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## ON THE NON-TRIVIALITY OF THE TORSION SUBGROUP OF THE ABELIANIZED JOHNSON KERNEL

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**ABSTRACT.** — The *Johnson kernel* is the subgroup of the mapping class group of a closed oriented surface that is generated by Dehn twists along separating simple closed curves. The rational abelianization of the Johnson kernel has been computed by Dimca, Hain and Papadima, and a more explicit form was subsequently provided by Morita, Sakasai and Suzuki. Based on these results, Nozaki, Sato and Suzuki used the theory of finite-type invariants of 3-manifolds to prove that the torsion subgroup of the abelianized Johnson kernel is non-trivial.

In this paper, we give a purely 2-dimensional proof of the non-triviality of this torsion subgroup and provide a lower bound for its cardinality. Our main tool is the action of the mapping class group on the Malcev Lie algebra of the fundamental group of the surface. Using the same infinitesimal techniques, we also provide an alternative diagrammatic description of the rational abelianized Johnson kernel, and we include in the results the case of an oriented surface with one boundary component.

**RÉSUMÉ.** — Le *noyau de Johnson* est le sous-groupe du groupe de difféotopie d'une surface fermée orientée qui est engendré par les twists de Dehn le long de courbes simples, fermées et séparantes. L'abélianisation rationnelle du noyau de Johnson a été calculée par Dimca, Hain et Papadima, et une forme plus explicite a ensuite été apportée par Morita, Sakasai et Suzuki. À partir de ces résultats, Nozaki, Sato et Suzuki ont utilisé la théorie des invariants de type fini des 3-variétés pour démontrer que le sous-groupe de torsion du noyau de Johnson abélianisé est non-trivial.

Dans cet article, nous donnons une démonstration purement 2-dimensionnelle de la non-trivialité de ce sous-groupe de torsion et nous produisons une borne inférieure pour son cardinal. Notre principal outil est l'action du groupe de difféotopie sur l'algèbre de Lie de Malcev du groupe fondamental de la surface. En utilisant les mêmes techniques infinitésimales, nous apportons une autre description, de nature diagrammatique, de l'abélianisation rationnelle du noyau de Johnson, et nous traitons aussi le cas d'une surface orientée avec une composante de bord.

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*Keywords:* Surface, mapping class group, Torelli group, Johnson kernel, Johnson homomorphisms.

2020 *Mathematics Subject Classification:* 57K20, 20F38, 20F34, 20F12, 20F14, 57K16.

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## 1. Introduction

Let  $\Sigma$  be a compact connected oriented surface with one boundary component, and denote by  $\mathcal{M} := \mathcal{M}(\Sigma)$  the mapping class group of  $\Sigma$ . Although its abelianization is trivial for  $g \geq 3$  [39], the group  $\mathcal{M}$  has remarkable subgroups with highly non-trivial abelianization. This is particularly manifest for the *Torelli group*, which is the subgroup  $\mathcal{I} := \mathcal{I}(\Sigma)$  of  $\mathcal{M}$  acting trivially on the homology  $H := H_1(\Sigma; \mathbb{Z})$  of the surface. In his fundamental works of the eighties (including [21, 22, 23]), Johnson proved that (like  $\mathcal{M}$ ) the group  $\mathcal{I}$  is finitely generated and that (unlike  $\mathcal{M}$ ) it has an interesting abelianization  $\mathcal{I}_{\text{ab}}$ . In fact, Johnson gave a full characterization of  $\mathcal{I}_{\text{ab}}$ , revealing that its torsion-free quotient is isomorphic to  $\Lambda^3 H$  and that its torsion subgroup is isomorphic to the space of quadratic boolean functions on the space of spin structures of  $\Sigma$ . The map  $\mathcal{I} \rightarrow \Lambda^3 H$  (corresponding to the canonical projection of  $\mathcal{I}_{\text{ab}}$  onto its torsion-free quotient) is the first  $\tau_1$  of a series of homomorphisms  $(\tau_k)_k$ , which are now referred to as the *Johnson homomorphisms* and are defined on the successive terms of the *Johnson filtration*  $(\mathcal{M}[k])_k$ . In particular, the subgroup  $\mathcal{K} := \mathcal{M}[2]$  of  $\mathcal{I} = \mathcal{M}[1]$  plays a very important role in Johnson's works: called the *Johnson kernel*,  $\mathcal{K}$  is generated by Dehn twists along separating simple closed curves. In the sequel, those generators will be referred to as “separating twists”.

Besides, Johnson did similar constructions and proved similar results for a closed oriented surface of genus  $g \geq 3$ . Here it will be convenient to think of it as the surface  $\widehat{\Sigma}$  obtained by gluing a 2-disk to  $\Sigma$ . Then, for simplicity, we shall denote by  $\widehat{\mathcal{M}}$  the mapping class group of  $\widehat{\Sigma}$ , by  $\widehat{\mathcal{I}}$  its Torelli group and by  $\widehat{\mathcal{K}}$  its Johnson kernel.

In the last decade, major advances on the abelianization of the Johnson kernel have been accomplished for the closed surface  $\widehat{\Sigma}$ . Firstly, Dimca and Papadima proved that the rational abelianization  $\widehat{\mathcal{K}}_{\text{ab}} \otimes \mathbb{Q}$  is finite-dimensional [7]. Later, using this result and Hain's description of the Malcev Lie algebra of  $\widehat{\mathcal{I}}$  [18], Dimca, Hain and Papadima computed this vector space [6]. More recently, Morita, Sakasai and Suzuki [36] could express this computation of  $\widehat{\mathcal{K}}_{\text{ab}} \otimes \mathbb{Q}$  in a more explicit form, involving two homomorphisms on  $\widehat{\mathcal{K}}$  that Morita introduced in the nineties: namely, a by-product  $\widehat{d}$  of the Casson invariant [31] and a “refined” version of the second Johnson homomorphism  $\tau_2$  [32]. As we shall see in Section 3, the results of [6, 36] can be adapted to the case of the bordered surface  $\Sigma$ : this will give us the opportunity to revisit these results and provide an alternative diagrammatic description of  $\mathcal{K}_{\text{ab}} \otimes \mathbb{Q}$ .

Using the computation of  $\widehat{\mathcal{K}}_{\text{ab}} \otimes \mathbb{Q}$  given in [36] and appealing to the theory of finite-type invariants of 3-manifolds, Nozaki, Sato and Suzuki [37] were able to show that the torsion subgroup of  $\widehat{\mathcal{K}}_{\text{ab}}$  is non-trivial. To be more explicit on their methods, let us mention that they use the *LMO homomorphism* (which is a universal rational finite-type invariant of homology cylinders [2, 17]), and that their arguments involve 3-dimensional surgery techniques (which are known as *clasper calculus* [12, 15]). Hence their proof requires a certain level of expertise in the theory of finite-type invariants, and it does not conclude with an explicit torsion element of  $\widehat{\mathcal{K}}_{\text{ab}}$ .

Regarding this result of Nozaki, Sato and Suzuki, our goal in this paper is two-fold. On the one hand, we provide explicit elements of the torsion subgroup of  $\widehat{\mathcal{K}}_{\text{ab}}$ , and we prove their non-triviality by purely 2-dimensional methods. Thus, we hope to make their result accessible to a wider audience, and to open the way towards a full computation of the torsion subgroup of  $\widehat{\mathcal{K}}_{\text{ab}}$ . On the other hand, we also deal with the case of the bordered surface  $\Sigma$ :

**THEOREM A.** — *For a compact connected oriented surface with 0 or 1 boundary component, of genus  $g \geq 6$ , the abelianization of the Johnson kernel has a non-trivial torsion subgroup.*

Our proof of Theorem A is based on the action of  $\mathcal{M}$  on the Malcev Lie algebra of the fundamental group  $\pi_1(\Sigma)$ . The possibility of such a proof is not so surprising, since Nozaki, Sato and Suzuki only use the tree-reduction of the LMO homomorphism in their arguments and, according to [27], the latter encodes in some way the action of the Torelli group  $\mathcal{I}$  on the Malcev Lie algebra of  $\pi_1(\Sigma)$ . (See Remark 2.9 and Remark 2.11 for the exact relationship with the arguments of [37].)

The proof of Theorem A, which is done in Section 4, can be summarized as follows. We use the diagrammatic description of the action of  $\mathcal{I}$  on the Malcev Lie algebra of  $\pi_1(\Sigma)$  that is given in [27]. From this *infinitesimal Dehn–Nielsen representation*, we derive in Section 2 a map  $R$  from the Johnson kernel  $\mathcal{K}$  to a torsion abelian group. Although this map  $R$  is only polynomial of degree 2, it restricts on the fourth term  $\mathcal{M}[4]$  of the Johnson filtration to a homomorphism (which is a reduction of  $\tau_4$ ). Then we exhibit an explicit element of  $\mathcal{M}[4]$ , which is not seen by the “core” of the Casson invariant  $d$  but is detected by the map  $R$ :

**THEOREM B.** — *Assume that  $g \geq 3$ . There exists a  $\varphi \in \mathcal{M}[4]$  such that  $d(\varphi) = 0$  and  $R(\varphi) \neq 0$ . Moreover, its extension  $\widehat{\varphi} \in \widehat{\mathcal{M}}[4]$  to the closed surface  $\widehat{\Sigma}$  also satisfies  $\widehat{d}(\widehat{\varphi}) = 0$  and  $R(\widehat{\varphi}) \neq 0$ .*

Since the kernel of Morita’s refinement of  $\tau_2$  is  $\mathcal{M}[4]$ , we deduce from Theorem B and the above-mentioned computation of  $\mathcal{K}_{\text{ab}} \otimes \mathbb{Q}$  that  $\varphi$  provides a non-trivial torsion element of  $\mathcal{K}_{\text{ab}}$ , thus proving Theorem A. Our proof of Theorem B is purely two-dimensional and involves a rather long computation of  $R(\varphi)$ . We also give a second proof of Theorem B, leading to another explicit element  $\varphi'$ : closer to the original arguments of [37], this proof is certainly less computational, but it is done in the 3-dimensional framework of homology cylinders and needs the techniques of clasper calculus, including results of [5, 11, 12, 15, 28]. Indeed, the restriction of  $R$  to  $\mathcal{M}[4]$  is the reduction of  $\tau_4$  that Conant, Schneiderman and Teichner considered in [5] under the name of “higher-order Sato–Levine invariant” (by analogy with the study of Milnor invariants of links).

We conclude the paper by giving in Section 4.4 lower bounds for the cardinalities of the torsion subgroups of  $\mathcal{K}_{\text{ab}}$  and  $\widehat{\mathcal{K}}_{\text{ab}}$ . These lower bounds are obtained by a rough estimation of the image of  $R$ , using the canonical action of the mapping class group on the abelianized Johnson kernel.

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## 2. The infinitesimal Dehn–Nielsen representation and the map $R$

The Dehn–Nielsen representation of the mapping class group is defined by its canonical action on the fundamental group of the surface. We review an infinitesimal version of the Dehn–Nielsen representation that has been introduced in [27]. Then we derive from this a quadratic map  $R$ , from the abelianized Johnson kernel to a torsion abelian group.

### 2.1. The space of tree diagrams

We first recall what is the target of the infinitesimal Dehn–Nielsen representation. A *tree diagram* is a finite, univalent, connected, acyclic graph whose trivalent vertices are oriented (i.e. edges are cyclically ordered around each trivalent vertex), and whose univalent vertices are colored

by  $H = H_1(\Sigma; \mathbb{Z})$ : the former are called *nodes* and the latter are called *leaves*. For example, here is a tree diagram with 3 nodes and 5 leaves:

$$(2.1) \quad \begin{array}{c} & & c & & \\ & b & | & d & \\ & \diagdown & \bullet & / & \\ a & & & & e \end{array} \quad (\text{where } a, b, c, d, e \in H).$$

Here, and henceforth, orientations at trivalent vertices are always given by the trigonometric orientation of the plane. Let  $\mathcal{T}(H)$  be the abelian group generated by tree diagrams modulo the following relations:

$$\begin{array}{ccc} \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ \text{---} \end{array} = - \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} + \begin{array}{c} \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \end{array} = 0 & \begin{array}{c} h_1 + h_2 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \end{array} = \begin{array}{c} h_1 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \end{array} + \begin{array}{c} h_2 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \end{array} \\ \text{AS} & \text{IHX} & \text{multilinearity} \end{array}$$

By defining the *degree* of a tree diagram to be the number of nodes, we turn  $\mathcal{T}(H)$  into a graded abelian group:

$$\mathcal{T}(H) = \bigoplus_{d=1}^{+\infty} \mathcal{T}_d(H).$$

The abelian group  $\mathcal{T}(H)$  has the structure of a Lie ring, which involves the intersection pairing  $\omega : H \times H \rightarrow \mathbb{Z}$  of  $\Sigma$ . Specifically, the bracket of two tree diagrams  $P$  and  $Q$  is given by

$$[P, Q] := (\text{sum of all ways of } \omega\text{-connecting one leaf of } P \text{ to one leaf of } Q)$$

where “ $\omega$ -connecting” an  $x$ -colored vertex  $u$  of  $P$  to a  $y$ -colored vertex  $v$  of  $Q$  results in the element of  $\mathcal{T}(H)$  obtained by gluing  $u$  to  $v$  and multiplying by  $\omega(x, y)$ .

Setting  $H^{\mathbb{Q}} := H_1(\Sigma; \mathbb{Q})$ , we define a graded  $\mathbb{Q}$ -vector space  $\mathcal{T}(H^{\mathbb{Q}})$  in a similar way by generators and relations. Note that we have  $\mathcal{T}(H^{\mathbb{Q}}) \simeq \mathcal{T}(H) \otimes \mathbb{Q}$ , and  $\mathcal{T}(H^{\mathbb{Q}})$  has the structure of a Lie algebra. This space or, to be more accurate its degree-completion  $\widehat{\mathcal{T}}(H^{\mathbb{Q}})$ , will be used in the next subsection as the target of the “infinitesimal” Dehn–Nielsen representation.

The abelian group  $\mathcal{T}(H)$  and the vector space  $\mathcal{T}(H^{\mathbb{Q}})$  appear in the study of Milnor invariants of links and Johnson homomorphisms for mapping class groups, and they constitute the “tree levels” of the theories of finite-type invariants (see for instance [11, 13, 14, 27]). Consequently, their structure has been much studied. Before reviewing their relevance for the study of mapping class groups, we now recall what is known about this structure in relation with free Lie rings.

Let  $\mathfrak{L} := \mathfrak{L}(H)$  be the Lie ring freely generated by  $H$  and, similarly, let  $\mathfrak{L}^{\mathbb{Q}} := \mathfrak{L}(H^{\mathbb{Q}})$  be the Lie algebra freely generated by  $H^{\mathbb{Q}}$ . For any integer

$k \geq 1$ , we denote by  $D_k(H)$  the kernel of the Lie bracket map  $H \otimes \mathfrak{L}_{k+1} \rightarrow \mathfrak{L}_{k+2}$  and we set  $D(H) := \bigoplus_k D_k(H)$ . There is a homomorphism of graded abelian groups

$$(2.2) \quad \eta : \mathcal{T}(H) \longrightarrow D(H)$$

which maps any tree diagram  $T$  to the sum

$$\sum_v \text{col}(v) \otimes \text{brack}(T_v)$$

over all leaves  $v$  of  $T$ : here  $\text{col}(v)$  denotes the color of  $v$ , and  $\text{brack}(T_v)$  is the iterated Lie bracket defined by the tree  $T$  rooted at  $v$ . In the example (2.1), the tree rooted at its  $d$ -colored vertex defines

$$\text{brack} \left( \begin{array}{c} \left( \begin{array}{c} e \qquad a \quad b \quad c \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \\ \text{root} \end{array} \right) \end{array} \right) = [e, [[a, b], c]].$$

The map  $\eta$  is known to be an isomorphism with rational coefficients

$$(2.3) \quad \eta^{\mathbb{Q}} : \mathcal{T}(H^{\mathbb{Q}}) \xrightarrow{\cong} D(H^{\mathbb{Q}})$$

(see, for instance, [14]), but it is not bijective with  $\mathbb{Z}$ -linear coefficients. Indeed  $D(H)$  is free abelian but, as we shall recall,  $\mathcal{T}(H)$  has 2-torsion in odd degrees.

To understand the lack of bijectivity of  $\eta$ , Levine considers the quasi-Lie ring  $\mathfrak{L}' := \mathfrak{L}'(H)$  freely generated by  $H$ . Recall from [26] that the definition of a “quasi-Lie” ring requires the bracket to be only skew-symmetric, instead of alternate as a Lie bracket should be. Levine proves that the canonical map  $\mathfrak{L}' \rightarrow \mathfrak{L}$  is an isomorphism in odd degree and that, in even degree, there is a short exact sequence

$$(2.4) \quad 0 \longrightarrow \mathfrak{L}_k \otimes \mathbb{Z}_2 \longrightarrow \mathfrak{L}'_{2k} \longrightarrow \mathfrak{L}_{2k} \longrightarrow 0$$

where the left-hand homomorphism is defined by  $x \otimes 1 \mapsto [x, x]$ . For any  $k \geq 1$ , let  $D'_k(H)$  be the kernel of the quasi-Lie bracket  $H \otimes \mathfrak{L}'_{k+1} \rightarrow \mathfrak{L}'_{k+2}$  and set  $D'(H) := \bigoplus_k D'_k(H)$ . As a consequence of (2.4), there are short exact sequences

$$(2.5) \quad 0 \longrightarrow D'_{2k}(H) \longrightarrow D_{2k}(H) \longrightarrow \mathfrak{L}_{k+1} \otimes \mathbb{Z}_2 \longrightarrow 0,$$

$$(2.6) \quad 0 \longrightarrow H \otimes \mathfrak{L}_{k+1} \otimes \mathbb{Z}_2 \longrightarrow D'_{2k+1}(H) \longrightarrow D_{2k+1}(H) \longrightarrow 0.$$

Levine observes that the map  $\eta$  can be defined similarly in the quasi-Lie case to get a surjective homomorphism

$$\eta' : \mathcal{T}(H) \longrightarrow \mathbf{D}'(H),$$

and the injectivity of  $\eta'$  is proved by Conant, Schneiderman and Teichner in [4]. Hence the following commutative diagram:

$$\begin{array}{ccc} \mathcal{T}(H) & \xrightarrow{\eta} & \mathbf{D}(H) \\ & \searrow \cong & \uparrow \\ & \eta' & \mathbf{D}'(H). \end{array}$$

Thus, the above results combine to the following statement:

**THEOREM 2.1** (Levine, Conant–Schneiderman–Teichner). — *For any integer  $k \geq 1$ , there are short exact sequences*

$$(2.7) \quad 0 \longrightarrow \mathcal{T}_{2k}(H) \xrightarrow{\eta} \mathbf{D}_{2k}(H) \xrightarrow{\varpi} \mathfrak{L}_{k+1} \otimes \mathbb{Z}_2 \longrightarrow 0,$$

$$(2.8) \quad 0 \longrightarrow H \otimes \mathfrak{L}_{k+1} \otimes \mathbb{Z}_2 \xrightarrow{\iota} \mathcal{T}_{2k+1}(H) \xrightarrow{\eta} \mathbf{D}_{2k+1}(H) \longrightarrow 0.$$

Furthermore, the homomorphism  $\varpi$  is uniquely determined on the free abelian group  $\mathbf{D}_{2k}(H)$  by the fact that

$$(2.9) \quad \varpi \left( \frac{1}{2} \eta(u \text{---} u) \right) = \text{brack}(u) \otimes 1$$

for any rooted tree  $u$ . (The map  $\iota$  can also be explicitly defined, but we shall not need (2.8).)

*Remark 2.2.* — Apart from the Lie algebra structure on  $\mathcal{T}(H)$  which needs the symplectic form  $\omega$  on  $H$ , all constructions and results of this subsection work for any free abelian group  $H$ .

## 2.2. The infinitesimal Dehn–Nielsen representation

The central ingredient to define the “infinitesimal” version of the Dehn–Nielsen representation is the following notion.

Let  $\pi := \pi_1(\Sigma, \star)$  be the fundamental group of  $\Sigma$  based at a point  $\star \in \partial\Sigma$ , and let  $\widehat{\mathfrak{L}}^{\mathbb{Q}}$  be the degree-completion of  $\mathfrak{L}^{\mathbb{Q}}$ . A *symplectic logansion* of  $\pi$  is a map  $\theta : \pi \rightarrow \widehat{\mathfrak{L}}^{\mathbb{Q}}$  with the following properties:

- for each  $x \in \pi$ , the Lie series  $\theta(x)$  starts in degree 1 with the class of  $x$  in  $\frac{\pi}{[\pi, \pi]} \otimes \mathbb{Q} \simeq H^{\mathbb{Q}}$ ;



- for all  $x, y \in \pi$ , we have  $\theta(xy) = \theta(x) \star \theta(y)$  where  $\star$  denotes the Baker–Campbell–Hausdorff product<sup>(1)</sup> in  $\widehat{\mathfrak{L}}^{\mathbb{Q}}$  induced by its Lie bracket;
- $\theta$  maps the class  $\zeta := [\partial\Sigma]$  of the oriented boundary of  $\Sigma$  to  $-\omega$  where  $\omega \in \Lambda^2 H^{\mathbb{Q}} \simeq \mathfrak{L}_2^{\mathbb{Q}}$  is the dual of the intersection pairing.

*Remark 2.3.* — A symplectic logansion is a “symplectic expansion” in the sense of [27], composed with the logarithm series to transform group-like elements into primitive elements.

For concrete computations, we shall use in Section 4 the following instance of a symplectic logansion.

*Example 2.4.* — Let  $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$  be the system of “meridians & parallels” shown in Figure 2.1, which defines a basis of the free group  $\pi$  and induces a basis  $(a_1, \dots, a_g, b_1, \dots, b_g)$  of the free abelian group  $H$ . According to [27, Example 2.19], there is a symplectic logansion  $\theta$  which, in degree  $\leq 4$ , is given by

$$\begin{aligned} \theta(\alpha_i) &= a_i - \frac{1}{2}[a_i, b_i] + \frac{1}{12}[[a_i, b_i], b_i] - \frac{1}{2} \sum_{j < i} [[a_j, b_j], a_i] \\ &\quad - \frac{1}{24}[a_i, [a_i, [a_i, b_i]]] + \frac{1}{4} \sum_{j < i} [[a_j, b_j], [a_i, b_i]] + (\text{deg} \geq 5), \\ \theta(\beta_i) &= b_i - \frac{1}{2}[a_i, b_i] + \frac{1}{12}[a_i, [a_i, b_i]] + \frac{1}{4}[[a_i, b_i], b_i] + \frac{1}{2} \sum_{j < i} [b_i, [a_j, b_j]] \\ &\quad - \frac{1}{24}[[[a_i, b_i], b_i], b_i] + \frac{1}{4} \sum_{j < i} [[a_j, b_j], [a_i, b_i]] + (\text{deg} \geq 5). \end{aligned}$$

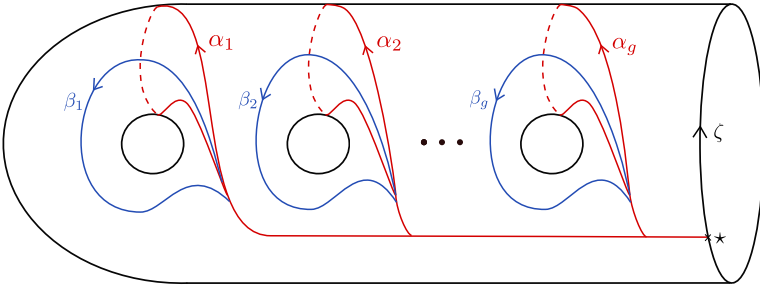


Figure 2.1. A system of “meridians & parallels” in the oriented surface  $\Sigma$

(1) Abbreviated to *BCH product* in the sequel.

To pursue towards the definition of the “infinitesimal” version of the Dehn–Nielsen representation, we introduce the following notations. Let  $\text{Aut}(\widehat{\mathfrak{L}}^{\mathbb{Q}})$  be the group of filtered automorphisms of  $\widehat{\mathfrak{L}}^{\mathbb{Q}}$ , and let  $\text{Aut}_{\omega}(\widehat{\mathfrak{L}}^{\mathbb{Q}})$  be the subgroup of  $\text{Aut}(\widehat{\mathfrak{L}}^{\mathbb{Q}})$  fixing  $\omega \in \Lambda^2 H^{\mathbb{Q}} \simeq \mathfrak{L}_2(H^{\mathbb{Q}})$ . As explained in [27, Section 3.1], the canonical action of  $\mathcal{M}$  on  $\pi$ , namely the *Dehn–Nielsen representation*

$$\rho : \mathcal{M} \longrightarrow \text{Aut}_{\zeta}(\pi),$$

is equivalent to an action of  $\mathcal{M}$  on the Malcev Lie algebra of  $\pi$  and, via the symplectic logansion  $\theta$ , the latter can equivalently be regarded as an action

$$\varrho^{\theta} : \mathcal{M} \longrightarrow \text{Aut}_{\omega}(\widehat{\mathfrak{L}}^{\mathbb{Q}})$$

of  $\mathcal{M}$  on the filtered Lie algebra  $\widehat{\mathfrak{L}}^{\mathbb{Q}}$ . Furthermore,  $\varrho^{\theta}$  maps the Torelli group  $\mathcal{I} \subset \mathcal{M}$  to the subgroup  $\text{IAut}_{\omega}(\widehat{\mathfrak{L}}^{\mathbb{Q}})$  of automorphisms that induce the identity at the graded level, and this is the source of a bijection

$$\log : \text{IAut}_{\omega}(\widehat{\mathfrak{L}}^{\mathbb{Q}}) \xrightarrow{\simeq} \text{Der}_{\omega}^{+}(\widehat{\mathfrak{L}}^{\mathbb{Q}})$$

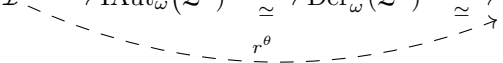
onto the space of derivations of  $\widehat{\mathfrak{L}}^{\mathbb{Q}}$  that strictly increase degrees and vanish on  $\omega$ .

It is well-known that  $\text{Der}_{\omega}^{+}(\mathfrak{L})$  (resp.  $\text{Der}_{\omega}^{+}(\mathfrak{L}^{\mathbb{Q}})$ ) is canonically isomorphic to  $\text{D}(H)$  (resp. to  $\text{D}(H^{\mathbb{Q}})$ ): specifically, a derivation  $d$  restricts to a homomorphism  $H \rightarrow \mathfrak{L}$  and, so, induces an element of  $H \otimes \mathfrak{L}$  using the isomorphism  $H \simeq H^*$  defined by  $h \mapsto \omega(h, -)$ ; then the symplectic condition  $d(\omega) = 0$  implies that this element of  $H \otimes \mathfrak{L}$  belongs to the subgroup  $\text{D}(H)$ . Thus, in the sequel, we will use interchangeably  $\text{Der}_{\omega}^{+}(\mathfrak{L})$  (resp.  $\text{Der}_{\omega}^{+}(\mathfrak{L}^{\mathbb{Q}})$ ) and  $\text{D}(H)$  (resp.  $\text{D}(H^{\mathbb{Q}})$ ). Recall that the space of derivations has a canonical Lie bracket given by

$$[d_1, d_2] := d_1 \circ d_2 - d_2 \circ d_1.$$

Clearly  $\text{Der}_{\omega}^{+}(\mathfrak{L}^{\mathbb{Q}})$  is a Lie subalgebra, and it turns out that the isomorphism (2.3) preserves the Lie brackets.

Finally, we define the *infinitesimal Dehn–Nielsen representation* of the Torelli group by the following composition of maps:

$$(2.10) \quad \mathcal{I} \xrightarrow{\varrho^{\theta}} \text{IAut}_{\omega}(\widehat{\mathfrak{L}}^{\mathbb{Q}}) \xrightarrow[\simeq]{\log} \text{Der}_{\omega}^{+}(\widehat{\mathfrak{L}}^{\mathbb{Q}}) \xrightarrow[\simeq]{(\eta^{\mathbb{Q}})^{-1}} \widehat{\mathcal{T}}(H^{\mathbb{Q}}).$$


The map  $r^\theta$  is a group homomorphism if we endow the target  $\widehat{\mathcal{T}}(H^\mathbb{Q})$  with the BCH product  $\star$  induced by its Lie bracket: thus, we have

$$(2.11) \quad \begin{aligned} r^\theta(fh) &= r^\theta(f) \star r^\theta(h) \\ &= r^\theta(f) + r^\theta(h) + \frac{1}{2}[r^\theta(f), r^\theta(h)] \\ &\quad + \frac{1}{12}[r^\theta(f), [r^\theta(f), r^\theta(h)]] + \frac{1}{12}[r^\theta(h), [r^\theta(h), r^\theta(f)]] + \dots \end{aligned}$$

for any  $f, h \in \mathcal{I}$ . See [27] for details about the above construction, and see [17] for a survey.

*Remark 2.5.* — The map  $r^\theta$  can be extended to the full mapping class group by setting

$$(2.12) \quad \forall f \in \mathcal{M}, \quad \widetilde{r}^\theta(f) := (\eta^\mathbb{Q})^{-1} \log(\varrho^\theta(f) \circ f_*^{-1}),$$

where  $f_* : H^\mathbb{Q} \rightarrow H^\mathbb{Q}$  is the automorphism induced by  $f$  in homology and is extended to an automorphism of  $\widehat{\mathfrak{L}}^\mathbb{Q}$  in the canonical way. Then the map  $\widetilde{r}^\theta : \mathcal{M} \rightarrow \widehat{\mathcal{T}}(H^\mathbb{Q})$  satisfies

$$(2.13) \quad \forall f, h \in \mathcal{M}, \quad \widetilde{r}^\theta(fh) = \widetilde{r}^\theta(f) \star (f_* \cdot \widetilde{r}^\theta(h))$$

where  $f_*$  acts on  $\widetilde{r}^\theta(h)$  by transforming the colors of all its leaves.

### 2.3. Truncations of the infinitesimal Dehn–Nielsen representation

We now consider the *Johnson filtration* of the mapping class group

$$\mathcal{M} \supset \underbrace{\mathcal{M}[1]}_{=\mathcal{I}} \supset \underbrace{\mathcal{M}[2]}_{=\mathcal{K}} \supset \dots \supset \mathcal{M}[k] \supset \mathcal{M}[k+1] \supset \dots$$

where  $\mathcal{M}[k]$  is defined as the kernel of the composition

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\rho} & \text{Aut}(\pi) \longrightarrow \text{Aut}(\pi/\Gamma_{k+1}\pi). \\ & \searrow & \nearrow \\ & & \rho_k \end{array}$$

(Here, and in the sequel, we denote by  $G = \Gamma_1 G \supset \Gamma_2 G \supset \dots$  the lower central series of a group  $G$ .) Recall from [32] that, for any  $k \geq 1$ , the restriction of  $\rho_{k+1}$  to  $\mathcal{M}[k]$  can be turned into a map

$$\tau_k : \mathcal{M}[k] \longrightarrow \text{D}_k(H) \subset H \otimes \mathfrak{L}_{k+1}$$

which is known as the *k-th Johnson homomorphism*. Since the abelian group  $\text{D}_k(H)$  is torsion-free, we do not lose any information by considering  $\tau_k$  with values in the vector space  $\text{D}_k(H^\mathbb{Q}) \simeq \text{D}_k(H) \otimes \mathbb{Q}$ . Then it is

equivalent to the degree  $k$  part of the infinitesimal Dehn–Nielsen representation  $r^\theta$ , in the sense that

$$(2.14) \quad (\eta_{\mathbb{Q}})^{-1} \circ \tau_k = r_k^\theta : \mathcal{M}[k] \longrightarrow \mathcal{T}_k(H^{\mathbb{Q}}).$$

In fact, a stronger version of (2.14) is known: the truncation

$$(2.15) \quad r_{[k,2k]}^\theta : \mathcal{M}[k] \longrightarrow \bigoplus_{d=k}^{2k-1} \mathcal{T}_d(H^{\mathbb{Q}})$$

of  $r^\theta$  on  $\mathcal{M}[k]$  to the degrees  $k, k+1, \dots, 2k-1$ , which encodes the restriction of  $\rho_{2k}$  to  $\mathcal{M}[k]$ , is equivalent to Morita’s “refinement”  $\tilde{\tau}_k$  of  $\tau_k$  [32]: see [27] for a precise statement and a proof.

We are also interested in larger truncations of the infinitesimal Dehn–Nielsen representation on  $\mathcal{M}[k]$ . Specifically, for any integer  $k \geq 1$ , we consider the map

$$r_{[k,3k]}^\theta : \mathcal{M}[k] \longrightarrow \bigoplus_{d=k}^{3k-1} \mathcal{T}_d(H^{\mathbb{Q}})$$

which is not a homomorphism anymore.

LEMMA 2.6. — *For all  $f, h \in \mathcal{M}[k]$ , we have*

$$r_{[k,3k]}^\theta(fh) = r_{[k,3k]}^\theta(f) + r_{[k,3k]}^\theta(h) + \frac{1}{2} \sum_{\substack{i \geq k, j \geq k \\ i+j < 3k}} [r_i^\theta(f), r_j^\theta(h)].$$

*Proof.* — Since  $r^\theta(f)$  and  $r^\theta(h)$  start in degree  $k$ , we deduce from (2.11) that

$$r^\theta(fh) = r^\theta(f) + r^\theta(h) + \frac{1}{2} [r^\theta(f), r^\theta(h)] + (\deg \geq 3k)$$

and the conclusion follows. □

By specializing Lemma 2.6 to  $k = 2$ , we obtain for any  $f, h \in \mathcal{K}$

$$\begin{aligned} r_{[2,5]}^\theta(fh) &= r_{[2,5]}^\theta(f) + r_{[2,5]}^\theta(h) + \frac{1}{2} [r_2^\theta(f), r_2^\theta(h)] \\ &\quad + \frac{1}{2} [r_2^\theta(f), r_3^\theta(h)] + \frac{1}{2} [r_3^\theta(f), r_2^\theta(h)] \end{aligned}$$

and, in particular,

$$(2.16) \quad r_4^\theta(fh) = r_4^\theta(f) + r_4^\theta(h) + \frac{1}{2} [(\eta^{\mathbb{Q}})^{-1} \tau_2(f), (\eta^{\mathbb{Q}})^{-1} \tau_2(h)].$$

*Remark 2.7.* — In the sequel, we will not mention anymore the isomorphism  $\eta^{\mathbb{Q}}$  when composed with a Johnson homomorphism. Said differently,

the values of the Johnson homomorphisms are considered either as derivations or  $\mathbb{Q}$ -linear combinations of tree diagrams, depending on the context. For instance, equation (2.16) simply writes  $r_4^\theta(fh) = r_4^\theta(f) + r_4^\theta(h) + \frac{1}{2}[\tau_2(f), \tau_2(h)]$  with this convention.

## 2.4. The quadratic map $R$ in the bordered case

Let  $\theta$  be a symplectic logansion of  $\pi$ . We consider the map

$$(2.17) \quad R^\theta : \mathcal{K} \longrightarrow \frac{\mathcal{T}_4(H^\mathbb{Q})}{\mathcal{T}_4(H)}, \quad f \longmapsto (r_4^\theta(f) \bmod 1),$$

and we simply denote it by  $R$  in the sequel. Here,  $\mathcal{T}_4(H)$  is viewed as a lattice in  $\mathcal{T}_4(H^\mathbb{Q})$  (since it is torsion-free as a consequence of (2.7)), and we refer to the congruence relation modulo  $\mathcal{T}_4(H)$  in  $\mathcal{T}_4(H^\mathbb{Q})$  as the congruence *modulo 1*.

LEMMA 2.8. — *The map  $R$  induces a map  $R_{\text{ab}}$  on the abelianization  $\mathcal{K}_{\text{ab}} = \mathcal{K}/[\mathcal{K}, \mathcal{K}]$ , which is polynomial of degree 2.*

*Proof.* — For any  $f \in \mathcal{K}$  and  $h \in [\mathcal{K}, \mathcal{K}]$ , we obtain from (2.16) and the nullity of  $\tau_2$  on  $[\mathcal{K}, \mathcal{K}]$  that  $r_4^\theta(fh) = r_4^\theta(f) + r_4^\theta(h)$ . Hence, to deduce that  $R(f) = R(fh)$ , it suffices to check that  $R(h) = 0$  and we can assume without loss of generality that  $h$  is a single commutator:  $h = [h', h'']$  with  $h', h'' \in \mathcal{K}$ . A straightforward computation, still based on (2.16), gives

$$r_4^\theta(h) = r_4^\theta([h', h'']) = [\tau_2(h'), \tau_2(h'')] \in \mathcal{T}_4(H^\mathbb{Q}).$$

Since we have  $\tau_2(\mathcal{K}) \subset \mathcal{D}_2(H)$ , it suffices to prove the following inclusion in  $\mathcal{T}_4(H^\mathbb{Q}) \simeq \mathcal{D}_4(H) \otimes \mathbb{Q}$ :

$$(2.18) \quad [\mathcal{D}_2(H), \mathcal{D}_2(H)] \subset \mathcal{T}_4(H).$$

On this purpose, we decompose the Lie bracket of  $\mathcal{T}(H^\mathbb{Q})$  as follows. Choose a symplectic basis  $(a_1, \dots, a_g, b_1, \dots, b_g)$  of  $H$ , and let  $\ell : H^\mathbb{Q} \times H^\mathbb{Q} \rightarrow H^\mathbb{Q}$  be the bilinear map defined by

$$\ell(a_i, b_j) := \delta_{ij}, \quad \ell(b_j, a_i) := 0, \quad \ell(a_i, a_j) := 0, \quad \ell(b_i, b_j) := 0.$$

Given any two trees  $P, Q \in \mathcal{T}(H^\mathbb{Q})$ , we set

$$P \triangleright Q := (\text{sum of all ways of } \ell\text{-connecting one leaf of } P \text{ to one leaf of } Q)$$

and we extend this to a bilinear map  $\triangleright : \mathcal{T}(H^\mathbb{Q}) \times \mathcal{T}(H^\mathbb{Q}) \rightarrow \mathcal{T}(H^\mathbb{Q})$ . Then, we have  $[P, Q] = P \triangleright Q - Q \triangleright P$  since  $\omega(x, y) = \ell(x, y) - \ell(y, x)$

for any  $x, y \in H^{\mathbb{Q}}$ . Thus, (2.18) will follow from the following inclusion in  $\mathcal{T}_4(H^{\mathbb{Q}}) \simeq \mathcal{D}_4(H) \otimes \mathbb{Q}$ :

$$(2.19) \quad \mathcal{D}_2(H) \triangleright \mathcal{D}_2(H) \subset \mathcal{T}_4(H).$$

It follows from (2.7) that  $\mathcal{D}_2(H)$  is generated by the following elements:

$$\begin{matrix} \wedge & & \wedge \\ a & b & c & d \end{matrix} \text{ with } a, b, c, d \in H, \quad \frac{1}{2} \begin{matrix} \wedge & & \wedge \\ u & v & u & v \end{matrix} \text{ with } u, v \in H.$$

Hence, to prove (2.19), it suffices to verify the following:

- (1)  $\begin{matrix} \wedge & & \wedge \\ a & b & c & d \end{matrix} \triangleright \begin{matrix} \wedge & & \wedge \\ a' & b' & c' & d' \end{matrix} \in \mathcal{T}_4(H)$ , for any  $a, b, c, d, a', b', c', d' \in H$ ;
- (2)  $\begin{matrix} \wedge & & \wedge \\ a & b & c & d \end{matrix} \triangleright \begin{matrix} \wedge & & \wedge \\ u & v & u & v \end{matrix} \in 2\mathcal{T}_4(H)$ , for any  $a, b, c, d, u, v \in H$ ;
- (2')  $\begin{matrix} \wedge & & \wedge \\ u & v & u & v \end{matrix} \triangleright \begin{matrix} \wedge & & \wedge \\ a & b & c & d \end{matrix} \in 2\mathcal{T}_4(H)$ , for any  $a, b, c, d, u, v \in H$ ;
- (3)  $\begin{matrix} \wedge & & \wedge \\ u & v & u & v \end{matrix} \triangleright \begin{matrix} \wedge & & \wedge \\ u' & v' & u' & v' \end{matrix} \in 4\mathcal{T}_4(H)$ , for any  $u, v, u', v' \in H$ .

(1) is obvious. (2) (resp. (2')) follows from the fact that each term resulting from the  $\triangleright$  operation here is repeated due to the symmetry of the right-hand (resp. left-hand) tree diagram. (3) is verified in a similar way.

Thus we have shown that  $R$  factorizes to a map  $R_{\text{ab}} : \mathcal{K}_{\text{ab}} \rightarrow \mathcal{T}(H^{\mathbb{Q}})/\mathcal{T}(H)$  verifying

$$R_{\text{ab}}(\{f\} + \{h\}) = R_{\text{ab}}(\{f\}) + R_{\text{ab}}(\{h\}) + \frac{1}{2}[\overline{\tau}_2(f), \overline{\tau}_2(h)]$$

where  $\overline{\tau}_2 : \mathcal{K}_{\text{ab}} \rightarrow \mathcal{D}_2(H)$  is the group homomorphism induced by  $\tau_2$ . The bilinearity of the Lie bracket in  $\mathcal{T}(H^{\mathbb{Q}})$  implies that  $R_{\text{ab}}$  is a polynomial map of degree 2 on the group  $\mathcal{K}_{\text{ab}}$ .  $\square$

*Remark 2.9.* — The map  $R_{\text{ab}}$  is *not* a group homomorphism, i.e. it is *not* of degree 1 as a polynomial map on the group  $\mathcal{K}_{\text{ab}}$ . Yet, using the operation  $\triangleright$  introduced in the proof of Lemma 2.8, we can instead of  $R$  consider the map  $R_{\circ}$  defined by

$$(2.20) \quad R_{\circ}(f) := \left( r_4^{\theta}(f) - \frac{1}{2}\tau_2(f) \triangleright \tau_2(f) \pmod{1} \right) \in \frac{\mathcal{T}_4(H^{\mathbb{Q}})}{\mathcal{T}_4(H)},$$

and deduce from (2.19) that  $R_{\circ}$  is a group homomorphism on  $\mathcal{K}$ . (Note that  $R_{\circ}$  depends on the choice of both a symplectic logansion of  $\pi$  and a symplectic basis of  $H$ .)

When  $\theta$  is the symplectic logansion defined by the LMO functor [27, Section 5.2], the homomorphism  $R_{\circ}$  is equal to the degree 4 part of the “mod 1 tree reduction” of  $\log_{\square} \widetilde{Z}^Y$ , which is considered by Nozaki, Sato and Suzuki [37]. (This equality is a consequence of [27, Theorem 5.13].)

The next lemma shows that the restriction of  $R$  to  $\mathcal{M}[4]$  is determined by the 4-th Johnson homomorphism (and, so, does not depend on the choice of  $\theta$ ).

LEMMA 2.10. — *We have the following commutative diagram*

$$\begin{array}{ccc}
 \mathcal{M}[4] & \xrightarrow{\tau_4} & D_4(H) \xrightarrow{\varpi} \mathfrak{L}_3 \otimes \mathbb{Z}_2 \\
 & \searrow R & \downarrow j \\
 & & \frac{\mathcal{T}_4(H^{\mathbb{Q}})}{\mathcal{T}_4(H)},
 \end{array}$$

where  $\varpi$  is the map given by (2.9) and  $j$  is defined by

$$j([a, [b, c]] \otimes 1) := \frac{1}{2} \begin{array}{c} c \quad \quad a \\ \diagdown \quad \diagup \\ \bullet \quad \quad \bullet \\ \diagup \quad \diagdown \\ b \quad \quad b \\ a \quad \quad c \end{array}$$

*Proof.* — The isomorphism  $\eta^{\mathbb{Q}} : \mathcal{T}_4(H^{\mathbb{Q}}) \rightarrow D_4(H^{\mathbb{Q}})$  induces an isomorphism between  $\mathcal{T}_4(H^{\mathbb{Q}})/\mathcal{T}_4(H)$  and  $D_4(H^{\mathbb{Q}})/\eta(\mathcal{T}_4(H))$ . The latter contains  $D_4(H)/\eta(\mathcal{T}_4(H))$  which, according to (2.7), is isomorphic to  $\mathfrak{L}_3 \otimes \mathbb{Z}_2$ . Then, we deduce from (2.17) and (2.14) that the restriction of  $R$  to  $\mathcal{M}[4]$  corresponds (through  $\eta^{\mathbb{Q}}$ ) to the composition

$$\mathcal{M}[4] \xrightarrow{\tau_4} D_4(H) \twoheadrightarrow \frac{D_4(H)}{\eta(\mathcal{T}_4(H))}.$$

Then we conclude thanks to the definition (2.9) of  $\varpi$ . □

### 2.5. The quadratic map $R$ in the closed case

We now consider the closed surface  $\widehat{\Sigma}$ , which is obtained from  $\Sigma$  by gluing a 2-disk. All the previous constructions of this section for  $\Sigma$  can be performed for  $\widehat{\Sigma}$ , but with extra technical difficulties which we outline.

First of all, we need to consider the subgroup  $I$  of  $\mathcal{T}(H)$  that is generated by trees showing an  $\omega$ -vertex, as shown below:

$$\begin{array}{c} | \\ \omega \end{array} := \sum_{i=1}^g \begin{array}{c} | \\ \bullet \\ \diagdown \quad \diagup \\ b_i \quad a_i \end{array}$$

It is easily deduced from the IHX relation that  $I$  is an ideal of the Lie ring  $\mathcal{T}(H)$  (see [16, Section 7]), hence the quotient

$$\widehat{\mathcal{T}}(H) := \frac{\mathcal{T}(H)}{I}$$

is a Lie ring. Besides, let  $\widehat{\mathfrak{L}}$  be the quotient of  $\mathfrak{L}$  by the ideal generated by  $\omega \in \mathfrak{L}_2$ , let  $\widehat{D}_d(H)$  be the kernel of the Lie bracket map  $H \otimes \widehat{\mathfrak{L}}_{d+1} \rightarrow \widehat{\mathfrak{L}}_{d+2}$  for any  $d \geq 1$ , and set

$$\text{OD}_d(H) := \frac{\widehat{D}_d(H)}{(\text{id}_H \otimes [\cdot, \cdot])(\omega \otimes \widehat{\mathfrak{L}}_d)}$$

where  $[\cdot, \cdot]$  denotes the Lie bracket map  $H \otimes \widehat{\mathfrak{L}}_d \rightarrow \widehat{\mathfrak{L}}_{d+1}$ . Then, similarly to (2.2), we have a homomorphism

$$(2.21) \quad \eta : \widehat{\mathcal{T}}_d(H) \longrightarrow \text{OD}_d(H)$$

which, just as (2.2), induces an isomorphism  $\eta^{\mathbb{Q}}$  for rational coefficients.

In the case of the closed surface  $\widehat{\Sigma}$ , the *infinitesimal Dehn–Nielsen representation* of the Torelli group [17] is the composition

$$(2.22) \quad \widehat{\mathcal{I}} \xrightarrow{\varrho^\theta} \text{IOut}(\widehat{\mathfrak{L}}^{\mathbb{Q}}) \xrightarrow[\simeq]{\log} \text{ODer}^+(\widehat{\mathfrak{L}}^{\mathbb{Q}}) \xrightarrow[\simeq]{(\eta^{\mathbb{Q}})^{-1}} \widehat{\mathcal{T}}(H^{\mathbb{Q}})$$

where, to simplify notations, we have omitted the  $\widehat{\cdot}$  decoration indicating degree-completions,  $\text{IOut}(\widehat{\mathfrak{L}}^{\mathbb{Q}})$  is the group of automorphisms of  $\widehat{\mathfrak{L}}^{\mathbb{Q}}$  (that induce the identity at the graded level) modulo inner automorphisms, and  $\text{ODer}^+(\widehat{\mathfrak{L}}^{\mathbb{Q}})$  is the Lie algebra of derivations of  $\widehat{\mathfrak{L}}^{\mathbb{Q}}$  (that strictly increase degrees) modulo inner derivations. In the definition of (2.22), we use the fact that the symplectic logansion  $\theta$  induces an isomorphism between the Malcev Lie algebra of  $\pi_1(\widehat{\Sigma}, \star)$  and (the degree-completion of)  $\widehat{\mathfrak{L}}^{\mathbb{Q}}$  [27, Proposition 2.18], and we have identified  $\text{OD}_d(H^{\mathbb{Q}})$  with  $\text{ODer}^+(\widehat{\mathfrak{L}}^{\mathbb{Q}})$  in the same way we have already identified  $D(H^{\mathbb{Q}})$  with  $\text{Der}_\omega^+(\mathfrak{L}^{\mathbb{Q}})$ . By construction, there is a commutative diagram

$$(2.23) \quad \begin{array}{ccc} \mathcal{I} & \xrightarrow{r^\theta} & \mathcal{T}(H^{\mathbb{Q}}) \\ \downarrow & & \downarrow \\ \widehat{\mathcal{I}} & \xrightarrow[r^\theta]{} & \widehat{\mathcal{T}}(H^{\mathbb{Q}}). \end{array}$$

According to [34], and similarly to the case of the bordered surface  $\Sigma$ , the mapping class group  $\widehat{\mathcal{M}}$  of  $\widehat{\Sigma}$  has a *Johnson filtration*

$$\widehat{\mathcal{M}} \supset \underbrace{\widehat{\mathcal{M}}[1]}_{=\widehat{\mathcal{I}}} \supset \underbrace{\widehat{\mathcal{M}}[2]}_{=\widehat{\mathcal{K}}} \supset \cdots \supset \widehat{\mathcal{M}}[k] \supset \widehat{\mathcal{M}}[k+1] \supset \cdots$$



and, for every integer  $k \geq 1$ , the  $k$ -th Johnson homomorphism takes the form

$$\tau_k : \widehat{\mathcal{M}}[k] \longrightarrow \text{OD}_k(H) \subset \frac{H \otimes \widehat{\mathfrak{L}}_{k+1}}{(\text{id}_H \otimes [\cdot, \cdot])(\omega \otimes \widehat{\mathfrak{L}}_k)}.$$

According to [1, Lemma 2], the abelian group  $\text{OD}_k(H)$  is torsion-free; hence, as in the case of  $\Sigma$ , there is no loss of information in tensoring  $\tau_k$  with  $\mathbb{Q}$  and the analogue of (2.14) holds true in the closed case too. As an analogue of Morita’s refinement  $\widetilde{\tau}_k$  of  $\tau_k$  in the closed case, we shall consider the homomorphism

$$(2.24) \quad r_{[k, 2k]}^\theta : \widehat{\mathcal{M}}[k] \longrightarrow \bigoplus_{d=k}^{2k-1} \widehat{\mathcal{T}}_d(H^\mathbb{Q}).$$

Similarly to the case of the surface  $\Sigma$ , we now define by the same formula (2.17) a map

$$(2.25) \quad R : \widehat{\mathcal{K}} \longrightarrow \frac{\widehat{\mathcal{T}}_4(H^\mathbb{Q})}{\widehat{\mathcal{T}}_4(H)}.$$

Here  $\widehat{\mathcal{T}}_4(H^\mathbb{Q})/\widehat{\mathcal{T}}_4(H)$  denotes the quotient of  $\widehat{\mathcal{T}}_4(H^\mathbb{Q})$  by the image of the canonical group homomorphism  $\widehat{\mathcal{T}}_4(H) \rightarrow \widehat{\mathcal{T}}_4(H^\mathbb{Q})$ : since it is not clear whether  $\widehat{\mathcal{T}}_4(H)$  is torsion-free (in contrast to the case of  $\Sigma$ ), this could be a slight abuse of notation. As a consequence of (2.23), we have the following commutative diagram:

$$(2.26) \quad \begin{array}{ccc} \mathcal{K} & \xrightarrow{R} & \frac{\mathcal{T}_4(H^\mathbb{Q})}{\mathcal{T}_4(H)} \\ \downarrow \cong & & \downarrow \cong \\ \widehat{\mathcal{K}} & \xrightarrow{R} & \frac{\widehat{\mathcal{T}}_4(H^\mathbb{Q})}{\widehat{\mathcal{T}}_4(H)}. \end{array}$$

Hence, by Lemma 2.8, we obtain a map  $R_{\text{ab}}$  on the abelianization  $\widehat{\mathcal{K}}_{\text{ab}} = \widehat{\mathcal{K}}/[\widehat{\mathcal{K}}, \widehat{\mathcal{K}}]$  which is polynomial of degree 2.

*Remark 2.11.* — The variation  $R_\circ : \mathcal{K} \rightarrow \mathcal{T}_4(H^\mathbb{Q})/\mathcal{T}_4(H)$  of  $R$  introduced in Remark 2.9 does not factorize to a group homomorphism from  $\widehat{\mathcal{K}}$  to  $\widehat{\mathcal{T}}_4(H^\mathbb{Q})/\widehat{\mathcal{T}}_4(H)$ . To produce from  $R_\circ$  a group homomorphism on  $\widehat{\mathcal{K}}$ , one would need to add extra relations to the quotient  $\widehat{\mathcal{T}}_4(H^\mathbb{Q})/\widehat{\mathcal{T}}_4(H)$ . These extra relations seem to be missing in the definition of the map induced by  $\log_{\square} \widetilde{Z}^Y$  that is considered in the proof of [37, Corollary 1.5]. (Indeed, the subspace of diagrammatic relations needed to have the LMO homomorphism  $\widetilde{Z}^Y$  defined on “closed” homology cylinders is an ideal for the “star

product” denoted by  $\star$  in [2], but it is not an ideal for the disjoint union operation  $\sqcup$ : hence  $\log_{\sqcup} \tilde{Z}^Y$  does not factorize to the monoid of “closed” homology cylinders.)

The next lemma shows that the restriction of  $R$  to  $\widehat{\mathcal{M}}[4]$  is determined by the 4-th Johnson homomorphism (and, so, does not depend on the choice of  $\theta$ ).

LEMMA 2.12. — *There is a non-trivial 2-torsion abelian group  $L$  that fits into a commutative diagram of the following form:*

$$\begin{array}{ccccc} \widehat{\mathcal{M}}[4] & \xrightarrow{\tau_4} & \widehat{\text{OD}}_4(H) & \xrightarrow{\varpi} & L \\ & \searrow R & & & \downarrow j \\ & & & & \frac{\widehat{\mathcal{T}}_4(H^{\mathbb{Q}})}{\widehat{\mathcal{T}}_4(H)}. \end{array}$$

*Proof.* — Set

$$L := \frac{\widehat{\text{OD}}_4(H)}{\eta(\widehat{\mathcal{T}}_4(H))}$$

and let  $\varpi : \widehat{\text{OD}}_4(H) \rightarrow L$  be the canonical projection. Like in the bordered case, we obtain that the restriction of  $R$  to  $\widehat{\mathcal{M}}[4]$  is equivalent to the composition

$$(2.27) \quad \widehat{\mathcal{M}}[4] \xrightarrow{\tau_4} \widehat{\text{OD}}_4(H) \xrightarrow{\varpi} L$$

through the isomorphism

$$\eta^{\mathbb{Q}} : \frac{\widehat{\mathcal{T}}_4(H^{\mathbb{Q}})}{\widehat{\mathcal{T}}_4(H)} \xrightarrow{\simeq} \frac{\widehat{\text{OD}}_4(H^{\mathbb{Q}})}{\eta(\widehat{\mathcal{T}}_4(H))}.$$

Let  $p : \mathfrak{L} \rightarrow \widehat{\mathfrak{L}}$  be the canonical projection, whose kernel is the ideal  $\langle\langle \omega \rangle\rangle$  generated by  $\omega$ . The map  $H \otimes \langle\langle \omega \rangle\rangle_5 \rightarrow \langle\langle \omega \rangle\rangle_6$  defined by the Lie bracket is surjective. Hence an application of the snake lemma to the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & D_4(H) & \longrightarrow & H \otimes \mathfrak{L}_5 & \longrightarrow & \mathfrak{L}_6 \longrightarrow 0 \\ & & \downarrow q & & \downarrow H \otimes p_5 & & \downarrow p_6 \\ 0 & \longrightarrow & \widehat{D}_4(H) & \longrightarrow & H \otimes \widehat{\mathfrak{L}}_5 & \longrightarrow & \widehat{\mathfrak{L}}_6 \longrightarrow 0 \end{array}$$

shows that the induced map  $q : D_4(H) \rightarrow \widehat{D}_4(H)$  is surjective, with kernel  $(H \otimes \langle\langle \omega \rangle\rangle_5) \cap D_4(H)$ . Hence we deduce that  $q$  induces an isomorphism

from

$$\frac{D_4(H)}{\eta(\mathcal{T}_4(H)) + (H \otimes \langle\langle\omega\rangle\rangle_5 + (\text{id} \otimes [\cdot, \cdot]) (\omega \otimes \mathfrak{L}_4)) \cap D_4(H)}$$

to  $L$ . But, by considering tree diagrams in  $\mathcal{T}_4(H)$  with  $\omega$ -vertices, we obtain the following identity of subgroups of  $H \otimes \mathfrak{L}_5$ :

$$\begin{aligned} \eta(\mathcal{T}_4(H)) + (H \otimes \langle\langle\omega\rangle\rangle_5 + (\text{id} \otimes [\cdot, \cdot]) (\omega \otimes \mathfrak{L}_4)) \cap D_4(H) \\ = \eta(\mathcal{T}_4(H)) + (H \otimes \langle\langle\omega\rangle\rangle_5) \cap D_4(H). \end{aligned}$$

Hence  $L$  is canonically isomorphic to

$$L' := \frac{D_4(H)}{\eta(\mathcal{T}_4(H)) + (H \otimes \langle\langle\omega\rangle\rangle_5) \cap D_4(H)} \quad \left( \simeq \frac{\widehat{D}_4(H)}{q\eta(\mathcal{T}_4(H))} \right).$$

Being a quotient of  $D_4(H)/\eta(\mathcal{T}_4(H)) \stackrel{(2.7)}{\simeq} \mathfrak{L}_3 \otimes \mathbb{Z}_2$ , the abelian group  $L'$  is 2-torsion. Let  $A := H/\langle b_1, \dots, b_g \rangle$ : using Remark 2.2, we see that the canonical projection  $H \rightarrow A$  induces a map  $m : D_4(H)/\eta(\mathcal{T}_4(H)) \rightarrow D_4(A)/\eta(\mathcal{T}_4(A))$ . According to (2.7),  $m$  is essentially the canonical map  $\mathfrak{L}_3(H) \otimes \mathbb{Z}_2 \rightarrow \mathfrak{L}_3(A) \otimes \mathbb{Z}_2$ , so that it is not zero. Thus, to conclude that  $L'$  is not trivial, it suffices to observe that the homomorphism  $m$  factorizes through  $L'$ .  $\square$

*Remark 2.13.* — It is likely that  $L \simeq \widehat{\mathfrak{L}}_3 \otimes \mathbb{Z}_2$ . Indeed, the composition of homomorphisms

$$\mathfrak{L}_3 \otimes \mathbb{Z}_2 \xrightarrow{j} \frac{\mathcal{T}_4(H^\mathbb{Q})}{\mathcal{T}_4(H)} \twoheadrightarrow \frac{\widehat{\mathcal{T}}_4(H^\mathbb{Q})}{\widehat{\mathcal{T}}_4(H)},$$

where  $j$  is the map of Lemma 2.10, factorizes through the quotient  $\widehat{\mathfrak{L}}_3 \otimes \mathbb{Z}_2$  of  $\mathfrak{L}_3 \otimes \mathbb{Z}_2$ . Hence we have a homomorphism  $\widehat{j} : \widehat{\mathfrak{L}}_3 \otimes \mathbb{Z}_2 \rightarrow \widehat{\mathcal{T}}_4(H^\mathbb{Q})/\widehat{\mathcal{T}}_4(H)$  such that

$$(2.28) \quad \begin{array}{ccc} \widehat{\mathfrak{L}}_3 \otimes \mathbb{Z}_2 & \xrightarrow{\widehat{j}} & \frac{\widehat{\mathcal{T}}_4(H^\mathbb{Q})}{\widehat{\mathcal{T}}_4(H)} \\ \swarrow & \dashrightarrow \exists & \nearrow j \\ \mathfrak{L}_3(A) \otimes \mathbb{Z}_2 & \longleftarrow & L \end{array}$$

where the bottom map is given by the last paragraph of the proof of Lemma 2.12, and the leftmost map is the canonical projection. It is expected that  $\widehat{j}$  is injective, which is equivalent to the identity  $\varpi((H \otimes \langle\langle\omega\rangle\rangle_5) \cap D_4(H)) = [\omega, H] \otimes \mathbb{Z}_2$ .

### 3. On the rational abelianization of the Johnson kernel

In this section, we review the computation of the rational abelianization of the Johnson kernel by Dimca, Hain and Papadima [6], in the more explicit form given by Morita, Sakasai and Suzuki [36]. Although these results are obtained in [6] and [36] only for  $\widehat{\mathcal{K}}_{\text{ab}} \otimes \mathbb{Q}$ , we shall see that they can be proved for  $\mathcal{K}_{\text{ab}} \otimes \mathbb{Q}$  too. Furthermore, we make the computation very explicit by using the infinitesimal Dehn–Nielsen representation.

#### 3.1. The closed case

In his study of the relationship between the Casson invariant and the structure of the Torelli group [30, 31], Morita introduced two fundamental homomorphisms

$$d : \mathcal{K} \longrightarrow \mathbb{Z} \quad \text{and} \quad d' : \mathcal{K} \longrightarrow \mathbb{Z}$$

of a very different nature. While  $d'$  is defined from  $\tau_2$  (and, so, is determined by the action of  $\mathcal{K}$  on  $\pi/\Gamma_4\pi$ ), the homomorphism  $d$  involves the Casson invariant in its definition (and, as a consequence of results in [18], it is not determined by the action of  $\mathcal{K}$  on  $\pi/\Gamma_k\pi$  for any fixed  $k$ ). Yet, both of them are invariant under the conjugacy action of  $\mathcal{M}$  on  $\mathcal{K}$ , and they have simple values on any separating twist  $T_\gamma$  of genus  $h$ :

$$d(T_\gamma) = 4h(h-1) \quad \text{and} \quad d'(T_\gamma) = h(2h+1).$$

Furthermore, Morita proved in [31, Theorem 5.7] that the linear combination

$$\widehat{d} := -\frac{1+2g}{12} \cdot d + \frac{g-1}{3} \cdot d'$$

vanishes on the kernel of the canonical map  $\mathcal{K} \rightarrow \widehat{\mathcal{K}}$ : hence there is an  $\widehat{\mathcal{M}}$ -invariant group homomorphism  $\widehat{d} : \widehat{\mathcal{K}} \rightarrow \mathbb{Z}$  satisfying  $\widehat{d}(T_\gamma) = h(g-h)$ .

In the case of the closed surface  $\widehat{\Sigma}$ , the rational abelianization of the Johnson kernel is determined by the following theorem, which results from [6, Theorem B] combined with [36, Theorem 1.4].

**THEOREM 3.1** (Dimca–Hain–Papadima, Morita–Sakasai–Suzuki). — *In genus  $g \geq 6$ , the group homomorphism*

$$(\widehat{d}, r_{[2,4]}^\theta) : \widehat{\mathcal{K}} \longrightarrow \mathbb{Z} \oplus \widehat{\mathcal{T}}_2(H^\mathbb{Q}) \oplus \widehat{\mathcal{T}}_3(H^\mathbb{Q})$$

*induces a linear embedding of  $\widehat{\mathcal{K}}_{\text{ab}} \otimes \mathbb{Q}$  into  $\mathbb{Q} \oplus \widehat{\mathcal{T}}_2(H^\mathbb{Q}) \oplus \widehat{\mathcal{T}}_3(H^\mathbb{Q})$ .*

Note that the truncation  $r_{[2,4]}^\theta$  of the infinitesimal Dehn–Nielsen representation plays the same role as the second Morita homomorphism  $\widetilde{\tau}_2$  in the statement of [36, Theorem 1.4].

### 3.2. The bordered case

Similarly to Theorem 3.1, we have the following result for the bordered surface  $\Sigma$ .

**THEOREM 3.2.** — *In genus  $g \geq 6$ , the group homomorphism*

$$(d, r_{[2,4]}^\theta) : \mathcal{K} \longrightarrow \mathbb{Z} \oplus \mathcal{T}_2(H^\mathbb{Q}) \oplus \mathcal{T}_3(H^\mathbb{Q})$$

*induces a linear embedding of  $\mathcal{K}_{\text{ab}} \otimes \mathbb{Q}$  into  $\mathbb{Q} \oplus \mathcal{T}_2(H^\mathbb{Q}) \oplus \mathcal{T}_3(H^\mathbb{Q})$ .*

To prove Theorem 3.2, we shall mainly adapt the proof of Theorem 3.1 given in [6] and [36]. So, as in the closed case, our arguments will require some fundamental results of Dimca & Papadima [7], Hain [18] (which impose the lower bound on the genus  $g$ ) and Putman [41]. But, because we were not able to completely “translate” the arguments of [6] from the closed case to the bordered case, we will also need the finite generation of  $\mathcal{K}$  which has been obtained more recently by Ershov & He [10] and Church, Ershov & Putman [3].

Thus, as done in [6, Section 2], we start with the following general situation:  $G$  is a group,  $G_{\text{abf}} := G_{\text{ab}} / \text{Tors}(G_{\text{ab}})$  is its torsion-free abelianization, and  $K$  is the kernel of the canonical projection  $G \rightarrow G_{\text{abf}}$ . We are interested in the  $\mathbb{Q}[G_{\text{abf}}]$ -module

$$K_{\text{ab}} \otimes \mathbb{Q}$$

where the action of  $G_{\text{abf}}$  on  $K_{\text{ab}}$  is induced by the conjugacy action of  $G$  on  $K$ . The  $I$ -adic filtration of the group algebra  $\mathbb{Q}[G_{\text{abf}}]$  induces a filtration on  $K_{\text{ab}} \otimes \mathbb{Q}$ , and the corresponding completion is denoted by

$$\widehat{K_{\text{ab}} \otimes \mathbb{Q}}.$$

In fact,  $\widehat{K_{\text{ab}} \otimes \mathbb{Q}}$  has a structure of  $\widehat{\mathbb{Q}[G_{\text{ab}}]}$ -module, where  $\widehat{\mathbb{Q}[G_{\text{ab}}]}$  denotes the  $I$ -adic completion of the group algebra  $\mathbb{Q}[G_{\text{ab}}]$ . (Note that  $\widehat{\mathbb{Q}[G_{\text{ab}}]}$  is isomorphic to  $\widehat{\mathbb{Q}[G_{\text{abf}}]}$ , since both groups  $G_{\text{ab}}$  and  $G_{\text{abf}}$  are abelian and they have the same rationalization.) Similarly, using the action of  $G_{\text{ab}}$  on  $G'_{\text{ab}}$  induced by the conjugacy action of  $G$  on  $G' = \Gamma_2 G$ , we can consider the completion  $\widehat{G'_{\text{ab}} \otimes \mathbb{Q}}$  of  $G'_{\text{ab}} \otimes \mathbb{Q}$  with respect to the filtration induced by the  $I$ -adic filtration of the group algebra  $\mathbb{Q}[G_{\text{ab}}]$ . It is proved in [6, Proposition 2.4], under the assumption

$$(3.1) \quad \text{“}G \text{ is finitely generated and } K/G' \text{ is finite”},$$

that the canonical map

$$(3.2) \quad \widehat{G'_{ab} \otimes \mathbb{Q}} \longrightarrow \widehat{K_{ab} \otimes \mathbb{Q}}$$

induced by the inclusion  $G' \hookrightarrow K$  is a filtered  $\widehat{\mathbb{Q}[G_{ab}]}$ -linear isomorphism.

To go further, let us recall that any group  $G$  has a *Malcev completion*  $M(G)$  and a *Malcev Lie algebra*  $\mathfrak{m}(G)$ : they are defined respectively as the group-like part and the primitive part of the complete Hopf algebra  $\widehat{\mathbb{Q}[G]}$ , which is the  $I$ -adic completion of the group algebra  $\mathbb{Q}[G]$ . Recall also that the group  $M(G)$  and the Lie algebra  $\mathfrak{m}(G)$  inherit filtrations from  $\widehat{\mathbb{Q}[G]}$ , and that they correspond each other through the formal exp and log series. Let  $\mathfrak{m}(G)'$  be the derived subalgebra of the complete Lie algebra  $\mathfrak{m}(G)$ , and let  $\mathfrak{m}(G)'_{ab}$  be its abelianization. The adjoint action of  $\mathfrak{m}(G)$  on itself induces an action of the abelian Lie algebra

$$\mathfrak{m}(G)/\mathfrak{m}(G)' \simeq G_{ab} \otimes \mathbb{Q}$$

on the vector space  $\mathfrak{m}(G)'_{ab}$ . Hence  $\mathfrak{m}(G)'_{ab}$  has also a structure of  $\widehat{S}(G_{ab} \otimes \mathbb{Q})$ -module, where  $\widehat{S}(G_{ab} \otimes \mathbb{Q})$  is the degree-completion of the symmetric algebra  $S(G_{ab} \otimes \mathbb{Q})$  generated by the vector space  $G_{ab} \otimes \mathbb{Q}$ . According to [8, Proposition 5.4], the canonical map  $\iota : G \rightarrow M(G)$  composed with  $\log : M(G) \rightarrow \mathfrak{m}(G)$  induces a  $\widehat{\mathbb{Q}[G_{ab}]}$ -linear isomorphism

$$(3.3) \quad \widehat{G'_{ab} \otimes \mathbb{Q}} \longrightarrow \mathfrak{m}(G)'_{ab}.$$

Here, the complete algebra  $\widehat{\mathbb{Q}[G_{ab}]}$  is identified with  $\widehat{S}(G_{ab} \otimes \mathbb{Q})$  via the expansion  $G_{ab} \rightarrow \widehat{S}(G_{ab} \otimes \mathbb{Q})$  defined by  $g \mapsto \exp(g) = \sum_{i \geq 0} g^{\otimes i} / i!$ . (Note that (3.3) shifts filtrations by 2 if  $\mathfrak{m}(G)'_{ab}$  has the filtration induced by that of  $\mathfrak{m}(G)$ .)

Assume now the following formality assumption on the group  $G$ :

$$(3.4) \quad \text{“There is an isomorphism of filtered Lie algebras } \mathfrak{m}(G) \rightarrow \widehat{\text{Gr}} \mathfrak{m}(G) \text{ which is the identity at the graded level.”}$$

Here  $\text{Gr} \mathfrak{m}(G)$  is the associated graded of  $\mathfrak{m}(G)$ , and  $\widehat{\text{Gr}} \mathfrak{m}(G)$  denotes its degree-completion. We recall that  $\text{Gr} \mathfrak{m}(G)$  is canonically isomorphic to the associated graded  $(\text{Gr} G) \otimes \mathbb{Q}$  of the lower central series of  $G$ . Thus, under the assumption (3.4), we get an isomorphism

$$(3.5) \quad \mathfrak{m}(G)'_{ab} \longrightarrow \widehat{\mathfrak{b}}(G)$$

where

$$\mathfrak{b}(G) := (\text{Gr} \mathfrak{m}(G))'_{ab} = \frac{(\text{Gr} \mathfrak{m}(G))'}{[(\text{Gr} \mathfrak{m}(G))', (\text{Gr} \mathfrak{m}(G))']}$$

is the *infinitesimal Alexander module* of  $G$ .

Hence, if we assume simultaneously (3.1) and (3.4), we can compose the inverse of the isomorphism (3.2) with (3.3) and next (3.5) to get a filtered  $\widehat{\mathbb{Q}[G_{\text{ab}}]}$ -linear isomorphism

$$(3.6) \quad \widehat{K_{\text{ab}}} \otimes \mathbb{Q} \longrightarrow \widehat{\mathfrak{b}}(G).$$

We now come back to the specific situation of the Torelli group  $\mathcal{I}$  of  $\Sigma$ .

*Proof of Theorem 3.2.* — The above discussion applies to the group  $G := \mathcal{I}$ . Indeed, according to the fundamental results of Johnson [21, 22, 23], we have  $K := \mathcal{K}$  in this case and (3.1) is satisfied for any  $g \geq 3$ . Furthermore, (3.4) is satisfied too when  $g \geq 3$  according to Hain [18]. Hence, we get a filtered isomorphism (3.6) between  $\widehat{K_{\text{ab}}} \otimes \mathbb{Q}$  and  $\widehat{\mathfrak{b}}(\mathcal{I})$ .

Similarly to the closed case [6], the most important part of the proof is the following claim:

$$(3.7) \quad “\mathcal{K}_{\text{ab}} \otimes \mathbb{Q} \text{ is a nilpotent } \mathbb{Q}[\mathcal{I}_{\text{abf}}]\text{-module}”$$

which is proved using the restricted characteristic variety of the Torelli group  $\mathcal{I}$ . For any finitely-generated group  $G$ , let  $\mathbb{T}_0(G) := \text{Hom}(G_{\text{abf}}, \mathbb{C}^*)$  and recall that the *restricted characteristic variety* of  $G$  is defined by

$$\mathcal{V}(G) := \{\rho \in \mathbb{T}_0(G) : H_1^p(G; \mathbb{C}) \neq 0\}.$$

As before, denote by  $K$  the kernel of the canonical projection  $G \rightarrow G_{\text{abf}}$ . It follows from a result of Dwyer & Fried [9], in the refined form of [38, Corollary 6.2], that  $\mathcal{V}(G)$  is finite if, and only if,  $H_1(K; \mathbb{Q})$  is finite-dimensional. Besides, it is now known from [3, 10] that  $\mathcal{K}$  is finitely generated in genus  $g \geq 4$  and, so,  $H_1(\mathcal{K}; \mathbb{Q})$  is finite-dimensional. (In the closed case, that  $H_1(\widehat{\mathcal{K}}; \mathbb{Q})$  is finite-dimensional was proved earlier in [7].) Hence  $\mathcal{V}(\mathcal{I})$  is finite.

To go further in the proof of the claim (3.7), we next show that any element of  $\mathcal{V}(\mathcal{I})$  has finite order in the group  $\mathbb{T}_0(\mathcal{I})$ . For that, we closely follow the proof of [6, Theorem 3.1] but with a small variation at the end of the argument. (This theorem can not be applied directly in our situation, because  $\Lambda^3 H^{\mathbb{Q}} \simeq \mathcal{I}_{\text{abf}} \otimes \mathbb{Q}$  is not irreducible as an  $\text{Sp}(H^{\mathbb{Q}})$ -module.) Let  $t \in \mathcal{V}(\mathcal{I})$  and assume that it has infinite order in  $\mathbb{T}_0(\mathcal{I})$ . The canonical action of the group  $\text{Sp}(H) \simeq \mathcal{M}/\mathcal{I}$  on  $\mathbb{T}_0(\mathcal{I})$  leaves  $\mathcal{V}(\mathcal{I})$  globally invariant. Since  $\mathcal{V}(\mathcal{I})$  is finite, the stabilizer  $D_t$  of  $t$  under this action is a finite-index subgroup of  $\text{Sp}(H)$ . By Borel’s density theorem,  $\text{Sp}(H)$  is Zariski-dense in  $\text{Sp}(H^{\mathbb{C}})$ , and so is  $D_t$  in  $\text{Sp}(H^{\mathbb{C}})$ . Let  $Z$  be the Zariski-closure of the subgroup of  $\mathbb{T}_0(\mathcal{I})$  generated by  $t$ . Since the algebraic group  $Z$  is infinite, its dimension is at least 1, and so is the dimension of its Lie algebra  $T_1 Z$ . Hence,  $T_1 Z$  constitutes in the tangent space  $T_1 \mathbb{T}_0(\mathcal{I}) = \text{Hom}(\Lambda^3 H^{\mathbb{C}}, \mathbb{C})$ , of

$\mathbb{T}_0(\mathcal{I})$  at the trivial character 1, a subspace which is fixed by  $D_t$ . Hence, by Zariski-density of  $D_t$  in  $\mathrm{Sp}(H^{\mathbb{C}})$ , we obtain that the  $\mathrm{Sp}(H^{\mathbb{C}})$ -invariant part of  $\mathrm{Hom}(\Lambda^3 H^{\mathbb{C}}, \mathbb{C})$  is not reduced to zero. But this is not possible, because we have  $\mathrm{Hom}(\Lambda^3 H^{\mathbb{C}}, \mathbb{C}) \simeq \Lambda^{2g-3} H^{\mathbb{C}}$  as  $\mathrm{Sp}(H^{\mathbb{C}})$ -modules and  $\Lambda^k H^{\mathbb{C}}$  has no  $\mathrm{Sp}(H^{\mathbb{C}})$ -invariant part if  $k$  is not even. We conclude that any  $t \in \mathcal{V}(\mathcal{I})$  is a torsion element of the group  $\mathbb{T}_0(\mathcal{I})$ . Since  $\mathcal{V}(\mathcal{I})$  is finite, we can thus find an integer  $m \geq 1$  such that

$$(3.8) \quad \forall t \in \mathcal{V}(\mathcal{I}), \quad t^m = 1 \in \mathbb{T}_0(\mathcal{I}).$$

Next, the proof of claim (3.7) is strictly the same as in [6, Proof of Theorem A] and we only highlight the main points for the reader’s convenience. Let  $\mathcal{I}(m)$  be the preimage by the canonical projection  $\mathcal{I} \rightarrow \mathcal{I}_{\mathrm{abf}}$  of the subgroup of  $\mathcal{I}_{\mathrm{abf}}$  consisting of elements that are divisible by  $m$ , and let  $\mathcal{K}(m)$  be the kernel of the canonical projection  $\mathcal{I}(m) \rightarrow \mathcal{I}(m)_{\mathrm{abf}}$ . According to [41, Theorem B], all subgroups of  $\mathcal{I}$  of finite index containing  $\mathcal{K}$  have the same first Betti numbers for  $g \geq 3$ . Hence [6, Lemma 3.4] can be applied to deduce that

- (i) the inclusion  $\mathcal{I}(m) \hookrightarrow \mathcal{I}$  induces an isomorphism  $\widehat{\mathcal{I}(m)'_{\mathrm{ab}} \otimes \mathbb{Q}} \rightarrow \widehat{\mathcal{I}'_{\mathrm{ab}} \otimes \mathbb{Q}}$ ,
- (ii) we have  $\mathcal{K}(m) = \mathcal{K}$ .

By a double application of (3.2), we get a  $\widehat{\mathbb{Q}[\mathcal{I}_{\mathrm{ab}}]}$ -linear isomorphism  $\widehat{\mathcal{I}'_{\mathrm{ab}} \otimes \mathbb{Q}} \simeq \widehat{\mathcal{K}_{\mathrm{ab}} \otimes \mathbb{Q}}$  and a  $\widehat{\mathbb{Q}[\mathcal{I}(m)_{\mathrm{ab}}]}$ -linear isomorphism  $\widehat{\mathcal{I}(m)'_{\mathrm{ab}} \otimes \mathbb{Q}} \simeq \widehat{\mathcal{K}(m)_{\mathrm{ab}} \otimes \mathbb{Q}}$ . By the statement (ii) above,  $\widehat{\mathcal{K}(m)_{\mathrm{ab}} \otimes \mathbb{Q}}$  is the completion  ${}^m \widehat{\mathcal{K}_{\mathrm{ab}} \otimes \mathbb{Q}}$  of  $\mathcal{K}_{\mathrm{ab}} \otimes \mathbb{Q}$  defined by the  $I$ -adic filtration of  $\mathbb{Q}[\mathcal{I}(m)_{\mathrm{abf}}]$  via the algebra homomorphism  $\mathbb{Q}[\mathcal{I}(m)_{\mathrm{abf}}] \rightarrow \mathbb{Q}[\mathcal{I}_{\mathrm{abf}}]$  induced by the inclusion  $\mathcal{I}(m) \hookrightarrow \mathcal{I}$ . But, if we identify  $\mathcal{I}(m)_{\mathrm{abf}}$  with  $\mathcal{I}_{\mathrm{abf}}$  as in the proof of [6, Lemma 3.4], that algebra homomorphism corresponds to the homomorphism  $\mathbb{Q}[\iota_m] : \mathbb{Q}[\mathcal{I}_{\mathrm{abf}}] \rightarrow \mathbb{Q}[\mathcal{I}_{\mathrm{abf}}]$  induced by the “multiply by  $m$ ” map  $\iota_m : \mathcal{I}_{\mathrm{abf}} \rightarrow \mathcal{I}_{\mathrm{abf}}$ . Hence, denoting by  ${}^m(\mathcal{K}_{\mathrm{ab}} \otimes \mathbb{Q})$  the vector space  $\mathcal{K}_{\mathrm{ab}} \otimes \mathbb{Q}$  with the structure of  $\mathbb{Q}[\mathcal{I}_{\mathrm{abf}}]$ -module defined by the algebra map  $\mathbb{Q}[\iota_m] : \mathbb{Q}[\mathcal{I}_{\mathrm{abf}}] \rightarrow \mathbb{Q}[\mathcal{I}_{\mathrm{abf}}]$ , we see that  ${}^m \widehat{\mathcal{K}_{\mathrm{ab}} \otimes \mathbb{Q}}$  is the completion of  ${}^m(\mathcal{K}_{\mathrm{ab}} \otimes \mathbb{Q})$  with respect to the filtration defined by the  $I$ -adic filtration of  $\mathbb{Q}[\mathcal{I}_{\mathrm{abf}}]$ . We deduce from the above statement (i) that the  $I$ -adic completion  $\widehat{\mathcal{K}_{\mathrm{ab}} \otimes \mathbb{Q}}$  of the  $\mathbb{Q}[\mathcal{I}_{\mathrm{abf}}]$ -module  $\mathcal{K}_{\mathrm{ab}} \otimes \mathbb{Q}$  is isomorphic to the  $I$ -adic completion  ${}^m \widehat{\mathcal{K}_{\mathrm{ab}} \otimes \mathbb{Q}}$  of the  $\mathbb{Q}[\mathcal{I}_{\mathrm{abf}}]$ -module  ${}^m(\mathcal{K}_{\mathrm{ab}} \otimes \mathbb{Q})$ .

Next, exactly as in the second paragraph of [6, p. 816] (which involves a result of [38] relating restricted character varieties to supports of Alexander



modules), we deduce from (3.8) that  ${}^m(\mathcal{K}_{\text{ab}} \otimes \mathbb{Q})$  is nilpotent as a  $\widehat{\mathbb{Q}[\mathcal{I}_{\text{abf}}]}$ -module. Thus the canonical map  ${}^m(\mathcal{K}_{\text{ab}} \otimes \mathbb{Q}) \rightarrow \widehat{{}^m\mathcal{K}_{\text{ab}} \otimes \mathbb{Q}}$  is injective, and so is the canonical map  $\mathcal{K}_{\text{ab}} \otimes \mathbb{Q} \rightarrow \widehat{\mathcal{K}_{\text{ab}} \otimes \mathbb{Q}}$ . In other words, the filtration of  $\mathcal{K}_{\text{ab}} \otimes \mathbb{Q}$  defined by the  $I$ -adic filtration of  $\widehat{\mathbb{Q}[\mathcal{I}_{\text{abf}}]}$  has trivial intersection; moreover, this filtration must stabilize since  $\mathcal{K}_{\text{ab}} \otimes \mathbb{Q}$  is finite-dimensional. Thus, we have proved claim (3.7). (Note that the arguments can be continued a little bit further, as in the closed case, to deduce that  $\mathcal{V}(\mathcal{I})$  is actually reduced to the trivial character.)

Since the canonical map  $\mathcal{K}_{\text{ab}} \otimes \mathbb{Q} \rightarrow \widehat{\mathcal{K}_{\text{ab}} \otimes \mathbb{Q}}$  is an isomorphism,  $\widehat{\mathcal{K}_{\text{ab}} \otimes \mathbb{Q}}$  is finite-dimensional so that, by (3.6), the graded module  $\mathfrak{b}(\mathcal{I})$  is concentrated in finitely many degrees. Therefore, in fine, there is a filtered  $\widehat{\mathbb{Q}[\mathcal{I}_{\text{ab}}]}$ -linear isomorphism

$$(3.9) \quad F : \mathcal{K}_{\text{ab}} \otimes \mathbb{Q} \longrightarrow \mathfrak{b}(\mathcal{I})$$

(which is induced by any isomorphism  $\mathfrak{m}(\mathcal{I}) \rightarrow \widehat{\text{Gr}} \mathfrak{m}(\mathcal{I})$  giving the identity at the graded level). Moreover, the least integer  $\ell$  such that  $\mathfrak{b}_k(\mathcal{I}) = 0$  for all  $k \geq \ell$  corresponds to the least integer  $\ell - 2$  such that  $I^{\ell-2}$  acts trivially on  $\mathcal{K}_{\text{ab}} \otimes \mathbb{Q}$ , where  $I$  is the augmentation ideal of  $\widehat{\mathbb{Q}[\mathcal{I}_{\text{ab}}]}$ .

We now follow the same strategy as in [36, Proof of Theorem 1.4] but, again, with some variations with respect to the closed case. We also give more explicit arguments. Since the graded Lie algebra  $\text{Gr} \mathfrak{m}(\mathcal{I})$  is generated by its degree 1 part, we have

$$(\text{Gr} \mathfrak{m}(\mathcal{I}))' = \text{Gr}_{\geq 2} \mathfrak{m}(\mathcal{I}).$$

By [36, Theorem 1.2 and Proposition 3.1], the canonical map between  $\text{Gr}_4 \mathfrak{m}(\mathcal{I}) \simeq (\Gamma_4 \mathcal{I} / \Gamma_5 \mathcal{I}) \otimes \mathbb{Q}$  and  $(\mathcal{M}[4] / \mathcal{M}[5]) \otimes \mathbb{Q}$  is an isomorphism for  $g \geq 6$ ; furthermore, [42, Lemma 4.4 and Theorem 4.7] implies that the Lie bracket map  $\Lambda^2((\mathcal{M}[2] / \mathcal{M}[3]) \otimes \mathbb{Q}) \rightarrow (\mathcal{M}[4] / \mathcal{M}[5]) \otimes \mathbb{Q}$  is surjective for  $g \geq 4$ ; therefore, the Lie bracket map  $\Lambda^2 \text{Gr}_2 \mathfrak{m}(\mathcal{I}) \rightarrow \text{Gr}_4 \mathfrak{m}(\mathcal{I})$  is surjective for  $g \geq 6$ . It follows that the degree 4 part of  $\mathfrak{b}(\mathcal{I})$  is trivial. Next, it can be proved by an induction on  $j \geq 4$  that  $\mathfrak{b}_j(\mathcal{I}) = 0$  using the fact that  $\text{Gr} \mathfrak{m}(G)$  is generated in degree 1: the argument is general, and the same as used in [36, Proof of Theorem 1.4] for the closed case. (In particular, we deduce that the integer  $\ell$  defined in the previous paragraph is equal to 4.) Thus we have obtained that  $\mathfrak{b}(\mathcal{I})$  is concentrated in degrees 2 and 3:

$$\mathfrak{b}(\mathcal{I}) = \mathfrak{b}_2(\mathcal{I}) \oplus \mathfrak{b}_3(\mathcal{I}) = \text{Gr}_2 \mathfrak{m}(G) \oplus \text{Gr}_3 \mathfrak{m}(G).$$

so that the isomorphism (3.9) can be viewed as an isomorphism:

$$F = (F_2, F_3) : \mathcal{K}_{\text{ab}} \otimes \mathbb{Q} \longrightarrow \left( \frac{\Gamma_2 \mathcal{I}}{\Gamma_3 \mathcal{I}} \otimes \mathbb{Q} \right) \oplus \left( \frac{\Gamma_3 \mathcal{I}}{\Gamma_4 \mathcal{I}} \otimes \mathbb{Q} \right).$$

Let now  $j : \mathcal{K} \rightarrow \mathcal{K}_{\text{ab}} \otimes \mathbb{Q}$  be the canonical homomorphism. The statement of Theorem 3.2 can be rephrased as follows:

$$(3.10) \quad \ker(j) = \ker(d) \cap \ker(r_{[2,4]}^\theta).$$

Since  $F$  is an isomorphism, we have  $\ker(j) = \ker(F \circ j)$ ; besides, since  $F$  arises from a formality isomorphism of  $\mathfrak{m}(\mathcal{I})$  as in (3.4), we have the equality  $\ker(F \circ j) = \sqrt{\Gamma_4 \mathcal{I}}$  in  $\mathcal{K}$ , where  $\sqrt{\Gamma_4 \mathcal{I}}$  denotes the radical of  $\Gamma_4 \mathcal{I}$  in  $\mathcal{I}$ . Hence we see that (3.10) is equivalent to

$$(3.11) \quad \sqrt{\Gamma_4 \mathcal{I}} = \ker(d) \cap \ker(r_{[2,4]}^\theta).$$

That  $\sqrt{\Gamma_4 \mathcal{I}}$  is contained in  $\ker(d) \cap \ker(r_{[2,4]}^\theta)$  is clear, since we have  $\Gamma_3 \mathcal{I} \subset \ker(d)$  and  $\mathcal{M}[4] = \ker(r_{[2,4]}^\theta)$ . To prove the converse, let  $k \in \ker(d) \cap \ker(r_{[2,4]}^\theta)$ . Recall that  $[\mathcal{I}, \mathcal{I}]$  is of finite index in  $\mathcal{K}$ : hence, to justify that  $k \in \sqrt{\Gamma_4 \mathcal{I}}$ , we can assume without loss of generality that  $k \in \Gamma_2 \mathcal{I}$ . We know from [18, 30, 31] that

$$(3.12) \quad (d, \tau_2) : \frac{\Gamma_2 \mathcal{I}}{\Gamma_3 \mathcal{I}} \otimes \mathbb{Q} \longrightarrow \mathbb{Q} \oplus \mathcal{T}_2(H^\mathbb{Q})$$

is an isomorphism for any  $g \geq 3$ . Therefore we have  $k \in \sqrt{\Gamma_3 \mathcal{I}}$ : let  $m \in \mathbb{N}$  be such that  $k^m \in \Gamma_3 \mathcal{I}$ . We have  $k^m \in \mathcal{M}[3]$  and  $\tau_3(k^m) = m r_3^\theta(k) = 0$ . But we also know from [35, Proposition 6.3] that  $\tau_3$  induces an embedding of  $(\Gamma_3 \mathcal{I} / \Gamma_4 \mathcal{I}) \otimes \mathbb{Q}$  into  $\mathcal{T}_3(H^\mathbb{Q})$ : therefore,  $k^m$  belongs to  $\sqrt{\Gamma_4 \mathcal{I}}$ , and so does  $k$ . This completes the proof of (3.10).  $\square$

### 3.3. Complements

Theorem 3.1 and Theorem 3.2 produce embeddings of the rational abelianized Johnson kernel into well-understood vector spaces, and these theorems will be enough for our purpose of proving Theorem A. Yet, for the sake of completeness, we now identify the images and the equivariance property of those embeddings. In the closed case, similar results were given in [6, Theorem B]. Here, we consider both the closed case and the bordered case.

The homomorphism  $d : \mathcal{K} \rightarrow \mathbb{Z}$  (resp.  $\widehat{d} : \widehat{\mathcal{K}} \rightarrow \mathbb{Z}$ ) is known to be  $\mathcal{M}$ -invariant (resp.  $\widehat{\mathcal{M}}$ -invariant). Thus, for the equivariance property of the embeddings of Theorem 3.1 and Theorem 3.2, we only have to understand how  $r_{[2,4]}^\theta$  behaves under conjugacy by the mapping class group. For this, we consider the following analogue of Morita's extension of  $\tau_1$  [33]:

$$(3.13) \quad \tau_1^\theta : \mathcal{M} \longrightarrow \Lambda^3 H^\mathbb{Q} \rtimes \text{Sp}(H^\mathbb{Q}), \quad f \longmapsto (\widetilde{r}_1^\theta(f), f_*).$$

Here  $\widehat{r}_1^\theta$  denotes the degree 1 part of the map  $\widehat{r}^\theta : \mathcal{M} \rightarrow \widehat{\mathcal{T}}(H^\mathbb{Q})$  defined by (2.12). It follows from (2.13) that  $\tau_1^\theta$  is a group homomorphism; its kernel is  $\mathcal{K}$ . It can be also checked that  $\tau_1^\theta$  induces a group homomorphism  $\tau_1^\theta : \widehat{\mathcal{M}} \rightarrow \frac{\Lambda^3 H^\mathbb{Q}}{\omega \wedge H^\mathbb{Q}} \rtimes \mathrm{Sp}(H^\mathbb{Q})$  whose kernel is  $\widehat{\mathcal{K}}$ . Besides, the target group  $\Lambda^3 H^\mathbb{Q} \rtimes \mathrm{Sp}(H^\mathbb{Q})$  acts on  $\mathcal{T}_2(H^\mathbb{Q}) \oplus \mathcal{T}_3(H^\mathbb{Q})$  by

$$\forall (w, \psi) \in \Lambda^3 H^\mathbb{Q} \times \mathrm{Sp}(H^\mathbb{Q}), \forall (a, b) \in \mathcal{T}_2(H^\mathbb{Q}) \oplus \mathcal{T}_3(H^\mathbb{Q}), \\ (w, \psi) \cdot (a, b) := (\psi \cdot a, \psi \cdot b + [w, \psi \cdot a]),$$

and we also have an induced action of  $\frac{\Lambda^3 H^\mathbb{Q}}{\omega \wedge H^\mathbb{Q}} \rtimes \mathrm{Sp}(H^\mathbb{Q})$  on  $\widehat{\mathcal{T}}_2(H^\mathbb{Q}) \oplus \widehat{\mathcal{T}}_3(H^\mathbb{Q})$ .

**PROPOSITION 3.3.** — *The map  $r_{[2,4[}^\theta : \mathcal{K} \rightarrow \mathcal{T}_2(H^\mathbb{Q}) \oplus \mathcal{T}_3(H^\mathbb{Q})$  (resp.  $r_{[2,4[}^\theta : \widehat{\mathcal{K}} \rightarrow \widehat{\mathcal{T}}_2(H^\mathbb{Q}) \oplus \widehat{\mathcal{T}}_3(H^\mathbb{Q})$ ) is equivariant over the group homomorphism  $\tau_1^\theta$ .*

*Proof.* — By the commutativity of (2.23), it suffices to prove the proposition in the bordered case. A straightforward computation gives

$$\begin{aligned} \forall f \in \mathcal{M}, \forall h \in \mathcal{I}, \quad r^\theta(fh f^{-1}) \\ &= (\eta^\mathbb{Q})^{-1} \log(\varrho^\theta(fh f^{-1})) \\ &= (\eta^\mathbb{Q})^{-1} \log((\varrho^\theta(f) f_*^{-1}) \circ (f_* \varrho^\theta(h) f_*^{-1}) \circ (\varrho^\theta(f) f_*^{-1})^{-1}) \\ &= e^{[\widehat{r}^\theta(f), -1]}(f_* \cdot r^\theta(h)) \\ &= f_* r^\theta(h) + [\widehat{r}^\theta(f), f_* r^\theta(h)] + \frac{1}{2} [\widehat{r}^\theta(f), [\widehat{r}^\theta(f), f_* r^\theta(h)]] + \dots \end{aligned}$$

which implies that

$$\forall f \in \mathcal{M}, \forall h \in \mathcal{K}, \quad r_{[2,4[}^\theta(fh f^{-1}) = f_* \cdot r_{[2,4[}^\theta(h) + [\widehat{r}_1^\theta(f), f_* \cdot r_2^\theta(h)]. \quad \square$$

To understand the image of  $r_{[2,4[}^\theta : \mathcal{K}_{\mathrm{ab}} \otimes \mathbb{Q} \rightarrow \mathcal{T}_2(H^\mathbb{Q}) \oplus \mathcal{T}_3(H^\mathbb{Q})$ , we need *Morita's trace*. This linear map  $\mathrm{Tr}_k : \mathcal{T}_k(H^\mathbb{Q}) \rightarrow S^k(H^\mathbb{Q})$  has been introduced in [32] for any odd  $k$ , and it is defined for  $k = 3$  by

$$\mathrm{Tr}_3 \left( \begin{array}{c} b & & c & & d \\ & \diagdown & | & / & \\ & a & \bullet & & e \end{array} \right) := 2\omega(e, a) bcd + 2\omega(a, d) ecb \\ + 2\omega(d, b) ace + 2\omega(b, e) dca.$$

A straightforward computation shows that it factorizes to  $\widehat{\mathrm{Tr}}_3 : \widehat{\mathcal{T}}_3(H^\mathbb{Q}) \rightarrow S^3(H^\mathbb{Q})$ .

PROPOSITION 3.4. — *The image of  $r_{[2,4]}^\theta$  is  $\mathcal{T}_2(H^\mathbb{Q}) \oplus \ker \text{Tr}_3$  (resp.,  $\widehat{\mathcal{T}}_2(H^\mathbb{Q}) \oplus \ker \widehat{\text{Tr}}_3$ ) in the bordered case (resp., in the closed case).*

*Proof.* — By works of Morita [35], we have the following isomorphisms:

$$\frac{\mathcal{M}[2]}{\mathcal{M}[3]} \otimes \mathbb{Q} \xrightarrow[\simeq]{\tau_2} \mathcal{T}_2(H^\mathbb{Q}) \quad \text{and} \quad \frac{\mathcal{M}[3]}{\mathcal{M}[4]} \otimes \mathbb{Q} \xrightarrow[\simeq]{\tau_3} \ker(\text{Tr}_3) \subset \mathcal{T}_3(H^\mathbb{Q}).$$

We deduce from these isomorphisms and the injectivity of  $r_{[2,4]}^\theta : \frac{\mathcal{M}[2]}{\mathcal{M}[4]} \otimes \mathbb{Q} \rightarrow \mathcal{T}_2(H^\mathbb{Q}) \oplus \mathcal{T}_3(H^\mathbb{Q})$  that

$$\dim(\mathcal{T}_2(H^\mathbb{Q}) \oplus \ker \text{Tr}_3) = \dim\left(\frac{\mathcal{M}[2]}{\mathcal{M}[4]} \otimes \mathbb{Q}\right) = \dim r_{[2,4]}^\theta(\mathcal{K}_{\text{ab}} \otimes \mathbb{Q}).$$

So, to conclude that  $r_{[2,4]}^\theta(\mathcal{K}_{\text{ab}} \otimes \mathbb{Q})$  is equal to  $\mathcal{T}_2(H^\mathbb{Q}) \oplus \ker \text{Tr}_3$ , it is enough to recall from [29, Proposition 7.3] that the former is contained in the latter. (Note that the triviality of  $\text{Tr}_3 \circ \tau_3$ , due to Morita [32, Theorem 6.11], is not enough to conclude here.)

The proof in the closed case follows the same lines.  $\square$

## 4. Proofs of Theorem A and Theorem B

In this section, we prove Theorem A and Theorem B. In fact, we give two proofs of Theorem B: the first one is purely 2-dimensional, while the second one uses 3-dimensional surgery techniques.

### 4.1. Proof of Theorem A

We first prove Theorem A assuming Theorem B. We only deal with the case of the bordered surface  $\Sigma$ , since the case of the closed surface  $\widehat{\Sigma}$  is proved exactly in the same way.

Theorem B asserts the existence of an element  $\varphi \in \mathcal{M}[4]$  such that  $d(\varphi) = 0$  and  $R(\varphi) \neq 0$ . That  $\varphi$  belongs to  $\mathcal{M}[4]$  implies that  $r_{[2,4]}^\theta(\varphi) = 0$ . Then we deduce from Theorem 3.2 that the class  $\{\varphi\} \in \mathcal{K}_{\text{ab}}$  is a torsion element. Furthermore, we have  $R_{\text{ab}}(\{\varphi\}) = R(\varphi) \neq 0$  and we conclude that  $\{\varphi\} \neq 0 \in \mathcal{K}_{\text{ab}}$ .

**4.2. Proof of Theorem B**

We shall exhibit an element  $\varphi \in \mathcal{M}[3]$  of the form

$$\varphi = [i, k], \quad \text{where } i \in \mathcal{I} \text{ and } k \in \mathcal{K}.$$

Recall that  $\mathcal{I}$  is generated by opposite Dehn twists  $T_{c^-}^{-1}T_{c^+}$  along pairs  $(c^-, c^+)$  of simple closed curves that cobound a subsurface of  $\Sigma$ : in short,  $T_{c^-}^{-1}T_{c^+}$  is called a *bounding pair map*. The above element  $i$  will be given in this generating system of  $\mathcal{I}$ , while  $k$  will be given as a product of separating twists.

It will turn out that  $\varphi$  actually belongs to  $\mathcal{M}[4]$  and satisfies the following:

$$(4.1) \quad R(\varphi) = \frac{1}{2} \begin{array}{c} a_2 \quad a_3 \quad a_3 \quad a_2 \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ a_1 \quad \quad \quad a_1 \end{array} \pmod{1}.$$

Since  $\varphi$  belongs to  $[\mathcal{M}, \mathcal{K}]$ , we have  $d(\varphi) = 0$ ; since we have  $R(\varphi) = j([a_3, [a_2, a_1]])$  and  $j$  is injective, we have  $R(\varphi) \neq 0$ . This proves Theorem B for the bordered surface  $\Sigma$ .

The extension  $\widehat{\varphi}$  of  $\varphi$  to  $\widehat{\Sigma}$  satisfies  $\widehat{d}(\widehat{\varphi}) = -\frac{1+2g}{12}d(\varphi)$  since  $\varphi \in \mathcal{M}[3]$ : therefore  $\widehat{d}(\widehat{\varphi}) = 0$ . Besides, it follows from (2.26) that

$$R(\widehat{\varphi}) = \widehat{j}([a_3, [a_2, a_1]] \otimes 1)$$

where the homomorphism  $\widehat{j}$  is introduced in Remark 2.13. Since  $[a_3, [a_2, a_1]] \otimes 1$  is a non-trivial element of  $\mathfrak{L}_3(A) \otimes \mathbb{Z}_2$ , the commutativity of (2.28) implies that  $R(\widehat{\varphi}) \neq 0$ . This proves Theorem B for the closed surface  $\widehat{\Sigma}$ .

The rest of the subsection is devoted to the construction of  $\varphi \in \mathcal{M}[4]$  and the proof of (4.1).

4.2.1. The element  $\varphi$  of  $\mathcal{M}[4]$

Let  $(c_1^+, c_1^-)$  and  $(c_2^+, c_2^-)$  be the pairs of curves in  $\Sigma$  shown in Figure 4.1: note that  $c_i^+$  and  $c_i^-$  cobound a subsurface of genus 1. Then consider the following product of bounding pair maps:

$$(4.2) \quad i := (T_{c_1^-}^{-1} T_{c_1^+}) \circ (T_{c_2^-}^{-1} T_{c_2^+})^{-1} = T_{c_1^-}^{-1} T_{c_2^-} \in \mathcal{I}.$$

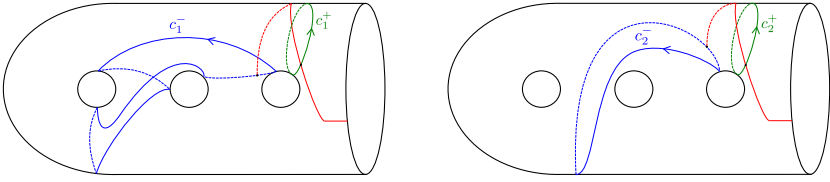


Figure 4.1. The bounding pairs  $(c_1^+, c_1^-)$  and  $(c_2^+, c_2^-)$  in  $\Sigma$

Besides, let  $\gamma_1, \gamma_2, \gamma_3, \gamma_4$  be the four curves shown in Figure 4.2: note that each of  $\gamma_1, \gamma_2$  bounds a subsurface of genus 2, and each of  $\gamma_3, \gamma_4$  bounds a subsurface of genus 1. Then consider the following product of separating twists:

$$k := T_{\gamma_1} T_{\gamma_2}^{-1} T_{\gamma_3}^{-1} T_{\gamma_4} \in \mathcal{K}.$$

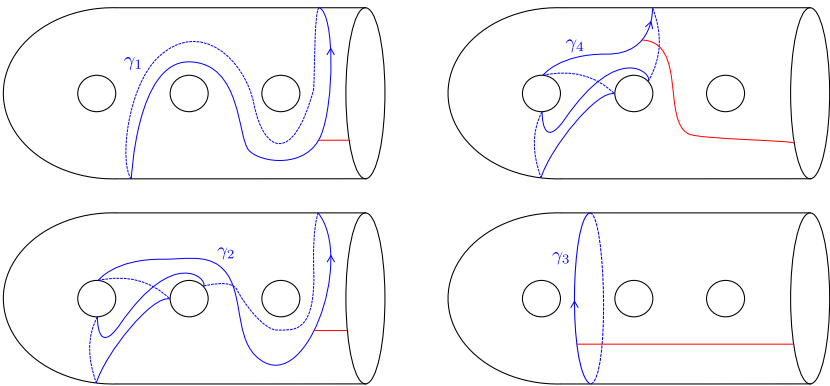
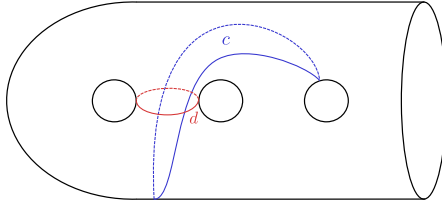


Figure 4.2. The separating simple closed curves  $\gamma_1, \gamma_2, \gamma_3$  and  $\gamma_4$  in  $\Sigma$

*Remark 4.1.* — The above elements  $i \in \mathcal{I}$  and  $k \in \mathcal{K}$  can be alternatively described as products of “commutators of simply intersecting pairs”, which participate to Putman’s infinite presentation of the Torelli group [40]. Recall that a *commutator of simply intersecting pair* is an element  $[T_c, T_d]$  where  $(c, d)$  is a pair of simple closed curves meeting at two points such that  $\omega(c, d) = 0$ . Then it can be checked that

$$i = [T_d, T_c^{-1}]$$

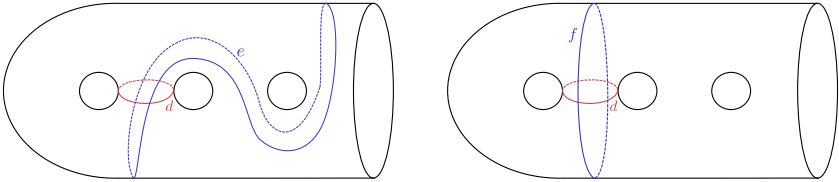
where the curves  $c = c_2^-$  and  $d$  are shown below:



Similarly, it can be verified that

$$k = [T_e, T_d] [T_f^{-1}, T_d]$$

where the curves  $e = \gamma_1$  and  $f = \gamma_3$  are shown below:



By construction,  $\varphi = [i, k]$  belongs to  $[\mathcal{I}, \mathcal{K}]$  and so to  $\mathcal{M}[3]$ . Hence we can consider  $\tau_3(\varphi)$ , the value of which can be deduced from the values of  $\tau_1(i)$  and  $\tau_2(k)$ . To compute  $\tau_1(i)$ , we use Johnson's formula [19, Lemma 4.B] for a bounding pair map (or, alternatively, we can use (4.12) below):

$$\tau_1(T_{c_1^-}^{-1} T_{c_1^+}) = -a_3 \wedge a_1 \wedge (b_1 - a_2), \quad \tau_1(T_{c_2^-}^{-1} T_{c_2^+}) = -a_3 \wedge a_1 \wedge b_1.$$

Hence

$$(4.3) \quad \tau_1(i) = -a_3 \wedge a_1 \wedge (b_1 - a_2) + a_3 \wedge a_1 \wedge b_1 = a_1 \wedge a_2 \wedge a_3 = \begin{matrix} a_2 \\ | \\ a_1 \text{---} a_3 \end{matrix}.$$

To compute now  $\tau_2(k)$ , we use Morita's formula [30, Proposition 1.1] for a separating twist (or, alternatively, we can specialize formula (4.7) below in degree 2):

$$\begin{aligned} \tau_2(T_{\gamma_1}) &= \frac{1}{2}(a_1 \wedge b_1 + a_3 \wedge b_3) \text{---} (a_1 \wedge b_1 + a_3 \wedge b_3), \\ \tau_2(T_{\gamma_2}) &= \frac{1}{2}(a_1 \wedge (b_1 - a_2) + a_3 \wedge b_3) \text{---} (a_1 \wedge (b_1 - a_2) + a_3 \wedge b_3), \\ \tau_2(T_{\gamma_3}) &= \frac{1}{2}(a_1 \wedge b_1) \text{---} (a_1 \wedge b_1), \\ \tau_2(T_{\gamma_4}) &= \frac{1}{2}(a_1 \wedge (b_1 - a_2)) \text{---} (a_1 \wedge (b_1 - a_2)). \end{aligned}$$

Hence

$$(4.4) \quad \tau_2(k) = \frac{1}{2} \begin{array}{c} b_3 \quad a_3 \\ | \quad | \\ \hline | \quad | \\ a_3 \quad b_3 \end{array} + \begin{array}{c} b_3 \quad a_1 \\ | \quad | \\ \hline | \quad | \\ a_3 \quad b_1 \end{array} - \frac{1}{2} \begin{array}{c} b_3 \quad a_3 \\ | \quad | \\ \hline | \quad | \\ a_3 \quad b_3 \end{array} - \begin{array}{c} b_3 \quad a_1 \\ | \quad | \\ \hline | \quad | \\ a_3 \quad b_1 - a_2 \end{array}$$

$$= \begin{array}{c} b_3 \quad a_1 \\ | \quad | \\ \hline | \quad | \\ a_3 \quad a_2 \end{array}.$$

We deduce that

$$(4.5) \quad \tau_3(\varphi) = [\tau_1(i), \tau_2(k)] = \left[ \begin{array}{c} a_2 \\ / \quad \backslash \\ a_1 \quad a_3 \end{array}, \begin{array}{c} b_3 \quad a_1 \\ | \quad | \\ \hline | \quad | \\ a_3 \quad a_2 \end{array} \right] = \begin{array}{c} a_2 \quad a_3 \quad a_2 \\ | \quad | \quad | \\ a_1 \quad a_1 \quad a_1 \end{array} = 0$$

where the last identity follows from the AS relation. We conclude that  $\varphi \in \mathcal{M}[4]$ .

#### 4.2.2. Beginning of the computation of $R(\varphi)$

Since  $\varphi$  belongs to  $\mathcal{M}[4]$ ,  $R(\varphi)$  is simply the reduction of  $\tau_4(\varphi)$  modulo 1. To compute this congruence class, we will use the infinitesimal Dehn–Nielsen representation  $r^\theta$  described in Section 2.2. We have

$$r^\theta(\varphi) = r^\theta([i, k]) = [r^\theta(i), r^\theta(k)]_\star$$

where  $[-, -]_\star$  denotes the commutator for the BCH product  $\star$  on  $\mathcal{T}(H^\mathbb{Q})$ . The latter is explicitly given by

$$\begin{aligned} [u, v]_\star &= u \star v \star (-u) \star (-v) \\ &= [u, v] + \frac{1}{2}[u, [u, v]] - \frac{1}{2}[v, [v, u]] + (\text{Lie brackets of length } \geq 4) \end{aligned}$$

for any  $u, v \in \mathcal{T}(H^\mathbb{Q})$ . Since  $u := r^\theta(i)$  and  $v := r^\theta(k)$  start in degrees 1 and 2, respectively, we deduce that

$$\begin{aligned} r_4^\theta(\varphi) &= [r_1^\theta(i), r_3^\theta(k)] + [r_2^\theta(i), r_2^\theta(k)] + \frac{1}{2}[r_1^\theta(i), [r_1^\theta(i), r_2^\theta(k)]] \\ &= [\tau_1(i), r_3^\theta(k)] + [r_2^\theta(i), \tau_2(k)] + \frac{1}{2}[\tau_1(i), [\tau_1(i), \tau_2(k)]] \end{aligned}$$

and, using (4.5), we obtain

$$(4.6) \quad r_4^\theta(\varphi) = [\tau_1(i), r_3^\theta(k)] + [r_2^\theta(i), \tau_2(k)].$$

Hence, we are led to compute  $r_2^\theta(i)$  and  $r_3^\theta(k)$  and, for that, we will use the logansion given in Example 2.4. In fact, since we are only interested in the value of  $r_4^\theta(\varphi)$  modulo 1, we will only determine the classes of  $r_2^\theta(i)$  and  $r_3^\theta(k)$  modulo  $\mathbb{Z}$ -linear combinations of trees.



4.2.3. Computation of  $r_3^\theta(k)$ 

The map  $r_{[2,3]}^\theta$  is a group homomorphism on  $\mathcal{K} = \mathcal{M}[2]$  (see [27]; this follows from Lemma 2.6 with  $k := 2$ ). In particular, we have

$$r_3^\theta(k) = r_3^\theta(T_{\gamma_1}) - r_3^\theta(T_{\gamma_2}) - r_3^\theta(T_{\gamma_3}) + r_3^\theta(T_{\gamma_4}).$$

Next, we shall compute  $r_3^\theta(T_{\gamma_i})$  for each  $i \in \{1, 2, 3, 4\}$  using the following formula, which is deduced in [25, Equation (5.4)] from the main result of [24]:

*For any separating simple closed curve  $\gamma$  in  $\Sigma$ , and any representative  $\tilde{\gamma} \in \pi$ , we have*

$$(4.7) \quad r^\theta(T_\gamma) = \frac{1}{2}\theta(\tilde{\gamma}) - \theta(\tilde{\gamma}) \in \widehat{\mathcal{T}}(H^\mathbb{Q}).$$

We orient and base the curves  $\gamma_i$  as shown in Figure 4.2 to get the following lifts:

$$(4.8) \quad \tilde{\gamma}_1 = [\alpha_3, \beta_3^{-1}]\beta_2[\alpha_1, \beta_1^{-1}]\beta_2^{-1}, \quad \tilde{\gamma}_4 = [\alpha_2\beta_2\beta_1^{-1}, \alpha_1^{-1}][\alpha_1^{-1}, \beta_2],$$

$$(4.9) \quad \tilde{\gamma}_2 = [\alpha_3, \beta_3^{-1}][\alpha_2\beta_2\beta_1^{-1}, \alpha_1^{-1}][\alpha_1^{-1}, \beta_2], \quad \tilde{\gamma}_3 = [\alpha_1, \beta_1^{-1}].$$

Then, a direct computation gives

$$\begin{aligned} \theta(\tilde{\gamma}_1) &= - \begin{array}{c} a_1 \quad b_1 \\ \diagdown \quad \diagup \\ \text{Y} \end{array} - \begin{array}{c} a_3 \quad b_3 \\ \diagdown \quad \diagup \\ \text{Y} \end{array} + \begin{array}{c} a_1 \quad b_1 \quad b_2 \\ \diagdown \quad \diagup \quad \diagup \\ \text{Y} \end{array} + (\text{deg} \geq 4) \\ \theta(\tilde{\gamma}_2) &= - \begin{array}{c} a_1 \quad b_1 \\ \diagdown \quad \diagup \\ \text{Y} \end{array} + \begin{array}{c} a_1 \quad a_2 \\ \diagdown \quad \diagup \\ \text{Y} \end{array} - \begin{array}{c} a_3 \quad b_3 \\ \diagdown \quad \diagup \\ \text{Y} \end{array} + \begin{array}{c} b_1 \quad a_1 \quad a_1 \\ \diagdown \quad \diagup \quad \diagup \\ \text{Y} \end{array} - \frac{1}{2} \begin{array}{c} a_2 \quad a_1 \quad a_1 \\ \diagdown \quad \diagup \quad \diagup \\ \text{Y} \end{array} - \frac{1}{2} \begin{array}{c} a_1 \quad a_2 \quad a_2 \\ \diagdown \quad \diagup \quad \diagup \\ \text{Y} \end{array} \\ &\quad + \frac{1}{2} \begin{array}{c} a_1 \quad b_1 \quad a_2 \\ \diagdown \quad \diagup \quad \diagup \\ \text{Y} \end{array} + \begin{array}{c} a_1 \quad b_1 \quad b_2 \\ \diagdown \quad \diagup \quad \diagup \\ \text{Y} \end{array} - \frac{1}{2} \begin{array}{c} b_2 \quad a_2 \quad a_1 \\ \diagdown \quad \diagup \quad \diagup \\ \text{Y} \end{array} - \begin{array}{c} a_1 \quad b_2 \quad a_2 \\ \diagdown \quad \diagup \quad \diagup \\ \text{Y} \end{array} + (\text{deg} \geq 4) \\ \theta(\tilde{\gamma}_3) &= - \begin{array}{c} a_1 \quad b_1 \\ \diagdown \quad \diagup \\ \text{Y} \end{array} + (\text{deg} \geq 4) \\ \theta(\tilde{\gamma}_4) &= - \begin{array}{c} a_1 \quad b_1 \\ \diagdown \quad \diagup \\ \text{Y} \end{array} + \begin{array}{c} a_1 \quad a_2 \\ \diagdown \quad \diagup \\ \text{Y} \end{array} + \begin{array}{c} b_1 \quad a_1 \quad a_1 \\ \diagdown \quad \diagup \quad \diagup \\ \text{Y} \end{array} - \frac{1}{2} \begin{array}{c} a_2 \quad a_1 \quad a_1 \\ \diagdown \quad \diagup \quad \diagup \\ \text{Y} \end{array} - \frac{1}{2} \begin{array}{c} a_1 \quad a_2 \quad a_2 \\ \diagdown \quad \diagup \quad \diagup \\ \text{Y} \end{array} \\ &\quad + \frac{1}{2} \begin{array}{c} a_1 \quad b_1 \quad a_2 \\ \diagdown \quad \diagup \quad \diagup \\ \text{Y} \end{array} + \begin{array}{c} a_1 \quad b_1 \quad b_2 \\ \diagdown \quad \diagup \quad \diagup \\ \text{Y} \end{array} - \frac{1}{2} \begin{array}{c} b_2 \quad a_2 \quad a_1 \\ \diagdown \quad \diagup \quad \diagup \\ \text{Y} \end{array} - \begin{array}{c} a_1 \quad b_2 \quad a_2 \\ \diagdown \quad \diagup \quad \diagup \\ \text{Y} \end{array} + (\text{deg} \geq 4). \end{aligned}$$

This computation can be performed either by hand or by using the SageMath code given in Appendix A. For instance,  $\theta(\tilde{\gamma}_4)$  is computed and displayed with the following command lines:

```
1 g4=theta(comm('a2+b2+b1-', 'a1-')+comm('a1-', 'b2+'))
2 display(g4)
```

We observe that  $r_3^\theta(T_{\gamma_3})$  is trivial, since  $\theta_3(\tilde{\gamma}_3) = 0$ , and that  $r_3^\theta(T_{\gamma_1})$  is a  $\mathbb{Z}$ -linear combination of trees, since the above formula for  $\theta_3(\tilde{\gamma}_1) = 0$  shows no fraction. Besides, we remark that  $\theta_2(\tilde{\gamma}_2) = \theta_2(\tilde{\gamma}_4) - [a_3, b_3]$  and  $\theta_3(\tilde{\gamma}_2) = \theta_3(\tilde{\gamma}_4)$ ; therefore we have

$$\begin{aligned}
 (4.10) \quad r_3^\theta(k) &\equiv -r_3^\theta(T_{\gamma_2}) + r_3^\theta(T_{\gamma_4}) \\
 &= -\theta_2(\tilde{\gamma}_2) \text{---} \theta_3(\tilde{\gamma}_2) + \theta_2(\tilde{\gamma}_4) \text{---} \theta_3(\tilde{\gamma}_4) \\
 &= [a_3, b_3] \text{---} \theta_3(\tilde{\gamma}_2) \\
 &\equiv \frac{1}{2}[a_3, b_3] \text{---} ([[a_2, a_1], a_1] + [[a_1, a_2], a_2] + [[b_1, a_1], a_2] + [[b_2, a_2], a_1])
 \end{aligned}$$

where the symbol “ $\equiv$ ” stands for a congruence modulo  $\mathbb{Z}$ -linear combinations of trees.

#### 4.2.4. Computation of $r_2^\theta(i)$

Lemma 2.6 with  $k := 1$  implies that

$$(4.11) \quad r_2^\theta(i) = r_2^\theta(P_1) - r_2^\theta(P_2) - \frac{1}{2}[\tau_1(P_1), \tau_1(P_2)] \quad \text{where } P_i := T_{c_i}^{-1} T_{c_i}^+.$$

We need the following formulas for bounding pair maps.

PROPOSITION 4.2. — *Let  $\gamma$  and  $\delta$  be elements of  $\pi$  representing two simple closed curves that cobound a subsurface of  $\Sigma$ , and set  $c := \gamma^{-1}\delta$ . Then we have*

$$(4.12) \quad r_1^\theta(T_\gamma T_\delta^{-1}) = -[\gamma] \text{---} [c]$$

and

$$(4.13) \quad r_2^\theta(T_\gamma T_\delta^{-1}) = -\frac{1}{2}[c] \text{---} [c] - \theta_2(\gamma) \text{---} [c] - [\gamma] \text{---} \theta_3(c)$$

where

$$[\gamma] \in \frac{\pi}{\Gamma_2\pi} \simeq H \quad \text{and} \quad [c] \in \frac{\Gamma_2\pi}{\Gamma_3\pi} \simeq \mathfrak{L}_2$$

are the leading terms of  $\theta(\gamma) = [\gamma] + \theta_2(\gamma) + \dots \in \widehat{\mathfrak{L}}^\mathbb{Q}$  and  $\theta(c) = [c] + \theta_3(c) + \dots \in \widehat{\mathfrak{L}}^\mathbb{Q}$ , respectively.

*Proof.* — There is also a version of (4.7) for the Dehn twist  $T_\gamma$  along any simple closed curve  $\gamma$ . Of course,  $T_\gamma$  does not belong to the Torelli group if  $\gamma$  is not separating, but the automorphism  $\varrho^\theta(T_\gamma)$  of  $\widehat{\mathfrak{X}}^\mathbb{Q}$  still has a logarithm and we can consider

$$r^\theta(T_\gamma) := (\eta^\mathbb{Q})^{-1}(\log \varrho^\theta(T_\gamma)) \in \widehat{\mathcal{T}}(H^\mathbb{Q})$$

by allowing tree diagrams to have no trivalent vertices and derivations to be of degree 0. (Hence the degree 0 part of  $\mathcal{T}(H^\mathbb{Q})$  is canonically isomorphic to  $S^2(H^\mathbb{Q})$ .) This generalization of formula (4.7) is proved exactly as in [25, Equation (5.4)] using the main result of [24].

Since  $T_\gamma$  and  $T_\delta$  commute, the automorphisms  $\varrho^\theta(T_\gamma)$  and  $\varrho^\theta(T_\delta)$  commute, and so do their logarithms. Hence we have

$$[r^\theta(T_\gamma), r^\theta(T_\delta)] = 0,$$

so that the BCH formula reduces to

$$r^\theta(T_\gamma T_\delta^{-1}) = r^\theta(T_\gamma) + r^\theta(T_\delta^{-1}),$$

and we deduce that

$$r^\theta(T_\gamma T_\delta^{-1}) = \frac{1}{2}\theta(\gamma) \text{---} \theta(\gamma) - \frac{1}{2}\theta(\delta) \text{---} \theta(\delta).$$

Besides, we have

$$\theta(\delta) = \theta(\gamma) \star \theta(c) = \theta(\gamma) + \theta(c) + \frac{1}{2}[\theta(\gamma), \theta(c)] + (\deg \geq 4)$$

since  $\theta(c)$  starts in degree 2. Thus we obtain

$$r^\theta(T_\gamma T_\delta^{-1}) = -\frac{1}{2}\theta(c) \text{---} \theta(c) - \theta(\gamma) \text{---} \theta(c) + (\deg \geq 3)$$

since we have  $\theta(\gamma) \text{---} [\theta(\gamma), \theta(c)] = 0$  by the AS relation. In particular, we get

$$r_1^\theta(T_\gamma T_\delta^{-1}) = -\theta_1(\gamma) \text{---} \theta_2(c)$$

and

$$r_2^\theta(T_\gamma T_\delta^{-1}) = -\frac{1}{2}\theta_2(c) \text{---} \theta_2(c) - \theta_2(\gamma) \text{---} \theta_2(c) - \theta_1(\gamma) \text{---} \theta_3(c). \quad \square$$

We now apply (4.13) to compute  $r_2^\theta(P_1)$  and  $r_2^\theta(P_2)$  modulo  $\mathbb{Z}$ -linear combinations of trees:

- (1) For  $P_1$ , we consider  $\gamma := \alpha_3$  and  $c := \beta_3 \tilde{\gamma}_4^{-1} \beta_3^{-1}$  where  $\tilde{\gamma}_4$  has been defined at (4.9). Then, a direct computation (using the SageMath code of Appendix A) gives

$$\begin{aligned} \theta(\gamma) &= \left| -\frac{a_3}{2} \begin{array}{c} a_3 \\ \diagdown \\ \diagup \\ b_3 \end{array} - \frac{1}{2} \begin{array}{c} a_3 \quad b_3 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \end{array} - \frac{1}{2} \begin{array}{c} a_1 \quad b_1 \quad a_3 \\ \diagdown \quad \diagup \quad \diagup \\ \diagup \quad \diagdown \quad \diagdown \\ \end{array} - \frac{1}{2} \begin{array}{c} a_2 \quad b_2 \quad a_3 \\ \diagdown \quad \diagup \quad \diagup \\ \diagup \quad \diagdown \quad \diagdown \\ \end{array} + \frac{1}{12} \begin{array}{c} a_3 \quad b_3 \quad b_3 \\ \diagdown \quad \diagup \quad \diagup \\ \diagup \quad \diagdown \quad \diagdown \\ \end{array} + (\text{deg} \geq 4), \\ \theta(c) &= \begin{array}{c} a_1 \quad b_1 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \end{array} - \begin{array}{c} a_1 \quad a_2 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \end{array} - \frac{1}{2} \begin{array}{c} a_1 \quad a_2 \quad a_1 \\ \diagdown \quad \diagup \quad \diagup \\ \diagup \quad \diagdown \quad \diagdown \\ \end{array} + \begin{array}{c} a_1 \quad b_1 \quad a_1 \\ \diagdown \quad \diagup \quad \diagup \\ \diagup \quad \diagdown \quad \diagdown \\ \end{array} + \frac{1}{2} \begin{array}{c} a_1 \quad a_2 \quad a_2 \\ \diagdown \quad \diagup \quad \diagup \\ \diagup \quad \diagdown \quad \diagdown \\ \end{array} + \frac{1}{2} \begin{array}{c} a_2 \quad b_2 \quad a_1 \\ \diagdown \quad \diagup \quad \diagup \\ \diagup \quad \diagdown \quad \diagdown \\ \end{array} \\ &\quad - \frac{1}{2} \begin{array}{c} a_1 \quad b_1 \quad a_2 \\ \diagdown \quad \diagup \quad \diagup \\ \diagup \quad \diagdown \quad \diagdown \\ \end{array} - \begin{array}{c} a_1 \quad b_1 \quad x \\ \diagdown \quad \diagup \quad \diagup \\ \diagup \quad \diagdown \quad \diagdown \\ \end{array} + \begin{array}{c} a_1 \quad a_2 \quad x \\ \diagdown \quad \diagup \quad \diagup \\ \diagup \quad \diagdown \quad \diagdown \\ \end{array} + (\text{deg} \geq 4) \quad \text{with } x := b_2 + b_3. \end{aligned}$$

Hence we get

$$\begin{aligned} r_2^\theta(P_1) &= \left( -\frac{1}{2} \begin{array}{c} b_1 \quad a_1 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ a_1 \quad b_1 \end{array} - \frac{1}{2} \begin{array}{c} a_2 \quad a_1 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ a_1 \quad a_2 \end{array} + \begin{array}{c} a_2 \quad a_1 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ a_1 \quad b_1 \end{array} \right) \\ &\quad + \left( \frac{1}{2} \begin{array}{c} b_3 \quad a_1 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ a_3 \quad b_1 \end{array} - \frac{1}{2} \begin{array}{c} b_3 \quad a_1 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ a_3 \quad a_2 \end{array} \right) \\ &\quad + \left( \frac{1}{2} \begin{array}{c} a_2 \quad a_1 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ a_1 \quad a_3 \end{array} - \begin{array}{c} b_1 \quad a_1 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ a_1 \quad a_3 \end{array} - \frac{1}{2} \begin{array}{c} a_2 \quad a_2 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ a_1 \quad a_3 \end{array} - \frac{1}{2} \begin{array}{c} b_2 \quad a_1 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ a_2 \quad a_3 \end{array} \right. \\ &\quad \left. + \frac{1}{2} \begin{array}{c} b_1 \quad a_2 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ a_1 \quad a_3 \end{array} + \begin{array}{c} b_1 \quad x \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ a_1 \quad a_3 \end{array} - \begin{array}{c} a_2 \quad x \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ a_1 \quad a_3 \end{array} \right) \\ &\equiv \frac{1}{2} \left( \begin{array}{c} a_1 \quad a_1 + a_2 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ a_2 \quad a_3 \end{array} + \begin{array}{c} a_1 \quad a_2 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ b_1 \quad a_3 \end{array} + \begin{array}{c} a_2 \quad a_1 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ b_2 \quad a_3 \end{array} \right. \\ &\quad \left. + \begin{array}{c} b_3 \quad a_1 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ a_3 \quad b_1 + a_2 \end{array} + \begin{array}{c} a_1 \quad b_1 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ b_1 \quad a_1 \end{array} + \begin{array}{c} a_1 \quad a_2 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ a_2 \quad a_1 \end{array} \right) \end{aligned}$$

where the symbol “ $\equiv$ ” stands for a congruence modulo  $\mathbb{Z}$ -linear combinations of trees.

- (2) For  $P_2$ , we consider now  $\gamma := \alpha_3$  and  $c := (\beta_3\beta_2)\tilde{\gamma}_3^{-1}(\beta_3\beta_2)^{-1}$  where  $\tilde{\gamma}_3$  has been defined at (4.8). Then, a direct computation (using the SageMath code of Appendix A) gives

$$\theta(\gamma) = \left| \begin{array}{c} a_3 \\ | \\ -\frac{1}{2} \end{array} \right. \begin{array}{c} a_3 \quad b_3 \\ \diagdown \quad \diagup \\ | \\ -\frac{1}{2} \end{array} \left| \begin{array}{c} a_1 \quad b_1 \quad a_3 \\ \diagdown \quad | \quad \diagup \\ | \\ -\frac{1}{2} \end{array} \right. \begin{array}{c} a_2 \quad b_2 \quad a_3 \\ \diagdown \quad | \quad \diagup \\ | \\ -\frac{1}{2} \end{array} \left| \begin{array}{c} a_3 \quad b_3 \quad b_3 \\ \diagdown \quad | \quad \diagup \\ | \\ +\frac{1}{12} \end{array} \right. + (\text{deg} \geq 4),$$

$$\theta(c) = \begin{array}{c} a_1 \quad b_1 \\ \diagdown \quad \diagup \\ | \\ - \end{array} \begin{array}{c} a_1 \quad b_1 \quad x \\ \diagdown \quad | \quad \diagup \\ | \\ + \end{array} + (\text{deg} \geq 4) \quad \text{with } x := b_2 + b_3.$$

Hence we get

$$r_2^\theta(P_2) = -\frac{1}{2} \begin{array}{c} b_1 \quad a_1 \\ | \quad | \\ \hline | \quad | \\ a_1 \quad b_1 \end{array} + \frac{1}{2} \begin{array}{c} b_3 \quad a_1 \\ | \quad | \\ \hline | \quad | \\ a_3 \quad b_1 \end{array} + \begin{array}{c} b_1 \quad x \\ | \quad | \\ \hline | \quad | \\ a_1 \quad a_3 \end{array}$$

$$\equiv \frac{1}{2} \left( \begin{array}{c} b_3 \quad a_1 \\ | \quad | \\ \hline | \quad | \\ a_3 \quad b_1 \end{array} + \begin{array}{c} b_1 \quad a_1 \\ | \quad | \\ \hline | \quad | \\ a_1 \quad b_1 \end{array} \right).$$

We now insert into (4.11) the above values of  $r_2^\theta(P_i)$  to get

$$(4.14) \quad r_2^\theta(i) \equiv \frac{1}{2} \left( \begin{array}{c} a_1 \quad a_1 + a_2 \\ | \quad | \\ \hline | \quad | \\ a_2 \quad a_3 \end{array} + \begin{array}{c} a_1 \quad a_2 \\ | \quad | \\ \hline | \quad | \\ b_1 \quad a_3 \end{array} + \begin{array}{c} a_2 \quad a_1 \\ | \quad | \\ \hline | \quad | \\ b_2 \quad a_3 \end{array} \right.$$

$$+ \begin{array}{c} b_3 \quad a_1 \\ | \quad | \\ \hline | \quad | \\ a_3 \quad a_2 \end{array} + \begin{array}{c} a_1 \quad a_2 \\ | \quad | \\ \hline | \quad | \\ a_2 \quad a_1 \end{array} \left. \right)$$

$$- \frac{1}{2} \left[ \begin{array}{c} a_3 \\ | \\ \diagdown \quad \diagup \\ a_1 \quad b_1 \end{array} - \begin{array}{c} a_3 \\ | \\ \diagdown \quad \diagup \\ a_1 \quad a_2 \end{array}, \begin{array}{c} a_3 \\ | \\ \diagdown \quad \diagup \\ a_1 \quad b_1 \end{array} \right]$$

$$\equiv \frac{1}{2} \left( \begin{array}{c} a_1 \quad a_1 + a_2 \\ | \quad | \\ \hline | \quad | \\ a_2 \quad a_3 \end{array} + \begin{array}{c} a_1 \quad a_2 \\ | \quad | \\ \hline | \quad | \\ b_1 \quad a_3 \end{array} + \begin{array}{c} a_2 \quad a_1 \\ | \quad | \\ \hline | \quad | \\ b_2 + a_3 \quad a_3 \end{array} \right.$$

$$+ \begin{array}{c} b_3 \quad a_1 \\ | \quad | \\ \hline | \quad | \\ a_3 \quad a_2 \end{array} + \begin{array}{c} a_1 \quad a_2 \\ | \quad | \\ \hline | \quad | \\ a_2 \quad a_1 \end{array} \left. \right).$$

4.2.5. End of the computation of  $R(\varphi)$

It follows from (4.3) and (4.10) that

$$\begin{aligned}
 & [\tau_1(i), r_3^\theta(k)] \\
 & \equiv \frac{1}{2} \left[ \begin{array}{c} a_2 \\ | \\ a_1 \text{---} a_3 \end{array}, \begin{array}{c} b_3 \quad a_1 \quad a_1 \\ | \quad | \quad | \\ a_3 \quad a_2 \end{array} + \begin{array}{c} b_3 \quad a_2 \quad a_2 \\ | \quad | \quad | \\ a_3 \quad a_1 \end{array} + \begin{array}{c} b_3 \quad a_2 \quad a_1 \\ | \quad | \quad | \\ a_3 \quad b_1 \end{array} + \begin{array}{c} b_3 \quad a_1 \quad a_2 \\ | \quad | \quad | \\ a_3 \quad b_2 \end{array} \right] \\
 & \equiv \frac{1}{2} \left( \begin{array}{c} a_2 \quad a_3 \quad a_1 \quad a_1 \\ | \quad | \quad | \quad | \\ a_1 \quad a_2 \end{array} + \begin{array}{c} a_2 \quad a_3 \quad a_2 \quad a_2 \\ | \quad | \quad | \quad | \\ a_1 \quad a_1 \end{array} + \begin{array}{c} a_2 \quad a_3 \quad a_2 \quad a_1 \\ | \quad | \quad | \quad | \\ a_1 \quad b_1 \end{array} + \begin{array}{c} a_2 \quad a_3 \quad a_1 \quad a_2 \\ | \quad | \quad | \quad | \\ a_1 \quad b_2 \end{array} \right. \\
 & \qquad \qquad \qquad \left. + \begin{array}{c} b_3 \quad a_2 \quad a_1 \quad a_2 \\ | \quad | \quad | \quad | \\ a_3 \quad a_3 \end{array} + \begin{array}{c} b_3 \quad a_1 \quad a_2 \quad a_3 \\ | \quad | \quad | \quad | \\ a_3 \quad a_1 \end{array} \right).
 \end{aligned}$$

Besides, it follows from (4.4) and (4.14) that

$$\begin{aligned}
 & [r_2^\theta(i), \tau_2(k)] \\
 & \equiv \frac{1}{2} \left[ \begin{array}{c} a_1 \quad a_1 + a_2 \\ | \quad | \\ a_2 \quad a_3 \end{array} + \begin{array}{c} a_1 \quad a_2 \\ | \quad | \\ b_1 \quad a_3 \end{array} + \begin{array}{c} a_2 \quad a_1 \\ | \quad | \\ b_2 + a_3 \quad a_3 \end{array}, \begin{array}{c} b_3 \quad a_1 \\ | \quad | \\ a_3 \quad a_2 \end{array} \right] \\
 & \equiv \frac{1}{2} \left( \begin{array}{c} a_1 \quad a_1 \quad a_3 \quad a_1 \\ | \quad | \quad | \quad | \\ a_2 \quad a_2 \end{array} + \begin{array}{c} a_1 \quad a_2 \quad a_3 \quad a_1 \\ | \quad | \quad | \quad | \\ a_2 \quad a_2 \end{array} + \begin{array}{c} a_1 \quad a_2 \quad a_3 \quad a_1 \\ | \quad | \quad | \quad | \\ b_1 \quad a_2 \end{array} + \begin{array}{c} b_3 \quad a_2 \quad a_1 \quad a_2 \\ | \quad | \quad | \quad | \\ a_3 \quad a_3 \end{array} \right. \\
 & \qquad \qquad \qquad \left. + \begin{array}{c} a_2 \quad a_1 \quad a_3 \quad a_1 \\ | \quad | \quad | \quad | \\ b_2 \quad a_2 \end{array} + \begin{array}{c} a_2 \quad a_1 \quad a_3 \quad a_1 \\ | \quad | \quad | \quad | \\ a_3 \quad a_2 \end{array} + \begin{array}{c} a_2 \quad a_3 \quad a_2 \quad a_1 \\ | \quad | \quad | \quad | \\ a_1 \quad a_3 \end{array} + \begin{array}{c} a_3 \quad a_1 \quad a_2 \quad a_1 \\ | \quad | \quad | \quad | \\ b_3 \quad a_3 \end{array} \right).
 \end{aligned}$$

We deduce from (4.6) that

$$r_4^\theta(\varphi) \equiv \frac{1}{2} \left( \begin{array}{c} a_2 \quad a_1 \quad a_3 \quad a_1 \\ | \quad | \quad | \quad | \\ a_3 \quad a_2 \end{array} + \begin{array}{c} a_2 \quad a_3 \quad a_2 \quad a_1 \\ | \quad | \quad | \quad | \\ a_1 \quad a_3 \end{array} \right) \equiv \frac{1}{2} \begin{array}{c} a_2 \quad a_3 \quad a_3 \quad a_1 \\ | \quad | \quad | \quad | \\ a_1 \quad a_2 \end{array}$$

where the last congruence follows from the IHX relation. Thus we have proved (4.1).

**4.3. Another proof of Theorem B**

Our second proof of Theorem B is based on 3-dimensional topology and, to be more specific, on the clasper calculus in homology cylinders. It is

inspired by the arguments of Nozaki, Sato and Suzuki [37] to prove Theorem A in the closed case. (See [37, Section 5.4] in connection to this.) But, in contrast to [37], our arguments do not involve any computation of the LMO homomorphism.

To be more specific, still assuming that the surface  $\Sigma$  has genus  $g \geq 3$ , we will show here the existence of an element  $\varphi' \in \Gamma_3 \mathcal{I}$  of the form

$$\varphi' = [\varphi_1, [\varphi_2, \varphi_3]]$$

where  $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{I}$  will be required to satisfy certain properties. It will follow from these properties that  $\varphi'$  belongs to  $\mathcal{M}[4]$  and satisfies

$$(4.15) \quad R(\varphi') = \frac{1}{2} \begin{array}{cccc} a_2 & a_1 & a_1 & a_2 \\ | & | & | & | \\ \hline & & & \\ | & & & | \\ a_3 & & & a_3 \end{array} \pmod{1}.$$

Then, Theorem B is proved with  $\varphi'$  exactly as we did in Section 4.2 with the first element  $\varphi$ .

Let  $\mathcal{C} := \mathcal{C}(\Sigma)$  be the monoid of *homology cobordisms* over  $\Sigma$ : the reader is referred to [17] for a survey. Recall that the mapping class group embeds into this monoid via the *mapping cylinder* construction

$$\mathbf{c} : \mathcal{M} \longrightarrow \mathcal{C}$$

and that most of the constructions outlined in Section 2.2, Section 2.3 for  $\mathcal{M}$  can be extended to  $\mathcal{C}$ . Thus, we have at our disposal the Johnson filtration

$$(4.16) \quad \mathcal{C} \supset \underbrace{\mathcal{C}[1]}_{\mathcal{IC}} \supset \underbrace{\mathcal{C}[2]}_{\mathcal{KC}} \supset \cdots \supset \mathcal{C}[k] \supset \mathcal{C}[k+1] \supset \cdots$$

and, for every  $k \geq 1$ , the  $k$ -th Johnson homomorphism  $\tau_k : \mathcal{C}[k] \rightarrow \mathcal{D}_k(H)$  which, by work of Garoufalidis and Levine [11], is surjective. The submonoid  $\mathcal{C}[1] = \mathcal{IC}$  consists of *homology cylinders* over  $\Sigma$ , and the infinitesimal Dehn–Nielsen representation  $r^\theta : \mathcal{IC} \rightarrow \widehat{\mathcal{T}}(H^\mathbb{Q})$  is defined on this monoid. Besides, the map  $R_\circ : \mathcal{K} \rightarrow \mathcal{T}_4(H^\mathbb{Q})/\mathcal{T}_4(H)$  that has been defined in Remark 2.9 as a variation of  $R$  extends to  $\mathcal{KC}$  by the same formula (2.20), and the same arguments show that the resulting map

$$R_\circ : \mathcal{KC} \longrightarrow \frac{\mathcal{T}_4(H^\mathbb{Q})}{\mathcal{T}_4(H)}$$

is a monoid homomorphism.

To go further, we shall need the  $Y_k$ -equivalence relations on  $\mathcal{C}$  that have been introduced by Goussarov [12] and Habiro [15]. Recall that two cobordisms  $M, M' \in \mathcal{C}$  are  $Y_k$ -*equivalent* if there is a (compact, connected, oriented) surface  $S \subset \text{int}(M)$  with one boundary component, and an element

$s \in \Gamma_k \mathcal{I}(S)$ , such that  $M'$  is obtained by cutting open  $M$  along  $S$  and gluing it back with  $s$ . We need the following facts about these equivalence relations (see the survey paper [17], and references therein):

- For every  $k \geq 1$ , the  $Y_k$ -equivalence relation is generated by surgeries along connected graph claspers with  $k$  nodes (using the terminology of [15]).
- Denoting by  $Y_k \mathcal{I}C$  the  $Y_k$ -equivalence class of the trivial cylinder  $U := \Sigma \times [-1, +1]$ , we obtain a decreasing sequence of submonoids

$$\mathcal{I}C = Y_1 \mathcal{I}C \supset Y_2 \mathcal{I}C \supset \dots \supset Y_k \mathcal{I}C \supset Y_{k+1} \mathcal{I}C \supset \dots$$

which is called the  $Y$ -filtration and is smaller than the Johnson filtration (4.16).

- For every  $k \geq 1$ , the quotient monoid  $\mathcal{I}C/Y_k$  is a group and, for all  $\ell, \ell' \geq 1$ , we have

$$\left[ \frac{Y_\ell \mathcal{I}C}{Y_k}, \frac{Y_{\ell'} \mathcal{I}C}{Y_k} \right] \subset \frac{Y_{\ell+\ell'} \mathcal{I}C}{Y_k}.$$

- The associated graded of the  $Y$ -filtration, i.e. the direct sum of abelian groups

$$\text{Gr}^Y \mathcal{I}C := \bigoplus_{k=1}^{+\infty} \frac{Y_k \mathcal{I}C}{Y_{k+1}},$$

is a graded Lie ring whose Lie bracket is induced by group commutators.

- The graded Lie ring  $\text{Gr}^Y \mathcal{I}C$  can be “approximated” by a space of *Jacobi diagrams* in the following way. A *Jacobi diagram* is a finite and univalent graph, whose trivalent vertices are oriented and whose univalent vertices are colored by the finite set

$$\{1^+, \dots, g^+\} \cup \{1^-, \dots, g^-\}.$$

The *degree* of a Jacobi diagram is the number of its trivalent vertices. Let  $\mathcal{A}^Y$  be the graded abelian group generated by Jacobi diagrams, subject to the AS and IHX relations as presented in Section 2.1. Equipped with the multiplication  $\star$  defined by

$$D \star E := \sum \left( \begin{array}{l} \text{all possible ways of gluing some of the } i^+ \text{-vertices of } D \\ \text{with some of the } i^- \text{-vertices of } E, \text{ for all } i \in \{1, \dots, g\} \end{array} \right),$$

the graded abelian group  $\mathcal{A}^Y$  is a graded ring. Furthermore, equipped with the bracket  $[D, E]_\star := D \star E - E \star D$ , the subgroup



$\mathcal{A}^{Y,c}$  of  $\mathcal{A}^Y$  spanned by connected Jacobi diagrams is a graded Lie ring. Then, there is a homomorphism of graded Lie rings [2, 11]

$$(4.17) \quad \psi : \mathcal{A}^{Y,c} \longrightarrow \mathrm{Gr}^Y \mathcal{IC}$$

defined by  $\psi(D) := (-1)^{\chi(D)} \cdot (U_{\bar{D}} \bmod Y_{k+1})$  for any connected Jacobi diagram  $D$  of degree  $k$ , where  $\chi(D)$  is the Euler characteristic of  $D$  and  $\bar{D}$  is a graph clasper in the trivial cylinder  $U$  “realizing”  $D$ : in particular, every univalent vertex of  $D$  is “realized” by a leaf of  $\bar{D}$  which is a push-off (in the interior of  $U$ ) of the framed curve  $\alpha_i \subset \Sigma \times \{-1\}$  (resp.  $\beta_i \subset \Sigma \times \{+1\}$ ) if the color of that vertex is  $i^-$  (resp.  $i^+$ ).

We can now prove the following.

LEMMA 4.3. — *The monoid homomorphism  $R_o : \mathcal{KC} \rightarrow \mathcal{T}_4(H^{\mathbb{Q}})/\mathcal{T}_4(H)$  factorizes to a group homomorphism  $R_o : \mathcal{KC}/Y_4 \rightarrow \mathcal{T}_4(H^{\mathbb{Q}})/\mathcal{T}_4(H)$ .*

*Proof.* — Let  $M \in \mathcal{KC}$  and let  $G \subset \mathrm{int}(M)$  be a connected graph clasper with 4 nodes. Since the  $Y_4$ -equivalence is generated by surgeries of the type  $M \rightsquigarrow M_G$ , it is enough to prove that

$$(4.18) \quad R_o(M_G) = R_o(M).$$

Let  $N_+$  be a collar neighborhood of the “top” boundary of the cobordism  $M$ . Since  $M$  is a homology cobordism, each leaf of  $G$  is homologous to a framed knot contained in  $N_+$ . Hence, by standard techniques of clasper calculus, we can find another connected graph clasper  $G' \subset N_+$  with 4 nodes such that

$$(4.19) \quad M_G \sim_{Y_5} M_{G'}.$$

Since the map  $R_o$  is determined by  $r_{[2,4]}^\theta$ , it is determined by the action of  $\mathcal{KC}$  on the 5-th nilpotent quotient  $\pi/\Gamma_6\pi$  of  $\pi$ . Hence  $R_o$  factorizes through the  $Y_5$ -equivalence, and we deduce from (4.19) that

$$R_o(M_G) = R_o(M_{G'}).$$

Identify  $N_+$  with the trivial cylinder  $U$  using the “top” boundary parametrization of  $M$ : then  $G' \subset N_+$  corresponds to yet another graph clasper  $G'' \subset U$  with 4 nodes. Thus we obtain

$$R_o(M_G) = R_o(M \circ U_{G''}) = R_o(M) + R_o(U_{G''}).$$

Besides,  $R_o(U_{G''}) = (\tau_4(U_{G''}) \bmod 1)$  is trivial, because  $\tau_4(U_{G''}) \in \mathcal{T}_4(H^{\mathbb{Q}})$  is 0 if the Jacobi diagram  $\underline{G''}$  underlying  $G''$  is looped and is equal to  $\underline{G''} \in \mathcal{T}_4(H)$  otherwise [11]. Thus, we conclude to (4.18).  $\square$

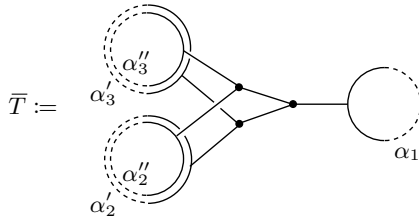
Consider the following Jacobi diagrams of degree 1:

$$T_1 := \begin{array}{c} 1^- \\ \diagup \quad \diagdown \\ 2^- \quad 3^- \end{array}, \quad T_2 := \begin{array}{c} 1^- \\ \diagup \quad \diagdown \\ 1^+ \quad 2^- \end{array}, \quad T_3 := \begin{array}{c} 2^- \\ \diagup \quad \diagdown \\ 2^+ \quad 3^- \end{array} \in \mathcal{A}_1^Y,$$

and note that

$$[T_1, [T_2, T_3]_{\star}]_{\star} = \underbrace{\begin{array}{c} 2^- \quad 1^- \quad 2^- \\ | \quad | \quad | \\ \hline 3^- \quad 3^- \end{array}}_{T:=} \in \mathcal{A}_3^Y.$$

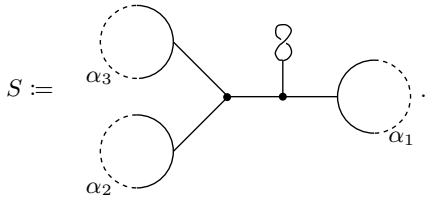
Consider now a graph clasper in the trivial cylinder  $U = \Sigma \times [-1, +1]$  of the following form:



where  $\alpha_1$  denotes a push-off of the framed curve  $\alpha_1 \subset \Sigma \times \{-1\}$  and, for  $i \in \{2, 3\}$ ,  $\alpha'_i, \alpha''_i$  denote parallel copies of a push-off of the framed curve  $\alpha_i \subset \Sigma \times \{-1\}$ ; note that

$$\tau_3(U_{\bar{T}}) = T|_{i \rightarrow \alpha_i} = 0 \quad (\text{by the AS relation})$$

so that  $U_{\bar{T}}$  belongs to  $\mathcal{C}[4]$ . Consider also a graph clasper in  $U$  of the following form:



By a result of Conant, Schneiderman and Teichner [5, proof of Lemma 33], the homology cylinders  $U_{\bar{T}}$  and  $U_S$  are, up to surgeries along graph claspers with 4 nodes, related by a 4-dimensional homology cobordism. Since  $R_o$  is invariant under 4-dimensional homology cobordism (because  $r^\theta$  is so) and since  $R_o$  is invariant under the  $Y_4$ -equivalence too (by Lemma 4.3), we deduce that

$$(4.20) \quad R_o(U_{\bar{T}}) = R_o(U_S).$$

It follows also from [5, Section 3.8] (or, alternatively, from [25, Theorem B] which is more general) that

$$(4.21) \quad \tau_4(U_S) = \frac{1}{2} \begin{array}{c} a_2 \quad a_1 \quad a_1 \quad a_2 \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ a_3 \quad \quad \quad a_3 \end{array} .$$

Besides, since the map (4.17) is a Lie homomorphism and  $\bar{T}$  is a “realization” of  $T$ , we have

$$[\psi_1(T_1), [\psi_1(T_2), \psi_1(T_3)]] = \psi_3(T) = -(U_{\bar{T}} \text{ mod } Y_4).$$

Next, since  $\mathbf{c} : \mathcal{I}/\Gamma_2\mathcal{I} \rightarrow \mathcal{IC}/Y_2\mathcal{IC}$  is an isomorphism in genus  $g \geq 3$  [28], we can find  $\varphi_i \in \mathcal{I}$  such that

$$(4.22) \quad \psi_1(T_i) = (\mathbf{c}(\varphi_i) \text{ mod } Y_2).$$

So, the mapping cylinder of the inverse of  $\varphi' := [\varphi_1, [\varphi_2, \varphi_3]] \in \Gamma_3\mathcal{I}$  is  $Y_4$ -equivalent to  $U_{\bar{T}}$ : in particular, we have  $\tau_3(\varphi') = -\tau_3(U_{\bar{T}}) = 0$  so that  $\varphi' \in \mathcal{M}[4]$ . Finally, thanks to Lemma 4.3, we conclude from (4.20) and (4.21) that

$$R_\circ(\varphi') = R_\circ(U_{\bar{T}}) = \frac{1}{2} \begin{array}{c} a_2 \quad a_1 \quad a_1 \quad a_2 \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ a_3 \quad \quad \quad a_3 \end{array} \text{ mod } 1.$$

*Remark 4.4.* — We can give an explicit example of an element  $\varphi_i \in \mathcal{I}$  satisfying (4.22) for  $i \in \{1, 2, 3\}$ , which leads to an explicit formula for  $\varphi' = [\varphi_1, [\varphi_2, \varphi_3]]$ . Indeed, according to [28, Theorem 1.3], the property (4.22) is equivalent to the double condition

$$(4.23) \quad \tau_1(\varphi_i) = \tau_1(\psi(T_i)) \in \Lambda^3 H \quad \text{and} \quad \beta(\varphi_i) = \beta(\psi(T_i)) \in B_{\leq 3},$$

where  $\beta$  denotes the Birman–Craggs homomorphism with values in the space  $B_{\leq 3}$  of cubic boolean functions on the space  $\text{Spin}(\Sigma)$  of spin structures on  $\Sigma$ . By [28, Lemma 4.22] and [28, Lemma 4.23], respectively, we have

$$\tau_1(\psi(T_1)) = \begin{array}{c} a_2 \\ | \\ a_1 \text{---} a_3 \end{array}, \quad \tau_1(\psi(T_2)) = \begin{array}{c} b_1 \\ | \\ a_1 \text{---} a_2 \end{array}, \quad \tau_1(\psi(T_3)) = \begin{array}{c} b_2 \\ | \\ a_2 \text{---} a_3 \end{array}$$

and

$$\beta(\psi(T_1)) = \overline{a_2} \cdot \overline{a_1} \cdot \overline{a_3}, \quad \beta(\psi(T_2)) = \overline{b_1} \cdot \overline{a_1} \cdot \overline{a_2}, \quad \beta(\psi(T_3)) = \overline{b_2} \cdot \overline{a_2} \cdot \overline{a_3}.$$

Here, identifying  $\text{Spin}(\Sigma)$  with the space  $\mathcal{Q}$  of quadratic functions  $H \otimes \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$  whose polar form is the mod 2 intersection form of the surface, we associate to any  $z \in H$  the affine boolean map  $\bar{z} : \mathcal{Q} \rightarrow \mathbb{Z}_2$  defined by

$\bar{z}(q) := q(z \otimes 1)$ . Thus, using [19, Lemma 4.B] and [20, Section 7], we see that the following instances of  $\varphi_1, \varphi_2, \varphi_3$  satisfy (4.23):

$$\varphi_1 := i(T_{d'}T_{d''}), \quad \varphi_2 := (T_{e^+}T_{e^-}^{-1})T_f, \quad \varphi_3 := (T_{u^+}T_{u^-}^{-1})T_v.$$

Here  $i$  is the product of two bounding pair maps defined by (4.2) and  $T_{d'}, T_{d''}$  are the separating twists along the curves  $d', d''$  given by Figure 4.3, thus defining  $\varphi_1$ ; also,  $\varphi_2$  and  $\varphi_3$  are defined as products of a bounding pair map and a separating twist, whose curves are also shown in Figure 4.3.

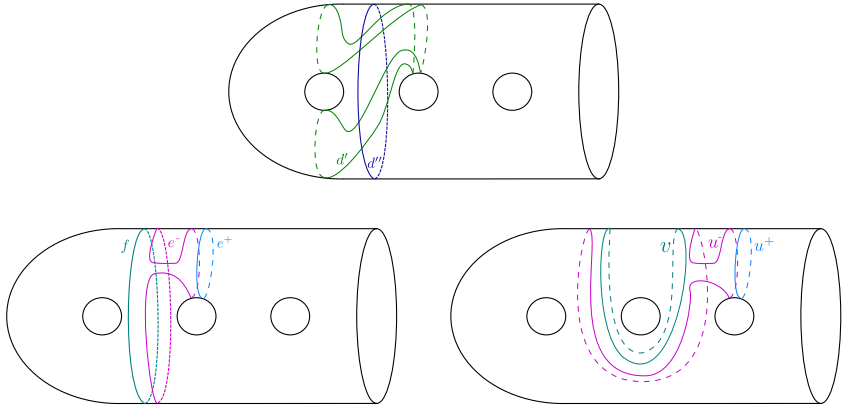


Figure 4.3. The curves defining instances of  $\varphi_1, \varphi_2$  and  $\varphi_3$

### 4.4. Complements

We conclude by discussing the size of the torsion subgroup of the abelianized Johnson kernel. For that, we will give a lower bound on the size of the image of the map  $R_{ab} : \mathcal{K}_{ab} \rightarrow \mathcal{T}_4(H^{\mathbb{Q}})/\mathcal{T}_4(H)$ :

$$(4.24) \quad |\text{Tors}(\mathcal{K}_{ab})| \geq |R_{ab}(\text{Tors}(\mathcal{K}_{ab}))|.$$

Note that the above inequality may be strict. Indeed we have the inclusions of groups

$$\{0\} \subset \frac{\Gamma_4\mathcal{I} \cdot \mathcal{K}'}{\mathcal{K}'} \subset \underbrace{\frac{\sqrt{\Gamma_4\mathcal{I}}}{\mathcal{K}'}}_{=\text{Tors}(\mathcal{K}_{ab})} \subset \frac{\mathcal{M}[4]}{\mathcal{K}'} \subset \mathcal{K}_{ab}$$

where the description of  $\text{Tors}(\mathcal{K}_{ab})$  has been justified just after (3.10). Since  $\tau_4$  maps  $\Gamma_4\mathcal{I}$  to  $\mathcal{T}_4(H) \subset \mathcal{T}_4(H^{\mathbb{Q}})$ , we deduce from Lemma 2.10 that  $R_{ab}$  vanishes on  $(\Gamma_4\mathcal{I} \cdot \mathcal{K}')/\mathcal{K}'$ . Hence, if  $\Gamma_4\mathcal{I}$  is not included in  $\mathcal{K}'$  (which the authors ignore), then  $R_{ab}$  is not enough to detect all the torsion of  $\mathcal{K}_{ab}$ .

*Remark 4.5.* — Actually, the authors do not even know whether  $R_{\text{ab}}$  has only 2-torsion.

To obtain a lower bound on the size of the image of  $R_{\text{ab}}$ , we use Lemma 2.10, the  $\mathcal{M}$ -equivariance property of  $\tau_4$  and formula (4.1):

$$(4.25) \quad |R_{\text{ab}}(\text{Tors}(\mathcal{K}_{\text{ab}}))| \geq |R_{\text{ab}}(\langle \varphi \rangle_{\mathcal{M}})| = |\varpi\tau_4(\langle \varphi \rangle_{\mathcal{M}})| \\ = |\langle [[a_1, a_2], a_3] \rangle_{\text{Sp}(H)}|.$$

Here  $\langle \varphi \rangle_{\mathcal{M}}$  is the  $\mathcal{M}$ -submodule of  $\mathcal{K}_{\text{ab}}$  generated by  $\{\varphi\}$  and  $\langle [[a_1, a_2], a_3] \rangle_{\text{Sp}(H)}$  is the  $\text{Sp}(H)$ -submodule of  $\mathfrak{L}_3 \otimes \mathbb{Z}_2$  generated by  $[[a_1, a_2], a_3]$ . (Here and in the sequel, we simply denote by  $x$  the element  $x \otimes 1$  of  $\mathfrak{L}_3 \otimes \mathbb{Z}_2$  defined by  $x \in \mathfrak{L}_3$ .)

LEMMA 4.6. — We have a split short exact sequence of  $\text{Sp}(H)$ -modules

$$0 \longrightarrow H \otimes \mathbb{Z}_2 \xrightarrow{\quad \begin{array}{c} [\omega, -] \\ \swarrow \quad \searrow \\ \mathfrak{L}_3 \otimes \mathbb{Z}_2 \end{array} \quad} \mathfrak{L}_3 \otimes \mathbb{Z}_2 \xrightarrow{\quad p \quad} \widehat{\mathfrak{L}}_3 \otimes \mathbb{Z}_2 \longrightarrow 0,$$

$\underbrace{\hspace{10em}}_{\zeta}$

where  $p$  is the canonical projection and  $\zeta : \mathfrak{L}_3 \otimes \mathbb{Z}_2 \rightarrow H \otimes \mathbb{Z}_2$  is defined by  $\zeta([[a, b], c]) = \omega(b, c)a + \omega(a, c)b$ .

*Proof.* — By definition of  $\widehat{\mathfrak{L}}$ , we have a short exact sequence  $0 \rightarrow H \rightarrow \mathfrak{L}_3 \rightarrow \widehat{\mathfrak{L}}_3 \rightarrow 0$  and, since  $\widehat{\mathfrak{L}}_3$  is torsion-free, this sequence remains exact after tensorization with  $\mathbb{Z}_2$ . Regarding  $\mathfrak{L}_3$  as a quotient of  $H^{\otimes 3}$ , we easily verify that  $\zeta$  is well-defined. Besides, we have  $\zeta([\omega, h]) = h$  for all  $h \in H$ , showing that the short exact sequence is split.  $\square$

LEMMA 4.7. — With the notations of Lemma 4.6 and for  $g \geq 3$ , we have  $\ker \zeta = \langle [[a_1, a_2], a_3] \rangle_{\text{Sp}(H)}$ .

*Proof.* — Set  $S := \langle [[a_1, a_2], a_3] \rangle_{\text{Sp}(H)}$ . The  $\text{Sp}(H)$ -equivariance of  $\zeta$  implies that  $S \subset \ker \zeta$ . To prove the converse inclusion, we consider the projection  $q : \mathfrak{L}_3 \otimes \mathbb{Z}_2 \rightarrow \ker \zeta$  corresponding to the split short exact sequence of Lemma 4.7; specifically, we have

$$q(x) = x + [\omega, \zeta(x)], \quad \forall x \in H.$$

Let  $Z = \{a_1, \dots, a_g, b_1, \dots, b_g\}$  and, for all  $z, z' \in Z$ , let us write  $z \perp z'$  if we have  $\{z, z'\} = \{a_i, b_i\}$  for some  $i$ . Clearly, for any  $z, z', z'' \in Z$  with  $z' \neq z''$ , we have

$$q(\langle [z', z''], z \rangle) = \begin{cases} \langle [z', z''], z \rangle & \text{if } (z \not\perp z' \text{ and } z \not\perp z'') \quad \text{(i)} \\ \langle [z', z''], z \rangle + [\omega, z''] & \text{if } z \perp z' \quad \text{(ii)} \\ \langle [z', z''], z \rangle + [\omega, z'] & \text{if } z \perp z'' \quad \text{(iii)}. \end{cases}$$

Since  $\mathfrak{L}_3 \otimes \mathbb{Z}_2$  is generated by the elements  $[[z', z''], z]$ , for all  $z, z', z'' \in Z$  with  $z' \neq z''$ , we deduce that  $\ker \zeta$  is generated by the above elements of types (i), (ii) and (iii). Hence, we are reduced to show that any element of type (i) and any element of type (ii) belong to  $S$ . For that, we will use the following elements of  $\text{Sp}(H)$ :

- for  $x \in H$ ,  $T_x$  is the transvection defined by  $T_x(h) = h + \omega(x, h)x$ ;
- for any  $r, s \in \{1, \dots, g\}$  with  $r \neq s$ ,  $E_{rs}$  exchanges  $a_r$  (resp.  $b_r$ ) with  $a_s$  (resp.  $b_s$ ) and fixes all other elements of  $Z$ ;
- for each  $r \in \{1, \dots, g\}$ ,  $F_r$  maps  $a_r$  (resp.  $b_r$ ) to  $-b_r$  (resp.  $a_r$ ) and fixes all other elements of  $Z$ .

We start by considering the elements of type (i). They can be of the following forms:

- $[[a_i, a_j], a_k]$  with  $(i, j, k$  pairwise disjoint) or  $(k = j$  and  $i \neq j)$ ,
- $[[a_i, b_j], a_k]$  with  $(i, j, k$  pairwise disjoint) or  $(k = i$  and  $i \neq j)$   
or  $(j = i$  and  $k \neq i)$ ,
- $[[b_i, b_j], a_k]$  with  $(i, j, k$  pairwise disjoint),
- $[[b_i, b_j], b_k]$  with  $(i, j, k$  pairwise disjoint) or  $(k = j$  and  $i \neq j)$ ,
- $[[b_i, a_j], b_k]$  with  $(i, j, k$  pairwise disjoint) or  $(k = i$  and  $i \neq j)$   
or  $(j = i$  and  $k \neq i)$ ,
- $[[a_i, a_j], b_k]$  with  $(i, j, k$  pairwise disjoint).

The last three forms can be derived from the first three forms by applying symplectic transformations of type  $F_r$ : hence it suffices to consider the first three forms. We have

$$T_{b_1} \cdot [[a_1, a_2], a_3] = [[a_1, a_2], a_3] + [[b_1, a_2], a_3]$$

and

$$T_{b_2} \cdot [[b_1, a_2], a_3] = [[b_1, a_2], a_3] + [[b_1, b_2], a_3],$$

hence we obtain that  $[[b_1, a_2], a_3] \in S$  and  $[[b_1, b_2], a_3] \in S$ . By using the transformations of type  $E_{rs}$ , we deduce that all  $[[a_i, a_j], a_k]$ ,  $[[a_i, b_j], a_k]$  and  $[[b_i, b_j], a_k]$  with  $i, j, k$  pairwise disjoint, belong to  $S$ . Furthermore, we have

$$T_{a_2+a_3} \cdot [[a_1, a_2], b_3] = [[a_1, a_2], b_3] + [[a_1, a_2], a_2] + [[a_1, a_2], a_3]$$

and

$$F_1 \cdot [[a_1, a_2], a_2] = [[b_1, a_2], a_2],$$

hence we obtain that  $[[a_1, a_2], a_2] \in S$  and  $[[b_1, a_2], a_2] \in S$ . By using the transformations of type  $E_{rs}$ , we deduce that  $[[a_i, a_j], a_j] \in S$  and  $[[a_i, b_j], a_i] \in S$  for all  $i \neq j$ . Finally, we have

$$T_{b_1} \cdot [[a_1, a_2], a_1] = [[a_1, a_2], a_1] + [[b_1, a_2], b_1] + \underbrace{[[b_1, a_2], a_1] + [[a_1, a_2], b_1]}_{= [[a_1, b_1], a_2]}$$

and we deduce that  $[[a_1, b_1], a_2] \in S$ : hence we have  $[[a_i, b_i], a_k] \in S$  for all  $i \neq k$ . Thus we have checked that all elements of type (i) belong to  $S$ .

We now consider the elements of type (ii). They can be of the following forms:

$$\begin{aligned} q([[a_i, b_i], b_i]), \quad q([[b_i, a_j], a_i]) \text{ with } i \neq j, \quad q([[a_i, a_j], b_i]) \text{ with } i \neq j, \\ q([[b_i, a_i], a_i]), \quad q([[a_i, b_j], b_i]) \text{ with } i \neq j, \quad q([[b_i, b_j], a_i]) \text{ with } i \neq j. \end{aligned}$$

The last three forms can be derived from the first three forms by applying symplectic transformations of type  $F_r$ : hence it suffices to consider the first three forms. In the sequel, we denote by  $\equiv$  the congruence modulo  $S$ . First, note that

$$q([[a_i, b_i], b_i]) = [[a_i, b_i], b_i] + [\omega, b_i] \equiv [[a_i, b_i], b_i] + [[a_i, b_i], b_i] = 0$$

where the congruence follows from the consideration of elements of type (i). Second, we have

$$q([[b_i, a_j], a_i]) = [[b_i, a_j], a_i] + [\omega, a_j] \equiv [[b_i, a_j], a_i] + [[a_j, b_j], a_j].$$

Let  $G_{ij}$  be the symplectic transformation of  $H$  that maps  $a_i$  to  $a_i + a_j$ ,  $b_i$  to  $b_i$ ,  $a_j$  to  $-a_j$ ,  $b_j$  to  $b_i - b_j$ , and fixes all other elements of  $S$ . Observe that

$$\begin{aligned} G_{ij} \cdot [[a_j, b_j], a_i] &= [[a_j, b_i + b_j], a_i + a_j] \\ &= [[a_j, b_i], a_i] + [[a_j, b_i], a_j] + [[a_j, b_j], a_i] + [[a_j, b_j], a_j] \\ &\equiv [[a_j, b_i], a_i] + [[a_j, b_j], a_j] \end{aligned}$$

where the congruence follows from the consideration of elements of type (i). Thus we deduce that  $q([[b_i, a_j], a_i]) \in S$ . Third, we have

$$\begin{aligned} q([[a_i, a_j], b_i]) &= [[a_i, a_j], b_i] + [\omega, a_j] \equiv [[a_i, a_j], b_i] + [[a_j, b_j], a_j] \\ &\equiv [[a_j, b_i], a_i] + [[a_j, b_j], a_j] \end{aligned}$$

where the last congruence follows from the Jacobi identity: hence we are back to the second form of elements of type (ii).  $\square$

We can now conclude by giving lower bounds on the sizes of the torsion parts of the abelianized Johnson kernels.

PROPOSITION 4.8. — Assume that  $g \geq 6$ . The cardinality of  $\text{Tors}(\mathcal{K}_{\text{ab}})$  is at least  $2^{\frac{8}{3}(g^3-g)}$ , and the cardinality of  $\text{Tors}(\widehat{\mathcal{K}}_{\text{ab}})$  is at least  $2^{\frac{1}{3}(g^3-4g)}$ .

*Proof.* — We deduce from Lemma 4.7 that  $\langle [a_1, a_2], a_3 \rangle_{\text{Sp}(H)}$  is isomorphic to  $(\mathfrak{L}_3(H)/H) \otimes \mathbb{Z}_2$  as a  $\mathbb{Z}_2$ -vector space. Hence, thanks to (4.24) and (4.25), we obtain the lower bound

$$|\text{Tors}(\mathcal{K}_{\text{ab}})| \geq 2^{\text{rk } \mathfrak{L}_3(H) - \text{rk } H}$$

in the bordered case. In the closed case, it follows from diagram (2.26) and Remark 2.13 that

$$|\text{Tors}(\widehat{\mathcal{K}}_{\text{ab}})| \geq 2^{\text{rk } \mathfrak{L}_3(A) - \text{rk}(A)}$$

where  $A$  is the quotient  $H/\langle b_1, \dots, b_g \rangle$ .  $\square$

*Remark 4.9.* — It is expected that the lower bounds of Proposition 4.8 are far from optimal (at least in the closed case).

## Appendix A. Computation of the symplectic logansion

```

1 # Choose the genus and the nilpotency class
2
3 g=3
4 N=3
5
6
7 # The free Lie algebra on 2g generators a1,...,ag,b1,...,bg of
   nilpotency class N
8
9 L=LieAlgebra(QQ,2*g,step=N)
10 a=[L.gen(i) for i in range(g)]
11 b=[L.gen(i+g) for i in range(g)]
12
13
14 # Values of the symplectic logansion "theta" up to order N
15
16 theta_a = [ a[i] - (1/2) * L[a[i],b[i]]
17 + (1/12) * L[L[a[i],b[i]],b[i]] - (1/24)* L[a[i],L[a[i],L[a[i],b[i]]]]
18 - (1/2) * L[ sum( L[a[j],b[j]] for j in range(i) ) , a[i] ]
19 + (1/4) * L[ sum( L[a[j],b[j]] for j in range(i) ) , L[a[i],b[i]] ]
20 for i in range(g) ]
21
22 theta_b = [ b[i] - (1/2) * L[a[i],b[i]] + (1/4) * L[L[a[i],b[i]],b[i]]
23 + (1/12) * L[a[i],L[a[i],b[i]]] - (1/24)* L[L[L[a[i],b[i]],b[i]],b[i]]
24 + (1/2) * L[ b[i], sum( L[a[j],b[j]] for j in range(i) ) ]
25 + (1/4) * L[ sum( L[a[j],b[j]] for j in range(i) ) , L[a[i],b[i]] ]

```



```

26 for i in range(g) ]
27
28
29 # Computation of theta from a string such a 'a1+b2-a1-' which encodes an
    element of the fundamental group
30
31 def theta(lis):
32     res = L(0)
33     for j in range(len(lis)/3):
34         index = int(lis[3*j+1])-1
35         if [lis[3*j],lis[3*j+2]]=='a,+': res = L.bch(res,theta_a
    [index])
36         if [lis[3*j],lis[3*j+2]]=='a,-': res = L.bch(res,-
    theta_a[index])
37         if [lis[3*j],lis[3*j+2]]=='b,+': res = L.bch(res,theta_b
    [index])
38         if [lis[3*j],lis[3*j+2]]=='b,-': res = L.bch(res,-
    theta_b[index])
39     return res
40
41
42 # Creation of strings corresponding to commutators in the fundamental
    group
43
44 def invert(lis):
45     w=''
46     for j in range(len(lis)/3):
47         if lis[3*j+2]=='+' :
48             w = lis[3*j]+lis[3*j+1]+'-'+w
49         else:
50             w = lis[3*j]+lis[3*j+1]+''+w
51     return w
52
53 def comm(a,b):
54     return a+b+invert(a)+invert(b)
55
56
57 # Checks that the logansion theta is symplectic up to degree N
58
59 boundary = ''
60 for i in range(g): boundary = boundary + 'b' + str(i+1) + '-' + 'a' +
    str(i+1) + '+' + 'b' + str(i+1) + '+' + 'a' + str(i+1) + '-'
61
62 omegatilde = theta(boundary)
63 omega = sum(L[a[i],b[i]] for i in range(g))
64
65 if omegatilde==omega:
66     print('OK: the expansion is symplectic up to order '+str(N))
67 else:
68     print('Warning: the given expansion is not symplectic up to the
    given order !')
```

```

69
70
71 # Display of an element of the nilpotent free Lie algebra
72
73 import sage.combinat.words.lyndon_word as lyndon_word
74
75 def transform(lis):
76     if lis in [1..g]:
77         return 'a'+str(lis)
78     if lis in [(g+1)..(2*g)]:
79         return 'b'+str(lis-g)
80     if len(lis)==1:
81         return transform(lis[0])
82     else:
83         return '[' + (transform([lis[0]])) + ',' + (transform(lis[1])) +']'
84
85 def display(x):
86     Llist = (L.basis()).list()
87     LW = LyndonWords(2*g,1).list()
88     for i in [2..N]:
89         LW = LW + LyndonWords(2*g,i).list()
90     if x == 0:
91         return '0'
92     else:
93         if x != x.leading_monomial():
94             z=x.leading_coefficient()
95             return display(x-z*x.leading_monomial())+'+'+str(z)+'*'+display(
x.leading_monomial())
96         else:
97             for j in range(len(Llist)):
98                 if x == Llist[j]:
99                     return transform(lyndon_word.standard_bracketing(LW[j]))

```

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