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ANALYSIS ON SOME LINEAR SETS
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0.

Let \( F \) be a compact subset of \((-\infty, \infty)\) and for each integer \( N \geq 1 \) let \( v_N = v(N; F) \) be the number of intervals \([kN^{-1}, (k+1)N^{-1}]\) meeting \( F \); \( F \) is called small provided \( \log v_N = o(\log N) \). The existence of small sets of « multiplicity » (\( M_0 \)-sets in [61, p. 344]) was proved in 1942 by Salem and used by Rudin [4, VIII]; a program somewhat analogous for locally compact abelian groups was completed by Varopoulos [5].

Does there exist a small set \( F \) with the property that both \( F \) and (say) \( F^2 = \{x^2 : x \in F\} \) are \( M_0 \)-sets? The construction of these sets doesn’t seem accessible by the method of Rudin and Salem [4], nor by the Brownian motion [3]. In this note an affirmative answer is given to a more general problem.

**Theorem 1.** — Let \( (h_n) \) be a sequence of real functions of class \( C^1(-\infty, \infty) \) with derivatives \( h_n' > 0 \). Then there is a small set \( F \) with the property that each \( h_n(F) \) is an \( M_0 \)-set.

Small sets occur naturally in the construction of independent sets [3, 4, 5]; after the metrical theory of Diophantine approximation a set \( F \) is called metrically independent if to each integer \( N \geq 1 \) and each \( \varepsilon \) in \((0, 1)\) there is a \( U_0 \) so that the simultaneous inequalities

\[
\left| \sum_{j=1}^{N} u_jx_j - v \right| < U^{-N-\varepsilon}, \quad U = \max (|u_1|, \ldots, |u_N|) > U_0 \\
|x_i - x_j| \geq \varepsilon \quad \text{for} \quad 1 \leq i < j \leq N
\]
have no solution in integers \( u_1, \ldots, u_N, \nu \) and members \( x_1, \ldots, x_N \) of \( F \). Compare [1, VII].

Uncountable metrically independent subsets could perhaps be constructed by classical arguments, for example that of Perron [1, p. 79] or Davenport [2].

**Theorem 2.** — The set \( F \) determined in Theorem 1 can be required to have the property that each \( h_n(F) \) be metrically independent.

**Theorems 1a, 2a.** — Theorems 1 and 2 remain true provided each \( h_n \) is monotone-continuous and \( h_n' > 0 \) almost everywhere.

1.

In the proof of Theorem 1 we require two arrays of independent random variables \((Y_{k,m})\) and \((\xi_{k,m})\) defined on a space \((\Omega, P)\) for \( 1 \leq k < \infty, 1 \leq m \leq k^8 \). Each \( Y_k \) is uniformly distributed upon \([0, 1]\) while

\[
P(\xi_{k,m} = 1) = \pi_k = k^{-1} = 1 - P(\xi_{k,m} = 0).
\]

Suppose that \( f \) is a measurable function on \((-\infty, \infty)\) and \(-1 \leq f \leq 1\), and let \( \mu = \pi_k E(f(Y)) \); elementary calculations show that

\[
E(e^{\xi_{k,m}/Y_k}e^{-t\xi_{k,m}}) \leq \exp \frac{1}{2} \pi_k t^2 \exp 0(\pi_k t^3)
\]

with an '0' uniform for \(-1 \leq f \leq 1, -1 \leq t \leq 1, 0 \leq \pi_k \leq 1\). Hence for any \( z > 0 \) and \( 1 > t > 0 \)

\[
P\left\{ \left| \sum_m \xi_{k,m} - k^5 \right| > zk^5 \right\} \leq 2 \exp -z k^5 t \exp \frac{1}{2} k^8 \pi_k t^2 \exp 0(\pi_k k^8 t^3).
\]

Choosing \( z = t = k^{-2} \) and using \( \pi_k = k^{-1} \) we obtain

\[
P\left\{ \left| \sum_m \xi_{k,m} - k^5 \right| \geq k^3 \right\} \leq 0(1) \exp - \frac{1}{2} k.
\]

Thus

**Lemma 1.** — \( \sum_{m=1}^{k^4} \xi_{k,m} = k^5 + 0(k^3) \) almost surely in \( \Omega \).

A sequence of random measures \( \lambda_k \) is now determined as
follows: for any function \( g \) on \((-\infty, \infty)\)
\[
\int g \, d\lambda_k = k^{-2}g(0) + k^{-5} \sum_m \xi_{n,m}g(e^{-k \log^k k}Y_{k,m}).
\]
Thus in every instance \( \lambda_k \geq 0 \) and \( \|\lambda_k\| \geq k^{-2} \); moreover \( \|\lambda_k\| = 1 + O(k^{-2}) \) almost surely. Because \( \sum e^{-k \log^k k} < \infty \) the convolution \( \lambda = \pi \ast \lambda_k \) converges, and \( F \) is defined to be its closed support. \( F \) is contained in at most
\[
\prod_{j=1}^k [j^5 + O(j^3)] = e^{O(k \log k)}
\]
intervals of length \( e^{-k \log^k k} \).
Because \((k + 1) \log^2 (k + 1)/k \log^2 k \to 1\), this is sufficient to obtain

**Lemma 2.** — \( F \) is almost surely a small set.

**Lemma 3.** — Let \( h \in C^1(-\infty, \infty) \) and \( h' > 0 \); let \((c_m), (u_m), (\nu_m)\) be sequences of real numbers such that
\[
|c_m| + |\nu_m| = o(1) \quad \text{and} \quad |u_m| \nu_m \to \infty.
\]
Then
\[
\lim_{m \to \infty} \int_0^1 \exp iu_m h(c_m + \nu_m t) \, dt = 0.
\]

**Proof.** — Let \( g \) denote the \( C^1 \) function inverse to \( h \), and let \( \nu_m > 0 \). The integral is transformed to
\[
J = \int_{\alpha_m}^{\beta_m} g'(y) \exp iu_m y \nu_m^{-1} \, dy,
\]
where \( \alpha_m = h(c_m) \), \( \beta_m = h(\nu_m + c_m) \). A further substitution \( y = y_1 + \pi u_m^{-1} \) yields
\[
J = \frac{1}{2} \int_{\alpha_m}^{\beta_m} g'(y) \exp iu_m y \nu_m^{-1} \, dy
- \frac{1}{2} \int_{\alpha_m - \pi u_m^{-1}}^{\beta_m - \pi u_m^{-1}} g'(y + \pi u_m^{-1}) \exp iu_m y \nu_m^{-1} \, dy.
\]
This tends to 0 because \( \beta_m - \alpha_m = o(\nu_m) \) and \( \nu_m^{-1} \nu_m^{-1} = o(1) \).

**Proof of Theorem 1.** — We show that for each function \( h_n \)
\[
\lim_{n \to \infty} \int \exp iuh_n(s) \lambda \, (ds) = 0, \quad \text{almost surely. Then} \quad h_n(F) \quad \text{is an}
Mo-set; because \( h_n(F) \) is compact it is enough to prove

\[
\lim_{r \to \infty} \int \exp \frac{1}{2} h_n(s) \lambda \, (ds) = 0, \quad r = 1, 2, 3, \ldots.
\]

To each integer \( r \geq 3 \) we attach the integer \( k(r) \) defined by \( k(r) \leq \log^\frac{1}{3} r < k(r) + 1 \) and write \( \lambda'_k = \prod_{j \neq k} \lambda_j \). Then

\[
\int \exp \frac{1}{2} h_n(s) \lambda \, (ds) = \int \int \exp \frac{1}{2} h_n(s + \omega) \lambda_k (ds) \lambda'_k (d\omega).
\]

For each real number \( \omega \) in the support of \( \lambda'_k \) let \( m(\omega) \) be the expected value of \( \int \exp \frac{1}{2} h_n(s + \omega) \lambda_k (ds) \). Then

\[
\left| \int \exp \frac{1}{2} h_n(s) \lambda \, (ds) \right| \leq \left| \int \int \exp \frac{1}{2} h_n(s + \omega) \lambda_k (ds) - m(\omega)| \lambda'_k (d\omega) + \| \lambda'_k \| \max |m(\omega)|.
\]

The second integral, say \( I \), can be handled by Jensen's inequality and the estimates at the beginning of 1. Let \(-1 < t < 1\) and \( \Phi(x) = e^{tx} \). Then

\[
E(\Phi(\| \lambda'_k \|^{-1} k^4 \text{Re I})) \leq 2 \exp \frac{1}{2} k^8 \exp \left( k^4 \right).
\]

Choosing \( t = k^{-\frac{1}{2}} \) we observe

\[
P\left( |\text{Re I}| > \| \lambda'_k \| k^{-\frac{1}{2}} \right) = P\left( \Phi(\| \lambda'_k \|^{-1} k^4 \text{Re I}) > \exp k^4 \right)
\]

\[
\leq 2 \exp \frac{1}{2} k^4 \exp 0(k^{7/2}) \exp - k^4.
\]

This is the general term of a convergent series, inasmuch as \( k = k(r) > -1 + \log^\frac{1}{3} r \). Thus, almost surely in \( \Omega \), for \( r > r_0 \)

\[
|\text{Re} \int \exp \frac{1}{2} h_n(s) \lambda \, (ds)| \leq k^{-\frac{1}{2}} \| \lambda'_k \| + \| \lambda'_k \| \max |m(\omega)|
\]

and of course a similar statement holds for the imaginary part of the integral. Now

\[
|m(\omega)| \leq k^{-2} + \left| \int_0^1 \exp \frac{1}{2} h_n(e^{-k \log^\frac{1}{3} t} + \omega) \, dt \right|
\]

with \( \omega = O(1) \) and \( k = k(r) \). To apply Lemma 3 we must
verify $r^2 e^{-k \log k} \to \infty$ but this is plain from $k(r) < \log^\frac{1}{3} r$.

Because $\max_k \|\lambda_k\| < \infty$ almost surely, the proof of Theorem 1 is complete.

2.

Theorem 2 requires the construction of a random function $\varphi$ in $C^\infty(-\infty, \infty)$. Let $\psi$ be a function in $C^\infty(-\infty, \infty)$ with the properties

(i) $\psi = 0$ on $[-\infty, -2]$, $\psi = 3$ on $[2, \infty]$,
(ii) $\psi' > 0$, and $\psi' > 1$ on $(-1, 1)$.

Let $(a_p)$ be a sequence of real numbers such that every real number belongs to infinitely many of the intervals $(a_p - p^{-1}, a_p + p^{-1})$. Finally, let $(Z_p)$ be a sequence of independent random variables on $(\Omega, P)$, uniformly distributed upon $[0, 1]$. We define

$$\varphi(x) = \sum_{p=1}^\infty e^{-p^\frac{1}{2}}(p^{-1}Z_p + p^\frac{1}{2}(x - a_p)) + x.$$ 

To each compact set $F$ and number $\delta > 0$ there are numbers $q_1$ and $q_2$ so that

$q_1 \geq 4$, $q_1^\frac{1}{2} \delta \geq 5$, $\bigcup_{p=q_1}^{q_2} (a_p - p^{-1}, a_p + p^{-1}) \ni F$.

**Theorem 3.** — Let $F$ be a small set and $h \in C^1(-\infty, \infty)$, $h' > 0$; then $h\varphi(F)$ is almost surely metrically independent.

For each integer $U \geq 1$ we can choose a subset $S(N, U)$ of $R^N$ so that every point in $F^N$ has distance $< U^{-3N}$ from some point in $S(N, U)$, while card $S(N, U) \leq v^N(NU^X; F)$.

Beginning with an inequality

$$\left| \sum_{j=1}^N u_j h\varphi(y_j) - \nu \right| < U^{-N - \varepsilon}, \quad |h\varphi(y_j) - h\varphi(y_i)| > \varepsilon \quad (i \neq j)$$

we conclude first that $|y_i - y_j| > \gamma$ for some fixed $\gamma > 0$.

Let $(z_1, \ldots, z_n)$ be the member of $S(N, U)$ associated to
\( (y_1, \ldots, y_n) \). Then
\[
(1) \quad \left| \sum_{j=1}^{N} u_j h\varphi(z_j) - \nu \right| < U^{-N-\varepsilon} + 0(U/U^{-3N}),
\]
|\( z_i - z_j | > \eta - 2U^{-3N}. \)

For large \( U \) we can find \( \delta < \eta - 2U^{-3N} \) and corresponding numbers \( q_1, q_2 \). Let \( q_1 \leq p \leq q_2, |z_i - a_p| < p^{-1}. \)

\[
\left| p^{-1}Z_p + p^{\frac{1}{2}}(z_i - a_p) \right| < p^{-1} + p^{-\frac{1}{2}} < 1, \\
\left| p^{-1}Z_p + p^{\frac{1}{2}}(Z_j - a_p) \right| > p^{\frac{1}{2}}\delta - p^{-1} - p^{-\frac{1}{2}} > 4, \quad \text{when } j \neq i.
\]

Therefore \( \frac{\partial}{\partial Z_p} \sum_{j=1}^{N} u_j h\varphi(z_j) = u_l \frac{\partial}{\partial Z_p} h\varphi(Z_l) \) exceeds \( \alpha|u_l| \) in modulus, with an \( \alpha > 0 \) independent of \( u_1, \ldots, u_n \). Hence the probability of the inequality (1) is \( 0(U^{-1}.U^{-N-\varepsilon}) \) for each \( (z_1, \ldots, z_N). \) The requirement \( U = \max (|u_1|, \ldots, |u_N|) \) determines \( 0(U^{N-1}) \) N-tuples and plainly \( \nu = 0(U) \). Because \( F \) is a small set \( \nu^{N}(NU^{3N}; F) = U^{o(1)} \) as \( U \to \infty \). Theorem 3 follows from this and \( \Sigma U^{-1+i}U^{o(1)} < \infty. \)

**Proof of Theorem 2.** — Here we use the fact that \( F \) and \( \varphi \) depend on independent \( \sigma \)-fields. \( F \) is almost surely small, whence each \( h_n \varphi(F) \) is almost surely metrically independent, by Theorem 3. By Theorem 1, each \( h_n \varphi(F) \) is almost surely an \( M_0 \)-set and Theorem 2 is proved.

### 3.

**Proof of Theorems 1a and 2a.** — According to a theorem of Marcinkiewicz [61I, pp. 73-77], to each \( \delta > 0 \) there exist functions \( g_n \) in \( C^1(-\infty, \infty) \) so that
\[
m(h_n \neq g_n) < \delta n^{-2}, \quad n = 1, 2, 3, \ldots.
\]

At almost all points of density of the set \( (h_n = g_n), g_n = h_n > 0 \). Passing to a perfect subset of the set \( (g_n > 0, g_n' = h_n, g_n = h_n) \), we can find a \( \tilde{g}_n \) in \( C^1(-\infty, \infty) \) such that
\[
m(h_n \neq \tilde{g}_n) < 2\delta n^{-2}, \quad n = 1, 2, 3, \ldots,
\]
\( \tilde{g}_n' > 0 \) everywhere.
We observe next that to each $\varepsilon > 0$ there is a constant $B(\varepsilon)$ so that for all Borel sets $S$

$$\int_{\Omega} \lambda(S) \, dP \leq \varepsilon + B(\varepsilon)m(S).$$

Thus to each $\varepsilon > 0$ we can choose functions $\tilde{g}_n$ by Marcin-
kiewicz' theorem, so that

$$P\{\lambda(x: \tilde{g}_n \varphi(x) \neq h_n \varphi(x) \text{ for some } n) > \varepsilon\} < \varepsilon.$$

In proving this implication it must be observed that $\varphi$ and $\lambda$ are stochastically independent and $\varphi' > 1$. Writing $G$ for the inner set in the last inequality, we know that $h_n \varphi(G' \cap F) = \tilde{g}_n \varphi(G' \cap F)$ is almost surely metrically inde-
pendent and that $h_n \varphi(G' \cap F)$ is almost surely an $M_0$-set, if only $\lambda(G' \cap F) > 0$; and this holds for $\|\lambda\| > \varepsilon$ excepting an event of probability $< \varepsilon$. Thus Theorems 1a and 2a are derived from Theorems 1 and 2.

BIBLIOGRAPHY


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