



ANNALES DE L'INSTITUT FOURIER

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Schrödinger representation**

Article à paraître, mis en ligne le 5 février 2025, 47 p.

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Les *Annales de l'Institut Fourier* sont membres du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

ON THE RIGIDITY OF INVARIANT NORMS ON THE p -ADIC SCHRÖDINGER REPRESENTATION

by Amit OPHIR

ABSTRACT. — Motivated by questions about \mathbb{C}_p -valued Fourier transform on the locally compact group $(\mathbb{Q}_p^d, +)$, we study invariant norms on the p -adic Schrödinger representation of the Heisenberg group. Our main result is a minimality and rigidity property for norms in a family of invariant norms parameterized by a Grassmannian. This family is the orbit of the sup norm under the action of the symplectic group, acting via intertwining operators. We also prove general fundamental properties of quotients of the universal unitary completion of cyclic algebraic representations. Combined with the rigidity property, we are able to show that the completion of the Schrödinger representation in any of the norms in that family satisfies a strong notion of irreducibility and a version of Schur’s lemma. Norms that can be formed as the maximum of a finite number of norms from that family are also studied. We conclude this paper with a list of open questions.

RÉSUMÉ. — Motivés par des questions sur la transformée de Fourier à valeurs dans \mathbb{C}_p sur le groupe localement compact $(\mathbb{Q}_p^d, +)$, nous étudions les normes invariantes dans la représentation de Schrödinger p -adique du groupe de Heisenberg. Notre principal résultat est une propriété de minimalité et de rigidité pour les normes dans une famille de normes invariantes paramétrées par une Grassmannienne. Cette famille est l’orbite de la norme uniforme sous l’action du groupe symplectique, agissant via des opérateurs d’entrelacement. Nous prouvons également des propriétés fondamentales générales des quotients de la complétion unitaire universelle des représentations algébriques cycliques. Combinées à la propriété de rigidité, nous sommes en mesure de montrer que la complétion de la représentation de Schrödinger dans chacune des normes de cette famille satisfait une notion forte d’irréductibilité et une version du lemme de Schur. Nous étudions également les normes qui peuvent être obtenues comme le maximum d’un nombre fini de normes de cette famille. Nous concluons cet article par une liste de questions ouvertes.

1. Introduction

Let p be a prime number and let \mathbb{Q}_p be the field of p -adic numbers. Let $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$ be the $2d + 1$ -dimensional Heisenberg group over \mathbb{Q}_p (Definition 2.1). A classification of the irreducible smooth representations of

Keywords: p -adic analysis, Schrödinger representations, invariant norms, q -analogs.
2020 Mathematics Subject Classification: 46S10, 22D12, 05A30.

$\mathcal{H}_{2d+1}(\mathbb{Q}_p)$ over an algebraically closed field C of characteristic zero is well known and we recall it in Section 2.1. This classification is analogous to the classical Stone–von Neumann theorem for Heisenberg groups over \mathbb{R} . Taking C to be the field of complex numbers, any irreducible representation of $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$ admits an invariant unitary structure ([16, Theorem 4]), and the classification of irreducible unitary representations of $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$ is similar to the classification of irreducible smooth representations.

In this work we take the field C to be the field \mathbb{C}_p , the completion of an algebraic closure of \mathbb{Q}_p . There are no inner products over \mathbb{C}_p . As a replacement, we study invariant (non-archimedean) norms on the smooth irreducible representations of $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$ over \mathbb{C}_p . The interaction between the p -adic topology of $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$ and the p -adic topology of \mathbb{C}_p leads to complications. For example, the Heisenberg groups do not admit a \mathbb{C}_p -valued Haar measure, hence we cannot use classical tools that rely on integration. Nevertheless, we are able to construct invariant norms with a surprising rigidity property (Theorem 4.3). Moreover, the completions with respect to these norms behave in certain ways like irreducible unitary representations (Theorem 4.5 and Proposition 4.6).

Part of our motivation comes from a p -adic Fourier transform on \mathbb{Q}_p^d .

1.1. Invariant norms on the smooth Schrödinger representation

For the sake of exposition we describe in the introduction the case $d = 1$, i.e. the 3-dimensional Heisenberg group, but study the general case in the rest of the paper. The main ideas are already apparent in the case $d = 1$.

The 3-dimensional Heisenberg group over \mathbb{Q}_p is the group of upper triangular unipotent matrices:

$$\mathcal{H}_3(\mathbb{Q}_p) = \left\{ \begin{pmatrix} 1 & a & t \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_3(\mathbb{Q}_p).$$

We denote its elements by $[a, b, t]$. Note that this construction is different than the one given in Definition 2.1 (with $d = 1$), but both constructions give rise to isomorphic topological groups.

Fix a non-trivial smooth character $\psi : (\mathbb{Q}_p, +) \rightarrow \mathbb{C}_p^\times$. Recall that ψ is a smooth character if $\ker(\psi)$ is open. Let $\mathcal{S}(\mathbb{Q}_p)$ denote the space of all functions $f : \mathbb{Q}_p \rightarrow \mathbb{C}_p$ which are compactly supported and locally constant. A function $f \in \mathcal{S}(\mathbb{Q}_p)$ is called a Schwartz function. The smooth

Schrödinger representation $\rho_\psi : \mathcal{H}_3(\mathbb{Q}_p) \rightarrow \mathrm{GL}(\mathcal{S}(\mathbb{Q}_p))$ attached to ψ is defined by

$$(\rho_\psi([a, b, t])f)(x) = \psi(t + bx) \cdot f(x + a).$$

The representation ρ_ψ is irreducible and the smooth Stone-von Neumann theorem says that, up to isomorphism, ρ_ψ is the unique smooth irreducible representation of $\mathcal{H}_3(\mathbb{Q}_p)$ with central character ψ ([16, Theorem 3], [11, Theorem 1.1], [13, Chapter 2, Theorem I.2.]).

A $\mathcal{H}_3(\mathbb{Q}_p)$ -invariant norm, or simply an invariant norm, on the smooth Schrödinger representation ρ_ψ is a non-archimedean norm $\|\cdot\|$ on $\mathcal{S}(\mathbb{Q}_p)$ such that

$$\|\rho_\psi(h)f\| = \|f\|,$$

for any $f \in \mathcal{S}(\mathbb{Q}_p)$ and any $h \in \mathcal{H}_3(\mathbb{Q}_p)$. The prominent example of a $\mathcal{H}_3(\mathbb{Q}_p)$ -invariant norm on $\mathcal{S}(\mathbb{Q}_p)$ is the sup norm:

$$\|f\|_\infty = \max_{x \in \mathbb{Q}_p} |f(x)|.$$

The completion of $\mathcal{S}(\mathbb{Q}_p)$ with respect to $\|\cdot\|_\infty$ is the space $\mathcal{C}_0(\mathbb{Q}_p)$ of \mathbb{C}_p -valued, continuous functions on \mathbb{Q}_p that go to zero at infinity.

By composing the sup norm with intertwining operators (defined in Section 2.2), we get new $\mathcal{H}_3(\mathbb{Q}_p)$ -invariant norms. In Section 4.1 we show that, up to homothety, these norms are parameterized by the projective line $\mathbb{P}^1(\mathbb{Q}_p)$. Thus, for each $\alpha \in \mathbb{P}^1(\mathbb{Q}_p)$ we attach an invariant norm $\|\cdot\|_\alpha$. When $\alpha = \infty \in \mathbb{P}^1(\mathbb{Q}_p)$, the norm $\|\cdot\|_\alpha$ is the sup norm.

Our main result is Theorem 4.3, which can be thought of as a minimality and rigidity result for the norms $\|\cdot\|_\alpha$.

THEOREM 1.1 (Theorem 4.3). — *Let $\alpha \in \mathbb{P}^1(\mathbb{Q}_p)$. Let $\|\cdot\|$ be an $\mathcal{H}_3(\mathbb{Q}_p)$ -invariant norm on $\mathcal{S}(\mathbb{Q}_p)$ that is dominated by $\|\cdot\|_\alpha$ (i.e. $\|\cdot\| \leq c \cdot \|\cdot\|_\alpha$ for some $c > 0$). Then there exists $r > 0$ such that $\|\cdot\| = r \cdot \|\cdot\|_\alpha$.*

Denote the completion of $\mathcal{S}(\mathbb{Q}_p)$ with respect to $\|\cdot\|_\alpha$ by $\mathcal{S}(\mathbb{Q}_p)_{\|\cdot\|_\alpha}$. In Theorem 4.5 we prove that $\mathcal{S}(\mathbb{Q}_p)_{\|\cdot\|_\alpha}$ is topologically irreducible. In fact, we show that the completions satisfy a stronger notion of irreducibility, a notion that we call *strong irreducibility* (Definition 3.13).

THEOREM 1.2 (Theorem 4.5). — *Let $\alpha \in \mathbb{P}^1(\mathbb{Q}_p)$.*

- (1) *The completion $\mathcal{S}(\mathbb{Q}_p)_{\|\cdot\|_\alpha}$ is a strongly irreducible Banach representation of $\mathcal{H}_3(\mathbb{Q}_p)$.*
- (2) *Let $\beta \in \mathbb{P}^1(\mathbb{Q}_p)$. The space of continuous $\mathcal{H}_3(\mathbb{Q}_p)$ -equivariant maps from $\mathcal{S}(\mathbb{Q}_p)_{\|\cdot\|_\alpha}$ to $\mathcal{S}(\mathbb{Q}_p)_{\|\cdot\|_\beta}$ is given by*

$$\mathrm{Hom}_{\mathcal{H}_3(\mathbb{Q}_p)} \left(\mathcal{S}(\mathbb{Q}_p)_{\|\cdot\|_\alpha}, \mathcal{S}(\mathbb{Q}_p)_{\|\cdot\|_\beta} \right) \simeq \begin{cases} \mathbb{C}_p & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}.$$

We remark that Any strongly irreducible Banach representation of a group G is topologically irreducible, but the converse is false. At the end of Section 3 we give an example of a topologically irreducible representation which is not strongly irreducible.

The following result is in the spirit of Schur’s lemma.

PROPOSITION 1.3 (Proposition 4.6). — *Let $\alpha \in \mathbb{P}^1(\mathbb{Q}_p)$. Let $(B, \|\cdot\|)$ be a topologically irreducible Banach representation of $\mathcal{H}_3(\mathbb{Q}_p)$ (see Definition 3.1). Assume that we are in one of the two following cases.*

- (1) *$F : B \rightarrow \mathcal{S}(\mathbb{Q}_p)_{\|\cdot\|_\alpha}$ is a nonzero continuous map of representations.*
- (2) *$F : \mathcal{S}(\mathbb{Q}_p)_{\|\cdot\|_\alpha} \rightarrow B$ is a nonzero continuous map of representations.*

Then F is an isomorphism. Moreover, there exists $r > 0$ such that by replacing $\|\cdot\|$ with $r\|\cdot\|$, F becomes an isometric isomorphism.

The main ingredients in the proofs of Theorem 1.2 and Proposition 1.3 are Theorem 1.1 and “soft” functional analysis results about Banach representations from Section 3.

In Section 5 we make a series of reduction steps and show that Theorem 4.3 follows from the following theorem, which is interesting in its own right.

THEOREM 1.4 (Theorem 4.4). — *Let $\|\cdot\|$ be a norm on $\mathcal{S}(\mathbb{Z}_p)$ that is dominated by the sup norm and invariant under translations and multiplication by the smooth characters of \mathbb{Z}_p . Then $\|\cdot\| = r \cdot \|\cdot\|_\infty$ for some $r > 0$.*

Here, $\mathcal{S}(\mathbb{Z}_p)$ is the space of all \mathbb{C}_p -valued, locally constant functions on \mathbb{Z}_p .

The proof of Theorem 1.4 is the most technical part of this work. It consists of showing that the sup norm satisfies a minimality property (Definition 3.2) with respect to translations and a maximality property (Definition 3.6) with respect to multiplication by smooth characters. The minimality is easy while the maximality is much deeper. Proving the maximality property is the goal of Section 6.

Remark 1.5. — The completions that we study of the Schrödinger representation are large in the sense that the reduction of their unit ball modulo the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$ is a non-admissible smooth representation over $\overline{\mathbb{F}}_p$ (the residue field of $\mathcal{O}_{\mathbb{C}_p}$).

1.2. p -adic Fourier transform on \mathbb{Q}_p

Let dt be the unique \mathbb{C}_p -valued Haar distribution on the space of Schwartz functions $\mathcal{S}(\mathbb{Q}_p)$, normalized such that $\int \mathbf{1}_{\mathbb{Z}_p}(t) dt = 1$, where $\mathbf{1}_{\mathbb{Z}_p}(t)$ is the characteristic function of \mathbb{Z}_p . The Haar distribution dt is not a measure and we cannot define the integral $\int f(t) dt$ for a general continuous function $f : \mathbb{Q}_p \rightarrow \mathbb{C}_p$, even if f is compactly supported.

Fix a non-trivial smooth character $\psi : (\mathbb{Q}_p, +) \rightarrow \mathbb{C}_p^\times$. The Fourier transform of $f \in \mathcal{S}(\mathbb{Q}_p)$ is defined by

$$(1.1) \quad \widehat{f}(x) = \int_{\mathbb{Q}_p} \psi(xt) f(t) dt.$$

We have that $\widehat{f} \in \mathcal{S}(\mathbb{Q}_p)$, for any $f \in \mathcal{S}(\mathbb{Q}_p)$. The Fourier transform is not continuous with respect to the sup norm. In [14], we showed that the Fourier transform is “as discontinuous as it can get” in the sense that the graph

$$\left\{ (f, \widehat{f}) \mid f \in \mathcal{S}(\mathbb{Q}_p) \right\}$$

is dense in $\mathcal{C}_0(\mathbb{Q}_p) \times \mathcal{C}_0(\mathbb{Q}_p)$. The Fourier transform is an intertwining operator in the sense of Section 2.2. This raises the question: is a similar result holds for other intertwining operators? If T_1 and T_2 are intertwining operators, one can use the above result from [14] together with the fact that the action of $\mathrm{Sp}_2(\mathbb{Q}_p) = \mathrm{SL}_2(\mathbb{Q}_p)$ on $\mathbb{P}^1(\mathbb{Q}_p)$ is doubly transitive to show that the set

$$\{(T_1(f), T_2(f)) \mid f \in \mathcal{S}(\mathbb{Q}_p)\}$$

is dense in $\mathcal{C}_0(\mathbb{Q}_p) \times \mathcal{C}_0(\mathbb{Q}_p)$, as long as T_1, T_2 do not lie in the same orbit of the upper triangular matrices.

Now consider three intertwining operators T_1, T_2, T_3 . Is

$$\{(T_1(f), T_2(f), T_3(f)) \mid f \in \mathcal{S}(\mathbb{Q}_p)\}$$

dense in $\mathcal{C}_0(\mathbb{Q}_p)^3$? Already this case seems to be beyond the reach of the methods of [14].

Using the rigidity of the norms $\|\cdot\|_\alpha$ (Theorem 1.1) we are able to answer this question in the affirmative for any finite number of intertwining operators (Proposition 7.9). In particular, we reprove the main result in [14].

1.3. Relation to previous work of J. Fresnel and B. de Mathan

Let $\psi : (\mathbb{Q}_p, +) \rightarrow \mathbb{C}_p^\times$ be a smooth character with $\ker(\psi) = \mathbb{Z}_p$. Although the Fourier transform (defined in (1.1)) is not continuous with respect to the sup norm, its restriction to the space $\mathcal{S}(\mathbb{Q}_p/\mathbb{Z}_p)$ is continuous. Here, $\mathcal{S}(\mathbb{Q}_p/\mathbb{Z}_p)$ is the space of Schwartz functions which are constant on cosets of \mathbb{Z}_p . If $f \in \mathcal{S}(\mathbb{Q}_p/\mathbb{Z}_p)$, the Schwartz function \widehat{f} is supported on \mathbb{Z}_p . Thus, the Fourier transform extends to a map \mathcal{F} :

$$\mathcal{F} : \mathcal{C}_0(\mathbb{Q}_p/\mathbb{Z}_p) \rightarrow \mathcal{C}(\mathbb{Z}_p)$$

given by

$$\mathcal{F}(f)(x) = \sum_{t \in \mathbb{Q}_p/\mathbb{Z}_p} \psi(tx)f(t).$$

Surprisingly, \mathcal{F} is not injective. This was proven independently by Fresnel and de Mathan [6, 7] and Amice and Escassut [1]. Fresnel and de Mathan proved the following stronger and deep result.

THEOREM 1.6 ([7, Theorem 2]). — *The Fourier transform \mathcal{F} is surjective, its the kernel $K = \ker(\mathcal{F})$ is nonzero, and the induced map*

$$\mathcal{C}_0(\mathbb{Q}_p/\mathbb{Z}_p)/K \rightarrow \mathcal{C}(\mathbb{Z}_p)$$

is an isometric isomorphism.

It follows from the results of Section 3 that Theorem 1.6 is equivalent to the property that the sup norm on $\mathcal{S}(\mathbb{Z}_p)$ is *locally maximal* (Definition 3.6) at $\mathbf{1}_{\mathbb{Z}_p}(x)$ with respect to multiplication by smooth characters. In Section 4.3 we give another proof of [7, Theorem 2] based on Theorem 4.4.

Remark 1.7.

- (1) Our methods are different than the methods in [7]. Our proof, using q -arithmetic, can be generalized to include the case where $\psi : (\mathbb{Q}_p, +) \rightarrow \mathbb{C}_p^\times$ is continuous but not smooth, and this case does not follow from [7]. These results will appear in a forthcoming paper.
- (2) The space $\mathcal{C}_0(\mathbb{Q}_p/\mathbb{Z}_p)$ is a Banach algebra with convolution, and $\mathcal{C}(\mathbb{Z}_p)$ is a Banach algebra with the pointwise multiplication. The Fourier transform is a homomorphism of Banach algebras. A deep result by Fresnel and de Mathan ([5, 8]) says that the kernel K of the Fourier transform contains nilpotent elements, and that the nilpotent elements are dense in K . We do not how to reproduce this result with our methods.

Acknowledgements

I wish to express my deep gratitude to Ehud de Shalit for many helpful discussions, for his encouragement, and for reading and commenting on the first draft of this paper. I would also like to thank the anonymous referee for valuable suggestions, including an argument that made Section 6.3 shorter and easier to read.

List of notation

Throughout this paper we fix a prime number p and an algebraic closure \mathbb{Q}_p^{al} of \mathbb{Q}_p . The absolute value $|\cdot|_p$ on \mathbb{Q}_p extends uniquely to \mathbb{Q}_p^{al} and we denote by \mathbb{C}_p the completion of \mathbb{Q}_p^{al} with respect to $|\cdot|_p$. The field \mathbb{C}_p is a complete non-archimedean normed field and algebraically closed. We denote by $\mathcal{O}_{\mathbb{C}_p}$ the ring $\{a \in \mathbb{C}_p \mid |a|_p \leq 1\}$.

Fix a positive integer d . The following notation will be used.

$\mathcal{H} = \mathcal{H}_{2d+1}(\mathbb{Q}_p)$ - The $2d + 1$ -dimensional Heisenberg group over \mathbb{Q}_p (see Definition 2.1).

$\mathrm{Sp}_{2d}(\mathbb{Q}_p)$ - The $2d$ -dimensional symplectic group.

P - The Siegel parabolic subgroup of $\mathrm{Sp}_{2d}(\mathbb{Q}_p)$ (defined in the discussion preceding Definition 4.2).

Gr - The right cosets space $\mathrm{P} \backslash \mathrm{Sp}_{2d}(\mathbb{Q}_p)$. It can be realised as the Grassmannian of maximal isotropic subspaces of the standard $2d$ -dimensional symplectic space.

$\mathcal{S} = \mathcal{S}(\mathbb{Q}_p^d)$ - The space of locally constant and compactly supported functions on \mathbb{Q}_p^d .

$\mathcal{S}(\mathbb{Z}_p^d)$ - The space of locally constant functions on \mathbb{Z}_p^d .

ψ - A fixed non-trivial smooth character $\psi : (\mathbb{Q}_p, +) \rightarrow \mathbb{C}_p^\times$.

ρ_ψ - the Schrödinger representation of \mathcal{H} on \mathcal{S} with central character ψ (defined in the discussion preceding Theorem 2.2).

$\mathcal{N}(V)^G$ - The set of norms on V which are invariant under the action of G . Here, V is a representation of a group G over \mathbb{C}_p .

$\mathcal{C}(\mathbb{Z}_p^d)$ - The Banach space of \mathbb{C}_p -valued continuous functions on \mathbb{Z}_p^d , with the sup norm.

$\mathcal{C}_0(\mathbb{Q}_p^d)$ - The Banach space of \mathbb{C}_p -valued continuous functions on \mathbb{Q}_p^d that goes to zero at infinity, with the sup norm. A function $f : \mathbb{Q}_p^d \rightarrow \mathbb{C}_p$ is said to go to zero at infinity if for any $\epsilon > 0$ there exists a compact subset $K \subset \mathbb{Q}_p^d$ such that $|f(x)|_p < \epsilon$ for any $x \in \mathbb{Q}_p^d \setminus K$.

$\mathcal{C}_0(\mathbb{Q}_p/\mathbb{Z}_p)$ - The Banach space of \mathbb{C}_p -valued functions on \mathbb{Q}_p that goes to zero at infinity and are constant on cosets of \mathbb{Z}_p .

2. A reminder on p -adic Heisenberg groups and Schrödinger representations

In this section we recall the classical theory of smooth irreducible representations of p -adic Heisenberg groups. This section is based on [11, 16].

In the introduction we used a construction of $\mathcal{H}_3(\mathbb{Q}_p)$ as a matrix group, but for the treatment of higher dimensional Heisenberg groups, and especially for introducing intertwining operators, we found it more convenient to use a different construction. The construction we will use throughout the paper is given in this section. Both constructions result in isomorphic groups, but some formulas (such as the one used to define the Schrödinger representations) take different forms than the ones given in the introduction.

2.1. The Heisenberg group over \mathbb{Q}_p and its smooth representations

Let $W = \mathbb{Q}_p^d \oplus \mathbb{Q}_p^d$ and denote by ω the symplectic form on W given by $\omega((x_1, y_1), (x_2, y_2)) = x_1 \cdot y_2 - y_1 \cdot x_2$, where $a \cdot b$, for $a, b \in \mathbb{Q}_p^d$, is the standard scalar product.

DEFINITION 2.1. — *The $2d + 1$ -dimensional Heisenberg group over \mathbb{Q}_p , denoted by $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$, is the topological group whose underlying set is $W \times \mathbb{Q}_p$ and the multiplication is given by*

$$[w_1, t_1] \cdot [w_2, t_2] = \left[w_1 + w_2, t_1 + t_2 + \frac{1}{2}\omega(w_1, w_2) \right].$$

Since d is fixed, we denote $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$ by \mathcal{H} . One easily verifies that the center of \mathcal{H} , which is also its commutator subgroup, is $Z := \{[0, t] \mid t \in \mathbb{Q}_p\}$, and that $\mathcal{H}/Z \simeq W = \mathbb{Q}_p^{2d}$. In particular, \mathcal{H} is nilpotent group of class two.

As a topological group, \mathcal{H} inherits a topology from the topology of \mathbb{Q}_p . This makes \mathcal{H} a totally disconnected and locally compact topological group.

Recall that a representation (V, π) of a totally disconnected group G over C is said to be smooth if the stabilizer $\text{Stab}_G(v)$ in G of any vector $v \in V$ is open. A smooth representation of G is called admissible if for any open

compact subgroup $K \subset G$ the sub-space V^K of vectors fixed by K is finite dimensional.

By Schur's lemma, if (V, π) is a smooth irreducible representation of \mathcal{H} , the center of \mathcal{H} acts on V via a character ψ , called *the central character* of ρ . We identify the center of \mathcal{H} with \mathbb{Q}_p and view ψ as a character $\psi : (\mathbb{Q}_p, +) \rightarrow \mathbb{C}_p^\times$. Since π is smooth, the kernel of ψ is an open subgroup of \mathbb{Q}_p and we say that ψ is a smooth character.

The classification of smooth irreducible representations of \mathcal{H} is well known, and we recall it. If ψ is trivial, the action of \mathcal{H} factors through an abelian quotient, and V is 1-dimensional. Assume that ψ is non-trivial. We construct a representation ρ_ψ , called the Schrödinger representation of \mathcal{H} , which has central character ψ .

Let $\mathcal{S} = \mathcal{S}(\mathbb{Q}_p^d)$ be the space of Schwartz functions, that is functions $f : \mathbb{Q}_p^d \rightarrow \mathbb{C}_p$ which are locally constant and compactly supported. It is an infinite dimensional vector space over \mathbb{C}_p . Define a representation ρ_ψ of \mathcal{H} on \mathcal{S} as follows. Let $w = (a, b)$ with $a, b \in \mathbb{Q}_p^d$. Then

$$(2.1) \quad (\rho_\psi([w, t])f)(x) = \psi \left(t + \frac{1}{2}a \cdot b + b \cdot x \right) \cdot f(x + a).$$

THEOREM 2.2 (Smooth Stone-von Neumann). — *Let ψ be a non-trivial smooth character of $(\mathbb{Q}_p, +)$.*

- (1) *The representation ρ_ψ is a smooth, irreducible and admissible representation of \mathcal{H} and has central character ψ .*
- (2) *Let (V, π) be a smooth representation of \mathcal{H} . Assume that the center of \mathcal{H} acts via the character ψ . Then V decomposes as a direct sum of sub-representations, each isomorphic to ρ_ψ .*

It is important to note that in the Schrödinger representation, the Heisenberg group acts on \mathcal{S} by translations and by multiplication by the smooth characters of \mathbb{Q}_p^d . A smooth character of \mathbb{Q}_p^d is a homomorphism $\alpha : (\mathbb{Q}_p^d, +) \rightarrow \mathbb{C}_p^\times$ with an open kernel. By definition, any translation appears as an action of an element of the Heisenberg group. It is also true that if $\alpha : \mathbb{Q}_p^d \rightarrow \mathbb{C}_p^\times$ is a smooth character, there exists an element $[0, b, 0]$ whose action is multiplication by α . Indeed, if ψ is a non-trivial smooth character of \mathbb{Q}_p , any smooth character of \mathbb{Q}_p^d is of the form $\psi \circ \lambda$, for some λ in the dual space of \mathbb{Q}_p^d .

2.2. Automorphisms of the Heisenberg group and intertwining operators on the Schrödinger representation

Let $J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$ and $\mathrm{Sp}_{2d}(\mathbb{Q}_p)$ be the symplectic group

$$\mathrm{Sp}_{2d}(\mathbb{Q}_p) = \{g \in \mathrm{GL}_{2d}(\mathbb{Q}_p) \mid gJg^t = J\}.$$

Thinking about the vectors in $\mathbb{Q}_p^d \oplus \mathbb{Q}_p^d$ as row vectors, an element $g \in \mathrm{Sp}_{2d}(\mathbb{Q}_p)$ acts on W by right multiplication: $w \mapsto wg$ and preserves the symplectic form ω . This defines a right action of the symplectic group on the Heisenberg group by automorphisms as follows:

$$[w, t] \cdot g = [wg, t].$$

These automorphisms are continuous and their restriction to the center $Z = \{[0, t] \mid t \in \mathbb{Q}_p\}$ is the identity.

Let $g \in \mathrm{Sp}_{2d}(\mathbb{Q}_p)$. Define a new representation $\rho_{g, \psi}$ of \mathcal{H} on \mathcal{S} by

$$\rho_{g, \psi}([w, t])f = \rho_\psi([w, t] \cdot g)f$$

for any $[w, t] \in \mathcal{H}$ and $f \in \mathcal{S}$. The representation $\rho_{g, \psi}$ is smooth, irreducible and has ψ as its central character. Thus, by the Stone–von Neumann theorem, $\rho_\psi \simeq \rho_{g, \psi}$, so there exists an invertible linear operator T_g on \mathcal{S} such that

$$\rho_\psi([w, t]) \circ T_g = T_g \circ \rho_{g, \psi}([w, t])$$

for any $[w, t] \in \mathcal{H}$. By Schur's lemma, T_g is unique up to a multiplicative constant. It also follows that $T_{g_1 \cdot g_2}$ is equal, up to a constant, to $T_{g_1} \circ T_{g_2}$, so $g \mapsto T_g$ is a projective representation, called the Weil representation. There is an explicit formula for the operators T_g . Let $g \in \mathrm{Sp}_{2d}(\mathbb{Q}_p)$ and write it as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d \in M_d(\mathbb{Q}_p)$ are square matrices.

PROPOSITION 2.3 ([11, Proposition 2.3]). — *Let g be as above and T_g an intertwining operator corresponding to g . There is a unique choice of a \mathbb{C}_p -valued Haar distribution $d\mu$ on $\mathrm{Im}(c)$ such that*

$$T_g(f)(x) = \int_{\mathrm{Im}(c)} \psi \left(\frac{1}{2}(xa) \cdot (xb) - (xb) \cdot y + \frac{1}{2}y \cdot (yd) \right) \cdot f(xa + y) d\mu(y).$$

Here, $\mathrm{Im}(c)$ is the space $\{vc \mid v \in \mathbb{Q}_p^d\}$.

Note that if $g = J$, the formula gives, up to normalization, the usual Fourier transform. If $c = 0$, we get the operation of multiplication by a quadratic exponential accompanied by the dilation $x \mapsto xa$.

3. Banach representations

The goal of this section is twofold. We introduce the terminology about norms and Banach representations that will be used throughout this paper, and we prove some fundamental properties of Banach representations that we will later need. We address two issues. The first is a notion of minimality of norms that we call weak minimality. The second consists of several characterizations of quotients of universal unitary completions of (algebraically) cyclic representations. Let G be an abstract group and V a representation of G over \mathbb{C}_p . We say that V is a cyclic representation of G if V is a cyclic $\mathbb{C}_p[G]$ -module. Under the assumptions that V is cyclic, \widehat{V} has a universal unitary completion in the sense of [4] that we denote by \widehat{V} . The quotients of \widehat{V} by closed sub-representations will play an important role in this paper, especially quotients by maximal sub-representations. We give two intrinsic characterizations of these quotients: in terms of a special type of norms which we call *locally maximal* and in terms of the existence of a special type of vectors which we call *strongly cyclic*.

3.1. General terminology and notation

Let V be vector space over \mathbb{C}_p . A norm on V is a map $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$ such that

- (1) $\|v\| = 0$ if and only if $v = 0$.
- (2) $\|a \cdot v\| = |a|_p \cdot \|v\|$ for any $v \in V$ and $a \in \mathbb{C}_p$.
- (3) $\|v_1 + v_2\| \leq \max(\|v_1\|, \|v_2\|)$.

If $\|\cdot\|$ satisfies only 2 and 3 we say that it is a seminorm.

Let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on V . We write $\|\cdot\|_1 \leq \|\cdot\|_2$ if $\|v\|_1 \leq \|v\|_2$ for any $v \in V$. We say that $\|\cdot\|_1$ is dominated by $\|\cdot\|_2$, and denote it by $\|\cdot\|_1 \preceq \|\cdot\|_2$ if there exists a constant $D > 0$ such that $\|\cdot\|_1 \leq D \cdot \|\cdot\|_2$. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if each dominates the other: $\|\cdot\|_1 \preceq \|\cdot\|_2$ and $\|\cdot\|_2 \preceq \|\cdot\|_1$. These two norms are called homothetic if there exists $c > 0$ such that $\|v\|_1 = c \cdot \|v\|_2$ for any $v \in V$.

If $v \in V$ is a nonzero vector, we say that $\|\cdot\|$ is normalized at v if $\|v\| = 1$. In any homothety class of norms there is exactly one norm that is normalized at v .

Given a norm $\|\cdot\|$ on V , we denote the completion of V with respect to $\|\cdot\|$ by $V_{\|\cdot\|}$.

Assume that a group G acts on V . A norm $\|\cdot\|$ on V is said to be G -invariant if $\|gv\| = \|v\|$ for any $v \in V$ and $g \in G$. When there is no

ambiguity about the group G , we will simply say that $\|\cdot\|$ is an invariant norm. We denote the set of norms on V by $\mathcal{N}(V)$ and by $\mathcal{N}(V)^G$ its subset of G -invariant norms.

In this paper the term *Banach representation* means the following.

DEFINITION 3.1. — *A Banach representation (over \mathbb{C}_p) of G is a pair $(B, \|\cdot\|)$ of a G -representation B and a G -invariant norm $\|\cdot\|$ such that B is complete with respect to $\|\cdot\|$.*

A morphism of Banach representations of G is a continuous G -equivariant map, but it need not be an isometry. In particular, isomorphic Banach representations of G are not necessarily isometric.

3.2. Weakly minimal norms

DEFINITION 3.2. — *Let $(B, \|\cdot\|)$ be a Banach representation of the group G and $v \in B$ a nonzero vector. We say that $\|\cdot\|$ is weakly minimal at v if the following holds.*

- *For any G -invariant norm $\|\cdot\|'$ on B such that $\|\cdot\|' \leq \|\cdot\|$ and $\|v\|' = \|v\|$, we have $\|\cdot\|' = \|\cdot\|$.*

Let $(B, \|\cdot\|)$ be a Banach space over \mathbb{C}_p . The values of $\|\cdot\|$ is the set of non-negative real numbers $\{\|v\| \mid v \in B\}$. Similarly, the values of the absolute value $|\cdot|_p$ on \mathbb{C}_p is the set $\{|a|_p \mid a \in \mathbb{C}_p\}$. Unlike Banach spaces over \mathbb{R} or \mathbb{C} , the values of a norm $\|\cdot\|$ of a \mathbb{C}_p -Banach space need not be equal to the values of the absolute value $|\cdot|_p$.

LEMMA 3.3. — *Let $(B, \|\cdot\|)$ be a Banach representation of G . Assume that the values of $\|\cdot\|$ are the same as the values of $|\cdot|_p$ on \mathbb{C}_p . Assume that $v \in B$ is a nonzero vector such that $\|v\| = 1$ and such that its image \bar{v} in the quotient*

$$\overline{B}(\|\cdot\|) := \{v \in B \mid \|v\| \leq 1\} / \{v \in B \mid \|v\| < 1\}$$

is contained in any nonzero sub-representation of $\overline{B}(\|\cdot\|)$. Then $\|\cdot\|$ is weakly minimal at v .

Proof. — Let $\|\cdot\|' \in \mathcal{N}(B)^G$ be a G -invariant norm such that $\|v\|' = 1$ and $\|\cdot\|' \leq \|\cdot\|$. The identity map $Id : B \rightarrow B$ induces a map $T : \overline{B}(\|\cdot\|) \rightarrow \overline{B}(\|\cdot\|')$. The kernel of T is a sub-representation of $\overline{B}(\|\cdot\|)$ that does not contain v , hence by assumption, this kernel is trivial. It follows that T is injective. Therefore, $\|w\|' = 1$ for any w with $\|w\| = 1$. Since the values of $\|\cdot\|$ are the same as the values of $|\cdot|_p$, $\|\cdot\|' = \|\cdot\|$. \square

PROPOSITION 3.4. — *Let G be a pro- p group. Let $\mathcal{C}(G)$ denote the space of continuous functions on G with values in \mathbb{C}_p and let $\|\cdot\|_\infty$ be the sup norm on $\mathcal{C}(G)$. Consider the action of G on $\mathcal{C}(G)$ by right translations. The sup norm is weakly minimal at $\mathbf{1}$, where $\mathbf{1}$ denotes the constant function $\mathbf{1}(x) = 1$.*

Proof. — Identify the quotient

$$\{f \in \mathcal{C}(G) \mid \|f\|_\infty \leq 1\} / \{f \in \mathcal{C}(G) \mid \|f\|_\infty < 1\}$$

with the space $\mathcal{S}(G, \overline{\mathbb{F}}_p)$ of locally constant functions on G with values in an algebraic closure $\overline{\mathbb{F}}_p$ of \mathbb{F}_p . By the previous lemma, it is enough to show that any nonzero sub-representation of $\mathcal{S}(G, \overline{\mathbb{F}}_p)$ contains the constant function $\mathbf{1}$. Here, $\overline{\mathbb{F}}_p$ is the residue field of $\mathcal{O}_{\mathbb{C}_p}$. Let $f \in \mathcal{S}(G, \overline{\mathbb{F}}_p)$ be nonzero and denote by V the sub-representation generated by f . As f is fixed by some open normal subgroup $N \subset G$, V is a cyclic representation of the finite group G/N . In particular, V is a finite dimensional representation of the finite p -group G/N over $\overline{\mathbb{F}}_p$. Thus, V contains a nonzero G -invariant vector ϕ . This ϕ is a nonzero constant function. \square

3.3. The universal unitary completion of a cyclic representation

Let V be a representation of G and assume that $v \in V$ is a cyclic vector. In addition, assume that $\mathcal{N}(V)^G$ is non-empty, i.e. there exists a G -invariant norm on V .

If $\|\cdot\| \in \mathcal{N}(V)^G$, its closed unit ball $\{w \in V \mid \|w\| \leq 1\}$ is an $\mathcal{O}_{\mathbb{C}_p}[G]$ -module that contains a nonzero multiple of any vector in V , but contains no \mathbb{C}_p -lines. Such an $\mathcal{O}_{\mathbb{C}_p}$ -module is called an integral structure. Conversely, any integral structure L defines a G -invariant norm, called the gauge of L , by

$$\|v\|_L = \inf \{|a|_p \mid v \in a \cdot L\}.$$

We stress the fact that in general L might not be equal to the closed unit ball nor to the open unit ball of $\|\cdot\|_L$, but lies strictly between them. For future use we record the following formulas for the closed and open unit balls of $\|\cdot\|_L$,

$$(3.1) \quad \begin{aligned} \{v \in V \mid \|v\|_L \leq 1\} &= \bigcap_{\substack{\lambda \in \mathbb{C}_p \\ |\lambda|_p > 1}} \lambda L, \\ \{v \in V \mid \|v\|_L < 1\} &= \bigcup_{\substack{\lambda \in \mathbb{C}_p \\ |\lambda|_p < 1}} \lambda L. \end{aligned}$$

Since \mathbb{C}_p is not discretely valued, two different invariant norms give rise to different integral structures, but two different integral structures might define the same norm. Nevertheless, the correspondence between invariant norms and integral structures inverts order.

The set $L_v := \mathcal{O}_{\mathbb{C}_p}[G] \cdot v$ is an integral structure. Indeed, it contains a multiple of any vector since v is cyclic, and it contains no lines because of the existence of an invariant norm. As L_v is the smallest integral structure that contains v , its corresponding norm, which we denote by $\|\cdot\|_v$, is normalized at v and is the maximal invariant norm normalized at v . This means that if $\|\cdot\| \in \mathcal{N}(V)^G$ is normalized at v , then $\|\cdot\| \leq \|\cdot\|_v$. We call the norm $\|\cdot\|_v$ *the maximal invariant norm at v* or *the maximal norm at v* for short.

If $v_1, v_2 \in V$ are two cyclic vectors of V , the norms $\|\cdot\|_{v_1}$ and $\|\cdot\|_{v_2}$ are equivalent. In particular the completion of V with respect to $\|\cdot\|_v$, where v is a cyclic vector, is independent of v as a topological vector space. We denote this completion by \widehat{V} and call it the universal unitary completion of V , or the universal completion for short. Note that this is a particular case of [4, Example A]. The universal completion of V has the following universal property: if $(B, \|\cdot\|)$ is a Banach representation of G and $T : V \rightarrow (B, \|\cdot\|)$ is G -equivariant, then T factors continuously through \widehat{V} .

Remark 3.5. — The assumption that $\mathcal{N}(V)^G \neq \emptyset$ is superfluous, is made for simplification and because this is the case that will appear later. If V does not have a G -invariant norm, $\mathcal{O}_{\mathbb{C}_p}[G] \cdot v$ contains \mathbb{C}_p -lines. The union of these lines is a sub-representation $W \subset V$ and the quotient $V' = V/W$ is cyclic and has an invariant norm. The universal completion of V is $\widehat{V'}$.

Any element of \widehat{V} can be written as

$$\sum_{g \in G} \lambda_g \cdot g(v)$$

where $(\lambda_g)_{g \in G} \subset \mathbb{C}_p$ is *summable*, meaning that for any $\epsilon > 0$, at most finitely many of the λ_g satisfy $|\lambda_g|_p \geq \epsilon$. Conversely, any series of this form converges in \widehat{V} .

DEFINITION 3.6. — *Let W be a representation of G and $\|\cdot\| \in \mathcal{N}(W)^G$. Let $w \in W$ a nonzero vector such that $\|\cdot\|$ is normalized at w . We say that $\|\cdot\|$ is locally maximal at w if the following property holds.*

- *For any $\|\cdot\|' \in \mathcal{N}(W)^G$ that is normalized at w and is dominated by $\|\cdot\|$ we have $\|\cdot\|' \leq \|\cdot\|$.*

For example, the norm $\|\cdot\|_v$ on V is a locally maximal norm at v . Consider the Schrödinger representation of $\mathcal{H}_3(\mathbb{Q}_p)$ on $\mathcal{S}(\mathbb{Q}_p)$, and the two norms

$\|\cdot\|_f, \|\cdot\|_g$, where $f(x) = \mathbf{1}_{\mathbb{Z}_p}(x)$ and $g(x) = \mathbf{1}_{p\mathbb{Z}_p}(x)$. Were $\|\cdot\|_f$ also locally maximal at g , then the two norms $\|\cdot\|_f$ and $\|\cdot\|_g$, being equivalent, would be homothetic. One can show that $\|g\|_f = p$ and $\|f\|_g = 1$, so the two norms are not homothetic, and $\|\cdot\|_f$ is locally maximal at f but not at g . Thus a norm which is locally maximal at one cyclic vector is in general not locally maximal at another one.

For another example, consider the space $\mathcal{C}_0(\mathbb{Q}_p^d)$ of \mathbb{C}_p -valued continuous functions on \mathbb{Q}_p^d that go to zero at infinity, and the action of the Heisenberg group \mathcal{H} on it by the formula given in the previous section. We will later show that the sup norm $\|\cdot\|_\infty$ is a locally maximal norm on $\mathcal{C}_0(\mathbb{Q}_p^d)$ at f , for any nonzero $f \in \mathcal{C}_0(\mathbb{Q}_p^d)$.

DEFINITION 3.7. — *Let $(B, \|\cdot\|)$ be a Banach representation of G and $v \in B$ a nonzero vector. We say that v is topologically cyclic if v generates (algebraically) a dense representation in B . We say that v is strongly cyclic if any $w \in V$ can be written as*

$$w = \sum_{g \in G} \lambda_g \cdot g(v),$$

where the $(\lambda_g)_{g \in G}$ is summable.

For example, if $v \in V$ is a cyclic vector, then v is strongly cyclic in \widehat{V} . In the end of this section we give an example of a topologically cyclic vector which is not a strongly cyclic vector.

We make the following two observations. Let $T : B' \rightarrow B$ be a map of Banach representations of G .

- Assume that the image of T contains a strongly cyclic vector. Then T is surjective.
- Assume that $v' \in B'$ is strongly cyclic and that T is surjective. Then $v = T(v')$ is strongly cyclic in B .

We begin with two lemmas. The first says that a quotient norm of a locally maximal one is locally maximal. The second says that strongly cyclic vectors give rise to locally maximal norms.

LEMMA 3.8. — *Let W be a representation of G , $\|\cdot\| \in \mathcal{N}(W)^G$ and $K \subset W$ a closed (with respect to $\|\cdot\|$) sub-representation. Assume that $\|\cdot\|$ is normalized and locally maximal at $w \in W$. Then the quotient norm on W/K is normalized and locally maximal at the image of w .*

Proof. — Let $\|\cdot\|'$ denote the quotient norm on W/K . Let $\|\cdot\|_2$ be a G -invariant norm on W/K that is normalized at the image of w and dominated by $\|\cdot\|'$. Using the quotient map $W \rightarrow W/K$ we view $\|\cdot\|_2$ also as a

semi-norm on W . Then $\max(\|\cdot\|_2, \|\cdot\|)$ is a G -invariant norm on W that is normalized at w and dominated by $\|\cdot\|$. Thus, $\|\cdot\|_2 \leq \|\cdot\|$, as semi-norms on W . Taking the quotient by K , we obtain $\|\cdot\|_2 \leq \|\cdot\|'$, as norms on W/K . \square

LEMMA 3.9. — *Assume that $(B, \|\cdot\|)$ is a Banach representation of G and that $0 \neq v \in B$ is a strongly cyclic vector. Then there exists a unique norm, which we denote by $\|\cdot\|_{v,B}$, which is normalized and locally maximal at v and is equivalent to $\|\cdot\|$. In addition, if $w \in B$ with $\|w\|_{v,B} = r$, then for any $\epsilon > 0$ there exists a summable sequence $(\lambda_g)_{g \in G}$ such that*

$$w = \sum_{g \in G} \lambda_g \cdot g(v)$$

and $\max_{g \in G} |\lambda_g|_p < (1 + \epsilon) \cdot r$.

Proof. — The uniqueness of a normalized and locally maximal norm at v is clear. In the rest of the proof we construct the norm $\|\cdot\|_{v,B}$ using an integral structure and show the additional property.

Let \bar{L} be the closure in B of

$$L = \left\{ \sum_{g \in G} \lambda_g \cdot g(v) \mid (\lambda_g)_{g \in G} \text{ is summable and } |\lambda_g|_p \leq 1 \text{ for all } g \in G \right\}.$$

Assume, for convenience, that $\|v\| = 1$. We first show that \bar{L} is an open integral structure. It is straightforward that \bar{L} is an integral structure, the only non-obvious part is that \bar{L} contains no \mathbb{C}_p -lines. This is true since L , and therefore \bar{L} , is contained in the closed unit ball of $\|\cdot\|$. We now show that \bar{L} is open. Since v is strongly cyclic, $B = \bigcup_{n=0}^{\infty} p^{-n} \cdot \bar{L}$. Since B is a complete metric space, it follows from Baire's category theorem that \bar{L} has a non-empty interior, and since \bar{L} is a topological subgroup, it must be open. Let $\|\cdot\|_{v,B}$ be the norm that corresponds to \bar{L} . Since \bar{L} is an open integral structure, $\|\cdot\|_{v,B}$ is a G -invariant norm and equivalent to $\|\cdot\|$.

Now we show that $\|\cdot\|_{v,B}$ is normalized at v and locally maximal at v . Since $v \in \bar{L}$, it follows that $\|v\|_{v,B} \leq 1$. Let $\|\cdot\|'$ be a G -invariant norm dominated by $\|\cdot\|_{v,B}$ and normalized at v . The closed unit ball of $\|\cdot\|'$ contains L , and since

$$\|\cdot\|' \preceq \|\cdot\|_{v,B} \preceq \|\cdot\|$$

its unit ball is closed in B . Thus, the unit ball of $\|\cdot\|'$ contains \bar{L} . Therefore, $\|\cdot\|' \leq \|\cdot\|_{v,B}$. Substituting v , we see that $\|v\|_{v,B} \geq 1$, so $\|\cdot\|_{v,B}$ is normalized at v and locally maximal at v .

Finally, we prove the additional property. Let $w \in B$ and let $\epsilon > 0$. We may assume that $(1 + \epsilon)^{-1} < \|w\|_{v,B} < 1$. By (3.1) (formula for the open

unit ball), $w \in \bar{L}$, so there exists $w_0 \in L$ such that $\|w - w_0\|_{v,B} < p^{-1}$. Similarly, there exists $w_1 \in pL$ such that $\|w - w_0 - p \cdot w_1\|_{v,B} < p^{-2}$. Continuing in this manner we obtain a sequence $(w_n)_{n=0}^\infty$, where $w_n \in p^n \cdot L$ for all n , and $w = \sum_{n=0}^\infty w_n$. Therefore, $w \in L$, so it can be written as $w = \sum_{g \in G} \lambda_g \cdot g(v)$ and $\max_{g \in G} |\lambda_g|_p < 1 < (1 + \epsilon) \cdot \|w\|_{v,B}$. \square

Remark 3.10. — Note that the last step in the proof can be modified slightly to show that $L = \bar{L}$. However, the closed unit ball of $\|\cdot\|_{v,B}$ might be strictly larger than L .

THEOREM 3.11. — *Let $(B, \|\cdot\|)$ be a Banach representation of G and $v \in B$ a nonzero vector. The following are equivalent.*

- (1) v is a strongly cyclic vector of B and $\|\cdot\| = \|\cdot\|_{v,B}$.
- (2) Let V be the (algebraic) sub-representation generated by v . Then the map $I : \widehat{V} \rightarrow B$ is surjective and if K is its kernel, the induced map $\widehat{V}/K \rightarrow B$ is an isometry when we equip \widehat{V} with the norm $\|\cdot\|_v$.
- (3) $\|\cdot\|$ is normalized at v and locally maximal at v .

Proof. — We will show (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1). Assume (1). Since v is strongly cyclic, the map $I : \widehat{V} \rightarrow B$ is surjective. Equip \widehat{V} with the norm $\|\cdot\|_v$ and let $\|\cdot\|'$ denote the quotient norm on \widehat{V}/K . Via I , we view $\|\cdot\|'$ as a norm on B . By the open mapping theorem, $\|\cdot\|'$ and $\|\cdot\| = \|\cdot\|_{v,B}$ are equivalent. By Lemma 3.8 and Lemma 3.9, these two norms are normalized and locally maximal at v . Thus, they are equal.

Assume (2). (3) Follows from Lemma 3.8.

Assume (3). We will prove (1). We assume that $\|\cdot\|$ is normalized and locally maximal at v . Let $w \in B$; we want to show that w is of the form $\sum_{g \in G} \lambda_g \cdot g(v)$, where $(\lambda_g)_{g \in G}$ is summable. We may assume that $\|w\| \leq 1$. Let $L = \mathcal{O}_{\mathbb{C}_p}[G] \cdot v$ and let D be the closed unit ball of $\|\cdot\|$. Then $L + p^2 \cdot D$ is an open integral structure in B that contains v . Its corresponding norm, that we denote by $\|\cdot\|'$, is dominated by $\|\cdot\|$ and satisfies $\|v\|' \leq 1$. Therefore, by (3), $\|\cdot\|' \leq \|\cdot\|$. By (3.1) (formula for the closed unit ball) it follows that

$$w \in D \subset \bigcap_{\substack{\lambda \in \mathbb{C}_p \\ |\lambda|_p > 1}} \lambda(L + p^2 \cdot D) \subset p^{-1}(L + p^2 \cdot D) = p^{-1}L + pD.$$

Thus, there exist $x_1 \in L$ and $d_1 \in D$ such that $w = p^{-1}x_1 + p \cdot d_1$. Repeating this process with d_1 instead of w , there exist $x_2 \in L$ and $d_2 \in D$ such that $d_1 = p^{-1}x_2 + p d_2$. Thus, $w = p^{-1}x_1 + p(p^{-1}x_2 + p d_2) = p^{-1}x_1 + x_2 + p^2 d_2$. Repeating this process, we obtain sequences $(x_n)_{n=1}^\infty \subset L$ and $(d_n)_{n=1}^\infty \subset D$

such that $d_n = p^{-1}x_{n+1} + pd_{n+1}$ for any $n \geq 0$. Thus, $w = \sum_{n=0}^{\infty} p^{n-1} \cdot x_n$ which is of the desired form. \square

COROLLARY 3.12. — *Let $(B, \|\cdot\|)$ be a Banach representation of G . The following are equivalent.*

- (1) *B is isomorphic to a quotient of a universal completion of a cyclic representation.*
- (2) *B has a strongly cyclic vector.*
- (3) *$\|\cdot\|$ is equivalent to a locally maximal norm with respect to some vector v .*
- (4) *There exists $v \in B$ such that any map $T : B' \rightarrow B$ of Banach representations of G such that v lies in its image is surjective.*

3.4. Strongly irreducible Banach representations

DEFINITION 3.13. — *Let $(B, \|\cdot\|)$ be a Banach representation of G . We say that B is strongly irreducible if any nonzero vector in B is strongly cyclic.*

Clearly, a strongly irreducible Banach representation is topologically irreducible. The converse is not true (see the example at the end of this section).

PROPOSITION 3.14. — *Let $(B, \|\cdot\|)$ be a Banach representation of G and $v \in B$ a strongly cyclic vector. Assume that $\|\cdot\|$ is normalized and locally maximal at v . Then, any $w \in B$ with $\|v - w\| < 1$ is also strongly cyclic.*

Proof. — First, note that $\|\cdot\|$ is equal to $\|\cdot\|_{v,B}$ from Lemma 3.9. Let $w \in B$ such that $\|v - w\| < 1$. By Theorem 3.11, it is enough to show that $\|\cdot\|$ is normalized and locally maximal at w . That $\|\cdot\|$ is normalized at w follows from the strong triangle inequality. To show that $\|\cdot\|$ is locally maximal at w , let $\|\cdot\|' \in \mathcal{N}(V)^G$ a norm that is dominated by $\|\cdot\|$ and normalized at w . By Lemma 3.9 we can write $v - w = \sum_{g \in G} \lambda_g \cdot g(v)$, where $(\lambda_g)_{g \in G}$ is summable and $\max_{g \in G} (\lambda_g) < 1$. Therefore, if $\|w\|' = 1$, then also $\|v\|' = 1$. Since $\|\cdot\|$ is locally maximal at v , $\|\cdot\|' \leq \|\cdot\|$. This proves that $\|\cdot\|$ is also locally maximal at w . \square

PROPOSITION 3.15. — *Let $(B, \|\cdot\|)$ be a Banach representation of G . The set of all strongly cyclic vectors in B is open (possibly empty).*

Proof. — Assume that the set of strongly cyclic vectors in B is not empty. Let v be a strongly cyclic vector in B . By the previous proposition, all the

vectors in the open unit ball around v with respect to $\|\cdot\|_{v,B}$ are strongly cyclic. By Lemma 3.9, the open unit ball of $\|\cdot\|_{v,B}$ is open in B . \square

COROLLARY 3.16. — *Let $(B, \|\cdot\|)$ be a Banach representation of G and assume that B contains a strongly cyclic vector. If B is not strongly irreducible, then B contains a nonzero proper closed sub-representation.*

Proof. — Let U be the open subset of strongly cyclic vectors in B . By assumption, U is not empty. Assume that B is not strongly irreducible and let $0 \neq w \in B$ be a vector that is not strongly cyclic. Denote by W the algebraic representation generated by w in B , then $W \subset B \setminus U$. Thus, the closure of W is a proper nonzero closed sub-representation of B . \square

THEOREM 3.17. — *Let $(B, \|\cdot\|)$ be a Banach representation of G . The following are equivalent.*

- (1) B is strongly irreducible.
- (2) B is topologically irreducible and there exists a strongly cyclic vector in B .
- (3) B is topologically irreducible and there exists a locally maximal norm at some vector $0 \neq v \in B$ on B , equivalent to $\|\cdot\|$.
- (4) Any nonzero G -equivariant bounded map $B' \rightarrow B$, where $(B', \|\cdot\|')$ is a Banach representation of G , is surjective.
- (5) $(B, \|\cdot\|)$ is isomorphic to a quotient of a universal completion of a cyclic representation of G by a maximal sub-representation.

Proof. — The implication (1) \Rightarrow (2) is trivial and the implication (2) \Rightarrow (1) follows from Corollary 3.16. The equivalence (2) \iff (3) follows from Theorem 3.11. Next we show (1) \iff (4). Assume (1). Let $(B', \|\cdot\|')$ be a Banach representation of G and $T : B' \rightarrow B$ a nonzero G -equivariant bounded map. Let $0 \neq v \in \text{Im}(T)$. Then v is strongly cyclic and by a previous observation, T is surjective. Assume (4). Let $v \in B$ nonzero. Let V be the algebraic representation generated by v in B . The map $I : \widehat{V} \rightarrow B$ is a nonzero G -equivariant bounded map, so by assumption I is surjective. Thus, by a previous observation, v is strongly cyclic in B . Finally, we prove (2) \iff (5). Assume (2). Let $v \in B$ be a strongly cyclic vector, let V be the algebraic representation generated by v and $I : \widehat{V} \rightarrow B$. Since v is strongly cyclic, I is surjective. If $W \subset \widehat{V}$ denotes the kernel of I , then \widehat{V}/W is isomorphic to B . Since B is topologically irreducible, W is a maximal closed sub-representation. Assume (5). As a quotient of a universal completion of a cyclic representation by a maximal sub-representation, B is topologically irreducible and contains a strongly cyclic vector. By Corollary 3.16, B is strongly cyclic. \square

PROPOSITION 3.18. — *Let $(B_i, \|\cdot\|_i)$, for $i = 1, \dots, n$, be pairwise non-isomorphic strongly irreducible Banach representations of G . Let B be the Banach representation $B = \bigoplus_{i=1}^n B_i$, equipped with the norm $\max(\|\cdot\|_1, \dots, \|\cdot\|_n)$. Then any $x = (x_1, \dots, x_n) \in B$ such that $x_i \neq 0$ for all i is strongly cyclic in B .*

Proof. — By induction on n . The case $n = 1$ is trivial. Assume that $n > 1$ and that the claim holds for any $1 \leq k < n$. By Theorem 3.11, it is enough to show that for any Banach representation $(B', \|\cdot\|')$ of G and any continuous map of representations $T : B' \rightarrow B$, if x lies in the image of T then T is surjective. Let $T : B' \rightarrow B$ be such a map. Denote by $P_1 : B \rightarrow B_1$ and $P_2 : B \rightarrow \bigoplus_{i=2}^n B_i$ the projections. Since x_1 lies in the image of $P_1 \circ T$ and (x_2, \dots, x_n) lies in the image of $P_2 \circ T$, it follows from the induction hypothesis that both $P_1 \circ T$ and $P_2 \circ T$ are surjective. Let K_1, K_2 be the kernels of $P_1 \circ T$ and $P_2 \circ T$ respectively. Then K_2 is not contained in K_1 , for otherwise we would have a nonzero map $\bigoplus_{i=2}^n B_i \rightarrow B_1$. Such a map would give a nonzero map between one of the B_i , for $i \geq 2$, and B_1 . Since both B_1 and B_i are strongly irreducible, such a map must be an isomorphism, contradicting the hypothesis. Therefore, the restriction of $P_1 \circ T$ to K_2 is a nonzero map, hence surjective since B_1 is strongly irreducible. It follows that B_1 is contained in the image of T . Similarly, for any $1 \leq i \leq n$, B_i is contained in the image of T . Thus, T is surjective. \square

We end this section with an example of a topologically irreducible Banach representation which is not strongly irreducible.

Example. — Let $\mathcal{C}(\mathbb{Z}_p)$ be the space of continuous functions on \mathbb{Z}_p with values in \mathbb{C}_p , equipped with the sup norm $\|\cdot\|_\infty$. We choose $q \in \mathbb{C}_p$, not a root of unity, such that $|q - 1|_p < 1$ and denote by G the group generated by translations and by multiplications by q^{nx} , $n \in \mathbb{Z}$. The sup norm is invariant under the action of G , so $\mathcal{C}(\mathbb{Z}_p)$ is a Banach representation of G .

As a representation of G , $\mathcal{C}(\mathbb{Z}_p)$ is topologically irreducible, as we now show. Let A be the linear span of the functions q^{nx} , $n \in \mathbb{Z}$. Then A is an algebra in $\mathcal{C}(\mathbb{Z}_p)$ that separates points and contains the constant functions. By the p -adic Stone–Weierstrass theorem ([10]), A is dense in $\mathcal{C}(\mathbb{Z}_p)$. Let $0 \neq f \in \mathcal{C}(\mathbb{Z}_p)$ and W be the closed sub-representation generated by f . We will show that $W = \mathcal{C}(\mathbb{Z}_p)$. We first show that W contains a nowhere vanishing function. The \mathbb{Z}_p -representation

$$\overline{W} = \{h \in W \mid \|h\|_\infty \leq 1\} / \{h \in W \mid \|h\|_\infty < 1\}$$

is nonzero and smooth. Let $0 \neq v \in \overline{W}$, and let E be the \mathbb{Z}_p -sub-representation generated by v . Then E is a finite dimensional representation of a finite

quotient of \mathbb{Z}_p over a field of characteristic p (the residue field of $\mathcal{O}_{\mathbb{C}_p}$). Thus, there exists a nonzero $v' \in E$ which is fixed by \mathbb{Z}_p . Let $g \in W$ a lift of v' . Then g is a nowhere vanishing function. Then $A \cdot g \subset W$ is dense in $\mathcal{C}(\mathbb{Z}_p)$, so $W = \mathcal{C}(\mathbb{Z}_p)$.

We show that the constant function $\mathbf{1}(x)$ is not a strongly cyclic vector, thus $\mathcal{C}(\mathbb{Z}_p)$ is not a strongly irreducible representation of G . Let $f(x) \in \mathcal{C}(\mathbb{Z}_p)$ and assume that it can be written as

$$f(x) = \sum_{n \in \mathbb{Z}} \lambda_n \cdot q^{nx},$$

where $\lim_{n \rightarrow \infty} \lambda_n = 0$. The Mahler expansion of $f(x)$ is

$$\begin{aligned} f(x) &= \sum_{n \in \mathbb{Z}} \lambda_n \cdot (q^n)^x = \sum_{n \in \mathbb{Z}} \lambda_n \cdot \sum_{k=0}^{\infty} (q^n - 1)^k \cdot \binom{x}{k} \\ &= \sum_{k=0}^{\infty} \left(\sum_{n \in \mathbb{Z}} \lambda_n \cdot (q^n - 1)^k \right) \cdot \binom{x}{k} = \sum_{k=0}^{\infty} b_k \cdot \binom{x}{k}. \end{aligned}$$

There exists $0 < \epsilon < 1$ such that $|q^n - 1|_p < \epsilon$ for all $n \in \mathbb{Z}$. Let $m = \max_{n \in \mathbb{Z}} |\lambda_n|_p$. Then the coefficients $(b_k)_{k=1}^{\infty}$ obey the asymptotic formula

$$|b_k|_p \leq m \cdot \epsilon^k.$$

In particular, the function $f \in \mathcal{C}(\mathbb{Z}_p)$ with Mahler expansion $f(x) = \sum_{k=0}^{\infty} p^k \cdot \binom{x}{p^k}$ is not of the form $\sum_{n \in \mathbb{Z}} \lambda_n \cdot q^{nx}$.

4. The main results

In this section and for the rest of this paper all the representations are assumed to be over \mathbb{C}_p . Fix a non-trivial smooth character $\psi : (\mathbb{Q}_p, +) \rightarrow \mathbb{C}_p^\times$, and let (ρ_ψ, \mathcal{S}) be the Schrödinger representation of the Heisenberg group $\mathcal{H} = \mathcal{H}_{2d+1}(\mathbb{Q}_p)$. In particular, the functions in $\mathcal{S} = \mathcal{S}(\mathbb{Q}_p^d)$ are valued in \mathbb{C}_p . The action of \mathcal{H} on \mathcal{S} is generated by translations and multiplication by smooth characters. An \mathcal{H} -invariant norm on \mathcal{S} is therefore a norm $\|\cdot\|$ on \mathcal{S} such that

$$\|f(x+a)\| = \|f(x)\|, \quad \|\chi(x) \cdot f(x)\| = \|f(x)\|$$

for any $f \in \mathcal{S}$, any $a \in \mathbb{Q}_p$ and any smooth character χ of \mathbb{Q}_p^d .

Our main results concern a family of \mathcal{H} -invariant norms on \mathcal{S} with a surprising rigidity. This family is the orbit of the sup norm by intertwining operators. In the first sub-section we define these norms and show that they are parameterized by a Grassmannian. In the second sub-section we state the main results. The proofs are given in the next sections.

4.1. A special family of \mathcal{H} -invariant norms parameterized by a Grassmannian

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in the symplectic group $\mathrm{Sp}_{2d}(\mathbb{Q}_p)$ and choose T_g , a corresponding intertwining operator. Recall that intertwining operators were defined in Section 2.2. If $\|\cdot\|$ is an \mathcal{H} -invariant norm on \mathcal{S} , the norm $f \mapsto \|T_g(f)\|$ is also \mathcal{H} -invariant. Indeed,

$$\|T_g([w, t]f)\| = \|[wg, t]T_g(f)\| = \|T_g(f)\|.$$

As the T_g are determined up to a constant, this defines a right action of $\mathrm{Sp}_{2d}(\mathbb{Q}_p)$ on the space $\mathcal{N}_H(\mathcal{S})^{\mathcal{H}}$ of homothety classes of \mathcal{H} -invariant norm on \mathcal{S} . If $x \in \mathcal{N}_H(\mathcal{S})^{\mathcal{H}}$ denotes the homothety class of the norm $\|\cdot\|$, then we denote by xg the homothety class of the norm $\|T_g(\cdot)\|$. Then $(xg_1)g_2 = x(g_1g_2)$ are both equal to the homothety class of the norm $\|T_{g_1}(T_{g_2}(\cdot))\|$.

An important example of an \mathcal{H} -invariant norm on \mathcal{S} is the sup norm:

$$\|f\|_{\infty} = \sup_{x \in \mathbb{Q}_p^d} |f(x)|_p.$$

In the following proposition we determine the stabilizer in $\mathrm{Sp}_{2d}(\mathbb{Q}_p)$ of the homothety class of the sup norm.

PROPOSITION 4.1. — *Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in the symplectic group $\mathrm{Sp}_{2d}(\mathbb{Q}_p)$ and T_g a corresponding intertwining operator. The norms $\|\cdot\|_{\infty}$ and $\|T_g(\cdot)\|_{\infty}$ are homothetic if and only if they are equivalent, if and only if $c = 0$.*

Proof. — If $c = 0$, Proposition 2.3 says that there exists $\lambda \in \mathbb{C}_p^{\times}$ such that

$$T_g(f)(x) = \lambda \cdot \psi \left(\frac{1}{2}(xa) \cdot (xb) \right) \cdot f(xa).$$

As a must be invertible, $\|f(xa)\|_{\infty} = |\lambda|_p \cdot \|f(x)\|_{\infty}$. Thus, $\|\cdot\|_{\infty}$ and $\|T_g(\cdot)\|_{\infty}$ are homothetic and therefore equivalent.

Assume that $c \neq 0$ and let $k \geq 1$ be the dimension of $\mathrm{Im}(c)$. Recall the definition of $\mathrm{Im}(c)$ from Proposition 2.3. Choose a basis v_1, \dots, v_k of $\mathrm{Im}(c)$ and complete it to a basis $v_1, \dots, v_k, v_{k+1}, \dots, v_d$ of \mathbb{Q}_p^d . Let U_n and V_n be the compact open sets in \mathbb{Q}_p^d and in $\mathrm{Im}(c)$ respectively, given by

$$U_n = \left\{ \sum_{i=1}^d \lambda_i v_i \mid \lambda_1, \dots, \lambda_d \in p^n \mathbb{Z}_p \right\},$$

$$V_n = \left\{ \sum_{i=1}^k \lambda_i v_i \mid \lambda_1, \dots, \lambda_k \in p^n \mathbb{Z}_p \right\}.$$

Denote by $f_n(x)$ the characteristic function of U_n . Note that $f_n(x) \in \mathcal{S}$. By Proposition 2.3, there exists a Haar distribution $d\mu$ on $\text{Im}(c)$ such that

$$T_g(f_n)(x) = \int_{\text{Im}(c)} \psi \left(\frac{1}{2}(xa) \cdot (xb) - (xb) \cdot y + \frac{1}{2}y \cdot (yd) \right) \cdot f_n(xa + y) d\mu(y).$$

We may assume that $\mu(V_0) = 1$. Substituting $x = 0$, we obtain

$$T_g(f_n)(0) = \int_{\text{Im}(c)} \psi \left(\frac{1}{2}y \cdot (yd) \right) \cdot f_n(y) d\mu(y) = \int_{V_n} \psi \left(\frac{1}{2}y \cdot (yd) \right) d\mu(y).$$

When n is sufficiently large, $\frac{1}{2}y \cdot (yd) \in \ker(\psi)$ for any $y \in V_n$, so

$$T_g(f_n)(0) = \int_{V_n} 1 d\mu(y) = p^{-nk}.$$

Thus, $\lim_{n \rightarrow \infty} \|T_g(f_n)\|_\infty = \infty$, whereas $\|f_n\|_\infty = 1$ for any n . Then $\|\cdot\|_g$ and $\|\cdot\|_\infty$ are not equivalent and therefore not homothetic. \square

Let P be the Siegel parabolic subgroup

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \text{Sp}_{2d}(\mathbb{Q}_p) \right\},$$

and denote $\text{Gr} = P \backslash \text{Sp}_{2d}(\mathbb{Q}_p)$. Then Gr is the Grassmannian of maximal isotropic subspaces of (W, ω) .

DEFINITION 4.2. — We denote the point that corresponds to P in Gr by ∞ . For any $\alpha = Pg \in \text{Gr}$ we denote by $\|\cdot\|_\alpha$ the unique \mathcal{H} -invariant norm in the homothety class of $\|T_g(\cdot)\|_\infty$ that is normalized at $\mathbf{1}_{\mathbb{Z}_p^d}(x)$.

4.2. The main results

Our deepest results are Theorem 4.3 and Theorem 4.4 below. Their proofs will occupy Section 5 and Section 6 of the paper.

THEOREM 4.3 (Rigidity). — Let $\alpha \in \text{Gr}$. If $\|\cdot\|$ is an \mathcal{H} -invariant norm on \mathcal{S} that is dominated by $\|\cdot\|_\alpha$, then $\|\cdot\| = r \cdot \|\cdot\|_\alpha$ for some constant $r > 0$.

In particular, each $\|\cdot\|_\alpha$ is locally maximal at every nonzero vector in the completion $\mathcal{S}_{\|\cdot\|_\alpha}$.

In order to prove Theorem 4.3 we will prove its \mathbb{Z}_p -analog. We denote by $\mathcal{S}(\mathbb{Z}_p^d)$ the space of locally constant \mathbb{C}_p -valued functions on \mathbb{Z}_p^d . The sup norm on $\mathcal{S}(\mathbb{Z}_p^d)$ is invariant under translations and under multiplication by the smooth characters of \mathbb{Z}_p^d . Here, as before, a smooth character of \mathbb{Z}_p^d is a homomorphism $\chi : (\mathbb{Z}_p^d, +) \rightarrow \mathbb{C}_p^\times$ with an open kernel.

THEOREM 4.4. — *Let $\|\cdot\|$ be a norm on $\mathcal{S}(\mathbb{Z}_p^d)$ that is dominated by the sup norm and invariant under translations and multiplication by the smooth characters of \mathbb{Z}_p^d . Then $\|\cdot\| = r \cdot \|\cdot\|_\infty$ for some $r > 0$.*

Compare Theorem 4.3 with the complex coefficients case: Denote by $\mathcal{S}^\mathbb{C}$ the space of complex-valued Schwartz functions on \mathbb{Q}_p^d , and let $\rho_\psi^\mathbb{C}$ be the complex Schrödinger representation of \mathcal{H} on $\mathcal{S}^\mathbb{C}$ (defined in the same way as in Section 2.1). We have a \mathcal{H} -invariant inner product on $\mathcal{S}^\mathbb{C}$:

$$\langle f, g \rangle = \int_{\mathbb{Q}_p^d} f(x) \overline{g(x)} \, dx,$$

where dx a \mathbb{C} -valued Haar measure on \mathbb{Q}_p^d . Since $\rho_\psi^\mathbb{C}$ is smooth and admissible, it follows from Schur's lemma that any other \mathcal{H} -invariant inner product on $\mathcal{S}^\mathbb{C}$ is a positive multiple of $\langle \cdot, \cdot \rangle$. Over \mathbb{C}_p , we do not have uniqueness of invariant norms, but each $\|\cdot\|_\alpha$, for $\alpha \in \text{Gr}$, is unique among the \mathcal{H} -invariant norms that it dominates.

The following proposition gives some basic properties of the completions of \mathcal{S} by a norm $\|\cdot\|_\alpha$.

THEOREM 4.5. — *Let $\alpha \in \text{Gr}$ and $\|\cdot\|_\alpha$ the corresponding norm.*

- (1) *The completion $\mathcal{S}_{\|\cdot\|_\alpha}$ is a strongly irreducible Banach representation of \mathcal{H} .*
- (2) *The smooth part of $\mathcal{S}_{\|\cdot\|_\alpha}$ is precisely \mathcal{S} .*
- (3) *Let $\beta \in \text{Gr}$. The space of continuous \mathcal{H} -equivariant maps from $\mathcal{S}_{\|\cdot\|_\alpha}$ to $\mathcal{S}_{\|\cdot\|_\beta}$ is given by*

$$\text{Hom}_{\mathcal{H}} \left(\mathcal{S}_{\|\cdot\|_\alpha}, \mathcal{S}_{\|\cdot\|_\beta} \right) \simeq \begin{cases} \mathbb{C}_p & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}.$$

Proof. — (1). By Theorem 4.3, the norm $\|\cdot\|_\alpha$ is locally maximal at f , for any $f \in \mathcal{S}_{\|\cdot\|_\alpha}$ with $\|f\|_\alpha = 1$. By Theorem 3.17 it is enough to show that $\mathcal{S}_{\|\cdot\|_\alpha}$ is topologically irreducible. Let W be a proper closed subrepresentation of $\mathcal{S}_{\|\cdot\|_\alpha}$. The quotient norm on $\mathcal{S}_{\|\cdot\|_\alpha}/W$ induces an invariant semi-norm $\|\cdot\|$ on \mathcal{S} that is dominated by $\|\cdot\|_\alpha$. Since \mathcal{S} is irreducible, $\|\cdot\|$ is a norm and by Theorem 4.3, $\|\cdot\| = r \cdot \|\cdot\|_\alpha$ for some $r > 0$, and $W = 0$.

(2). We first prove the claim for $\mathcal{S}_{\|\cdot\|_\infty} = \mathcal{C}_0(\mathbb{Q}_p^d)$. It is clear that any function in \mathcal{S} has an open stabilizer in \mathcal{H} . Let $f \in \mathcal{C}_0(\mathbb{Q}_p^d)$ with an open stabilizer in \mathcal{H} . Recall the action of the Heisenberg group given in (2.1). Then

$$f(x) = \psi(b \cdot x) \cdot f(x + a)$$

for all $a, b \in \mathbb{Z}_p^n$, for some large enough n . Letting $b = 0$, we see that f is constant on cosets of \mathbb{Z}_p^d . Letting $a = 0$, we see that f is supported on $p^{-n+k}\mathbb{Z}_p^d$, where $\ker(\psi) = p^k\mathbb{Z}_p$. For the general case, let $g \in \mathrm{Sp}_{2d}(\mathbb{Q}_p)$ such that $\alpha = Pg$, and let T_g be a corresponding intertwining operator, normalized such that $\|\cdot\|_\alpha = \|T_g(\cdot)\|_\infty$ on \mathcal{S} . Then T_g extends to an isometric isomorphism $T_g : \mathcal{S}_{\|\cdot\|_\alpha} \rightarrow \mathcal{C}_0(\mathbb{Q}_p^d)$. Although T_g is not \mathcal{H} -equivariant, it satisfies

$$T_g([w, t]f) = [wg, t]T_g(f).$$

In particular, $T_g(f)$ is a smooth vector in $\mathcal{C}_0(\mathbb{Q}_p^d)$ if and only if f is a smooth vector in $\mathcal{S}_{\|\cdot\|_\alpha}$.

(3) Let $T : \mathcal{S}_{\|\cdot\|_\alpha} \rightarrow \mathcal{S}_{\|\cdot\|_\beta}$ be a continuous \mathcal{H} -equivariant map. By the previous part, the restriction of T to $\mathcal{S} \subset \mathcal{S}_{\|\cdot\|_\alpha}$ is an \mathcal{H} -equivariant map $T' : \mathcal{S} \rightarrow \mathcal{S}$. By Schur's lemma for smooth representations, T' is multiplication by a constant. If this constant is nonzero, it means that $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$, considered on \mathcal{S} , are homothetic. By Proposition 4.1 this could only be the case if $\alpha = \beta$. Thus, if $\alpha \neq \beta$, $T = 0$. If $\alpha = \beta$ then by continuity, T is a multiplication by a scalar. \square

PROPOSITION 4.6. — *Let $\alpha \in \mathrm{Gr}$. Let $(B, \|\cdot\|)$ be a topologically irreducible Banach representation of \mathcal{H} (see Definition 3.1). Assume that we are in one of the two following cases.*

- (1) $F : B \rightarrow \mathcal{S}_{\|\cdot\|_\alpha}$ is a nonzero continuous map of representations.
- (2) $F : \mathcal{S}_{\|\cdot\|_\alpha} \rightarrow B$ is a nonzero continuous map of representations.

Then F is an isomorphism. Moreover, there exists $r > 0$ such that by replacing $\|\cdot\|$ with $r \cdot \|\cdot\|$, F becomes an isometric isomorphism.

Proof. — Let $(B, \|\cdot\|)$ be an irreducible Banach representation of \mathcal{H} . Assume we are in the first case, and let $F : B \rightarrow \mathcal{S}_{\|\cdot\|_\alpha}$ be a continuous map of representations. Assume that F is nonzero. Since B is topologically irreducible, the kernel of F is zero, so F is injective. By Theorem 4.5 and Theorem 3.17, F is surjective. Thus, F is an isomorphism. The norm $\|F^{-1}(\cdot)\|$ is an \mathcal{H} -invariant norm on $\mathcal{S}_{\|\cdot\|_\alpha}$ that is dominated by $\|\cdot\|_\alpha$. By Theorem 4.3, there exists $r > 0$ such that $r \cdot \|F^{-1}(\cdot)\| = \|\cdot\|_\alpha$. Replacing $\|\cdot\|$ by $r \cdot \|\cdot\|$, F becomes an isometry.

Assume we are in the second case and let $F : \mathcal{S}_{\|\cdot\|_\alpha} \rightarrow B$ a continuous map of representations. Assume that F is nonzero. The norm $\|F(\cdot)\|$ is an \mathcal{H} -invariant norm on $\mathcal{S}_{\|\cdot\|_\alpha}$ that is dominated by $\|\cdot\|_\alpha$. By Theorem 4.3, there exists $r > 0$ such that $r \cdot \|F(\cdot)\| = \|\cdot\|_\alpha$. Replacing $\|\cdot\|$ by $r \cdot \|\cdot\|$, F becomes an isometry. In particular, F is injective. The image of F is

therefore a closed sub-representation of B , and since B is topologically irreducible, F is surjective. \square

4.3. A new proof of a results of Fresnel and de Mathan

Let $\psi : (\mathbb{Q}_p, +) \rightarrow \mathbb{C}_p^\times$ be a smooth character with $\ker(\psi) = \mathbb{Z}_p$. Recall that the Fourier transform \mathcal{F} on \mathbb{Q}_p (see Section 1.2) is not continuous with respect to the sup norm. The restriction of \mathcal{F} to the space $\mathcal{S}(\mathbb{Q}_p/\mathbb{Z}_p)$ of Schwartz functions on \mathbb{Q}_p which are constant on cosets of \mathbb{Z}_p , is continuous. If $f \in \mathcal{S}(\mathbb{Q}_p/\mathbb{Z}_p)$, then $\mathcal{F}(f)$ is a Schwartz function supported on \mathbb{Z}_p . Thus, the restriction of the Fourier transform extends to a map between the completions

$$\mathcal{F} : \mathcal{C}_0(\mathbb{Q}_p/\mathbb{Z}_p) \rightarrow \mathcal{C}(\mathbb{Z}_p),$$

given by

$$\mathcal{F}(f)(x) = \sum_{t \in \mathbb{Q}_p/\mathbb{Z}_p} \psi(tx) \cdot f(t).$$

In [7] Fresnel and de Mathan proves the following theorem.

THEOREM 4.7 ([7, Theorem 2]). — *The Fourier transform \mathcal{F} is surjective and is not injective. Moreover, if K denotes its kernel, the induced map*

$$\mathcal{C}_0(\mathbb{Q}_p/\mathbb{Z}_p)/K \rightarrow \mathcal{C}(\mathbb{Z}_p)$$

is a surjective isometry.

Proof. — Let $\mathcal{H}(\mathbb{Z}_p)$ be the following subgroup of the Heisenberg group $\mathcal{H}_3(\mathbb{Q}_p)$,

$$\mathcal{H}(\mathbb{Z}_p) = \{[a, b, t] \mid a \in \mathbb{Z}_p\}.$$

Then $\mathcal{H}(\mathbb{Z}_p)$ acts on $\mathcal{C}(\mathbb{Z}_p)$ by the usual rule

$$([a, b, t]f)(x) = \psi(t + bx) \cdot f(x + a),$$

namely, by translations and multiplication by smooth characters. The group $\mathcal{H}(\mathbb{Z}_p)$ also acts on $\mathcal{C}_0(\mathbb{Q}_p/\mathbb{Z}_p)$ by the rule

$$([a, b, t]g)(x) = \psi(t - ab + ax) \cdot g(x - b).$$

It is easy to verify that

$$\mathcal{F} : \mathcal{C}_0(\mathbb{Q}_p/\mathbb{Z}_p) \rightarrow \mathcal{C}(\mathbb{Z}_p)$$

is a continuous homomorphism of Banach representations of $\mathcal{H}(\mathbb{Z}_p)$. By Theorem 4.4, the sup norm on $\mathcal{C}(\mathbb{Z}_p)$ is locally maximal with respect to any $f \in \mathcal{C}(\mathbb{Z}_p)$ with $\|f\|_\infty = 1$. It follows from Theorem 3.11 that any

nonzero $f \in \mathcal{C}(\mathbb{Z}_p)$ is strongly cyclic, hence that $\mathcal{C}(\mathbb{Z}_p)$ is a strongly irreducible representation. By Theorem 3.17, \mathcal{F} is surjective. If \mathcal{F} were also injective, it would be, by the open mapping theorem, an isomorphism of Banach spaces. To see that this is not true, consider the characteristic functions $\phi_n(x) := \mathbf{1}_{p^{-n}\mathbb{Z}_p}(x) \in \mathcal{C}_0(\mathbb{Q}_p/\mathbb{Z}_p)$. Then $\|\phi_n\|_\infty = 1$, while $\|\mathcal{F}(\phi_n)\|_\infty = \|p^n \cdot \mathbf{1}_{p^n\mathbb{Z}_p}\|_\infty = p^{-n}$. Therefore, \mathcal{F} is not injective. Finally, denoting the kernel of \mathcal{F} by K , we have the induced isomorphism of Banach representations

$$\mathcal{C}(\mathbb{Q}_p/\mathbb{Z}_p)_0/K \rightarrow \mathcal{C}(\mathbb{Z}_p).$$

By Theorem 4.4, there exists a real number $r > 0$ such that by taking the norm $r \cdot \|\cdot\|_\infty$ on $\mathcal{C}(\mathbb{Z}_p)$, the above isomorphism is an isometry. To show that $r = 1$, it is enough to show that the image of ϕ_0 in the quotient $\mathcal{C}_0(\mathbb{Q}_p/\mathbb{Z}_p)/K$ has norm 1. Note that ϕ_0 is a strongly cyclic vector in $\mathcal{C}_0(\mathbb{Q}_p/\mathbb{Z}_p)$, and that $\|\cdot\|_\infty$ is normalized and locally maximal at ϕ_0 . Thus, by Proposition 3.14, the open unit ball around ϕ_0 in $\mathcal{C}_0(\mathbb{Q}_p/\mathbb{Z}_p)$ consists of strongly cyclic vectors. In particular, all elements of K are at distance at least 1 from ϕ_0 . It follows that the image of ϕ_0 in the quotient has norm 1. Therefore, $r = 1$. \square

Remark 4.8. — In [7], Fresnel and de Mathan first show that \mathcal{F} is not injective by constructing nonzero elements in the kernel of \mathcal{F} . These elements have some special properties which then enable them to show that \mathcal{F} is surjective.

5. Reduction steps

In this section we show that Theorem 4.3 (rigidity of $\|\cdot\|_\alpha$ on \mathbb{Q}_p^d) follows from Theorem 4.4 (rigidity of the sup norm on \mathbb{Z}_p^d), and also that Theorem 4.4 follows from the particular case of Theorem 4.4 when $d = 1$.

PROPOSITION 5.1. — *If Theorem 4.3 holds for the sup norm then it holds for $\|\cdot\|_\alpha$ for any $\alpha \in \text{Gr}$.*

Proof. — Let $\text{Pg} = \alpha \in \text{Gr}$, where $g \in \text{Sp}_{2d}(\mathbb{Q}_p)$, and let T_g be an intertwining operator such that $\|\cdot\|_\alpha = \|T_g(\cdot)\|_\infty$. Let $\|\cdot\|$ be an \mathcal{H} -invariant norm on \mathcal{S} , dominated by $\|\cdot\|_\alpha$.

The operators T_g and $(T_g)^{-1}$ act on $\mathcal{N}(\mathcal{S})^{\mathcal{H}}$ and preserve order. In particular, $\|(T_g)^{-1}(\cdot)\|$ is an \mathcal{H} -invariant norm, dominated by the sup norm. By assumption, $\|(T_g)^{-1}(\cdot)\| = r \cdot \|\cdot\|_\infty$ for some $r > 0$. Thus, $\|\cdot\| = r \cdot \|\cdot\|_\alpha$. \square

Next we show that Theorem 4.3 for the sup norm follows from Theorem 4.4.

PROPOSITION 5.2. — *Assume that Theorem 4.4 holds. Let $\|\cdot\|$ be an \mathcal{H} -invariant norm on \mathcal{S} that is dominated by the sup norm. Then $\|\cdot\| = r \cdot \|\cdot\|_\infty$ for some $r > 0$.*

Proof. — For any $n \in \mathbb{N}$ we denote $V_n = \mathcal{S}(p^{-n}\mathbb{Z}_p^d)$ and think about V_n as the subspace of $\mathcal{S}(\mathbb{Q}_p^d)$ of functions supported on the disc $p^{-n}\mathbb{Z}_p^d$. The restriction of $\|\cdot\|$ to V_n is invariant under translations by $p^{-n}\mathbb{Z}_p^d$ and multiplication by smooth characters. By Theorem 4.4 and an obvious change of variables, there exists $r_n > 0$ such that $\|f\| = r_n \cdot \|f\|_\infty$ for any $f \in V_n$. The function $\mathbf{1}_{\mathbb{Z}_p^d}(x)$ lies in any of the V_n , so the numbers $(r_n)_{n \in \mathbb{N}}$ must be equal to the same r . Then $\|f\| = r \cdot \|f\|_\infty$ for any compactly supported function f . \square

PROPOSITION 5.3. — *If Theorem 4.4 holds for \mathbb{Z}_p then it holds for \mathbb{Z}_p^d for any d .*

Proof. — The proof is by induction, the case $d = 1$ being assumed to be true. Let $d > 1$ and assume that Theorem 4.4 holds for $d - 1$. Let $\|\cdot\|$ be a norm on $\mathcal{S}(\mathbb{Z}_p^d)$ that is invariant under translations and multiplication by smooth characters, dominated by the sup norm and normalized on $\mathbf{1}_{\mathbb{Z}_p^d}(x)$. By Proposition 3.4 it is enough to show that $\|\cdot\| \leq \|\cdot\|_\infty$. The latter follows if we show that for any n , $\|\mathbf{1}_{p^n\mathbb{Z}_p^d}(x)\| = 1$, where $\mathbf{1}_{p^n\mathbb{Z}_p^d}(x)$ is the characteristic function $p^n\mathbb{Z}_p^d$.

Let $0 < n \in \mathbb{N}$. Let P_d be the projection $\alpha : \mathbb{Z}_p^d \rightarrow \mathbb{Z}_p$ given by $\alpha(a_1, \dots, a_d) = a_d$, and denote by P_d^* the induced map $P_d^* : \mathcal{S}(\mathbb{Z}_p) \rightarrow \mathcal{S}(\mathbb{Z}_p^d)$. It is easy to see that the norm $\|P_d^*(\cdot)\|$ on $\mathcal{S}(\mathbb{Z}_p)$ is invariant under translations and multiplication by smooth characters, dominated by the sup norm and normalized at $\mathbf{1}_{\mathbb{Z}_p}(x)$. Thus, $\|P_d^*(f)\| = \|f\|_\infty$ for any $f \in \mathcal{S}(\mathbb{Z}_p)$. In particular,

$$\|P_d^*(\mathbf{1}_{p^n \cdot \mathbb{Z}_p}(x))\| = 1,$$

Note that

$$P_d^*(\mathbf{1}_{p^n \cdot \mathbb{Z}_p}(x)) = \mathbf{1}_{\mathbb{Z}_p^{d-1} \times (p^n \cdot \mathbb{Z}_p)}(x).$$

Now, consider the projection $\beta : \mathbb{Z}_p^{d-1} \times (p^n \cdot \mathbb{Z}_p) \rightarrow \mathbb{Z}_p^{d-1}$ given by $\beta(a_1, \dots, a_{d-1}, a_d) = (a_1, \dots, a_{d-1})$, and the induced map $\beta^* : \mathcal{S}(\mathbb{Z}_p^{d-1}) \rightarrow \mathcal{S}(\mathbb{Z}_p^d)$. The norm $\|\beta^*(\cdot)\|$ on $\mathcal{S}(\mathbb{Z}_p^{d-1})$ is invariant under translations and multiplication by smooth characters and dominated by the sup norm. Since

$$\beta^*(\mathbf{1}_{\mathbb{Z}_p^{d-1}}(x)) = \mathbf{1}_{\mathbb{Z}_p^{d-1} \times (p^n \cdot \mathbb{Z}_p)}(x)$$

and since $\|\mathbf{1}_{\mathbb{Z}_p^{d-1} \times (p^n \cdot \mathbb{Z}_p)}(x)\| = 1$, we deduce that $\|\beta^*(\cdot)\|$ is normalized on $\mathbf{1}_{\mathbb{Z}_p^{d-1}}$. Therefore $\|\beta^*(\cdot)\| = \|\cdot\|_\infty$. In particular,

$$\left\| \beta^* \left(\mathbf{1}_{p^n \cdot \mathbb{Z}_p^{d-1}}(x) \right) \right\| = 1.$$

Note that

$$\beta^* \left(\mathbf{1}_{p^n \cdot \mathbb{Z}_p^{d-1}}(x) \right) = \mathbf{1}_{p^n \cdot \mathbb{Z}_p^d}(x).$$

Thus, $\|\mathbf{1}_{p^n \cdot \mathbb{Z}_p^d}(x)\| = 1$. □

It remains to prove Theorem 4.4 for \mathbb{Z}_p . This is done in the next section.

6. Proof of Theorem 4.4 for \mathbb{Z}_p

In this section we prove Theorem 4.4 for \mathbb{Z}_p . We begin by noting that in the formulation of Theorem 4.4, the space $\mathcal{S}(\mathbb{Z}_p)$ can be replaced by $\mathcal{C}(\mathbb{Z}_p)$, which is its completion with respect to the sup norm. Working with $\mathcal{C}(\mathbb{Z}_p)$ allows us to use functions, such as polynomials, which are not in $\mathcal{S}(\mathbb{Z}_p)$.

Clearly, Theorem 4.4 follows if we know that $\|\cdot\|_\infty$ is both weakly minimal and locally maximal at $\mathbf{1}_{\mathbb{Z}_p}(x)$. That the sup norm is weakly minimal at $\mathbf{1}_{\mathbb{Z}_p}(x)$ follows from Proposition 3.4. Thus, it remains to show that $\|\cdot\|_\infty$ is locally maximal at $\mathbf{1}_{\mathbb{Z}_p}(x)$.

The proof uses two main ingredients:

- (1) The **growth modulus** of a norm. This is a real valued function associated with norms on $\mathcal{C}(\mathbb{Z}_p)$ that are dominated by the sup norm.
- (2) The **q -Mahler bases**. To each $q \in \mathbb{C}_p$ with $|q - 1|_p < 1$, there corresponds a basis of $\mathcal{C}(\mathbb{Z}_p)$ called the q -Mahler basis which shares some nice properties with the Mahler basis: $\left\{ \binom{x}{n} \mid n \geq 0 \right\}$. The q -Mahler bases can be viewed as a family of deformations of the Mahler basis.

6.1. The growth modulus of a norm

The beginning of this section is an adaptation of [15, Chapter 6, part 1.4].

Let $(a_n)_{n=0}^\infty$ be a bounded sequence of non-negative real numbers. The growth modulus associated with the sequence $(a_n)_{n=0}^\infty$ is the function

$$r \mapsto \sup_n a_n r^n$$

defined on the interval $[0, 1]$. It is a continuous, non-decreasing and convex function (part of the **Classical Lemma** in [15, p. 292]).

We say that a real number $0 < r < 1$ is regular with respect to the sequence $(a_n)_{n=0}^\infty$ if there exists n such that $a_n r^n > a_m r^m$ for any $m \neq n$. Otherwise, we call r a critical value (with respect to the sequence).

We remark that if r is a regular value and $a_n r^n > a_m r^m$ for any $m \neq n$, there exists some interval containing r on which the growth modulus is equal to $a_n x^n$. In particular the growth modulus is smooth at the regular values.

The fundamental lemma about critical values is the following.

LEMMA 6.1. — *Assume that $(a_n)_{n=0}^\infty$ is not the zero sequence. The set of critical values is discrete in $[0, 1)$.*

Proof. — See the discussion that follows the **Classical Lemma** in [15]. □

Until the end of this subsection, fix a norm $\|\cdot\|$ on $\mathcal{C}(\mathbb{Z}_p)$ that is dominated by the sup norm and normalized at $\mathbf{1}_{\mathbb{Z}_p}(x)$. Let $\binom{x}{n}$ be the n^{th} binomial polynomial. Since $\|\cdot\|$ is dominated by $\|\cdot\|_\infty$, and $\left\|\binom{x}{n}\right\|_\infty = 1$ for all $n \geq 0$, the sequence $(\left\|\binom{x}{n}\right\|)_{n \geq 0}$ is bounded. We define the growth modulus of the norm $\|\cdot\|$ to be the growth modulus of that sequence. We denote the growth modulus of $\|\cdot\|$ by $G_{\|\cdot\|}(r)$. Explicitly, $G_{\|\cdot\|}(r) : [0, 1] \rightarrow \mathbb{R}$ is the function

$$G_{\|\cdot\|}(r) = \sup_{n \geq 0} \left(\left\|\binom{x}{n}\right\| \cdot r^n \right).$$

We call $r \in [0, 1]$ a regular (resp. critical) value for the norm $\|\cdot\|$ if it is regular (resp. critical) with respect to the sequence $(\left\|\binom{x}{n}\right\|)_{n \geq 0}$.

The connection between the growth modulus of $\|\cdot\|$ and the study of the norm itself comes from the work of Mahler. We recall the basic facts about the Mahler basis.

THEOREM 6.2 (Mahler [12]). — *Any $f \in \mathcal{C}(\mathbb{Z}_p)$ can be written as*

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot \binom{x}{n}$$

where $\lim_{n \rightarrow \infty} a_n = 0$ and the sum converges to f in the sup norm. Moreover, $\|f\|_\infty = \max_n |a_n|_p$.

The following proposition immediately follows.

PROPOSITION 6.3. — *Let M be the smallest number such that $\|\cdot\| \leq M \cdot \|\cdot\|_\infty$. Then*

$$M = G_{\|\cdot\|}(1) = \sup_{n \geq 0} \left\|\binom{x}{n}\right\|.$$

We note that by Proposition 3.4, if $\|\cdot\|$ is a translation invariant norm on $\mathcal{C}(\mathbb{Z}_p)$ which is dominated by $\|\cdot\|_\infty$ and normalized at $\mathbf{1}_{\mathbb{Z}_p}(x)$, then $\|\cdot\| = \|\cdot\|_\infty$ if and only if $M = 1$.

We conclude this subsection with the following proposition.

PROPOSITION 6.4. — *Let $\|\cdot\|$ be a norm on $\mathcal{C}(\mathbb{Z}_p)$ dominated by the sup norm. Let $q \in \mathbb{C}_p$ with $r := |q - 1|_p < 1$. Then $\|q^x\| \leq G_{\|\cdot\|}(r)$. Moreover, if r is a regular value for the norm $\|\cdot\|$ then*

$$\|q^x\| = G_{\|\cdot\|}(r).$$

Proof. — Using the Mahler expansion of the function q^x and the non-archimedean triangle inequality,

$$(6.1) \quad \|q^x\| = \left\| \sum_{k=0}^{\infty} (q-1)^k \binom{x}{k} \right\| \leq \sup_{k \geq 0} \left(|q-1|_p^k \left\| \binom{x}{k} \right\| \right) = G_{\|\cdot\|}(r).$$

If $r = |q - 1|_p$ is a regular value, there exists $n \geq 0$ such that

$$r^n \left\| \binom{x}{n} \right\| > r^m \left\| \binom{x}{m} \right\|$$

for any $m \neq n$, and therefore we have an equality instead of inequality in (6.1). \square

Example. —

- (1) The growth modulus of the sup norm is constant $G_{\|\cdot\|}(r) = 1$.
- (2) Assume that $\|\cdot\|$ is invariant under multiplication by smooth characters, normalized at $\mathbf{1}_{\mathbb{Z}_p}(x)$, and that $G_{\|\cdot\|}(1) > 1$. Then, for N large enough and ζ a root of unity of order p^N , $r = |\zeta - 1|_p$ is a critical value for the norm $\|\cdot\|$. Indeed, for N large enough, $G_{\|\cdot\|}(r) > 1$ while $\|\zeta^x\| = 1$, so r is a critical value by the previous proposition.

Remark 6.5. — In general, $\|q^x\|$ is not a function of $|1 - q|_p$, i.e. it might be the case that $\|q_1^x\| \neq \|q_2^x\|$ while $|q_1 - 1|_p = |q_2 - 1|_p$.

6.2. q -Mahler bases

We briefly recall the q -analog terminology, the q -Mahler bases and the expansion formula for exponents in these bases. This subsection is self contained. For a more thorough exposition to the q -analog formalism and its properties we refer to [2, 9]. For more on the q -analog of the Mahler basis see [3].

Let q be an indeterminate. The q -analog of the natural number n is the following expression in $\mathbb{Z}[q]$

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}.$$

The q -analog of the factorial of n is

$$[n]_q! = [1]_q \cdot [2]_q \cdot \cdots \cdot [n]_q$$

and the q -binomial coefficients, also known as Gaussian binomial coefficients, are defined by the analogous formula

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!}$$

whenever $0 \leq k \leq n$, and zero otherwise. The q -Pochhammer symbol is the expression

$$(a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i).$$

When $a = q$ we get

$$(q; q)_n = \prod_{i=1}^n (1 - q^i).$$

By expanding the terms in the definition, it is easy to verify that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

The q -Pascal identity

$$(6.2) \quad \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q = \begin{bmatrix} n \\ k+1 \end{bmatrix}_q + q^{n-k} \cdot \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

implies, by induction, that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q with integer coefficients.

From now on q will not be an indeterminate but an element in \mathbb{C}_p such that $|q - 1|_p < 1$. The map $n \mapsto \begin{bmatrix} n \\ k \end{bmatrix}_q$ is continuous with respect to the p -adic topologies on \mathbb{Z} and on \mathbb{C}_p , and therefore extends to a map $x \mapsto \begin{bmatrix} x \\ k \end{bmatrix}_q$ that lies in $\mathcal{C}(\mathbb{Z}_p)$.

Since for any $x \in \mathbb{N}$ the expression $\begin{bmatrix} x \\ k \end{bmatrix}_q$ is a polynomial with integral coefficients in q , we have $|\begin{bmatrix} x \\ k \end{bmatrix}_q| \leq 1$. By continuity, $\|\begin{bmatrix} x \\ k \end{bmatrix}_q\| \leq 1$. Substituting $x = k$ we see that

$$\left\| \begin{bmatrix} x \\ k \end{bmatrix}_q \right\|_{\infty} = 1.$$

Note that if q is not a root of unity the term $(q; q)_k$ is nonzero for any k , so

$$\begin{bmatrix} x \\ k \end{bmatrix}_q = \frac{(1 - q^{x-(k-1)}) \cdot (1 - q^{x-(k-2)}) \cdots (1 - q^x)}{(1 - q)(1 - q^2) \cdots (1 - q^k)}.$$

We will need two results about q -binomial functions. The first is the q -analog of Mahler's theorem. The second is the expansion of an exponent ζ^x with respect to the q -Mahler basis. Both results appear in [3], the first is a combination of Theorem 3.3 and Theorem 4.1, and the second is the example at the beginning of page 14. For completeness we will prove both results.

THEOREM 6.6. — *Let $q \in \mathbb{C}_p$ with $|q - 1|_p < 1$. Then for any function $f \in \mathcal{C}(\mathbb{Z}_p)$ there exists a unique sequence $(a_n)_{n=0}^\infty$ of numbers in \mathbb{C}_p such that the series*

$$\sum_{k=0}^\infty a_k \begin{bmatrix} x \\ k \end{bmatrix}_q$$

converges in the sup norm to f (in particular $\lim_{k \rightarrow \infty} a_k = 0$). Moreover,

$$\|f\|_\infty = \max_{k \geq 0} |a_k|_p.$$

Proof. — Consider the operator $T = \frac{\Delta}{q^x}$ on $\mathcal{C}(\mathbb{Z}_p)$, where Δ is the forward difference operator. Thus,

$$Tf(x) = \frac{f(x+1) - f(x)}{q^x}.$$

We begin by showing that for any $f \in \text{Con}(\mathbb{Z}_p)$, the sequence $(T^n f(0))_{n=0}^\infty$ converges to zero. Afterwards we will construct the sequence $(a_n)_{n=0}^\infty$ from $(T^n f(0))_{n=0}^\infty$.

Recall that $|q - 1|_p < 1$, and denote $r = |q - 1|_p$. Denote $r = |q - 1|_p$ and recall the assumption that $r < 1$. We consider the quotient space

$$W = \{f \in \mathcal{C}(\mathbb{Z}_p) \mid \|f\|_\infty \leq 1\} / \{f \in \mathcal{C}(\mathbb{Z}_p) \mid \|f\|_\infty \leq r\}.$$

Its elements can be realized as locally constant functions on \mathbb{Z}_p with values in $\mathcal{O}_{\mathbb{C}_p} / (q - 1)\mathcal{O}_{\mathbb{C}_p}$. Since the operator T is norm reducing, i.e. $\|T(f)\|_\infty \leq \|f\|_\infty$ for any $f \in \mathcal{C}(\mathbb{Z}_p)$, T induces an operator on W . Since the image of q^x in W is the constant function 1, the operator T reduces in W to the forward difference operator Δ . If $v \in W$, there exists some number N such that $\Delta^{p^N} v = 0$. Thus, for any functions $f \in \mathcal{C}(\mathbb{Z}_p)$ there exists $N > 0$ such that

$$\|T^{p^N} f\|_\infty \leq r \cdot \|f\|_\infty.$$

Together with the fact that T is norm reducing, it follows that

$$\lim_{n \rightarrow \infty} \|T^n f\|_\infty = 0.$$

In particular, $\lim_{n \rightarrow \infty} T^n f(0) = 0$.

By rearranging the q -Pascal identity (6.2) we get

$$\frac{q^{\binom{k}{2}} [n+1]_q - q^{\binom{k}{2}} [n]_q}{q^n} = q^{\binom{k-1}{2}} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q.$$

Continuity with respect to n implies that

$$T \left(q^{\binom{k}{2}} \begin{bmatrix} x \\ k \end{bmatrix}_q \right) = q^{\binom{k-1}{2}} \begin{bmatrix} x \\ k-1 \end{bmatrix}_q$$

for any $k \geq 1$. When $k = 1$ the function $q^0 \begin{bmatrix} x \\ 0 \end{bmatrix}_q$ is just the constant function 1, and clearly $T(1) = 0$.

Let $f \in \mathcal{C}(\mathbb{Z}_p)$ and denote $a_n = q^{\binom{n}{2}} (T^n f)(0)$. The series

$$h(x) = \sum_{k=0}^{\infty} a_k \begin{bmatrix} x \\ k \end{bmatrix}_q$$

converges in $\mathcal{C}(\mathbb{Z}_p)$, and we have

$$T^n h(0) = T^n f(0),$$

for any $n \geq 0$. Since $h(0) = f(0)$, it follows that $h(n) = f(n)$ for any $n \geq 0$. By continuity we must have $h = f$. Thus,

$$f(x) = \sum_{k=0}^{\infty} (T^k f)(0) q^{\binom{k}{2}} \begin{bmatrix} x \\ k \end{bmatrix}_q.$$

This formula implies that $\|f\|_\infty \leq \max_k |a_k|_p$. The inequality in the other direction follows from the fact that T is norm-reducing, so

$$|a_k|_p = |(T^n f)(0)|_p \leq \|T^n f\|_\infty \leq \|f\|_\infty. \quad \square$$

DEFINITION 6.7. — We denote

$$[\zeta, q]_k = (\zeta - 1)(\zeta - q^1) \dots (\zeta - q^{k-1}) = (-1)^k \cdot q^{\binom{k}{2}} \cdot (\zeta; q^{-1})_k,$$

for $k > 0$ and $[\zeta, q]_0 = 1$.

COROLLARY 6.8. — Let $\zeta, q \in \mathbb{C}_p$ with $|q - 1|_p < 1$ and $|\zeta - 1|_p < 1$. Then

$$\zeta^x = \sum_{k=0}^{\infty} [\zeta, q]_k \cdot \begin{bmatrix} x \\ k \end{bmatrix}_q.$$

Proof. — We have $T^0(\zeta^x)(0) = \zeta^0 = 1 = [\zeta, q]_0$. Compute

$$T(\zeta^x) = \frac{\zeta^{x+1} - \zeta^x}{q^x} = (\zeta - 1) \left(\frac{\zeta}{q} \right)^x.$$

By induction:

$$T^k(\zeta^x) = (\zeta - 1) \left(\frac{\zeta}{q} - 1 \right) \dots \left(\frac{\zeta}{q^{k-1}} - 1 \right) \left(\frac{\zeta}{q^k} \right)^x.$$

By the proof of Theorem 6.6, the coefficient of $\left[\begin{smallmatrix} x \\ k \end{smallmatrix} \right]_q$ in the expansion of ζ^x is

$$\begin{aligned} q^{\binom{k}{2}} \cdot (T^k f)(0) &= q^{\binom{k}{2}} (\zeta - 1) \left(\frac{\zeta}{q} - 1 \right) \dots \left(\frac{\zeta}{q^{k-1}} - 1 \right) \left(\frac{\zeta}{q^k} \right)^0 \\ &= (\zeta - 1)(\zeta - q) \dots (\zeta - q^{k-1}). \end{aligned} \quad \square$$

6.3. The p -adic valuation of $(\zeta; \zeta)_n$ when ζ is a root of unity

Fix $N \in \mathbb{N}$ and let ζ be a primitive p^N th root of unity in \mathbb{C}_p . In this subsection we study the p -adic valuation of the expression

$$(\zeta; \zeta)_n = (1 - \zeta)(1 - \zeta^2) \dots (1 - \zeta^n)$$

for $1 \leq n < p^N$. We will use the following notation.

DEFINITION 6.9.

- (1) We denote by $\ell_p : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ the real logarithm to the base p .
- (2) For $a \in \mathbb{C}_p$, we denote its valuation by $v_p(a) = -\ell_p(|a|_p)$.

It will be convenient to denote $\lambda = v_p(1 - \zeta) = \frac{1}{p^N - 1(p - 1)}$.

PROPOSITION 6.10. — Let $1 \leq n < p^N$, and let d be the largest integer such that $p^d \leq n$. Then

$$\lambda^{-1} v_p((\zeta; \zeta)_n) = \sum_{k=0}^d p^k \left(\left\lfloor \frac{n}{p^k} \right\rfloor - \left\lfloor \frac{n}{p^{k+1}} \right\rfloor \right).$$

Proof. — Let $1 \leq i \leq n$ and write $i = p^k u$, where $p \nmid u$. Note that $k \leq d$. Then

$$v_p(1 - \zeta^i) = \frac{1}{p^{N-k}(p-1)} = p^k \lambda.$$

There are exactly $\left\lfloor \frac{n}{p^k} \right\rfloor - \left\lfloor \frac{n}{p^{k+1}} \right\rfloor$ numbers between 1 and n that are divisible by p^k but not by p^{k+1} . Therefore,

$$v_p((\zeta; \zeta)_n) = \sum_{i=1}^n v_p(1 - \zeta^i) = \sum_{k=0}^d \lambda p^k \cdot \left(\left\lfloor \frac{n}{p^k} \right\rfloor - \left\lfloor \frac{n}{p^{k+1}} \right\rfloor \right). \quad \square$$

PROPOSITION 6.11. — Let $1 \leq n < p^N$, and let d be largest integer such that $p^d \leq n$. Then

$$(6.3) \quad \lambda^{-1}v_p((\zeta; \zeta)_n) \geq \frac{p-1}{p}dn$$

Proof. — Rewrite (6.3) as

$$\lambda^{-1}v_p((\zeta; \zeta)_n) = n + \sum_{k=1}^d (p^k - p^{k-1}) \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Note that

$$p^d - 1 = \sum_{k=1}^d (p^k - p^{k-1}).$$

Then

$$\begin{aligned} n + \sum_{k=1}^d (p^k - p^{k-1}) \left\lfloor \frac{n}{p^k} \right\rfloor &= n - (p^d - 1) + \sum_{k=1}^d (p^k - p^{k-1}) \left(\left\lfloor \frac{n}{p^k} \right\rfloor + 1 \right) \\ &\geq n + 1 - p^d + \sum_{k=1}^d (p^k - p^{k-1}) \frac{n}{p^k} \\ &= n + 1 - p^d + \sum_{k=1}^d \left(1 - \frac{1}{p} \right) n \\ &= n + 1 - p^d + \frac{p-1}{p}dn. \end{aligned}$$

The claim follows from the assumption that $n \geq p^d$. \square

PROPOSITION 6.12. — Assume that $N > 2$, and let $p^2 \leq n < p^N$. Then

$$\lambda^{-1}v_p((\zeta; \zeta)_n) \geq \frac{1}{4}n\ell_p(n).$$

Proof. — For any prime p , $\frac{p-1}{p} \geq \frac{1}{2}$. Let d be the largest integer such that $p^d \leq n$. Then $d+1 > \ell_p(n) \geq 2$, so

$$d > \ell_p(n) - 1 \geq \frac{1}{2}\ell_p(n).$$

Together with Equation (6.3),

$$\lambda^{-1}v_p((\zeta; \zeta)_n) \geq \frac{p-1}{p}dn \geq \frac{1}{2}dn \geq \frac{1}{4}n\ell_p(n). \quad \square$$

6.4. Completing the proof of Theorem 4.4 for \mathbb{Z}_p

Let $\|\cdot\|$ be a norm on $\mathcal{C}(\mathbb{Z}_p)$, dominated by the sup norm, normalized at $\mathbf{1}_{\mathbb{Z}_p}(x)$ and invariant under multiplication by smooth characters of \mathbb{Z}_p . We let M be the smallest positive number such that $\|\cdot\| \leq M \cdot \|\cdot\|_\infty$. Let $G_{\|\cdot\|}(r)$ be the growth modulus of $\|\cdot\|$. We need to show that $M = 1$. Supposing that $G_{\|\cdot\|}(1) > 1$ and reaching a contradiction is enough thanks to Proposition 6.3.

By the assumption that $G_{\|\cdot\|}(1) > 1$, the continuity of $G_{\|\cdot\|}(r)$ and the density of regular values (Lemma 6.1), there exists $h \in \mathbb{C}_p$ such that $s := |h|_p < 1$ is a regular value for the norm $\|\cdot\|$ and such that $G_{\|\cdot\|}(s) > 1$. We may also assume that $s \geq p^{-1/(p-1)}$. The last assumption can be written as $\ell_p(\frac{1}{s}) \leq \frac{1}{p-1}$. Recall that ℓ_p denotes the real logarithm to base p . We fix such h and denote $s = |h|_p$. We remark that h and s depend only on the norm $\|\cdot\|$.

From now on, ζ denotes a primitive $p^{N\text{th}}$ root of unity and N is assumed to be very large (in a way that will be made explicit below). We denote $\lambda = v_p(1 - \zeta)$ and let $q = \zeta + h$.

Thus, h is fixed and ζ is at our disposal, close as we wish to the circumference of the unit disc around 1, and q varies with ζ at a fixed distance s from it.

The idea of the proof is to use the expansion

$$\zeta^x = \sum_{k=0}^{\infty} [\zeta, q]_k \cdot \begin{bmatrix} x \\ k \end{bmatrix}_q$$

to show, under the assumption that N is very large, that

$$(6.4) \quad \left\| [\zeta, q]_1 \cdot \begin{bmatrix} x \\ 1 \end{bmatrix}_q \right\| > \left\| [\zeta, q]_k \cdot \begin{bmatrix} x \\ k \end{bmatrix}_q \right\|$$

for any $k \neq 1$. Then, by the strong triangle inequality,

$$\|\zeta^x\| = \left\| [\zeta, q]_1 \cdot \begin{bmatrix} x \\ 1 \end{bmatrix}_q \right\| > \left\| [\zeta, q]_0 \cdot \begin{bmatrix} x \\ 0 \end{bmatrix}_q \right\| = \|\mathbf{1}(x)\| = 1$$

which is a contradiction to the assumption that $\|\cdot\|$ is invariant under multiplication by ζ^x and normalized at $\mathbf{1}_{\mathbb{Z}_p}(x)$.

The proof of (6.4) will be divided into three cases. The first, $k = 0$, is the easiest. The second and third cases are when $1 < k < \frac{1}{\lambda} \ell_p(\frac{1}{\sqrt{s}})$ and $k \geq \frac{1}{\lambda} \ell_p(\frac{1}{\sqrt{s}})$ respectively. In each of these cases we will need to use different types of inequalities.

PROPOSITION 6.13. — We have that $\|q^x\| = G_{\|\cdot\|}(s)$ and $\|q^{ax}\| \leq \|q^x\|$ for any $a \in \mathbb{Z}_p$.

Proof. — Write

$$q^x = (\zeta + h)^x = \zeta^x \left(1 + \frac{h}{\zeta}\right)^x.$$

Since $\|\cdot\|$ is invariant under multiplication by smooth characters

$$\left\| \zeta^x \left(1 + \frac{h}{\zeta}\right)^x \right\| = \left\| \left(1 + \frac{h}{\zeta}\right)^x \right\|.$$

Since $|h/\zeta|_p = |h|_p = s$ is a regular value for the norm $\|\cdot\|$, we have, by Proposition 6.4, an equality

$$\left\| \left(1 + \frac{h}{\zeta}\right)^x \right\| = G_{\|\cdot\|}(s).$$

Thus, $\|q^x\| = G_{\|\cdot\|}(s)$. To show that $\|q^{ax}\| \leq \|q^x\|$ we use the same trick. Write

$$\begin{aligned} \|q^{ax}\| &= \|(\zeta + h)^{ax}\| = \left\| \zeta^{ax} \cdot \left(1 + \frac{h}{\zeta}\right)^{ax} \right\| \\ &= \left\| \left(1 + \frac{h}{\zeta}\right)^{ax} \right\| = \|(1 + h')^x\|, \end{aligned}$$

where $h' = (1 + h/\zeta)^a - 1$. Then $|h'|_p \leq |h|_p$. Since $G_{\|\cdot\|}(r)$ is monotone increasing, and by Proposition 6.4,

$$\|q^{ax}\| = \|(1 + h')^x\| \leq G(|h'|_p) \leq G(|h|_p) = \|q^x\|. \quad \square$$

PROPOSITION 6.14. — Assume that $|1 - \zeta|_p > s$.

(1) Let $s < r < 1$. Then for any $1 \leq i \leq \frac{1}{\lambda} \ell_p(\frac{1}{r})$

$$|1 - q^i|_p = |1 - \zeta^i|_p \geq r.$$

(2) For any $1 < k \leq \frac{1}{\lambda} \ell_p(\frac{1}{\sqrt{s}})$

$$|[\zeta, q]_k|_p \leq \sqrt{s} \cdot |(q; q)_k|_p$$

Proof. — First, recall our assumption that $\ell_p(\frac{1}{s}) \leq \frac{1}{p-1}$. Then

$$\frac{1}{\lambda} \ell_p\left(\frac{1}{r}\right) \leq \frac{1}{\lambda} \ell_p\left(\frac{1}{s}\right) \leq \frac{1}{(p-1)\lambda} = p^{N-1},$$

for any $s < r < 1$. In particular, any indices i and k that appear in this proof are in $\{0, 1, \dots, p^N - 1\}$, so the expressions ζ^i, ζ^k are not equal to 1. Second, note that if N is not large enough, the interval $[1, \frac{1}{\lambda} \ell_p(\frac{1}{r})]$ may be empty.

(1) For any $i \geq 1$, $|\zeta^i - q^i|_p \leq |\zeta - q|_p = s < r$. The condition $i \leq \frac{1}{\lambda} \ell_p(\frac{1}{r})$ is equivalent to

$$r \leq p^{-\lambda i} = |\zeta - 1|_p^i.$$

Write $i = ap^k$ with $p \nmid a$. Then

$$|1 - \zeta^i|_p = |1 - \zeta^{p^k}|_p = |1 - \zeta|_p^{p^k} \geq |1 - \zeta|_p \geq r.$$

Thus,

$$|1 - q^i|_p = |(\zeta^i - q^i) + (1 - \zeta^i)|_p = |1 - \zeta^i|_p \geq r.$$

(2) We use part (1) with $r = \sqrt{s} > s$. Then,

$$|1 - q^i|_p = |1 - \zeta^i|_p \geq \sqrt{s} > s,$$

for all $1 \leq i \leq \frac{1}{\lambda} \ell_p(\frac{1}{\sqrt{s}})$. If in addition $i > 1$, then by writing $\zeta - q^i = (\zeta - q) + q(1 - q^{i-1})$ we see that

$$|\zeta - q^i|_p = |q(1 - q^{i-1})|_p = |1 - q^{i-1}|_p.$$

Let $1 < k \leq \frac{1}{\lambda} \ell_p(\frac{1}{\sqrt{s}})$. Then

$$\begin{aligned} \frac{[\zeta, q]_k}{(q; q)_k} &= \frac{(\zeta - 1)(\zeta - q)(\zeta - q^2) \dots (\zeta - q^{k-1})}{(1 - q)(1 - q^2)(1 - q^3) \dots (1 - q^k)} \\ &= \frac{(\zeta - 1)(\zeta - q)}{(1 - q^{k-1})(1 - q^k)} \cdot \left(\frac{\zeta - q^2}{1 - q} \right) \cdot \left(\frac{\zeta - q^3}{1 - q^2} \right) \dots \left(\frac{\zeta - q^{k-1}}{1 - q^{k-2}} \right) \end{aligned}$$

(Note that $(q; q)_k \neq 0$). Using the equality $|\zeta - q^i|_p = |1 - q^{i-1}|_p$ for any $2 \leq i \leq k - 1$ we see that

$$\frac{|[\zeta, q]_k|_p}{|(q; q)_k|_p} = \frac{|(\zeta - 1)|_p |(\zeta - q)|_p}{|(1 - q^{k-1})|_p |(1 - q^k)|_p}.$$

By part (1), $|1 - q^k|_p \geq \sqrt{s}$ and $|1 - q^{k-1}|_p \geq \sqrt{s}$. Moreover, since one of k or $k - 1$ is not divisible by p , the p -adic absolute value of one of them is equal to $|1 - q|_p$. The assumption that $|\zeta - 1|_p > s$ implies that $|\zeta - 1|_p = |q - 1|_p$. Thus,

$$\frac{|[\zeta, q]_k|_p}{|(q; q)_k|_p} \leq \frac{|q - 1|_p \cdot s}{|q - 1|_p \cdot \sqrt{s}} = \sqrt{s}. \quad \square$$

Proof of (6.4). — Let N be a positive integer, ζ a primitive $p^{N\text{th}}$ root of unity, and denote

$$\alpha = \frac{1}{2\lambda} \ell_p \left(\frac{1}{\sqrt{s}} \right) = \frac{1}{2} p^{N-1} (p-1) \ell_p \left(\frac{1}{\sqrt{s}} \right).$$

We assume that N is large enough so that the following conditions hold.

(1) $|1 - \zeta|_p > s$.

- (2) $\alpha > p^2$.
 (3) $\frac{\lambda}{4}\alpha\ell_p(\alpha) \geq \ell_p(\frac{M}{\sqrt{s}})$. Note that $\frac{\lambda}{4}\alpha\ell_p(\alpha) = A \cdot \ell_p(\frac{1}{\lambda}) + B$ where $A = \frac{1}{8}\ell_p(\frac{1}{\sqrt{s}}) > 0$ and $B = A \cdot \ell_p(\frac{1}{2}\ell_p(\frac{1}{\sqrt{s}}))$. Note that B and $\ell_p(\frac{M}{\sqrt{s}})$ do not depend on ζ .

Under these assumptions we will show that

$$\left\| [\zeta, q]_1 \cdot \begin{bmatrix} x \\ 1 \end{bmatrix}_q \right\| > \left\| [\zeta, q]_k \cdot \begin{bmatrix} x \\ k \end{bmatrix}_q \right\|$$

for any $k \neq 1$.

We begin by showing that $\| \begin{bmatrix} x \\ 1 \end{bmatrix}_q \| = \|q^x\| > 1$. Indeed, by the assumption that $|\zeta - 1| > s$ we have that $|\zeta - 1| = |q - 1|$. Thus,

$$\left\| [\zeta, q]_1 \cdot \begin{bmatrix} x \\ 1 \end{bmatrix}_q \right\| = \left\| (\zeta - 1) \frac{1 - q^x}{1 - q} \right\| = \|1 - q^x\|.$$

By Proposition 6.13, $\|q^x\| = G_{\|\cdot\|}(s) > 1$. Therefore,

$$\|1 - q^x\| = \|q^x\| > 1.$$

Assume that $k = 0$. Then

$$\left\| [\zeta, q]_0 \cdot \begin{bmatrix} x \\ 0 \end{bmatrix}_q \right\| = \|\mathbf{1}_{\mathbb{Z}_p}(x)\| = 1.$$

Assume $1 < k \leq \frac{1}{\lambda}\ell_p(\frac{1}{\sqrt{s}})$. By part (2) of Proposition 6.14 and by Proposition 6.13,

$$\begin{aligned} & \left\| [\zeta, q]_k \cdot \begin{bmatrix} x \\ k \end{bmatrix}_q \right\| \\ &= \frac{|[\zeta, q]_k|_p}{|(q; q)_k|_p} \cdot \|(q^x - 1)(q^x - q) \dots (q^x - q^{k-1})\| \leq \sqrt{s} \cdot \|q^x\| < \|q^x\| \\ &= \left\| [\zeta, q]_1 \cdot \begin{bmatrix} x \\ 1 \end{bmatrix}_q \right\|. \end{aligned}$$

Assume that $k > \frac{1}{\lambda}\ell_p(\frac{1}{\sqrt{s}})$. Let m be an integer with

$$\frac{1}{2\lambda}\ell_p\left(\frac{1}{\sqrt{s}}\right) \leq m < \frac{1}{\lambda}\ell_p\left(\frac{1}{\sqrt{s}}\right).$$

Such an integer exists, since $\frac{1}{\lambda}\ell_p(\frac{1}{\sqrt{s}}) > 2p^2 > 4$. As $k > m$, $|[\zeta, q]_k|_p \leq |[\zeta, q]_m|_p$. By the second and first parts of Proposition 6.14 we have

$$|[\zeta, q]_m|_p \leq \sqrt{s} \cdot |(q; q)_m|_p = \sqrt{s} \cdot |(\zeta; \zeta)_m|_p.$$

Since $m > p^8$ we can apply Proposition 6.12, and together with the assumption that $m \geq \alpha$ we get

$$\ell_p(|(\zeta; \zeta)_m|_p) \leq -\frac{\lambda}{4} m \ell_p(m) \leq -\frac{\lambda}{4} \alpha \ell_p(\alpha) \leq -\ell_p\left(\frac{M}{\sqrt{s}}\right).$$

In the last inequality we used our assumption (3). Then $|(\zeta; \zeta)_m|_p \leq \frac{\sqrt{s}}{M}$. Therefore,

$$|[\zeta, q]_k|_p \leq \sqrt{s} \cdot |(\zeta; \zeta)_m|_p \leq \sqrt{s} \cdot \frac{\sqrt{s}}{M} = \frac{s}{M}.$$

Finally,

$$\left\| [\zeta, q]_k \cdot \begin{bmatrix} x \\ k \end{bmatrix}_q \right\| \leq |[\zeta, q]_k|_p \cdot M \leq s < 1 < \left\| [\zeta, q]_1 \cdot \begin{bmatrix} x \\ 1 \end{bmatrix}_q \right\|.$$

This completes the proof of (6.4), hence of Theorem 4.4 for \mathbb{Z}_p . □

7. Further discussion about \mathcal{H} -invariant norms on \mathcal{S}

This section is motivated by the search for other minimal invariant norms on \mathcal{S} . In addition, Proposition 7.9 is a generalization of the discontinuity of the Fourier transform proved in [14] to finite families of intertwining operators.

By now we have constructed two types of \mathcal{H} -invariant norms on \mathcal{S} : the family of minimal norms $\{\|\cdot\|_\alpha \mid \alpha \in \text{Gr}\}$, and, for each nonzero $f \in \mathcal{S}$, the maximal invariant norm at f which we denoted by $\|\cdot\|_f$. The latter belong to the maximal equivalence class of \mathcal{H} -invariant norms. Given any subset $I \subset \text{Gr}$, we can form the norm $\sup_{\alpha \in I} \|\cdot\|_\alpha$. The supremum exists since all the $\|\cdot\|_\alpha$, being normalized at $\mathbf{1}_{\mathbb{Z}_p^d}(x)$, are bounded from above by the maximal invariant norm at $\mathbf{1}_{p^d \mathbb{Z}_p}(x)$. In this section we consider finite families $I \subset \text{Gr}$ and the norms

$$\|\cdot\|_I := \max_{\alpha \in I} \|\cdot\|_\alpha,$$

and answer the question: are there new minimal norms that lie below $\|\cdot\|_I$? We show that the answer is negative. In fact, we will show the following.

THEOREM 7.1.

- (1) *Let $I, J \subset \text{Gr}$ be distinct finite subsets. Then $\|\cdot\|_I$ and $\|\cdot\|_J$ are not equivalent.*

- (2) Let $\|\cdot\|$ be an \mathcal{H} -invariant norm which is dominated by $\|\cdot\|_I$, where $I \subset \text{Gr}$ is finite. Then there exists $J \subseteq I$ such that $\|\cdot\|$ is equivalent to $\|\cdot\|_J$.
- (3) If $I_1, I_2 \subset \text{Gr}$ are finite and disjoint, there does not exist any \mathcal{H} -invariant norm on \mathcal{S} which is dominated by both $\|\cdot\|_{I_1}$ and by $\|\cdot\|_{I_2}$.

Clearly, (1) implies that the $J \subset I$ in (2) is unique, and (1) and (2) imply (3). If L_α is the unit ball of $\|\cdot\|_\alpha$, the meaning of (3) is that if we put $L_I = \bigcap_{\alpha \in I} L_\alpha$, then $L_{I_1} + L_{I_2} = \mathcal{S}$.

We will also show that any norm of the form $\|\cdot\|_I$, where $I \subset \text{Gr}$ is finite, is equivalent to a norm which is locally maximal at some vector.

To prove these results, we introduce a notion of independence of norms.

7.1. Independence of norms

The setting in this sub-section is general. Let V be a vector space over \mathbb{C}_p .

PROPOSITION 7.2. — *Let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on V . The following are equivalent.*

- (1) *There exists no (nonzero) seminorm on V which is dominated by both $\|\cdot\|_1$ and $\|\cdot\|_2$.*
- (2) *The diagonal map*

$$V \rightarrow V_{\|\cdot\|_1} \oplus V_{\|\cdot\|_2}$$

has a dense image, where the norm on the right hand side is $(v, w) \mapsto \max(\|v\|_1, \|w\|_2)$.

- (3) *Let L_1, L_2 be the closed unit balls of $\|\cdot\|_1, \|\cdot\|_2$ respectively. Then $L_1 + L_2 = V$.*

Proof. — We will show that each of (1) and (2) is equivalent to (3). If To show that (1) and (3) are equivalent, note that the gauge of $L_1 + L_2$ is either zero, if $L_1 + L_2 = V$, or defines a nonzero seminorm $\|\cdot\|'$ on V . The seminorm $\|\cdot\|'$ is dominated by both $\|\cdot\|_1$ and $\|\cdot\|_2$, and any seminorm that is dominated by both $\|\cdot\|_1$ and $\|\cdot\|_2$ is also dominated by $\|\cdot\|'$. From this it follows that (1) and (3) are equivalent.

We now show that (2) and (3) are equivalent. It is easy to see that (2) is equivalent to the statement that for any $w \in V$ and $\epsilon > 0$ there exists $v \in V$ such that $\|v - w\|_1 < \epsilon$ and $\|v\|_2 < \epsilon$. This statement is equivalent to the claim that any $w \in V$ can be written as $w = v_1 + v_2$ with $\|v_1\|_1 < \epsilon$ and $\|v_2\|_2 < \epsilon$, and this is equivalent to (3). \square

DEFINITION 7.3. — We say that two norms $\|\cdot\|_1, \|\cdot\|_2$ on V are independent if one of the equivalent conditions of the previous proposition is satisfied.

The following Proposition is left as an exercise to the reader.

PROPOSITION 7.4. — Assume that V is an irreducible representation of a group G , and $\|\cdot\|_1, \|\cdot\|_2 \in \mathcal{N}(V)^G$. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are independent if and only if there exists no G -invariant norm on V that is dominated by both $\|\cdot\|_1$ and $\|\cdot\|_2$.

DEFINITION 7.5. — We say that the norms $\|\cdot\|_1, \dots, \|\cdot\|_n$ on V are independent if for any $1 \leq i \leq n$ the two norms:

$$\|\cdot\|_i \quad \text{and} \quad \max_{\substack{1 \leq j \leq n \\ j \neq i}} \|\cdot\|_j$$

are independent.

Note that if $\|\cdot\|_1, \dots, \|\cdot\|_n$ are independent, so is any subset of them.

PROPOSITION 7.6. — Let $\|\cdot\|_1, \dots, \|\cdot\|_n$ be norms on V . The following are equivalent.

- (1) $\|\cdot\|_1, \dots, \|\cdot\|_n$ are independent.
- (2) The diagonal embedding

$$V \xrightarrow{\Delta} \bigoplus_{i=1}^n V_{\|\cdot\|_i}$$

has a dense image.

- (3) For any two disjoint sets $I, J \subset \{1, 2, \dots, n\}$ the norms

$$\max_{i \in I} \|\cdot\|_i \quad \text{and} \quad \max_{j \in J} \|\cdot\|_j$$

are independent.

Proof. — As (1) is a particular case of (3), it remains to show (1) \Rightarrow (2) \Rightarrow (3). We will prove this by induction on n . The case $n = 2$ is essentially Proposition 7.2.

Assume (1). By the assumption and Proposition 7.2, the diagonal map

$$V \rightarrow V_{\|\cdot\|_1} \oplus V_{\max_{1 < i \leq n} \|\cdot\|_i}$$

has a dense image. The norms $\|\cdot\|_2, \dots, \|\cdot\|_n$ are also independent and by the induction hypothesis the map

$$V_{\max_{1 < i \leq n} \|\cdot\|_i} \rightarrow \bigoplus_{1 < i \leq n} V_{\|\cdot\|_i}$$

is an isomorphism. Thus, $V \rightarrow \bigoplus_{1 \leq i \leq n} V_{\|\cdot\|_i}$ also has a dense image.

Assume (2) and let $I, J \subset \{1, \dots, n\}$ be non-empty disjoint subsets. We may assume that $I \cup J = \{1, \dots, n\}$. Denote $\|\cdot\|_I = \max_{i \in I} \|\cdot\|_i$ and $\|\cdot\|_J = \max_{j \in J} \|\cdot\|_j$. Consider the maps:

$$V \xrightarrow{\Delta} V_{\|\cdot\|_I} \oplus V_{\|\cdot\|_J} \rightarrow \bigoplus_{1 \leq i \leq n} V_{\|\cdot\|_i}.$$

By the induction hypothesis, the second arrow is an isometric isomorphism. The first arrow Δ therefore has a dense image, so (3) follows from Proposition 7.2. \square

7.2. Proofs of the claims in this section

PROPOSITION 7.7. — *Let $I \subset \text{Gr}$ be a finite subset. The norms $\{\|\cdot\|_\alpha \mid \alpha \in I\}$ are independent.*

Proof. — The proof is by induction on the size of the set I . If $|I| = 1$ there is nothing to prove. Assume that $|I| = n > 1$. Let $\alpha \in I$, we need to show that the two norms

$$\|\cdot\|_\alpha \quad \text{and} \quad \|\cdot\|_{I \setminus \{\alpha\}} := \max_{\beta \in I \setminus \{\alpha\}} \|\cdot\|_\beta$$

are independent. By Theorem 4.3 and Proposition 7.4 it is enough to prove that $\|\cdot\|_{I \setminus \{\alpha\}}$ does not dominate $\|\cdot\|_\alpha$. Suppose, for a contradiction, that $\|\cdot\|_\alpha \preceq \|\cdot\|_{I \setminus \{\alpha\}}$. By the induction hypothesis, there is an isometry

$$\mathcal{S}_{I \setminus \{\alpha\}} \xrightarrow{\sim} \bigoplus_{\beta \in I \setminus \{\alpha\}} \mathcal{S}_{\|\cdot\|_\beta}.$$

Thus, we obtain a nonzero map

$$\bigoplus_{\beta \in I \setminus \{\alpha\}} \mathcal{S}_{\|\cdot\|_\beta} \rightarrow \mathcal{S}_{\|\cdot\|_\alpha}.$$

Then there exists $\beta \in I \setminus \{\alpha\}$ such that the reduced map $\mathcal{S}_{\|\cdot\|_\beta} \rightarrow \mathcal{S}_{\|\cdot\|_\alpha}$ is nonzero. By Theorem 4.5 we have $\alpha = \beta$, a contradiction. \square

COROLLARY 7.8. — *Let $I \subset \text{Gr}$ be a finite subset. The norm $\|\cdot\|_I$ is equivalent to a locally maximal norm (with respect to some vector).*

Proof. — Since the norms $\{\|\cdot\|_\alpha \mid \alpha \in I\}$ are independent, it follows by Proposition 7.6 that

$$\mathcal{S}_{\|\cdot\|} \simeq \bigoplus_{\alpha \in I} \mathcal{S}_{\|\cdot\|_\alpha}$$

are isomorphic Banach representations (and even isometrically isomorphic). By Theorem 4.5, the spaces $\{\mathcal{S}_{\|\cdot\|_\alpha} \mid \alpha \in I\}$ are pairwise non-isomorphic. By Proposition 3.18, $\bigoplus_{\alpha \in I} \mathcal{S}_{\|\cdot\|_\alpha}$ has a strongly cyclic vector, and by Theorem 3.11, $\|\cdot\|$ is equivalent to a locally maximal norm. \square

Proof of Theorem 7.1. — As already noted, (3) follows from (1) and (2). (1) follows from the fact that the $\|\cdot\|_\alpha$ are independent (Proposition 7.7) and by Proposition 7.6. We prove (2) by induction on the size of I . When $|I| = 1$ the claim follows from Theorem 4.3. Assume that $|I| = n > 1$, and that the claim is true for all subsets of Gr of size $< n$. Let $\|\cdot\|$ be an \mathcal{H} -invariant norm on \mathcal{S} that is dominated by $\|\cdot\|_I$. Then $\|\cdot\|$ extends to an \mathcal{H} -invariant seminorm on the completion $\mathcal{S}_{\|\cdot\|_I}$, which, by the independence of the $\|\cdot\|_\alpha$, is isometrically isomorphic to $\bigoplus_{\alpha \in I} \mathcal{S}_{\|\cdot\|_\alpha}$ via the diagonal embedding. By Proposition 3.18, the kernel of $\|\cdot\|$ is of the form $\bigoplus_{\alpha \in K} \mathcal{S}_{\|\cdot\|_\alpha}$ for some subset $K \subset I$. Using the diagonal embedding, this means that $\|\cdot\|$ is already dominated by $\|\cdot\|_{I \setminus K}$. If K is non-empty, then $|I \setminus K| < |I|$ and the claim is true by the induction hypothesis. Assume that K is empty. Then $\|\cdot\|$ is a norm on $\bigoplus_{\alpha \in I} \mathcal{S}_{\|\cdot\|_\alpha}$. We want to show that in this case $\|\cdot\|$ is equivalent to $\|\cdot\|_I$. Choose $\alpha \in I$ and denote $J = I \setminus \{\alpha\}$. By Theorem 4.3, the restriction of $\|\cdot\|$ to the component $\mathcal{S}_{\|\cdot\|_\alpha}$ is of the form $r_\alpha \cdot \|\cdot\|_\alpha$. Similarly, the seminorm on $\mathcal{S}_{\|\cdot\|_\alpha}$, obtained from $\|\cdot\|$ by taking the quotient of $\bigoplus_{\alpha \in I} \mathcal{S}_{\|\cdot\|_\alpha}$ by $\bigoplus_{\beta \in J} \mathcal{S}_{\|\cdot\|_\beta}$ is of the form $s_\alpha \cdot \|\cdot\|_\alpha$. Clearly, $s_\alpha \leq r_\alpha$. We claim that $0 < s_\alpha$. By the induction hypothesis, the restriction of $\|\cdot\|$ to the component $\bigoplus_{\beta \in J} \mathcal{S}_{\|\cdot\|_\beta}$ is equivalent to $\|\cdot\|_J$. It follows that $\bigoplus_{\beta \in J} \mathcal{S}_{\|\cdot\|_\beta}$ is a closed subspace of $\bigoplus_{\alpha \in I} \mathcal{S}_{\|\cdot\|_\alpha}$ with respect to the topology induced by $\|\cdot\|$. Therefore, $s_\alpha \cdot \|\cdot\|_\alpha$ is a norm, so $s_\alpha > 0$. This is true for any $\alpha \in I$, so

$$\max_{\alpha \in I} (s_\alpha \cdot \|\cdot\|_\alpha) \leq \|\cdot\|' \leq \max_{\alpha \in I} (r_\alpha \cdot \|\cdot\|_\alpha),$$

which shows that $\|\cdot\|$ is equivalent to $\|\cdot\|_I$. \square

PROPOSITION 7.9. — *Let $g_1, \dots, g_n \in \text{Sp}_{2d}(\mathbb{Q}_p)$ such that all the cosets $\text{Pg}_1, \dots, \text{Pg}_n$ are distinct. Let T_1, \dots, T_n be intertwining operators corresponding to g_1, \dots, g_n respectively. The set*

$$(7.1) \quad \{(T_1(f), \dots, T_n(f)) \mid f \in \mathcal{S}(\mathbb{Q}_p^d)\}$$

is dense in $\mathcal{C}_0(\mathbb{Q}_p^d)^n$.

Proof. — For each $1 \leq i \leq n$, let $\alpha_i = g_i \text{P} \in \text{Gr}$. Consider the maps

$$\mathcal{S}(\mathbb{Q}_p^d) \xrightarrow{\Delta} \bigoplus_{i=1}^n \mathcal{S}(\mathbb{Q}_p^d)_{\|\cdot\|_{\alpha_i}} \xrightarrow{\beta} \bigoplus_{i=1}^n \mathcal{C}_0(\mathbb{Q}_p^d),$$

where $\beta(v_1, \dots, v_n) = (T_1(v_1), \dots, T_n(v_n))$. Note that β is not a map of representations, but it is an isomorphism of Banach spaces. By Proposition 7.7 and Proposition 7.6, $\beta \circ \Delta$ has dense image in $\mathcal{C}_0(\mathbb{Q}_p^d)$. The image of $\beta \circ \Delta$ is exactly (7.1). \square

7.3. Open questions

We conclude with some open questions that we find interesting.

QUESTION 7.10. — *Does there exist an \mathcal{H} -invariant norm on \mathcal{S} which does not dominate any of the norms $\|\cdot\|_\alpha$, for $\alpha \in \text{Gr}$?*

We find this question especially interesting, regardless of the answer. If the answer is negative, the spaces $\{\mathcal{S}_{\|\cdot\|_\alpha} \mid \alpha \in \text{Gr}\}$ form a complete list of the irreducible completions of \mathcal{S} . If the answer is positive, constructing such norms will require new ideas that could be useful in the study of Banach representations of p -adic groups. In the latter case, we also ask

QUESTION 7.11. — *Does there exist another \mathcal{H} -invariant norm on \mathcal{S} , the completion by which is an (strongly) irreducible Banach representation?*

The last section gives a complete picture of those norms which are dominated by some $\|\cdot\|_I$, for a finite subset $I \subset \text{Gr}$. When I is not finite, we can still define the norm $\|\cdot\|_I$ as before. Now it seems reasonable to consider the topology of Gr .

QUESTION 7.12. — *Let I_1, I_2 be closed and disjoint subsets of Gr .*

- (1) *Are the norms $\|\cdot\|_{I_1}$ and $\|\cdot\|_{I_2}$ independent?*
- (2) *Is there a simple description of the completion $\mathcal{S}_{\|\cdot\|_{I_1}}$ in terms of the completions $\mathcal{S}_{\|\cdot\|_\alpha}$ for $\alpha \in I_1$?*

Finally, taking $I = \text{Gr}$, we ask

QUESTION 7.13. — *Does the norm $\sup_{\alpha \in \text{Gr}} \|\cdot\|_\alpha$ belong to the maximal equivalence class of \mathcal{H} -invariant norms on \mathcal{S} ?*

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Manuscrit reçu le 13 juin 2021,
révisé le 10 janvier 2023,
accepté le 16 mai 2023.

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