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CONSTRUCTIONS OF q-HYPERBOLIC KNOTS

by Efstratia KALFAGIANNI & Joseph M. MELBY (*)

ABSTRACT. — We use Dehn surgery methods to construct infinite families of hyperbolic knots in the 3-sphere satisfying a weak form of the Turaev–Viro invariants volume conjecture. The results have applications to a conjecture of Andersen, Masbaum and Ueno about quantum representations of surface mapping class groups. We obtain an explicit family of pseudo-Anosov mapping classes acting on surfaces of any genus and with one boundary component that satisfy the conjecture.

RÉSUMÉ. — Nous utilisons des méthodes de chirurgie de Dehn pour construire des familles infinies de noeuds hyperboliques dans S^3 vérifiant une forme faible de la conjecture du volume pour les invariants de Turaev–Viro. Ces résultats ont des applications à la conjecture d'Andersen–Masbaum–Ueno sur les représentations quantiques des groupes de difféotopie des surfaces. Nous obtenons une famille explicite de difféotopies pour les surfaces de chaque genre à une composante de bord qui vérifient la conjecture

1. Introduction

The Turaev–Viro invariants of a compact 3-manifold M are a family of $\mathbb{R}$-valued homeomorphism invariants $TV_r(M;q)$ parameterized by an integer $r \geqslant 3$ depending on a $2r$-th root of unity q. They were originally defined in terms of triangulations of compact 3-manifolds [31] and were later related to skein-theoretic quantum invariants such as the Reshetikhin–Turaev and colored Jones invariants [5, 6, 18, 27]. In this paper, we are primarily concerned with the geometric data recovered from the asymptotic behavior of the Turaev–Viro invariants at the root $q = e^{\frac{2\pi i}{r}}$.

Keywords: annulus presentation, Dehn surgery, double twist knot, hyperbolic knot, pseudo-Anosov mapping class, q-hyperbolic knot, quantum representation, Turaev–Viro invariants.
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For a compact 3-manifold M, that is closed or has toroidal boundary, let

$$lTV(M) := \liminf_{r\to\infty,\ r \text{ odd}} \frac{2\pi}{r} \log \left|\mathrm{TV}_r\left(M; q = e^{\frac{2\pi i}{r}}\right)\right|,$$

A 3-manifold M with $lTV(M) > 0$ is called *q-hyperbolic.* We will say that a knot K is q-hyperbolic if the complement $M_K := \overline{S^3 \setminus n(K)}$ is q-hyperbolic, where $n(K)$ is a tubular neighborhood of K.

Chen and Yang [11] conjectured that if M is hyperbolic, with volume $\mathrm{vol}(M)$, then $lTV(M) = \mathrm{vol}(M) > 0$. A related weaker conjecture, which was stated and studied by Detcherry and Kalfagianni [15, 16, 17], is the following:

Conjecture 1.1 (Exponential Growth Conjecture). — *Let M be a compact, oriented 3-manifold with empty or toroidal boundary with Gromov norm $\|M\|$. Then, M is q-hyperbolic if and only if $\|M\| > 0$.*

By the geometrization theorem, a compact, prime, oriented 3-manifold M with empty or toroidal boundary can be cut along a canonical collection of tori into pieces that are either Seifert fibered manifolds or hyperbolic. Moreover, M has positive Gromov norm precisely when this decomposition contains hyperbolic pieces. In this language, Conjecture 1.1 asserts that M is q-hyperbolic if and only if its geometric decomposition contains hyperbolic manifolds. One direction of the conjecture, namely that if M is q-hyperbolic, then $\|M\| > 0$, follows from the main result of [16]. The other direction was shown in [15] to imply a conjecture of Andersen, Masbaum, and Ueno [4] on the geometric content of quantum representations of mapping class groups of surfaces.

The purpose of this paper is to give constructions of the first infinite families of hyperbolic knots in the 3-sphere that are shown to be q-hyperbolic. The only knot complements in the 3-sphere for which the asymptotic behavior of the Turaev–Viro invariants has been explicitly understood is the figure-eight and all of its 2-cables [14, 18]. The only hyperbolic knot among these is the figure-eight knot. On the other hand, the volume conjecture of [11] has been proved for all hyperbolic 3-manifolds that are obtained by Dehn filling the figure-eight knot complement [25, 32]. Our constructions combine these results, with a result of [16] about the behavior of the Turaev–Viro invariants under Dehn-filling, and with several Dehn surgery techniques. Our results also have new applications to the conjecture of [4].

1.1. Main results

Given a knot K, let μ, λ denote a set of canonical generators for the homology group $H_1(\partial(n(K)))$. For a simple closed curve s on $\partial(n(K))$, we denote by $[s] = p\mu + q\lambda$ its class in $H_1(\partial(n(K)))$, where p, q are relatively prime integers. Recall that s is completely determined, up to isotopy, by the fraction $p/q \in \mathbb{Q} \cup \{\infty\}$. We will use $M_K(p/q)$ to denote the 3-manifold obtained by Dehn-filling M_K along the slope s determined by p/q.

For $m, n \in \mathbb{Z}$, let the *double twist knot* $D(m, n)$ with m vertical half-twists and n horizontal half-twists be as in Figure 1.1. For example, $D(2, -2)$ is the figure-eight knot and $D(2, 2)$ is the left-handed trefoil. With the exception of the unknot and the two trefoils, the double twist knots are hyperbolic. We show the following:

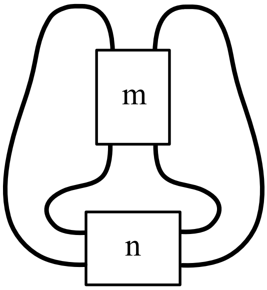

Figure 1.1. A double twist knot $D(m, n)$ diagram with m vertical half-twists and n horizontal half-twists.

THEOREM 1.2. — *For any integer $n \neq 0, -1$, the following are true:*

(a) *The knots $D_n := D(2n, -3)$ and $D'_n := D(2n, -2)$ are q-hyperbolic.*
(b) *The 3-manifolds $M_n := M_{D_n}(4n + 1)$ and $M'_n := M_{D'_n}(1)$ are hyperbolic and q-hyperbolic.*
(c) *We have*

$$lTV(M_{D_n}) \geqslant \mathrm{vol}(M_n), \quad \text{and} \quad lTV(M_{D'_n}) \geqslant \mathrm{vol}(M'_n).$$

Using Theorem 1.2, we may conclude that many low-crossing knots are q-hyperbolic. We refer to Tables 5.1 and 5.2 in Section 5.

The knots $D(2n, -3)$ are fibered when $n < -1$ and the monodromies of their fibrations provide explicit families of pseudo-Anosov mapping classes acting on surfaces with a single boundary component that satisfy the AMU conjecture. See Theorem 4.3. These are the first examples known to satisfy this conjecture that are constructed as monodromies of fibered knots in

S^3. The examples of [15] are coming from monodromies of fibered links of multiple components, while the examples [17] come from monodromies of fibered knots in closed q-hyperbolic 3-manifolds.

A slope p/q is called *non-characterizing* for a knot $K \subset S^3$ if there is a knot K' that is not equivalent to K and such that $M_K(p/q)$ is homeomorphic to $M_{K'}(p/q)$. The articles [1, 2, 3] give constructions of knots that admit infinitely many non-characterizing slopes. Combining their techniques and results with Theorem 1.2, we are able to construct new infinite families of q-hyperbolic knots.

Theorem 1.3. — *There is an infinite set of knots $\mathcal{K}$ such that:*

(a) *Every knot in $\mathcal{K}$ is q-hyperbolic.*
(b) *For every $K \in \mathcal{K}$, $M_K(-7)$ is homeomorphic to $M_{4_1}(-7/2)$ and it is q-hyperbolic.*
(c) *We have*

$$lTV(M_K) \geqslant \mathrm{vol}(M_{4_1}(-7/2)) \approx 1.649610.$$

(d) *No two knots in $\mathcal{K}$ are equivalent.*

To apply the methods of [1] one needs to start with a knot K_0 that admits an "annulus presentation". Then, for any non-zero $n \in \mathbb{N}$, one applies a certain operation called an "n-fold annulus twist" repeatedly to generate a family of knots $\mathcal{K}$, so that for any $K \in \mathcal{K}$ we have $M_K(n) \cong M_{K_0}(n)$. The method of the proof of Theorem 1.3 is as follows: First we show that the six crossing knot 6_2 is q-hyperbolic and that the 3-manifold $M_{6_2}(-7)$, obtained by -7-surgery on 6_2, is homeomorphic to $M_{4_1}(-7/2)$ and is q-hyperbolic. Then we verify that the knot 6_2 has an "annulus presentation" to which we apply a "-7-fold annulus twist" inductively, to generate a family of knots $\mathcal{K}$. In this case, the annulus presentation of 6_2 is nice in a certain sense (see Remark 3.7), and we are able to argue that the resulting knots have mutually distinct Alexander polynomials. The reader is referred to Section 3 for the definitions of annulus presentations and annulus twists and for the details of our construction. The annulus twisting technique also applies to each of the knots $D'_n := D(2n, -2)$ to produce families of q-hyperbolic knots. However, in this case we don't know whether the resulting knots are necessarily distinct. We have the following:

Theorem 1.4. — *For any $|n| > 1$, let $D'_n := D(2n, -2)$. There is a sequence knots $\{K^i_n\}_{i \in \mathbb{N}}$ such that, for any $i \in \mathbb{N}$,*

(a) *the knot K^i_n is q-hyperbolic;*

(b) *the 3-manifold* $M_{K^i_n}(1)$ *is homeomorphic to* $M_{D'_n}(1)$ *and it is* q*-hyperbolic.*

1.2. Organization

The paper is organized as follows: We give a proof of Theorem 1.2 in Section 2. In Section 3, first we recall the definitions and results from [1, 2, 3] relevant here, and then we prove Theorems 1.3 and 1.4. We apply our results to the conjecture of [4] in Section 4. Finally, in Section 5 we list all knots up to ten crossings, and all knots from the SnapPy census of hyperbolic cusped 3-manifolds with triangulation complexity at most nine, that can be shown to be q-hyperbolic using our methods.

Acknowledgments

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2. q-hyperbolic double twist knots

In this section, we will show the q-hyperbolicity of two families of knots in S^3 which share hyperbolic Dehn surgeries with the figure-eight knot.

Suppose M is a compact 3-manifold with empty or toroidal boundary. If M is hyperbolic, by Mostow rigidity the volume of a hyperbolic metric is a topological invariant of M denoted by $\mathrm{vol}(M)$. If M is disconnected the total volume is the sum of volumes over all connected components. In general, by the geometrization theorem, M admits a unique decomposition along tori into manifolds with toroidal boundary that are Seifert fibered spaces or hyperbolic. Let H_M denote the union of the hyperbolic components in the geometric decomposition of M. For the purposes of this paper we define the Gromov norm of M by

$$\|M\| := \frac{\mathrm{vol}(H_M)}{v_{\mathrm{tet}}},$$

where $v_{\mathrm{tet}} = 1.01494\ldots$ is the volume of a regular ideal tetrahedron and $\mathrm{vol}(H_M)$ denotes the total volume of H_M. By work of Thurston [29], the

Gromov norm of 3-manifolds with toroidal boundary does not increase under Dehn-filling. That is, if M is a 3-manifold with toroidal boundary, and M' is obtained by Dehn-filling of some components of ∂M, then $\|M'\| \leqslant \|M\|$.

The asymptotics of the Turaev–Viro invariants have an analogous property, as shown by Detcherry–Kalfagianni [16].

Theorem 2.1 ([16], Corollary 5.3). — *Let M be a compact oriented 3-manifold with nonempty toroidal boundary and let M' be a manifold obtained from M by Dehn-filling some of the boundary components. Then*

$$lTV(M') \leqslant lTV(M).$$

In particular, if M' is q-hyperbolic then M is q-hyperbolic.

Let K be a knot in the 3-sphere with complement M_K. Recall that isotopy classes of simple closed curves on ∂M_K are in one to one correspondence with slopes $p/q \in \mathbb{Q} \cup \{1/0\}$. Slopes of the form $p/1$ will be denoted by p. Given a slope p/q, let $M_K(p/q)$ denote the 3-manifold obtained by p/q-surgery along K (i.e. $M_K(p/q)$ is obtained by a Dehn-filling of M_K along the simple closed curve of slope p/q on ∂M_K). If K is hyperbolic and $M_K(p/q)$ is not hyperbolic, we say that p/q is an *exceptional slope* of K.

Let M_{4_1} denote the complement of figure-eight knot 4_1. The following is well known:

Proposition 2.2. — *The set of the exceptional slopes of the figure-eight knot is $E_{4_1} := \{0, 1/0, \pm1, \pm2, \pm3, \pm4\}$. Thus for any $p/q \notin E_{4_1}$ the 3-manifold $M_{4_1}(p/q)$ is hyperbolic.*

The asymptotics of the Turaev–Viro invariants of hyperbolic manifolds obtained by surgery on the figure-eight knot are well understood. Ohtsuki [25] proved that hyperbolic manifolds obtained by integral surgeries on 4_1 satisfy the volume conjecture, and the result was extended to rational surgeries by Wong and Yang [32]. Hence we have the following:

Theorem 2.3 ([25, 32]). — *For any non-exceptional slope p/q of the knot 4_1 we have*

$$lTV(M_{4_1}(p/q)) = \mathrm{vol}(M_{4_1}(p/q)),$$

and hence, in particular, $M_{4_1}(p/q)$ is q-hyperbolic.

Recall that for $m, n \in \mathbb{Z}$, we denote by $D(m, n)$ the double twist knot shown in Figure 1.1.

Next we construct two families of q-hyperbolic double twist knots. The families are parametrized by integers n, and are denoted by

$$\{D_n := D(2n,-3)\}_{n\geqslant 0} \text{ and } \{D'_n := D(2n,-2)\}_{n\geqslant 0}.$$

Theorem 1.2. — *For any integer $n \neq 0, -1$, the following are true:*

(a) *The knots $D_n := D(2n,-3)$ and $D'_n := D(2n,-2)$ are q-hyperbolic.*
(b) *The 3-manifolds $M_n := M_{D_n}(4n+1)$ and $M'_n := M_{D'_n}(1)$ are hyperbolic and q-hyperbolic.*
(c) *We have*

$$lTV(M_{D_n}) \geqslant \mathrm{vol}(M_n), \quad \textit{and} \quad lTV(M_{D'_n}) \geqslant \mathrm{vol}(M'_n).$$

For the proof of Theorem 1.2 we will need the following lemma:

Lemma 2.4. — *For any $n \in \mathbb{Z}$ we have the following:*

(a) *The 3-manifold $M_{4_1}((-4n-1)/n)$ is homeomorphic to $M_{D_n}(4n+1)$.*
(b) *The 3-manifold $M_{4_1}(-1/n)$ is homeomorphic to $M_{D'_n}(1)$.*

Proof. — For $n = 0$ both (a) and (b) are trivially true: For, both D_0, D'_0 are the trivial knot and we have: $M_{4_1}(1/0) \cong M_{D_0}(1) = M_{D'_0}(1) \cong S^3$.

Next suppose that $n \neq 0$. Part (a) follows from the fact that the 3-manifold $M_{4_1}(-(4n+1)/n)$ is related to $M_{D_n}(4n+1)$ by a sequence of Kirby–Rolfsen–Rourke moves. These moves are well known to preserve 3-manifolds up to homeomorphism. See for example [28, Chapter 9]. The particular sequence of moves required in this case is shown in Figure 2.1.

To describe the moves required in more detail, let us recall that, as is customary in the Kirby–Rolfsen–Rourke calculus, one indicates the 3-manifold M obtained by Dehn-filling along a link L in S^3 by a diagram of L with each component labeled by the surgery slope used for the component. For components where the surgery coefficient is $1/0$, we will omit the label (such surgery is called ∞-surgery and it produces back S^3). Such a diagram is called a *surgery diagram* of M.

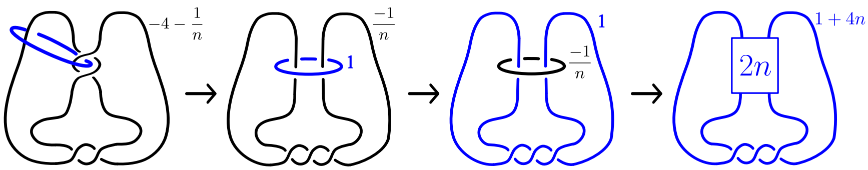

Figure 2.1. Kirby–Rolfsen–Rourke calculus moves showing that $M_{4_1}((-4n-1)/n)$ is homeomorphic to $M_{D_n}(4n+1)$.

(i) The 3-manifold $M_{4_1}(-(4n+1)/n)$ has a surgery diagram consisting of a knot diagram for 4_1 labeled by $(-4n-1)/n = -4 - 1/n$. In the leftmost panel of Figure 2.1, we have inserted an unknotted component U, shown in blue, on which the surgery coefficient is $1/0 = \infty$ and such that it has linking number ± 2 with the figure-eight knot component. That is $|\mathrm{lk}(U, 4_1)| = 2$. A -1-twist along U produces the second surgery diagram in the sequence. Note that the surgery coefficient of the component corresponding to 4_1 has now changed to $-(4n+1)/n + (\mathrm{lk}(U, 4_1))^2 = -1/n$. This operation is also known as a *blow up*.

(ii) The surgery diagram shown in the third panel of Figure 2.1 is obtained by that of the second panel by ambient isotopy that interchanges the two components of the underlying link.

(iii) Finally, performing n-twists on the component labelled by $-1/n$ gives the rightmost panel of Figure 2.1, which represents a surgery diagram of $M_{D_n}(4n+1)$. The operation of performing this $(-1/n)$-surgery on an unknotted component is also known as a *blow down*.

For part (b), a similar sequence of Kirby–Rolfsen–Rourke calculus moves, shown in Figure 2.2, proves that $M_{4_1}(-1/n)$ is homeomorphic $M_{D'_n}(1)$. Note that this time the inserted unknotted component U, drawn in blue in the leftmost panel of Figure 2.2, has zero linking number with 4_1. That is, $\mathrm{lk}(U, 4_1) = 0$. In this case, the surgery coefficient of the component corresponding to 4_1 is unchanged under the blow up operation since $-1/n + (\mathrm{lk}(U, 4_1))^2 = -1/n$. ☐

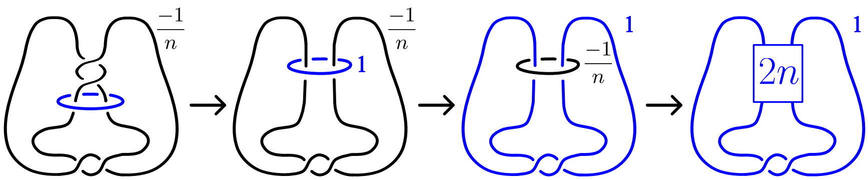

Figure 2.2. Kirby–Rolfsen–Rourke calculus moves showing that $M_{4_1}(-1/n)$ is homeomorphic to $M_{D'_n}(1)$.

We are now ready to give the proof of Theorem 1.2:

Proof of Theorem 1.2. — By Lemma 2.4, for any $n \in \mathbb{Z}$, the 3-manifold $M_n := M_{D_n}(4n+1)$ is obtained by $-(4n+1)/n$-surgery along the knot 4_1. Since $n \neq 0, -1$, by Proposition 2.2, the slope $-(4n+1)/n$ is not exceptional for 4_1. Hence M_n is hyperbolic. Similarly, since $M'_n := M_{D'_n}(1)$ is also obtained by a $-1/n$-surgery along 4_1, it is hyperbolic for $n \neq 0, \pm 1$.

By Theorem 2.3, we conclude that the manifolds $M_{D_n}(4n+1)$ and $M_{D'_n}(1)$ are q-hyperbolic for $n \neq \pm 1$. Hence, part (b) of the statement follows.

Theorem 2.1 implies that the growth rates of the Turaev–Viro invariants of the unfilled twist knot complements M_{D_n} and $M_{D'_n}$ are bounded below by the growth rates of M_n and M'_n respectively. That is we have

$$0 < lTV(M_n) \leqslant lTV(M_{D_n}) \text{ and } 0 < lTV(M'_n) \leqslant lTV(M_{D'_n}).$$

Hence, by their definitions, the double twist knots D_n and D'_n are also q-hyperbolic, concluding the proof of part (a).

Now we prove part (c): Since M_n and M'_n are hyperbolic 3-manifolds obtained by surgery of 4_1, by Theorem 2.3, $lTV(M_n) = \mathrm{vol}(M_n)$ and $lTV(M'_n) = \mathrm{vol}(M'_n)$. Combining these equations with the last displayed inequalities gives part (c). □

3. Non-characterizing slopes and q-hyperbolicity

A slope p/q is called *non-characterizing* for a knot $K \subset S^3$ if there is a knot K' that is not equivalent to K and such that $M_K(p/q)$ is homeomorphic to $M_{K'}(p/q)$. For the viewpoint of this paper, non-characterizing slopes are useful in the following sense: If we know that $M_K(p/q)$ is q-hyperbolic then, arguing as in the proof Theorem 1.2, we conclude that

$$0 < lTV(M_K(p/q)) = lTV(M_{K'}(p/q)) \leqslant lTV(M_{K'}),$$

and hence K' is a q-hyperbolic knot.

In the articles, [1, 2, 3], the authors provide constructions of knots that admit many non-characterizing slopes. The techniques of these papers apply to many double twist knots to conclude that they admit non-characterizing slopes. On the other hand, these knots can be seen to be q-hyperbolic by Theorem 1.2. Using this approach, one starts with a double twist knot, say K, to which both the techniques of [1, 2, 3] and Theorem 1.2 apply, and builds a family of q-hyperbolic knots that have a common surgery with K.

To illustrate this, we note that the knot 6_2 is isotopic to the double twist knot $D(-4,-3)$; we will write $6_2 = D(-4,-3)$. See Section 5 for more details. By Theorem 1.2, $M_{6_2}(-7) \cong M_{4_1}(-7/2)$ and 6_2 is q-hyperbolic. We will use the approach discussed above to prove the following theorem stated in the Introduction:

Theorem 1.3. — *There is an infinite set of knots $\mathcal{K}$ such that:*

(a) *Every knot in $\mathcal{K}$ is q-hyperbolic.*
(b) *For every $K \in \mathcal{K}$, $M_K(-7)$ is homeomorphic to $M_{4_1}(-7/2)$ and it is q-hyperbolic.*
(c) *We have*

$$lTV(M_K) \geqslant \mathrm{vol}(M_{4_1}(-7/2)) \approx 1.649610.$$

(d) *No two knots in $\mathcal{K}$ are equivalent.*

In order to prove Theorem 1.3 and to discuss further applications of the techniques of [1, 2, 3] in constructions of q-hyperbolic knots, we need some preparation.

3.1. Annulus presentations and twists

We begin by recalling the notion of *annulus presentations* of knots and the operation of *annulus twists* for knots admitting annulus presentations. The latter operation takes a surgery presentation along a particular class of knots and returns a different knot which shares a surgery with the original knot.

Definition 3.1. — *We will say that a knot $K \subset S^3$ admits an annulus presentation if it can be constructed in the following way:*

(1) *Start with a standardly embedded annulus $A \subset \mathbb{R}^2 \cup \{\infty\} \subset S^3$ together with an unknotted curve c that is disjoint from A that bounds a disc Σ whose interior intersects ∂A twice; once for each component of ∂A. Consider c as a framed knot with framing ± 1.*
(2) *Consider an embedded band $b : I \times I \to S^3$ such that*
  (i) $b(I \times I) \cap \partial A = b(\partial I \times I)$,
  (ii) $b(I \times I) \cap intA$ *consists of ribbon singularities,*
  (iii) $A \cup b(I \times I)$ *is an immersed orientable surface, and*
  (iv) $b(I \times I) \cap c = \emptyset$,
  where $I = [0,1]$. See the right hand side panel of Figure 3.1 for an illustration of an annulus presentation (A, b, c).
(3) *Performing the ± 1 surgery on c (i.e. blowing down along c) transforms the curve $(\partial A \setminus b(\partial I \times I)) \cup b(I \times \partial I)$ into a knot that is isotopic to K in S^3.*

Remark 3.2. — We note that the definition of annulus presentation differs slightly across the literature. Namely, in [3], the authors use a more

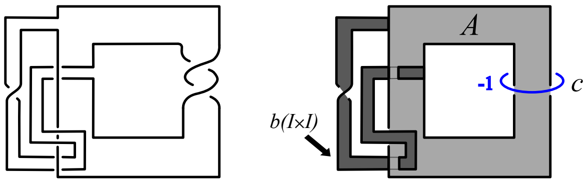

Figure 3.1. Annulus presentation of the knot 6_2 *in* S^3*.*

general definition of annulus presentation that allows the annulus A to be any embedding. They define a *special* annulus presentation equivalently to the above definition except that the presentation includes the single full crossing (either positive or negative) in the Hopf band resulting from surgery along the (± 1)-framed unknotted component c. Here we use the definition given by Abe–Jong–Omae–Takeuchi [2] and Abe–Jong–Luecke–Osoinach [1]. Note that in [2], the authors use the term *band presentation* rather than annulus presentation.

To continue, note that given an annulus presentation (A, b, c), the complement of the annulus $A \subset \mathbb{R}^2 \cup \{\infty\}$ consists of two disk components D and D'. Take D to be the component corresponding to the finite region in $\mathbb{R}^2$ (see the leftmost panel of Figure 3.2) and assume that $\infty \in D'$.

Definition 3.3. — *The annulus presentation* (A, b, c) *is called* simple *if we have* $b(I \times I) \cap \operatorname{int} D = \emptyset$*.*

The middle panel of Figure 3.2 illustrates a simple annulus presentation of the knot 6_2 while the rightmost panel illustrates a non-simple annulus presentation for the knot 5_2.

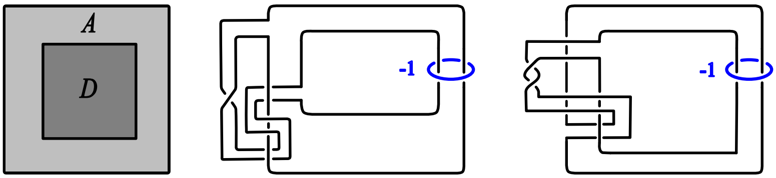

Figure 3.2. Left: *One of the two connected components,* D*, of* $\mathbb{R}^2 \cup \{\infty\} \setminus \operatorname{int} A$*.* Middle: *Simple annulus presentation of the knot* 6_2*.* Right: *Non-simple annulus presentation of the knot* 5_2*.*

The following lemma of [2] gives a family of knots which admit annulus presentations. In particular, the double twist knots $D'_n = D(2n, -2)$, including those listed in Table 5.2, satisfy the assumptions of Lemma 3.4.

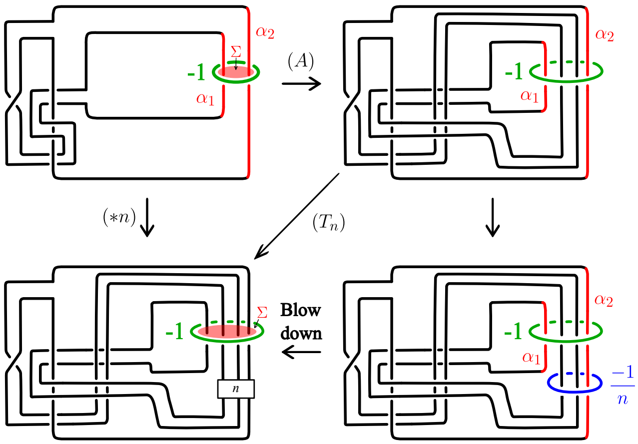

Figure 3.3. Top row: *Simple annulus presentation of* 6_2 *and the annulus twist (A).* Bottom row: *Introduction of* $(-1/n)$*-framed component and blow down.*

Lemma 3.4 ([2, Lemma 2.2]). — *If K is a knot with unknotting number one, then K admits an annulus presentation.*

Abe and Tagami [3] give a tabulation of all prime knots with 8 or fewer crossings admitting an annulus presentation (see Table 1 of [3]).

We now define an operation known as an *n-fold annulus twist*. We refer the reader to [1, 2] for further details of this construction. This operation can be applied to a knot K with an annulus presentation (A, b, c), and surgery slope given by an integer $n \in \mathbb{Z}$, to produce another knot K', with annulus presentation (A, b', c), so that the 3-manifold $M_K(n) \cong M_{K'}(n)$.

Definition 3.5. — *Let K be a knot with annulus presentation (A, b, c) with $\partial A = l_1 \sqcup l_2$, and let $n \in \mathbb{Z}$. We define the n-*fold annulus twist operation*, denoted by $(*n)$, as follows:*

(1) *First apply an annulus twist (A). This involves performing Dehn surgery on l_1 and l_2 along slopes 1 and -1, respectively, and gives rise to a homeomorphism of the complement $M_{l_1 \sqcup l_2}$. An example is illustrated in the top row of Figure 3.3. Note that in the leftmost panel we have two vertical arcs $\alpha_1, \alpha_2 \subset \partial A$ that intersect the interior of a disk Σ bounded by the -1 framed unknot c exactly twice. After the operation (A) is applied, the disk Σ is intersected by four vertical arcs, two of which are between α_1 and α_2.*

(2) *Apply the operation* (T_n)*, which is defined by*
   (a) *adding another* $(-1/n)$*-framed unknot engulfing all but* α_1 *of the vertical arcs going through* c*. An illustration is given in the rightmost panel of the second row of Figure 3.3.*
   (b) *blowing down along the* $(-1/n)$*-framed component, as shown in the leftmost panel of the second row of Figure 3.3.*

An important property of the n-fold annulus twist operation is the following result of Abe-Jong-Luecke-Osoinach [1].

Theorem 3.6 ([1, Theorem 3.10]). — *Let* K *be a knot with an annulus presentation and* K' *be the knot obtained by the* n*-fold twist* $(*n)$*. Then the 3-manifold* $M_K(n)$ *is homeomorphic to* $M_{K'}(n)$*. That is we have*

$$M_K(n) \cong M_{K'}(n).$$

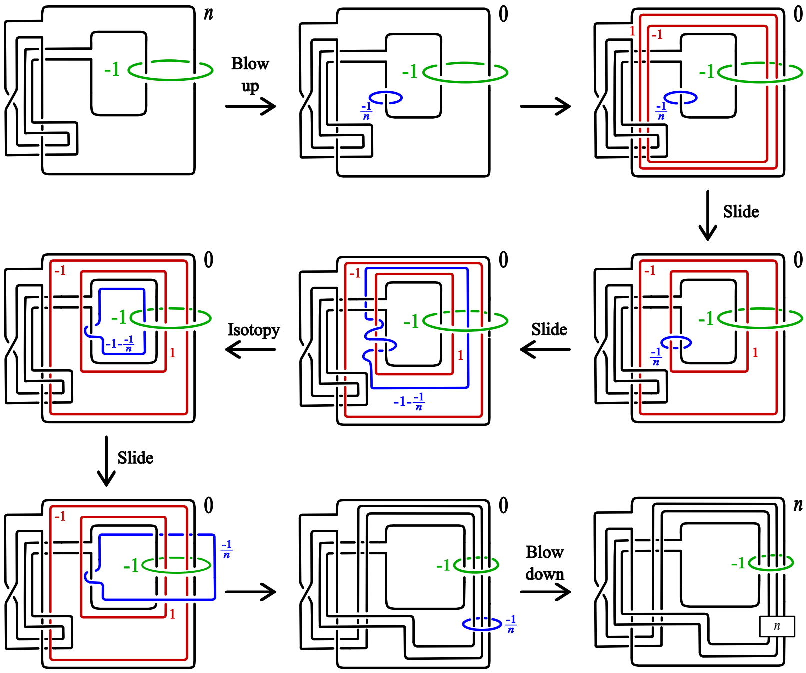

Figure 3.4. A proof that $M_K(n) \cong M_{K'}(n)$ *for* $K = 6_2$ *starting with an annulus presentation of* K *in the top-left and ending with an annulus presentation of* $K' = (*n)K$ *in the bottom-right.*

A proof of Theorem 3.6 for the knot $K = 6_2$ is given in Figure 3.4, which is summarized as follows:

(i) First we perform a blow up operation, which changes the framing of the n-framed component to 0 and introduces a $(-1/n)$-framed component as shown in the middle panel of the first row.

(ii) After introducing 1 and -1-framed components (in red) in the right most panel of the first row, we slide the 1-framed component across the -1-framed component to get the right most panel of the second row. Note that the 1 and -1-framed components (in red) in the right most panel of the second row correspond to the boundary components of the annulus A and give the surgery description for the move (A).

(iii) Next we slide the $(-1/n)$-framed component across both the 1-framed component (in red) and the -1-framed component (in green) to get the left most panel of the third row.

(iv) To get from the left most panel to the middle panel of the third row, we perform surgery on the red 1 and -1-framed components, corresponding to the annulus twist (A), and isotope the $(-1/n)$-framed component. Finally, we blow down, which introduces n full positive twists and changes the framing from 0 to n. This isotopy and blow down correspond to the operation (T_n). Hence the sequence of operations in the third row contains an (A) move and a (T_n) move.

Remark 3.7. — If a knot K admits an annulus presentation and a knot K' is obtained from K by an n-fold annulus twist $(*n)$, then, in general, K' can be far more complicated than K. However, if K admits a simple annulus presentation, then the annulus presentation of K' is also simple and is not quite as complicated.

Since the n-fold annulus twist operation on a knot produces another knot which also admits an annulus presentation, this operation can be iterated. Indeed, Theorem 3.6 implies that for any knot K which admits an annulus presentation and any integer $n \neq 0$, there is a set $\mathcal{K} = \{K_i\}_{i \in \mathbb{N}}$ of knots such that

$$\cdots M_{K_i}(n) \cong M_{K_{i-1}}(n) \cong \cdots \cong M_{K_1}(n) \cong M_K(n).$$

In general, we don't know that the knots K_i are necessarily distinct, so the set $\mathcal{K}$ may be finite. However, we will see in the proof of Theorem 1.3 that in the case of 6_2, iterating the twist operation produces an infinite sequence of mutually distinct knots.

3.2. Applications to q-hyperbolicity

In order to prove Theorem 1.3, we recall some definitions from Section 3.3.1 of [1]. There the authors use the surgery description of the infinite cyclic covering $\tilde{E}(K)$ of the exterior $E(K)$ of a knot K to distinguish knots obtained by applying the operation $(*n)$ iteratively, provided that the annulus presentation of the knot to begin with is "good" in the sense of Definition 3.8 below.

Let K be a knot with a simple annulus presentation (A, b, c). If we ignore the (-1)-framed loop c, the knot $U := (\partial A \setminus b(\partial I \times I)) \cup b(I \times \partial I)$ is trivial in S^3. Consider the link $U \cup c$ in S^3. The component U bounds an immersed disk with ribbon singularities while the component c bounds an embedded disk Σ whose interior is pierced twice by U. We may isotope $U \cup c$ so that the immersed disk bounded by U becomes an embedded flat disk, denoted by D, contained in $\mathbb{R}^2 \subset (\mathbb{R}^2 \cup \{\infty\})$. This isotopy, which we denote by ϕ, will gradually shrink the band $b(I \times I)$ till it is eliminated, and will introduce ribbon singularities between the (isotopic image of c) and the interior of D. Abusing our notation, we will continue to denote the image of c under ϕ by c, and we will continue to use Σ to denote the image of Σ under ϕ. Note that after the isotopy Σ may become an immersed disk. We also note that in Figure 3.5 (middle panel) the disk Σ after the isotopy is not entirely depicted shaded; we only indicate the shading in a small portion we wish to highlight in Figure 3.6 below where we show the ribbon singularities of a disk Σ after isotopy of $U \cup c$ that makes U bound a flat disk D.

Fix orientations on U and c and cut the complement of U after the isotopy in S^3 along the flat disk D. This gives a solid cylinder $D \times [-1, 1]$. We will denote the two copies of D resulting from this cutting by D_{-1} and D_1. The cutting separates the oriented loop c (after the isotopy) into a set

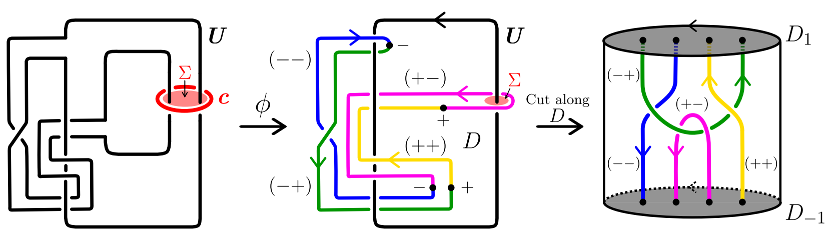

Figure 3.5. The isotopy ϕ applied to the simple annulus presentation of 6_2. In the collar $D \times [-1, 1]$, the $(+-)$ arc (in pink) and the $(-+)$ arc (in green) have linking number -1 relative to $D_{-1} \sqcup D_1$.

$\mathcal{A}$ of oriented arcs with endpoints on $D_{\pm 1}$, and the endpoints of each arc $\alpha \in \mathcal{A}$ may be labelled by "+" (resp. "−") according to whether the algebraic intersection number of α with the disk D is positive (resp. negative). This categorizes α as one of four types: $(++), (--), (+-)$, and $(-+)$. An illustration of the process for the 6_2 knot is shown in Figure 3.5. We refer the reader to [1, Section 3.3.1] for further details of this construction.

We need the following definition of [1].

Definition 3.8 ([1, Definition 3.14]). — *A simple annulus presentation (A, b, c) is* good *if $b(I \times \partial I) \cap \operatorname{int} A \neq \emptyset$ and the set of arcs $\mathcal{A}$ in $D \times [-1, 1]$ obtained by cutting along D satisfies the following up to isotopy.*

1. *$\mathcal{A}$ contains exactly one $(+-)$ arc and exactly one $(-+)$ arc, and the linking number of these arcs $\operatorname{rel}(D_{-1} \sqcup D_1)$ is ± 1.*
2. *For $\alpha \in \mathcal{A}$, if $\alpha \cap \operatorname{int} \Sigma \neq \emptyset$, then α is of type $(++)$ (resp. $(--)$) and the sign of each intersection point in $\alpha \cap \Sigma$ is $+$ (resp. $-$). Note that here we refer to the immersed Σ after the isotopy ϕ, and so the arcs $\mathcal{A}$ intersect $\operatorname{int} \Sigma$.*

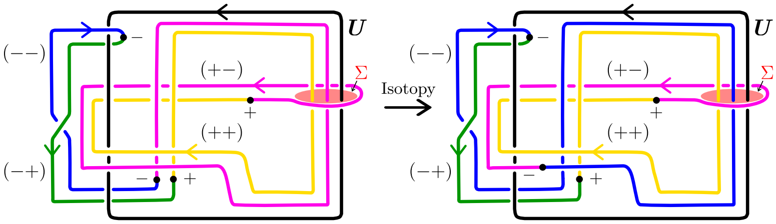

Figure 3.6. Left: *The isotopy ϕ applied to the simple annulus presentation of $(A)6_2$. In this form, Part (2) of Definition 3.8 fails.* Right: *Good annulus presentation of $(A)6_2$.*

To illustrate and motivate Definition 3.8, we apply the isotopy ϕ shown in Figure 3.5 for the simple annulus presentation of 6_2 to the simple annulus presentation obtained by applying the annulus twist (A) shown in the top row of Figure 3.3. The left panel of Figure 3.6 shows the result of applying ϕ to $(A)6_2$. However, since there is a $(+-)$ arc intersecting the interior of Σ, it does not satisfy Part (2) of Definition 3.8.

To remedy this, we apply an isotopy to move the $(-)$ intersection point between the disk D and the $(+-)$ and $(--)$ arcs, as shown in the right panel of Figure 3.6. This isotopy moves this intersection through Σ and results in a diagram satisfying Part (2) of Definition 3.8. In particular, since the only arcs intersecting $\operatorname{int} \Sigma$ are of type $(++)$ and $(--)$, the diagram in the right panel of Figure 3.6 corresponds to a good annulus presentation. After this isotopy, further applications of the annulus twist (A) leave the $(+-)$ and $(-+)$ arcs fixed since they are now disjoint from $\operatorname{int}\Sigma$.

The importance of having a good annulus presentation for a knot K lies in the fact that, as shown in [1], the number of intersection points between $(++)$ arcs and $\operatorname{int}\Sigma$ determines the degree of its Alexander polynomial. As Figure 3.6 illustrates, each annulus twist increases the number of such intersections with $\operatorname{int}\Sigma$, hence increasing the degree of the Alexander polynomial.

The following lemma of [1] will be used in the proof of Theorem 1.3.

LEMMA 3.9 ([1, Lemma 3.12]). — *Let $n \in \mathbb{Z}$ and suppose the knot K admits a good annulus presentation. Let K' be the knot obtained by applying the operation $(*n)$ to K. Then, we have the following:*

(a) *The knot K' also admits a good annulus presentation.*
(b) *If $\Delta_K(t)$ and $\Delta_{K'}(t)$ denote the Alexander polynomials of K and K', respectively, then $\deg \Delta_K(t) < \deg \Delta_{K'}(t)$.*

Remark 3.10. — As shown in [1], if a knot K admits a good annulus presentation, then its Alexander polynomial $\Delta_K(t)$ is monic.

We may now prove Theorem 1.3.

Proof of Theorem 1.3. — As noted earlier the knot $K_0 := 6_2$ admits a simple annulus presentation. We will consider K_0 with framing -7 and apply the sequence of moves in Figure 3.4 to obtain a knot K_1 with simple annulus presentation and such that $M_{K_1}(-7) \cong M_{6_2}(-7) \cong M_{4_1}(-7/2)$. See Theorem 3.6. Since as discussed earlier $M_{6_2}(-7)$ is q-hyperbolic, we obtain that K_1 is q-hyperbolic and $M_{K_1}(-7)$ is q-hyperbolic. Now we can repeat the process for the knot K_1, and apply Theorem 3.6 again to obtain a knot K_2 with simple annulus presentation and such that $M_{K_2}(-7) \cong M_{K_1}(-7)$. Inductively, we create a set $\mathcal{K} = \{K_i\}_{i\in\mathbb{N}}$ of knots with simple annulus presentations such that

$$\cdots M_{K_i}(-7) \cong M_{K_{i-1}}(-7) \cong \cdots \cong M_{K_1}(-7) \cong M_{6_2}(-7) \cong M_{4_1}(-7/2).$$

By construction, each K_i and $M_{K_i}(-7)$ are q-hyperbolic, hence the collection $\mathcal{K}$ satisfies parts (a)–(b) of the statement of the theorem.

By Theorem 2.3, $1.649610 \approx \operatorname{vol}(M_{4_1}(-7/2)) = lTV(M_{4_1}(-7/2))$. Combining this with part (b) and Theorem 2.1, we get

$$lTV(M_{K_i}) \geqslant lTV(M_{4_1}(-7/2)) = \operatorname{vol}(M_{4_1}(-7/2)) \approx 1.649610,$$

obtaining part (c) of the theorem statement.

Next we claim that the knots $K_i \in \mathcal{K}$ are distinct. Let U be the trivial knot in S^3 obtained by ignoring the (-1)-framed loop c in the simple annulus presentation of K_0 (see the middle panel of Figure 3.2). Let D be the disk bounded by U, and let Σ be the disk bounded by c. Figure 3.5 gives the isotopy that flattens D so that it is contained in $\mathbb{R}^2 \subset (\mathbb{R}^2 \cup \{\infty\})$. Let $D_{-1} = D \times \{-1\}$ and $D_1 = D^2 \times \{1\}$ be copies of D in the bundle $D^2 \times [-1,1]$ obtained by cutting along D. Note that the resulting set of oriented arcs $\mathcal{A}$ shown in the right panel of Figure 3.5 contains exactly one $(+-)$ arc and exactly one $(-+)$ arc. Relative to $D_{-1} \sqcup D_1 \subset D \times [-1,1]$, the $(+-)$ arc (in pink) links with the $(-+)$ arc (in green) with linking number -1. Moreover, $\alpha \cap \Sigma = \emptyset$ for any arc $\alpha \in \mathcal{A}$. By Definition 3.8, K_0 admits a good annulus presentation.

By Lemma 3.9, every $K_i \in \mathcal{K}$ admits a good annulus presentation, and the Alexander polynomials of this family satisfy

$$\deg \Delta_{K_0}(t) < \deg \Delta_{K_1}(t) < \cdots < \deg \Delta_{K_{i-1}}(t) < \deg \Delta_{K_i}(t) < \cdots$$

This establishes part (d) of the statement of the theorem. □

The above argument applies to any q-hyperbolic knot which admits a good annulus presentation and an integer q-hyperbolic Dehn-filling. For any such knot, analogously to 6_2, one may apply the same procedure to produce an infinite family of distinct q-hyperbolic knots with homeomorphic n-surgeries. For instance, the method can be applied to the knot 8_{20}. See Section 5 for more details. Hence we have the following:

Theorem 3.11. — *Suppose that K is a knot that admits a good annulus presentation and such that $M_K(n)$ is q-hyperbolic for some $0 \neq n \in Z$. Then there is an infinite family $\{K_i\}_{i\in\mathbb{N}}$ of distinct q-hyperbolic knots, such that $M_{K_i}(n) \cong M_K(n)$, for any $i \in \mathbb{N}$.*

We now turn our attention to the q-hyperbolic knots $D'_n = D(2n,-2)$ and their q-hyperbolic fillings $D'_n(1)$. It is known that these double twist knots have unknotting number 1, hence admit an annulus presentation by Lemma 3.4. This gives rise to the following theorem.

Theorem 1.4. — *For any $|n| > 1$, let $D'_n := D(2n,-2)$. There is a sequence of knots $\{K^i_n\}_{i\in\mathbb{N}}$ such that for any $i \in \mathbb{N}$, the following are true:*

(a) *the knot K^i_n is q-hyperbolic;*
(b) *the 3-manifold $M_{K^i_n}(1)$ is homeomorphic to $M_{D'_n}(1)$ and it is q-hyperbolic.*

Proof. — Fix $|n| > 1$. The double twist knot $D'_n := D(2n, -2)$ has unknotting number 1 and hence by Lemma 3.4, it admits an annulus presentation. Let $K^0_n := D'_n$. By Theorem 3.6 there is a sequence $\{K^i_n\}_{i\in\mathbb{N}}$ such that

$$\cdots M_{K^i_n}(1) \cong M_{K^{i-1}_n}(1) \cong \cdots \cong M_{K^1_n}(1) \cong M_{D'_n}(1).$$

Since $D'_n(1)$ is q-hyperbolic, by Theorem 1.2, each manifold $K^i_n(1)$ is q-hyperbolic, and (b) follows. Furthermore, by Theorem 2.1, each knot K^i_n is also q-hyperbolic, proving part (a). □

Remark 3.12. — We note that the knots considered in Theorem 1.4 are obtained by iteratively applying 1-fold annulus twists. While each knot D'_n admits an annulus presentation, they do not have monic Alexander polynomials. Indeed, for $n \in \mathbb{Z}$, we have

$$\Delta_{D'_n}(t) \doteq nt - (2n+1) + nt^{-1},$$

where $\doteq$ is taken up to multiplication by $\pm t^k$. By Remark 3.10, for $|n| > 1$, the knot D'_n does not admit a good annulus presentation. This means the knots resulting from 1-fold annulus twists may not be distinct from D'_n, so the resulting sequence $\{K^n_i\}_{i\in\mathbb{N}}$ may only be a finite family of distinct q-hyperbolic knots.

4. An application to quantum representations

In this section, we discuss an application to a conjecture of Andersen, Masbaum, and Ueno known as the AMU conjecture [4] on quantum representations of mapping class groups of surfaces.

Let $\Sigma_{g,n}$ be a compact oriented surface of genus g with n boundary components. Let $\mathrm{Mod}(\Sigma_{g,n})$ denote its mapping class group, the group of orientation-preserving homeomorphisms of $\Sigma_{g,n}$ fixing the boundary pointwise. The $SO(3)$-Witten–Reshetikhin–Turaev TQFTs [26, 30] give families of finite-dimensional projective representations of $\mathrm{Mod}(\Sigma_{g,n})$.

Fix an odd integer $r \geqslant 3$, which we refer to as the *level*, and let $I_r = \{0, 2, \ldots, r-3\}$ be the set of non-negative even integers less than $r-2$. Fix a primitive $2r$th root of unity ζ_{2r} and a coloring c of the components of $\partial\Sigma_{g,n}$ by elements of I_r. Using the skein-theoretic framework of Blanchet, Habegger, Masbaum, and Vogel [7], this gives a finite dimensional $\mathbb{C}$-vector space $RT_r(\Sigma_{g,n}, c)$ and a respresentation

$$\rho_{r,c} : \mathrm{Mod}(\Sigma_{g,n}) \to \mathbb{P}\,\mathrm{Aut}(RT_r(\Sigma_{g,n}, c)),$$

called the $SO(3)$-quantum representation of $\mathrm{Mod}(\Sigma_{g,n})$ at level r.

The Nielsen–Thurston classification implies that mapping classes $\phi \in \mathrm{Mod}(\Sigma_{g,n})$ are either periodic, reducible, or pseudo-Anosov, and the geometry of the mapping torus $M_\phi = \Sigma_{g,n} \times I/(x \sim \phi(x))$ of ϕ is determined by this classification. The AMU conjecture [4] relates the Nielsen–Thurston classification of mapping classes to their quantum representations.

Conjecture 4.1 (AMU Conjecture, [4]). — *Let $\phi \in \mathrm{Mod}(\Sigma_{g,n})$ be a pseudo-Anosov mapping class. Then for any big enough level r, there is a choice of coloring c of the components of $\partial\Sigma_{g,n}$ such that $\rho_{r,c}(\phi)$ has infinite order.*

Remark 4.2. — Note that if a mapping class $\phi \in \mathrm{Mod}(\Sigma_{g,n})$ satisfies the AMU conjecture, then any mapping class that is a conjugate of a power of ϕ also satisfies the conjecture.

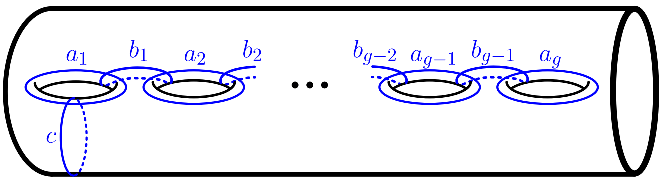

Figure 4.1. The curves $c, a_1, b_1, \cdots, b_{g-1}, a_g$ on $\Sigma_{g,1}$.

For a simple closed curve $a \subset \Sigma_{g,n}$ let $\tau_a \in \mathrm{Mod}(\Sigma_{g,n})$ denote the mapping class represented by a Dehn twist along a and τ_a^{-1} denote the inverse mapping class.

On the surface of genus g and with one boundary component $\Sigma_{g,1}$, consider the simple closed curves $c, a_1, b_1, \cdots, b_{g-1}, a_g$ shown in Figure 4.1 and the mapping classes

$$\phi_g = \tau_c \tau_{a_1} \tau_{b_1}^{-1} \tau_{a_2} \cdots \tau_{b_{g-1}}^{-1} \tau_{a_g}, \quad \text{and} \quad \phi'_g = \tau_c^{-1} \tau_{a_1}^{-1} \tau_{b_1} \tau_{a_2}^{-1} \cdots \tau_{b_{g-1}} \tau_{a_g}^{-1}.$$

Theorem 4.3. — *For $g \geqslant 1$, the mapping classes $\phi_g, \phi'_g \in \mathrm{Mod}(\Sigma_{g,1})$ are pseudo-Anosov and they satisfy the AMU conjecture.*

Given $\phi \in \mathrm{Mod}(\Sigma_{g,1})$, the mapping torus

$$T_\phi = \Sigma_{g,1} \times [-1,1]/_{(x,1)\sim(\phi(x),-1)}$$

is a 3-manifold which fibers over S^1 with fiber $\Sigma_{g,1}$ and monodromy ϕ. By [15, Theorem 1.2], if T_ϕ is q-hyperbolic, then ϕ satisfies the AMU conjecture. To prove Theorem 4.3, we will show that each of T_{ϕ_g} and $T_{\phi'_g}$ is homeomorphic to the complement of a q-hyperbolic double twist knot.

4.1. Fibered double twist knots

The knot $D(m,n)$ is the two-bridge knot associated with the rational number

$$\frac{n}{mn-1} = [m,-n] = \frac{1}{m-\frac{1}{n}}.$$

In general, we define the continued fraction expansion (CFE) by

$$[a_1,a_2,\ldots,a_k] := \cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_k}}}}.$$

We note that a CFE for a rational number is not unique, hence a two-bridge knot can have multiple associated CFEs. The following properties of double twist knots will be useful:

(i) $D(m,n) = D(n,m)$ are equivalent knots.

(ii) For a double twist knot $D(m,n)$ with CFE $[a_1,\ldots,a_k]$, its mirror image is $D^*(m,n) := D(-m,-n)$ and has CFE $[-a_1\ldots,-a_k]$.

We recall the following well known lemma that can be found, for example, in [22].

Lemma 4.4. — *A two-bridge knot is fibered if and only if it has a CFE of the form $[a_1,\ldots,a_k]$ such that $|a_i| = 2$ for $i = 1,\ldots,k$ and k is even.*

As shown in [22], every fibered two-bridge knot can be identified with the boundary of the Murasugi sum of a sequence of right and left Hopf bands determined by the entries in its CFE. The monodromy of the left (resp. right) Hopf band is the left (resp. right) Dehn twist, and the monodromy of a fibered two-bridge knot with CFE $[a_1,\ldots,a_k]$ (with $|a_i| = 2$) is given by the product of k Dehn twists corresponding to each Hopf band in the Murasugi sum. In this case, the resulting fiber is the surface of genus $\frac{k}{2}$ with one boundary component.

Proposition 4.5. — *For any integer $g > 0$, the double twist knot $D(3,2g) \subset S^3$ is fibered with monodromy $\phi_g \in \mathrm{Mod}(\Sigma_{g,1})$.*

Proof. — Let $n \leqslant -1$ be an integer, and set $g := -n$. By the properties of twist knots, we have $D(2n,-3) = D^*(3,2g)$. The knot $D(3,2g)$ is the two-bridge knot associated to $[3,-2g] = \frac{-2g}{-6g+1}$. By Lemma 4.4, $D(2n,-3)$ is fibered if and only if $D(3,2g)$ has a CFE of the form $[a_1,\ldots,a_k]$ with

$|a_i| = 2$. We will show that $D(3,2g)$ has a CFE $[2,2,-2,2,\dots,-2,2]$ of length $2g$.

We note this CFE alternates sign beginning with the second term. We assume

$$\frac{-2g}{-6g+1} = [2,2,-2,2,\dots,(-1)^{2g-1}2,(-1)^{2g}2] = \frac{1}{2+A_g}, \tag{4.1}$$

where $A_g := [2,-2,2,\dots,(-1)^{2g-1}2,(-1)^{2g}2]$ of length $2g-1$, and proceed by induction. For $D(3,2(g+1))$, we have

$$\begin{aligned}[2,2,-2,\dots,(-1)^{2g+2}2] &= \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{-2+A_g}}} \\ &= \frac{-2(g+1)}{-6(g+1)+1} \\ &= [3,-2(g+1)],\end{aligned}$$

where the second line follows from the identity $A_g = [3,-2g]-2$. This establishes the claim, which implies that the double twist knot $D(2n,-3)$ is fibered for $n \leqslant -1$.

Following [22], the knot $D(3,2g)$ can be identified with the boundary of the Murasugi sum of $2g$ Hopf bands. The monodromy is then a product of left and right Dehn twists corresponding to the sign of each entry of the CFE $[2,2,-2,2,\dots,(-1)^{2g-1}2,(-1)^{2g}2]$. These Dehn twists correspond to the collection of curves on $\Sigma_{g,1}$ shown in Figure 4.1, and the monodromy $\phi_g = \tau_c\tau_{a_1}\tau_{b_1}^{-1}\tau_{a_1}\cdots\tau_{b_{g-1}}^{-1}\tau_{a_g}$. □

4.2. Proof of Theorem 4.3

By Proposition 4.5, the knot $D(3,2g)$, for $g>0$, is fibered. Since these knots are hyperbolic (see for example [19]), by the work of Thurston the mapping class ϕ_g is pseudo-Anosov [29]. By Theorem 1.2, these knots are q-hyperbolic. The mirror image $D(-2g,-3) = D^*(3,2g)$ is also hyperbolic, q-hyperbolic, and fibered with monodromy $\phi'_g = \tau_c^{-1}\tau_{a_1}^{-1}\tau_{b_1}\tau_{a_1}^{-1}\cdots\tau_{b_{g-1}}\tau_{a_1}^{-1}$. Hence, by [15, Theorem 1.2], ϕ_g and ϕ'_g satisfy the AMU conjecture. □

5. Low crossing knots and low volume 3-manifolds

Tables 5.1 and 5.2 give the twist knots $D_n = D(2n, -3)$ and $D'_n = D(2n, -2)$ up to 10 crossings, respectively. By Lemma 2.4, all of these share surgeries with 4_1.

We identify these knots by giving an alternating projection realizing the crossing numbers in conjunction with Rolfsen's tabulation of low-crossing knots [28]. We note that for the knot $D(2n, -3)$, the resulting diagram with $2n + 3$ crossings corresponding to Figure 1.1 is alternating when $n \geqslant 1$, allowing us to identify the odd crossing knots of Table 5.1. To identify the even crossing knots of Table 5.1, we see in Figure 5.1 that, after applying Reidemeister moves, we obtain an alternating diagram for $D(-2n, -3)$ with $2n + 2$ crossings.

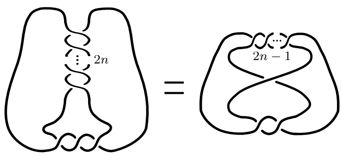

Figure 5.1. Alternating diagram of $D(-2n, -3)$ realizing the even crossing knots of Table 5.1.

Similarly, the original diagram for $D(2n, -2)$ is also alternating with $2n + 2$ crossings for $n \geqslant 1$, allowing us to identify the even crossing knots of Table 5.2. Figure 5.2 gives an alternating diagram for $D(-2n, -2)$ with $2n + 1$ crossings, realizing the odd crossing knots of Table 5.2.

Table 5.1. Low-crossing knots $D_n = D(2n, -3)$.

n	-4	-3	-2	-1	1	2	3
D_n	10_2	8_2	6_2	4_1	5_2	7_3	9_3

By Proposition 4.5, for $n \leqslant -1$ the knot $D(2n, -3)$ is fibered with genus $|n|$. By Table 5.1, for $n = -4, -3, -2, -1$, the knot $D(2n, -3)$ is identified as the corresponding knot shown in the table. Indeed the knots $10_2, 8_2, 6_2$, and 4_1 are known to be fibered of genus, $4, 3, 2$, and 1, respectively [24].

Remark 5.1. — The manifold $M_{4_1}(-5)$, which is homeomorphic to $M_{5_2}(5)$ by Lemma 2.4 and Table 5.1, is known as the Meyerhoff manifold. It is the second-smallest volume closed orientable hyperbolic 3-manifold, with volume approximately 0.9814.

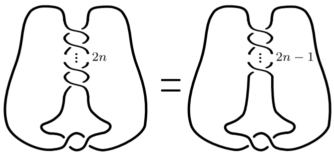

Figure 5.2. Alternating diagram of $D(-2n,-2)$ realizing the odd crossing knots of Table 5.2.

Table 5.2. Low-crossing knots $D'_n = D(2n,-2)$.

n	-4	-3	-2	-1	1	2	3	4
D'_n	9_2	7_2	5_2	3_1	4_1	6_1	8_1	10_1

Futer, Purcell, and Schleimer recently wrote a software package [20], in conjunction with the article [21], for testing the cosmetic surgery conjecture. At our request, they extended the code to allow for testing whether pairs of cusped 3-manifolds have common Dehn fillings as well as identifying those fillings. Running the code for knots up to 12 crossings, as well as on SnapPy's census of the 1267 hyperbolic knot complements that can be triangulated with fewer than 10 tetrahedra [13], they verified the data given in Tables 5.1 and 5.2 and identified many additional knots which share surgeries with 4_1. See [21, Theorem 5.1]

Tables 5.3 and 5.4 list all the knot complements from the SnapPy census of hyperbolic cusped 3-manifolds that admit triangulations with at most nine tetrahedra and have shared Dehn fillings with the complement of 4_1. The information on the tables is recorded as follows:

Column 1 presents the knot K with the notation used in the SnapPy census, while Column 2 gives the approximate volume of M_K. Column 3 gives the surgery slopes a/b, p/q, with $M_K(a/b) \cong M_{4_1}(p/q)$ and Column 4 gives the approximate volume of that manifold. All of these knots, many of which are twisted torus knots, have known diagrams, but they may be complicated and require hundreds of crossings. Using the tables of [8, 9, 10], we may identify some of the examples from the knot tables. Note that since 4_1 is amphicheiral, we have $M_{4_1}(-p/q) \cong M_{4_1}(p/q)$. Hence the slopes p/q in the tables can be, equivalently, be replaced with its negative.

The following estimate of the quantity $lTV(M_K)$ applies to all the knots in Tables 5.3 and 5.4:

Table 5.3. Knots in the SnapPy census of cusped hyperbolic 3-manifolds that share surgeries with 4_1.

K	vol(M_K)	Slopes a/b, p/q	vol($M_K(a/b)$)	Knot
$K2_1$	2.029883	-	-	4_1
$K3_2$	2.828122	5, -5	0.981369	5_2
		1, 1/2	1.398509	
$K4_1$	3.163963	$1, -1/2$	1.398509	6_1
$K4_2$	3.331744	1, 1/3	1.731983	7_2
$K5_2$	3.427205	$1, -1/3$	1.731983	8_1
$K5_3$	3.486660	1, 1/4	1.858138	9_2
$K5_9$	4.056860	$-2, 2/3$	1.737124	10_{132}
$K5_{12}$	4.124903	3, 3/2	1.440699	8_{20}
$K5_{13}$	4.124903	1, 1/3	1.731983	$11n_{38}$
$K5_{19}$	4.400833	$-7, -7/2$	1.649610	6_2
$K5_{20}$	4.592126	$9, -9/2$	1.752092	7_3
$K6_1$	3.526196	$1, -1/4$	1.858138	10_1
$K6_2$	3.553820	1, 1/5	1.918602	$11a_{247}$
$K6_8$	4.293750	$-3, 3/5$	1.921026	
$K6_9$	4.307917	$-2, 2/5$	1.919520	
$K6_{23}$	4.935243	$-11, -11/3$	1.876053	8_2
$K6_{24}$	4.994856	$13, -13/3$	1.903695	9_3
$K6_{37}$	5.413307	7, 7/3	1.805827	$15n_{41127}$
$K7_1$	3.573883	$1, -1/5$	1.918602	$12a_{803}$
$K7_2$	3.588914	1, 1/6	1.952062	$13a_{3143}$
$K7_{10}$	4.354670	$-4, 4/7$	1.973762	
$K7_{11}$	4.359783	$-3, 3/7$	1.973161	
$K7_{41}$	4.933530	$-5, 5/4$	1.873482	
$K7_{44}$	4.993457	7, 7/5	1.932061	
$K7_{45}$	5.114841	$-15, -15/4$	1.946574	10_2
$K7_{46}$	5.140207	$17, -17/4$	1.957888	$11a_{364}$
$K7_{95}$	5.860539	11, 11/2	1.822675	10_{128}
$K7_{96}$	5.860539	13, 13/3	1.903695	$11n_{57}$
$K7_{98}$	5.904086	14, 14/3	1.915331	$12n_{243}$
$K7_{129}$	6.922634	$-7, 7/3$	1.805827	

Table 5.4. Knots in the SnapPy census of cusped hyperbolic 3-manifolds that share surgeries with 4_1.

K	vol(M_K)	Slopes a/b, p/q	vol($M_K(a/b)$)	Knot
$K8_1$	3.60046726278	1, −1/6	1.9520620754135	$14a_{12741}$
$K8_2$	3.6095391745	1, 1/7	1.9724601973306	$15a_{54894}$
$K8_9$	4.3790606712	−5, 5/9	1.9957717794010	
$K8_{10}$	4.38145643736	−4, 4/9	1.9954776244141	
$K8_{61}$	5.07001608898	9, 9/7	1.9788631982608	
$K8_{62}$	5.0827080657	11, 11/8	1.9914466741922	
$K8_{64}$	5.1955903246	−19, 19/5	1.9776430099735	$12a_{722}$
$K8_{65}$	5.2086109485	−21, 21/5	1.983357467405	$13a_{4874}$
$K8_{96}$	5.75222662008	11, 11/5	1.9478817102192	
$K8_{105}$	5.8281487245	−16, 16/7	1.9891579197851	
$K8_{133}$	6.1411744018	22, 22/5	1.9859441335531	
$K8_{135}$	6.1504206159	23, 23/5	1.9883610027459	$T(7,9,6,-6,5,-1)$
$K8_{143}$	6.2597017011	−13, 13/4	1.9334036965515	
$K8_{145}$	6.27237250941	1, 1/2	1.3985088841508	$14n_{18212}$
$K8_{268}$	7.26711903086	9, 9/4	1.9026876676640	
$K9_1$	3.61679304740	−1, −1/7	1.9724601973306	
$K9_2$	3.62268440821	1, 1/8	1.9857927453641	
$K9_8$	4.3912243457	−6, 6/11	2.0069885249369	
$K9_9$	4.39253386353	5, 5/11	2.0068241855029	
$K9_{83}$	5.1043901461	13, 13/10	2.004926648441	
$K9_{85}$	5.1089909300	−15, 15/11	2.0095023855854	
$K9_{93}$	5.23864536794	23, −23/6	1.9940644235057	$14a_{12197}$
$K9_{94}$	5.24618858374	25, −25/6	1.9973474789782	$15a_{85258}$
$K9_{152}$	5.8653629974	20, 20/9	2.004886373798	
$K9_{155}$	5.8812168764	−25, 25/11	2.0133867882020	
$K9_{242}$	6.2152290434	31, 31/7	2.007727892627	
$K9_{244}$	6.21858163948	−32, 32/7	2.0085996110216	
$K9_{282}$	6.5328202770	−21, 21/4	1.9754820965797	
$K9_{296}$	6.6272713527	27, 27/5	1.9965186652378	
$K9_{299}$	6.6445653099	19, 19/3	1.9565702867106	
$K9_{435}$	7.2356793751	−3, 3/4	1.8634426716184	

Proposition 5.2. — *Suppose that K is a hyperbolic knot in S^3 such that M_K admits a triangulation with t tetrahedra. Suppose, moreover, that $M_K(a/b) \cong M_{4_1}(p/q)$, for some slopes $a/b, p/q \in \mathbb{Q}$, where p/q is a non-exceptional slope of 4_1. Then we have*

$$\mathrm{vol}(M_K(a/b)) \leqslant lTV(M_K) \leqslant v_{\mathrm{oct}} \cdot t,$$

where $v_{\mathrm{oct}} \approx 3.6638$ is the volume of the ideal regular octahedron.

Proof. — The upper bound follows at once from [5, Corollary 3.9].

By Theorem 2.3, $\mathrm{vol}(M_{4_1}(p/q)) = lTV(M_{4_1}(p/q))$ and, by assumption, $M_K(a/b) \cong M_{4_1}(p/q)$. Combining these with Theorem 2.1, leads to the lower bound of $lTV(M_K)$. □

Remark 5.3. — The four hyperbolic knots with the smallest volumes are 4_1, 5_2, 6_1, and the $(-2, 3, 7)$-pretzel knot. In [11], Chen and Yang gave computational evidence for the Turaev–Viro invariant volume conjecture for each of these knots. By Lemma 2.4, 5_2 and 6_1 share surgeries with 4_1, hence are q-hyperbolic by Theorem 1.2. However, the $(-2, 3, 7)$-pretzel knot was shown not to share any surgeries with 4_1 using [21, Theorem 5.1]. Similarly the knot 6_3 was shown not to share any surgeries with 4_1, making it the only hyperbolic knot with up to six crossings for which q-hyperbolicity cannot be decided with the methods of this paper.

Remark 5.4. — By Table 5.3, the knot $K5_{12} = 8_{20}$ shares a surgery with 4_1. In particular, $M_{4_1}(3/2) \cong M_{8_{20}}(3)$, which has volume ≈ 1.440699. In addition, as shown in [1], 8_{20} admits a good annulus presentation. This means an analogous version of Theorem 1.3 also holds for 8_{20}.

Remark 5.5. — Table 5.3 includes the knots $8_{20}, 10_{132}, 11n_{38}$, and $11n_{57}$. According to KnotInfo [24], the complements $M_{8_{20}}$, $M_{10_{132}}$, $M_{11n_{38}}$, and $M_{11n_{57}}$ are also fibered, so their associated monodromies (as well as powers of conjugates of those mapping classes) satisfy he AMU conjecture.

Note added on proofs

Since this paper was written, proofs of the Tuarev-Viro invariants volume conjecture for 3-manifolds obtained by surgery on the twists knot $D(2, n)$ were announced [12, 23]. The methods of this paper can be applied to these families of 3-manifolds to produce additional families of q-hyperbolic knots and 3-manifolds.

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