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THE TEICHMÜLLER–RANDERS METRIC

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Abstract. — In this paper, we introduce a new asymmetric weak metric on the Teichmüller space of a closed orientable surface with (possibly empty) punctures. This new metric, which we call the Teichmüller–Randers metric, is an asymmetric deformation of the Teichmüller metric and is obtained by adding to the infinitesimal form of the Teichmüller metric a differential 1-form. We study basic properties of the Teichmüller–Randers metric. In the case when the 1-form is exact, any Teichmüller geodesic between two points is also a unique Teichmüller–Randers geodesic between them. A particularly interesting case is when the differential 1-form is the differential of the logarithm of the extremal length function associated with a measured foliation. We show that in this case the Teichmüller–Randers metric is incomplete in any Teichmüller disc, and we give a characterisation of geodesic rays with bounded length in this disc in terms of their directing measured foliations.

Résumé. — Dans cet article, nous introduisons une nouvelle métrique asymétrique sur l'espace de Teichmüller d'une surface fermée orientable avec ou sans perforations que nous appelons la métrique de Teichmüller–Randers. C'est une déformation asymétrique de la métrique de Teichmüller obtenue en ajoutant une forme différentielle de degré 1 à la forme infinitésimale de cette dernière. Nous étudions les propriétés de base de cette nouvelle métrique. Nous démontrons que dans le cas où la forme différentielle ajoutée est exacte, toute géodésique entre deux points pour la métrique de Teichmüller est aussi une géodésique unique pour la métrique de Teichmüller–Randers. Un cas particulièrement intéressant est celui où la forme différentielle est la différentielle du logarithme de la fonction longueur extrémale associée à un feuilletage mesuré. Nous montrons que dans ce cas la métrique de Teichmüller–Randers restreinte à un disque de Teichmüller quelconque n'est pas complète et nous caractérisons les rayons géodésiques de longueur bornée dans ce disque.

Keywords: Thurston metric, Teichmüller space, Teichmüller disc, Finsler manifold, Randers metric, Teichmüller–Randers metric, extremal length.
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1. Introduction

A Randers metric is a deformation of a Riemannian or Finsler metric obtained by adding to its infinitesimal form a differential 1-form. In [17], in the case where the surface is a torus, we exhibited a natural family of Randers metrics which connects the Teichmüller metric on the Teichmüller space of that surface to its Thurston asymmetric metric. It is natural to study now the same kind of deformation of the Teichmüller metric on the Teichmüller space $\mathcal{T}_{g,m}$ of a general closed orientable surface $\Sigma_{g,m}$ of genus g with m punctures, and this is what we do in the present paper. It turns out that the metrics in this family are interesting to study in this general setting and this is what we propose to show in this paper.

In its original form given in [19], a Randers metric is associated with an n-dimensional Riemannian manifold (M, g) and a 1-form ω on M satisfying $\|\omega\|_g < 1$ at every point of M. In this situation, the associated *Randers metric* is a Finsler asymmetric metric on M defined infinitesimally by $F(v) = g(v, v)^{1/2} + \omega(v)$. Randers metrics have applications in the physical world, and they have been widely studied since their appearance. The same construction also works when the original metric is not Riemannian, but Finsler, like in the case we study here.

In this paper we study a Randers deformation of the infinitesimal form of the Teichmüller metric κ on $\mathcal{T}_{g,m}$, which we call the *Teichmüller–Randers metric* associated with a real 1-form ω, defined by

$$\kappa^{\omega}(x; v) = \kappa(x; v) + \omega(v) \tag{1.1}$$

where x is a point in Teichmüller space and v a tangent vector at x.

In a natural way, the lengths of differentiable arcs on $\mathcal{T}_{g,m}$ can be defined using this metric, and the distance δ_T^{ω} between two points is set to be the infimum of the lengths of arcs connecting them. The Teichmüller–Randers distance δ_T^{ω} may take negative values for a general 1-form ω, but it gives a Finsler metric when the Teichmüller norm $\|\omega\|_T(x)$ of ω at x (i.e., the supremum of the value of ω on the tangent vectors at x with Teichmüller norm $\leqslant 1$) is less than 1 at every point x of $\mathcal{T}_{g,m}$. The Teichmüller–Randers metric is a weak metric when the Teichmüller norm of ω is 1 (see Section 2.1).

We have already introduced and studied the Teichmüller–Randers metric in the case of the torus. In [2], Belkhirat, Papadopoulos and Troyanov showed that Thurston's asymmetric metric coincides with the weak distance on the upper-half plane $\mathbb{H}$ defined by

$$\delta(\zeta_1, \zeta_2) = \log \sup_{x\in\mathbb{R}} \left| \frac{\zeta_2 - x}{\zeta_1 - x} \right| \tag{1.2}$$

for $\zeta_1, \zeta_2 \in \mathbb{H}$ if we identify the Teichmüller space of a torus T with the hyperbolic plane by choosing a generator system a, b of $\pi_1(T)$, and consider normalised flat structures on T such that a has length 1. In [17], we showed that this weak distance is indeed a Finsler metric and that it is also given by the formula

$$\mathrm{d}s_{\mathrm{hyp}} + \frac{1}{2}\mathrm{d}\log \mathrm{Im}(\zeta), \tag{1.3}$$

where $\mathrm{d}s_{\mathrm{hyp}}$ is the hyperbolic metric on $\mathbb{H}$ of constant curvature -4. Since the Teichmüller metric coincides with the hyperbolic metric in this setting, the weak distance in (1.2) is nothing but the Teichmüller–Randers metric given by (1.3), that is, associated with the 1-form

$$\omega = -\frac{1}{2}\mathrm{d}\log \mathrm{Im}(\zeta)^{-1}.$$

For $0 \leqslant t \leqslant 1$, we define δ_t to be the weak metric defined by the Finsler norm $\mathrm{d}s_{\mathrm{hyp}} + \frac{t}{2}\mathrm{d}\log \mathrm{Im}(\zeta)$. Then $\{\delta_t\}$ constitutes a continuous family of Teichmüller–Randers metrics joining the hyperbolic metric and δ. We note that the 1-form $\omega = -\frac{1}{2}\mathrm{d}\log \mathrm{Im}(\zeta)^{-1}$ is exact and $\mathrm{Im}(\zeta)^{-1}$ coincides with the extremal length of the isotopy class of simple closed curves corresponding to a.

We now turn to stating our main theorems. Before that, we recall that the Teichmüller distance is a uniquely geodesic metric, and that any geodesic extends to a holomorphic disc called a Teichmüller disc. Namely, for any two points in $\mathcal{T}_{g,m}$, there is a holomorphic (or anti-holomorphic) isometry $(\mathbb{H}, d_{\mathrm{hyp}}) \to (\mathcal{T}_{g,m}, d_T)$ whose image contains the two points, and this image, which is unique, is called a Teichmüller disc. A Teichmüller disc is determined by a holomorphic quadratic differential q, hence we denote it by $\mathbb{D}_q$ (see Section 4.2).

For a measured foliation F on $\Sigma_{g,m}$, we denote by $\mathrm{Ext}_x(F)$ the function on $\mathcal{T}_{g,m}$ taking a point x to the extremal length of a measured foliation F at x, and by $q_{F,x}$ the Hubbard–Masur differential on x for F (see Section 2.2). We shall show the following three main theorems.

Theorem 1.1 (Geodesics of the Teichmüller–Randers metric). — *Let F be a measured foliation on $\Sigma_{g,m}$, and set*

$$\omega = -\frac{1}{2}\mathrm{d}\log \mathrm{Ext}_{(\cdot)}(F).$$

Then the following hold:

(1) *For any $0 \leqslant t \leqslant 1$, the (asymmetric) metric space $(\mathcal{T}_{g,m}, \delta_T^{t\omega})$ is a uniquely geodesic space such that the Teichmüller geodesics are the geodesics.*

(2) *For any* $x \in \mathcal{T}_{g,m}$ *and for any* $0 \leqslant t \leqslant 1$*, the Teichmüller disc defined by* $q_{F,x}$ *coincides with the image of an isometric embedding of* $(\mathbb{H}, \delta_t)$ *into* $(\mathcal{T}_{g,m}, \delta_T^{t\omega})$.

Theorem 1.2 (Isometric discs). — *Suppose that there is an isometry* $\phi\colon (\mathbb{H}, \delta) \to (\mathcal{T}_{g,m}, \delta_T^{\omega})$ *where* ω *is exact and satisfies* $\|\omega\|_T \leqslant 1$ *in a neighbourhood of the image of* ϕ*. Then, there is a measured foliation* F *on* $\Sigma_{g,m}$ *such that* ϕ *is a holomorphic or anti-holomorphic isometry onto the Teichmüller disc associated with* $q_{F,x}$ *with* $x = \phi(i)$*, and such that* $\omega = -\frac{1}{2}\mathrm{d}\log \mathrm{Ext}_{(\cdot)}(F)$ *holds on the image of that isometry.*

From Theorem 1.1 and Theorem 1.2, for a fixed measured foliation F, we have a characteristic property of the geometry of the weak distance δ_T^{ω} with $\omega = -\frac{1}{2}\mathrm{d}\log \mathrm{Ext}_{(\cdot)}(F)$ on the Teichmüller disc defined by the Hubbard–Masur differential for F. Since, by Theorem 1.1, any Teichmüller disc is totally geodesic with respect to δ_T^{ω}, it is natural to ask how the weak distance δ_T^{ω} behaves on Teichmüller discs that are different from the one associated with $q_{F,x}$.

Theorem 1.3. — *Let* F *be a measured foliation on* $\Sigma_{g,m}$*, and set* $\omega = -\frac{1}{2}\mathrm{d}\log \mathrm{Ext}_{(\cdot)}(F)$*. For any* $x \in \mathcal{T}_{g,m}$ *and for any measured foliation* G *on* $\Sigma_{g,m}$*, we have the following.*

(1) *If* $q_{G,x}$ *is not a complex constant multiple of* $q_{F,x}$*, then the restriction of* δ_T^{ω} *to the Teichmüller disc* $\mathbb{D}_{q_{G,x}}$ *is a weak non-negative distance function which separates any two points.*

(2) *The following two conditions are equivalent:*
- $i(F, G) \neq 0$.
- *The Teichmüller geodesic ray directed by* $q_{G,x}$ *has bounded length with respect to* δ_T^{ω}.

In particular, the restriction of δ_T^{ω} *to every Teichmüller disc is incomplete.*

We note that the weak distance δ on $\mathbb{H}$, which corresponds to the Teichmüller space of a torus, does not separate points. Theorem 1.3 gives a generalisation of this torus case. See Section 2.1 for more details.

As can be seen in the definition, our Teichmüller–Randers metric depends on the choice of the (projective class) of a measured foliation F. Because of this, our metric is not invariant under the entire mapping class group. Still, if we consider the family of metrics making x and F vary, then the resulting family is invariant under the action of the mapping class group.

Besides the theorems stated above, we shall also discuss the extension of the Hamilton–Krushkal condition (see Theorem 3.1), and the Teichmüller–Randers cometric on the cotangent space (see Theorem 3.3).

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2. Preliminaries

2.1. Weak metric

A *weak metric* δ on a set X is a map $\delta\colon X \times X \to \mathbb{R}$ satisfying the following.

(1) $\delta(x,x) = 0$ for every x in X;
(2) $\delta(x,y) \geqslant 0$ for every x and y in X;
(3) $\delta(x,y) + \delta(y,z) \geqslant \delta(x,z)$ for every x, y and z in X.

A weak metric δ is said to *separate points* if $\delta(x_1,x_2) = 0$ for $x_1,x_2 \in X$ implies $x_1 = x_2$, and to be *complete* if for any sequence (x_n) in X satisfying $\delta(x_n,x_{n+m}) \to 0$ as $n,m \to \infty$, the sequence (x_n) converges in X (see [3, I.1]). (Notice that since the metric is not symmetric, the order of the arguments in $\delta(x_1,x_2)$ is important.)

In [2], the following weak metric was introduced on $\mathbb{H}$. First, for $\zeta_1 \neq \zeta_2 \in \mathbb{H}$, we set

$$M(\zeta_1,\zeta_2) = \sup_{x\in\mathbb{R}} \left| \frac{\zeta_2 - x}{\zeta_1 - x} \right|. \tag{2.1}$$

For $\zeta_1 = \zeta_2$, we set $M(\zeta_1,\zeta_2) = 1$. We set

$$\delta(\zeta_1,\zeta_2) = \log M(\zeta_1,\zeta_2) \quad (\zeta_1,\zeta_2 \in \mathbb{H}).$$

Then, δ is an asymmetric weak metric on $\mathbb{H}$. Furthermore, δ does not separate points of $\mathbb{H}$. Indeed, when $\zeta_1 = y_1 i$, $\zeta_2 = y_2 i \in \mathbb{H}$ with $y_1 > y_2$, we have $\delta(\zeta_1,\zeta_2) = 0$. In particular, δ is not complete (see [2, Proposition 1]). The distance between ζ_1 and $\zeta_2 \in \mathbb{H}$ is explicitly given by

$$\delta(\zeta_1,\zeta_2) = \log\left(\frac{|\zeta_2 - \overline{\zeta_1}| + |\zeta_2 - \zeta_1|}{|\zeta_1 - \overline{\zeta_1}|} \right)$$

(see [2]). Hence, any hyperbolic geodesic ray tending to a point on $\mathbb{R} \subset \partial\mathbb{H}$ has bounded length, and the length of a geodesic ray is infinite only if it goes upward in the vertical direction.

We note that when we identify $\mathbb{H}$ with the Teichmüller space of a torus, the ideal boundary $\partial\mathbb{H}$ is naturally thought of as the Thurston boundary, which is the space of projective measured foliations on the torus (see [6]). Using this identification, we see that the intersection number $i(F_x, F_\infty)$ is zero if and only if $x = \infty$, where $[F_x]$ is the projective measured foliation corresponding to an arbitrary $x \in \partial\mathbb{H}$. This corresponds to the condition of Theorem 1.3 in the case of the torus.

2.2. Teichmüller theory

We review some infinitesimal Teichmüller theory. We refer the reader to [10] for more details.

2.2.1. Teichmüller space

Let $\Sigma_{g,m}$ be an orientable surface of type (g, m), that is, of genus g with m points deleted. The integers g and m may take all non-negative values except that if $g = 0$ we assume $m \geqslant 4$ and if $g = 1$ we assume that $m \geqslant 1$. A *marked Riemann surface* (X, f) of type (g, m) is a pair of an analytically finite Riemann surface X of type (g, m) and an orientation-preserving homeomorphism $f \colon \Sigma_{g,m} \to X$. Two marked Riemann surfaces (X_1, f_1) and (X_2, f_2) are said to be *Teichmüller equivalent* if there is a conformal mapping $h \colon X_1 \to X_2$ such that $h \circ f_1$ is homotopic to f_2. The set $\mathcal{T}_{g,m}$ of Teichmüller equivalence classes of marked Riemann surfaces of type (g, m) is called the *Teichmüller space* of analytically finite Riemann surfaces of type (g, m). The *Teichmüller distance* d_T on $\mathcal{T}_{g,m}$ is defined by

$$d_T(x, y) = \frac{1}{2} \log \inf_h K(h)$$

where h ranges over all quasi-conformal maps $h \colon X_1 \to X_2$ homotopic to $f_2 \circ f_1^{-1}$ and where $K(h)$ denotes the maximal quasiconformal dilatation of h. The Teichmüller space is known to be a complex manifold which is biholomorphically equivalent to a bounded domain in $\mathbb{C}^{3g-3+m}$. Furthermore, the Teichmüller distance is complete, uniquely geodesic, and coincides with the Kobayashi distance.

2.2.2. Infinitesimal theory

For a Riemann surface X, let $L^\infty(X)$ be the complex Banach space of bounded measurable $(-1, 1)$-forms $\mu = \mu(z)(\mathrm{d}\bar{z}/\mathrm{d}z)$ on X with the norm

$$\|\mu\|_\infty = \operatorname{ess.sup}\big\{|\mu(z)| \,\big|\, z \in X\big\}.$$

A form in $L^\infty(X)$ is called a *Beltrami differential*.

Let $A^2(X)$ be the Banach space of holomorphic quadratic differentials $\varphi = \varphi(z)\mathrm{d}z^2$ on X with the norm

$$\|\varphi\|_1 = \int_X |\varphi(z)|\mathrm{d}x\mathrm{d}y.$$

There is a natural pairing between Beltrami differentials and holomorphic quadratic differentials, defined as follows.

$$L^\infty(X) \times A^2(X) \ni (\mu, \varphi) \longmapsto \int_X \mu\varphi. \tag{2.2}$$

Let $N^\infty(X) \subset L^\infty(X)$ be the subspace orthogonal to $A^2(X)$ with respect to the pairing. Namely,

$$N^\infty(X) := \left\{ \mu \in L^\infty(X) \,\middle|\, \int_X \mu\varphi = 0,\ \forall\, \varphi \in A^2(X) \right\}.$$

Two Beltrami differentials μ and ν are said to be *infinitesimally Teichmüller equivalent* if $\mu - \nu \in N^\infty(X)$. The (holomorphic) tangent space $T_x\mathcal{T}_{g,m}$ at $x = (X, f) \in \mathcal{T}_{g,m}$ is canonically identified with the quotient space $L^\infty(X)/N^\infty(X)$. Hence, for $x = (X, f) \in \mathcal{T}_{g,m}$, the pairing (2.2) descends to a non-degenerate pairing

$$T_x\mathcal{T}_{g,m} \times A^2(X) \ni (v, \varphi) \longmapsto \langle v, \varphi\rangle = \int_X \mu\varphi$$

where $v = [\mu]$ with $\mu \in L^\infty(X)$. From this observation, the space $A^2(X)$ is canonically identified with the (holomorphic) cotangent space $T_x^*\mathcal{T}_{g,m}$ at $x\in\mathcal{T}_{g,m}$. A real 1-form ω is *presented by* $\psi\in A^2(X)$ at $x=(X,f)\in\mathcal{T}_{g,m}$ if

$$\omega(v) = \mathrm{Re}\langle v, \psi\rangle$$

for all $v \in T_x\mathcal{T}_{g,m}$. Notice that in general any real 1-form on a complex manifold is the real part of a $(1,0)$-form.

Teichmüller defined in [23, 24] a metric d_T on the Teichmüller space, which is called the Teichmüller metric today, and proved that it is induced by a Finsler structure. The norm on each tangent space of the Teichmüller space takes the form of

$$\kappa\,(x; v) = \sup\bigl\{\mathrm{Re}\langle v, \varphi\rangle \bigm| \varphi \in A^2(X), \|\varphi\|_1 = 1\bigr\}$$

for $v \in T_x\mathcal{T}_{g,m}$ (see Royden [20, 21]). The distance as a Finsler space coincides with the Teichmüller distance defined above.

We call a Beltrami differential μ on X a *Teichmüller differential* when it has a form $\mu = c\overline{\varphi}/|\varphi|$ for some $\varphi \in A^2(X) - \{0\}$. It is known that the Beltrami differential μ above is *infinitesimally extremal* in the sense that $\|v\|_\infty \geqslant \|\mu\|_\infty$ for any Beltrami differential v such that $\langle v, \phi\rangle = \langle \mu, \phi\rangle$

for all $\phi \in A^2(X)$. Fix a basepoint $x_0 = (X_0, f_0)$ in $\mathcal{T}_{g,m}$. For $t \geqslant 0$ and $\varphi \in A^2(X_0)$, we denote by $F_t \colon X_0 \to X_t = F_t(X_0)$ a quasi-conformal map with the property that $\bar{\partial} F_t = \tanh(t)(\overline{\varphi}/|\varphi|)\partial F_t$. Then, a path

$$r_\varphi \colon [0,\infty) \ni t \longmapsto (X_t, F_t \circ f_0) \in \mathcal{T}_{g,m}$$

constitutes a geodesic ray with respect to d_T. We call such a geodesic a *Teichmüller geodesic ray* emanating from x_0. It is known that for any $x \in \mathcal{T}_{g,m} - \{x_0\}$, there is a unique Teichmüller geodesic ray passing through x and emanating from x_0. Furthermore,

$$(0,\infty) \times \left\{\varphi \in A^2(X_0) \,\middle|\, \|\varphi\|_1 = 1\right\} \ni (t,\varphi) \longmapsto r_\varphi(t) \in \mathcal{T}_{g,m} - \{x_0\}$$

is a homeomorphism.

Unless (g,m) is either $(1,1)$ or $(0,4)$, the Teichmüller metric is not Riemannian. This was known to Teichmüller, but we can prove it just by using the fact that the group of linear isometries of a tangent (or cotangent) space of any Riemannian metric is an orthogonal group, whereas by [4, 5, 21] the linear isometry group of a tangent/cotangent space with respect to the Teichmüller metric is a union of finite copies of S^1.

The following might be well known and follows from the discussion in the proof of [8, Lemma 3]. For completeness, we give a brief proof.

Lemma 2.1 (Derivative of the Teichmüller norm). — *Take $x = (X,f) \in \mathcal{T}_{g,m}$ and $v_0 \in T_x\mathcal{T}_{g,m} - \{0\}$. Suppose that v_0 is represented by the Teichmüller differential $\beta\overline{\alpha_0}/|\alpha_0|$ (with $\|\alpha_0\|_1 = 1$, $\beta > 0$). Then*

$$\left.\frac{d}{\mathrm{d}t}\right|_{t=0} \kappa\left(x; v_0 + tv\right) = \mathrm{Re}\langle v, \alpha_0\rangle$$

for any $v \in T_x\mathcal{T}_{g,m}$.

Proof. — For $t \in \mathbb{R}$, we take $\alpha_t \in A^2(X)$ with $\|\alpha_t\|_1 = 1$ such that

$$\kappa\left(x; v_0 + tv\right) = \mathrm{Re}\langle v_0 + tv, \alpha_t\rangle.$$

We claim that $\mathbb{R} \ni t \mapsto \alpha_t \in A^2(X)$ is well-defined and continuous. Indeed, by the Lebesgue dominated convergence theorem, the map

(2.3)
$$L_1 \colon A^2(X) \ni \alpha \longmapsto \ell_\alpha = \left[A^2(X) \ni \varphi \longmapsto \|\alpha\|_1 \int_X \frac{\bar{\alpha}}{|\alpha|}\varphi\right] \in A^2(X)^*$$

is continuous with respect to the weak topology on $A^2(X)^*$. Since $A^2(X)$ is finite-dimensional, the weak topology on $A^2(X)^*$ coincides with the topology derived from the dual norm (the operator norm). One can see that $\mathrm{Re}(\ell_\alpha(\psi)) \leqslant \|\alpha\|_1\|\psi\|_1$ for all $\psi \in A^2(X)$, and $\mathrm{Re}(\ell_\alpha(\varphi)) = \|\alpha\|_1\|\varphi\|_1$ if and only if $\varphi = \alpha$. Hence, α_t is well-defined for $t \in \mathbb{R}$, and the map defined in (2.3) is injective and proper. Since $A^2(X)$ and $A^2(X)^*$ are homeomorphic to the Euclidean space $\mathbb{R}^{6g-6+2m}$, from the invariance of domains, (2.3) is homeomorphic. Since

$$L_2\colon T_x\mathcal{T}_{g,m} \ni v \longmapsto \big[\varphi \longmapsto \langle v,\varphi\rangle\big] \in A^2(X)^* \tag{2.4}$$

is a complex linear isomorphism, and continuous with respect to the Teichmüller metric and the dual norm, we see that the map $\mathbb{R} \ni t \mapsto \alpha_t = L_1^{-1} \circ L_2(v_0 + tv) \in A^2(X)$ is continuous.

Then,

$$\begin{aligned}\kappa\,(x; v_0 + tv) - \kappa\,(x; v_0) &= \mathrm{Re}\langle v_0 + tv, \alpha_t\rangle - \mathrm{Re}\langle v_0, \alpha_0\rangle \\ &\geqslant \mathrm{Re}\langle v_0 + tv, \alpha_0\rangle - \mathrm{Re}\langle v_0, \alpha_0\rangle = t\ \mathrm{Re}\langle v, \alpha_0\rangle\end{aligned}$$

and

$$\begin{aligned}\kappa\,(x; v_0 + tv) - \kappa\,(x; v_0) &\leqslant \mathrm{Re}\langle v_0 + tv, \alpha_t\rangle - \mathrm{Re}\langle v_0, \alpha_t\rangle \\ &= t\,\mathrm{Re}\langle v, \alpha_0\rangle + t\,\mathrm{Re}\langle v, \alpha_t - \alpha_0\rangle \\ &= t\ \mathrm{Re}\langle v, \alpha_0\rangle + o(t)\end{aligned}$$

as $t \to 0$. Therefore,

$$\big|\kappa\,(x; v_0 + tv) - \kappa\,(x; v_0) - t\,\mathrm{Re}\langle v, \alpha_0\rangle\big| = o(t)$$

as $t \to 0$. □

2.2.3. Measured foliations

Let $\mathcal{S}$ be the set of homotopy classes of non-contractible and non-peripheral simple closed curves on $\Sigma_{g,m}$. Let $\mathcal{WS}$ be the set of formal scalar products $\{t\alpha \,|\, t \geqslant 0, \alpha \in \mathcal{S}\}$, which we call the set of *weighted simple closed curves* on $\Sigma_{g,m}$. Consider the embedding

$$\mathcal{WS} \ni t\alpha \longmapsto \big[\mathcal{S} \ni \beta \longmapsto t\,i(\alpha,\beta)\big] \in \mathbb{R}^{\mathcal{S}}_{\geqslant 0},$$

where i denotes the geometric intersection number. We equip the function space $\mathbb{R}^{\mathcal{S}}_{\geqslant 0}$ with the pointwise convergence topology. The closure $\mathcal{MF}$ of the image of $\mathcal{WS}$ in $\mathbb{R}^{\mathcal{S}}_{\geqslant 0}$ is called the space of *measured foliations* on $\Sigma_{g,m}$. For $F \in \mathcal{MF}$, we call the value $F(\alpha)$ the *intersection number* of F with α, and denote it by $i(F,\alpha)$. Set $i(F, t\alpha) = t\,i(F,\alpha)$ for $t\alpha \in \mathcal{WS}$. It is known that the intersection number $i(\cdot,\cdot)$ on $\mathcal{MF} \times \mathcal{WS}$ extends continuously to a function $i(\cdot,\cdot)$ on $\mathcal{MF} \times \mathcal{MF}$ which satisfies $i(F,G) = i(G,F)$ for $F, G \in \mathcal{MF}$.

2.2.4. Hubbard–Masur differentials and extremal length

Let $x = (X, f)$ be a point in $\mathcal{T}_{g,m}$. For $q = q(z)\mathrm{d}z^2 \in A^2(X)$, we set

$$v(q)(\alpha) = \inf_{\alpha' \in f(\alpha)} \int_{\alpha'} \left|\mathrm{Re}\left(\sqrt{q(z)}\mathrm{d}z\right)\right|$$

for $\alpha \in \mathcal{S}$. Regarding $v(q)$ as contained in $\mathbb{R}^{\mathcal{S}}_{\geqslant 0}$, we call it the *vertical foliation* of q. It is known that $v(q) \in \mathcal{MF}$.

For $x = (X, f) \in \mathcal{T}_{g,m}$ and $F \in \mathcal{MF}$, there is a unique quadratic differential $q_{F,x} \in A^2(X)$ such that $i(F, \alpha) = v(q)(\alpha)$ for all $\alpha \in \mathcal{S}$. We call the differential $q_{F,x}$ the *Hubbard–Masur differential* for F on x. The norm

$$\mathrm{Ext}_x(F) = \int_X |q_{F,x}(z)|\mathrm{d}x\mathrm{d}y$$

is called the *extremal length* of F on x. The extremal length function

$$\mathcal{T}_{g,m} \times \mathcal{MF} \ni (x, F) \longmapsto \mathrm{Ext}_x(F)$$

is continuous. When $F \in \mathcal{MF}$ is fixed, the extremal length function is of class C^1. The following formula, called the *Gardiner formula*, is known:

$$\mathrm{d}\,\mathrm{Ext}_{\cdot}(F) = -2\,\mathrm{Re} \int_X \mu q_{F,x} \tag{2.5}$$

for $v = [\mu] \in T_x\mathcal{T}_{g,m} \cong L^\infty(X)/N^\infty(X)$ (cf. [7]). Notice that the minus sign in the right-hand side of (2.5) comes from the fact that $q_{F,x}$ has F as the vertical foliation, while Gardiner considers the horizontal foliations when he concludes the formula (2.5).

2.3. Teichmüller–Randers metric

For a given n-dimensional Riemannian manifold (M, g) and a 1-form ω on M with $\|\omega\|_g < 1$ at every point of M, the associated *Randers metric* is a Finsler metric on M defined by the functional $F(v) = g(v, v)^{1/2} + \omega(v)$ on the tangent space of M. Although in the general literature Randers metrics refer to deformations of Riemannian metrics by 1-forms, we can consider such deformations for Finsler (symmetric) metrics in the same way. Furthermore, even in the case when $\|\omega\|_g = 1$, the Randers metric makes sense as a weak Finsler metric. In this paper, we study Randers-type deformations of the Teichmüller metric κ on $\mathcal{T}_{g,m}$ which we explained in the previous subsection, by taking the 1-form ω as presented in the introduction:

$$\kappa^\omega(x; v) = \kappa(x; v) + \omega(v). \tag{2.6}$$

As a 1-form ω, we shall consider in particular the form expressed as

$$-\frac{1}{2}\mathrm{d}\log \mathrm{Ext}_{(\cdot)}(F)$$

for a measured foliation F on $\Sigma_{g,m}$. This metric depends on the choice of F, but only on the projective class of F since we are taking log in the second term. This metric can be regarded as a generalisation of the weak Finsler metric which we studied in [17].

2.4. References to background material

We now give some references for the background material which we briefly presented in this section. The Teichmüller metric was introduced and studied thoroughly by Teichmüller in his paper [23] (see its English translation [24]). In this paper there is a long section (§25) on the Finsler nature of this metric. In the same section, Teichmüller introduced and studied what are now called Teichmüller discs (holomorphic images of the hyperbolic plane, defined by quadratic differentials), which he calls complex geodesics. As modern introductions to Teichmüller theory, we refer the reader to [8] and [10]. For the theory of measured foliations and measured foliation spaces, we refer the reader to [6], and for a comprehensive introduction to extremal length, to [16] for instance. For Randers metric, we refer the reader to Randers's original paper [18].

3. Extension of the Hamilton–Krushkal condition

In this section, we discuss the infinitesimal extremal property for our Teichmüller–Randers metric.

Let X be a Riemann surface, and fix $\varphi_0 \in A^2(X)$. We consider the following functional on the space $L^\infty(X)$ of Beltrami differentials:

$$\beta(\mu,\varphi_0) = \sup\left\{ \left|\int_X \mu\varphi\right| + \mathrm{Re}\int_X \mu\varphi_0 \;\middle|\; \varphi \in A^2(X), \|\varphi\|_1 = 1 \right\} \tag{3.1}$$

for $\mu \in L^\infty(X)$. It immediately follows from the definition that

$$\beta(\mu,\varphi_0) \leqslant \|\mu\|_\infty + \mathrm{Re}\int_X \mu\varphi_0 \tag{3.2}$$

for all $\mu \in L^\infty(X)$. We say that a Beltrami differential μ is *infinitesimally φ_0-extremal* if

$$\beta(\mu,\varphi_0) = \|\mu\|_\infty + \mathrm{Re}\int_X \mu\varphi_0,$$

and that μ satisfies the *Hamilton condition* if

$$\sup\left\{\left|\int_X \mu\varphi\right| \;\middle|\; \varphi \in A^2(X), \|\varphi\|_1 \leqslant 1\right\} = \|\mu\|_\infty.$$

It is known that $\mu \in L^\infty(X)$ is *infinitesimally Teichmüller extremal* in the sense that $\|\mu - \nu\|_\infty \geqslant \|\mu\|_\infty$ for all $\nu \in N^\infty(X)$ if and only if it satisfies the Hamilton condition [9, 12].

Theorem 3.1 (Extension of the Hamilton–Krushkal condition). — *Let X be a Riemann surface and φ_0 a holomorphic quadratic differential on X.*

(1) *If two Beltrami differentials $\mu, \nu \in L^\infty(X)$ are infinitesimally Teichmüller equivalent, then $\beta(\mu, \varphi_0) = \beta(\nu, \varphi_0)$.*
(2) *For a Beltrami differential $\mu \in L^\infty(X)$, the following three conditions are equivalent:*
   (a) *μ is* infinitesimally φ_0-extremal*;*
   (b) *μ is* infinitesimally Teichmüller extremal*;*
   (c) *μ satisfies the Hamilton condition.*

Proof.

(1). — The assumption that μ and ν are Teichmüller equivalent means by definition that $\int_X \mu\varphi = \int_X \nu\varphi$ for all $\varphi \in A^2(X)$. Hence, we have

$$\left|\int_X \mu\varphi\right| + \mathrm{Re}\int_X \mu\varphi_0 = \left|\int_X \nu\varphi\right| + \mathrm{Re}\int_X \nu\varphi_0$$

for all $\varphi \in A^2(X)$, which implies $\beta(\mu, \varphi_0) = \beta(\nu, \varphi_0)$.

(2). — We only need to show the equivalence between conditions (2a) and (2b), for the equivalence of the condition (2c) with (2a) and (2b) follows immediately then. Suppose that μ is infinitesimally φ_0-extremal. Then for $\nu \in N^\infty(X)$, we have

$$\begin{aligned}\|\mu\|_\infty + \mathrm{Re}\int_X \mu\varphi_0 &= \beta(\mu, \varphi_0) = \beta(\mu - \nu, \varphi_0) \\ &\leqslant \|\mu - \nu\|_\infty + \mathrm{Re}\int_X (\mu - \nu)\varphi_0 = \|\mu - \nu\|_\infty + \mathrm{Re}\int_X \mu\varphi_0,\end{aligned}$$

hence $\|\mu\|_\infty \leqslant \|\mu - \nu\|_\infty$. This means that μ is infinitesimally Teichmüller extremal.

Conversely, suppose that μ is infinitesimally Teichmüller extremal. Then, by definition, there exists a sequence (φ_n) in $A^2(X)$ such that $\left|\int_X \mu\varphi_n\right| \to \|\mu\|_\infty$. Therefore,

$$\|\mu\|_\infty + \operatorname{Re}\int_X \mu\varphi_0 = \lim_{n\to\infty}\left|\int_X \mu\varphi_n\right| + \operatorname{Re}\int_X \mu\varphi_0 \leqslant \beta(\mu,\varphi_0).$$

Combining this with (3.2), we see that μ is infinitesimally φ_0-extremal. □

In the case of analytically finite Riemann surfaces, an infinitesimally Teichmüller extremal Beltrami differential is a Teichmüller–Beltrami differential, and vice versa. Hence we have the following.

Corollary 3.2 (Analytically finite case). — *Let X be an analytically finite Riemann surface and φ_0 a holomorphic quadratic differential on X. Then, for $\mu \in L^\infty(X)$, the following two conditions are equivalent.*

(1) *μ is infinitesimally φ_0-extremal;*
(2) *μ is a Teichmüller–Beltrami differential. Namely, there are $\psi \in A^2(X) - \{0\}$ and $c \geqslant 0$ such that $\mu = c\overline{\psi}/|\psi|$.*

3.1. Teichmüller–Randers cometric

Let $x = (X, f)$ be a point in $\mathcal{T}_{g,m}$, and ω a 1-form on $\mathcal{T}_{g,m}$ with $\|\omega\|_T(x) \leqslant 1$. We define the *Teichmüller–Randers cometric* G_ω on the space of holomorphic quadratic differentials, which is identified with the cotangent space as $A^2(X) \cong T_x^*\mathcal{T}_{g,m}$, by

$$G_\omega(\varphi) = \sup_{\kappa^\omega(x;v)=1} |\langle v,\varphi\rangle| = \sup_{\kappa^\omega(x;v)=1} \operatorname{Re}\langle v,\varphi\rangle$$

for $\varphi \in A^2(X)$. This is dual to the Teichmüller–Randers metric. When $\omega = 0$, it is known that

$$G_0(\varphi) = \|\varphi\|_1.$$

More generally, if $\|\omega\|_T(x) < 1$ for all $x \in \mathcal{T}_{g,m}$, then G_ω defines a norm. Even if $\|\omega\|_T(x) = 1$ for some $x \in \mathcal{T}_{g,m}$, as we have seen above, G_ω defines an (asymmetric) weak norm on $A^2(X)$, whereas G_ω is not a norm then. Indeed, assuming $\|\omega\|_T(x) = 1$, take a tangent vector $v \in T_x\mathcal{T}_{g,m}$ with $\kappa(x;v) = 1$ and $\omega(v) = -1$. Then, $\kappa^\omega(x;v) = 0$ by definition of κ^ω. Take $\alpha \in A^2(X)$ such that $\operatorname{Re}\langle v,\alpha\rangle = \|\alpha\|_1 = 1$, and $(v_n) \subset T_x\mathcal{T}_{g,m}$ converging to v. Then, we have

$$\frac{\operatorname{Re}\langle v_n,\alpha\rangle}{\kappa^\omega(x;v_n)} \longrightarrow \infty$$

as $n \to \infty$, which shows that G_ω is not a norm. For this reason, when we discuss the dual G_ω, we always assume that $\|\omega\|_T(x) < 1$.

Theorem 3.3 (Teichmüller–Randers cometric). — *Let $x = (X, f)$ be a point in $\mathcal{T}_{g,m}$. Suppose that ω is represented by $\psi \in A^2(X) \cong T_x^*\mathcal{T}_{g,m}$ at x and that $\|\omega\|_T(x) = \|\psi\|_1 < 1$. Then,*

$$G_\omega(\varphi) = \inf\left\{t > 0 \,\middle|\, \left\|\frac{\varphi}{t} - \psi\right\|_1 \leqslant 1\right\}. \tag{3.3}$$

In particular, if $\varphi \neq 0$, then we have

$$\left\|\frac{\varphi}{G_\omega(\varphi)} - \psi\right\|_1 = 1.$$

Proof. — By Corollary 3.2, for a Teichmüller–Beltrami differential $\mu = \overline{\alpha}/|\alpha|$ ($\alpha \in A^2(X)$), we have

$$\kappa^\omega\left(x; [\mu]\right) = 1 + \mathrm{Re}\int_X \frac{\overline{\alpha}}{|\alpha|}\psi. \tag{3.4}$$

We may assume that neither φ nor ψ is 0, for our claim evidently holds if one of them is 0. From the definition of G_ω and (3.4), we have

$$G_\omega(\varphi) = \sup_{\kappa(x;v)=1} \frac{\mathrm{Re}\langle v, \varphi\rangle}{1 + \mathrm{Re}\langle v, \psi\rangle}. \tag{3.5}$$

If v is represented by the Teichmüller–Beltrami differential $\overline{\varphi}/|\varphi|$, the function in the supremum in the right-hand side of (3.5) is positive. Hence we have $G_\omega(\varphi) > 0$.

Let v_0 be a tangent vector represented by a Teichmüller–Beltrami differential $\overline{\alpha_0}/|\alpha_0|$ ($\|\alpha_0\|_1 = 1$) which attains the supremum in the right-hand side of (3.5). Then

$$G_\omega(\varphi) = \frac{\mathrm{Re}\langle v_0, \varphi\rangle}{1 + \mathrm{Re}\langle v_0, \psi\rangle}.$$

We note that $\mathrm{Re}\langle v_0, \varphi\rangle > 0$ and $|\langle v_0, \psi\rangle| < 1$ since $G_\omega(\varphi) > 0$ and $\|\psi\|_1 < 1$. For $v \in T_x\mathcal{T}_{g,m}$, we can compute

$$\frac{\mathrm{Re}\langle v_0 + tv, \varphi\rangle}{1 + \mathrm{Re}\langle v_0 + tv, \psi\rangle} = G_\omega(\varphi) + tG_\omega(\varphi)\,\mathrm{Re}\left\langle v, \frac{\varphi}{\mathrm{Re}\langle v_0, \varphi\rangle} - \frac{\psi}{1 + \mathrm{Re}\langle v_0, \psi\rangle}\right\rangle + o(t)$$

as $t \to 0$. As remarked above, $G_\omega(\varphi) > 0$. Since the left-hand side of the above equality attains the supremum in $\{v \mid \kappa(x; v) = 1\}$ at v_0, by Lemma 2.1, we have

$$\mathrm{Re}\left\langle v, \frac{\varphi}{\mathrm{Re}\langle v_0, \varphi\rangle} - \frac{\psi}{1 + \mathrm{Re}\langle v_0, \psi\rangle}\right\rangle = 0$$

for all $v \in T_x\mathcal{T}_{g,m}$ with $\mathrm{Re}\langle v, \alpha_0\rangle = 0$. This means that there exists $t \in \mathbb{R}$ such that

$$\frac{\varphi}{\mathrm{Re}\langle v_0, \varphi\rangle} - \frac{\psi}{1 + \mathrm{Re}\langle v_0, \psi\rangle} = t\alpha_0.$$

By taking pairing with v_0 on both sides, we get

$$\frac{\mathrm{Re}\langle v_0,\varphi\rangle}{\mathrm{Re}\langle v_0,\varphi\rangle}-\frac{\mathrm{Re}\langle v_0,\psi\rangle}{1+\mathrm{Re}\langle v_0,\psi\rangle}=t\|\alpha_0\|_1=t,$$

which means $t=(1+\mathrm{Re}\langle v_0,\psi\rangle)^{-1}$. Thus we obtain

$$\frac{\varphi}{G_\omega(\varphi)}-\psi=\frac{1+\mathrm{Re}\langle v_0,\psi\rangle}{\mathrm{Re}\langle v_0,\varphi\rangle}\varphi-\psi=\alpha_0,$$

which implies the desired equalities. □

3.2. The case when ω is exact

Assume that ω is a continuous exact form, that is, $\omega=\mathrm{d}h_\omega$ for some C^1-function h_ω on $\mathcal{T}_{g,m}$. Then, the length of any C^1-path $\gamma\colon[a,b]\to\mathcal{T}_{g,m}$ is expressed as

$$\begin{aligned}\int_a^b\kappa^\omega\left(\gamma(t);\dot\gamma(t)\right)\mathrm{d}t&=\int_a^b\Big(\kappa\left(\gamma(t);\dot\gamma(t)\right)+\omega\big(\dot\gamma(t)\big)\Big)\mathrm{d}t\\&=\int_a^b\kappa\left(\gamma(t);\dot\gamma(t)\right)\mathrm{d}t+h_\omega\big(\gamma(b)\big)-h_\omega\big(\gamma(a)\big)\end{aligned}$$

with respect to the Teichmüller–Randers metric κ^ω. Therefore, taking the infimum on the lengths of paths connecting $x_1\in\mathcal{T}_{g,m}$ to $x_2\in\mathcal{T}_{g,m}$, the weak metric δ_T^ω associated with the Teichmüller–Randers metric satisfies

$$\delta_T^\omega(x_1,x_2)=d_T(x_1,x_2)+h_\omega(x_2)-h_\omega(x_1).\tag{3.6}$$

This equality implies the following proposition.

Proposition 3.4. — *For any continuous exact form ω on $\mathcal{T}_{g,m}$, any Teichmüller geodesic is a unique geodesic with respect to the Teichmüller–Randers distance δ_T^ω.*

This gives a generalisation of Theorem 2.1 in [17]. We also note that in the case when ω is exact, the symmetrisation of the weak metric associated with the Teichmüller–Randers metric coincides with the Teichmüller distance. Indeed, we have

$$\begin{aligned}&S(\delta_T^\omega)(x_1,x_2)=\frac{1}{2}\big(\delta_T^\omega(x_1,x_2)+\delta_T^\omega(x_2,x_1)\big)\\&=\frac{1}{2}\Big(d_T(x_1,x_2)+\big(h_\omega(x_2)-h_\omega(x_1)\big)+d_T(x_2,x_1)+\big(h_\omega(x_1)-h_\omega(x_2)\big)\Big)\\&=\frac{1}{2}\big(d_T(x_1,x_2)+d_T(x_2,x_1)\big)\\&=d_T(x_1,x_2).\end{aligned}$$

Proof of Proposition 3.4. — By (3.6), we see immediately that every Teichmüller geodesic is also a geodesic with respect to δ^{ω}_{T}. It remains to check the uniqueness of geodesics. Let $x_1, x_2 \in \mathcal{T}_{g,m}$ and $\gamma\colon [a,b] \to \mathcal{T}_{g,m}$ be a C^1-path connecting x_1 to x_2. If γ is not a Teichmüller geodesic, by the uniqueness of Teichmüller geodesics, we have

$$\begin{aligned} &d_T(x_1,x_2) + h_\omega(x_2) - h_\omega(x_1) \\ &\qquad < \int_a^b \kappa\left(\gamma(t);\dot\gamma(t)\right)\mathrm{d}t + h_\omega(x_2) - h_\omega(x_1) = \int_a^b \kappa^\omega\left(\gamma(t);\dot\gamma(t)\right)\mathrm{d}t, \end{aligned}$$

which implies that γ is not a geodesic with respect to δ^{ω}_{T} either. □

4. Proof of theorems

4.1. Teichmüller discs

Let $x = (X, f)$ be a point in $\mathcal{T}_{g,m}$. For $q \in A^2(X)$ and $\lambda \in \mathbb{H}$, we define

$$\mu_{\lambda,q} := \frac{\lambda - i}{\lambda + i}\frac{\bar q}{|q|}. \tag{4.1}$$

Let $f_{\lambda,q}$ be a quasi-conformal map on X with $\bar\partial f_{\lambda,q} = \mu_{\lambda,q}\partial f_{\lambda,q}$, and set $X_{\lambda,q}$ to be the image of $f_{\lambda,q}$. The *Teichmüller disc* associated with q, which is denoted by $\mathbb{D}_q$, is a holomorphic disc in $\mathcal{T}_{g,m}$ defined by

$$\phi_q\colon \mathbb{H} \ni \lambda \longmapsto x(\lambda,q) := (X_{\lambda,q}, f_{\lambda,q}) \in \mathcal{T}_{g,m}. \tag{4.2}$$

The following lemma shows basic properties of Teichmüller discs.

Lemma 4.1. — *For $x = (X, f) \in \mathcal{T}_{g,m}$ and a measured foliation F on S, we have the following.*

(1) *The extremal length function satisfies*

$$\mathrm{Ext}_{x(\lambda,q_{F,x})}(F) = \frac{1}{\mathrm{Im}(\lambda)}\mathrm{Ext}_x(F). \tag{4.3}$$

(2) *For the Teichmüller disc defined as in (4.2) for $q = q_{F,x}$, the image of any vertical geodesic line in $\mathbb{H}$ is the Teichmüller geodesic defined by holomorphic quadratic differentials whose vertical foliations are F.*

(3) *For any $\lambda \in \mathbb{H}$, the unit tangent vector $v_\lambda = (\phi_{q_{F,x}})_*\,(2i\,\mathrm{Im}(\lambda)\,\partial/\partial\lambda)$ is represented by*

$$\frac{\overline{q_{F,\phi(\lambda)}}}{|q_{F,\phi(\lambda)}|}.$$

Proof. — The assertions follow from the discussion by Marden and Masur in [13, §1.3]. We review some details for the convenience of the reader.

(1). — We shall only show (4.3) for $\alpha \in \mathcal{S}$. Since the weighted simple closed curves are dense in $\mathcal{MF}$ and $\mathcal{MF} \ni F \mapsto q_{F,x} \in A^2(X)$ is continuous, we can then conclude (4.3) for general measured foliations by taking limits.

One of the characterisations of the extremal length of α is that it is the reciprocal of the modulus of the "characteristic annulus" of $q_{\alpha,x}$, that is, the maximal (open) annulus formed by closed leaves of the vertical trajectories of $q_{\alpha,x}$ (see also [22, §20.3]). By the discussion by Marden and Masur in [13, §1.3], the extremal length $\mathrm{Ext}_{x(\lambda,q_{F,x})}(\alpha)$ satisfies

$$\mathrm{Ext}_{x(\lambda,q_{F,x})}(\alpha) = \frac{1}{1+\mathrm{Re}(\lambda')}\,\mathrm{Ext}_x(\alpha) \tag{4.4}$$

where λ' is a complex number satisfying $\mathrm{Re}(\lambda') > -1$ and

$$\frac{\lambda - i}{\lambda + i} = \frac{\lambda'}{2+\lambda'}.$$

Since $\mathrm{Re}(\lambda') = \mathrm{Re}\,(-1 - i\lambda) = -1 + \mathrm{Im}(\lambda)$, we obtain (4.3) from (4.4) in the case when $F = \alpha \in \mathcal{S}$.

(2). — Let A be the characteristic annulus of $q_{\alpha,x}$. The Teichmüller map $f_{\lambda,q}$ defined by $\mu_{\lambda,q_{\alpha,x}}$ is expressed as a map h_λ defined by

$$h_\lambda(z) = z|z|^{-i\lambda-1} = z|z|^{\mathrm{Im}(\lambda)-1-i\,\mathrm{Re}(\lambda)}$$

on the characteristic annulus $A \cong \{1 < |z| < r\}$ with $r = \exp(2\pi/\,\mathrm{Ext}_x(\alpha))$. The image $h_\lambda(\{1 < |z| < r\}) = \{1 < |z| < r^{\mathrm{Im}(\lambda)}\}$ corresponds to the characteristic annulus of the terminal quadratic differential $q_{\alpha,x(\lambda,q_{\alpha,x})}$. Therefore, the deformation along the vertical line in $\mathbb{H}$ passing through $\lambda \in \mathbb{H}$ is the Teichmüller geodesic associated with the differential $q_{\alpha,x(\lambda,q_{\alpha,x})}$.

(3). — Let $v_\lambda \in T_{x(\lambda,q_{F,x})}\mathcal{T}_{g,m}$ be the unit tangent vector in $\mathbb{D}_{q_{F,x}}$ at $x(\lambda, q_{F,x})$ as given in the statement (3). Then v_λ is represented by a Teichmüller–Beltrami differential $\overline{\psi}/|\psi|$ with $\psi \in A^2(X_{\lambda,q_{F,x}})$. From (1) above and the Gardiner formula (2.5),

$$
\begin{aligned}
-\operatorname{Re}\int_{X_{\lambda,q_{F,x}}}\frac{\overline{\psi}}{|\psi|}\frac{q_{F,x(\lambda,q_{F,x})}}{\|q_{F,x(\lambda,q_{F,x})}\|_1} &= \frac{1}{2}\mathrm{d}\log\operatorname{Ext}_{x(\cdot,q_{F,x})}(F)[v_\lambda]\\
&= \operatorname{Re}\left(2i\operatorname{Im}(\lambda)\frac{d}{d\lambda}\log\operatorname{Ext}_{x(\cdot,q_{F,x})}(F)\right)\\
&= \operatorname{Re}\left(2i\operatorname{Im}(\lambda)\cdot\left(-\frac{1}{2i\operatorname{Im}(\lambda)}\right)\right)\\
&= -1,
\end{aligned}
$$

which means that $\psi = q_{F,x(\lambda,q_{F,x})}$. $\square$

4.2. Proof of Theorem 1.1

Part (1) follows from Proposition 3.4. Let x be a point in $\mathcal{T}_{g,m}$ and let $\phi\colon \mathbb{H}\to\mathcal{T}_{g,m}$ be the Teichmüller disc defined by $q_{F,x}$ with $\phi(i)=x$. Since ω is exact, by Proposition 3.4, for any two points $\zeta_1,\zeta_2\in\mathbb{H}$, the hyperbolic geodesic $\gamma\colon [a,b]\to\mathbb{H}$ connecting ζ_1 to ζ_2 is mapped to a geodesic with respect to δ_T^ω connecting $x_1=\phi(\zeta_1)$ to $x_2=\phi(\zeta_2)$. From (1.3) and Lemma 4.1, we have

$$
\begin{aligned}
\delta_T^\omega(x_1,x_2) &= \int_a^b\Big(\kappa\left(\phi(\gamma(s));\phi_*\circ\dot\gamma(t)\right)+\omega\big(\phi_*\circ\dot\gamma(t)\big)\Big)\mathrm{d}t\\
&= d_{\mathrm{hyp}}(\zeta_1,\zeta_2)-\frac{1}{2}\int_{\phi(\gamma)}\mathrm{d}\log\operatorname{Ext}_{\phi\circ\gamma(t)}(F)\\
&= d_{\mathrm{hyp}}(\zeta_1,\zeta_2)+\frac{1}{2}\log\operatorname{Ext}_{x_1}(F)-\frac{1}{2}\log\operatorname{Ext}_{x_2}(F)\\
&= d_{\mathrm{hyp}}(\zeta_1,\zeta_2)+\frac{1}{2}\log\frac{1}{\operatorname{Im}(\zeta_1)}-\frac{1}{2}\log\frac{1}{\operatorname{Im}(\zeta_2)}\\
&= \int_\gamma\left(\mathrm{d}s_{\mathrm{hyp}}+\frac{1}{2}\mathrm{d}\log\operatorname{Im}(\zeta)\right)\\
&= \delta(\zeta_1,\zeta_2),
\end{aligned}
$$

which implies Part (2) of Theorem 1.1.

4.3. Proof of Theorem 1.2

Let $\phi\colon \mathbb{H}\to\mathcal{T}_{g,m}$ be an isometry as in the statement. We may assume that ω is exact on $\mathcal{T}_{g,m}$ by modifying it outside a neighbourhood of $\phi(\mathbb{H})$.

Then, there is a C^1-function h_ω on $\mathcal{T}_{g,m}$ such that $\mathrm{d}h_\omega = \omega$. We set $f_\omega = h_\omega \circ \phi$.

Take two points $\zeta_1 = \xi_1 + i\eta_1$ and $\zeta_2 = \xi_2 + i\eta_2 \in \mathbb{H}$. Let $\gamma\colon [0, s_0] \to \mathbb{H}$ be a hyperbolic geodesic connecting ζ_1 to ζ_2. By Proposition 3.4, $\phi \circ \gamma\colon [0, s_0] \to \mathcal{T}_{g,m}$ is a Teichmüller geodesic, and since ϕ is an isometry, we obtain

$$
\begin{aligned}
d_{\mathrm{hyp}}(\zeta_1, \zeta_2) + \frac{1}{2}\log\frac{\eta_2}{\eta_1} &= \delta(\zeta_1, \zeta_2) \\
&= \delta_T^\omega\big(\phi(\zeta_1), \phi(\zeta_2)\big) \\
&= \int_0^{s_0}\Big(\kappa\left(\phi\big(\gamma(t)\big); \phi_* \circ \dot{\gamma}(t)\right) + \omega\big(\phi_* \circ \dot{\gamma}(t)\big)\Big)\mathrm{d}t \\
&= d_T\big(\phi(\zeta_1), \phi(\zeta_2)\big) + f_\omega(\zeta_2) - f_\omega(\zeta_1).
\end{aligned}
\tag{4.5}
$$

Case 1 (horizontal lines). — Suppose that $\eta_1 = \eta_2$. Since both d_{hyp} and d_T are symmetric, from (4.5), we obtain

$$
f_\omega(\zeta_1) = f_\omega(\zeta_2), \quad \text{and hence} \quad d_T\big(\phi(\zeta_1), \phi(\zeta_2)\big) = d_{\mathrm{hyp}}(\zeta_1, \zeta_2). \tag{4.6}
$$

Case 2 (vertical lines). — Suppose $\xi_1 = \xi_2$ and $\eta_1 > \eta_2$. In this case, the geodesic γ is a vertical segment from ζ_1 to ζ_2. Since $\delta(\zeta_1, \zeta_2) = 0$ in this case, from (4.5), we have

$$
f_\omega(\zeta_1) - f_\omega(\zeta_2) = d_T\big(\phi(\zeta_1), \phi(\zeta_2)\big). \tag{4.7}
$$

For $\xi \in \mathbb{R}$, let $L_\xi = \{\xi + \eta i \mid \eta > 0\}$. Then by (4.7), we see that $\phi(L_\xi)$ is a geodesic with respect to d_T. Take a measured foliation F_ξ on S such that $\phi(L_\xi)$ is the Teichmüller geodesic defined by the Hubbard–Masur differential for F_ξ. To describe this more precisely, fix $\eta_0 > 0$ and set $x(\xi) = \phi(\xi + i\eta_0) \in \mathcal{T}_{g,m}$. Let $x(\xi + i\eta)$ be the image of the Teichmüller map from $x(\xi)$ with the Beltrami differential

$$
\tanh(t)\frac{\overline{q_{F_\xi, x(\xi)}}}{|q_{F_\xi, x(\xi)}|}, \tag{4.8}
$$

where $t = t(\eta)$ satisfies $|t| = d_T(x(\xi + i\eta), x(\xi))$, $t > 0$ if $\eta > \eta_0$, and $t \leqslant 0$ otherwise. Then we have $\phi(L_\xi) = \{x(\xi + i\eta) \mid \eta > 0\}$. By the Gardiner formula (2.5), $\mathrm{Ext}_{x(\xi+i\eta)}(F_\xi)$ decreases as η increases. Hence, from the Kerckhoff formula, we have

$$
\mathrm{Ext}_{x(\xi+i\eta)}(F_\xi) = \begin{cases} e^{-2d_T(x(\xi+i\eta), x(\xi))}\,\mathrm{Ext}_{x(\xi)}(F_\xi) & (\eta \geqslant \eta_0), \\ e^{2d_T(x(\xi+i\eta), x(\xi))}\,\mathrm{Ext}_{x(\xi)}(F_\xi) & (\eta \leqslant \eta_0). \end{cases}
$$

Now, for any η, η', take $\eta_3 > 0$ smaller than $\min\{\eta, \eta'\}$. From (4.7), we can compute as follows:

$$
\begin{aligned}
\int_{\gamma'} \omega &= f_\omega(\xi + i\eta') - f_\omega(\xi + i\eta) \\
&= \big(f_\omega(\xi + i\eta') - f_\omega(\xi + i\eta_3)\big) - \big(f_\omega(\xi + i\eta) - f_\omega(\xi + i\eta_3)\big) \\
&= \frac{1}{2}\log\frac{\mathrm{Ext}_{x(\xi+i\eta_3)}(F_\xi)}{\mathrm{Ext}_{x(\xi+i\eta')}(F_\xi)} - \frac{1}{2}\log\frac{\mathrm{Ext}_{x(\xi+i\eta_3)}(F_\xi)}{\mathrm{Ext}_{x(\xi+i\eta)}(F_\xi)} \\
&= -\frac{1}{2}\log\frac{\mathrm{Ext}_{x(\xi+i\eta')}(F_\xi)}{\mathrm{Ext}_{x(\xi+i\eta)}(F_\xi)} \\
&= -\frac{1}{2}\int_{\gamma'} \mathrm{d}\log \mathrm{Ext}_{(\cdot)}(F_\xi),
\end{aligned} \tag{4.9}
$$

where γ' is the image under ϕ of the vertical segment from $\xi + i\eta$ to $\xi + i\eta'$ in $\mathbb{H}$. We note that by (4.8) or Lemma 4.1 (3), the tangent vector $v_y \in T_y\mathcal{T}_{g,m}$ along $\phi(L_\xi)$ at $y \in \phi(L_\xi)$ has unit length with respect to the Teichmüller metric, and is given by the Beltrami differential

$$
\frac{\overline{q_{F_\xi,x}}}{|q_{F_\xi,x}|}. \tag{4.10}
$$

Hence we obtain

$$
\left|-\frac{1}{2}\mathrm{d}\log\mathrm{Ext}_y(F_\xi)[v_y]\right| = \frac{1}{2}\cdot\frac{2}{\|q_{F,x}\|}\,\mathrm{Re}\int_X \frac{\overline{q_{F_\xi,x}}}{|q_{F_\xi,x}|}q_{F_\xi,x} = 1.
$$

Since $\|\omega\|_T \leqslant 1$ on the image $\phi(\mathbb{H})$, from (4.9), we conclude that we have

$$
\omega = -\frac{1}{2}\mathrm{d}\log\mathrm{Ext}_{(\cdot)}(F_\xi) \tag{4.11}
$$

on L_ξ.

Case 3 (general case). — We take $\zeta_1 = \xi_1 + i\eta_1$ and $\zeta_2 = \xi_2 + i\eta_2 \in \mathbb{H}$ to be arbitrary. Set $\zeta_3 = \xi_2 + i\eta_1$. By (4.3) we have

$$
\mathrm{Ext}_{\phi(\xi+i\eta_1)}(F_\xi) = \frac{\eta_2}{\eta_1}\mathrm{Ext}_{\phi(\xi+i\eta_2)}(F_\xi) \tag{4.12}
$$

for all $\xi \in \mathbb{R}$ and $\eta_1, \eta_2 > 0$, and $\mathrm{Ext}_{\phi(\zeta_3)}(F_{\xi_2}) = \mathrm{Ext}_{\phi(\zeta_1)}(F_{\xi_1})$. Combining this with the argument in Case 2, we have

$$
\begin{aligned}
f_\omega(\zeta_2) - f_\omega(\zeta_1) &= \big(f_\omega(\zeta_2) - f_\omega(\zeta_3)\big) + \big(f_\omega(\zeta_3) - f_\omega(\zeta_1)\big) \\
&= -\frac{1}{2}\log\mathrm{Ext}_{\phi(\zeta_2)}(F_{\xi_2}) + \frac{1}{2}\log\mathrm{Ext}_{\phi(\zeta_1)}(F_{\xi_2}) \\
&= -\frac{1}{2}\log\frac{\eta_1}{\eta_2}.
\end{aligned}
$$

Then, from (4.5), we conclude that $d_T(\phi(\zeta_1), \phi(\zeta_2)) = d_{\mathrm{hyp}}(\zeta_1, \zeta_2)$ for any ζ_1, $\zeta_2 \in \mathbb{H}$. Hence, $\phi\colon (\mathbb{H}, d_{\mathrm{hyp}}) \to (\mathcal{T}_{g,m}, d_T)$ is an isometry. Theorem 1.1 in [1] shows that in this situation, ϕ is either holomorphic or anti-holomorphic, and the image is the Teichmüller disc. As shown in Lemma 4.1, $F_{\xi_1} = F_{\xi_2}$ for all $\xi_1, \xi_2 \in \mathbb{R}$. Setting $F = F_\xi$ ($\xi \in \mathbb{R}$), we see that the image $\phi(\mathbb{H})$ is the Teichmüller disc defined by the Hubbard–Masur differential for F.

Consider $\zeta = \xi + i\eta \in \mathbb{H}$ and L_ξ defined above. By (4.6), the derivative of f_ω at ζ in the horizontal direction is constantly zero. As shown in Lemma 4.1 (3), the image $v \in T_{\phi(\zeta)}\mathcal{T}_{g,m}$ of the unit tangent vector $2i\,\mathrm{Im}(\zeta)(\partial/\partial\zeta) \in T_\zeta\mathbb{H}$ to L_ξ at ζ is represented by the Teichmüller–Beltrami differential

$$\frac{\overline{q_{F,\phi(\zeta)}}}{|q_{F,\phi(\zeta)}|}.$$

Hence, by the Gardiner formula (2.5), the derivative of the function

$$\mathbb{H} \ni \zeta \longmapsto -\frac{1}{2}\log \mathrm{Ext}_{\phi(\zeta)}(F)$$

is also zero in the horizontal direction in $\mathbb{H}$. As a consequence, by (4.11),

$$\omega = -\frac{1}{2}\mathrm{d}\log \mathrm{Ext}_{(\cdot)}(F)$$

on the image $\phi(\mathbb{H})$.

4.4. Proof of Theorem 1.3

Let x be a point in $\mathcal{T}_{g,m}$, and G a measured foliation on S.

(1). — Suppose that $\alpha q_{F,x} \neq q_{G,x}$ for any complex number α, and hence $\mathbb{D}_{q_{F,x}} \cap \mathbb{D}_{q_{G,x}} = \{x\}$.

Then we claim the following.

Claim. — *For any $y \in \mathbb{D}_{q_{G,x}}$, we have $\mathbb{D}_{q_{F,y}} \cap \mathbb{D}_{q_{G,y}} = \{y\}$.*

Proof of the claim. — Otherwise, there is $y \in D_{q_{G,x}}$ such that $\mathbb{D}_{q_{F,y}}$ and $\mathbb{D}_{q_{G,y}}$ share at least two points. By the uniqueness of the Teichmüller geodesic, $\mathbb{D}_{q_{F,y}}$ and $\mathbb{D}_{q_{G,y}}$ share a common Teichmüller geodesic line passing through these two points. Since $\mathbb{D}_{q_{F,y}}$ and $\mathbb{D}_{q_{G,y}}$ are holomorphic discs, by the identity theorem, $\mathbb{D}_{q_{F,y}} = \mathbb{D}_{q_{G,y}}$. Since both x and y lie in $\mathbb{D}_{q_{G,x}} = \mathbb{D}_{q_{F,y}}$, from the discussion in Lemma 4.1 (or the discussion in [13, §1.3]), we have $\mathbb{D}_{q_{F,y}} = \mathbb{D}_{q_{F,x}}$ and $\mathbb{D}_{q_{G,x}} = \mathbb{D}_{q_{G,y}}$. Therefore, we obtain

$$\mathbb{D}_{q_{G,x}} = \mathbb{D}_{q_{G,y}} = \mathbb{D}_{q_{F,y}} = \mathbb{D}_{q_{F,x}},$$

which contradicts our assumption. □

Let y be a point in $\mathbb{D}_{q_{G,x}}$, and v_y the unit tangent vector to $\mathbb{D}_{q_{G,x}}$ at y represented by

$$\frac{\overline{q_{G,y}}}{|q_{G,y}|}$$

(cf. Lemma 4.1 (3)). We note that by the claim, $q_{G,y}$ is not a complex scalar multiple of $q_{F,y}$. Hence,

$$\left|-\frac{1}{2}\mathrm{d}\log \mathrm{Ext}_y(F)[v_y]\right| = \left|\mathrm{Re}\int_{X_{\lambda,q_{G,y}}} \frac{\overline{q_{G,y}}}{|q_{G,y}|}\frac{q_{F,x(\lambda,q_{F,y})}}{\|q_{F,y}\|_1}\right| < 1.$$

Therefore, for any compact set K in $\mathbb{D}_{q_{G,x}}$, there is a constant $C_K < 1$ such that

$$\left|-\frac{1}{2}\mathrm{d}\log \mathrm{Ext}_y(F)[v_y]\right| \leqslant C_K$$

for all $y \in K$.

Let x_1 and x_2 be distinct points on $\mathbb{D}_{q_{G,x}}$, and $\gamma \subset \mathbb{D}_{q_{G,x}}$ the Teichmüller geodesic containing x_1 and x_2. From the above discussion, we have

$$\left|\frac{1}{2}\log \mathrm{Ext}_{x_1}(F) - \frac{1}{2}\log \mathrm{Ext}_{x_2}(F)\right| < d_T(x_1,x_2)$$

and

$$\delta^{\omega}_T(x_1,x_2) = d_T(x_1,x_2) + \frac{1}{2}\log \mathrm{Ext}_{x_1}(F) - \frac{1}{2}\log \mathrm{Ext}_{x_2}(F) > 0,$$

which implies that δ^{ω}_T separates two points in $\mathbb{D}_{q_{G,x}}$.

(2). — Let $r_G = r_{G,x}\colon [0,\infty) \to \mathcal{T}_{g,m}$ be the Teichmüller geodesic ray defined by $q_{G,x}$ with arclength parameterisation. By [15, Lemma 1], the function

$$[0,\infty) \ni t \longmapsto e^{-\delta^{\omega}_T(x,r_G(t))} = e^{-t}\left(\frac{\mathrm{Ext}_{r_G(t)}(F)}{\mathrm{Ext}_{x_0}(F)}\right)^{1/2}$$

is non-increasing and tends to

$$\frac{\mathcal{E}(F)}{\mathrm{Ext}_{x_0}(F)^{1/2}}$$

as $t \to \infty$ where $\mathcal{E}$ is some continuous function defined on $\mathcal{MF}$ (see also [14, Theorem 1.1]). Let $G = G_1 + \cdots + G_m$ be the decomposition of G into indecomposable components (for details, see [14]). In [25, Corollary 1], Walsh showed that the limit function $\mathcal{E}$ is expressed as

$$\mathcal{E}(H) = \sqrt{\sum_{i=1}^{m}\frac{i(G_i,H)^2}{i\big(G_i,\mathcal{H}(q_{G,x})\big)}}$$

for $H \in \mathcal{MF}$, where $\mathcal{H}(q_{G,x})$ is the horizontal foliation of $q_{G,x}$. Therefore, $\mathcal{E}(H) = 0$ if and only if $i(G,H) = 0$. This means that $\delta_T^\omega(x, r_G(t))$ is uniformly bounded in terms of $t \geqslant 0$ if and only if $i(F,G) \neq 0$.

Finally, we prove the incompleteness of the restriction of δ_T^ω to any Teichmüller disc. Let x be a point in $\mathcal{T}_{g,m}$ and G a measured foliation on S. By [11, Theorem 2], the vertical foliation G_θ of $e^{i\theta}q_{G,x}$ is uniquely ergodic for almost every θ. Therefore, $i(F,G_\theta) \neq 0$ for almost every θ. It follows that almost all Teichmüller geodesic rays emanating from x in $\mathbb{D}_{q_{G,x}}$ have bounded length with respect to the distance δ_T^ω, and in particular, the restriction of δ_T^ω to $\mathbb{D}_{q_{G,x}}$ is incomplete.

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