \in \mathbb{R}$

$$\|g_i^\nu(\hbar, s)\|_{\mathcal{L}(L^2, L^2 \otimes \mathfrak{H}^0)} \leq M_1(1 + |s|) \|\xi\|_{X^0} \|\chi_i\|_{\mathfrak{H}^{1/2}},$$

$$\|\omega^{-1/2} g_i^\nu(\hbar, s)\|_{\mathcal{L}(L^2, L^2 \otimes \mathfrak{H}^0)} \leq M_1(1 + |s|) \|\xi\|_{X^0} \|\chi_i\|_{\mathfrak{H}^0},$$

for some constant $M_1 > 0$. On the other hand, θ satisfies for some constant M_2 ,

$$\|\theta(\hbar, s)\|_{\mathcal{L}(L^2(\mathbb{R}^{dn}))} \leq M_2(1 + |s|)^2 \|\xi\|_{X^0}^2,$$

uniformly with respect to $\hbar \in (0, 1)$ and $s \in \mathbb{R}$. Combining the latter estimates on g_i^ν and θ with the aid of Corollary A.5 and the assumptions (A_0) on the potential V and (A_1) – $(A_2^{(1)})$ on χ_i , one shows the estimates (4.35) and (4.36). \square

4.4. Characteristic equation

In this subsection, we take the classical limit $\hbar \rightarrow 0$ in the Duhamel formula previously derived in Proposition 4.7. Recall that $(\tilde{\varrho}_\hbar(t))_{\hbar \in (0,1)}$ and $(\varrho_\hbar(t))_{\hbar \in (0,1)}$ are given in (4.25).

LEMMA 4.11. — *Let $\sigma \in [\frac{1}{2}, 1]$ and assume (A_0) , (A_1) and $(A_2^{(\sigma)})$. Let $(\varrho_\hbar)_{\hbar \in (0,1)}$ be a family of density matrices on \mathcal{H} satisfying the assumptions $(S_0^{(1)})$, $(S_1^{(1)})$ and $(S_2^{(\sigma)})$. Then the families of states $(\varrho_\hbar(t))_{\hbar \in (0,1)}$ and $(\tilde{\varrho}_\hbar(t))_{\hbar \in (0,1)}$ defined according to (4.25) satisfy the same assumptions $(S_0^{(1)})$, $(S_1^{(1)})$ and $(S_2^{(\sigma)})$ uniformly for any $t \in \mathbb{R}$ in an arbitrary compact time interval.*

Proof. — First, note that

$$\begin{aligned}\mathrm{Tr}[\tilde{\varrho}_{\hbar}(t)\hat{H}_{\hbar}^0] &= \mathrm{Tr}[\varrho_{\hbar}(t)\hat{H}_{\hbar}^0], \\ \mathrm{Tr}[\tilde{\varrho}_{\hbar}(t)d\Gamma(|k|^{2\sigma})] &= \mathrm{Tr}[\varrho_{\hbar}(t)d\Gamma(|k|^{2\sigma})], \\ \mathrm{Tr}[\tilde{\varrho}_{\hbar}(t)\hat{N}_{\hbar}] &= \mathrm{Tr}[\varrho_{\hbar}(t)\hat{N}_{\hbar}].\end{aligned}$$

Thus, it is enough to bound only the right-hand sides. By spectral decomposition,

$$\varrho_{\hbar}(t) = \sum_{m \in \mathbb{N}} \lambda_{\hbar}(m) |e^{-i\frac{t}{\hbar}\hat{H}_{\hbar}} \psi_{\hbar}(m)\rangle \langle e^{-i\frac{t}{\hbar}\hat{H}_{\hbar}} \psi_{\hbar}(m)|,$$

and

$$\mathrm{Tr}[\varrho_{\hbar}(t)\hat{N}_{\hbar}] = \sum_{m \in \mathbb{N}} \lambda_{\hbar}(m) \|(\hat{N}_{\hbar})^{1/2} e^{-i\frac{t}{\hbar}\hat{H}_{\hbar}} \psi_{\hbar}(m)\|^2.$$

Hence using Corollary 3.11 one shows for some constants $K, C > 0$,

$$\mathrm{Tr}[\varrho_{\hbar}(t)\hat{N}_{\hbar}] \leq K (\mathrm{Tr}[\varrho_{\hbar}\hat{N}_{\hbar}] + \mathrm{Tr}[\varrho_{\hbar}(\hat{H}_{\hbar}^0 + 1)]) e^{C|t|},$$

uniformly in $\hbar \in (0, 1)$. Similarly, $\varrho_{\hbar}(t)$ satisfies the assumption $(S_2^{(\sigma)})$ by Corollary 3.11 and the spectral decomposition. Now, using Corollary 3.3 one deduces

$$\begin{aligned}\mathrm{Tr}[\varrho_{\hbar}(t)\hat{H}_{\hbar}^0] &= \sum_{m \in \mathbb{N}} \lambda_{\hbar}(m) \|(\hat{H}_{\hbar}^0)^{1/2} e^{-i\frac{t}{\hbar}\hat{H}_{\hbar}} \psi_{\hbar}(m)\|^2 \\ &\lesssim \sum_{m \in \mathbb{N}} \lambda_{\hbar}(m) \|(\hat{H}_{\hbar} + a)^{1/2} \psi_{\hbar}(m)\|^2 \\ (4.39) \quad &\lesssim \sum_{m \in \mathbb{N}} \lambda_{\hbar}(m) \|(\hat{H}_{\hbar}^0 + b)^{1/2} \psi_{\hbar}(m)\|^2 \\ &\lesssim \mathrm{Tr}[\varrho_{\hbar}(\hat{H}_{\hbar}^0 + 1)].\end{aligned}$$

On the other hand, thanks to Lemma 3.122 the estimate

$$(4.40) \quad \mathrm{Tr}[\varrho_{\hbar}(t)\hat{q}^2] \leq K_1 \mathrm{Tr}[\varrho_{\hbar}(\hat{H}_{\hbar}^0 + \hat{q}^2 + 1)] e^{c_1|t|}$$

holds true for some constant $K_1, c_1 > 0$ and uniformly in $\hbar \in (0, 1)$. Finally, applying Lemma 3.121 for the state $\varrho_{\hbar}(t)$ with the help of its spectral decomposition, one proves the estimate

$$\begin{aligned}\mathrm{Tr}[\tilde{\varrho}_{\hbar}(t)\hat{q}^2] &\leq K_1 \mathrm{Tr}[\varrho_{\hbar}(t)(\hat{H}_{\hbar}^0 + \hat{q}^2 + 1)] e^{c_1|t|} \\ &\leq K_2 \mathrm{Tr}[\varrho_{\hbar}(\hat{H}_{\hbar}^0 + \hat{q}^2 + 1)] e^{c_2|t|}.\end{aligned}$$

For the last inequality we have used (4.39) and (4.40). \square

LEMMA 4.12. — *Let $(\varrho_{\hbar})_{\hbar \in (0,1)}$ be a family of density matrices on \mathcal{H} satisfying $(S_0^{(1)})$, $(S_1^{(1)})$ and $(S_2^{(\sigma)})$ for some $\sigma \in [\frac{1}{2}, 1]$. Then for any compact time interval J there exists $C > 0$ such that for any Wigner measures $\mu_t \in \mathcal{M}(\varrho_{\hbar}(t), \hbar \in (0, 1))$ and $\tilde{\mu}_t \in \mathcal{M}(\tilde{\varrho}_{\hbar}(t), \hbar \in (0, 1))$, and for all $t \in J$,*

- (1) $\int_{X^0} \|u\|_{X^\sigma}^2 d\mu_t < C,$
- (2) $\int_{X^0} \|u\|_{X^\sigma}^2 d\tilde{\mu}_t < C.$

Proof. — The result is a consequence of Lemma 4.11 and a general argument related to Wick calculus and semiclassical analysis elaborated in [10, Lemma 3.12]. In fact, according to the latter reference if one verifies that a family of density matrices $(\rho_{\hbar})_{\hbar \in (0,1)}$ on \mathcal{H} satisfies

$$\mathrm{Tr} \left[\rho_{\hbar} \hat{N}_{\hbar} \right] \leq C, \quad \mathrm{Tr} \left[\rho_{\hbar} d\Gamma(|k|^{2\sigma}) \right] \leq C, \quad \mathrm{Tr} \left[\rho_{\hbar} (\hat{p}^2 + \hat{q}^2) \right] \leq C,$$

for some constant $C > 0$ uniformly in $\hbar \in (0, 1)$, then it holds true that for any Wigner measure $\mu \in \mathcal{M}(\rho_{\hbar}, \hbar \in (0, 1))$,

$$\int_{X^0} \|\alpha\|_{\mathfrak{H}^0}^2 d\mu \leq C, \quad \int_{X^0} \|\alpha\|_{\mathfrak{H}^\sigma}^2 d\mu \leq C, \quad \int_{X^0} (p^2 + q^2) d\mu \leq C.$$

Thus, applying such result for $\varrho_{\hbar}(t)$ and $\tilde{\varrho}_{\hbar}(t)$ with the help of the uniform estimates in Lemma 4.11, one proves the claimed bounds. \square

LEMMA 4.13. — *Let $(\varrho_{\hbar})_{\hbar \in (0,1)}$ be a family of density matrices satisfying $(S_0^{(\delta)})$. Then for any compact time interval J there exists $C > 0$ such that for all $\xi \in X^{1/2}$ and all $\hbar \in (0, 1)$,*

$$(4.41) \quad \left| \mathrm{Tr} [\mathcal{W}_{\hbar}(\xi) \tilde{\varrho}_{\hbar}(t)] - \mathrm{Tr} [\mathcal{W}_{\hbar}(\xi) \tilde{\varrho}_{\hbar}(s)] \right| \leq C |t - s| (1 + \|\xi\|_{X^0})^2$$

for any $t, s \in J$.

Proof. — Applying Proposition 4.7 and Lemma 4.10, one concludes

$$\begin{aligned} & \left| \mathrm{Tr} [\mathcal{W}_{\hbar}(\xi) \tilde{\varrho}_{\hbar}(t)] - \mathrm{Tr} [\mathcal{W}_{\hbar}(\xi) \tilde{\varrho}_{\hbar}(t_0)] \right| \\ & \lesssim \int_{t_0}^t \left\| (\hat{H}_{\hbar}^0 + 1)^{-1/2} (M(s, \hbar, \xi) + \hbar R(s, \hbar, \xi)) (\hat{H}_{\hbar}^0 + 1)^{-1/2} \right\|_{\mathcal{L}(\mathcal{H})} ds. \end{aligned}$$

Note that in the last inequality we have also used Lemma 4.5. Now, using the bounds (4.35) and (4.36) one obtains for any $t, t_0 \in J$,

$$\begin{aligned} \left| \mathrm{Tr} [\mathcal{W}_{\hbar}(\xi) \tilde{\varrho}_{\hbar}(t)] - \mathrm{Tr} [\mathcal{W}_{\hbar}(\xi) \tilde{\varrho}_{\hbar}(t_0)] \right| & \leq c \int_{t_0}^t (1 + |s|)^2 (1 + \|\xi\|_{X^0})^2 ds \\ & \leq C |t - t_0| (1 + \|\xi\|_{X^0})^2. \end{aligned} \quad \square$$

LEMMA 4.14. — *Let $(\varrho_{\hbar})_{\hbar \in (0,1)}$ be a family of density matrices satisfying $(S_0^{(1)})$ and $(S_1^{(1)})$. Then for any compact time interval J there exists $C > 0$ such that for all $\xi_1, \xi_2 \in X^{1/2}$, all $\hbar \in (0,1)$ and all $t \in J$,*

$$\begin{aligned} & \left| \text{Tr}[\mathcal{W}_{\hbar}(\xi_1)\tilde{\varrho}_{\hbar}(t)] - \text{Tr}[\mathcal{W}_{\hbar}(\xi_2)\tilde{\varrho}_{\hbar}(t)] \right| \\ & \leq C\|\xi_1 - \xi_2\|_{X^0}(\|\xi_1\|_{X^0} + \|\xi_2\|_{X^0} + 1). \end{aligned}$$

Proof. — We shall use the following estimate:

$$\begin{aligned} & \|(\mathcal{W}_{\hbar}(\xi_1) - \mathcal{W}_{\hbar}(\xi_2))(\hat{N}_{\hbar} + 1)^{-1/2}\|_{\mathcal{L}(\mathcal{H})} \\ & \leq c\|\xi_1 - \xi_2\|_{X^0}(\|\xi_1\|_{X^0} + \|\xi_2\|_{X^0} + 1). \end{aligned}$$

See for instance [8, Lemma 3.1] for a proof of such inequality. This yields

$$\begin{aligned} & \left| \text{Tr}[(\mathcal{W}_{\hbar}(\xi_1) - \mathcal{W}_{\hbar}(\xi_2))\tilde{\varrho}_{\hbar}(t)] \right| \\ & \leq \|(\mathcal{W}_{\hbar}(\xi_1) - \mathcal{W}_{\hbar}(\xi_2))(\hat{N}_{\hbar} + 1)^{-1/2}\|_{\mathcal{L}(\mathcal{H})} \|(\hat{N}_{\hbar} + 1)^{1/2}\tilde{\varrho}_{\hbar}(t)\|_{\mathcal{L}^1(\mathcal{H})}. \end{aligned}$$

Now, assumptions $(S_0^{(1)})$ – $(S_1^{(1)})$ and Lemma 4.11 imply that there exists a constant $c > 0$ such that for all times $t \in J$,

$$(4.42) \quad \text{Tr}[\tilde{\varrho}_{\hbar}(t)(\hat{N}_{\hbar} + 1)] \leq c,$$

uniformly with respect to $\hbar \in (0,1)$. (4.42) ensures that

$$\left\| (\hat{N}_{\hbar} + 1)^{1/2}\tilde{\varrho}_{\hbar}(t) \right\|_{\mathcal{L}^1(\mathcal{H})}$$

is uniformly bounded for $t \in J$ and $\hbar \in (0,1)$. Then the proof is complete. \square

4.4.1. Extraction

For each time $t_0 \in \mathbb{R}$, it is possible thanks to Definition 1.1 and Proposition 4.1 to extract a sequence $(\hbar_k)_{k \in \mathbb{N}}$ — which may depend on the time t_0 — such that $\mathcal{M}(\tilde{\varrho}_{\hbar_k}, k \in \mathbb{N})$ is a singleton. However, for our analysis we need to find a sequence $(\hbar_k)_{k \in \mathbb{N}}$ such that for all times $\mathcal{M}(\tilde{\varrho}_{\hbar_k}(t), k \in \mathbb{N})$ is a singleton. Such property is proved below and it is based on a diagonal extraction procedure and Prokhorov's theorem. In particular, we shall use the following weak narrow convergence topology on $\mathfrak{P}(X^\sigma)$. Let $(e_n)_{n \in \mathbb{N}}$ be a O.N.B of the Hilbert space X^σ and define the norm,

$$(4.43) \quad \|u\|_{X_w^\sigma}^2 = \sum_{n \in \mathbb{N}} \frac{1}{n^2} |\langle u, e_n \rangle_{X^\sigma}|^2,$$

on X^σ .

DEFINITION 4.15. — A curve $\mathbb{R} \ni t \mapsto \mu_t \in \mathfrak{P}(X^\sigma)$ is said to be weakly narrowly continuous if and only if the real-valued maps,

$$\mathbb{R} \ni t \mapsto \int_{X^\sigma} \psi \, d\mu_t \in \mathbb{R},$$

are continuous for every bounded continuous function ψ on $(X^\sigma, \|\cdot\|_{X_w^\sigma})$.

A sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\mathfrak{P}(X^\sigma)$ is said to be weakly narrowly convergent to $\mu \in \mathfrak{P}(X^\sigma)$ if and only if

$$\lim_{n \rightarrow \infty} \int_{X^\sigma} \psi \, d\mu_n = \int_{X^\sigma} \psi \, d\mu,$$

for every bounded continuous function ψ on $(X^\sigma, \|\cdot\|_{X_w^\sigma})$.

PROPOSITION 4.16. — Let $\sigma \in [\frac{1}{2}, 1]$ and assume that (A_0) , (A_1) and $(A_2^{(\sigma)})$ are satisfied. Let $(\varrho_{\hbar})_{\hbar \in (0,1)}$ be a family of density matrices in \mathcal{H} satisfying the assumptions $(S_0^{(1)})$, $(S_1^{(1)})$ and $(S_2^{(\sigma)})$. For any sequence $(\hbar_n)_{n \in \mathbb{N}}$ in $(0, 1)$ such that $\hbar_n \rightarrow 0$, there exist a subsequence $(\hbar_{n_k})_{k \in \mathbb{N}}$ and a family of probability measures $(\tilde{\mu}_t)_{t \in \mathbb{R}}$ such that for all $t \in \mathbb{R}$,

$$\mathcal{M}(\tilde{\varrho}_{\hbar_{n_k}}(t), k \in \mathbb{N}) = \{\tilde{\mu}_t\}.$$

Moreover, for every compact time interval J there exists a constant $C > 0$ such that for all times $t \in J$,

$$(4.44) \quad \int_{X^0} \|u\|_{X^\sigma}^2 \, d\tilde{\mu}_t(u) < C.$$

Proof. — Denote for $\xi = (p, q, \alpha) \in X^0$,

$$G_{\hbar}(t, \xi) = \text{Tr} \left[\tilde{\varrho}_{\hbar}(t) \mathcal{W}_{\hbar}(2\pi q, -2\pi p, \sqrt{2}\pi\alpha) \right].$$

For every compact interval J there exists $C > 0$ such that for every $t, s \in J$ and every $\xi_1, \xi_2 \in X^0$,

$$(4.45) \quad \begin{aligned} & |G_{\hbar}(t, \xi_1) - G_{\hbar}(s, \xi_2)| \\ & \leq |G_{\hbar}(t, \xi_1) - G_{\hbar}(s, \xi_1)| + |G_{\hbar}(s, \xi_1) - G_{\hbar}(s, \xi_2)| \\ & \leq C|t - s| (1 + \|\xi_1\|_{X^0})^2 + C\|\xi_1 - \xi_2\|_{X^0} (\|\xi_1\|_{X^0} + \|\xi_2\|_{X^0} + 1). \end{aligned}$$

The last inequality is a consequence of Lemmas 4.13 and 4.14. Now take a countable dense set $(t_j)_{j \in \mathbb{N}}$ in \mathbb{R} . Then by Proposition 4.1 for t_1 there exist a sequence $(\hbar_\ell)_{\ell \in \mathbb{N}}$ and a probability measure $\tilde{\mu}_{t_1}$ such that

$$\mathcal{M}(\tilde{\varrho}_{\hbar_\ell}(t_1), \ell \in \mathbb{N}) = \{\tilde{\mu}_{t_1}\}.$$

Repeating the same argument for each t_2, \dots, t_j, \dots and extracting at each time a subsequence from the previous one, one obtains the sequences $(\hbar_{\psi_2(\ell)})_{\ell \in \mathbb{N}}, \dots, (\hbar_{\psi_j(\ell)})_{\ell \in \mathbb{N}}, \dots$ such that

$$\mathcal{M}(\tilde{\varrho}_{\hbar_{\psi_j(\ell)}}(t_j), \ell \in \mathbb{N}) = \{\tilde{\mu}_{t_j}\},$$

and such that $\psi_k : \mathbb{N} \rightarrow \mathbb{N}$ are successive extractions. Hence by the diagonal argument there exists a sequence — given precisely by $(\hbar_{\psi_\ell(\ell)})_{\ell \in \mathbb{N}}$ — still denoted by $(\hbar_\ell)_{\ell \in \mathbb{N}}$ for simplicity such that for all $j \in \mathbb{N}$,

$$\mathcal{M}(\tilde{\varrho}_{\hbar_\ell}(t_j), \ell \in \mathbb{N}) = \{\tilde{\mu}_{t_j}\}.$$

So, according to Definition 1.1 this means that for any $\xi \in X^0$ and $j \in \mathbb{N}$,

$$\begin{aligned} G(t_j, \xi) &= \lim_{\ell \rightarrow \infty} \text{Tr}[\tilde{\varrho}_{\hbar_\ell}(t_j) \mathcal{W}_{\hbar_\ell}(2\pi q, -2\pi p, \sqrt{2\pi}\alpha)] \\ &= \int_{X^0} e^{2\pi i \text{Re}\langle \xi, u \rangle_{X^0}} d\tilde{\mu}_{t_j}(u). \end{aligned}$$

Consider the family $(\tilde{\varrho}_{\hbar_\ell}(t_j))_{\ell \in \mathbb{N}}$ with the diagonal extracted sequence $(\hbar_\ell)_{\ell \in \mathbb{N}}$ constructed above; then the estimate (4.41) yields,

$$|G(t_j, \xi) - G(t_{j'}, \xi)| \leq C|t_j - t_{j'}|(1 + \|\xi\|_{X^0})^2.$$

So, by a density argument and completeness one extends the function $G(\cdot, \xi)$ to the function on \mathbb{R} such that

$$G(t, \xi) = \lim_{t_j \rightarrow t} G(t_j, \xi).$$

The limit is independent of the choice of the sequence $t_j \rightarrow t$ and satisfies as in (4.45) the estimate,

$$\begin{aligned} |G(t, \xi_1) - G(s, \xi_2)| \\ \leq C|t - s|(1 + \|\xi_1\|_{X^0})^2 + C\|\xi_1 - \xi_2\|_{X^0}(\|\xi_1\|_{X^0} + \|\xi_2\|_{X^0} + 1). \end{aligned}$$

Moreover, by the triangle inequality and (4.45) one deduces that for all $t \in \mathbb{R}$ and $\xi \in X^0$,

$$(4.46) \quad \lim_{\ell \rightarrow \infty} G_{\hbar_\ell}(t, \xi) = \lim_{\ell \rightarrow \infty} \text{Tr}[\tilde{\varrho}_{\hbar_\ell}(t) \mathcal{W}_{\hbar_\ell}(2\pi q, -2\pi p, \sqrt{2\pi}\alpha)] = G(t, \xi).$$

Remark that by Lemma 4.122 one verifies that for all t_j in an arbitrary compact interval,

$$(4.47) \quad \int_{X^0} \|u\|_{X^\sigma}^2 d\tilde{\mu}_{t_j} < \infty.$$

Now, we claim that for all $t \in \mathbb{R}$ there exists $\tilde{\mu}_t \in \mathfrak{P}(X^0)$ such that for all $\xi \in X^0$,

$$G(t, \xi) = \int_{X^0} e^{2\pi i \operatorname{Re}\langle \xi, u \rangle_{X^0}} d\tilde{\mu}_t(u).$$

Of course, the above statement holds true for all t_j and we need to extend to every time t . On the other hand, we know that the limit (4.46) exists for all times and so it is enough to identify $G(t, \xi)$ as the characteristic function of a given probability measure over X^0 . In fact, thanks to the bound (4.47) one notices that the family $(\tilde{\mu}_{t_j})_{j \in \mathbb{N}}$ is tight in $\mathfrak{P}(X^0)$, where X^0 is endowed with the norm $\|\cdot\|_{X_w^0}$. Hence by Prokhorov's theorem such a family is sequentially compact with respect to the weak narrow convergence topology on $\mathfrak{P}(X^0)$ (see, e.g., [2]). Therefore, for any $t \in \mathbb{R}$ and $(t_{\psi(j)})_{j \in \mathbb{N}}$ a subsequence such that $\lim_{j \rightarrow \infty} t_{\psi(j)} = t$ there exists a subsequence — still denote the same notation — so that $\tilde{\mu}_{\psi(j)}$ converges towards a Borel probability measure $\tilde{\mu}_t$. Consequentially, for every time t and for every $\xi \in X^0$,

$$\lim_{j \rightarrow \infty} G(t_{\psi(j)}, \xi) = G(t, \xi) = \int_{X^0} e^{2\pi i \operatorname{Re}\langle \xi, u \rangle_{X^0}} d\tilde{\mu}_t,$$

since the complex-valued function $u \mapsto e^{2\pi i \operatorname{Re}\langle \xi, u \rangle_{X^0}}$ is bounded continuous relatively to the norm $\|\cdot\|_{X_w^0}$. Hence from (4.46) one deduces for all $t \in \mathbb{R}$ and $\xi = (p, q, \alpha) \in X^0$,

$$\lim_{\ell \rightarrow \infty} \operatorname{Tr} \left[\tilde{\varrho}_{\hbar_\ell}(t) \mathcal{W}_{\hbar_\ell}(2\pi q, -2\pi p, \sqrt{2\pi}\alpha) \right] = \int_{X^0} e^{2\pi i \operatorname{Re}\langle \xi, u \rangle_{X^0}} d\tilde{\mu}_t. \quad \square$$

4.4.2. Convergence

We are in position to prove convergence of the Duhamel formula (4.27) when $\hbar \rightarrow 0$ and to derive the characteristic equation satisfied by the Wigner measures $(\tilde{\mu}_t)_{t \in \mathbb{R}}$.

LEMMA 4.17. — *Suppose the same hypothesis as in Proposition 4.16. Then for all $t, t_0 \in \mathbb{R}$ and all $\xi = (z, \alpha) \in X^0$,*

$$\begin{aligned} \tilde{\mu}_t(e^{i\Im m\langle \cdot, z \rangle} e^{\sqrt{2}i \operatorname{Re}\langle \alpha, \cdot \rangle_{\mathcal{H}^0}}) &= \tilde{\mu}_{t_0}(e^{i\Im m\langle \cdot, z \rangle} e^{\sqrt{2}i \operatorname{Re}\langle \alpha, \cdot \rangle_{\mathcal{H}^0}}) \\ &\quad - i \lim_{k \rightarrow \infty} \int_{t_0}^t \operatorname{Tr} [M(s, \hbar_{n_k}, \xi) \mathcal{W}_{\hbar_{n_k}}(\xi) \tilde{\varrho}_{\hbar_{n_k}}(s)] ds, \end{aligned}$$

where the term $M(s, \hbar, \xi)$ is given explicitly in Lemma 4.10.

Proof. — Apply the Duhamel's formula of Proposition 4.7 with aid of Lemma 4.10. \square

The main step next is to use the dominated convergence theorem and to calculate the limit

$$\lim_{k \rightarrow \infty} \text{Tr}[M(s, \hbar_{n_k}, \xi) \mathcal{W}_{\hbar_{n_k}}(\xi) \tilde{\varrho}_{\hbar_{n_k}}(s)],$$

explicitly as integrals with respect to the measures $\tilde{\mu}_s$. This leads to an equation describing the time-evolution of characteristic functions of $\tilde{\mu}_s$.

LEMMA 4.18. — *Suppose the same hypothesis as in Proposition 4.16. Then for all $\xi = (p_0, q_0, \alpha_0) \equiv (z_0, \alpha_0) \in X^0$,*

$$(4.48) \quad \begin{aligned} \mathcal{G}(s, \xi) &= \lim_{k \rightarrow \infty} \text{Tr}[M(s, \hbar_{n_k}, \xi) \mathcal{W}_{\hbar_{n_k}}(\xi) \tilde{\varrho}_{\hbar_{n_k}}(s)] \\ &= \int_{X^0} m(s, \xi) e^{i\Im m\langle q+ip, z_0 \rangle} e^{\sqrt{2}i\Re\langle \alpha_0, \alpha \rangle_{S^0}} d\tilde{\mu}_s(u), \end{aligned}$$

with the variable $u = (p, q, \alpha) \in X^0$ and

$$(4.49) \quad \begin{aligned} m(s, \xi) &= \sum_{i=1}^n \frac{1}{2m_i} \left[\frac{i}{\sqrt{2}} \sum_{\nu=1}^d (\langle \alpha_0, F_i^\nu(s) \rangle \langle \alpha, F_i^\nu(s) \rangle - \langle F_i^\nu(s), \alpha_0 \rangle \langle F_i^\nu(s), \alpha \rangle) \right] + \text{h.c.} \\ &\quad - \sum_{i=1}^n \frac{1}{m_i} \left[\frac{i}{\sqrt{2}} \sum_{\nu=1}^d \langle \alpha, F_i^\nu(s) \rangle \langle F_i^\nu(s), \alpha_0 \rangle \right] + \text{h.c.} \\ &\quad - \sum_{i=1}^n \frac{1}{m_i} \left[\frac{i}{\sqrt{2}} \sum_{\nu=1}^d \langle \alpha_0, F_i^\nu(s) \rangle p_i^\nu - \sum_{\nu=1}^d \langle \alpha, F_i^\nu(s) \rangle p_{0,i}^\nu \right] + \text{h.c.} \\ &\quad - \sum_{j=1}^n \nabla_{x_j} V \left(q_1 + s \frac{p_1}{m_1}, \dots, q_n + s \frac{p_n}{m_n} \right) \cdot \left(q_{0,j} + s \frac{p_{0,j}}{m_j} \right), \end{aligned}$$

where

$$F_i^\nu(s) = F_i^\nu(s)(p, q, k) = \left(\frac{\chi_i(k)}{\sqrt{2|k|}} \epsilon_\lambda^\nu(k) e^{-2\pi i k \cdot (q_i + s \frac{p_i}{m_i}) + i s |k|} \right)_{\lambda=1, \dots, d-1}.$$

Proof. — One needs to compute the limit for each term in the right-hand side of (4.34). However, there are essentially three types:

$$\begin{aligned} \mathcal{Y}_1 &= \nabla_{x_j} V \left(\hat{q}_1 + s \frac{\hat{p}_1}{m_1}, \dots, \hat{q}_n + s \frac{\hat{p}_n}{m_n} \right) \cdot \left(q_{0,j} + s \frac{p_{0,j}}{m_j} \right), \\ \mathcal{Y}_2 &= \langle \alpha_0, r_i^\nu(s) \rangle \hat{p}_i^\nu, \\ \mathcal{Y}_3 &= \langle \alpha_0, r_i^\nu(s) \rangle a^*(r_i^\nu(s)), \end{aligned}$$

and the others are similar. So, we only provide here the limits for $\mathcal{Y}_1 - \mathcal{Y}_3$. Note that

$$(4.50) \quad \mathcal{Y}_1 = e^{i \frac{s}{\hbar} \hat{H}_\hbar^0} f(\hat{q}) e^{-i \frac{s}{\hbar} \hat{H}_\hbar^0}$$

with $f(x) = \nabla_{x_j} V(x) \cdot \left(q_{0,j} + s \frac{p_{0,j}}{m_j} \right)$. Hence one obtains

$$\begin{aligned} \text{Tr}[\mathcal{Y}_1 \mathcal{W}_{\hbar_{n_k}}(\xi) \tilde{\varrho}_{\hbar_{n_k}}(s)] &= \text{Tr}[f(\hat{q}) e^{-i \frac{s}{\hbar} \hat{H}_\hbar^0} \mathcal{W}_{\hbar_{n_k}}(\xi) \tilde{\varrho}_{\hbar_{n_k}}(s) e^{i \frac{s}{\hbar} \hat{H}_\hbar^0}] \\ &= \text{Tr}[f(\hat{q}) \mathcal{W}_{\hbar_{n_k}}(\Phi_s^0(\xi)) \varrho_{\hbar_{n_k}}(s)]. \end{aligned}$$

On the other hand, one knows according to Lemma C.4 that $(\varrho_{\hbar_{n_k}}(s))_{k \in \mathbb{N}}$ admits a unique Wigner measure μ_s and $(\Phi_s^0)_\# \tilde{\mu}_s = \mu_s$ (Φ_s^0 is the linear free flow defined on X^0 by (C.11)). So, since $V \in \mathcal{C}^2(\mathbb{R}^{dn})$ and ∇V is bounded, one can use Lemma C.1 and equation (C.2b), and deduce

$$\begin{aligned} \lim_{k \rightarrow \infty} \text{Tr}[f(\hat{q}) \mathcal{W}_{\hbar_{n_k}}(\Phi_s^0(\xi)) \varrho_{\hbar_{n_k}}(s)] \\ = \int_{X^0} e^{i \Im \langle z, \Phi_s^0(z_0) \rangle} e^{\sqrt{2} i \Re \langle \Phi_s^0(\alpha_0), \alpha \rangle_{\mathcal{H}^0}} f(q) d\mu_s(u) \\ = \int_{X^0} e^{i \Im \langle z, z_0 \rangle} e^{\sqrt{2} i \Re \langle \alpha_0, \alpha \rangle_{\mathcal{H}^0}} f(\Phi_s^0(q)) d\tilde{\mu}_s(u). \end{aligned}$$

Note that here $\Phi_s^0(z_0)$, $\Phi_s^0(\alpha)$ and $\Phi_s^0(q)$ designate respectively the action of Φ_s^0 on those components. Hence we recover the claimed limit related to (i) by replacing f and Φ_s^0 with their values given respectively in (4.50) and (C.11).

For the second term \mathcal{Y}_2 , one writes

$$\begin{aligned} \langle \alpha_0, r_i^\nu(s) \rangle &= \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \overline{\alpha_0(k)} \frac{\chi_i(k)}{\sqrt{2|k|}} \epsilon_\lambda^\nu(k) e^{-2\pi i k \cdot (\hat{q}_i + s \frac{\hat{p}_i}{m_i}) + i s |k|} dk \\ &= \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} g_\lambda^i(k) \hat{T}(-2\pi k, \frac{2\pi s}{m_i} k) dk, \end{aligned}$$

where $g_\lambda^i(k) = \overline{\alpha_0(k)} \frac{\chi_i(k)}{\sqrt{2|k|}} \epsilon_\lambda^\nu(k) e^{i s |k|}$ belongs to $L^1(\mathbb{R}^d)$. In particular, Fubini's theorem yields

$$\begin{aligned} \text{Tr}[\mathcal{Y}_2 \mathcal{W}_{\hbar_{n_k}}(\xi) \tilde{\varrho}_{\hbar_{n_k}}(s)] \\ = \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} g_\lambda^i(k) \text{Tr} \left[\hat{T} \left(-2\pi k, \frac{2\pi s}{m_i} k \right) \hat{p}_i^\nu \mathcal{W}_{\hbar_{n_k}}(\xi) \tilde{\varrho}_{\hbar_{n_k}}(s) \right] dk \\ = \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} g_\lambda^i(k) \text{Tr} \left[\hat{p}_i^\nu \hat{T} \left(-2\pi k, \frac{2\pi s}{m_i} k \right) \mathcal{W}_{\hbar_{n_k}}(\xi) \tilde{\varrho}_{\hbar_{n_k}}(s) \right] dk + \mathcal{O}(\hbar_{n_k}). \end{aligned}$$

The last identity is a consequence of the commutation relations (4.3). So, using the dominated convergence theorem with the aid of the assumption $(S_0^{(1)})$ and Lemma C.1 and equation (C.2a) with Fubini's theorem one deduces

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \operatorname{Tr} [\mathcal{Y}_2 \mathcal{W}_{\hbar_{n_k}}(\xi) \tilde{\varrho}_{\hbar_{n_k}}(s)] \\
&= \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} g_{\lambda}^i(k) \lim_{k \rightarrow \infty} \operatorname{Tr} \left[\hat{p}_i^{\nu} \mathcal{W}_{\hbar_{n_k}} \left(p_0 - 2\pi k, q_0 + \frac{2\pi s}{m_i} k, \alpha_0 \right) \tilde{\varrho}_{\hbar_{n_k}}(s) \right] dk \\
&= \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} g_{\lambda}^i(k) \left(\int_{X^0} p_i^{\nu} e^{2iQ(k,p,q)} e^{2i\Im \langle q+ip, z_0 \rangle} e^{\sqrt{2}i\Re \langle \alpha_0, \alpha \rangle_{\mathfrak{S}^0}} d\tilde{\mu}_s(u) \right) dk \\
&= \int_{X^0} \langle \alpha_0, F_i^{\nu}(s) \rangle p_i^{\nu} e^{2i\Im \langle q+ip, z_0 \rangle} e^{\sqrt{2}i\Re \langle \alpha_0, \alpha \rangle_{\mathfrak{S}^0}} d\tilde{\mu}_s(u)
\end{aligned}$$

with

$$Q(k, p, q) = -\pi k \cdot \left(q_i + s \frac{p_i}{m_i} \right).$$

For the last term \mathcal{Y}_3 , one writes

$$\begin{aligned}
& \operatorname{Tr} [\mathcal{Y}_3 \mathcal{W}_{\hbar_{n_k}}(\xi) \tilde{\varrho}_{\hbar_{n_k}}(s)] \\
&= \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} g_{\lambda}^i(k) \operatorname{Tr} \left[\underbrace{\hat{T} \left(-2\pi k, \frac{2\pi s}{m_i} k \right) a^*(r_i^{\nu}(s)) \mathcal{W}_{\hbar_{n_k}}(\xi) \tilde{\varrho}_{\hbar_{n_k}}(s)}_{=\mathcal{T}(s, \xi)} \right] dk
\end{aligned}$$

Hence the dominated convergence theorem and Lemma C.3 give

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \mathcal{T}(s, \xi) \\
&= \int_{X^0} e^{i\Im \langle q+ip, z_0 \rangle} e^{\sqrt{2}i\Re \langle \alpha_0, \alpha \rangle_{\mathfrak{S}^0}} e^{-2\pi i k \cdot (q_i + s \frac{p_i}{m_i})} \langle \alpha, F_i^{\nu}(s) \rangle_{\mathfrak{S}^0} d\tilde{\mu}_s(u).
\end{aligned}$$

So, by Fubini's theorem

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \operatorname{Tr} [\mathcal{Y}_3 \mathcal{W}_{\hbar_{n_k}}(\xi) \tilde{\varrho}_{\hbar_{n_k}}(s)] \\
&= \int_{X^0} e^{i\Im \langle q+ip, z_0 \rangle} e^{\sqrt{2}i\Re \langle \alpha_0, \alpha \rangle_{\mathfrak{S}^0}} \langle \alpha_0, r_i^{\nu}(s) \rangle \langle \alpha, F_i^{\nu}(s) \rangle_{\mathfrak{S}^0} d\tilde{\mu}_s(u)
\end{aligned}$$

with the variable $u = (p, q, \alpha) \in X^0$ and $\xi = (p_0, q_0, \alpha_0) \in X^0$. \square

So, we obtain below the intended *characteristic equation* as a consequence of Lemmas 4.17 and 4.18.

PROPOSITION 4.19. — *Let $\sigma \in [\frac{1}{2}, 1]$ and assume that (A_0) , (A_1) and $(A_2^{(\sigma)})$ are satisfied. Let $(\varrho_{\hbar})_{\hbar \in (0,1)}$ be a family of density matrices in*

\mathcal{H} satisfying the assumptions $(S_0^{(1)})$, $(S_1^{(1)})$ and $(S_2^{(\sigma)})$. Then the Wigner measures $(\tilde{\mu}_t)_{t \in \mathbb{R}}$ provided by Proposition 4.16 satisfy the following characteristic equation for all $t, t_0 \in \mathbb{R}$ and all $\xi = (z, \alpha) \in X^0$,

$$(4.51) \quad \begin{aligned} \tilde{\mu}_t(e^{i\Im m\langle \cdot, z \rangle} e^{\sqrt{2}i\Re e\langle \alpha, \cdot \rangle_{S^0}}) \\ = \tilde{\mu}_{t_0}(e^{i\Im m\langle \cdot, z \rangle} e^{\sqrt{2}i\Re e\langle \alpha, \cdot \rangle_{S^0}}) - i \int_{t_0}^t \mathcal{G}(s, \xi) ds \end{aligned}$$

where the term $\mathcal{G}(s, \xi)$ is given explicitly in Lemma 4.18 by (4.48)–(4.49).

5. Proofs of the main results

We reconsider now the Newton–Maxwell equation (1.7) or equivalently (2.1) as a non-autonomous initial value problem over the Hilbert spaces X^σ for $\sigma \in [\frac{1}{2}, 1]$,

$$(5.1) \quad \begin{cases} \frac{d}{dt}u(t) = \vartheta(t, u(t)), \\ u(0) = u_0, \end{cases}$$

where $u_0 \in X^\sigma$. Here the non-autonomous vector field ϑ is defined in terms of the nonlinearity $G : X^\sigma \rightarrow X^\sigma$ of the Newton–Maxwell equation

$$(5.2a) \quad G(u)_{p_i} = \frac{1}{m_i} \sum_{\nu=1}^d (p_i^\nu - A_i^\nu(q_i, \alpha)) \nabla_{q_i} A_i^\nu(q_i, \alpha) - \nabla_{q_i} V(q),$$

$$(5.2b) \quad G(u)_{q_i} = -\frac{1}{m_i} A_i(q_i, \alpha),$$

$$(5.2c) \quad G(u)_{\alpha_\lambda}(k) = i \sum_{i=1}^n \frac{1}{m_i} \frac{\chi_i(k)}{\sqrt{2|k|}} (p_i - A_i(q_i, \alpha)) \cdot \epsilon_\lambda(k) e^{-2\pi i k \cdot q_i}.$$

and the free flow $\Phi_t^0 : X^\sigma \rightarrow X^\sigma$ given by

$$(5.3) \quad \Phi_t^0(p, q, \alpha) = \left((p_i)_{i=1, \dots, n}; \left(q_i + t \frac{p_i}{m_i} \right)_{i=1, \dots, n}; (e^{-it\omega} \alpha_\lambda)_{\lambda=1, \dots, d-1} \right)$$

as follows,

$$(5.4) \quad \vartheta(t, u) = \Phi_{-t}^0 \circ G \circ \Phi_t^0(u).$$

In other words, we have written here the Newton–Maxwell dynamical system in its interaction representation. Note that the nonlinearity G is very similar to F given in (2.2) and differs from the latter only by the term $\frac{p_i}{m_i}$ in (2.2b).

As a consequence of Proposition 2.5, ϑ inherits the following properties from F .

LEMMA 5.1. — *Let $\sigma \in [\frac{1}{2}, 1]$ and assume that (A_0) , (A_1) and $(A_2^{(\sigma)})$ are satisfied. Then the vector field $\vartheta : \mathbb{R} \times X^\sigma \rightarrow X^\sigma$ is continuous and bounded on bounded subsets of $\mathbb{R} \times X^\sigma$.*

Moreover, the strong solutions to the Newton–Maxwell equation (1.7) are related to the above initial value problem (5.1) according to the following relation. In fact, assume the hypothesis of Lemma 5.1 is satisfied. Then if

$$\tilde{u} = (\tilde{p}, \tilde{q}, \tilde{\alpha}) \in \mathcal{C}(I, X^\sigma) \cap \mathcal{C}^1(I, X^{\sigma-1})$$

is a strong solution to the Newton–Maxwell equation (1.7) or equivalently (2.1), then

$$u = (\tilde{p}, \tilde{q} - t \left(\frac{\tilde{p}_i}{m_i} \right), e^{it\omega} \tilde{\alpha}) \in \mathcal{C}^1(I, X^\sigma)$$

is a solution of the non-autonomous initial value problem (5.1). Reciprocally, if

$$u = (p, q, \alpha) \in \mathcal{C}^1(I, X^\sigma)$$

is a solution of the non-autonomous initial value problem (5.1), then

$$(5.5) \quad \tilde{u} = \left(p, q + t \left(\frac{p_i}{m_i} \right), e^{-it\omega} \alpha \right) \in \mathcal{C}(I, X^\sigma) \cap \mathcal{C}^1(I, X^{\sigma-1})$$

is a strong solution to the Newton–Maxwell equation (1.7) or (2.1). Furthermore, recalling the free flow Φ_t^0 on X^σ , the strong solution \tilde{u} in (5.5) takes the expression below for all $t \in I$,

$$\tilde{u}(t) = \Phi_t^0(u(t)).$$

Additionally, thanks to the statement in Lemma 5.1 the initial value problem (5.1) is equivalent to the Duhamel formula,

$$(5.6) \quad u(t) = u_0 + \int_0^t \vartheta(s, u(s)) \, ds, \quad \forall t \in I.$$

The above discussion is summarized in the proposition below.

PROPOSITION 5.2. — *Let $\sigma \in [\frac{1}{2}, 1]$ and assume that (A_0) , (A_1) and $(A_2^{(\sigma)})$ are satisfied. Let I be a bounded open interval containing the origin. Then the statements below are equivalent:*

- (1) $u \in \mathcal{C}^1(I, X^\sigma)$ is a strong solution of the initial value problem (5.1),
- (2) $u \in \mathcal{C}(I, X^\sigma)$ satisfies the Duhamel formula (5.6) for all times $t \in I$,
- (3) the curve $t \in I \mapsto \Phi_t^0(u(t)) \in \mathcal{C}(I, X^\sigma) \cap \mathcal{C}^1(I, X^{\sigma-1})$ is a strong solution to the Newton–Maxwell equation (1.7) or equivalently (2.1).

LEMMA 5.3. — *Let $\sigma \in [\frac{1}{2}, 1]$ and assume that (A_0) , (A_1) and $(A_2^{(\sigma)})$ are satisfied. Let $(\varrho_{\hbar})_{\hbar \in (0,1)}$ be a family of density matrices in \mathcal{H} satisfying the assumptions $(S_0^{(1)})$, $(S_1^{(1)})$ and $(S_2^{(\sigma)})$. Then for every compact time interval J there exists a constant $C > 0$ such that for all $u = (p, q, \alpha) \in X^\sigma$ and $t \in J$,*

$$(5.7a) \quad \|\vartheta(t, u)\|_{X^\sigma} \leq C_T (|p|^2 + \|\alpha\|_{L^2}^2 + 1) ,$$

$$(5.7b) \quad \leq C_T (\|u\|_{X^0}^2 + 1) .$$

Moreover, if $(\tilde{\mu}_t)_{t \in \mathbb{R}}$ is the family of Wigner measures provided by Proposition 4.16 then for any bounded open interval I ,

$$(5.8) \quad \int_I \int_{X^\sigma} \|\vartheta(t, u)\|_{X^\sigma} d\tilde{\mu}_t(u) dt < \infty .$$

Proof. — The estimate (5.7) is a consequence of the inequalities (2.3) while (5.8) follows from Proposition 4.16 and (5.7). \square

We recast below the characteristic equation of Proposition 4.19 in a form more suitable for applying the general arguments in Appendix B.

COROLLARY 5.4. — *Let $\sigma \in [\frac{1}{2}, 1]$ and assume that (A_0) , (A_1) and $(A_2^{(\sigma)})$ are satisfied. Let $(\varrho_{\hbar})_{\hbar \in (0,1)}$ be a family of density matrices in \mathcal{H} satisfying the assumptions $(S_0^{(1)})$, $(S_1^{(1)})$ and $(S_2^{(\sigma)})$. Then the characteristic equation given in Proposition 4.19 takes the simple form:*

$$(5.9) \quad \begin{aligned} & \tilde{\mu}_t(e^{2\pi i \Re \langle y, \cdot \rangle_{X^\sigma}}) \\ &= \tilde{\mu}_{t_0}(e^{2\pi i \Re \langle y, \cdot \rangle_{X^\sigma}}) + 2\pi i \int_{t_0}^t \tilde{\mu}_s(e^{2\pi i \Re \langle y, u \rangle_{X^\sigma}} \Re \langle \vartheta(s, u), y \rangle_{X^\sigma}) ds , \end{aligned}$$

for all $t, t_0 \in \mathbb{R}$ and $y \in X^\sigma$.

Proof. — The scalar product over X^σ can be written as a duality bracket such that for all $u_1, u_2 \in X^\sigma$,

$$\langle u_1, u_2 \rangle_{X^\sigma} = \langle u_1, \Lambda^\sigma u_2 \rangle_{X^0},$$

where Λ is the unbounded operator over X^0 defined by

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + |k|^2 \end{bmatrix} .$$

In particular, one has for every $y \in X^{2\sigma}$ and every $\tilde{\xi} = \Lambda^\sigma y \in X^0$,

$$\Re \langle y, u \rangle_{X^\sigma} = \Re \langle \tilde{\xi}, u \rangle_{X^0},$$

$$\Re \langle \vartheta(s, u), y \rangle_{X^\sigma} = \Re \langle \vartheta(s, u), \tilde{\xi} \rangle_{X^0},$$

Moreover, one can verify that

$$(5.10) \quad \begin{aligned} \mathcal{G}(s, \xi) &= -2\pi \tilde{\mu}_s(e^{2\pi i \Re \langle \tilde{\xi}, u \rangle_{X^0}} \Re \langle \vartheta(s, u), \tilde{\xi} \rangle_{X^0}) \\ &= -2\pi \tilde{\mu}_s(e^{2\pi i \Re \langle y, u \rangle_{X^\sigma}} \Re \langle \vartheta(s, u), y \rangle_{X^\sigma}), \end{aligned}$$

if one chooses

$$\tilde{\xi} = \left(\frac{z_0}{2i\pi}, \frac{\alpha_0}{\sqrt{2\pi}} \right),$$

while $\xi = (z_0, \alpha_0)$, where the quantity $\mathcal{G}(s, \xi)$ is given in (4.48). Therefore, the characteristic equation (4.51) yields (5.9) for all $y \in X^{2\sigma}$. The latter equation (5.9) extends then to all $y \in X^\sigma$ by the dominated convergence theorem and the bound (5.8). So, to finish the proof it is enough to justify (5.10). In fact, this is equivalent to proving

$$(5.11) \quad m(s, \xi) = -2\pi \Re \langle \vartheta(s, u), \tilde{\xi} \rangle_{X^0},$$

where $m(s, \xi)$ is given in (4.49). Indeed, the latter equality follows from the identity

$$m(s, \xi) = -\Im \langle (G \circ \Phi_t^0(u))_z, (\Phi_t^0(\xi))_z \rangle - \sqrt{2} \Re \langle (G \circ \Phi_t^0(u))_\alpha, (\Phi_t^0(\xi))_\alpha \rangle.$$

Hence using the symplectic and euclidian invariance of the free particle and field dynamics respectively, one finds that

$$\begin{aligned} m(s, \xi) &= -\Im \langle (\Phi_{-t}^0 \circ G(\Phi_t^0(u)))_z, z_0 \rangle - \sqrt{2} \Re \langle (\Phi_{-t}^0 \circ G(\Phi_t^0(u)))_\alpha, \alpha_0 \rangle \\ &= -\Im \langle (\vartheta(s, u))_z, z_0 \rangle - \sqrt{2} \Re \langle (\vartheta(s, u))_\alpha, \alpha_0 \rangle \\ &= -2\pi \Re \left\langle (\vartheta(s, u))_z, \frac{z_0}{2\pi i} \right\rangle - 2\pi \Re \left\langle (\vartheta(s, u))_\alpha, \frac{\alpha_0}{\sqrt{2\pi}} \right\rangle \\ &= -2\pi \Re \langle \vartheta(s, u), \tilde{\xi} \rangle. \end{aligned} \quad \square$$

In the sequel, we establish some regularity of the family $(\tilde{\mu}_t)_{t \in \mathbb{R}}$ with respect to time.

LEMMA 5.5. — *The Wigner measures $(\tilde{\mu}_t)_{t \in \mathbb{R}}$ provided by Proposition 4.16 satisfy:*

- (1) $\tilde{\mu}_t$ concentrates on X^σ , i.e., $\tilde{\mu}_t(X^\sigma) = 1$.
- (2) $\mathbb{R} \ni t \mapsto \tilde{\mu}_t \in \mathfrak{P}(X^\sigma)$ is weakly narrowly continuous.

Proof. — On one hand, the bound (4.44) gives

$$\int_{X^0} \|\alpha\|_{\dot{\mathfrak{H}}^\sigma}^2 d\tilde{\mu}_t < \infty.$$

On the other, Markov's inequality implies

$$\tilde{\mu}_t(\{(p, q, \alpha) \in X^0 : \|\alpha\|_{\dot{\mathfrak{H}}^\sigma} \geq \varepsilon\}) \leq \frac{1}{\varepsilon^2} \tilde{\mu}_t(\|\alpha\|_{\dot{\mathfrak{H}}^\sigma}^2).$$

So, taking $\varepsilon \rightarrow \infty$ yields

$$\tilde{\mu}_t(\{(p, q, \alpha) \in X^0 : \alpha \notin \dot{\mathcal{H}}^\sigma\}) = 0.$$

Therefore, the probability measures $\tilde{\mu}_t$ are concentrated on X^σ . To prove the weak narrow continuity of $(\tilde{\mu}_t)_{t \in \mathbb{R}}$, we claim that it is enough to show that the function

$$(5.12) \quad \mathbb{R} \ni t \mapsto \int_{X^\sigma} e^{2\pi i \Re \langle y, u \rangle_{X^\sigma}} d\tilde{\mu}_t(u),$$

is continuous for every $y \in X^\sigma$. In fact, suppose that the statement is true. One remarks that for every arbitrary compact interval J the family $(\tilde{\mu}_t)_{t \in J}$ satisfies the bound

$$\sup_{t \in J} \int_{X^\sigma} \|u\|_{X_w^\sigma}^2 d\tilde{\mu}_t(u) < \infty,$$

since the estimate (4.44) holds true and $\|u\|_{X_w^\sigma} \leq \|u\|_{X^\sigma}^2$. So, thanks to the above bound one deduces that the family $(\tilde{\mu}_t)_{t \in J}$ is tight in $\mathfrak{P}(X^\sigma)$, where X^σ is endowed here with the norm $\|\cdot\|_{X_w^\sigma}$. Therefore, applying the Prokhorov's theorem one deduces that for any arbitrary compact interval J the family $(\tilde{\mu}_t)_{t \in J}$ is sequentially compact for the weak narrow convergence topology. In particular, take $t_0 \in \mathbb{R}$ and ψ a bounded continuous function on $(X^\sigma, \|\cdot\|_{X_w^\sigma})$. Then the steps below follow:

- (1) For any sequence $t_j \rightarrow t_0$ one can extract a subsequence denoted by $(t_{j_k})_{k \in \mathbb{N}}$ such that $\tilde{\mu}_{t_{j_k}}$ weakly narrowly converges to a provability measure μ .
- (2) Thanks to the above convergence and the continuity of the function (5.12), one deduces that the characteristic function of μ_{t_0} and μ coincide and hence $\mu = \mu_{t_0}$. Note that we have used that the map $u \mapsto e^{2\pi i \Re \langle y, u \rangle_{X^\sigma}}$ is bounded and continuous over $(X^\sigma, \|\cdot\|_{X_w^\sigma})$.
- (3) Hence one concludes that for every sequence $(t_j)_{j \in \mathbb{N}}$ there exists a subsequence $(t_{j_k})_{k \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow \infty} \int_{X^\sigma} \psi(u) d\tilde{\mu}_{t_{j_k}} = \int_{X^\sigma} \psi(u) d\tilde{\mu}_{t_0}.$$

- (4) So there exists a unique limit point and therefore the sequence $(\tilde{\mu}_{t_j})_{j \in \mathbb{N}}$ converges weakly narrowly to $\tilde{\mu}_{t_0}$.

Now, the continuity of the functions (5.12) is a consequence of the characteristic equation (5.9) and the estimate (5.7) which yield

$$\begin{aligned}
 (5.13) \quad & \left| \tilde{\mu}_t(e^{2\pi i \Re \langle y, \cdot \rangle_{X^\sigma}}) - \tilde{\mu}_{t_0}(e^{2\pi i \Re \langle y, \cdot \rangle_{X^\sigma}}) \right| \\
 & \lesssim \left| \int_{t_0}^t \|y\|_{X^\sigma} \int_{X^\sigma} \|\vartheta(t, u)\|_{X^\sigma} d\tilde{\mu}_s(u) ds \right| \\
 & \leq C \left| \int_{t_0}^t \int_{X^\sigma} (\|u\|_{X^0}^2 + 1) d\tilde{\mu}_s(u) ds \right| \leq C|t - t_0|
 \end{aligned}$$

for any t, t_0 in an arbitrary compact interval. In the last inequality (5.13), we have used the bound (4.44) in Proposition 4.16. \square

Next we apply the general results of the Appendix B. At this point, we suggest the reading of Appendix B. In particular, the spaces of smooth cylindrical functions $\mathcal{C}_{0,cyl}^\infty$ and the gradient ∇ are defined there. The proposition below relates the characteristic equation in (5.9) to the Liouville equation introduced in Appendix B.

PROPOSITION 5.6. — *The family of Wigner measures $(\tilde{\mu}_t)_{t \in \mathbb{R}}$ defined in Proposition 4.16 satisfies the Liouville equation*

$$(5.14) \quad \int_I \int_{X^\sigma} \{ \partial_t \phi(t, u) + \Re \langle \vartheta(t, u), \nabla_{X^\sigma} \phi(t, u) \rangle_{X^\sigma} \} d\mu_t(u) dt = 0$$

for any open bounded time interval I containing the origin with $\phi \in \mathcal{C}_{0,cyl}^\infty(I \times X^\sigma)$.

Proof. — It is enough to apply Proposition B.2 which establishes an equivalence between Liouville equation and characteristic equation. Indeed, the assumptions required to apply such result are true:

- (1) $\mathbb{R} \ni t \mapsto \tilde{\mu}_t$ is weakly narrowly continuous in $\mathfrak{P}(X^\sigma)$ according to Lemma 5.5.
- (2) The bound

$$\int_I \int_{X^\sigma} \|\vartheta(t, u)\|_{X^\sigma} d\tilde{\mu}_t(u) dt < \infty ,$$

holds true for any bounded open interval I thanks to Lemma 5.3 with the bound (5.8). \square

Proof of Theorem 1.3. — Let $\sigma \in [\frac{1}{2}, 1]$ and assume that (A_0) , (A_1) and $(A_2^{(\sigma)})$ are satisfied. Consider for $u_* \in X^\sigma$ the family of coherent states defined in (4.9):

$$(5.15) \quad \varrho_{\hbar} = \mathcal{C}_{\hbar}(u_*).$$

Then according to Lemma 4.2 one verifies that such family $\{\varrho_h\}_{h \in (0,1)}$ satisfies the assumptions $(S_0^{(1)})$, $(S_1^{(1)})$ and $(S_2^{(\sigma)})$. Hence one can apply Propositions 4.16 and 5.6, and one concludes that there exists $\{\tilde{\mu}_t\}_{t \in \mathbb{R}}$ a weakly narrowly continuous curve in $\mathfrak{P}(X^\sigma)$ satisfying the bound (5.8) and the Liouville equation (5.14).

So, we have at our disposal all the ingredients to apply Theorem B.8 on the existence of global generalized flow for abstract initial value problems. Indeed, the condition 1 in Theorem B.8 is satisfied according to the above discussion while the uniqueness condition in Theorem B.82 holds true by Proposition 2.6. Hence one concludes that the ensemble of initial data

$$\mathcal{G} = \left\{ u_0 \in X^\sigma : \begin{array}{l} \text{there exists a global strong solution } u \text{ of} \\ \text{Newton–Maxwell equation satisfying (5.1)} \end{array} \right\}$$

is a Borel set and is total with respect to the probability measure $\tilde{\mu}_0$ which is the unique Wigner measure of the coherent state family in (5.15). Therefore, by Lemma 4.2 one deduces that

$$\tilde{\mu}_0(\mathcal{G}) = \delta_{u_*}(\mathcal{G}) = 1,$$

and consequently any $u_* \in X^\sigma$ belongs to \mathcal{G} . This implies that for any initial condition $u_0 \in X^\sigma$ there exists a unique global solution $u \in \mathcal{C}(\mathbb{R}, X^\sigma)$ of the Newton–Maxwell equation (5.1). Moreover, the generalized flow map

$$(5.16) \quad \begin{aligned} \tilde{\Phi}_t : X^\sigma &\longrightarrow X^\sigma \\ u_0 &\longmapsto u(t) \end{aligned}$$

is well defined and Borel measurable. Finally, using Proposition 5.2 which gives the equivalence between the two formulations (5.1) and (1.7) of the Newton–Maxwell equation, one shows the existence and uniqueness of global solutions for the latter with a generalized global flow defined by

$$(5.17) \quad \Phi_t(u_0) = \Phi_t^0 \circ \tilde{\Phi}_t(u_0),$$

where Φ_t^0 is the free flow given in (5.3) and $\tilde{\Phi}_t$ is the global flow of the Newton–Maxwell initial value problem (5.1) in the interaction representation given by (5.16). \square

Proof of Theorem 1.4 (The case $\delta = 1$ and singleton Wigner set). — Let $\sigma \in [\frac{1}{2}, 1]$ and assume that (A_0) , (A_1) and $(A_2^{(\sigma)})$ are satisfied. Consider a family of density matrices $(\varrho_h)_{h \in (0,1)}$ on \mathcal{H} satisfying the assumptions

$(S_0^{(0)})$, $(S_1^{(1)})$ and $(S_2^{(\sigma)})$. Recall that

$$\begin{aligned}\varrho_{\hbar}(t) &= e^{-i\frac{t}{\hbar}\hat{H}_{\hbar}} \varrho_{\hbar} e^{i\frac{t}{\hbar}\hat{H}_{\hbar}}, \\ \widetilde{\varrho}_{\hbar}(t) &= e^{i\frac{t}{\hbar}\hat{H}_{\hbar}^0} \varrho_{\hbar}(t) e^{-i\frac{t}{\hbar}\hat{H}_{\hbar}^0}.\end{aligned}$$

For $t_0 \in \mathbb{R}$ pick $\widetilde{\mu}_{t_0}$ a Wigner measure of the family $(\widetilde{\varrho}_{\hbar}(t_0))_{\hbar \in (0,1)}$. The existence of such Wigner measure $\widetilde{\mu}_{t_0}$ is a consequence of Proposition 4.1 and Lemma 4.11. Moreover, according to Definition 1.1 this implies that there exists a sequence $(\hbar_k)_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} \hbar_k = 0$ and

$$\mathcal{M}(\widetilde{\varrho}_{\hbar_k}(t_0), k \in \mathbb{N}) = \{\widetilde{\mu}_{t_0}\}.$$

So, by Proposition 4.16 there exists a subsequence (still denoted by $(\hbar_k)_{k \in \mathbb{N}}$) such that for each $t \in \mathbb{R}$,

$$\mathcal{M}(\widetilde{\varrho}_{\hbar_k}(t), k \in \mathbb{N}) = \{\widetilde{\mu}_t\}.$$

Furthermore, by Proposition 4.19 the curve $\mathbb{R} \ni t \mapsto \widetilde{\mu}_t \in \mathfrak{P}(X^\sigma)$ satisfies the characteristic equation (4.51) or equivalently (5.9). Therefore, using Proposition 5.6 one deduces that $\mathbb{R} \ni t \mapsto \widetilde{\mu}_t$ is a measure-valued solution of the Liouville equation (5.14). So applying the probabilistic representation in Theorem B.6 one obtains the existence of a measure

$$\eta \in \mathfrak{P}(\mathcal{X}),$$

with $\mathcal{X} \equiv X^\sigma \times \mathcal{C}(\bar{I}, X^\sigma)$, concentrated on the ensemble of solutions of the Newton–Maxwell equation (5.1), i.e.,

$$\eta(\mathcal{F}_I) = 1,$$

where

$$\mathcal{F}_I = \{(u_0, u) \in \mathcal{X} : u \text{ satisfies (5.1) on } I \text{ with the initial condition } u_0\}.$$

Moreover, Theorem B.62 implies that for any bounded Borel functions $\psi : X^\sigma \rightarrow \mathbb{R}$

$$(5.18) \quad \int_{X^\sigma} \psi(u) d\widetilde{\mu}_t = \int_{\mathcal{F}_I} \psi(\Xi_t(u_0, u)) d\eta(u_0, u).$$

Thanks to the proof of Theorem 1.3, the generalized global flow $\widetilde{\Phi}_t$ in (5.16) of the Newton–Maxwell equation (5.1) is already constructed. Hence one proves that the following identity holds true on the set \mathcal{F}_I

$$\Xi_t(u_0, u) = \widetilde{\Phi}_t(\Xi_0(u_0, u)) = \widetilde{\Phi}_t(u_0).$$

Recall that Ξ_t is the evaluation map defined in (B.8). Thus, the equality (5.18) yields

$$\begin{aligned} \int_{X^\sigma} \psi(u) d\tilde{\mu}_t(u) &= \int_{\mathcal{X}} \psi \circ \tilde{\Phi}_t(\Xi_0(u_0, u)) d\eta(u_0, u) \\ &= \int_{X^\sigma} \psi \circ \tilde{\Phi}_t(u) d\tilde{\mu}_0(u), \end{aligned}$$

since $(\Xi_0)_\# \eta = \tilde{\mu}_0$ and $\tilde{\Phi}_t$ is Borel measurable. So, one concludes that for all $t \in I$,

$$\tilde{\mu}_t = (\tilde{\Phi}_t)_\# \tilde{\mu}_0.$$

In particular, since any Wigner measure μ_{t_0} of the family $(\tilde{\varrho}_h(t_0))_{h \in (0,1)}$ is equal to $(\tilde{\Phi}_{t_0})_\# \tilde{\mu}_0$, one concludes that such family of density matrices admits a unique Wigner measure, i.e.,

$$\mathcal{M}(\tilde{\varrho}_h(t_0), h \in (0, 1)) = \{(\tilde{\Phi}_{t_0})_\# \tilde{\mu}_0\}.$$

Finally, using Lemmas 4.11 and C.4 and the fact that the time t_0 is arbitrary one concludes for all times $t \in \mathbb{R}$,

$$\begin{aligned} \mathcal{M}(\varrho_h(t), h \in (0, 1)) &= \{(\Phi_t^0)_\# \tilde{\mu}_t, \tilde{\mu}_t \in \mathcal{M}(\tilde{\varrho}_h(t), h \in (0, 1))\} \\ &= \{(\Phi_t^0 \circ \tilde{\Phi}_t)_\# \tilde{\mu}_0\} \\ &= \{(\Phi_t)_\# \mu_0\}, \end{aligned}$$

since $\Phi_t = \Phi_t^0 \circ \tilde{\Phi}_t$ is the Newton–Maxwell global flow and $\tilde{\mu}_0 = \mu_0$ as a consequence of $\tilde{\varrho}_h(0) = \varrho_h(0) = \varrho_h$. \square

Proof of Theorem 1.4 (The case $\delta > 0$ and general Wigner set). — In this paragraph, we indicate how to consider a general family of density matrices $(\varrho_h)_{h \in (0,1)}$ without assuming (1.20) and on the other hand how to weaken the assumptions $(S_0^{(1)})$ and $(S_1^{(1)})$ in Theorem 1.4 and replace them by the ones given in $(S_0^{(\delta)})$ and $(S_1^{(\delta)})$ for some $\delta \in (0, 1]$. The two arguments (1) and (2) below are independent:

(1). — Suppose that $(\varrho_h)_{h \in (0,1)}$ is a family of density matrices on \mathcal{H} satisfying $(S_0^{(1)})$, $(S_1^{(1)})$ and $(S_2^{(\sigma)})$. According to Definition 1.1, if $\mu_0 \in \mathcal{M}(\varrho_h, h \in (0, 1))$ then there exists a sequence $(h_k)_{k \in \mathbb{N}}$, $h_k \rightarrow 0$, such that

$$\mathcal{M}(\varrho_{h_k}, k \in \mathbb{N}) = \{\mu_0\}.$$

Hence thanks to the previous results so far proved one concludes that for all $t \in \mathbb{R}$,

$$\mathcal{M}(\varrho_{h_k}(t), k \in \mathbb{N}) = \{(\Phi_t)_\# \mu_0\},$$

and consequently, one proves

$$\{(\Phi_t)_\# \mu_0, \mu_0 \in \mathcal{M}(\varrho_{\hbar}, \hbar \in (0, 1))\} \subset \mathcal{M}(\varrho_{\hbar}(t), \hbar \in (0, 1)).$$

Reciprocally, for any $t \in \mathbb{R}$ take a $\mu_t \in \mathcal{M}(\varrho_{\hbar}(t), \hbar \in (0, 1))$ then there exists a sequence $(\hbar_k)_{k \in \mathbb{N}}, \hbar_k \rightarrow 0$, such that

$$\mathcal{M}(\varrho_{\hbar_k}(t), k \in \mathbb{N}) = \{\mu_t\}.$$

So, applying the previous results so far proved one obtains

$$\mathcal{M}(e^{i\frac{t}{\hbar}\hat{H}_{\hbar}} \varrho_{\hbar_k}(t) e^{-i\frac{t}{\hbar}\hat{H}_{\hbar}}, k \in \mathbb{N}) = \mathcal{M}(\varrho_{\hbar_k}, k \in \mathbb{N}) = \{(\Phi_{-t})_\# \mu_t\}.$$

Thus, we conclude that $\mu_t = (\Phi_t)_\# \mu_0$ for some $\mu_0 \in \mathcal{M}(\varrho_{\hbar}, \hbar \in (0, 1))$ and prove the opposite inclusion. Note that we have used implicitly Lemma 4.11.

(2). — Suppose that $(\varrho_{\hbar})_{\hbar \in (0,1)}$ is a family of density matrices on \mathcal{H} satisfying $(S_0^{(\delta)})$, $(S_1^{(\delta)})$ and $(S_2^{(\sigma)})$. Then we will use an approximation argument: Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ be a smooth cutoff function such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in a neighborhood of the origin. Denote

$$\chi_R = \chi\left(\frac{\cdot}{R}\right)$$

and let $A = \hat{N}_{\hbar} + \hat{q}^2 + \hat{H}_{\hbar}^0$. Then the family of density matrices

$$\varrho_{\hbar,R} = \frac{\chi_R(A) \varrho_{\hbar} \chi_R(A)}{\text{Tr}[\chi_R(A) \varrho_{\hbar} \chi_R(A)]}$$

approximates ϱ_{\hbar} in the trace norm as $R \rightarrow \infty$ for fixed \hbar . Moreover, let

$$m(R) = \sup_{\hbar \in (0,1), t \in \mathbb{R}} \|\varrho_{\hbar}(t) - \varrho_{\hbar,R}(t)\|_{\mathcal{L}^1(\mathcal{H})},$$

where $\varrho_{\hbar,R}(t) = e^{-i\frac{t}{\hbar}\hat{H}_{\hbar}} \varrho_{\hbar,R} e^{i\frac{t}{\hbar}\hat{H}_{\hbar}}$. Thanks to the assumptions $(S_0^{(\delta)})$ and $(S_1^{(\delta)})$ one deduces

$$\lim_{R \rightarrow 0} m(R) = 0.$$

Furthermore, $(S_0^{(1)})$, $(S_1^{(1)})$ and $(S_2^{(\sigma)})$ are satisfied by the family of density matrices $(\varrho_{\hbar,R})_{\hbar \in (0,1)}$. In fact, $d\Gamma(|k|)$, \hat{N}_{\hbar} and $d\Gamma(|k|^{2\sigma})$ commute with A which leads to $(S_1^{(1)})$ and $(S_2^{(\sigma)})$ while for $(S_0^{(1)})$ one uses the commutation relations. Hence applying the so far proved results above one deduces that for all $t \in \mathbb{R}$,

$$\mathcal{M}(\varrho_{\hbar,R}(t), \hbar \in (0, 1)) = \{(\Phi_t)_\# \mu_{0,R}, \mu_{0,R} \in \mathcal{M}(\varrho_{\hbar,R}, \hbar \in (0, 1))\}.$$

On the other hand for each $t \in \mathbb{R}$ and R sufficiently large, according to Definition 1.1 from any sequence $(\hbar_k)_{k \in \mathbb{N}}$, $\hbar_k \rightarrow 0$ as $k \rightarrow \infty$, one can extract a subsequence — still denoted the same — such that

$$(5.19) \quad \mathcal{M}(\varrho_{\hbar_k}(t), k \in \mathbb{N}) = \{\mu_t\}, \quad \mathcal{M}(\varrho_{\hbar_k, R}(t), k \in \mathbb{N}) = \{(\Phi_t)_\# \mu_{0, R}\},$$

$$(5.20) \quad \mathcal{M}(\varrho_{\hbar_k}, k \in \mathbb{N}) = \{\mu_0\}, \quad \mathcal{M}(\varrho_{\hbar_k, R}, k \in \mathbb{N}) = \{\mu_{0, R}\}.$$

Let $|\nu|$ be the total variation of measure ν . So, applying the general result on the comparison of Wigner measures in [9, Proposition 2.10] yields

$$|\mu_t - (\Phi_t)_\# \mu_{0, R}| \leq \liminf_{k \rightarrow \infty} \|\varrho_{\hbar_k}(t) - \varrho_{\hbar_k, R}(t)\|_{\mathcal{L}^1(\mathcal{H})} \leq m(R),$$

$$|\mu_0 - \mu_{0, R}| \leq \liminf_{k \rightarrow \infty} \|\varrho_{\hbar_k} - \varrho_{\hbar_k, R}\|_{\mathcal{L}^1(\mathcal{H})} \leq m(R),$$

where the left-hand side denotes the total variation of the signed measures $\mu_t - (\Phi_t)_\# \mu_{0, R}$ and $\mu_0 - \mu_{0, R}$. Therefore, the triangle inequality implies

$$(5.21) \quad |\mu_t - (\Phi_t)_\# \mu_0| \leq |\mu_t - (\Phi_t)_\# \mu_{0, R}| + |\mu_{0, R} - \mu_0| \leq 2m(R),$$

since in particular the total variation of $(\Phi_t)_\# \mu_{0, R} - (\Phi_t)_\# \mu_0$ and those of $\mu_{0, R} - \mu_0$ are equal. So, taking $R \rightarrow \infty$ in (5.21) one deduces that

$$(5.22) \quad \mu_t = (\Phi_t)_\# \mu_0.$$

Therefore, picking any $\mu_t \in \mathcal{M}(\varrho_{\hbar}(t), \hbar \in (0, 1))$ and using the above argument one shows that

$$\mathcal{M}(\varrho_{\hbar}(t), \hbar \in (0, 1)) \subset \{(\Phi_t)_\# \mu_0, \mu_0 \in \mathcal{M}(\varrho_{\hbar}, \hbar \in (0, 1))\}.$$

While the opposite inclusion is justified by taking any $\mu_0 \in \mathcal{M}(\varrho_{\hbar}, \hbar \in (0, 1))$ and then choosing and extracting sequences $(\hbar_k)_{k \in \mathbb{N}}$ such that (5.19)–(5.20) are satisfied; thus arriving to the conclusion (5.22) for some $\mu_t \in \mathcal{M}(\varrho_{\hbar}(t), \hbar \in (0, 1))$. This ends the proof of Theorem 1.4. \square

As an illustration of our method, we give further quantitative propagation estimates satisfied by the Newton–Maxwell flow derived from the quantum dynamics of Pauli–Fierz.

PROPOSITION 5.7. — *Assume (A_0) , (A_1) and $(A_2^{(1)})$. Then for any probability measure $\mu_0 \in \mathfrak{P}(X^1)$ satisfying*

$$(5.23) \quad \int_{X^1} \|u\|_{X^1}^2 d\mu_0(u) < \infty, \quad \int_{X^1} (p^4 + \|\alpha\|_{\dot{H}^{1/2}}^4 + \|\alpha\|_{L^2}^4) d\mu_0(u) < \infty,$$

there exists a constant $c > 0$ such that for all times $t \in \mathbb{R}$,

$$(5.24) \quad \int_{X^1} p(t)^4 + \|\alpha(t)\|_{\dot{H}^{1/2}}^4 d\mu_0(u) \leq c, \quad \int_{X^1} \|\alpha(t)\|_{L^2}^4 d\mu_0(u) \leq c e^{c|t|},$$

where $(p(t), q(t), \alpha(t))$ is the solution of the Newton–Maxwell equation (1.7) at time t starting from the initial condition $u = (p, q, \alpha) \in X^1$.

Proof. — Consider the following family of superposed coherent states defined in (4.17),

$$\varrho_{\hbar} = \int_{X^1} \mathcal{C}_{\hbar}(u) d\mu_0(u).$$

According to Lemma 4.3, $(\varrho_{\hbar})_{\hbar \in (0,1)}$ satisfies the assumptions $(S_0^{(1)})$, $(S_1^{(1)})$ and $(S_2^{(1)})$. So thanks to Theorems 1.3 and 1.4, one concludes

$$(5.25) \quad \mathcal{M}(\varrho_{\hbar}(t), \hbar \in (0, 1)) = \{\mu_t\} = \{(\Phi_t)_{\#} \mu_0\}.$$

On the other hand, there exists a constant $c > 0$ such that for all $\hbar \in (0, 1)$ and $t \in \mathbb{R}$,

$$(5.26) \quad \mathrm{Tr}[\varrho_{\hbar}(t) (\hat{H}_{\hbar}^0)^2] \leq c,$$

$$(5.27) \quad \mathrm{Tr}[\varrho_{\hbar}(t) (\hat{N}_{\hbar})^2] \leq c e^{c|t|}.$$

The above inequalities are consequences of Lemma 3.2, Corollary 3.7 and the fact that (5.26)–(5.27) hold true at time $t = 0$. Indeed, a direct computation yields

$$\mathrm{Tr}[\varrho_{\hbar} (\hat{H}_{\hbar}^0)^2] \leq 2 \int_{X^1} (p^4 + \hbar \|\alpha\|_{\mathfrak{H}^1}^2 + \|\alpha\|_{\mathfrak{H}^{1/2}}^4) d\mu_0 < \infty,$$

$$\mathrm{Tr}[\varrho_{\hbar} (\hat{N}_{\hbar})^2] = \int_{X^1} (\hbar \|\alpha\|_{L^2}^2 + \|\alpha\|_{L^2}^4) d\mu_0 < \infty.$$

So, applying the argument in [9, Lemma 3.12] one deduces

$$(5.28) \quad \int_{X^1} (p^4 + \|\alpha\|_{\mathfrak{H}^{1/2}}^4) d\mu_t \leq \liminf_{\hbar \rightarrow 0} \mathrm{Tr}[\varrho_{\hbar}(t) (\hat{H}_{\hbar}^0)^2] \leq c,$$

$$(5.29) \quad \int_{X^1} \|\alpha\|_{\mathfrak{H}^{1/2}}^4 d\mu_t \leq \liminf_{\hbar \rightarrow 0} \mathrm{Tr}[\varrho_{\hbar}(t) (\hat{N}_{\hbar})^2] \leq c e^{c|t|}.$$

Hence (5.24) are consequences of (5.25) and (5.28)–(5.29). \square

Appendix A. Uniform energy-field estimates

In this appendix we recall some well-known energy-field estimates which we extend to \hbar -dependent operators and verify that such inequalities still hold true uniformly in $\hbar \in (0, 1)$.

In the sequel, we use the following shorthand notations:

$$\mathfrak{H} = L^2(\mathbb{R}^d, \mathbb{C}^{d-1})$$

Consider a separable Hilbert space \mathcal{H} , and bounded operators $v \in \mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathfrak{H})$ such that there exist maps

$$(A.1) \quad k \mapsto v^\nu(k) = (v_1^\nu(k), \dots, v_{d-1}^\nu(k)), \quad \forall \nu \in \{1, \dots, d\}$$

defined for a.e. $k \in \mathbb{R}^d$, which satisfy that $v_\lambda^\nu(k) \in \mathcal{L}(\mathcal{H})$ for a.a. $k \in \mathbb{R}^d$ and for all $\phi \in \mathcal{H}$:

$$(v\phi)_\lambda^\nu(k) = v_\lambda^\nu(k)\phi,$$

$$\|v\phi\|_{\mathcal{H} \otimes \mathfrak{H}}^2 = \sum_{\lambda=1}^{d-1} \sum_{\nu=1}^d \int_{\mathbb{R}^d} \|v_\lambda^\nu(k)\phi\|_{\mathcal{H}}^2 dk.$$

In particular, the given representation (A.1) defines an operator $v^\nu \in \mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathfrak{H}^0)$, with

$$\|v^\nu\|_{\mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathfrak{H}^0)}^2 = \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \|v_\lambda^\nu(k)\|_{\mathcal{L}(\mathcal{H})}^2 dk,$$

$$\|v\|_{\mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathfrak{H})}^2 = \sum_{\nu=1}^d \|v^\nu\|_{\mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathfrak{H}^0)}^2.$$

This leads to the following definition.

DEFINITION A.1. — *Let $v \in \mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathfrak{H})$ be as above. Then $\widehat{a}(v^\nu)$ and $\widehat{a}^*(v^\nu)$, $\nu \in \{1, \dots, d\}$, are the densely defined closed operators on $\mathcal{H} \otimes \Gamma_s(\mathfrak{H})$ whose actions are defined by*

$$\widehat{a}(v^\nu) = \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} (v_\lambda^\nu)^*(k) \widehat{a}_\lambda(k) dk,$$

$$\widehat{a}^*(v^\nu) = \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} v_\lambda^\nu(k) \widehat{a}_\lambda^*(k) dk.$$

Here $(v_\lambda^\nu)^*(k) \in \mathcal{L}(\mathcal{H})$ is the adjoint of $v_\lambda^\nu(k)$.

If v and w are such that for a.a. $k \in \mathbb{R}^d$ and all $\lambda, \lambda' \in \{1, \dots, d-1\}$ and $\nu \in \{1, \dots, d\}$, $[(v_\lambda^\nu)^*(k), w_{\lambda'}^\nu(k')] = 0$, then the d -dimensional vectors $\widehat{a}(v)$ and $\widehat{a}^*(w)$, whose components are $\widehat{a}(v^\nu)$ and $\widehat{a}^*(w^\nu)$, respectively, satisfy generalized canonical commutation relations on a dense set, as stated in the following lemma, whose proof is straightforward.

LEMMA A.2. — *If for a.a. $(k, k') \in \mathbb{R}^{2d}$ and all $\lambda, \lambda' \in \{1, \dots, d-1\}$ and $\nu \in \{1, \dots, d\}$,*

$$[(v_\lambda^\nu)^*(k), w_{\lambda'}^\nu(k')] = 0,$$

then $\widehat{a}(v^\nu)$ and $\widehat{a}^*(w^\nu)$ satisfy

$$[\widehat{a}(v), \widehat{a}^*(w)] = \sum_{\nu=1}^d [\widehat{a}(v^\nu), \widehat{a}^*(w^\nu)] = \hbar \sum_{\lambda=1}^{d-1} \sum_{\nu=1}^d \int_{\mathbb{R}^d} (v_\lambda^\nu)^*(k) w_\lambda^\nu(k) dk.$$

Let us adopt the following shorthand notation $\llbracket v^*, w \rrbracket = 0$:

$$\begin{aligned} \llbracket v^*, w \rrbracket = 0 &\iff [(v_\lambda^\nu)^*(k), w_{\lambda'}^\nu(k')] = 0 \text{ for a.a. } (k, k') \in \mathbb{R}^{2d}, \\ &\quad \forall \lambda, \lambda' \in \{1, \dots, d-1\}, \forall \nu \in \{1, \dots, d\}. \end{aligned}$$

In addition, let us denote by ω the operator of multiplication by $|k|$. Given $v \in \mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathfrak{H})$, with associated representation $k \mapsto v^\nu(k)$ for a.a. $k \in \mathbb{R}^d$, then $\omega^\beta v$ has $k \mapsto |k|^\beta v^\nu(k)$ as associated representation.

LEMMA A.3. — *Let $v \in \mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathfrak{H})$, with associated representation $k \mapsto v^\nu(k)$ for a.a. $k \in \mathbb{R}^d$. If $\llbracket v^*, v \rrbracket = 0$, then, uniformly with respect to $\hbar \in (0, 1)$:*

$$\begin{aligned} (1) \quad & \|\widehat{a}(v^\nu) (\widehat{H}_f + 1)^{-\frac{1}{2}}\|_{\mathcal{L}(\mathcal{H} \otimes \Gamma_s(\mathfrak{H}))} \leq \|\omega^{-\frac{1}{2}} v^\nu\|_{\mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathfrak{H}^0)}, \\ (2) \quad & \|\widehat{a}^*(v^\nu) (\widehat{H}_f + 1)^{-\frac{1}{2}}\|_{\mathcal{L}(\mathcal{H} \otimes \Gamma_s(\mathfrak{H}))} \\ & \leq \|\omega^{-\frac{1}{2}} v^\nu\|_{\mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathfrak{H}^0)} + \|v^\nu\|_{\mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathfrak{H}^0)}. \end{aligned}$$

Remark A.4. — Analogous estimates hold if we take $\omega \equiv 1$, and replace \widehat{H}_f with the number operator \widehat{N}_\hbar defined in (1.14).

Proof. — Let us start proving (1). By definition, one obtains

$$\|\widehat{a}(v^\nu) \Psi\|^2 = \sum_{n \in \mathbb{N}} \|\widehat{a}(v^\nu) \Psi^{(n)}\|^2 = \sum_{n \in \mathbb{N}} \hbar n \|(v^\nu)^* \otimes 1^{(n-1)} \Psi^{(n)}\|^2.$$

Hence multiplying and dividing by $\omega^{\frac{1}{2}}$ (acting on the first variable), using the symmetry of $\Psi^{(n)}$, and $\|(v^\nu)^* \omega^{-\frac{1}{2}}\|_{\mathcal{L}(\mathcal{H} \otimes \mathfrak{H}^0, \mathcal{H})} = \|\omega^{-\frac{1}{2}} v^\nu\|_{\mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathfrak{H}^0)}$ one obtains

$$\begin{aligned} \sum_{\nu=1}^d \|\widehat{a}(v^\nu) \Psi\|^2 &\leq \sum_j \|(v^\nu)^* \omega^{-\frac{1}{2}}\|_{\mathcal{L}(\mathcal{H} \otimes \mathfrak{H}^0, \mathcal{H})}^2 \sum_{n \in \mathbb{N}} \hbar n \|\omega^{\frac{1}{2}} \otimes 1^{(n-1)} \Psi^{(n)}\|^2 \\ &\leq \|\omega^{-\frac{1}{2}} v^\nu\|_{\mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathfrak{H}^0)} \|(\widehat{H}_f + 1)^{\frac{1}{2}} \Psi\|^2. \end{aligned}$$

To prove (2), one uses the canonical commutation relations:

$$\begin{aligned}
& \|\widehat{a}^*(v^\nu)\Psi\|^2 \\
&= \langle \Psi, \widehat{a}(v^\nu)\widehat{a}^*(v^\nu)\Psi \rangle \\
&= \langle \Psi, \widehat{a}^*(v^\nu)\widehat{a}(v^\nu)\Psi \rangle + \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \langle \Psi, (v_\lambda^\nu)^*(k) v_\lambda^\nu(k) \Psi \rangle dk \\
&\leq \|\omega^{-\frac{1}{2}} v^\nu\|_{\mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathfrak{H}^0)} \|(\widehat{H}_f + 1)^{\frac{1}{2}} \Psi\|^2 + \|v^\nu\|_{\mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathfrak{H}^0)}^2 \|\Psi\|^2. \quad \square
\end{aligned}$$

We apply the general estimates of Lemma A.3 to the Pauli–Fierz Hamiltonian. To this extent, let us identify

$$\mathcal{H} \equiv L^2(\mathbb{R}^{dn})$$

and eventually take

$$v_\lambda^\nu(k, x_j) \equiv |k|^{-\frac{1}{2}} \epsilon_\lambda^\nu(k) \chi_i(k) e^{-2\pi i k \cdot x_j},$$

acting as a x_j -multiplicative operator on $L^2(\mathbb{R}^{dn})$ for each $k \in \mathbb{R}^d$, where $x_j \in \mathbb{R}^d$ is the j^{th} particle position.

COROLLARY A.5. — *Let $v, w \in \mathcal{L}(L^2, L^2 \otimes \mathfrak{H})$ such that $\llbracket v^*, v \rrbracket = \llbracket w^*, w \rrbracket = 0$ and let $\|\cdot\|_{\mathcal{L}} = \|\cdot\|_{\mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathfrak{H})}$. Then the following bounds hold uniformly with respect to $\hbar \in (0, 1)$:*

$$\begin{aligned}
(1) \quad & \|(\widehat{H}_f + 1)^{-\frac{1}{2}} \widehat{a}^*(v) \cdot \widehat{a}^*(w) (\widehat{H}_f + 1)^{-\frac{1}{2}}\|_{\mathcal{L}(\mathcal{H})} \\
& \leq \|\omega^{-\frac{1}{2}} v\|_{\mathcal{L}} (\|\omega^{-\frac{1}{2}} w\|_{\mathcal{L}} + \|w\|_{\mathcal{L}}), \\
(2) \quad & \|(\widehat{H}_f + 1)^{-\frac{1}{2}} \widehat{a}(v) \cdot \widehat{a}(w) (\widehat{H}_f + 1)^{-\frac{1}{2}}\|_{\mathcal{L}(\mathcal{H})} \\
& \leq (\|\omega^{-\frac{1}{2}} v\|_{\mathcal{L}} + \|v\|_{\mathcal{L}}) \|\omega^{-\frac{1}{2}} w\|_{\mathcal{L}} \\
(3) \quad & \|(\widehat{H}_f + 1)^{-\frac{1}{2}} \widehat{a}^*(v) \cdot \widehat{a}(w) (\widehat{H}_f + 1)^{-\frac{1}{2}}\|_{\mathcal{L}(\mathcal{H})} \leq \|\omega^{-\frac{1}{2}} v\|_{\mathcal{L}} \|\omega^{-\frac{1}{2}} w\|_{\mathcal{L}} \\
(4) \quad & \|(\widehat{H}_f + 1)^{-\frac{1}{2}} \widehat{a}(v) \cdot \widehat{a}^*(w) (\widehat{H}_f + 1)^{-\frac{1}{2}}\|_{\mathcal{L}(\mathcal{H})} \\
& \leq (\|\omega^{-\frac{1}{2}} v\|_{\mathcal{L}} + \|v\|_{\mathcal{L}}) (\|\omega^{-\frac{1}{2}} w\|_{\mathcal{L}} + \|w\|_{\mathcal{L}}) \\
(5) \quad & \|(\widehat{H}_h^0 + 1)^{-\frac{1}{2}} \widehat{p} \cdot \widehat{a}^*(v) (\widehat{H}_h^0 + 1)^{-\frac{1}{2}}\|_{\mathcal{L}(\mathcal{H})} \leq \|\omega^{-\frac{1}{2}} v\|_{\mathcal{L}} + \|v\|_{\mathcal{L}} \\
(6) \quad & \|(\widehat{H}_h^0 + 1)^{-\frac{1}{2}} \widehat{a}^*(v) \cdot \widehat{p} (\widehat{H}_h^0 + 1)^{-\frac{1}{2}}\|_{\mathcal{L}(\mathcal{H})} \leq \|\omega^{-\frac{1}{2}} v\|_{\mathcal{L}} + \|v\|_{\mathcal{L}}.
\end{aligned}$$

Appendix B. Measure theoretical techniques

In this appendix, we introduce a general abstract framework where a natural relationship between characteristic equations, Liouville equations

and initial value problems is established and a generalized global flow is constructed for the latter. Our discussion sums up and simplifies several technical results proved in [6, 7, 57], where we then use in a subtle way to construct the global dynamics of the Newton–Maxwell equation. In particular, this part is a crucial step for proving our main Theorems 1.3 and 1.4. It is also worth noting that the techniques involved here are related to the close subject of Gross–Pitaevskii and NLS hierarchies (see [22, 23, 52, 58], and references therein). Firstly, we recall the notion of Liouville equations over Hilbert spaces and state their equivalence to characteristic equations. Secondly, we provide a probabilistic representation to measure-valued solutions of Liouville equations and finally we deduce the existence of a Borel global flow for the initial value problem.

Consider a *separable* Hilbert space \mathcal{H} and a *continuous* vector field $v : \mathbb{R} \times \mathcal{H} \rightarrow \mathcal{H}$ such that it is bounded on bounded sets. In all the sequel I denotes a bounded open interval containing the origin.

B.1. Initial value problem

The following equation defines a non-autonomous initial value problem:

$$(IVP) \quad \begin{cases} \frac{d}{dt} u(t) = v(t, u(t)), \\ u(0) = u_0. \end{cases}$$

Here $u_0 \in \mathcal{H}$. A curve $t \in I \rightarrow u(t) \in \mathcal{H}$ is a strong solution to the above initial value problem if and only if $u \in \mathcal{C}^1(I, \mathcal{H})$ and satisfies (IVP) for some $u_0 \in \mathcal{H}$ and for all $t \in I$. In such case the Duhamel formula,

$$(B.1) \quad u(t) = u(0) + \int_0^t v(s, u(s)) \, ds,$$

holds true for all $t \in I$. Reciprocally, any curve $u \in \mathcal{C}(I, \mathcal{H})$ satisfying (B.1) is a strong solution of the initial value problem (IVP). One of our goals is to establish the existence of global strong solutions to (IVP) under suitable assumptions. The method we describe here is quite different from Cauchy–Lipschitz and fixed-point theories since it uses instead measure theoretical techniques. Note that the vector field $v : \mathbb{R} \times \mathcal{H} \rightarrow \mathcal{H}$ is only assumed to be continuous, hence standard arguments of nonlinear analysis are not applicable.

B.2. Liouville equation

Consider \mathcal{H} as a real Hilbert space endowed with the real scalar product $\Re\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Define the space of smooth cylindrical functions on \mathcal{H} , denoted by $\mathcal{C}_{0,cyl}^{\infty}(\mathcal{H})$, as the set of all functions of the form

$$\phi = \psi \circ \pi,$$

where $\psi \in \mathcal{C}_0^{\infty}(\mathbb{R}^m)$, $m \in \mathbb{N}$, and $\pi : \mathcal{H} \rightarrow \mathbb{R}^m$ the projection of the form:

$$(B.2) \quad \pi : u \longmapsto \pi(u) = (\Re\langle u, e_1 \rangle_{\mathcal{H}}, \dots, \Re\langle u, e_m \rangle_{\mathcal{H}}),$$

where (e_1, \dots, e_m) is an arbitrary orthonormal family of \mathcal{H} . Similarly, one defines for a bounded open interval I , the space of smooth cylindrical functions $\mathcal{C}_{0,cyl}^{\infty}(I \times \mathcal{H})$ as the set of functions ϕ such that :

$$\forall (t, u) \in I \times \mathcal{H}, \quad \phi(t, u) = \psi(t, \pi(u)),$$

for some $\psi \in \mathcal{C}_0^{\infty}(I \times \mathbb{R}^m)$, $m \in \mathbb{N}$, and π as in (B.2).

DEFINITION B.1. — *Let I be a bounded open interval containing the origin. A family of Borel probability measures $\{\mu_t\}_{t \in I}$ on \mathcal{H} is a measure-valued solution of the Liouville equation associated to the vector field $v : \mathbb{R} \times \mathcal{H} \rightarrow \mathcal{H}$ if and only if for all $\phi \in \mathcal{C}_{0,cyl}^{\infty}(I \times \mathcal{H})$:*

$$(B.3) \quad \int_I \int_{\mathcal{H}} \{ \partial_t \phi(t, u) + \Re\langle v(t, u), \nabla_{\mathcal{H}} \phi(t, u) \rangle_{\mathcal{H}} \} d\mu_t(u) dt = 0,$$

where $\nabla_{\mathcal{H}}$ denotes the real differential in the Hilbert space \mathcal{H} .

The following result establishes an equivalence between Liouville equations and characteristic equations. A proof can be taken from [57, Proposition 4.2].

PROPOSITION B.2 (Equivalence). — *Let $v : \mathbb{R} \times \mathcal{H} \rightarrow \mathcal{H}$ be a continuous vector field such that it is bounded on bounded sets. Let $I \ni t \rightarrow \mu_t$ a weakly narrowly continuous curve in $\mathfrak{P}(\mathcal{H})$ such that the following bound holds true:*

$$(B.4) \quad \int_I \int_{\mathcal{H}} \|v(t, u)\|_{\mathcal{H}} d\mu_t(u) dt < \infty.$$

Then the two assertions below are equivalent:

- (1) $(\mu_t)_{t \in I}$ is a solution of the Liouville equation (B.3).
- (2) $(\mu_t)_{t \in I}$ solves the following characteristic equation: $\forall t \in I, \forall y \in \mathcal{H}$,

$$(B.5) \quad \mu_t(e^{2\pi i \Re\langle y, \cdot \rangle_{\mathcal{H}}}) = \mu_0(e^{2\pi i \Re\langle y, \cdot \rangle_{\mathcal{H}}}) + 2\pi i \int_0^t \mu_s(e^{2\pi i \Re\langle y, u \rangle_{\mathcal{H}}} \Re\langle v(s, u), y \rangle_{\mathcal{H}}) ds.$$

B.3. Borel properties

Next we state some convenient Borel properties of the ensemble of solutions and initial conditions for the initial value problem (IVP). Define the space

$$\mathcal{X} = \mathcal{H} \times \mathcal{C}(\bar{I}, \mathcal{H}),$$

endowed with the norm

$$\|(u_0, u)\|_{\mathcal{X}} = \|u_0\|_{\mathcal{H}} + \sup_{t \in \bar{I}} \|u(t)\|_{\mathcal{H}}.$$

LEMMA B.3. — *The ensemble of solutions*

$$(B.6) \quad \mathcal{F}_I = \left\{ (u_0, u) \in \mathcal{X} : \begin{array}{l} u \text{ satisfies (IVP) on } I \\ \text{with the initial condition } u_0 \end{array} \right\}$$

is a Borel subset of \mathcal{X} .

Proof. — The maps

$$\begin{aligned} T_t : \mathcal{X} &\longrightarrow \mathcal{H} \\ (u_0, u) &\longmapsto u(t) - u_0 - \int_0^t v(s, u(s)) ds, \end{aligned}$$

are continuous since the vector field v is continuous, and then bounded on bounded sets. Thus, for all $t \in I$,

$$T_t^{-1}(\{0\})$$

is a Borel set. Take a dense countable set $\{t_i\}_{i \in \mathbb{N}}$ in I , one concludes that

$$\mathcal{F}_I = \bigcap_{i \in \mathbb{N}} T_{t_i}^{-1}(\{0\}),$$

since each u is a continuous curve over \bar{I} and the Duhamel formula (B.1) holds true. \square

The following result of measure theory is useful. A proof can be found in [55, Theorem 3.9].

LEMMA B.4. — *Let X_1 and X_2 be two complete separable metric spaces and $E_1 \subset X_1$, $E_2 \subset X_2$. Suppose that E_1 is a Borel set. Let ψ be a measurable injective map of E_1 into X_2 such that $\psi(E_1) = E_2$. Then E_2 is a Borel set of X_2 .*

LEMMA B.5. — *The ensemble of initial conditions*

$$(B.7) \quad \mathcal{G}_I = \{u_0 \in \mathcal{H} : \exists u \in \mathcal{C}(\bar{I}, \mathcal{H}) \text{ such that } (u_0, u) \in \mathcal{F}_I\}$$

is a Borel subset of \mathcal{H} .

Proof. — Note that the space $\mathcal{C}(\bar{I}, \mathcal{H})$ endowed with the sup norm is a complete separable metric space. Here the separability follows by Stone–Weierstrass theorem and Dini’s theorem. Apply Lemma B.4 with $X_1 = (\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, $X_2 = (\mathcal{H}, \|\cdot\|_{\mathcal{H}})$, the map

$$\begin{aligned} \psi = \wp : \quad \mathcal{X} &\longrightarrow \mathcal{H} \\ (u_0, u) &\longmapsto u_0, \end{aligned}$$

and $E_1 = \mathcal{F}_I$, $E_2 = \mathcal{G}_I$. Then one concludes that

$$\wp(\mathcal{F}_I) = \mathcal{G}_I$$

is a Borel subset of \mathcal{H} . □

A crucial tool for constructing a generalized global flow is the following probabilistic representation of measure-valued solutions to Liouville equations. Such general result is proved in [6, Proposition 4.1] (see also [2]) and justifies the existence of a probability measure η over \mathcal{H} concentrated on strong solutions of the initial value problem (5.1). Define for $t \in I$ the evaluation map Ξ_t over \mathcal{X} as,

$$(B.8) \quad \Xi_t : (u_0, u) \in \mathcal{X} \longmapsto u(t) \in \mathcal{H}.$$

THEOREM B.6 (Probabilistic representation). — *Let*

$$t \in I \longmapsto \mu_t \in \mathfrak{P}(\mathcal{H})$$

be a weakly narrowly continuous curve in $\mathfrak{P}(\mathcal{H})$ satisfying the bound (B.4) and the Liouville equation (B.3). Then there exists a Borel probability measure η over the space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ satisfying:

- (1) *The measure η concentrates on the set \mathcal{F}_I , i.e.,*

$$\eta(\mathcal{F}_I) = 1.$$

- (2) *For all $t \in I$,*

$$\mu_t = (\Xi_t)_\# \eta.$$

COROLLARY B.7. — *Let $t \in I \rightarrow \mu_t \in \mathfrak{P}(\mathcal{H})$ be a weakly narrowly continuous curve in $\mathfrak{P}(\mathcal{H})$ satisfying the bound (B.4) and the Liouville equation (B.3). Suppose that for each $u_0 \in \mathcal{H}$, the initial value problem (IVP) admits at most one strong solution on I . Then the ensemble of initial conditions \mathcal{G}_I in (B.7) is total with respect to μ_0 , i.e.,*

$$\mu_0(\mathcal{G}_I) = 1.$$

Proof. — Using Theorem B.6 one concludes

$$\mu_0(\mathcal{G}_I) = (\Xi_0)_\# \eta(\mathcal{G}_I) = \eta(\Xi_0^{-1}(\mathcal{G}_I)) = \eta(\mathcal{F}_I) = 1,$$

since $\Xi_0^{-1}(\mathcal{G}_I) \supset \mathcal{F}_I$. \square

B.4. Existence of global flows

The following result ensures the existence of global solutions for initial value problems under weak assumptions. In particular, we will apply such result to the Newton–Maxwell equation while proving Theorem 1.3.

THEOREM B.8 (Generalized global flow). — *Consider the initial value problem (IVP) with \mathcal{H} a separable Hilbert space and $v : \mathbb{R} \times \mathcal{H} \rightarrow \mathcal{H}$ a continuous vector field bounded on bounded sets. Suppose (1) and (2) below:*

- (1) *There exists a weakly narrowly continuous curve $t \in \mathbb{R} \rightarrow \mu_t \in \mathfrak{P}(\mathcal{H})$ satisfying the bound (B.4) and the Liouville equation (B.3) on any bounded open interval I containing the origin.*
- (2) *There exists at most one solutions of the initial value problem (IVP) over any bounded open interval I containing the origin.*

Then for μ_0 -almost all initial conditions u_0 in \mathcal{H} there exists a unique global strong solution to the initial value problem (IVP). Moreover, the ensemble of initial conditions

$$(B.9) \quad \mathcal{G} = \left\{ u_0 \in \mathcal{H} : \begin{array}{l} \text{there exists a global strong solution } u \\ \text{of (IVP) with the initial condition } u_0 \end{array} \right\}$$

is a Borel subset of \mathcal{H} satisfying $\mu_0(\mathcal{G}) = 1$ and for any time $t \in \mathbb{R}$ the map

$$\begin{aligned} \Phi_t : \quad \mathcal{G} &\longrightarrow \mathcal{H} \\ u_0 &\longmapsto u(t), \end{aligned}$$

is Borel measurable.

Proof. — Consider the increasing sequence of intervals $I_n = (-n, n)$. Then $\{\mathcal{G}_{I_n}\}_{n \in \mathbb{N}}$ is a decreasing sequence of measurable sets. So, using the uniqueness of strong solutions one concludes that

$$\mathcal{G} = \bigcap_{n \in \mathbb{N}} \mathcal{G}_{I_n}.$$

Furthermore by Corollary B.7,

$$\mu_0(\mathcal{G}) = \lim_{n \rightarrow \infty} \mu_0(\mathcal{G}_n) = 1.$$

To prove the measurability of the maps Φ_t , recall that the projection $\wp|_{\mathcal{F}_I}$ defined in the proof of Lemma B.5 is injective. So, \wp is a bijection from \mathcal{F}_I onto \mathcal{G}_I and its inverse map

$$\begin{aligned}\wp^{-1} : \mathcal{G}_I &\longrightarrow \mathcal{F}_I \\ u_0 &\longmapsto (u_0, u),\end{aligned}$$

is measurable thanks to Lemma B.4. Hence for $t \in \mathbb{R}$ and I containing t , the composition $\Phi_t = \Xi_t \circ \wp^{-1} \circ I$:

$$\begin{aligned}\Phi_t : \mathcal{G} &\xrightarrow{I} \mathcal{G}_I \xrightarrow{\wp^{-1}} \mathcal{F}_I \xrightarrow{\Xi_t} \mathcal{H} \\ u_0 &\longmapsto u_0 \longmapsto (u_0, u) \longmapsto u(t)\end{aligned}$$

yields a well-defined and measurable map. \square

Appendix C. Semiclassical calculus

We state some specific results on convergence towards Wigner measures which are related to semiclassical analysis in finite and infinite dimensions. For a related general discussion, we refer the reader to the work [8]. In particular, Lemmas C.1 and C.2 are key arguments in establishing the convergence towards the characteristic equations in Section 4.4 under optimal regularity conditions on the density matrices $(\varrho_{\hbar})_{\hbar \in (0,1)}$.

LEMMA C.1. — *Let $(\hbar_j)_{j \in \mathbb{N}}$ be a sequence such that $\hbar_j \rightarrow 0$ and $(\varrho_{\hbar_j})_{j \in \mathbb{N}}$ a family of density matrices on \mathcal{H} satisfying $(S_0^{(1)})$ and $(S_1^{(1)})$. Suppose that*

$$(C.1) \quad \mathcal{M}(\varrho_{\hbar_j}, j \in \mathbb{N}) = \{\mu\}$$

for some Borel probability measure $\mu \in \mathfrak{P}(X^0)$. Then the following assertions hold true for any $\xi = (z_0, \alpha_0) \in X^0$ and for any $f \in \mathcal{C}^1(\mathbb{R}^{dn})$ such that ∇f is bounded,

$$(C.2a) \quad \begin{aligned} \lim_{j \rightarrow \infty} \operatorname{Tr} [\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) f(\widehat{p})] \\ = \int_{X^0} e^{i\Im \langle z, z_0 \rangle} e^{\sqrt{2}i\Re \langle \alpha_0, \alpha \rangle_{\mathcal{H}^0}} f(p) d\mu(u), \end{aligned}$$

$$(C.2b) \quad \begin{aligned} \lim_{j \rightarrow \infty} \operatorname{Tr} [\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) f(\widehat{q})] \\ = \int_{X^0} e^{i\Im \langle z, z_0 \rangle} e^{\sqrt{2}i\Re \langle \alpha_0, \alpha \rangle_{\mathcal{H}^0}} f(q) d\mu(u), \end{aligned}$$

where the variable $u = (z, \alpha) \in X^0$.

Proof. — The arguments for (C.2a) and (C.2b) are similar. So, we consider for instance only the first limit. Note that $\text{Tr} [\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) f(\widehat{p})]$ is well-defined thanks to Lemma 4.6, the assumptions $(S_0^{(1)})$ and $(S_1^{(1)})$ and Taylor expansion for f which yields that there exists $c > 0$ such that

$$(C.3) \quad |f(x)| \leq c \langle x \rangle, \quad \forall x \in \mathbb{R}^{dn}.$$

Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{dn})$ be a smooth function such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ in a neighborhood of the origin and $\int_{\mathbb{R}^{dn}} \chi(x) dx = 1$. Define respectively a mollifier and a cutoff function for all $R > 0$ as

$$\begin{aligned} \chi_R(\cdot) &= R^{-dn} \chi(R\cdot), \\ \chi^R(\cdot) &= \chi(R^{-1}\cdot). \end{aligned}$$

For simplicity, we denote

$$(C.4) \quad e^{i\Im \langle z, z_0 \rangle} e^{\sqrt{2}i\Re \langle \alpha_0, \alpha \rangle_{S^0}} = e^{iQ(\xi, u)}.$$

We use an approximation argument based on the inequality:

$$\left| \text{Tr} [\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) f(\widehat{p})] - \int_{X^0} e^{iQ(\xi, u)} f(p) d\mu \right| \leq A + B + C,$$

where

$$\begin{aligned} A &= \left| \text{Tr} [\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) (f - \chi_R * f \chi^R)(\widehat{p})] \right|, \\ B &= \left| \text{Tr} [\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) \chi_R * f \chi^R(\widehat{p})] - \int_{X^0} e^{iQ(\xi, u)} (\chi_R * f \chi^R)(p) d\mu \right|, \\ C &= \left| \int_{X^0} e^{iQ(\xi, u)} (f - \chi_R * f \chi^R)(p) d\mu \right|. \end{aligned}$$

Now, we estimate A , B and C . Using Lemma 4.6 and assumptions $(S_0^{(1)})$ and $(S_1^{(1)})$, one obtains

$$\begin{aligned} A &\leq \|(\widehat{p}^2 + 1)^{1/2} \varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) (\widehat{p}^2 + 1)^{1/2}\|_{\mathcal{L}^1(\mathcal{H})} \\ &\quad \|(\widehat{p}^2 + 1)^{-1} (f - \chi_R * f \chi^R)(\widehat{p})\|_{\mathcal{L}(L^2(\mathbb{R}^{dn}))} \\ &\lesssim \sup_{x \in \mathbb{R}^{dn}} \left| \frac{f(x) - (\chi_R * f \chi^R)(x)}{|x|^2 + 1} \right| \\ &\lesssim \sup_{x \in \mathbb{R}^{dn}} \left| \frac{f(x) - f \chi^R(x)}{|x|^2 + 1} \right| + \sup_{x \in \mathbb{R}^{dn}} \left| \frac{f(x) - \chi_R * f(x)}{|x|^2 + 1} \right| \xrightarrow{R \rightarrow 0} 0. \end{aligned}$$

On the other hand, Lemma 4.12 shows that the Wigner measure μ satisfies the bound $\int_{X^0} (p^2 + q^2) d\mu(u) < \infty$. Hence one obtains

$$C \leq \sup_{p \in \mathbb{R}^{dn}} \left| \frac{f(p) - (\chi_R * f\chi^R)(p)}{p^2 + 1} \right| \int_{X^0} (p^2 + 1) d\mu(u) \xrightarrow{R \rightarrow 0} 0.$$

We estimate B . Since $g^R = \chi_R * f\chi^R \in \mathcal{C}_0^\infty(\mathbb{R}^{dn})$, then one can write using the Fourier Transform and the Heisenberg–Weyl operators,

$$\begin{aligned} g^R(\widehat{p}) &= \int_{\mathbb{R}^{dn}} \mathcal{F}(g^R)(k) e^{2\pi i k \cdot \widehat{p}} dk \\ &= \int_{\mathbb{R}^{dn}} \mathcal{F}(g^R)(k) \widehat{T}(-2\pi k, 0) dk \\ &= \int_{\mathbb{R}^{dn}} \mathcal{F}(g^R)(k) \mathcal{W}_{\hbar_j}(-2\pi k, 0, 0) dk, \end{aligned}$$

with absolutely convergent Bochner integrals in $\mathcal{L}(\mathcal{H})$. Hence by Fubini's theorem for all $\xi = (p, q, \alpha) \in X^0$,

$$\begin{aligned} \text{Tr}[\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) g^R(\widehat{p})] &= \int_{\mathbb{R}^{dn}} \mathcal{F}(g^R)(k) \text{Tr}[\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) \mathcal{W}_{\hbar_j}(-2\pi k, 0, 0)] dk \\ &= \int_{\mathbb{R}^{dn}} \mathcal{F}(g^R)(k) \text{Tr}[\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(-2\pi k + p, q, \alpha)] e^{-i\pi \hbar_j k \cdot p} dk. \end{aligned}$$

So, the dominated convergence theorem, Definition 1.1 and Fubini's theorem yield

$$\begin{aligned} \lim_{j \rightarrow \infty} \text{Tr}[\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) g^R(\widehat{p})] &= \int_{\mathbb{R}^{dn}} \mathcal{F}(g^R)(k) \int_{X^0} e^{iQ(\xi, u)} e^{2\pi i k \cdot p} d\mu(u) dk \\ &= \int_{X^0} g^R(p) e^{iQ(\xi, u)} d\mu(u). \end{aligned}$$

Therefore, the left-hand side of A is arbitrary small when choosing R sufficiently large and $j \rightarrow \infty$. \square

LEMMA C.2. — *Let $(\hbar_j)_{j \in \mathbb{N}}$ be a sequence such that $\hbar_j \rightarrow 0$ and $(\varrho_{\hbar_j})_{j \in \mathbb{N}}$ a family of density matrices on \mathcal{H} satisfying $(S_0^{(1)})$, $(S_1^{(1)})$. Suppose that*

$$(C.5) \quad \mathcal{M}(\varrho_{\hbar_j}, j \in \mathbb{N}) = \{\mu\}$$

for some Borel probability measure $\mu \in \mathfrak{P}(X^0)$. Then for all $\xi = (z_0, \alpha_0) \in X^0$ and for all $f \in \mathfrak{H}^0$,

$$(C.6a) \quad \lim_{j \rightarrow \infty} \text{Tr}[\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) \widehat{a}^*(f)] = \int_{X^0} e^{i\Im\langle z, z_0 \rangle} e^{\sqrt{2}i\Re\langle \alpha_0, \alpha \rangle_{\mathfrak{H}^0}} \langle \alpha, f \rangle_{\mathfrak{H}^0} d\mu(u),$$

$$(C.6b) \quad \lim_{j \rightarrow \infty} \text{Tr}[\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) \widehat{a}(f)] = \int_{X^0} e^{i\Im\langle z, z_0 \rangle} e^{\sqrt{2}i\Re\langle \alpha_0, \alpha \rangle_{\mathfrak{H}^0}} \langle f, \alpha \rangle_{\mathfrak{H}^0} d\mu(u),$$

where the variable $u = (p, q, \alpha) \in X^0$ and $z = q + ip$.

Proof. — We use the same notation as (C.4). By linearity, it suffices to prove the following limits,

$$(C.7a) \quad \lim_{j \rightarrow \infty} \text{Tr}[\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) \widehat{\phi}(f)] = 2 \int_{X^0} e^{iQ(\xi, u)} \Re\langle \alpha, f \rangle_{\mathfrak{H}^0} d\mu(u),$$

$$(C.7b) \quad \lim_{j \rightarrow \infty} \text{Tr}[\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) \widehat{\pi}(f)] = 2i \int_{X^0} e^{iQ(\xi, u)} \Im\langle f, \alpha \rangle_{\mathfrak{H}^0} d\mu(u),$$

where $\widehat{\phi}$ and $\widehat{\pi}$ are the fields operators defined as

$$(C.8) \quad \widehat{\phi}(f) = \widehat{a}^*(f) + \widehat{a}(f), \quad \widehat{\pi}(f) = i\widehat{a}^*(f) - i\widehat{a}(f).$$

The arguments for (C.7a) and (C.7b) are similar; so we will consider only the first one. Recall that $\widehat{\phi}(\alpha_0)$ is a self-adjoint operator satisfying $W(f) = e^{\frac{i}{\sqrt{2}}\widehat{\phi}(f)}$. We will use an approximation argument based on the inequality:

$$\left| \text{Tr}[\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) \widehat{\phi}(f)] - 2 \int_{X^0} e^{iQ(\xi, u)} \Re\langle \alpha, f \rangle_{\mathfrak{H}^0} d\mu \right| \leq A + B + C,$$

where

$$\begin{aligned} A &= \left| \text{Tr} \left[\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) \left(\widehat{\phi}(f) - \frac{W(\sqrt{2}sf) - I}{is} \right) \right] \right| \\ B &= \left| \text{Tr} \left[\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) \frac{W(\sqrt{2}sf) - I}{is} \right] - \int_{X^0} e^{iQ(\xi, u)} \frac{e^{2si\Re\langle \alpha, f \rangle_{\mathfrak{H}^0}} - 1}{is} d\mu \right| \\ C &= \left| \int_{X^0} e^{iQ(\xi, u)} \left(\frac{e^{2si\Re\langle \alpha, f \rangle_{\mathfrak{H}^0}} - 1}{is} - 2\Re\langle \alpha, f \rangle_{\mathfrak{H}^0} \right) d\mu \right| \end{aligned}$$

The dominated convergence theorem and Lemma 4.12 imply that C tends to zero as $s \rightarrow 0$ since the assumptions $(S_0^{(1)})$ and $(S_1^{(1)})$ lead the bound

$$\int_{X^0} \|\alpha\|_{\mathfrak{H}^0}^2 d\mu < \infty.$$

On the other hand, for any $s \in \mathbb{R} \setminus \{0\}$, B vanishes when $\hbar_j \rightarrow 0$ as a consequence of Definition 1.1. So, it remains to prove that A can be made arbitrary small uniformly in \hbar_j by choosing s sufficiently close to 0. In fact, one has $A \leq DE$, where

$$D = \|(\hat{N}_{\hbar} + 1)^{1/2} \varrho_{\hbar} \mathcal{W}_{\hbar_j}(\xi) (\hat{N}_{\hbar} + 1)^{1/2}\|_{\mathcal{L}^1(\mathcal{H})},$$

$$E = \left\| (\hat{N}_{\hbar} + 1)^{-1/2} \left(\hat{\phi}(f) - \frac{W(\sqrt{2}s f) - I}{is} \right) (\hat{N}_{\hbar} + 1)^{-1/2} \right\|_{\mathcal{L}(\mathcal{H})}.$$

So, according to Lemma 4.4 and assumption $(S_1^{(1)})$, D is uniformly bounded with respect to \hbar_j . Moreover, Taylor expansion yields

$$\hat{\phi}(f) - \frac{W(\sqrt{2}s f) - I}{is} = \frac{1}{s} \int_0^s \hat{\phi}(f) (I - W(\sqrt{2}\tau f)) d\tau,$$

and on the other hand the uniform inequality [8, Lemma 3.1] gives

$$\|(I - W(\sqrt{2}\tau f))(\hat{N}_{\hbar} + 1)^{-1/2}\|_{\mathcal{L}(\mathcal{H})} \lesssim |\tau| \|f\|_{\mathfrak{H}^0}.$$

Hence by Lemma A.3 and the above inequalities one obtains

$$E \lesssim \left| \frac{1}{s} \int_0^s |\tau| d\tau \right| \xrightarrow{s \rightarrow 0} 0,$$

uniformly with respect to \hbar_j . □

LEMMA C.3. — *Let $(\hbar_j)_{j \in \mathbb{N}}$ be a sequence such that $\hbar_j \rightarrow 0$ as $j \rightarrow \infty$ and $(\varrho_{\hbar_j})_{j \in \mathbb{N}}$ a family of density matrices on \mathcal{H} satisfying $(S_0^{(1)})$ and $(S_1^{(1)})$. Suppose that*

$$(C.9) \quad \mathcal{M}(\varrho_{\hbar_j}, j \in \mathbb{N}) = \{\mu\}$$

for some Borel probability measure $\mu \in \mathfrak{P}(X^0)$. Then for all $\xi = (z_0, \alpha_0) \in X^0$, $i = 1, \dots, n$ and $f \in \mathfrak{H}^0$,

$$\begin{aligned} & \lim_{j \rightarrow \infty} \text{Tr} \left[\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) \hat{a}^*(e^{-2\pi i k \cdot \hat{q}_i} f) \right] \\ &= \int_{X^0} e^{i\Im \langle z, z_0 \rangle} e^{\sqrt{2}i\Re \langle \alpha_0, \alpha \rangle_{\mathfrak{H}^0}} \langle \alpha, e^{-2\pi i k \cdot q_i} f \rangle_{\mathfrak{H}^0} d\mu(u), \\ & \lim_{j \rightarrow \infty} \text{Tr} \left[\varrho_{\hbar_j} \mathcal{W}_{\hbar_j}(\xi) \hat{a}(e^{-2\pi i k \cdot \hat{q}_i} f) \right] \\ &= \int_{X^0} e^{i\Im \langle z, z_0 \rangle} e^{\sqrt{2}i\Re \langle \alpha_0, \alpha \rangle_{\mathfrak{H}^0}} \langle e^{-2\pi i k \cdot q_i} f, \alpha \rangle_{\mathfrak{H}^0} d\mu(u), \end{aligned}$$

where the variable $u = (p, q, \alpha) \in X^0$ and $z = q + ip$.

Proof. — It is a consequence of Lemma C.2. □

LEMMA C.4. — Let $(\varrho_{\hbar})_{\hbar \in (0,1)}$ be a family of density matrices on \mathcal{H} satisfying $(S_0^{(1)})$ and $(S_1^{(1)})$. Define

$$(C.10) \quad \tilde{\varrho}_{\hbar}(t) = e^{-i\frac{t}{\hbar}\hat{H}_{\hbar}^0} \varrho_{\hbar} e^{i\frac{t}{\hbar}\hat{H}_{\hbar}^0},$$

then the family $(\tilde{\varrho}_{\hbar}(t))_{\hbar \in (0,1)}$ satisfies the assumption (4.10) with $\delta = 1$ and

$$\mathcal{M}(\varrho_{\hbar}, \hbar \in (0, 1)) = \{(\Phi_t^0)_\# \tilde{\mu}, \tilde{\mu} \in \mathcal{M}(\tilde{\varrho}_{\hbar}(t), \hbar \in (0, 1))\},$$

where Φ_t^0 is the free flow defined on X^0 as

$$(C.11) \quad \Phi_t^0(p, q, \alpha) = \left((p_i)_{i=1, \dots, n}; \left(q_i + t \frac{p_i}{m_i} \right)_{i=1, \dots, n}; (e^{-it\omega} \alpha_\lambda)_{\lambda=1, \dots, d-1} \right).$$

Proof. — We can assume that all the masses $m_i = 1$. First notice that

$$\mathrm{Tr}[\tilde{\varrho}_{\hbar}(t) (\hat{N}_{\hbar} + \hat{H}_{\hbar}^0)] = \mathrm{Tr}[\varrho_{\hbar} (\hat{N}_{\hbar} + \hat{H}_{\hbar}^0)],$$

since \hat{H}_{\hbar}^0 strongly commutes with the operator \hat{N}_{\hbar} . So, one only needs to verify that

$$\mathrm{Tr}[\tilde{\varrho}_{\hbar}(t) \hat{q}^2] < \infty.$$

The last statement follows by Lemma 3.12 and the spectral decomposition of ϱ_{\hbar} . Hence this ensures that the family $(\tilde{\varrho}_{\hbar}(t))_{\hbar \in (0,1)}$ satisfies the assumption (4.10) with $\delta = 1$. Now take $\mu \in \mathcal{M}(\varrho_{\hbar}, \hbar \in (0, 1))$. Then there exists a sequence \hbar_k such that $\hbar_k \rightarrow 0$ and

$$\mathcal{M}(\varrho_{\hbar_k}, k \in \mathbb{N}) = \{\mu\}.$$

In particular, Definition 1.1 implies that for any $\xi = (p_0, q_0, \alpha_0) \equiv (z_0, \alpha_0) \in X^0$,

$$\lim_{k \rightarrow \infty} \mathrm{Tr}[\varrho_{\hbar_k} \mathcal{W}_{\hbar_k}(\xi)] = \int_{X^0} e^{i\Im m \langle z, z_0 \rangle} e^{\sqrt{2}i\Re \langle \alpha_0, \alpha \rangle_{\mathcal{S}^0}} d\mu(u),$$

where $u = (z, \alpha)$. So, the latter identity yields,

$$\begin{aligned} \mathrm{Tr}[\tilde{\varrho}_{\hbar_k}(t) \mathcal{W}_{\hbar_k}(\xi)] &= \mathrm{Tr} \left[\varrho_{\hbar_k} e^{i\frac{t}{\hbar_k} \hat{H}_{\hbar_k}^0} \mathcal{W}_{\hbar_k}(\xi) e^{-i\frac{t}{\hbar_k} \hat{H}_{\hbar_k}^0} \right] \\ &= \mathrm{Tr}[\varrho_{\hbar_k} \mathcal{W}_{\hbar_k}(\xi_t)], \end{aligned}$$

with $\xi_t = (p_0, q_0 + tp_0, e^{-it\omega} \alpha_0) \equiv ((q_0 + tp_0) + ip_0, e^{-it\omega} \alpha_0) \in X^0$. Hence one obtains

$$\begin{aligned} \lim_{k \rightarrow \infty} \operatorname{Tr} [\tilde{\varrho}_{\hbar k}(t) \mathcal{W}_{\hbar k}(\xi)] &= \int_{X^0} e^{i\Im \langle z, z_0(t) \rangle} e^{\sqrt{2}i\Re \langle \alpha_0(t), \alpha \rangle_{\mathfrak{H}^0}} d\mu(u) \\ &= \int_{X^0} e^{i\Im \langle z, z_0 \rangle} e^{\sqrt{2}i\Re \langle \alpha_0, \alpha \rangle_{\mathfrak{H}^0}} d(\Phi_{-t}^0)_\# \mu(u), \end{aligned}$$

where $u = (z, \alpha)$, $z_0(t) = (q_0 + tp_0) + ip_0$ and $\alpha_0(t) = e^{-it\omega} \alpha_0$. Note that we have used for all $u = (z, \alpha) \equiv (p, q, \alpha) \in X^0$ and $t \in \mathbb{R}$,

$$\begin{aligned} \Im \langle q + ip, z_0(t) \rangle &= \Im \langle q - tp + ip, z_0 \rangle, \\ \Re \langle \alpha, \alpha_0(t) \rangle_{\mathfrak{H}^0} &= \Re \langle e^{it\omega} \alpha, \alpha_0 \rangle_{\mathfrak{H}^0}. \end{aligned}$$

Therefore $\mathcal{M}(\tilde{\varrho}_{\hbar k}(t), k \in \mathbb{N}) = \{\tilde{\mu}\}$ and $\tilde{\mu} = (\Phi_{-t}^0)_\# \mu$ hold true. This proves

$$\mathcal{M}(\varrho_{\hbar}, \hbar \in (0, 1)) \subset \{(\Phi_t^0)_\# \tilde{\mu}, \tilde{\mu} \in \mathcal{M}(\tilde{\varrho}_{\hbar}, \hbar \in (0, 1))\}.$$

On the other hand, starting from $\tilde{\mu} \in \mathcal{M}(\tilde{\varrho}_{\hbar}(t), \hbar \in (0, 1))$ and mimicking the same argument one shows the opposite inclusion. \square

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