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CHAINS OF REPRODUCING KERNEL HILBERT SPACES GENERATED BY UNIMODULAR FUNCTIONS

by Masatoshi SUZUKI (*)

ABSTRACT. — We present a method to construct a chain of reproducing kernel Hilbert spaces controlled by a first-order system of differential equations from a given unimodular function satisfying several conditions. One of the applications of that method is a conditional but richly general solution to the inverse problem of recovering the structure Hamiltonian from a given de Branges space.

RÉSUMÉ. — Nous présentons une méthode pour construire une chaîne d'espaces de Hilbert à noyau reproduisant contrôlés par un système d'équations différentielles du premier ordre à partir d'une fonction unimodulaire donnée satisfaisant plusieurs conditions. L'une des applications de cette méthode est une solution conditionnelle mais richement générale au problème inverse de la récupération de l'hamiltonien de structure à partir d'un espace de Branges donné.

1. Introduction

A first-order system of differential equations called a canonical system defined by a positive-semidefinite 2×2 symmetric matrix-valued function $H(t)$ gives rise to an entire function E in the Hermite–Biehler class, which is a generalization of the exponential functions. The inverse problem of recovering $H(t)$ from a given function in the Hermite–Biehler class is difficult in general, but has been the subject of many studies because of its significance and wide applications. In this context, the construction of $H(t)$ is naturally discussed assuming that E belongs to the Hermite–Biehler class. However, at times, we need a way to construct $H(t)$ without relying on such an assumption.

Keywords: de Branges spaces, inverse problem, structure Hamiltonians, reproducing kernel Hilbert spaces, unimodular functions.
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One important example is the entire function $E_\xi(z) := \xi(1/2 - iz) + \xi'(1/2 - iz)$, where $\xi(s) := 2^{-1}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ for the Riemann zeta function $\zeta(s)$ and the gamma function $\Gamma(s)$. The entire function E_ξ belongs to the Hermite–Biehler class if and only if the Riemann hypothesis is true. Therefore, if there is a method to construct $H(t)$ corresponding to E_ξ without assuming the Riemann hypothesis, that method could be applied to the study of the Riemann hypothesis. Such a strategy was realized in [20, 21], resulting in a necessary and sufficient condition for the Riemann hypothesis formulated in terms of canonical systems.

However, the method of [20] is applicable only when the corresponding $H(t)$ is diagonal. Thus, for example, it cannot be applied to a Dirichlet L-function of a non-real Dirichlet character. The first purpose of this paper is to solve that problem and make it applicable to non-diagonal $H(t)$. The second purpose is to extend the range of applications of the theory by axiomatically rearranging the method of [19, 20, 22], which assumed conditions for concrete integral kernels. This makes it possible, for example, to handle the examples given in Section 3 in a unified manner. For these two purposes, we discuss associating a unimodular function with a chain of reproducing kernel Hilbert spaces. We explain a more specialized and technical outline in the following.

A typical source of chains of reproducing kernel Hilbert spaces is an entire function E of the Hermite–Biehler class $\overline{\mathbb{HB}}$ which consists of all entire functions satisfying

$$\left|E^\sharp(z)\right| < |E(z)|$$

in the upper half-plane $\mathbb{C}_+ = \{z \mid \Im(z) > 0\}$, where

$$F^\sharp(z) = \overline{F(\bar{z})}.$$

We denote by $\mathbb{HB}$ the subspace of $\overline{\mathbb{HB}}$ consisting of functions that have no zeros on $\mathbb{R}$. First E defines the de Branges space $\mathcal{H}(E)$, which is a reproducing kernel Hilbert space consisting of entire functions. It is well-known that the set of all de Branges subspaces $\mathcal{H}(E_t)$ of $\mathcal{H}(E)$ is totally ordered by set-theoretical inclusion and the generators $E_t \in \overline{\mathbb{HB}}$ are controlled by a canonical system, which is a system of differential equations of the form

$$-\frac{\partial}{\partial t}\begin{bmatrix} A(t,z) \\ B(t,z) \end{bmatrix} = z \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} H(t) \begin{bmatrix} A(t,z) \\ B(t,z) \end{bmatrix} \tag{1.1}$$

on an interval $t \in I \subset \mathbb{R}$ parametrized by $z \in \mathbb{C}$, where $H(t)$ is a positive-semidefinite 2×2 symmetric matrix for almost all $t \in I$, $A(t,z) = (E_t(z) + E_t^\sharp(z))/2$, and $-iB(t,z) = (E_t(z) - E_t^\sharp(z))/2$; see Woracek [25] for example.

The matrix-valued function H corresponding to E as above is called the *structure Hamiltonian* of the de Branges space $\mathcal{H}(E)$, which is unique up to a reparameterization of t and the normalization $E(0) = 1$.

The inverse problem of recovering the structure Hamiltonian of $\mathcal{H}(E)$ from E was studied by many authors after the work of de Branges (cf. [20, Section 1]), and recently, a complete characterization of structure Hamiltonians of de Branges spaces was obtained by Romanov and Woracek [16]. However, each known method of constructing H has its advantages and disadvantages, depending on its applications. In particular, as already mentioned above, the method of [20] can be applied only to diagonal Hamiltonians that are often referred to as Kreĭn's strings by the correspondence explained after Theorem 2.8. Therefore, it can be applied only to the study of so-called self-dual zeta functions.

In this paper, the above disadvantage of [20] is removed within a rather broad framework of constructing a chain of reproducing kernel Hilbert spaces from a function E which is not necessarily an entire function. Such an extension of the method of constructing H would be interesting in its own right. Moreover, Hamiltonians that cannot be obtained as structure Hamiltonians of de Branges spaces can be systematically obtained from our method. For example, let $M(z)$ be a meromorphic function on $\mathbb{C}$ having no zeros in $\mathbb{C}_+ \cup \mathbb{R}$, and let us define the spaces

$$\mathcal{J}_t(M) := e^{izt}M(z)H^2(\mathbb{C}_+) \cap \left(e^{-izt}M^\sharp(z)H^2(\mathbb{C}_-)\right)$$

for real numbers t, where $\mathbb{C}_- = \{z \mid \Im(z) < 0\}$ is the lower half-plane and $H^2(\mathbb{C}_\pm)$ are Hardy spaces on $\mathbb{C}_\pm$, respectively. Then each $\mathcal{J}_t(M)$ is a reproducing kernel Hilbert space consisting of meromorphic functions on $\mathbb{C}$. If M is an entire function $E \in \overline{\mathbb{HB}}$, we have $\mathcal{J}_t(M) = \mathcal{H}(E)$ for $t = 0$. More generally, $\mathcal{J}_t(M)$ for $t = 0$ is isomorphic to the model space $H^2(\mathbb{C}_+) \ominus (M^\sharp/M)H^2(\mathbb{C}_+)$ if $M^\sharp/M$ is an inner function in $\mathbb{C}_+$. The model space is isomorphic to a de Branges space if $M^\sharp/M$ is a meromorphic inner function in $\mathbb{C}_+$. The theory of de Branges spaces and model spaces is studied actively by numerous researchers by its importance in connection with various topics of complex and harmonic analysis (cf. Garcia, Mashreghi, and Ross [6], and also Chalendar, Fricain, and Timotin [4], Havin and Mashreghi [7]). As detailed in Section 2, the reproducing kernel of $\mathcal{J}_t(M)$ has the form

$$J(t; z, w) = \frac{\overline{A(t,z)}B(t,w) - A(t,w)\overline{B(t,z)}}{\pi(w - \bar{z})}$$

under appropriate conditions for M. Through this formula of the reproducing kernel, we see that the reproducing kernel Hilbert space $\mathcal{J}_t(M)$ is generally different from de Branges spaces $\mathcal{H}(E)$ generated by $E \in \overline{\mathbb{HB}}$ and de Branges–Rovnyak spaces $\mathcal{H}(b)$ generated by $b \in L^\infty(\mathbb{R})$. On the other hand, the above spaces ordered by inclusion $\mathcal{J}_t(M) \supset \mathcal{J}_s(M)$ for $t \leqslant s$, therefore there exists $t_0 \leqslant +\infty$ such that $\mathcal{J}_t(M) \neq \{0\}$ for every $t < t_0$ if $\mathcal{J}_t(M) \neq \{0\}$ for some $t < \infty$. The chain of spaces $\mathcal{J}_t(M)$, $t < t_0$, is controlled by a system of differential equations in the sense that there exists a 2×2 symmetric matrix-valued function $H_M(t)$ such that the functions $A(t,z)$ and $B(t,z)$ in the reproducing kernel $J(t;z,w)$ above satisfy the system (1.1) with $H(t) = H_M(t)$ for $t < t_0$ and $z \in \mathbb{C}_+$. Moreover, we find that $\lim_{t \to t_0} J(t;z,w) = 0$ if assuming some additional conditions for M. If M is an entire function belonging to $\overline{\mathbb{HB}}$, $\mathcal{J}_t(M) = \mathcal{H}(E_t)$ for $0 \leqslant t < t_0$, and $H_M(t)$ is a structure Hamiltonian of $\mathcal{H}(M)$, but $H_M(t)$ is generally not a structure Hamiltonian of a de Branges space, because $\mathcal{J}_t(M)$ is not a de Branges space in general.

Briefly stated, the method detailed in the next section is to define an abstract conjugation on $L^2(\mathbb{R})$ from a unimodular function on $\mathbb{R}$ such that it defines a family of reproducing kernel Hilbert spaces as a natural family of conjugation invariant subspaces of $L^2(\mathbb{R})$. The above spaces $\mathcal{J}_t(M)$ are obtained as Fourier transforms of such invariant subspaces.

The paper is organized as follows. In Section 2, we describe the precise settings and state the main results Theorems 2.2–2.8. Furthermore, we explain the relation with Kreĭn's inverse spectral theory for strings. In Section 3, we present some non-trivial examples of unimodular functions that satisfy some of the assumptions in the main theorems. In Section 4, we prove Theorem 2.2. In Section 5, we prove Theorems 2.3, 2.5, and 2.6. The most essential new compared to the previous works [20, 22] is the proof of Theorem 2.3. In Section 6, we prove Theorem 2.7. Then Theorem 2.8 follows as a corollary. In Section 7, we describe sufficient conditions for the sixth and the seventh of the eight assumptions in Section 2 as a complement to the main results.

2. Results

In this and subsequent sections, u represents a unimodular function in $L^1_{\mathrm{loc}}(\mathbb{R})$, that is, u is a locally integrable function on $\mathbb{R}$ satisfying $|u(z)| = 1$ for almost every $z \in \mathbb{R}$. For technical reasons, we introduce the following conditions for unimodular functions u and denote by $U^1_{\mathrm{loc}}(\mathbb{R})$ the set of all u satisfying them:

(U1) the value $u(0)$ is defined, $u(0) \neq 0$, and u is Hölder continuous at $z = 0$ with exponent $1/2 < \alpha \leqslant 1$: $|u(z) - u(0)| \ll |z|^\alpha$ as $|z| \to 0$;

(U2) there exists a domain D of $\mathbb{C}$, which contains $\mathbb{R}$ and is closed under complex conjugation, and a meromorphic function U on $D \setminus \mathbb{R}$ such that $u(z)$ is the non-tangential limit of U at z approaching from both half-planes $\mathbb{C}_+$ and $\mathbb{C}_-$ for almost all $z \in \mathbb{R}$. Then we often identify u with U.

Condition (U1) means that u is equal to such a function almost everywhere. The reason why the domain D in (U2) is assumed to be symmetric for the real line is that if U is a meromorphic function on a domain $D \subset \mathbb{C}_+$, then it extends to $\overline{D} \subset \mathbb{C}_-$ by $U(z) = 1/U^\sharp(z)\,(= 1/\overline{U(\bar{z})})$. We say that a unimodular function $u \in U^1_{\mathrm{loc}}(\mathbb{R})$ is *symmetric* if

$$u^\sharp(z) = u(-z) \quad \text{for} \quad z \in D. \tag{2.1}$$

Unimodular functions of the form $u = M^\sharp/M$ with a meromorphic function M satisfying $M^\sharp(z) = M(-z)$ or $M^\sharp(z) = -M(-z)$ are typical examples of symmetric ones.

2.1. Construction of the first-order differential systems

First, we construct systems of differential equations of type (1.1) from unimodular functions satisfying several conditions. Let F and F^{-1} be the Fourier transform and Fourier inverse transform on $L^2(\mathbb{R})$, respectively:

$$(\mathsf{F}f)(z) = \int f(x)e^{izx}\,dx, \quad \left(\mathsf{F}^{-1}F\right)(x) = \frac{1}{2\pi}\int F(z)e^{-izx}\,dz,$$

where $\int$ means integration on $\mathbb{R}$ and will always be used in this sense. Define the operations $\mathsf{J}^\sharp$ and $\mathsf{J}_\sharp$ for functions by

$$\left(\mathsf{J}^\sharp F\right)(z) := F^\sharp(z) = \overline{F(\bar{z})}, \quad (\mathsf{J}_\sharp f)(x) := \overline{f(-x)}$$

so that they satisfy the commutative relation

$$\mathsf{J}^\sharp\mathsf{F} = \mathsf{F}\mathsf{J}_\sharp.$$

Let M_m be the operator of multiplication by $m \in L^\infty(\mathbb{R})$, that is, $(\mathsf{M}_m F)(z) = m(z)F(z)$. For a unimodular function $u \in L^1_{\mathrm{loc}}(\mathbb{R})$, we define the map $\mathsf{K} = \mathsf{K}_u : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\mathsf{K} = \mathsf{F}^{-1}\mathsf{M}_u\mathsf{J}^\sharp\mathsf{F} = \mathsf{F}^{-1}\mathsf{M}_u\mathsf{F}\mathsf{J}_\sharp. \tag{2.2}$$

Then K is ($\mathbb{C}$-)antilinear (also called conjugate linear), that is, $\mathsf{K}(af + bg) = \bar{a}\mathsf{K}f + \bar{b}\mathsf{K}g$ for $f, g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{C}$. For the properties of antilinear operators, see Huhtanen [8] and Uhlmann [24], for example. The operator K satisfies $(\mathsf{F}\mathsf{K}f)(z) = u(z)(\mathsf{F}f)^\sharp(z)$ for $z \in \mathbb{R}$ by definition. Also, K is isometric, because F is isometric up to scaling, $\mathsf{J}^\sharp$ and $\mathsf{J}_\sharp$ are clearly isometric, and M_u is isometric for a unimodular function u. Further $\mathsf{K}^2 = 1$, since $\mathsf{F}\mathsf{K}^2 f(z) = (\mathsf{F}\mathsf{K}(\mathsf{K}f))(z) = u(z)(\mathsf{F}\mathsf{K}f)^\sharp(z) = u(z)u^\sharp(z)(\mathsf{F}f)(z)$ and $u(z)u^\sharp(z) = |u(z)|^2 = 1$ for $z \in \mathbb{R}$. We summarize the above properties of K as follows recalling that, for an antilinear operator T, the adjoint T^* is defined by $\langle \mathsf{T}f, g\rangle = \overline{\langle f, \mathsf{T}^* g\rangle} = \langle \mathsf{T}^* g, f\rangle$, where $\langle f, g\rangle = \int f(x)\overline{g(x)}\, dx$ for $f, g \in L^2(\mathbb{R})$.

Proposition 2.1. — *For a unimodular function u in $L^1_{\mathrm{loc}}(\mathbb{R})$, the map $\mathsf{K} = \mathsf{K}_u$ in (2.2) is an antilinear isometric involution on $L^2(\mathbb{R})$, in other words, K is an abstract conjugation on $L^2(\mathbb{R})$. Hence, in particular, K is self-adjoint: $\mathsf{K} = \mathsf{K}^*$.*

For some special unimodular function u, the operator K is represented as an integral operator with a continuous integral kernel (see [21, Theorem 2.1], for example). If we allow the integral kernel to be a tempered distribution, K is always an integral operator as follows. Every u in $L^1_{\mathrm{loc}}(\mathbb{R})$ can be regarded as a tempered distribution on $\mathbb{R}$ by $(u, g) = \int u(x)g(x)dx$, $g \in S(\mathbb{R})$, where $S(\mathbb{R})$ is the Schwartz space. Therefore, there exists a tempered distribution k on $\mathbb{R}$ such that $u = \mathsf{F}k$, since the Fourier transform F extends to the space of tempered distributions $S'(\mathbb{R})$ as a bijection (so $k = \mathsf{F}^{-1}u$). Then the operator K in (2.2) is expressed as the integral operator

$$(\mathsf{K}f)(x) = (k * \mathsf{J}_\sharp f)(x) = \int k(x+y)\overline{f(y)}\, dy$$

by the product rule of the Fourier transform, where $*$ stands for the additive convolution. In some cases, k can be regarded as a function, but it never belongs to $L^1(\mathbb{R})$ by the Riemann–Lebesgue theorem. The tempered distribution $k = \mathsf{F}^{-1}u$ is real-valued if and only if u is symmetric, that is, u satisfies (2.1).

For $t \in \mathbb{R}$, we define the compression $\mathsf{K}[t] : L^2(-\infty, t) \to L^2(-\infty, t)$ of K by

$$\mathsf{K}[t] := \mathsf{P}_t \mathsf{K}|_{L^2(-\infty,t)},$$

where P_t is the orthogonal projection from $L^2(\mathbb{R})$ to $L^2(-\infty, t)$. Then, $\mathsf{K}[t] = \mathsf{P}_t\mathsf{K}\mathsf{P}_t$ on $L^2(-\infty, t)$. Since K is isometric on $L^2(\mathbb{R})$, the inequality of operator norm $\|\mathsf{K}[t]\|_{\mathrm{op}} \leqslant 1$ always holds. Now, we introduce the following condition on u in $L^1_{\mathrm{loc}}(\mathbb{R})$:

(O1) $\|\mathsf{K}[t]\|_{\mathrm{op}} < 1$ for some $t \in \mathbb{R}$, where $\|\cdot\|_{\mathrm{op}}$ is the operator norm for operators on $L^2(-\infty,t)$. Note that if $\|\mathsf{K}[t]\|_{\mathrm{op}}<1$ for one t then it holds for all smaller t's, since $\|\mathsf{K}[s]\|_{\mathrm{op}} \leqslant \|\mathsf{K}[t]\|_{\mathrm{op}}$ for $s<t$ by definition of the operator norm.

Henceforth, we suppose that u belongs to the subspace $U^1_{\mathrm{loc}}(\mathbb{R})$ ($\subset L^1_{\mathrm{loc}}(\mathbb{R})$) in order that $\mathsf{K}[t]1$ is defined as a function belonging to $L^2(-\infty,t)$ (cf. Proposition 4.1 below) and for other technical reasons. If $\|\mathsf{K}[t]\|_{\mathrm{op}}<1$, the equations $(1+\mathsf{K}[t])\varphi=-\mathsf{K}[t]1$ and $(1-\mathsf{K}[t])\psi=\mathsf{K}[t]1$ for $\varphi,\psi\in L^2(-\infty,t)$ have unique solutions. Using the solutions φ and ψ, we define the functions Φ and Ψ on $\mathbb{R}$ by $\Phi:=1-\mathsf{K}(\varphi+\mathsf{P}_t1)$ and $\Psi:=1+\mathsf{K}(\psi+\mathsf{P}_t1)$. Then they solve equations

$$\Phi+\mathsf{K}\mathsf{P}_t\Phi=1, \tag{2.3}$$

$$\Psi-\mathsf{K}\mathsf{P}_t\Psi=1. \tag{2.4}$$

We find that the solutions Φ and Ψ satisfy a certain system of partial differential equations if assuming the following technical conditions (Proposition 4.5 below). To state such conditions, we extend the action of K to the space of tempered distributions $S'(\mathbb{R})$ by using (2.2) (see Section 4.1 for details). Then the conditions are stated as follows:

(O2) For the above solutions $\Phi(t,x)$ and $\Psi(t,x)$ of (2.3) and (2.4), derivatives $(\partial/\partial t)\Phi$, $(\partial/\partial t)\Psi$, $(\partial/\partial t)\mathsf{F}\Phi$, $(\partial/\partial t)\mathsf{F}\Psi$ with respect to t are defined as tempered distributions of x, and the commutativity

$$\frac{\partial}{\partial t}\mathsf{F}\Phi=\mathsf{F}\frac{\partial}{\partial t}\Phi, \qquad \frac{\partial}{\partial t}\mathsf{F}\Psi=\mathsf{F}\frac{\partial}{\partial t}\Psi,$$

hold, respectively, whenever $\|\mathsf{K}[t]\|_{\mathrm{op}}<1$;

(O3) The values of $\Phi(t,x)$ and $\Psi(t,x)$ at $x=t$ are well-defined and nonzero, whenever $\|\mathsf{K}[t]\|_{\mathrm{op}}<1$;

(O4) $\mathsf{P}_t\mathsf{K}\delta_t$ is defined as a function belonging to $L^2(-\infty,t)$, or else the kernels of $(1\pm\mathsf{K}[t]):\mathsf{P}_tS'(\mathbb{R})\to\mathsf{P}_tS'(\mathbb{R})$ are zero, whenever $\|\mathsf{K}[t]\|_{\mathrm{op}}$ <1,

where $\delta_t(x):=\delta(x-t)$ for the Dirac distribution δ at the origin, $(\mathsf{P}_tf)(x)=\mathbf{1}_{(-\infty,0)}(x-t)f(x)$ for a tempered distribution f, and the kernels of $(1\pm\mathsf{K}[t])$ are considered as $\mathbb{R}$-linear maps. Condition (O4) guarantees that the solutions of (2.3) and (2.4) are unique in $S'(\mathbb{R})$.

Now we introduce two functions $\widetilde{A}(t,z)$ and $\widetilde{B}(t,z)$ by

$$\begin{aligned}\widetilde{A}(t,z)&:=-\frac{iz}{2}(\mathsf{F}(1-\mathsf{P}_t)\Psi)(z),\\ -i\widetilde{B}(t,z)&:=-\frac{iz}{2}(\mathsf{F}(1-\mathsf{P}_t)\Phi)(z),\end{aligned} \tag{2.5}$$

where the Fourier transforms are taken as tempered distributions. They play a central role in all of the following results in this section.

Theorem 2.2. — *Let $u \in U^1_{\mathrm{loc}}(\mathbb{R})$ and define $\mathsf{K} = \mathsf{K}_u$ as above. Suppose that conditions* (O1), (O2), (O3), (O4) *are satisfied, and let T be a real number such that $\|\mathsf{K}[t]\|_{\mathrm{op}} < 1$ for all $t < T$. Then*

(1) *$\widetilde{A}(t,z)$ and $\widetilde{B}(t,z)$ are defined by* (2.5) *as tempered distributions on $\mathbb{R}$ for each fixed $t < T$;*

(2) *$\widetilde{A}(t,z)$ and $\widetilde{B}(t,z)$ extend to meromorphic functions on $(\mathbb{C}_+ \cup D)\backslash\mathbb{R}$, and they are holomorphic on $\mathbb{C}_+$ for each fixed $t < T$, where D is the domain for u in* (U2)*;*

(3) *the limit equations $\lim_{z\to x}\widetilde{A}(t,z) = \widetilde{A}(t,x)$ and $\lim_{z\to x}\widetilde{B}(t,z) = \widetilde{B}(t,x)$ hold for almost all $x \in \mathbb{R}$ if z tends to x non-tangentially either in $\mathbb{C}_+$ or $\mathbb{C}_-$;*

(4) *$\widetilde{A}(t,z)$ and $\widetilde{B}(t,z)$ satisfy the functional equations*

$$\widetilde{A}(t,z) = u(z)\widetilde{A}^\sharp(t,z), \quad \widetilde{B}(t,z) = u(z)\widetilde{B}^\sharp(t,z) \tag{2.6}$$

for $z \in D$;

(5) *$\widetilde{A}(t,z)$ and $\widetilde{B}(t,z)$ are continuous with respect to t for fixed $z \in \mathbb{C}_+ \cup D$ except for their (isolated) singularities;*

(6) *$\widetilde{A}(t,z)$ and $\widetilde{B}(t,z)$ satisfy the first order system*

$$-\frac{\partial}{\partial t}\begin{bmatrix}\widetilde{A}(t,z)\\ \widetilde{B}(t,z)\end{bmatrix} = z\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}H(t)\begin{bmatrix}\widetilde{A}(t,z)\\ \widetilde{B}(t,z)\end{bmatrix} \tag{2.7}$$

for $t < T$ and $z \in \mathbb{C}_+ \cup D$, where $H(t)$ is the matrix-valued function defined by

$$H(t) = H_u(t) := \begin{bmatrix}\alpha(t) & \beta(t)\\ \beta(t) & \gamma(t)\end{bmatrix} \tag{2.8}$$

and

$$\begin{aligned}
\alpha(t) &= \frac{|\Phi(t,t)|^2}{\Re(\Phi(t,t)\overline{\Psi(t,t)})} = \frac{1}{\Re(\Psi(t,t)/\Phi(t,t))},\\
\beta(t) &= \frac{\Im(\Phi(t,t)\overline{\Psi(t,t)})}{\Re(\Phi(t,t)\overline{\Psi(t,t)})} = \frac{\Im(\Phi(t,t)/\Psi(t,t))}{\Re(\Phi(t,t)/\Psi(t,t))} = -\frac{\Im(\Psi(t,t)/\Phi(t,t))}{\Re(\Psi(t,t)/\Phi(t,t))},\\
\gamma(t) &= \frac{|\Psi(t,t)|^2}{\Re(\Phi(t,t)\overline{\Psi(t,t)})} = \frac{1}{\Re(\Phi(t,t)/\Psi(t,t))}.
\end{aligned} \tag{2.9}$$

(7) *$H(t)$ in* (6) *belongs to $\mathrm{SL}_2(\mathbb{R}) \cap \mathrm{Sym}_2(\mathbb{R})$ for all $t < T$. If u is symmetric, $H(t)$ is diagonal.*

Notation $\alpha(t)$, $\beta(t)$, $\gamma(t)$ in (2.8) and (2.9) correspond to $\alpha'(t)$, $\beta'(t)$, $\gamma'(t)$ in de Branges' book [2]. The first-order differential system in (6) has a different range of complex parameter z from usual theory of canonical systems. Such systems are called *lacunary* canonical systems in [22].

2.2. Chains of reproducing kernel Hilbert spaces

Second, we describe that the system of differential equations (2.7) controls the structure of a chain of reproducing kernel Hilbert spaces if assuming further conditions for a given unimodular function.

For $\mathsf{K} = \mathsf{K}_u$ and each $t \in \mathbb{R}$, we denote by $\mathcal{V}_t(u)$ the space of all functions $f \in L^2(\mathbb{R})$ such that both f and $\mathsf{K}f$ have their supports in $[t, \infty)$:

$$\mathcal{V}_t(u) := L^2(t,\infty) \cap \mathsf{K}L^2(t,\infty).$$

By definition, $\mathcal{V}_t(u)$ is a conjugation-invariant subspace of $L^2(\mathbb{R})$ with respect to the conjugation K. We do not need any of the conditions (O1)–(O4) to define $\mathcal{V}_t(u)$, but the following condition is necessary for the discussion about $\mathcal{V}_t(u)$ to be meaningful:

(O5) $\mathcal{V}_t(u)$ is non-zero for some $t \in \mathbb{R}$.

If $\mathcal{V}_t(u)$ is non-zero, $\mathsf{F}(\mathcal{V}_t(u))$ is a reproducing kernel Hilbert space consisting of functions on $\mathbb{C}_+ \cup D$ that are holomorphic on $\mathbb{C}_+$ and meromorphic on $D \cap \mathbb{C}_-$. The reproducing kernel $j(t; z, w)$ of $\mathsf{F}(\mathcal{V}_t(u))$ is expressed as $j(t; z, w) = \frac{1}{2\pi}\langle Y_w^t, Y_z^t\rangle$ using the vector $Y_z^t \in \mathcal{V}_t(u)$ satisfying $\langle f, \overline{Y_z^t}\rangle = \mathsf{F}f(z)$ for any $f \in \mathcal{V}_t(u)$ (see Section 5 for details). More specifically, $j(t; z, w)$ has the following explicit formula consisting of functions defined in (2.5).

THEOREM 2.3. — *Let $u \in U^1_{\mathrm{loc}}(\mathbb{R})$. Suppose that conditions* (O1), (O2), (O3), (O4), (O5) *are satisfied, and let $t \in \mathbb{R}$ such that $\|\mathsf{K}[t]\|_{\mathrm{op}} < 1$ and $\mathcal{V}_t(u) \neq 0$. Let $j(t; z, w)$ be the reproducing kernel of $\mathsf{F}(\mathcal{V}_t(u))$. Then,*

$$j(t; z, w) = \frac{1}{2\pi}\left\langle Y_w^t, Y_z^t\right\rangle = \frac{\overline{\widetilde{A}(t,z)}\widetilde{B}(t,w) - \widetilde{A}(t,w)\overline{\widetilde{B}(t,z)}}{\pi(w-\bar{z})} \tag{2.10}$$

holds for $z, w \in \mathbb{C}_+ \cup D$, where D is the domain for u in (U2).

Remark 2.4. — A formula similar to (2.10) is proved in [20, Section 4.2], but the assumed set of conditions in Theorem 2.3 is quite different from that in [20]. See also the comments after Theorem 2.8.

As is clear from the definition, the spaces $\{\mathcal{V}_t(u)\}_{t\in\mathbb{R}}$ are totally ordered by set-theoretical inclusion $\mathcal{V}_s(u) \subset \mathcal{V}_t(u)$ for $t < s$ and the inclusion is an isometric embedding as a Hilbert space. However, we should note that, unlike de Branges spaces, not all K-invariant subspaces of $\mathcal{V}_t(u)$ necessarily have the shape of $\mathcal{V}_s(u)$ $(t < s)$, and therefore, not all K-invariant subspaces of $\mathcal{V}_t(u)$ are necessarily totally ordered.

If $\mathcal{V}_t(u) = \{0\}$ for some $t \in \mathbb{R}$, then $\mathcal{V}_s(u) = \{0\}$ for all $s \geqslant t$ by definition. Therefore, it makes sense to consider the value

$$t_0 = t_0(u) := \sup\{t \in \mathbb{R} \mid \mathcal{V}_t(u) \neq \{0\}\}.$$

Under condition (O5), t_0 is determined as a finite real number or $+\infty$. To use the above results in the study of the chain of spaces $\{\mathsf{F}(\mathcal{V}_t(u))\}_{t<t_0}$, we introduce the following conditions:

- (O6) $\|\mathsf{K}_u[t]\|_{\mathrm{op}} < 1$ for every $t < t_0$;
- (O7) $\Re(\Phi(t,t)\overline{\Psi(t,t)}) > 0$ for every $t < t_0$.

Note that (O1) is automatically satisfied assuming (O5) and (O6).

Let $H^\infty = H^\infty(\mathbb{C}_+)$ be the space of all bounded analytic functions in $\mathbb{C}_+$. A function $\theta \in H^\infty$ is called an *inner function* in $\mathbb{C}_+$ if $\lim_{y\to 0+} |\theta(x+iy)| = 1$ for almost all $x \in \mathbb{R}$ with respect to the Lebesgue measure. An inner function θ defines a measurable unimodular function on $\mathbb{R}$ by taking a nontangential limit at a point of $\mathbb{R}$ and extends to the lower half-plane by setting $\theta(z) := 1/\theta^\sharp(z)$ for $z \in \mathbb{C}_-$, in particular, θ is meromorphic on $\mathbb{C}\backslash\mathbb{R}$. If an inner function θ in $\mathbb{C}_+$ extends to a meromorphic function on $\mathbb{C}$, it is called a *meromorphic inner function* in $\mathbb{C}_+$. For an inner function θ, the space

$$\mathcal{K}(\theta) := H^2(\mathbb{C}_+) \ominus \theta H^2(\mathbb{C}_+)$$

defined as the orthogonal complement, is called a *model subspace*.

Theorem 2.5. — *Let $u \in U^1_{\mathrm{loc}}(\mathbb{R})$. Suppose that conditions* (O2), (O3), (O4), (O5), (O6), (O7) *are satisfied. Then the function defined by*

$$\theta(t,z) := \frac{\widetilde{A}(t,z) + i\widetilde{B}(t,z)}{\widetilde{A}(t,z) - i\widetilde{B}(t,z)} \tag{2.11}$$

is an inner function in $\mathbb{C}_+$ and $\mathsf{F}(\mathcal{V}_t(u)) = \mathcal{K}(\theta(t,z))$ for every $t < t_0$.

Because of the connection with the theory of de Branges spaces, we are particularly interested in unimodular functions of the form $u(z) =$

$M^\sharp(z)/M(z)$ for some meromorphic function $M(z)$ on $\mathbb{C}$. Using the functions $\widetilde{A}(t,z)$ and $\widetilde{B}(t,z)$ in (2.5), we define

$$\begin{aligned} A(t,z) :=& M(z)\widetilde{A}(t,z), \quad B(t,z) := M(z)\widetilde{B}(t,z), \\ & E(t,z) := A(t,z) - iB(t,z). \end{aligned} \tag{2.12}$$

With the assumptions in Theorem 2.5, $\Theta(t,z) := E^\sharp(t,z)/E(t,z)$ extends to a meromorphic inner function for every $t < t_0$. In general, if Θ is a meromorphic inner function, there exists $E \in \mathbb{HB}$ such that $\Theta = E^\sharp/E$ and the model subspace $\mathcal{K}(\Theta)$ is isometrically isomorphic to the de Branges space $\mathcal{H}(E)$ by the map $F \mapsto EF$ ([7, Sections 2.3 and 2.4]), where $\mathbb{HB}$ is the subspace consisting of functions that have no zeros on $\mathbb{R}$ as before. Therefore, if $\Theta(\tau,z)$ is a meromorphic inner function, it is expected that $H(t)$ in Theorem 2.2 is nothing but the structure Hamiltonian of the de Branges space $\mathcal{H}(E(\tau,z))$. To realize this expectation, we introduce one more condition:

(O8) $\mathcal{V}_{t_0}(u) = \{0\}$ if $t_0 < \infty$.

Note that, when $t_0 < \infty$, both $\mathcal{V}_{t_0}(u) = \{0\}$ and $\mathcal{V}_{t_0}(u) \neq \{0\}$ can occur. See examples in Section 3.

Theorem 2.6. — *Let $u \in U^1_{\mathrm{loc}}(\mathbb{R})$. Suppose that $u = M^\sharp/M$ for some meromorphic function M on $\mathbb{C}$ that is holomorphic on $\mathbb{C}_+ \cup \mathbb{R}$ and has no zeros in $\mathbb{C}_+$. Suppose that* (O2), (O3), (O4), (O5), (O6), (O7), (O8) *are satisfied. Define $E(t,z)$ by* (2.5) *and* (2.12) *for $t < t_0$. Then, for any $t_1 < t_0$, $E(t_1,z)$ is an entire function of $\overline{\mathbb{HB}}$ and $H(t)$ on $[t_1,t_0)$ defined by* (2.8) *and* (2.9) *is the structure Hamiltonian of the de Branges space $\mathcal{H}(E(t_1,z))$.*

The asymptotic behavior of the reproducing kernel of $\mathcal{H}(E(t,z))$ as $t \to t_0$ is clarified in the proof of the theorem, but the asymptotic behavior of $E(t,z)$ as $t \to t_0$ is more difficult and will not be studied in this paper. See [2, Theorem 41] and [14, Theorems 1.34, 3.15, 4.20] for results on the asymptotic behavior of $E(t,z)$.

2.3. Specialization to de Branges spaces

Further specializes situation in the previous subsection. In Theorem 2.6, there is no direct relation between $u(z)$ and $E(t,z)$, but if u is a restriction of a meromorphic inner function $\Theta = E^\sharp/E$, then the chain of spaces $\mathsf{F}(\mathcal{V}_t(u))$ for $t \geqslant 0$ is isomorphic to the chain of de Branges subspaces of the de Branges space $\mathcal{H}(E)$. We state it after giving a result on $u = \theta$, which is an inner function but not necessarily a meromorphic inner function.

Theorem 2.7. — *Suppose that $u \in U^1_{\mathrm{loc}}(\mathbb{R})$ is a restriction of an inner function θ in $\mathbb{C}_+$ and that* (O1), (O2), (O3), (O4) *are satisfied. Then*

$$\widetilde{A}(0,z) = \frac{1}{2}(1+\theta(z)), \qquad -i\widetilde{B}(0,z) = \frac{1}{2}(1-\theta(z)) \tag{2.13}$$

for $z \in \mathbb{C}$. In particular, $\mathsf{F}(\mathcal{V}_0(u)) = \mathcal{K}(\theta)$.

From this result, it can be understood that Theorem 2.2 solves the direct problem for the lacunary canonical system on $[0, t_0)$ associated with the particular Hamiltonian $H(t)$ defined by (2.8) and (2.9) and equality (2.13) as the initial condition at zero by providing the explicit solution $(\widetilde{A}(t,z), \widetilde{B}(t,z))$.

If u is a restriction of an inner function, $\mathsf{K}[t] = 0$ for nonpositive t by Proposition 6.2 below. Further, we have $\Phi(t,t) = \Psi(t,t) = 1$ if $\Phi(t,x)$ and $\Psi(t,x)$ are continuous at $x = t$, and therefore $H(t)$ is the identity matrix for nonpositive t. In this sense, the nontrivial range of t for $u = \theta$ is $0 < t < t_0$, and it is actually meaningful as follows.

For a meromorphic inner function $\Theta = E^\sharp/E$ with $E \in \mathbb{HB}$, we define $A(t,z)$ and $B(t,z)$ by (2.5) and (2.12) with $M = E$. Then $E(0,z) = E(z)$ by Theorem 2.7. Therefore, as a corollary of Theorem 2.6, we obtain the following result which solves the inverse problem of finding the structure Hamiltonian from a given generator E of a de Branges space (see also Proposition 6.2):

Theorem 2.8. — *Let Θ be a meromorphic inner function such that $\Theta = E^\sharp/E$ for some $E \in \mathbb{HB}$. Define K for $u = \Theta$ by* (2.2) *and suppose that* (O2), (O3), (O4), (O6), (O7), (O8) *are satisfied. Then $H(t)$ on $[0, t_0)$ defined by* (2.8) *and* (2.9) *is the structure Hamiltonian of the de Branges space $\mathcal{H}(E)$.*

Note that there are many de Branges spaces to which Theorem 2.8 does not apply, as in the example in Section 3.2.

2.4. Comparison with Kreĭn's method

Here, we clarify the relation between the method proposed in the present paper and Kreĭn's inverse spectral theory of strings on a half-line ([10, 11], see also [9]). For details on the relation between the spectral theory of strings, canonical spaces, and de Branges spaces, see Dym and McKean [5] and Langer and Winkler [13].

Let $\Theta = E^\sharp/E$ be as in Theorem 2.8. If Θ is symmetric, then the structure Hamiltonian H of the de Branges space $\mathcal{H}(E)$ is diagonal and regular (limit circle) at $t = 0$. Such a Hamiltonian is associated with a Kreĭn's string $S[m, L]$ consisting of its length L ($0 < L \leqslant \infty$) and a nondecreasing right-continuous function $m(x)$ defined on $[0, L)$, the mass distribution. From a diagonal Hamiltonian $H(t) = \mathrm{diag}(1/\gamma(t), \gamma(t))$ on $[0, t_0)$, a string $S[m, L]$ is obtained by defining

$$x = f(t) := \int_0^t \frac{1}{\gamma(s)}\, ds, \quad L := \int_0^{t_0} \frac{1}{\gamma(s)}\, ds, \quad m(x) := \int_0^{f^{-1}(x)} \gamma(s)\, ds.$$

From the obtained string, the original $H(t)$ is restored by defining

$$g(x) := \int_0^x \sqrt{m'(y)}\, dy, \quad \gamma(t) := \sqrt{m'(g^{-1}(t))}.$$

This correspondence is a bit more general, but we have described it in a limited situation for simplicity. The Titchmarsh–Weyl function Q_S of the string $S[m, L]$ is related to the Titchmarsh–Weyl function Q_H of $H(t)$ as

$$z\, Q_S(z^2) = Q_H(z) \left(:= i\, \frac{1 - \Theta(z)}{1 + \Theta(z)} \right),$$

and the former admits the representation

$$Q_S(z) = b + \int_0^\infty \frac{d\tau(\lambda)}{\lambda - z},$$

where b is a nonnegative constant and τ is a measure on $[0, \infty)$ called the principal spectral measure of the string $S[m, L]$.

In [10, 11], Kreĭn announced the method to recover the string from the principal spectral measure. In the following outline, $b = 0$ is assumed, and all assumptions on the principal spectral measure τ (or its transfer function) are omitted. First, we introduce the transfer function

$$\Phi(t) = \int_0^\infty \frac{1 - \cos(t\sqrt{\lambda})}{\lambda}\, d\tau(\lambda) \tag{2.14}$$

on $[0, \infty)$, then consider the family of integral equations

$$2\Phi'(0) q(x) + \int_{-t}^{t} \Phi''(x - y) q(y)\, dy = 1, \quad 0 \leqslant t \leqslant t_0. \tag{2.15}$$

Under appropriate conditions for $\Phi(t)$, this equation has unique integrable solution $x \mapsto q(x; t)$ on $[-t, t]$ for each $0 \leqslant t \leqslant t_0$. Using the solution $q(x; t)$, we set

$$p(t) := \frac{d}{dt}\int_{-t}^{t} q(x;t)\,dx = \frac{2q(t,t)^2}{q(0,0)}. \tag{2.16}$$

Then, we get $m(f(t)) = \int_0^t p(s)\,ds$ with $f(t) = \int_0^t 1/p(s)\,ds$. This implies

$$\gamma(t) = p(t). \tag{2.17}$$

In this way, $H(t)$ is restored from the spectral measure of a given string. The similarity between (2.15), (2.16), (2.17) and (2.3), (2.4), (2.9) (or, the more direct matches are of [22]) is remarkable. It is even more striking if comparing the Fourier transforms

$$\int_0^\infty \Phi(t)e^{izt}\,dt = -\frac{1}{z^2}\left(\frac{1-\Theta(z)}{1+\Theta(z)}\right) \quad \text{for large} \quad \Im z > 0$$

and $(\mathsf{F}k)(z) = \Theta(z)$ for the kernel $k = \mathsf{F}^{-1}\Theta$ of the operator K (ignoring the twist $\mathsf{J}_\sharp$). For the reasons mentioned above, we may say that the method in the present paper is a generalization of a variant of Kreĭn's theory in which the integral kernel is replaced by a different type. If we mention the differences, depending on the choice of integral kernels k and Φ, the operator K is isometric, but $f \mapsto \int \Phi''(x-y)f(y)\,dy$ is generally not. Also, some technical differences occur in the proof depending on whether the integral kernel is additive type $k(x+y)$ or difference type $\Phi''(x-y)$. On the other hand, if the generator E of the de Branges space $\mathcal{H}(E)$ and the kernel $k = \mathsf{F}^{-1}\Theta$ of the operator K are not directly related as in the example of Section 3.3, the relation with Kreĭn's theory becomes indirect.

If we only aim to recover $H(t)$ from τ (or Θ), which method is more useful will depend on the ease of handling with functions $\Phi(t)$ and $k(x)$, but if we also take into account the recovery of the solution of the canonical system, there are more differences between the two methods. In Kreĭn's theory, the solution of (2.15) also generates the independent solutions $\phi(x,\lambda)$ and $\psi(x,\lambda)$ of the differential equation $(py')' + \lambda py = 0$ as

$$\phi(t;\lambda) = \frac{1}{p(t)}\frac{d}{dt}\int_0^t q(s;t)\cos(s\sqrt{\lambda})\,ds,$$
$$\psi(t;\lambda) = \frac{1}{p(t)}\frac{d}{dt}\int_0^t q(s;t)\,\omega(s,\sqrt{\lambda})\,ds,$$

where $\kappa\omega(t,\kappa) = \sin(\kappa t) + \int_0^t H(t-s)\sin(\kappa s)\,ds$ and $\phi(0;\lambda) = 1$, $\phi'(0;\lambda) = 0$, $\psi(0;\lambda) = 0$, $\psi'(0;\lambda) = 1$. Using these solutions, the solution of the canonical system associated with $H(t)$ is obtained as

$$\begin{bmatrix} A(t,z) \\ B(t,z) \end{bmatrix} = \begin{bmatrix} A(z)\psi'\left(t,z^2\right) - z^{-1}B(z)\phi'\left(t,z^2\right) \\ B(z)\phi\left(t,z^2\right) - zA(z)\psi\left(t,z^2\right) \end{bmatrix}$$

such that $E(z) = A(0,z) - iB(0,z)$ holds, where $A = (E + E^\sharp)/2$ and $B = i(E - E^\sharp)/2$. As this, the solution of the canonical system given by Theorems 2.2 and 2.8 can also be obtained from Kreĭn's theory, but the formula for the solution by (2.5) and (2.12) is somewhat direct and simpler. This difference is the same compared to the theory in [12] that deals with diagonal and non-diagonal H.

As the above, in the case of diagonal H, there are both similarities and differences between our method and Kreĭn's theory for strings. The advantages of our method are that it can be generalized to non-diagonal H in a different way than [12], and that it can deal with chains of spaces that are not necessarily related to canonical systems.

2.5. Comparison with previous work

To conclude this section, we comment on the difference of the operator K between this paper and [19, 20, 22]. The first difference is that K is antilinear in this paper, which is an essential ingredient that enables us the construction of non-diagonal H, whereas in the latter, K was linear. Second, in this paper, K is defined as the composition of several unitary operators as in (2.2), whereas in the latter, K was defined as the integral operator having the integral kernel defined by $K(x) = \frac{1}{2\pi}\int_{\Im(z)=c} u(z)e^{-izx}dz$ for large $c > 0$. This second difference is reflected in the conditions assumed in the results, and each has advantages and disadvantages. As an example of what makes a significant difference, we take up the equality $\mathsf{K}^2 = 1$ and the support condition of K, both are important in various discussions. In the definition of this paper, $\mathsf{K}^2 = 1$ is obvious, but it is nontrivial that the support of $k(x)$ is contained in $[0,\infty)$, which corresponds to u being an inner function. On the other hand, in [19, 20, 22], it is obvious from the settings that $K(x)$ is supported in $[0,\infty)$, but $\mathsf{K}^2 = 1$ is nontrivial and it relates whether u is inner. Besides these, if $K(x)$ is discontinuous or distribution, the definition in this paper is more convenient. In any case, two different definitions of K can be related as in [20, Theorem 5.1].

3. Examples

We provide several concrete examples of unimodular functions satisfying (a part of) the conditions assumed in the results in Section 2. Those examples may help readers understand the meaning or necessity of conditions (O1)–(O8).

3.1. Paley–Wiener spaces

Let $E(z) = \exp(-iaz)$ with $a > 0$ and put $u(z) = E^\sharp(z)/E(z) = \exp(2iaz)$. Then $u \in U^1_{\mathrm{loc}}(\mathbb{R})$. We have $k(x) = \delta(x-2a)$, so $(\mathsf{K}f)(x) = \overline{f(2a-x)}$, and therefore $\mathcal{V}_t(u) = L^2(t,\infty) \cap L^2(-\infty, 2a-t)$. Hence (O5) is satisfied, $t_0 = a$, $\mathcal{V}_t(u) = L^2(t, 2a-t)$ for $t < a$, and also (O8) is satisfied: $\mathcal{V}_a(u) = \{0\}$. We have $\mathsf{K}[t] = 0$ for $t < a$ from the support condition of $k(x)$, so (O1), (O4), and (O6) are satisfied, and $\Phi(t,x) = \Psi(t,x) = 1$ for $x < t$ if $t < a$. Therefore, (2.3) and (2.4) are solved as $\Phi(t,x) = 1 - \mathsf{KP}_t 1(x) = 1 - \mathbf{1}_{[0,\infty)}(x-(2a-t))$ and $\Psi(t,x) = 1 + \mathsf{KP}_t 1(x) = 1 + \mathbf{1}_{[0,\infty)}(x-(2a-t))$. Hence (O2), (O3), and (O7) are satisfied with $\Phi(t,t) = \Psi(t,t) = 1$, and

$$A(t,z) = \cos((a-t)z), \quad -iB(t,z) = -i\sin((a-t)z).$$

On the other hand, $Y^t_z(x) = \mathbf{1}_{(t,2a-t)}(x)e^{izx}$ and

$$J(t;z,w) = \frac{\overline{E(z)}E(w)}{2\pi}\langle Y^t_w, Y^t_z\rangle = \frac{\sin((a-t)(w-\bar{z}))}{\pi(w-\bar{z})}.$$

This shows that equality (2.10) holds. See also [18, 23], where the case that E is an exponential polynomial with real coefficients is studied, and explicit formulas for $\phi^\pm(t,x)$, $A(t,z)$, and $B(t,z)$ are stated, although $\Phi(t,x)$ and $\Psi(t,x)$ are not specified.

3.2. One dimensional de Branges space

Let $E(z) = 1 - iz$ and put $u(z) = E^\sharp(z)/E(z) = (1+iz)/(1-iz)$. Then $u \in U^1_{\mathrm{loc}}(\mathbb{R})$. We have $k(x) = -\delta(x) + 2e^{-x}\mathbf{1}_{(0,\infty)}(x)$ and easily find that $\mathcal{V}_t(u) = \{0\}$ for $t > 0$, $\mathcal{V}_0(u) = \mathbb{C}\,e^{-x}\mathbf{1}_{(0,\infty)}$, and $\mathcal{V}_t(u) = L^2(t,-t) + \mathbb{C}\,e^{-x}\mathbf{1}_{(-t,\infty)}$ for $t < 0$. Hence (O5) is satisfied and $t_0 = 0$, but (O8) is *not* satisfied. We have $\mathsf{K}[t] = 0$ for $t < 0$, so (O1), (O4), and (O6) are satisfied, and $\Phi(t,x) = \Psi(t,x) = 1$ for $x < t$ if $t < 0$. Therefore, (2.3) and (2.4) are solved as $\Phi(t,x) = 1 - \mathsf{KP}_t 1(x) = 1 - \mathbf{1}_{(-t,\infty)}(x)(1 - 2e^{-x-t})$

and $\Psi(t,x) = 1+\mathsf{K}\mathsf{P}_t 1(x) = 1+\mathbf{1}_{(-t,\infty)}(x)(1-2e^{-x-t})$. Hence (O2), (O3), and (O7) are satisfied with $\Phi(t,t) = \Psi(t,t) = 1$, and

$$A(t,z) = \cos(tz) + z\sin(tz), \quad -iB(t,z) = -i(z\cos(tz) - \sin(tz)).$$

On the other hand, we have

$$Y_z^t(x) = \frac{2i}{z+i} e^{-it(z-i)} \cdot e^{-x}\mathbf{1}_{(-t,\infty)}(x) + e^{izx}\mathbf{1}_{(t,-t)}(x),$$

and

$$\begin{aligned} J(t;z,w) &= \frac{\overline{E(z)}E(w)}{2\pi} \left\langle Y_w^t, Y_z^t \right\rangle \\ &= \frac{e^{-it(w-\bar{z})}}{\pi} - (w+i)(\bar{z}-i)\frac{\sin(t(w-\bar{z}))}{\pi(w-\bar{z})} \end{aligned}$$

for $t \leqslant 0$. We can check that equality (2.10) holds.

For $0 < t < 1$, we define $A(t,z) = 1$ and $B(t,z) = z(1-t)$. Then, they satisfy

$$-\frac{d}{dt}\begin{bmatrix} A(t,z) \\ B(t,z) \end{bmatrix} = z\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A(t,z) \\ B(t,z) \end{bmatrix}, \quad 0 < t < 1, \quad z \in \mathbb{C}$$

and $J(t;z,w) = (1/\pi)(1-t)$, and hence $J(t;z,w) \to 0$ as $t \to 1$. In other words, the structure Hamiltonian of $\mathcal{H}(E)$ is $H(t) = \mathrm{diag}(1,0)$ on $t \in (0,1)$, but it cannot be obtained from the method in Section 2.

3.3. De Branges spaces related to the Hankel transform of order zero

The example described here is based on the results of Burnol [3]. The section numbers in this part refers to that in [3]. Let $M(z) = \Gamma(\frac{1}{2} - iz)$ and put $u(z) = M^\sharp(z)/M(z) = \Gamma(\frac{1}{2}+iz)/\Gamma(\frac{1}{2}-iz)$. Then $u \in U^1_{\mathrm{loc}}(\mathbb{R})$ and u is meromorphic on $\mathbb{C}$ but not inner in $\mathbb{C}_+$. We have $k(x) = e^{x/2}J_0(2e^{x/2})$ and $u(z) = \int_{-\infty}^{\infty} k(x)e^{izx}\,dx$ for $-1/4 < \Im(z) < 1/2$, where $J_0(z)$ is the Bessel function of the first kind of order zero. The convergence of the Fourier integral is conditional if $\Im(z) \leqslant 1/4$. It is proved that (O1) and (O5) are satisfied for all $t \in \mathbb{R}$ in § 5, and hence (O6) and (O8) are also satisfied. We put $a = e^t$, $b = e^x$, and

$$\phi^{\pm}(t,x) = \sqrt{ab}\left(1 \pm \frac{\partial}{\partial b}\right) I_0\left(2\sqrt{a(a-b)}\right),$$

where $I_0(w)$ is the modified Bessel function of the first kind of order zero. Then $\phi^{\pm}(t,x)$ are real-valued and satisfy (5.1) and (5.2) below by discussions in § 7. Define

$$\Phi(t,x) = 1 - e^{-2e^t}\int_{-\infty}^{x} \phi^-(t,y)dy, \quad \Psi(t,x) = 1 + e^{2e^t}\int_{-\infty}^{x} \phi^+(t,y)dy.$$

Then they satisfy (2.3) and (2.4), and it is easily find that (O2) is satisfied. Also, the equalities $\Phi(t,t) = e^{-2e^t}$ and $\Psi(t,t) = e^{2e^t}$ show that (O3) and (O6) are satisfied. The asymptotic formula $(\mathsf{K}\delta_t)(x) = k(x+t) \sim \exp(x/2)$ as $x \to -\infty$ shows that (O4) is satisfied. Further, we have

$$A(t,z) = e^{2e^t+(t/2)}\left(K_{\frac{1}{2}-iz}(2e^t) + K_{\frac{1}{2}+iz}(2e^t)\right),$$
$$-iB(t,z) = e^{-2e^t+(t/2)}\left(K_{\frac{1}{2}-iz}(2e^t) - K_{\frac{1}{2}+iz}(2e^t)\right),$$

where $K_\nu(z)$ is the modified Bessel function of the second kind of order ν. The explicit formula for Y_z^t cannot be found in [3], and in fact, that is complicated to write down here, but the explicit formula for $\langle Y_w^t, Y_z^t\rangle$ can be found in § 7. For $z=0$, we have $Y_0^t(x) = \mathbf{1}_{(t,\infty)}(x)\Phi(t,x)$, which is implicitly dealt with in § 6. Anyway, we find the explicit formula for $J(t;z,w)$ via the second equality of (2.10).

3.4. De Branges spaces arising from L-functions in the Selberg class

The examples described here are based on the results of [21]. Let $\mathcal{S}$ be the Selberg class of Dirichlet series $L(s)$. Typical examples of elements in $\mathcal{S}$ are number-theoretic zeta- and L-functions such as the Riemann zeta-function and Dirichlet L-functions. For every $L \in \mathcal{S}$, there exists a product $\gamma_L(s)$ of shifts of the gamma function, a real number Q_L, and a complex number c_L of the unit modulus such that $\xi_L(s) := c_L\, Q_L^s \gamma_L(s) L(s)$ satisfies the functional equation $\xi_L(s) = \xi_L^\sharp(1-s)$. We define

$$\Theta_L^{\omega,\nu}(z) = \left(\frac{\xi_L(1/2-\omega-iz)}{\xi_L(1/2+\omega-iz)}\right)^{\nu}$$

for $L \in \mathcal{S}$, $\omega \in \mathbb{R}_{>0}$, and $\nu \in \mathbb{Z}_{>0}$. Then $\Theta_L^{\omega,\nu}$ is a meromorphic inner function for all $\omega \geqslant 1/2$ and $\nu \in \mathbb{Z}_{>0}$ unconditionally, and for all $0 < \omega < 1/2$ and $\nu \in \mathbb{Z}_{>0}$ if assuming the Grand Riemann Hypothesis for L ([21, Proposition 2.2]).

We suppose that $\omega\nu d_L > 1$, where d_L is the degree of L, and that $\Theta_L^{\omega,\nu}$ is a meromorphic inner function. Then, by [20, Theorem 5.1] and [21, Proposition 4.1], K defined by (2.2) for $u = \Theta_L^{\omega,\nu}$ equals to the integral operator having the continuous kernel $K(x+y)$ defined by $K(x) = \frac{1}{2\pi}\int_{\Im(z)=c}\Theta_L^{\omega,\nu}(z)e^{-izx}\,dz$ for $c > 1/2+\omega$. Further, (O2), (O3), (O4) are satisfied by [20, Section 2.4], [21, Proposition 4.1], and [22, Proposition 2.3]. And (O6) is satisfied by a similar argument as in the proof of [21, Lemma 5.2, Proposition 5.1]. Since Proposition 7.4 below can be applied, $\Phi(t,t) = \Psi(t,t) = 1$ for $t<0$, and $\Phi(t,t)$, $\Psi(t,t)$ are continuous for $t \geqslant 0$. Therefore, (O7) is satisfied. Finally, since $t_0 = \infty$ by [20, Lemma 4.1] and [21, Proposition 4.1], (O8) is also satisfied. Hence, Theorem 2.8 can be applied to $u = \Theta_L^{\omega,\nu}$. Note that [21] deals only with the case that $L(s)$ and $Q_L^s\gamma_L(s)L(s)$ take real-values on the real line, but essentially does not affect the discussions that prove the results referred to the above, and lead to the same results for general $L \in \mathcal{S}$.

On the other hand, we may also generalize the unimodular function

$$u(z) = \exp\left[-2\eta\frac{\xi'}{\xi}\left(\frac{1}{2}-iz\right)\right]$$

studied in [19] to functions in the Selberg class, where ξ is the Riemann xi-function and η is a positive real number.

4. Proof of Theorem 2.2

4.1. Extension of K to $S'(\mathbb{R})$

Let $(f,g) := \int f(x)g(x)\,dx$ be the pairing for $f \in S'(\mathbb{R})$ and $g \in S(\mathbb{R})$. Then the Fourier transform F, the multiplication operator M_u for $u \in L^\infty(\mathbb{R})$, and $\mathsf{J}^\sharp$ extend to $S'(\mathbb{R})$ by $(\mathsf{F}f,g) := (f,\mathsf{F}g)$, $(\mathsf{M}_u f,g) := (f,\mathsf{M}_u g)$, and $(\mathsf{J}^\sharp f,g) = \overline{(f,\mathsf{J}^\sharp g)}$ for $f \in S'(\mathbb{R})$ and $g \in S(\mathbb{R})$, respectively, since $S(\mathbb{R})$ is closed under these operations. Therefore, K extends to an antilinear involution on $S'(\mathbb{R})$ by (2.2).

We also use the Hermitian pairing $\langle f,g\rangle := (f,\bar{g}) = \overline{(\bar{f},g)}$ for $f \in S'(\mathbb{R})$ and $g \in S(\mathbb{R})$. Then, K is self-adjoint with respect to this Hermitian paring. In fact, we have $\overline{\mathsf{F}^{-1}h_1} = (2\pi)^{-1}\mathsf{F}\mathsf{J}^\sharp h_1$, $\mathsf{J}^\sharp\mathsf{F}h_2 = 2\pi\mathsf{F}^{-1}\overline{h_2}$, and therefore

$$\langle f,\mathsf{K}g\rangle = \left(f,\mathsf{F}\mathsf{J}^\sharp\mathsf{M}_u\mathsf{F}^{-1}\bar{g}\right) = \overline{(\mathsf{F}^{-1}\mathsf{M}_u\mathsf{J}^\sharp\mathsf{F}f,\bar{g})} = \overline{\langle\mathsf{K}f,g\rangle}.$$

Proposition 4.1. — *Let $u \in U^1_{\mathrm{loc}}(\mathbb{R})$. Then $\mathsf{K}\mathsf{P}_t 1 \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $\mathsf{K}[t]1 \in L^2(-\infty,t)$.*

Proof. — We calculate $\mathsf{K}\mathsf{P}_t 1$ as a tempered distribution. For $g \in S(\mathbb{R})$,

$$\begin{aligned}\langle \mathsf{K}\mathsf{P}_t 1, g\rangle &= \langle \mathsf{K}g, \mathsf{P}_t 1\rangle = \left(\mathsf{F}^{-1}\mathsf{M}_u\mathsf{J}^\sharp\mathsf{F}g, \mathsf{P}_t 1\right) = \left(\mathsf{M}_u\mathsf{J}^\sharp\mathsf{F}g, \mathsf{F}^{-1}\mathsf{P}_t 1\right) \\ &= \left((\mathsf{M}_u - u(0))\mathsf{J}^\sharp\mathsf{F}g, \mathsf{F}^{-1}\mathsf{P}_t 1\right) + u(0)\left(\mathsf{F}^{-1}\mathsf{J}^\sharp\mathsf{F}g, \mathsf{P}_t 1\right).\end{aligned}$$

In the second term of the right-hand side, $\mathsf{F}^{-1}\mathsf{J}^\sharp\mathsf{F}g = \mathsf{F}^{-1}\mathsf{F}\mathsf{J}_\sharp g = \mathsf{J}_\sharp g$ by $\mathsf{J}^\sharp\mathsf{F} = \mathsf{F}\mathsf{J}_\sharp$. Thus

$$\left(\mathsf{F}^{-1}\mathsf{J}^\sharp\mathsf{F}g, \mathsf{P}_t 1\right) = (\mathsf{J}_\sharp g, \mathsf{P}_t 1) = \int \overline{g(-x)}\mathsf{P}_t 1(x)\,dx = \langle (1-\mathsf{P}_{-t})1, g\rangle.$$

To calculate the first term of the right-hand side, we recall the Fourier transform of unit step functions

$$\mathsf{F}^{-1}\mathsf{P}_t 1(z) = \frac{e^{-itz}}{2}\left(\delta(z) - \frac{1}{\pi i}\,\mathrm{p.\,v.}\frac{1}{z}\right),$$

where the distribution p. v.$(1/x)$ is defined by

$$\left(\mathrm{p.\,v.}\frac{1}{x}, g(x)\right) = \lim_{\epsilon\to 0}\int_{|x|\geqslant\epsilon}\frac{g(x)}{x}\,dx$$

for $g \in S(\mathbb{R})$. Therefore,

$$\begin{aligned}&\left((\mathsf{M}_u - u(0))\mathsf{J}^\sharp\mathsf{F}g, \mathsf{F}^{-1}\mathsf{P}_t 1\right) \\ &\qquad= \left((u(z)-u(0))\int \overline{g(-x)}e^{izx}\,dx,\ -\frac{1}{2\pi i}e^{-itz}\mathrm{p.\,v.}\frac{1}{z}\right) \\ &\qquad= \frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{|z|>\epsilon}\left(\frac{u(z)-u(0)}{-iz}\int\overline{g(-x)}e^{izx}\,dx\right)e^{-itz}\,dz \\ &\qquad= \int\left(\frac{1}{2\pi}\int\left[\frac{u(z)-u(0)}{-iz}e^{-itz}\right]e^{-izx}\,dz\right)\overline{g(x)}\,dx \\ &\qquad= \int h(x)\overline{g(x)}\,dx = \langle h, g\rangle,\end{aligned}$$

where

$$h(x) := \frac{1}{2\pi}\int\left[\frac{u(z)-u(0)}{-iz}e^{-itz}\right]e^{-izx}\,dz \in L^2(\mathbb{R})$$

and the integral converges in the L^2 sense. (Hölder continuity of u at $z=0$ is used here.) The above calculation is justified by the Cauchy–Schwarz inequality and Fubini's theorem. Hence,

$$\mathsf{K}\mathsf{P}_t 1 = u(0)(1-\mathsf{P}_{-t})1 + h, \quad h \in L^2(\mathbb{R}). \tag{4.1}$$

This shows the desired results, since $L^2(\mathbb{R}) \subset L^1_{\mathrm{loc}}(\mathbb{R})$. □

4.2. Proof of (1), (2), (3), (4)

Proof. — Recall the solutions $\Phi = 1-\mathsf{K}\varphi-\mathsf{K}\mathsf{P}_t 1$ and $\Psi = 1+\mathsf{K}\psi+\mathsf{K}\mathsf{P}_t 1$ of (2.3) and (2.4) introduced in Section 2.1, where $\varphi = -(1+\mathsf{K}[t])^{-1}\mathsf{K}[t]1$ and $\psi = (1-\mathsf{K}[t])^{-1}\mathsf{K}[t]1$. It is not hard to see $\varphi = \mathsf{P}_t\Phi - \mathsf{P}_t 1$ and $\psi = \mathsf{P}_t\Psi - \mathsf{P}_t 1$. By Proposition 4.1, Φ and Ψ are tempered distributions at least. Therefore, $\mathsf{F}(1-\mathsf{P}_t)\Phi$ and $\mathsf{F}(1-\mathsf{P}_t)\Psi$ are always defined as tempered distributions, thus (1) is proved.

To prove (2) and (3), we show that $\mathsf{F}(1-\mathsf{P}_t)\Phi$ and $\mathsf{F}(1-\mathsf{P}_t)\Psi$ are defined as holomophic functions on $\mathbb{C}_+$. First, $(\mathsf{F}(1-\mathsf{P}_t)1)(z) = e^{itz}/(-iz)$ for $\Im(z)>0$. Second, $\mathsf{F}(1-\mathsf{P}_t)\mathsf{K}\psi$ is defined for $\Im(z)>0$, since $(1-\mathsf{P}_t)\mathsf{K}\psi \in L^2(t,\infty)$. Third,

$$(\mathsf{F}(1-\mathsf{P}_t)\mathsf{K}\mathsf{P}_t 1)(z) = u(0)\frac{e^{i|t|z}}{-iz} + (\mathsf{F}(1-\mathsf{P}_t)h)(z), \quad h \in L^2(\mathbb{R})$$

for $\Im(z)>0$ by (4.1). Hence $\widetilde{A}(t,z)$ and $\widetilde{B}(t,z)$ are defined by (2.5) and holomorphic on $\mathbb{C}_+$. Moreover, $\lim_{z\to x}\widetilde{A}(t,z) = \widetilde{A}(t,x)$ and $\lim_{z\to x}\widetilde{B}(t,z) = \widetilde{B}(t,x)$ hold for almost all $x\in\mathbb{R}$, where z tends to x non-tangentially in $\mathbb{C}_+$, since $\lim_{z\to x}\mathsf{F}f(z) = \mathsf{F}f(x)$ for $f\in L^2(t,\infty)$ for almost all $x\in\mathbb{R}$.

For (4), we extend $\widetilde{A}(t,z)$ and $\widetilde{B}(t,z)$ across the real line as follows. First, we observe that $\mathsf{K}1 = u(0)$ as a tempered distribution, because

$$\begin{aligned}\langle \mathsf{K}1, g\rangle &= \langle \mathsf{K}g, 1\rangle = \left\langle \mathsf{F}^{-1}\mathsf{F}\mathsf{K}g, 1\right\rangle = \left(\mathsf{F}^{-1}\mathsf{F}\mathsf{K}g, 1\right) = \left(\mathsf{F}\mathsf{K}g, \mathsf{F}^{-1}1\right)\\ &= (\mathsf{F}\mathsf{K}g,\delta) = \ \mathsf{F}\mathsf{K}g(0) = u(0)(\mathsf{F}g)^\sharp(0) = u(0)\overline{\mathsf{F}g(0)} = u(0)\langle 1, g\rangle\end{aligned}$$

for $g\in S(\mathbb{R})$. On the other hand, $(1-\mathsf{P}_t)\Psi = 1-\mathsf{P}_t\Psi+\mathsf{K}\mathsf{P}_t\Psi$ by (2.4). Therefore,

$$\mathsf{K}(1-\mathsf{P}_t)\Psi = \mathsf{K}1 - \mathsf{K}\mathsf{P}_t\Psi + \mathsf{P}_t\Psi = (u(0)+1) - (1-\mathsf{P}_t)\Psi.$$

Using this and $(\mathsf{F}\frac{\partial}{\partial x}f)(z) = (-iz)(\mathsf{F}f)(z)$, we have

$$\begin{aligned}2\widetilde{A}(t,z) &= (-iz)(\mathsf{F}(1-\mathsf{P}_t)\Psi)(z) = \left(\mathsf{F}\tfrac{\partial}{\partial x}(1-\mathsf{P}_t)\Psi\right)(z)\\ &= -\left(\mathsf{F}\tfrac{\partial}{\partial x}\mathsf{K}(1-\mathsf{P}_t)\Psi\right)(z) = \left(\mathsf{F}\mathsf{K}\tfrac{\partial}{\partial x}(1-\mathsf{P}_t)\Psi\right)(z)\\ &= u(z)\left(J^\sharp\mathsf{F}\tfrac{\partial}{\partial x}(1-\mathsf{P}_t)\Psi\right)(z) = 2\,u(z)\widetilde{A}^\sharp(t,z)\end{aligned}$$

for $z\in\mathbb{R}$. On the right-hand side,

$$\begin{aligned}2\widetilde{A}^\sharp(t,z) &= \left(\mathsf{J}^\sharp\mathsf{F}\tfrac{\partial}{\partial x}(1-\mathsf{P}_t)\Psi\right)(z) = \left(\mathsf{F}\tfrac{\partial}{\partial x}\mathsf{J}_\sharp(1-\mathsf{P}_t)\Psi\right)(z)\\ &= \left(\mathsf{F}\tfrac{\partial}{\partial x}\mathsf{P}_{-t}\Psi\right)(z) = (-iz)\left(\mathsf{F}\mathsf{P}_{-t}\mathsf{J}_\sharp\Psi\right)(z).\end{aligned}$$

Here $(\mathsf{F}\mathsf{P}_{-t}\mathsf{J}_\sharp\Psi)(z)$ on the right-hand side is defined for $\Im(z)\leqslant 0$ by $\Psi = 1+\mathsf{K}\psi+\mathsf{K}\mathsf{P}_t 1$ and (4.1). Hence $\widetilde{A}(t,z)$ extends from $\mathbb{C}_+\cup\mathbb{R}$ to $\mathbb{C}_+\cup D$, and (2.6) holds in D. Analytic continuation and functional equation for $\widetilde{B}(t,z)$ are proved in a similar argument.

If the (distribution) kernel $k = \mathsf{F}^{-1}u$ of K has support in $[0,\infty)$, we easily find that φ and ψ have support in $[-t,t]$ or zero (cf. Section 6.1), and thus $\mathsf{F}\varphi$ and $\mathsf{F}\psi$ are entire functions, since a tempered distribution is a higher derivative of a continuous function. If the analytic continuations of $\mathsf{F}\varphi$ and $\mathsf{F}\psi$ beyond $\mathbb{C}_+ \cup \mathbb{R}$ are proved in this way, the analytic continuations of $\widetilde{A}(t,z)$ and $\widetilde{B}(t,z)$ are proved as follows. We have $(1-\mathsf{P}_t)\Psi = (1-\mathsf{P}_t)1 - \psi + \mathsf{K}\mathsf{P}_t 1 + \mathsf{K}\psi$ with $\psi = (\mathsf{P}_t\Psi - \mathsf{P}_t 1)$ by (2.4). Hence

$$(-iz)(\mathsf{F}(1-\mathsf{P}_t)\Psi)(z) = \Big[e^{itz} + iz(\mathsf{F}\psi)(z)\Big] + u(z)\Big[e^{itz} + iz(\mathsf{F}\psi)(z)\Big]^\sharp$$

for $z \in \mathbb{C}_+ \cup \mathbb{R}$. This formula gives the analytic continuation of $\widetilde{A}(t,z)$ according to the extended domain of $(\mathsf{F}\psi)(z)$ (and the domain of u). □

4.3. Proof of (5)

For $t < s$, we have

$$\Phi(t,x) - \Phi(s,x) = \mathsf{K}(\mathsf{P}_s - \mathsf{P}_t)1 + \mathsf{K}\Big[(1+\mathsf{K}[t])^{-1}\mathsf{K}[t]1 - (1+\mathsf{K}[s])^{-1}\mathsf{K}[s]1\Big]$$

by $\Phi(t,x) = 1 - \mathsf{K}\mathsf{P}_t 1 + \mathsf{K}(1+\mathsf{K}[t])^{-1}\mathsf{K}[t]1$ (see the lines before (2.3)). The first term on the right-hand side tends to zero as $s \to t$ in $L^2(\mathbb{R})$, since K is isometric. We find that the second term on the right-hand side also tends to zero as $s \to t$ in $L^2(\mathbb{R})$ by the second resolvent equation

$$(1+\mathsf{K}[t])^{-1}\mathsf{K}[t] - (1+\mathsf{K}[s])^{-1}\mathsf{K}[s] = (1+\mathsf{K}[t])^{-1}(\mathsf{K}[t]-\mathsf{K}[s])(1+\mathsf{K}[s])^{-1}.$$

Therefore, $\|\Phi(t,x) - \Phi(s,x)\|_{L^2(\mathbb{R})} \to 0$ as $s \to t$. The same is true for $\Psi(t,x)$. Therefore, $\|z^{-1}(\widetilde{A}(t,z) - \widetilde{A}(s,z))\|_{H^2(\mathbb{C}_+)} \to 0$ and $\|z^{-1}(\widetilde{B}(t,z) - \widetilde{B}(s,z))\|_{H^2(\mathbb{C}_+)} \to 0$ as $s \to t$ by definition (2.5). The latter implies that $\widetilde{A}(t,z) - \widetilde{A}(s,z) \to 0$ and $\widetilde{B}(t,z) - \widetilde{B}(s,z) \to 0$ as $s \to t$ pointwisely, since the norm convergence in the reproducing kernel Hilbert space $H^2(\mathbb{C}_+)$ implies the pointwise convergence. □

4.4. Auxiliary results necessary to prove (6) and (7)

LEMMA 4.2. — *Let Φ and Ψ be nonzero complex numbers. Then the pair of equations*

$$\begin{cases} \Psi = -i\beta\Psi + \gamma\Phi, \\ \Phi = \alpha\Psi + i\beta\Phi \end{cases}$$

for real numbers α, β, γ *has the unique solution*

$$\begin{aligned}\alpha &= \frac{|\Phi|^2}{\Re(\Phi\overline{\Psi})} = \frac{1}{\Re(\Psi/\Phi)}, \quad \gamma = \frac{|\Psi|^2}{\Re(\Phi\overline{\Psi})} = \frac{1}{\Re(\Phi/\Psi)}, \\ \beta &= \frac{\Im(\Phi\overline{\Psi})}{\Re(\Phi\overline{\Psi})} = \frac{\Im(\Phi/\Psi)}{\Re(\Phi/\Psi)} = -\frac{\Im(\Psi/\Phi)}{\Re(\Psi/\Phi)}\end{aligned} \tag{4.2}$$

such that the symmetric matrix $\left[\begin{smallmatrix}\alpha & \beta \\ \beta & \gamma\end{smallmatrix}\right]$ *belongs to* $\mathrm{SL}_2(\mathbb{R})$. *Moreover,* $\left[\begin{smallmatrix}\alpha & \beta \\ \beta & \gamma\end{smallmatrix}\right]$ *is positive or negative definite according to the sign of* $\Re(\Phi\overline{\Psi})$.

Proof. — The given equation is equivalent to the linear equation

$$\begin{bmatrix} 0 & \Im(\Psi) & \Re(\Phi) & -\Re(\Psi) \\ \Re(\Psi) & -\Im(\Phi) & 0 & -\Re(\Phi) \\ 0 & -\Re(\Psi) & \Im(\Phi) & -\Im(\Psi) \\ \Im(\Psi) & \Re(\Phi) & 0 & -\Im(\Phi) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The kernel of the matrix on the left-hand side is one-dimensional since Φ and Ψ are non-zero. Hence the solution is unique. It can be confirmed by direct calculation that (4.2) solves the given equation and that $\alpha\gamma - \beta^2 = 1$. The eigenvalues of $\left[\begin{smallmatrix}\alpha & \beta \\ \beta & \gamma\end{smallmatrix}\right]$ are

$$\frac{|\Phi|^2 + |\Psi|^2 \pm \sqrt{(|\Phi|^2 + |\Psi|^2)^2 - 4(\Re(\Phi\overline{\Psi}))^2}}{2\,\Re(\Phi\overline{\Psi})}.$$

These are nonzero real numbers for nonzero Φ and Ψ, because

$$\left(|\Phi|^2 + |\Psi|^2\right)^2 - 4\left(\Re(\Phi\overline{\Psi})\right)^2 = |\Phi - \Psi|^2 |\Phi + \Psi|^2 \geqslant 0.$$

Hence the eigenvalues are both positive or both negative depending on the sign of $\Re(\Phi\overline{\Psi})$. □

LEMMA 4.3. — *The following commutative relations hold in* $S'(\mathbb{R})$*:*

$$\frac{\partial}{\partial x}\mathsf{K} = -\mathsf{K}\frac{\partial}{\partial x}, \tag{4.3}$$

$$\frac{\partial}{\partial x}\mathsf{P}_t f(x) = -\delta(x-t)f(x) + \mathsf{P}_t\frac{\partial}{\partial x}f(x). \tag{4.4}$$

Proof. — By the formula $\left(\mathsf{F}\frac{\partial}{\partial x}f\right)(z) = (-iz)(\mathsf{F}f)(z)$ for a tempered distribution f,

$$\mathsf{F}\frac{\partial}{\partial x}\mathsf{K}f(z) = -iz(\mathsf{F}\mathsf{K}f)(z) = -u(z)\left[-iz(\mathsf{F}f)(z)\right]^\sharp = -\mathsf{F}\mathsf{K}\frac{\partial}{\partial x}f(z).$$

Hence (4.3) holds in $S'(\mathbb{R})$. Because $\mathsf{P}_t f(x) = \mathbf{1}_{(-\infty,0)}(x-t)f(x)$, (4.4) is shown as

$$\frac{\partial}{\partial x}\mathsf{P}_t f(x) = \frac{\partial}{\partial x}\left(\mathbf{1}_{(-\infty,0)}(x-t)f(x)\right) = -\delta(x-t)f(x) + \mathbf{1}_{(-\infty,0)}(x-t)\frac{\partial}{\partial x}f(x)$$

by the product rule for derivatives. □

Lemma 4.4. — *Condition* (O2) *implies*

$$\frac{\partial}{\partial t}\mathsf{P}_t X = \delta(x-t)X + \mathsf{P}_t\frac{\partial}{\partial t}X, \tag{4.5}$$

$$\frac{\partial}{\partial t}\mathsf{K}X = \mathsf{K}\frac{\partial}{\partial t}X, \qquad \frac{\partial}{\partial t}\mathsf{K}\mathsf{P}_t X = \mathsf{K}\frac{\partial}{\partial t}\mathsf{P}_t X \tag{4.6}$$

for $X \in \{\Psi(t,x), \Phi(t,x)\}$*, where* $\Psi(t,x)$ *and* $\Phi(t,x)$ *are functions in* (2.3) *and* (2.4)*, and the projection* P_t *acts on* $X = X(t,x)$ *as a function of* x*.*

Proof. — Equation (4.5) is shown in a similar argument as for (4.4). The first equation of (4.6) is shown as follows by (2.2) and (O2):

$$\mathsf{F}\mathsf{K}\frac{\partial}{\partial t}\Phi = \mathsf{M}_u\mathsf{F}\mathsf{J}_\sharp\frac{\partial}{\partial t}\Phi = \mathsf{M}_u\frac{\partial}{\partial t}\mathsf{F}\mathsf{J}_\sharp\Phi = \mathsf{F}\mathsf{F}^{-1}\frac{\partial}{\partial t}\mathsf{M}_u\mathsf{F}\mathsf{J}_\sharp\Phi = \mathsf{F}\frac{\partial}{\partial t}\mathsf{F}^{-1}\mathsf{M}_u\mathsf{F}\mathsf{J}_\sharp\Phi = \mathsf{F}\frac{\partial}{\partial t}\mathsf{K}\Phi.$$

The second equation of (4.6) is obtained from the first equation as follows. Applying $\partial/\partial t$ to (2.3), we have $(\partial/\partial t)\Phi + (\partial/\partial t)\mathsf{K}\mathsf{P}_t\Phi = 0$. On the other hand, applying $\partial/\partial t$ to (2.3) after acting K, we have (i) $(\partial/\partial t)\mathsf{K}\Phi + (\partial/\partial t)\mathsf{P}_t\Phi = 0$. Thus $\mathsf{K}(\partial/\partial t)\Phi + (\partial/\partial t)\mathsf{P}_t\Phi = 0$ by the first equation of (4.6), and therefore (ii) $(\partial/\partial t)\Phi + \mathsf{K}(\partial/\partial t)\mathsf{P}_t\Phi = 0$. Comparing (i) and (ii), we obtain the second equation. The same is true for Ψ. □

Proposition 4.5. — *Assume that* (O1), (O2), (O3) *and* (O4) *are satisfied. Then the solutions* $\Psi(t,x)$ *and* $\Phi(t,x)$ *of* (2.3) *and* (2.4) *satisfy the differential system*

$$-\frac{\partial}{\partial t}\begin{bmatrix}\Psi(t,x)\\ i\Phi(t,x)\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} H(t)\left(i\frac{\partial}{\partial x}\right)\begin{bmatrix}\Psi(t,x)\\ i\Phi(t,x)\end{bmatrix}, \tag{4.7}$$

where partial derivatives $\partial/\partial t$ *and* $\partial/\partial x$ *are taken in the sense of distribution and* $H(t)$ *is defined by* (2.8) *and* (2.9)*.*

Proof. — First, we apply $(\partial/\partial t)$ to both sides of (2.4). Then,

$$(\partial/\partial t)\Psi - \mathsf{K}(\partial/\partial t)\mathsf{P}_t\Psi = 0$$

by (4.6), and further $(\partial/\partial t)\Psi - \overline{\Psi(t,t)}\mathsf{K}\delta_t - \mathsf{KP}_t(\partial/\partial t)\Psi = 0$ by (O3) and (4.5), where $\delta_t(x) = \delta(x-t)$. Hence,

$$\frac{\partial}{\partial t}\Psi - \mathsf{KP}_t\frac{\partial}{\partial t}\Psi = \overline{\Psi(t,t)}\mathsf{K}\delta_t. \tag{4.8}$$

Second, we apply $(\partial/\partial x)$ to both sides of (2.4). Then,

$$(\partial/\partial x)\Psi + \mathsf{K}(\partial/\partial x)\mathsf{P}_t\Psi = 0$$

by (4.3), and further $(\partial/\partial x)\Psi - \overline{\Psi(t,t)}\mathsf{K}\delta_t + \mathsf{KP}_t(\partial/\partial x)\Psi = 0$ by (O3) and (4.4). Hence,

$$\frac{\partial}{\partial x}\Psi + \mathsf{KP}_t\frac{\partial}{\partial x}\Psi = \overline{\Psi(t,t)}\mathsf{K}\delta_t, \tag{4.9}$$

and therefore,

$$i\beta(t)\frac{\partial}{\partial x}\Psi - \mathsf{KP}_t(i\beta(t))\frac{\partial}{\partial x}\Psi = \overline{(-i\beta(t))\Psi(t,t)}\mathsf{K}\delta_t \tag{4.10}$$

for any real number $\beta(t)$. Third, we apply $(\partial/\partial x)$ to both sides of (2.3). Then, $(\partial/\partial x)\Phi - \mathsf{K}(\partial/\partial x)\mathsf{P}_t\Phi = 0$ by (4.3), and further $(\partial/\partial x)\Phi + \overline{\Phi(t,t)}\mathsf{K}\delta_t - \mathsf{KP}_t(\partial/\partial x)\Phi = 0$ by (O3) and (4.4). Hence,

$$\frac{\partial}{\partial x}\Phi - \mathsf{KP}_t\frac{\partial}{\partial x}\Phi = -\overline{\Phi(t,t)}\mathsf{K}\delta_t, \tag{4.11}$$

and therefore,

$$-\gamma(t)\frac{\partial}{\partial x}\Phi + \mathsf{KP}_t\gamma(t)\frac{\partial}{\partial x}\Phi = \overline{\gamma(t)\Phi(t,t)}\mathsf{K}\delta_t \tag{4.12}$$

for any real number $\gamma(t)$. Adding (4.10) and (4.12),

$$(1-\mathsf{KP}_t)\left[i\beta(t)\frac{\partial}{\partial x}\Psi - \gamma(t)\frac{\partial}{\partial x}\Phi\right] = \overline{((-i\beta(t))\Psi(t,t) + \gamma(t)\Phi(t,t))}\mathsf{K}\delta_t.$$

The right-hand side is equal to $\overline{\Psi(t,t)}\mathsf{K}\delta_t$ by Lemma 4.2 if we take $\beta(t)$ and $\gamma(t)$ as in (2.9). Comparing the obtained equality with (4.8), we have

$$\frac{\partial}{\partial t}\Psi(t,x) = i\beta(t)\frac{\partial}{\partial x}\Psi(t,x) - \gamma(t)\frac{\partial}{\partial x}\Phi(t,x)$$

by (O4). Hence the first line of (4.7) is obtained. A similar argument gives

$$\frac{\partial}{\partial t}\Phi + \mathsf{KP}_t\frac{\partial}{\partial t}\Phi = -\overline{\Phi(t,t)}\mathsf{K}\delta_t. \tag{4.13}$$

This and $\Phi(t,t) = \alpha(t)\Psi(t,t) + i\beta(t)\Phi(t,t)$ with (2.9) lead to

$$\frac{\partial}{\partial t}\Phi(t,x) = -\alpha(t)\frac{\partial}{\partial x}\Psi(t,x) - i\beta(t)\frac{\partial}{\partial x}\Phi(t,x)$$

by Lemma 4.2. Hence the second line of (4.7) is obtained. □

4.5. Proof of (6) and (7)

Proof. — For (6), we obtain (2.7) by applying the projection $(1-\mathsf{P}_t)$ to both sides of (4.7), then taking their Fourier transform, and finally extend them to $\mathbb{C}_+ \cup D$, since F and $\partial/\partial t$ are commutative by (O2).

The first half of (7) follows from Lemma 4.2. If u is symmetric, $k = \mathsf{F}^{-1}u$ is real-valued, since $\bar{k} = \mathsf{F}^{-1}(u^\sharp(-z))$. Therefore, if Φ solves (2.3), then $\overline{\Phi}$ also solves it. Hence $\Phi = \overline{\Phi}$ by (O4) and thus $H(t)$ is diagonal by definition (2.9). □

5. Proof of results in Section 2.2

5.1. Auxiliary results required for the proof

To calculate the reproducing kernel of $\mathsf{F}(\mathcal{V}_t(u))$ explicitly, we study the solutions of the equations

$$\phi^+ + \mathsf{K}\mathsf{P}_t\phi^+ = \mathsf{K}\delta_t, \tag{5.1}$$

$$\phi^- - \mathsf{K}\mathsf{P}_t\phi^- = \mathsf{K}\delta_t \tag{5.2}$$

for $\phi^\pm$ in the space of tempered distributions $S'(\mathbb{R})$, where $\delta_t(x) = \delta(x-t)$ as before.

Proposition 5.1. — *Suppose that $\|\mathsf{K}[t]\|_{\mathrm{op}} < 1$ (as an operator on $L^2(-\infty,t)$) and that* (O3), (O4) *are satisfied. Then equations* (5.1) *and* (5.2) *have unique solutions in $S'(\mathbb{R})$.*

Proof. — From the shape of equations, solutions of (5.1) and (5.2) are uniquely determined by their projection to $(-\infty,t)$. Hence if $\mathsf{P}_t\mathsf{K}\delta_t \in L^2(-\infty,t)$, the assumption $\|\mathsf{K}[t]\|_{\mathrm{op}} < 1$ guarantees the existence and uniqueness of the solutions under the condition $\mathsf{P}_t\phi^\pm \in L^2(-\infty,t)$. If $\mathsf{P}_t\mathsf{K}\delta_t$ does not belong to $L^2(-\infty,t)$, the latter half of (O4) guarantees the uniqueness of the solutions in $S'(\mathbb{R})$. The existence of the solutions is shown by constructing them concretely. From (4.9) and (4.11), we find that $\phi^\pm$ defined by

$$\begin{aligned}\phi^+ = \phi^+(t,x) = {} & \frac{\Re(\Phi(t,t))}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\frac{\partial}{\partial x}\Psi(t,x)\\ & - i\,\frac{\Im(\Psi(t,t))}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\frac{\partial}{\partial x}\Phi(t,x),\end{aligned} \tag{5.3}$$

$$\begin{aligned}(5.4)\qquad \phi^- = \phi^-(t,x) = i\,&\frac{\Im(\Phi(t,t))}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\frac{\partial}{\partial x}\Psi(t,x)\\ &- \frac{\Re(\Psi(t,t))}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\frac{\partial}{\partial x}\Phi(t,x)\end{aligned}$$

solve the equations (5.1) and (5.2), respectively. □

If (O2) is added to the assumptions of Proposition 5.1, we find that $\phi^\pm$ defined by

$$\begin{aligned}(5.5)\qquad \phi^+ = \phi^+(t,x) = -&\frac{\Re(\Psi(t,t))}{\Re(\overline{\Phi(t,t)}\Psi(t,t))}\frac{\partial}{\partial t}\Phi(t,x)\\ &+ i\frac{\Im(\Phi(t,t))}{\Re(\overline{\Phi(t,t)}\Psi(t,t))}\frac{\partial}{\partial t}\Psi(t,x),\end{aligned}$$

$$\begin{aligned}(5.6)\qquad \phi^- = \phi^-(t,x) = -i&\frac{\Im(\Psi(t,t))}{\Re(\overline{\Phi(t,t)}\Psi(t,t))}\frac{\partial}{\partial t}\Phi(t,x)\\ &+ \frac{\Re(\Phi(t,t))}{\Re(\overline{\Phi(t,t)}\Psi(t,t))}\frac{\partial}{\partial t}\Psi(t,x)\end{aligned}$$

also solve equations (5.1) and (5.2), respectively, by (4.8) and (4.13). Therefore, by comparing the right-hand sides of (5.3) and (5.5), and (5.4) and (5.6), we get the system (4.7) again. Also, the following holds immediately from Proposition 5.1.

Proposition 5.2. — *Suppose that* $\|\mathsf{K}[t]\|_{\mathrm{op}} < 1$ *and that* (O3), (O4) *are satisfied. Let* a_0, a_1, b_0, b_1 *be real numbers. Then,* $f + \mathsf{KP}_t f = (a_0 + ia_1)\mathsf{K}\delta_t$ *for some* $f \in S'(\mathbb{R})$ *implies* $f = a_0\phi_t^+ + ia_1\phi_t^-$, *and* $g - \mathsf{KP}_t g = (b_0 + ib_1)\mathsf{K}\delta_t$ *for some* $g \in S'(\mathbb{R})$ *implies* $g = b_0\phi_t^- + ib_1\phi_t^+$. *Conversely,* $f + \mathsf{KP}_t f = (a_0 + ia_1)\mathsf{K}\delta_t$ *and* $g - \mathsf{KP}_t g = (b_0 + ib_1)\mathsf{K}\delta_t$ *imply*

$$\phi^+(t,x) = \frac{1}{\Re((a_0+ia_1)(b_0-ib_1))}\left(b_0 f(t,x) - ia_1 g(t,x)\right),$$

$$\phi^-(t,x) = \frac{1}{\Re((a_0+ia_1)(b_0-ib_1))}\left(-ib_1 f(t,x) + a_0 g(t,x)\right).$$

For $t \in \mathbb{R}$, we identify $\mathsf{F}L^2(t,\infty)$ with $e^{itz}H^2(\mathbb{C}_+)$ as usual by the Poisson integral formula. Then, $\mathsf{F}L^2(t,\infty) = e^{itz}H^2(\mathbb{C}_+)$ is a reproducing kernel Hilbert space consisting of holomorphic functions on $\mathbb{C}_+$. In particular, the evaluation $F \mapsto F(z)$ is continuous for all $z \in \mathbb{C}_+$. The reproducing kernel is $\frac{1}{2\pi}\langle(1-\mathsf{P}_t)e_w, (1-\mathsf{P}_t)e_z\rangle = ie^{it(w-\bar{z})}/(2\pi(w-\bar{z}))$, where $e_z(x) = \exp(izx)$.

Lemma 5.3. — *Let* $t \in \mathbb{R}$. *If* $\mathcal{V}_t(u) \neq \{0\}$, *then there exists* $Y_z^t \in \mathcal{V}_t(u)$ *for each* $z \in \mathbb{C}_+$ *such that* $\langle f, \overline{Y_z^t}\rangle = (\mathsf{F}f)(z)$ *holds for all* $f \in \mathcal{V}_t(u)$. *Actually,* Y_z^t *is the orthogonal projection of* $(1-\mathsf{P}_t)e_z \in L^2(\mathbb{R})$ *to* $\mathcal{V}_t(u)$.

Proof. — $\mathsf{F}(\mathcal{V}_t(u))$ is a reproducing kernel Hilbert space consisting of functions on $\mathbb{C}_+$, since it is a closed subspace of $\mathsf{F}L^2(t,\infty)$ by definition. Therefore, $\mathsf{F}f \mapsto \mathsf{F}f(z)$ is continuous on $\mathsf{F}(\mathcal{V}_t(u))$ for $z \in \mathbb{C}_+$. Hence $f \mapsto \mathsf{F}f(z)$ is a linear continuous functional on $\mathcal{V}_t(u)$, and thus Y_z^t exists by the Riesz representation theorem. If we have the decomposition $(1-\mathsf{P}_t)e_z = P_z^t + Q_z^t$ with $P_z^t \in \mathcal{V}_t(u)$ and $Q_z^t \in \mathcal{V}_t(u)^\perp$,

$$\begin{aligned}\int f(x)e^{izx}\,dx &= \int f(x)P_z^t(x)\,dx + \int f(x)Q_z^t(x)\,dx \\ &= \int f(x)P_z^t(x)\,dx + 0 = \left\langle f, \overline{P_z^t}\right\rangle\end{aligned}$$

for all $f \in \mathcal{V}_t(u)$. Thus Y_z^t coincides with the orthogonal projection P_z^t of $(1-\mathsf{P}_t)e_z$. □

Lemma 5.4. — *Let $u \in U^1_{\mathrm{loc}}(\mathbb{R})$ with $D \cap \mathbb{C}_- \neq \emptyset$ and let $t \in \mathbb{R}$. If $\mathcal{V}_t(u) \neq \{0\}$, each function $F \in \mathsf{F}(\mathcal{V}_t(u))$ extends to a function on $\mathbb{C}_+ \cup D$ and is meromorphic on $D \cap \mathbb{C}_-$. If u is holomorphic in a neighborhood of the interval $(a,b) \subset \mathbb{R}$, all $F \in \mathsf{F}(\mathcal{V}_t(u))$ are holomorphic there. Moreover, $\langle f, \overline{Y_z^t}\rangle = (\mathsf{F}f)(z)$ holds for all $f \in \mathcal{V}_t(u)$ and $z \in \mathbb{C}_+ \cup D$.*

Proof. — Let $F = \mathsf{F}f$ for $f \in \mathcal{V}_t(u)$. Then $F = \mathsf{F}\mathsf{K}(\mathsf{K}f) = \mathsf{M}_u\mathsf{J}^\sharp\mathsf{F}\mathsf{K}f = \mathsf{M}_u\mathsf{F}\mathsf{J}_\sharp\mathsf{K}f$. On the right-hand side, $\mathsf{F}\mathsf{J}_\sharp\mathsf{K}f$ is defined and $F \mapsto F(z)$ is continuous on $\mathsf{F}(\mathcal{V}_t(u))$ for $z \in \mathbb{C}_- \cup \mathbb{R}$, since $\mathsf{J}_\sharp\mathsf{K}f \in L^2(-\infty,-t)$. We have $\lim_{z\to x} u(z) = u(x)$ and $\lim_{z\to x}(\mathsf{F}f)(z) = (\mathsf{F}f)(x)$ for almost all $x \in \mathbb{R}$ if z tends to x non-tangentially inside $\mathbb{C}_+$ and $\mathbb{C}_-$. Hence F is holomorphic in a neighborhood of $(a,b) \subset \mathbb{R}$ if u is holomorphic there. The evaluation $f \mapsto \mathsf{F}f(z)$ is continuous for $z \in \mathbb{C}_+$ and $z \in \mathbb{C}_- \cap D$, and therefore it is also continuous for almost all $z \in \mathbb{R}$ by the Banach–Steinhaus theorem. Hence there exists $Y_z^t \in \mathcal{V}_t(u)$ such that $\langle f, \overline{Y_z^t}\rangle = (\mathsf{F}f)(z)$ for $z \in \mathbb{C}_-$ and for almost all $z \in \mathbb{R}$. □

Lemma 5.5. — *Let $t \in \mathbb{R}$. Suppose that $\|\mathsf{K}[t]\|_{\mathrm{op}} < 1$. Then,*

$$\mathcal{V}_t(u)^\perp = L^2(-\infty,t) + \mathsf{K}\left(L^2(-\infty,t)\right). \tag{5.7}$$

Proof. — It is proved by almost the same argument as the proof of [20, Lemma 4.2]. □

Proposition 5.6. — *Let $t \in \mathbb{R}$ and let $e_z(x) = \exp(izx)$ for $z \in \mathbb{C}_+$. Suppose that $\|\mathsf{K}[t]\|_{\mathrm{op}} < 1$ and $\mathcal{V}_t(u) \neq \{0\}$. Then the equations*

$$\left(a_z^t - \mathsf{P}_t e_z\right) + \mathsf{K}\mathsf{P}_t\left(a_z^t - \mathsf{P}_t e_z\right) = (1-\mathsf{P}_t)e_z + \mathsf{K}(1-\mathsf{P}_t)e_z, \tag{5.8}$$

$$\left(b_z^t - \mathsf{P}_t e_z\right) - \mathsf{K}\mathsf{P}_t\left(b_z^t - \mathsf{P}_t e_z\right) = (1-\mathsf{P}_t)e_z - \mathsf{K}(1-\mathsf{P}_t)e_z \tag{5.9}$$

for functions $a_z^t = a_z^t(x)$ *and* $b_z^t = b_z^t(x)$ *on* $\mathbb{R}$ *have unique solutions with conditions* $a_z^t - \mathsf{P}_t e_z \in L^2(\mathbb{R})$ *and* $b_z^t - \mathsf{P}_t e_z \in L^2(\mathbb{R})$*. Moreover,*

$$Y_z^t = (1 - \mathsf{P}_t)\frac{1}{2}\left(a_z^t + b_z^t\right). \tag{5.10}$$

Remark 5.7. — If $\mathsf{K}e_z$ makes sense, equations (5.8) and (5.9) are simplified as

$$a_z^t + \mathsf{K}\mathsf{P}_t a_z^t = e_z + \mathsf{K}e_z, \qquad b_z^t - \mathsf{K}\mathsf{P}_t b_z^t = e_z - \mathsf{K}e_z,$$

and their solutions are considered according to the class of $\mathsf{K}e_z$.

Proof. — Since $(1 - \mathsf{P}_t)e_z \in L^2(\mathbb{R})$ for $z \in \mathbb{C}_+$, $\mathsf{K}(1 - \mathsf{P}_t)e_z$ belongs to $L^2(\mathbb{R})$. Therefore, (5.8) and (5.9) are equations for functions $(a_z^t - \mathsf{P}_t e_z)$ and $(b_z^t - \mathsf{P}_t e_z)$ in $L^2(\mathbb{R})$. Multiplying by P_t on both sides and then substituting the obtained formulas for $\mathsf{P}_t(a_z^t - \mathsf{P}_t e_z)$ and $\mathsf{P}_t(b_z^t - \mathsf{P}_t e_z)$ into (5.8) and (5.9), we find that

$$\begin{aligned} a_z^t &= e_z + \mathsf{K}(1 - \mathsf{P}_t)e_z - \mathsf{K}(1 + \mathsf{K}[t])^{-1}\mathsf{P}_t\mathsf{K}(1 - \mathsf{P}_t)e_z, \\ b_z^t &= e_z - \mathsf{K}(1 - \mathsf{P}_t)e_z - \mathsf{K}(1 - \mathsf{K}[t])^{-1}\mathsf{P}_t\mathsf{K}(1 - \mathsf{P}_t)e_z \end{aligned} \tag{5.11}$$

are unique solutions of (5.8) and (5.9) with conditions $a_z^t - \mathsf{P}_t e_z$, $b_z^t - \mathsf{P}_t e_z \in L^2(\mathbb{R})$. Formulas in (5.11) show that both $(1 - \mathsf{P}_t)a_z^t$ and $(1 - \mathsf{P}_t)b_z^t$ belong to $L^2(t, \infty)$.

Let us prove the formula (5.10). By (5.7), there exists unique vectors u_z^t and v_z^t in $L^2(-\infty, t)$ such that

$$(1 - \mathsf{P}_t)e_z = Y_z^t + u_z^t + \mathsf{K}v_z^t. \tag{5.12}$$

Put $U_z^t = (a_z^t + b_z^t)/2 - \mathsf{P}_t e_z$ and $V_z^t = (a_z^t - b_z^t)/2$. Then, $U_z^t + \mathsf{K}\mathsf{P}_t V_z^t = (1 - \mathsf{P}_t)e_z$ and $V_z^t + \mathsf{K}\mathsf{P}_t U_z^t = \mathsf{K}(1 - \mathsf{P}_t)e_z$ by (5.8) and (5.9). By multiplying K on both sides of the second equation, $(1 - \mathsf{P}_t)e_z = \mathsf{K}(1 - \mathsf{P}_t)V_z^t + \mathsf{P}_t U_z^t + \mathsf{K}\mathsf{P}_t V_z^t$. Therefore, $Y_z^t = \mathsf{K}(1 - \mathsf{P}_t)V_z^t$, $u_z^t = \mathsf{P}_t U_z^t$ and $v_z^t = \mathsf{P}_t V_z^t$. Moreover,

$$\begin{aligned} Y_z^t &= \mathsf{K}(1 - \mathsf{P}_t)V_z^t = \mathsf{K}V_z^t - \mathsf{K}\mathsf{P}_t V_z^t \\ &= \big((1 - \mathsf{P}_t)e_z - \mathsf{P}_t U_z^t\big) + \big(U_z^t - (1 - \mathsf{P}_t)e_z\big) = (1 - \mathsf{P}_t)U_z^t. \end{aligned}$$

Hence, Y_z^t belongs to $\mathcal{V}_t(u)$ and formula (5.10) follows from the uniqueness of decomposition (5.12). □

Proposition 5.8. — *Let $t \in \mathbb{R}$ and $z \in \mathbb{C}_+$. Suppose that $\|\mathsf{K}[t]\|_{\mathrm{op}} < 1$ and that* (O3), (O4) *are satisfied. Then the following equality holds:*

$$\left(\frac{\partial}{\partial x} - iz\right)\left(a_z^t(x) + b_z^t(x)\right) = \frac{1}{2}\Big(\left(a_z^t(t) + b_z^t(t)\right) - \overline{\left(a_z^t(t) - b_z^t(t)\right)}\Big)\phi^+(t,x) - \frac{1}{2}\Big(\left(a_z^t(t) + b_z^t(t)\right) + \overline{\left(a_z^t(t) - b_z^t(t)\right)}\Big)\phi^-(t,x). \tag{5.13}$$

Proof. — Equation (5.13) is proved by showing that both sides satisfy the same equation having a unique solution. By taking the sum and difference of (5.8) and (5.9), we obtain

$$\left(a_z^t + b_z^t - 2\mathsf{P}_t e_z\right) + \mathsf{K}\mathsf{P}_t\left(a_z^t - b_z^t\right) = 2(1 - \mathsf{P}_t)e_z, \tag{5.14}$$

$$\left(a_z^t - b_z^t\right) + \mathsf{K}\mathsf{P}_t\left(a_z^t + b_z^t - 2\mathsf{P}_t e_z\right) = 2\,\mathsf{K}(1 - \mathsf{P}_t)e_z, \tag{5.15}$$

respectively. Substituting $(a_z^t + b_z^t - 2\mathsf{P}_t e_z)$ with $(a_z^t + b_z^t - 2e_z)$ in (5.15), since it dose not change the equation, then multiplying by $\mathsf{K}\mathsf{P}_t$ and subtracting from (5.14) yields

$$\left(a_z^t + b_z^t\right) - \mathsf{K}\mathsf{P}_t\mathsf{K}\mathsf{P}_t\left(a_z^t + b_z^t - 2e_z\right) = 2e_z - 2\mathsf{K}\mathsf{P}_t\mathsf{K}(1 - \mathsf{P}_t)e_z. \tag{5.16}$$

Using (4.3) and (4.4), repeatedly, the derivative of the second term of the left-hand side and the derivative of the right-hand side with respect to x are calculated as

$$\frac{\partial}{\partial x}\mathsf{K}\mathsf{P}_t\mathsf{K}\mathsf{P}_t\left(a_z^t + b_z^t - 2e_z\right) = \mathsf{K}\delta_t\mathsf{P}_t\mathsf{K}\mathsf{P}_t\left(a_z^t + b_z^t - 2e_z\right) - \mathsf{K}\mathsf{P}_t\mathsf{K}\delta_t\left(a_z^t + b_z^t - 2e_z\right) + \mathsf{K}\mathsf{P}_t\mathsf{K}\mathsf{P}_t\frac{\partial}{\partial x}\left(a_z^t + b_z^t - 2e_z\right)$$

and

$$\frac{\partial}{\partial x}\Big(2e_z - 2\mathsf{K}\mathsf{P}_t\mathsf{K}(1 - \mathsf{P}_t)e_z\Big) = 2iz\left(e_z - \mathsf{K}\mathsf{P}_t\mathsf{K}(1 - \mathsf{P}_t)e_z\right) - 2\mathsf{K}\mathsf{P}_t\mathsf{K}\delta_t e_z - 2\mathsf{K}\left(\delta_t\mathsf{K}(1 - \mathsf{P}_t)e_z\right),$$

respectively. Therefore,

$$\frac{\partial}{\partial x}\left(a_z^t + b_z^t\right) - \mathsf{K}\mathsf{P}_t\mathsf{K}\mathsf{P}_t\frac{\partial}{\partial x}\left(a_z^t + b_z^t - 2e_z\right) = 2iz\Big(e_z - \mathsf{K}\mathsf{P}_t\mathsf{K}(1 - \mathsf{P}_t)e_z\Big) - 2\mathsf{K}\mathsf{P}_t\mathsf{K}\delta_t e_z - 2\mathsf{K}(\delta_t\mathsf{K}(1 - \mathsf{P}_t)e_z) + \mathsf{K}\delta_t\mathsf{P}_t\mathsf{K}\mathsf{P}_t\left(a_z^t + b_z^t - 2e_z\right) - \mathsf{K}\mathsf{P}_t\mathsf{K}\delta_t\left(a_z^t + b_z^t - 2e_z\right).$$

Multiplying $(-iz)$ by (5.16) and then adding to this,

$$\left(\frac{\partial}{\partial x}-iz\right)\left(a_z^t+b_z^t\right)-\mathsf{KP}_t\mathsf{KP}_t\left(\frac{\partial}{\partial x}-iz\right)\left(a_z^t+b_z^t-2e_z\right)$$
$$=-2\mathsf{KP}_t\mathsf{K}\delta_t e_z-2\mathsf{K}\left(\delta_t\mathsf{K}(1-\mathsf{P}_t)e_z\right)$$
$$+\mathsf{K}\delta_t\mathsf{P}_t\mathsf{KP}_t\left(a_z^t+b_z^t-2e_z\right)-\mathsf{KP}_t\mathsf{K}\delta_t\mathsf{P}_t\left(a_z^t+b_z^t-2e_z\right).$$

Using (5.15), the right-hand side is calculated as

$$-\overline{(a_z^t(t)-b_z^t(t))}\mathsf{K}\delta_t-\left(a_z^t(t)+b_z^t(t)\right)\mathsf{KP}_t\mathsf{K}\delta_t,$$

and e_z in the second term on the left-hand side can be removed, since $(\partial/\partial x-iz)e_z=0$. Hence we obtain

$$(1-\mathsf{KP}_t\mathsf{KP}_t)\left(\frac{\partial}{\partial x}-iz\right)\left(a_z^t+b_z^t\right)=-\overline{(a_z^t(t)-b_z^t(t))}\mathsf{K}\delta_t-\left(a_z^t(t)+b_z^t(t)\right)\mathsf{KP}_t\mathsf{K}\delta_t. \tag{5.17}$$

On the other hand, we have

$$\phi^+-\mathsf{KP}_t\mathsf{KP}_t\phi^+=\mathsf{K}\delta_t-\mathsf{KP}_t\mathsf{K}\delta_t,\qquad \phi^--\mathsf{KP}_t\mathsf{KP}_t\phi^-=\mathsf{K}\delta_t+\mathsf{KP}_t\mathsf{K}\delta_t$$

by (5.1) and (5.2). By taking a linear combination of these two equations,

$$(1-\mathsf{KP}_t\mathsf{KP}_t)\left((b_0+ia_1)\phi^+-(a_0+ib_1)\phi^-\right)$$
$$=-\overline{(a-b)}\mathsf{K}\delta_t-(a+b)\mathsf{KP}_t\mathsf{K}\delta_t$$

for $a=a_0+ia_1$, $b=b_0+ib_1$ with $a_0,a_1,b_0,b_1\in\mathbb{R}$. Taking a and b as $a_z^t(t)$ and $b_z^t(t)$, $((b_0+ia_1)\phi^+-(a_0+ib_1)\phi^-)$ satisfies the same equation as (5.17), so (O4) leads (5.13), since $1-\mathsf{KP}_t\mathsf{KP}_t=(1-\mathsf{KP}_t)(1+\mathsf{KP}_t)$. ☐

Proposition 5.9. — *Let $t\in\mathbb{R}$ and $z\in\mathbb{C}_+$. Suppose that $\|\mathsf{K}[t]\|_{\mathrm{op}}<1$ and that* (O3), (O4) *are satisfied. Then the following equalities hold:*

$$\frac{1}{2}\left[\left(a_z^t(t)+b_z^t(t)\right)-\overline{(a_z^t(t)-b_z^t(t))}\right]=e^{izt}-\int_t^\infty\overline{\phi^-(t,x)}e^{izx}\,dx, \tag{5.18}$$

$$\frac{1}{2}\left[\left(a_z^t(t)+b_z^t(t)\right)+\overline{(a_z^t(t)-b_z^t(t))}\right]=e^{izt}+\int_t^\infty\overline{\phi^+(t,x)}e^{izx}\,dx. \tag{5.19}$$

Proof. — It is sufficient to show that

$$\left(a_z^t(t)+b_z^t(t)\right)=2e^{izt}+\int_t^\infty\overline{(\phi^+(t,x)-\phi^-(t,x))}e^{izx}\,dx, \tag{5.20}$$

$$\overline{(a_z^t(t)-b_z^t(t))}=\int_t^\infty\overline{(\phi^+(t,x)+\phi^-(t,x))}e^{izx}\,dx, \tag{5.21}$$

because (5.18) and (5.19) are obtained from the difference and sum of (5.20) and (5.21), respectively. In this proof, we use the paring symbol $\langle f,g\rangle=$

$\int f(x)\overline{g(x)}\,dx$ for simplification of the description, if the right-hand side makes sense. In particular, we describe the point evaluation of f by $f(t) = \langle f, \delta_t\rangle$ and $\overline{f(t)} = \langle \delta_t, f\rangle$. Note that K is self-adjoint with respect to this paring:

$$\langle \mathsf{K}f, g\rangle = \langle \mathsf{K}g, f\rangle. \tag{5.22}$$

First, we prove (5.20). Taking the pairing of both sides of $(a_z^t + b_z^t - 2\mathsf{P}_t e_z) = 2(1-\mathsf{P}_t)e_z - \mathsf{K}\mathsf{P}_t(a_z^t - b_z^t)$ and $\mathsf{P}_t\mathsf{K}\delta_t - \mathsf{P}_t\mathsf{K}\mathsf{P}_t\phi_t^+ = \mathsf{P}_t\phi^+$ obtained from (5.14) and (5.1), respectively, we obtain

$$\left\langle \left(a_z^t + b_z^t - 2\mathsf{P}_t e_z\right), \mathsf{P}_t\left(\mathsf{K}\delta_t - \mathsf{K}\mathsf{P}_t\phi^+\right)\right\rangle \\ = \left\langle 2(1-\mathsf{P}_t)e_z - \mathsf{K}\mathsf{P}_t\left(a_z^t - b_z^t\right), \mathsf{P}_t\phi^+\right\rangle.$$

Using the orthogonality of P_t and $(1-\mathsf{P}_t)$ on the right-hand side,

$$\left\langle \left(a_z^t + b_z^t - 2\mathsf{P}_t e_z\right), \mathsf{P}_t\mathsf{K}\delta_t\right\rangle - \left\langle \left(a_z^t + b_z^t - 2\mathsf{P}_t\right), \mathsf{P}_t\mathsf{K}\mathsf{P}_t\phi^+\right\rangle \\ = -\left\langle \mathsf{K}\mathsf{P}_t\left(a_z^t - b_z^t\right), \mathsf{P}_t\phi^+\right\rangle. \tag{5.23}$$

By a similar argument, we obtain

$$\left\langle \left(a_z^t + b_z^t - 2\mathsf{P}_t e_z\right), \mathsf{P}_t\mathsf{K}\delta_t\right\rangle + \left\langle \left(a_z^t + b_z^t - 2\mathsf{P}_t\right), \mathsf{P}_t\mathsf{K}\mathsf{P}_t\phi^-\right\rangle \\ = -\left\langle \mathsf{K}\mathsf{P}_t\left(a_z^t - b_z^t\right), \mathsf{P}_t\phi^-\right\rangle \tag{5.24}$$

from (5.2) and (5.14). On the other hand, taking the pairing of (5.1) and the equality $(a_z^t - b_z^t) = 2\,\mathsf{K}(1-\mathsf{P}_t)e_z - \mathsf{K}\mathsf{P}_t(a_z^t + b_z^t - 2\mathsf{P}_t e_z)$ obtained from (5.15),

$$\left\langle \mathsf{P}_t\mathsf{K}\delta_t, \left(a_z^t - b_z^t\right)\right\rangle - \left\langle \mathsf{P}_t\mathsf{K}\mathsf{P}_t\phi^+, \left(a_z^t - b_z^t\right)\right\rangle \\ = 2\left\langle \mathsf{P}_t\phi^+, \mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle - \left\langle \mathsf{P}_t\phi^+, \mathsf{K}\mathsf{P}_t\left(a_z^t + b_z^t - 2\mathsf{P}_t e_z\right)\right\rangle. \tag{5.25}$$

By a similar argument, we obtain

$$\left\langle \mathsf{P}_t\mathsf{K}\delta_t, \left(a_z^t - b_z^t\right)\right\rangle + \left\langle \mathsf{P}_t\mathsf{K}\mathsf{P}_t\phi^-, \left(a_z^t - b_z^t\right)\right\rangle \\ = 2\left\langle \mathsf{P}_t\phi^-, \mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle - \left\langle \mathsf{P}_t\phi^-, \mathsf{K}\mathsf{P}_t\left(a_z^t + b_z^t - 2\mathsf{P}_t e_z\right)\right\rangle \tag{5.26}$$

from (5.2) and (5.15).

On the other hand, we have

$$\left(a_z^t + b_z^t\right)(t) = \left\langle a_z^t + b_z^t, \delta_t\right\rangle = 2e_z(t) - \left\langle \mathsf{P}_t\mathsf{K}\delta_t, \left(a_z^t - b_z^t\right)\right\rangle, \tag{5.27}$$

$$\overline{\left(a_z^t - b_z^t\right)(t)} = 2\left\langle \delta_t, \mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle - \left\langle \left(a_z^t + b_z^t - 2\mathsf{P}_t\right), \mathsf{P}_t\mathsf{K}\delta_t\right\rangle \tag{5.28}$$

by the convention of the symbol, (5.14), and (5.15). Therefore,

$$(5.29)\quad \left(a_z^t + b_z^t\right)(t) - \overline{\left(a_z^t - b_z^t\right)(t)} = 2e_z(t) - 2\left\langle \delta_t, \mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle \\ + \left\langle \left(a_z^t + b_z^t - 2\mathsf{P}_t e_z\right), \mathsf{P}_t\mathsf{K}\delta_t\right\rangle - \left\langle \mathsf{P}_t\mathsf{K}\delta_t, \left(a_z^t - b_z^t\right)\right\rangle .$$

The right-hand side is equal to

$$(5.30)\quad 2e_z(t) - 2\left\langle \delta_t, \mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle - \frac{1}{2}\left\langle \mathsf{K}\mathsf{P}_t\left(a_z^t - b_z^t\right), \mathsf{P}_t\left(\phi^+ + \phi^-\right)\right\rangle \\ + \frac{1}{2}\left\langle \left(a_z^t + b_z^t - 2\mathsf{P}_t e_z\right), \mathsf{P}_t\mathsf{K}\mathsf{P}_t\left(\phi^+ - \phi^-\right)\right\rangle \\ - \left\langle \mathsf{P}_t\left(\phi^+ + \phi^-\right), \mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle \\ + \frac{1}{2}\left\langle \mathsf{P}_t\left(\phi^+ + \phi^-\right), \mathsf{K}\mathsf{P}_t\left(a_z^t + b_z^t - 2\mathsf{P}_t e_z\right)\right\rangle \\ - \frac{1}{2}\left\langle \mathsf{P}_t\mathsf{K}\mathsf{P}_t\left(\phi^+ - \phi^-\right), \left(a_z^t - b_z^t\right)\right\rangle$$

by using (5.23) and (5.24) to the third term and (5.25) and (5.26) to the fourth term of the right-hand side. Using (5.22), we find that the sum of the fourth and sixth terms of (5.30) is equal to $\langle (a_z^t + b_z^t - 2\mathsf{P}_t), \mathsf{P}_t\mathsf{K}\mathsf{P}_t\phi^+\rangle$ and that the sum of the third and seventh terms of (5.30) is equal to $-\langle \mathsf{P}_t\mathsf{K}\mathsf{P}_t\phi^+, (a_z^t - b_z^t)\rangle$. Therefore, (5.29) is

$$(5.31)\quad \left(a_z^t + b_z^t\right)(t) - \overline{\left(a_z^t - b_z^t\right)(t)} \\ = 2e_z(t) - 2\left\langle \delta_t, \mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle - \left\langle \mathsf{P}_t\left(\phi^+ + \phi^-\right), \mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle \\ - \left\langle \mathsf{P}_t\mathsf{K}\mathsf{P}_t\phi^+, \left(a_z^t - b_z^t\right)\right\rangle + \left\langle \left(a_z^t + b_z^t - 2\mathsf{P}_t e_z\right), \mathsf{P}_t\mathsf{K}\mathsf{P}_t\phi^+\right\rangle .$$

The sum of the fourth and fifth terms of the right-hand side is equal to $\langle \delta_t, \mathsf{K}\mathsf{P}_t(a_z^t + b_z^t - 2\mathsf{P}_t e_z)\rangle$ by (5.22) and (5.23), and that is equal to $2\langle \delta_t, \mathsf{K}(1-\mathsf{P}_t)e_z\rangle - \overline{(a_z^t - b_z^t)(t)}$ by (5.15). As a result, we obtain

$$\left(a_z^t + b_z^t\right)(t) = 2e_z(t) - \left\langle \mathsf{P}_t\left(\phi^+ + \phi^-\right), \mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle$$

from (5.31). The second term of the right-hand side is calculated as

$$\left\langle \mathsf{P}_t\left(\phi^+ + \phi^-\right), \mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle = \left\langle (1-\mathsf{P}_t)e_z, \mathsf{K}\mathsf{P}_t\left(\phi^+ + \phi^-\right)\right\rangle \\ = -\left\langle (1-\mathsf{P}_t)e_z, \left(\phi^+ - \phi^-\right)\right\rangle$$

by (5.22) and $(\phi^+ - \phi^-) + \mathsf{K}\mathsf{P}_t(\phi^+ + \phi^-) = 0$ obtained from (5.1) and (5.2). Hence we obtain (5.20).

Next, we prove (5.21). The following process looks similar to the proof of (5.20), but actually different. We have

$$(5.32)\quad \begin{aligned}\left(a_z^t+b_z^t\right)(t)+\overline{\left(a_z^t-b_z^t\right)(t)}=&\ 2e_z(t)+2\left\langle\delta_t,\mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle\\&+\frac{1}{2}\left\langle\mathsf{KP}_t\left(a_z^t-b_z^t\right),\mathsf{P}_t\left(\phi^++\phi^-\right)\right\rangle\\&-\frac{1}{2}\left\langle\left(a_z^t+b_z^t-2\mathsf{P}_te_z\right),\mathsf{P}_t\mathsf{KP}_t\left(\phi^+-\phi^-\right)\right\rangle\\&-\left\langle\mathsf{P}_t\left(\phi^++\phi^-\right),\mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle\\&+\frac{1}{2}\left\langle\mathsf{P}_t\left(\phi^++\phi^-\right),\mathsf{KP}_t\left(a_z^t+b_z^t-2\mathsf{P}_te_z\right)\right\rangle\\&-\frac{1}{2}\left\langle\mathsf{P}_t\mathsf{KP}_t\left(\phi^+-\phi^-\right),\left(a_z^t-b_z^t\right)\right\rangle\end{aligned}$$

by using (5.25) and (5.26) on the second term of the right-hand side of (5.27), and using (5.23) and (5.24) on the second term of the right-hand side of (5.28). The right-hand side of (5.32) is equal to

$$(5.33)\quad \begin{aligned}&2e_z(t)+2\left\langle\delta_t,\mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle-\left\langle\mathsf{P}_t\left(\phi^++\phi^-\right),\mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle\\&\quad+\left\langle\mathsf{P}_t\mathsf{KP}_t\phi^-,\left(a_z^t-b_z^t\right)\right\rangle+\left\langle\left(a_z^t+b_z^t-2\mathsf{P}_te_z\right),\mathsf{P}_t\mathsf{KP}_t\phi^-\right\rangle\end{aligned}$$

by (5.22) as well as the proof of (5.20).

By taking the paring of $\mathsf{P}_t\phi^-$ and $(a_z^t-b_z^t)=2\mathsf{K}(1-\mathsf{P}_t)e_z-\mathsf{KP}_t(a_z^t+b_z^t-2\mathsf{P}_te_z)$ obtained from (5.15),

$$\begin{aligned}&\left\langle\mathsf{P}_t\phi^-,\left(a_z^t-b_z^t\right)\right\rangle\\&\qquad=2\left\langle\mathsf{P}_t\phi^-,\mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle-\left\langle\mathsf{P}_t\phi^-,\mathsf{KP}_t\left(a_z^t+b_z^t-2\mathsf{P}_te_z\right)\right\rangle.\end{aligned}$$

Using (5.22) on the second term of the right-hand side, the equality becomes

$$\begin{aligned}&\left\langle\left(a_z^t+b_z^t-2\mathsf{P}_te_zz\right),\mathsf{P}_t\mathsf{KP}_t\phi^-z\right\rangle\\&\qquad=2\left\langle\mathsf{P}_t\phi^-,\mathsf{K}(1-\mathsf{P}_t)e_zz\right\rangle-\left\langle\mathsf{P}_t\phi^-,\left(a_z^t-b_z^tz\right)z\right\rangle.\end{aligned}$$

Substituting this in the last term of (5.33),

$$\begin{aligned}&\left(a_z^t+b_z^tz\right)(t)+\overline{\left(a_z^t-b_z^tz\right)(t)}=2e_z(t)+2\left\langle\delta_t,\mathsf{K}(1-\mathsf{P}_t)e_zz\right\rangle\\&-\left\langle\mathsf{P}_t\left(\phi^+-\phi^-z\right),\mathsf{K}(1-\mathsf{P}_t)e_zz\right\rangle+\left\langle\mathsf{P}_t\mathsf{KP}_t\phi^--\mathsf{P}_t\phi^-,\left(a_z^t-b_z^tz\right)z\right\rangle.\end{aligned}$$

The last term of the right-hand side is equal to $-\langle\mathsf{KP}_t(a_z^t-b_z^t),\delta_t\rangle$ by (5.2) and (5.22). Then $-\langle\mathsf{KP}_t(a_z^t-b_z^t),\delta_t\rangle=\langle(a_z^t+b_z^t),\delta_t\rangle-2\langle e_z,\delta_t\rangle$ by arranging the left-hand side of the equality $\langle(a_z^t+b_z^t-2\mathsf{P}_te_z),\delta_t\rangle-2\langle(1-\mathsf{P}_t)e_z,\delta_t\rangle=$

$-\langle \mathsf{KP}_t(a_z^t - b_z^t), \delta_t\rangle$ obtained from (5.14). Therefore,

$$\begin{aligned}&\left(a_z^t + b_z^t\right)(t) + \overline{\left(a_z^t - b_z^t\right)(t)}\\&\quad = 2\left\langle \delta_t, \mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle - \left\langle \mathsf{P}_t\left(\phi^+ - \phi^-\right), \mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle + \left(a_z^t + b_z^t\right)(t),\end{aligned}$$

and thus

$$\overline{\left(a_z^t - b_z^t\right)(t)} = 2\left\langle \delta_t, \mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle - \left\langle \mathsf{P}_t\left(\phi^+ - \phi^-\right), \mathsf{K}(1-\mathsf{P}_t)e_z\right\rangle.$$

The second term of the right-hand side is equal to $\langle (1-\mathsf{P}_t)e_z, \mathsf{KP}_t(\phi^+ - \phi^-)\rangle$, and it is further equal to $2\langle (1-\mathsf{P}_t)e_z, \mathsf{K}\delta_t\rangle - \langle (1-\mathsf{P}_t)e_z, (\phi^+ + \phi^-)\rangle$, since $\mathsf{KP}_t(\phi^+ - \phi^-) = 2\mathsf{K}\delta_t - (\phi^+ + \phi^-)$ by (5.1) and (5.2). Hence, we obtain (5.21). □

Proposition 5.10. — *Let $t \in \mathbb{R}$ and $z \in \mathbb{C}_+$. Suppose that $\|\mathsf{K}[t]\|_{\mathrm{op}} < 1$ and that* (O3), (O4) *are satisfied. Then the following equalities hold:*

$$\begin{aligned}&\frac{1}{2}\left[e^{izt} + \int_t^\infty \phi^+(t,x)e^{izx}\,dx\right]\\&\quad = \frac{\Re(\Phi(t,t))}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\widetilde{A}(t,z) - \frac{\Im(\Psi(t,t))}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\cdot i\cdot\left(-i\widetilde{B}(t,z)\right),\end{aligned} \tag{5.34}$$

$$\begin{aligned}&\frac{1}{2}\left[e^{izt} - \int_t^\infty \phi^-(t,x)e^{izx}\,dx\right]\\&\quad = -\frac{\Im(\Phi(t,t))}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\cdot i\cdot\widetilde{A}(t,z) + \frac{\Re(\Psi(t,t))}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\left(-i\widetilde{B}(t,z)\right).\end{aligned} \tag{5.35}$$

Proof. — By definition (2.5), (4.4), and the derivative rule for F,

$$\begin{aligned}2\widetilde{A}(t,z) &= \Psi(t,t)e^{izt} + \int_t^\infty \frac{\partial}{\partial x}\Psi(t,x)e^{izx}\,dx,\\-2i\widetilde{B}(t,z) &= \Phi(t,t)e^{izt} + \int_t^\infty \frac{\partial}{\partial x}\Phi(t,x)e^{izx}\,dx.\end{aligned}$$

Combining these with (5.3) and (5.4),

$$\begin{aligned}&\int_t^\infty \phi^+(t,x)e^{izx}\,dx\\&\quad = \frac{\Re(\Phi(t,t))}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\left(2\widetilde{A}(t,z)\right) - \frac{\Im(\Psi(t,t))}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\left(2\widetilde{B}(t,z)\right) - e^{izt},\end{aligned}$$

$$\begin{aligned}&\int_t^\infty \phi^-(t,x)e^{izx}\,dx\\&= \frac{\Im(\Phi(t,t))}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\left(2i\widetilde{A}(t,z)\right) - \frac{\Re(\Psi(t,t))}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\left(-2i\widetilde{B}(t,z)\right) + e^{izt}.\end{aligned}$$

Hence we obtain (5.34) and (5.35). □

5.2. Proof of Theorem 2.3

The first equality of (2.10) is shown in the same way as in the proof of [20, Theorem 4.1]. To prove the second equality of (2.10), we calculate the Fourier transform of Y_z^t. We have

$$\begin{aligned}
&-i(z+w)\int_0^\infty Y_z^t(x)e^{iwx}\,dx\\
&\quad= -i(z+w)\int_t^\infty \frac{1}{2}\left(a_z^t(x)+b_z^t(x)\right)e^{iwx}\,dx\\
&\quad= \int_t^\infty \frac{1}{2}\left(\frac{\partial}{\partial x}-iz\right)\left(a_z^t(x)+b_z^t(x)\right)e^{iwx}\,dx+\frac{1}{2}\left(a_z^t(t)+b_z^t(t)\right)e^{iwt}
\end{aligned}$$

by (5.10) for $z,w\in\mathbb{C}_+$. The right-hand side is calculated as

$$\begin{aligned}
&\frac{1}{2}b_z^t(t)e^{iwt}+\frac{1}{4}\left(\left(a_z^t(t)+b_z^t(t)\right)-\overline{\left(a_z^t(t)-b_z^t(t)\right)}\right)\int_t^\infty \phi^+(t,x)e^{iwx}\,dx\\
&+\frac{1}{2}a_z^t(t)e^{iwt}-\frac{1}{4}\left(\left(a_z^t(t)+b_z^t(t)\right)+\overline{\left(a_z^t(t)-b_z^t(t)\right)}\right)\int_t^\infty \phi^-(t,x)e^{iwx}\,dx\\
&\quad=\frac{1}{4}\left[\left(a_z^t(t)+b_z^t(t)\right)-\overline{\left(a_z^t(t)-b_z^t(t)\right)}\right]\left[e^{iwt}+\int_t^\infty \phi^+(t,x)e^{iwx}\,dx\right]\\
&\qquad+\frac{1}{4}\left[\left(a_z^t(t)+b_z^t(t)\right)+\overline{\left(a_z^t(t)-b_z^t(t)\right)}\right]\left[e^{iwt}-\int_t^\infty \phi^-(t,x)e^{iwx}\,dx\right]
\end{aligned}$$

by (5.13). Therefore,

$$\begin{aligned}
&-i(z+w)\int_0^\infty Y_z^t(x)e^{iwx}\,dx\\
&\quad=\frac{1}{2}\left[e^{izt}-\int_t^\infty \overline{\phi^-(t,x)}e^{izx}\,dx\right]\left[e^{iwt}+\int_t^\infty \phi^+(t,x)e^{iwx}\,dx\right]\\
&\qquad+\frac{1}{2}\left[e^{izt}+\int_t^\infty \overline{(\phi^+(t,x))}e^{izx}\,dx\right]\left[e^{iwt}-\int_t^\infty \phi^-(t,x)e^{iwx}\,dx\right]
\end{aligned}$$

by (5.18) and (5.19). Hence,

$$\frac{1}{2\pi}\int_0^\infty Y_z^t(x)e^{iwx}\,dx=\frac{\overline{\widetilde{A}(t,-\bar z)}\widetilde{B}(t,w)-\overline{\widetilde{B}(t,-\bar z)}\widetilde{A}(t,w)}{\pi(z+w)}$$

by (5.34) and (5.35). This implies the second equality of (2.10) for $z,w\in\mathbb{C}_+$, because

$$\langle Y_w^t, Y_z^t\rangle=\int_t^\infty Y_w^t(x)Y_{-\bar z}^t(x)\,dx=\int_t^\infty Y_{-\bar z}^t(x)e^{iwx}\,dx$$

by definition of the vector Y_z^t. Equality (2.10) extends to $z, w \in \mathbb{C}_+ \cup D$ by analytic continuation, since the second and third terms of (2.10) extend to $z, w \in \mathbb{C}_+ \cup D$ by Lemma 5.4 and Theorem 2.2, respectively. □

5.3. Proof of Theorem 2.5

In preparation for the proof of Theorems 2.5 and 2.6, we state one result related to the condition (O8).

PROPOSITION 5.11. — *Let $u \in U^1_{\mathrm{loc}}(\mathbb{R})$. Suppose that* (O2), (O3), (O4), (O5), (O6) *are satisfied. Then,*

(1) *if $t_0 = \infty$, $\lim_{t \to t_0} \|Y_z^t\|_{L^2(\mathbb{R})} = 0$ for every $z \in \mathbb{C}_+$;*
(2) *if $t_0 < \infty$, then $\mathcal{V}_{t_0}(u) = \{0\}$ if and only if $\lim_{t \to t_0} \|Y_z^t\|_{L^2(\mathbb{R})} = 0$ for almost every $z \in \mathbb{C}_+$.*

Proof. — (1) can be proved in the same way as in [20, Section 4.3], but here is a simpler proof. From the decomposition $(1 - \mathsf{P}_t)e_z = Y_z^t + J_z^t$ with $Y_z^t \in \mathcal{V}_t$ and $J_z^t \in \mathcal{V}_t^\perp$, we have $\|Y_z^t\|_{L^2(\mathbb{R})} \leqslant \|(1 - \mathsf{P}_t)e_z\|_{L^2(\mathbb{R})} = (2\,\Im(z))^{-1/2}\, e^{-t\cdot\Im(z)}$. Hence $\lim_{t \to \infty} \|Y_z^t\|_{L^2(\mathbb{R})} = 0$.

We prove (2). If $t < s$, $(f, Y_z^t) = (\mathsf{F}f)(z) = (f, Y_z^s)$ for every $f \in \mathcal{V}_s(u)$, since $\mathcal{V}_t(u) \supset \mathcal{V}_s(u)$. Therefore, $(f, Y_z^t - Y_z^s) = 0$ for every $f \in \mathcal{V}_s(u)$, that is, $Y_z^t - Y_z^s \in \mathcal{V}_s(u)^\perp$. Thus, $Y_z^t = (Y_z^t - Y_z^s) + Y_z^s$ is an orthogonal decomposition. In particular, $\|Y_z^t\|^2 = \|Y_z^t - Y_z^s\|^2 + \|Y_z^s\|^2$, so $\|Y_z^t\|$ is non-increasing with respect to t. Hence $\lim_{t \to t_0} \|Y_z^t\|$ exists. On the other hand, $\|Y_z^t\|^2 - \|Y_z^s\|^2 = \|Y_z^t - Y_z^s\|^2 \geqslant 0$ shows that the convergence of the norm $\|Y_z^t\|$ implies the convergence of Y_z^t. Hence $\lim_{t \to t_0} Y_z^t$ exists in L^2 sense. Now we suppose $t < t_0$ and put $Y = \lim_{t \to t_0} Y_z^t$. Then $\langle f, \overline{Y}\rangle = \mathsf{F}f(z)$ for every $f \in \mathcal{V}_{t_0}(u)$, since $\langle f, \overline{Y_z^t}\rangle = \mathsf{F}f(z)$. Hence $Y = Y_z^{t_0}$ and (2) holds. □

We have

$$
\begin{aligned}
2\pi i(\bar{z} - w) j(t; z, w) = &\overline{\left(\widetilde{A}(t,z) - i\widetilde{B}(t,z)\right)}\left(\widetilde{A}(t,w) - i\widetilde{B}(t,w)\right) \\
&- \overline{\left(\widetilde{A}(t,z) + i\widetilde{B}(t,z)\right)}\left(\widetilde{A}(t,w) + i\widetilde{B}(t,w)\right)
\end{aligned}
$$

by direct calculation. Thus

$$
j(t; z, z) = \frac{\left|\widetilde{A}(t,z) - i\widetilde{B}(t,z)\right|^2 - \left|\widetilde{A}(t,z) + i\widetilde{B}(t,z)\right|^2}{2\pi i(\bar{z} - z)}
$$

for $z \in \mathbb{C}_+$. On the other hand, we obtain

$$j(t; z, z) = \left(\lim_{t \to t_0} j(t, z, z)\right) + \frac{1}{\pi} \int_t^{t_0} \left[\widetilde{A}(u, z)\ \widetilde{B}(u, z)\right] H(u) \overline{\begin{bmatrix} \widetilde{A}(u, z) \\ \widetilde{B}(u, z) \end{bmatrix}}\, du$$

from the first order system (2.7) as in the proof of [20, (2.40)]. The first equality of (2.10) shows that $\lim_{t \to t_0} j(t, z, z) = (1/2\pi) \lim_{t \to t_0} \|Y_z^t\|^2$, and the right-hand side exists by the proof of Proposition 5.11. Hence $j(t; z, z) \geqslant 0$ for $z \in \mathbb{C}_+$ by (O7), which implies $|\theta(t, z)| \leqslant 1$ for $z \in \mathbb{C}_+$ by definition (2.11). On the other hand,

$$\theta(t, z)^\sharp = \frac{\widetilde{A}(t, z) - i\widetilde{B}(t, z)}{\widetilde{A}(t, z) + i\widetilde{B}(t, z)} = \frac{1}{\theta(t, z)}$$

as a function of z by definition (2.11) and (2.6). Thus, $|\theta(t, z)| = 1$ for real z. Hence $\theta(t, z)$ is inner. For an inner function θ, the reproducing kernel of the model space $\mathcal{K}(\theta)$ is $(1/2\pi i)(1 - \overline{\theta(z)}\theta(w))/(\bar{z} - w)$. We confirm that the reproducing kernel of $\mathsf{F}(\mathcal{V}_t(u))$ equals to the reproducing kernel of $\mathcal{K}(\theta(t, z))$ by direct calculation. □

5.4. Analytic properties of $A(t, z)$ and $B(t, z)$

In the cases of $u(z) = M^\sharp(z)/M(z)$, we defined $A(t, z)$, $B(t, z)$, and $E(t, z)$ by (2.12). Then they are entire functions satisfying

$$A(t, z) = A^\sharp(t, z), \quad B(t, z) = B^\sharp(t, z)$$

by (2.6) and Theorem 2.2. Therefore, $E^\sharp(t, z) = A(t, z) + iB(t, z)$ and

$$A(t, z) = \frac{1}{2}\left(E(t, z) + E^\sharp(t, z)\right), \quad B(t, z) = \frac{i}{2}\left(E(t, z) - E^\sharp(t, z)\right).$$

If $M^\sharp(z) = \varepsilon M(-z)$ for a sign $\varepsilon \in \{\pm 1\}$, then u is symmetric, and therefore Φ and Ψ are real-valued. Hence,

$$\begin{aligned} E^\sharp(t, z) &= M^\sharp(z) \frac{iz}{2} \int_t^\infty (\Phi(t, x) + \Psi(t, x)) e^{-izx}\, dx \\ &= \varepsilon M(-z) \frac{-i(-z)}{2} \int_t^\infty (\Phi(t, x) + \Psi(t, x)) e^{i(-z)x}\, dx = \varepsilon E(t, -z). \end{aligned}$$

Therefore, $A(t, z)$ is even and $B(t, z)$ is odd if $\varepsilon = +1$, and $A(t, z)$ is odd and $B(t, z)$ is even if $\varepsilon = -1$.

5.5. Conformity of the axiom of de Branges spaces

Proposition 5.12. — *Suppose that $u(z) = M^\sharp(z)/M(z)$ for some meromorphic function $M(z)$ on $\mathbb{C}$ such that it is holomorphic on $\mathbb{C}_+ \cup \mathbb{R}$ and has no zeros in $\mathbb{C}_+$. Further, suppose that* (O2), (O3), (O4), (O5), (O6) *are satisfied. Then $M(z)\mathsf{F}(\mathcal{V}_t(u))$ is a de Branges space for every $t < t_0$.*

Proof. — We show that $\mathcal{H} := M(z)\mathsf{F}(\mathcal{V}_t(u))$ is a Hilbert space consisting of entire functions and satisfies the axiom of the de Branges spaces:

(dB1) For each $z \in \mathbb{C} \setminus \mathbb{R}$ the point evaluation $\Phi \mapsto \Phi(z)$ is a continuous linear functional on $\mathcal{H}$;

(dB2) If $\Phi \in \mathcal{H}$, $\Phi^\sharp$ belongs to $\mathcal{H}$ and $\|\Phi\|_{\mathcal{H}} = \|\Phi^\sharp\|_{\mathcal{H}}$;

(dB3) If $w \in \mathbb{C} \setminus \mathbb{R}$, $\Phi \in \mathcal{H}$ and $\Phi(w) = 0$,

$$\frac{z-\bar{w}}{z-w}\Phi(z) \in \mathcal{H} \quad \text{and} \quad \left\| \frac{z-\bar{w}}{z-w}\Phi(z) \right\|_{\mathcal{H}} = \|\Phi\|_{\mathcal{H}},$$

where the Hilbert space structure is the one induced from $\mathcal{V}_t(u)$ that is equivalent to $\langle F, G\rangle_{\mathcal{H}} = \int_{\mathbb{R}} F(z)\overline{G(z)}|M(z)|^{-2}dz$ for $F, G \in \mathcal{H}$.

Let $\Phi(z) = M(z)(\mathsf{F}f)(z) \in \mathcal{H}$ with $f \in \mathcal{V}_t(u)$. First, we prove that $\mathcal{H}$ consists of entire functions. We see that $\Phi(z)$ is holomorphic on $\mathbb{C}_+$ by $f \in L^2(t,\infty)$ and is holomorphic on $\mathbb{C}_-$ by $\Phi(z) = M^\sharp(z)(\mathsf{F}\mathsf{J}_\sharp\mathsf{K}f)(z)$ and $\mathsf{J}_\sharp\mathsf{K}f \in L^2(-\infty,-t)$. Moreover, $\lim_{z\to x}(\mathsf{F}f)(z) = (\mathsf{F}f)(x)$ and

$$\lim_{z \to x}(\mathsf{F}\mathsf{J}_\sharp\mathsf{K}f)(z) = \lim_{z \to x}(\mathsf{F}\mathsf{K}f)^\sharp(z) = u^\sharp(x)(\mathsf{F}f)(x)$$

for almost all $x \in \mathbb{R}$, where z is allowed to tends to x non-tangentially from $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. Hence $\Phi(z)$ is also holomorphic in a neighborhood of each point of $\mathbb{R}$.

We confirm (dB1). For $z \in \mathbb{C}_+$, $\Phi \mapsto \Phi(z) = M(z)\int_t^\infty f(x)e^{izx}dx$ is a continuous linear functional. On the other hand, for $z \in \mathbb{C}_-$, $\Phi \mapsto \Phi(z) = M^\sharp(z)\int_{-\infty}^{-t}\overline{(\mathsf{K}f)(-x)}e^{izx}\,dx$ is also a continuous linear functional. (Moreover, for $z \in \mathbb{R}$, the continuity follows from the Banach–Steinhaus theorem.)

We confirm (dB2). We have $\Phi^\sharp(z) = M(z)(\mathsf{F}\mathsf{K}f)(z)$. Since $\mathsf{K}f \in \mathcal{V}_t(u)$, $\Phi^\sharp$ belongs to $\mathcal{H}$. Since K is isometric, the equality of norms in (dB2) holds.

We confirm (dB3). The equality of norms in (dB3) is trivial by the definition of the norm of $\mathcal{H}$. From (dB2), it is sufficient to consider only the case of $w \in \mathbb{C}_+$. Suppose that $\Phi(w) = 0$ for $w \in \mathbb{C}_+$. Then $(\mathsf{F}f)(w) = 0$, since $M(z)$ has no zeros in $\mathbb{C}_+$. We put $f_w(x) = f(x) - i(w-\bar{w})\int_0^{x-t} f(x-y)e^{-iwy}dy$. Then we easily find that $f_w \in L^2(t,\infty)$ and $(\mathsf{F}f_w)(z) = ((z-\bar{w})/(z-w))(\mathsf{F}f)(z)$ for $z \in \mathbb{C}_+$. Hence we complete the proof if it is shown that $\mathsf{K}f_w$ has support in $[t,\infty)$, since $\mathsf{K}f_w \in L^2(\mathbb{R})$ by $f_w \in L^2(t,\infty)$. We put $g_w(x) = (\mathsf{K}f)(x) - i(\bar{w}-w)\int_0^{x-t}(\mathsf{K}f)(x-y)e^{-i\bar{w}y}dy$. Then g_w has support in $[t,\infty)$ by $\mathsf{K}f \in L^2(t,\infty)$ and $(\mathsf{F}g_w)(z) = ((z-w)/(z-\bar{w}))(\mathsf{FK}f)(z) = (\mathsf{FK}f_w)(z)$ for $z \in \mathbb{C}_+$. Hence $g_w = \mathsf{K}f_w$ and the proof is completed. □

5.6. Proof of Theorem 2.6

Proof. — As mentioned in Section 5.4, $A(t,z)$, $B(t,z)$, and $E(t,z)$ are entire functions with the assumptions of Theorem 2.6. Also, they satisfy the system of differential equation (1.1) for $t < t_0$ and $z \in \mathbb{C}$ by Theorem 2.2. Further, we have $\Theta(t,z) = E^\sharp(t,z)/E(t,z)$ for $\Theta(t,z)$ defined by (2.11), since $E^\sharp(t,z) = A(t,z) + iB(t,z)$. Thus $E(t,z) \in \overline{\mathbb{HB}}$ by Theorem 2.5 and therefore the de Branges space $\mathcal{H}(E(t,z))$ is defined. On the other hand, $M(z)\mathsf{F}(\mathcal{V}_t(u))$ is also a de Branges space by Proposition 5.12. Let $J(t;z,w)$ be the reproducing kernel of $M(z)\mathsf{F}(\mathcal{V}_t(u))$. Then,

$$J(t;z,w) = \overline{M(z)}M(w)j(t;z,w) = \frac{\overline{A(t,z)}B(t,w) - A(t,w)\overline{B(t,z)}}{\pi(w-\bar{z})}$$

for every $t < t_0$ by Theorem 2.3. The right-hand side is nothing but the reproducing kernel of $\mathcal{H}(E(t,z))$ ([20, Section 3.2]). Hence $M(z)\mathsf{F}(\mathcal{V}_t(u)) = \mathcal{H}(E(t,z))$ for every $t < t_0$. To conclude that $H(t)$ on $[t_1,t_0)$ is the structure Hamiltonian of $\mathcal{H}(E(t_1,z))$, it remains to show $\lim_{t\to t_0} J(t;z,w) = 0$, but this follows from (O8) by Proposition 5.11. □

6. Proof of results in Section 2.3

6.1. Properties of K for an inner function u

To describe the properties of the operator $\mathsf{K} = \mathsf{K}_u$ when u is an inner function in $\mathbb{C}_+$, we recall the following result on inner functions [15, Theorems 1.1 and 1.2]:

Proposition 6.1. — *A unimodular function u in $L^1_{\mathrm{loc}}(\mathbb{R})$ is the non-tangential limit of an inner function θ in $\mathbb{C}_+$ if and only if the tempered distribution $k = \mathsf{F}^{-1}u$ has support in $[0,\infty)$.*

Therefore, if u is an inner function in $\mathbb{C}_+$, $\mathsf{K}f$ has support in $[-t,\infty)$ for every $f \in \mathsf{P}_t S'(\mathbb{R})$, since

$$\begin{aligned}\operatorname{supp}\mathsf{K}f = \operatorname{supp}(k * \mathsf{J}_\sharp f) &\subset \overline{\operatorname{supp} k + \operatorname{supp}\mathsf{J}_\sharp f}\\ &\subset \overline{[0,\infty) + [-t,\infty)} = [-t,\infty).\end{aligned}$$

PROPOSITION 6.2. — *Suppose that u is an inner function θ in $\mathbb{C}_+$, such that it is not equal to 1 as a function. Then,*

(1) $\mathsf{K}[t] = 0$ *as an operator for nonpositive t. In particular,* (O1) *holds;*
(2) $H(t)$ *defined by* (2.8) *and* (2.9) *is the identity matrix for all negative t:*
(3) $\mathcal{V}_t(u) \neq \{0\}$ *for nonpositive t. In particular,* (O5) *holds;*
(4) $\mathsf{F}(\mathcal{V}_0(u)) = \mathcal{K}(\theta)$*. In particular, if $\theta = E^\sharp/E$ for some $E \in \mathbb{HB}$, we have $E(z)\mathsf{F}(\mathcal{V}_0(u)) = \mathcal{H}(E)$.*

Proof. — For $f \in L^2(-\infty, t)$, $\mathsf{K}f$ belongs to $L^2(-t, \infty)$ by Proposition 6.1, and therefore $\mathsf{K}[f]f = 0$ for all $f \in L^2(-\infty, t)$, that is, (1) holds. For negative t, (2.3) and (2.4) are easily solved as $\Phi = 1 - \mathsf{KP}_t 1$ and $\Psi = 1 + \mathsf{KP}_t 1$ by (1). Here $\mathsf{KP}_t 1$ has support in $[-t, \infty)$, thus $\Phi(t,t) = \Psi(t,t) = 1$, which implies (2).

To prove $\mathcal{V}_0(u) \neq \{0\}$, it is sufficient to show that $\mathsf{K}L^2(-\infty, 0)$ is a proper subspace of $L^2(0,\infty)$ by (5.7). The latter is true because the space of Fourier transforms $\mathsf{F}(\mathsf{K}L^2(-\infty,0)) = \theta H^2(\mathbb{C}_+)$ is a proper subspace of $H^2(\mathbb{C}_+) = \mathsf{F}(L^2(0,\infty))$. For negative t, we have $\mathsf{K}L^2(-\infty,t) \subset L^2(-t,\infty) \subsetneq L^2(t,\infty)$. Therefore, $\mathcal{V}_t(u)^\perp$ is a proper subspace of $L^2(\mathbb{R})$, and hence (3) holds.

The proof of [20, Lemma 4.1] can be applied to prove (4) by replacing $\Theta(-z)$ and $(\mathsf{F}f)(-z)$, etc. in [20] with $\theta^\sharp(z)$ and $(\mathsf{F}f)^\sharp(z)$, etc. in this paper. The difference on the definition of the operator K does not affect the argument of the proof. □

For any of (O2), (O3), (O4), (O6), (O7), (O8), we do not know whether it holds for a general inner function. However, for (O6) and (O7), we can provide sufficient conditions as in Section 7. As in the example in Section 3.2, (O8) may not hold in general even if u is a meromorphic inner function.

6.2. Proof of Theorem 2.7

Note that $\widetilde{A}(0,z)$ and $\widetilde{B}(0,z)$ are defined, since $\mathsf{K}[0] = 0$ by the proof of Proposition 6.2. We have $(1-\mathsf{P}_0)\Phi + \mathsf{P}_0\Phi + \mathsf{KP}_0\Phi = (1-\mathsf{P}_0)1 + \mathsf{P}_0 1$ by (2.3). Acting P_0 to both sides gives $\mathsf{P}_0\Phi = \mathsf{P}_0 1$, since $\mathsf{KP}_0\Phi$ has support in $[0,\infty)$. Thus $(1-\mathsf{P}_0)\Phi = (1-\mathsf{P}_0)1 - \mathsf{KP}_0 1$. We calculate $\mathsf{KP}_0 1$. For $g \in S(\mathbb{R})$, if we write $G(z) = (\mathsf{F}g)(z)$,

$$\begin{aligned}\langle \mathsf{KP}_0 1, g\rangle = \langle \mathsf{K}g, \mathsf{P}_0 1\rangle &= \int \left(\frac{1}{2\pi}\int_{\Im z=0} u(z)G^\sharp(z)e^{-izx}dz\right)\mathsf{P}_0 1(x)\,dx \\ &= \int \left(\frac{1}{2\pi}\int_{\Im z=\delta>0} \theta(z)G^\sharp(z)e^{-izx}dz\right)\mathsf{P}_0 1(x)\,dx,\end{aligned}$$

since u is bounded in $\mathbb{C}_+ \cup \mathbb{R}$. Then, we have

$$\mathsf{KP}_0 1(x) = u(0)(1 - \mathsf{P}_0)1(x) + \frac{1}{2\pi} \int_{\Im(z)=\delta>0} \frac{\theta(z) - u(0)}{-iz} e^{-izx} dz$$

in a way similar to the proof of Proposition 4.1. Because the second term of the right-hand side belongs to $L^2(\mathbb{R})$, $(\mathsf{F}(1 - \mathsf{P}_0)\mathsf{KP}_0 1)(z)$ is defined for $z \in \mathbb{C}_+$. Hence $(\mathsf{F}(1 - \mathsf{P}_0)\Phi)(z)$ is defined for $z \in \mathbb{C}_+$. Therefore,

$$\frac{2}{-iz}\left(-i\widetilde{B}(0,z)\right) = \mathsf{F}(1 - \mathsf{P}_0)\Phi = \mathsf{F}(1 - \mathsf{P}_0)1 - \mathsf{FKP}_0 1 = \frac{1}{-iz}(1 - \theta(z)).$$

The case of $\widetilde{A}(0,z)$ is shown by a similar argument. □

7. Complementary results

7.1. A sufficient condition for (O6)

Proposition 7.1. — *Suppose that u is an inner function θ in $\mathbb{C}_+$ and continuous on $\mathbb{R}$. Then $\mathsf{K}[t]$ is compact for all $t \in \mathbb{R}$.*

Remark 7.2. — Even if u is not an inner function, it is possible for $\mathsf{K}[t]$ to be a compact operator. For example, $u(z) = \Gamma(1/2 + iz)/\Gamma(1/2 - iz)$ is not an inner function, but $\mathsf{K}[t]$ is compact for every $t \in \mathbb{R}$ because the kernel $k(x) = e^{x/2}J_0(2e^{x/2})$ satisfies the Hilbert–Schmidt condition on $(-\infty, t] \times (-\infty, t]$. (Also, it is proved that $\mathsf{K}[t]$ is a limit of finite rank operators in [3, Section 5]).

Proof. — Only the case of positive t needs to be proved by Proposition 6.2 (1). We find that $\mathsf{K}[t](L^2(-\infty, -t)) = \{0\}$ and $\mathsf{K}[t](L^2(-t,t)) \subset L^2(-t,t)$ by Proposition 6.1. Therefore, it suffices to prove that the restriction $\mathsf{K}[t]|_{L^2(-t,t)}$ is compact. For $f \in L^2(-t,t)$,

$$(\mathsf{K}[t]f)(x) = \mathbf{1}_{[0,2t]}(x-t) \int_0^{2t} k((x-t) - y + 2t)\overline{f(-y+t)}dy.$$

Therefore, the restriction of $\mathsf{K}[t]$ to $L^2(-t,t)$ is a composition of the translation $f(x) \mapsto f(x-t)$, inversion $f(x) \mapsto f(-x)$, conjugation $f(x) \mapsto \overline{f(x)}$ and the operator $\mathsf{F}^{-1}A_\phi\mathsf{F}$ defined by $A_\phi F := \mathsf{Q}_{2t}\mathsf{M}_\phi F$, where

$$\phi(z) = \int_0^\infty k(x+2t)e^{izx}dx = e^{-2itz}\theta(z) - \int_{-2t}^0 k(x+2t)e^{izx}dx$$

and Q_{2t} is the projection from $L^2(0,\infty)$ to the Paley–Wiener space $\mathsf{F}L^2(0,2t)$. That is, A_ϕ is the truncated Toeplitz operator on $\mathsf{F}L^2(0,2t)$. Note that

$\int_{-2t}^{0} k(x+2t)e^{izx}dx$ is entire, since k is a tempered distribution, which is a higher derivative of a continuous function. Then A_ϕ is compact if θ is continuous on $\mathbb{R}$ by [4, Theorem 5.1] (see also [1, Remark 3.5]). Hence the restriction $\mathsf{K}[t]|_{L^2(-t,t)}$ is compact. □

Proposition 7.3. — *Suppose that u is an inner function θ in $\mathbb{C}_+$ and is continuous on $\mathbb{R}$, and there are no entire functions F and G of exponential type such that $\theta = G/F$. Then $\|\mathsf{K}[t]\|_{\mathrm{op}} < 1$ for all $t \in \mathbb{R}$. In particular, $t_0 = \infty$ and* (O6) *is satisfied.*

Proof. — Only the case of positive t needs to be proved by Proposition 6.2 (1). Let $t > 0$. By applying the argument in the proof of [20, Theorem 5.2], it is shown that 1 is not an eigenvalue of $\mathsf{K}[t]$, since differences in the definition and properties of K do not affect the argument. Therefore, according to the general theory of antilinear operators ([8, 24]), if $|\lambda| = 1$, then λ is not an eigenvalue of $\mathsf{K}[t]$. Since $\|\mathsf{K}[t]\|_{\mathrm{op}} \leqslant 1$, $\|\mathsf{K}[t]\|_{\mathrm{op}} \neq 1$ implies $\|\mathsf{K}[t]\|_{\mathrm{op}} < 1$. Suppose that $\|\mathsf{K}[t]\|_{\mathrm{op}} = 1$. Then $\|\mathsf{K}[t]^2\|_{\mathrm{op}} = \|\mathsf{K}[t]\|_{\mathrm{op}}^2 = 1$, since $\mathsf{K}[t]$ is self-adjoint. Since $\mathsf{K}[t]$ is compact, $\mathsf{K}[t]^2$ is a linear compact operator. Therefore, 1 or -1 is an eigenvalue of $\mathsf{K}[t]^2$. Hence every complex number of absolute value one is an eigenvalue of $\mathsf{K}[t]$. This is a contradiction. □

7.2. A sufficient condition for (O7)

We can often easily handle the values of $\Phi(t,t)$ and $\Psi(t,t)$ for large negative t as in the case of u is an inner function or $u = \Gamma(\frac{1}{2}+iz)/\Gamma(\frac{1}{2}-iz)$. In such cases, the smoothness of $\Phi(t,x)$ and $\Psi(t,x)$ around the diagonal $x = t$ lead to the positive definiteness of $H(t)$ defined by (2.8) and (2.9). In stating the following proposition, we refer to [17, Definition 6.9] for the values of distributions.

Proposition 7.4. — *Let $u \in U^1_{\mathrm{loc}}(\mathbb{R})$. Suppose that* (O1), (O2), (O3), (O4) *are satisfied. Further, we suppose that there is an interval I such that*

1. *$\|\mathsf{K}[t]\|_{\mathrm{op}} < 1$ for $t \in I$;*
2. *the derivatives $\frac{d}{dt}\Phi(t,t)$ and $\frac{d}{dt}\Psi(t,t)$ are defined as a distribution on I;*
3. *the distributions $\frac{\partial}{\partial t}\Phi(t,x)$, $\frac{\partial}{\partial x}\Phi(t,x)$, $\frac{\partial}{\partial t}\Psi(t,x)$, $\frac{\partial}{\partial x}\Psi(t,x)$ for x have values at $x = t$ for almost all $t \in I$;*
4. *all $t \mapsto \frac{\partial}{\partial t}\Phi(t,t)$, $t \mapsto \frac{\partial}{\partial x}\Phi(t,t)$, $t \mapsto \frac{\partial}{\partial t}\Psi(t,t)$, $t \mapsto \frac{\partial}{\partial x}\Psi(t,t)$ define distributions on I and satisfy*

$$\frac{d}{dt}\Phi(t,t) = \frac{\partial\Phi}{\partial t}(t,t) + \frac{\partial\Phi}{\partial x}(t,t), \quad \frac{d}{dt}\Psi(t,t) = \frac{\partial\Psi}{\partial t}(t,t) + \frac{\partial\Psi}{\partial x}(t,t). \tag{7.1}$$

Then $\Re(\Phi(t,t)\overline{\Psi(t,t)})$ *is a constant on* I.

Proof. — Adding (5.3) and (5.6),

$$\phi^+ + \phi^- = \frac{\Re(\Phi(t,t))}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\left(\frac{\partial}{\partial t}\Psi(t,x) + \frac{\partial}{\partial x}\Psi(t,x)\right) - i\,\frac{\Im(\Psi(t,t))}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\left(\frac{\partial}{\partial t}\Phi(t,x) + \frac{\partial}{\partial x}\Phi(t,x)\right).$$

On the other, adding (5.4) and (5.5),

$$\phi^+ + \phi^- = -\frac{\Re(\Psi(t,t))}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\left(\frac{\partial}{\partial t}\Phi(t,x) + \frac{\partial}{\partial x}\Phi(t,x)\right) + i\,\frac{\Im(\Phi(t,t))}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\left(\frac{\partial}{\partial t}\Psi(t,x) + \frac{\partial}{\partial x}\Psi(t,x)\right).$$

Therefore,

$$\frac{\overline{\Psi(t,t)}}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\Phi(t,x) + \frac{\overline{\Phi(t,t)}}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\Psi(t,x) = 0.$$

Using this and (7.1), we have

$$\begin{aligned}\frac{d}{dt}&\log\Re(\Phi(t,t)\overline{\Psi(t,t)}) \\ &= \frac{\overline{\Phi(t,t)}}{\Re(\Phi(t,t)\overline{\Psi(t,t)})}\frac{d}{dt}\Psi(t,t) + \frac{\overline{\Psi(t,t)}}{\Re(\Phi(t,t)\overline{\Psi(t,t)})}\frac{d}{dt}\Phi(t,t) \\ &\quad + \overline{\left(\frac{\overline{\Phi(t,t)}}{\Re(\overline{\Psi(t,t)}\Phi(t,t))}\frac{d}{dt}\Psi(t,t) + \frac{\overline{\Psi(t,t)}}{\Re(\Phi(t,t)\overline{\Psi(t,t)})}\frac{d}{dt}\Phi(t,t)\right)} \\ &= 0.\end{aligned}$$

Hence $\Re(\Phi(t,t)\overline{\Psi(t,t)})$ is a constant. □

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