ANNALES DE L'INSTITUT FOURIER

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A pleasant exercise: Langlands' Boulder Lemma

A PLEASANT EXERCISE: LANGLANDS' BOULDER LEMMA

by Jean-Pierre LABESSE

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ABSTRACT. — In the Proceedings of the AMS Boulder conference in 1965 Langlands states a combinatorial lemma involving families of characteristic functions attached to ordered partitions of an obtuse basis in a finite dimensional euclidean vector space. Langlands does not give any indication about the proof of the lemma which is said to be a "pleasant exercise". Since we did not find a proof in the literature we decided to give one. We believe it of some interest for the history of the subject.

RÉSUMÉ. — Dans les Proceedings de la conférence de l'AMS à Boulder en 1965 Langlands énonce un lemme combinatoire qui utilise des familles de fonctions caractéristiques attachées à des partitions ordonnées d'une base obtuse dans un espace vectoriel euclidien de dimension finie. Langlands ne donne aucune indication sur la preuve de ce lemme qu'il déclare être un « plaisant exercice ». Comme il ne semble pas exister de preuve dans la littérature nous avons décidé d'en donner une. Nous pensons qu'elle peut être intéressante au moins du point de vue de l'histoire du domaine.

1. Introduction

A combinatorial lemma

Some months ago Bill Casselman observed that there does not seem to exist any proof in the literature for a combinatorial lemma stated by Langlands in [6, Section 8] we shall call the Boulder lemma. This is the first occurence of a family of combinatorial statements that became, after Langlands article [8] and Arthur's early contributions ([1, 2, 3]), a ubiquitous tool for harmonic analysis on reductive groups over local fields and, over adèles, for the Trace Formula. In [6, Section 9] the Boulder lemma is

used by Langlands to define and control avatars, denoted E'', of Eisenstein series arising from cuspidal forms and he gives a formula for the scalar product of two such E''-series. This is a generalization of a formula due to Selberg in rank one. It allows Langlands to outline a more direct proof than in [7] of the meromorphic continuation of those Eisenstein series in arbitrary rank. Langlands also anticipated that the scalar product formula would be an essential tool to establish the Trace Formula for reductive groups of arbitrary rank. In [3, Lemma 4.2] Arthur proves a formula for the scalar product of two truncated Eisenstein series (arising from cuspidal forms as above) and attributes the formula to Langlands. This implicitly assumes that series E'' are nothing but truncated Eisenstein series. This is not immediate since ordered partitions used by Langlands never occur in Arthur's statement. At least from an historical point of view, it seems useful to give a proof for the Boulder lemma, to relate it to the modern treatment of the combinatorics and, in particular, to establish the equivalence alluded to above.

The contents

We state the Boulder lemma with Langlands' notation as Proposition 2.1. Our proof, given in Section 4, relies on matrix equations established in Section 3 that are variants of combinatorial identities mainly due to Arthur inspired by Langlands work. Our Proposition 3.4 generalizes the well known "Langlands' combinatorial lemma" (see [1, Lemma 2.3] and [2, Lemma 6.3]). This combinatorial lemma appears here as Corollary 3.6; we observe that a particular case of it (when the parameter Λ is in the positive chamber) is a key ingredient for the Langlands classification of admissible irreducible representations of reductive groups over local fields. It appears, I believe for the first time, in [8, page 156]. In fact this paper was circulated, as a preprint, a long time before its actual publication (see also [4, § 6, Chapter IV]). Now, Corollary 3.5 is essentially equivalent to Proposition 4.1 which together with Corollary 3.6 yields a proof of the Boulder lemma. As another corollary of Proposition 4.1 we show in Section 5 that Langlands' series E'' do coincide with Arthur's truncated Eisenstein series and hence that Langlands' formula for the scalar product of E''-series is equivalent to the formula for the scalar product of truncated Eisenstein series proved by Arthur. We believe that Arthur knew about Proposition 4.1 since this is implicit in his attribution to Langlands of the scalar product formula.

Acknowledgments

I am indebted to Bill Casselman for suggesting that we look at these questions. We discussed preliminary results toward a proof of the Boulder combinatorial lemma and also some related historical questions, in particular the early proofs for the scalar product formula of two truncated Eisenstein series by Selberg and others, in relative rank one, using Maaß-Selberg relations. This would be the subject of another historical study and will not be discussed here. I also thank J.-L. Waldspurger for comments on a preliminary version and suggesting a simpler proof of Corollary 3.5 which eventually was recognized as a particular case of Proposition 3.4.

2. The Boulder lemma

Ordered partitions

Consider a euclidean vector space V of dimension $p \geqslant 1$. The euclidean structure allows to identify V with its dual V'. Let Δ be a basis and $\widehat{\Delta}$ the dual basis. We denote by

$$D : \lambda \in \Delta \mapsto D(\lambda) = \mu \in \widehat{\Delta}$$

the canonical bijection between Δ and $\widehat{\Delta}$. Let $\mathfrak{p}$ be an ordered partition of Δ built out of $r = r(\mathfrak{p})$ non empty disjoint subsets $F_{\mathfrak{p}}^u$ with cardinals $a_{\mathfrak{p}}^u$ with $1 \leqslant u \leqslant r$ so that

$$\Delta = \bigcup_{u=1}^{u=r} F_{\mathfrak{p}}^u \qquad \text{and} \qquad \sum_{\mathfrak{p}} a_{\mathfrak{p}}^u = p .$$

This is equivalent to be given an increasing filtration of Δ by subsets $E_{\mathfrak{p}}^v$

$$\emptyset = E_{\mathfrak{p}}^0 \subsetneq E_{\mathfrak{p}}^1 \subsetneq E_{\mathfrak{p}}^2 \subsetneq \cdots \subsetneq E_{\mathfrak{p}}^r = \Delta \qquad \text{where} \qquad E_{\mathfrak{p}}^v = \bigcup_{u \leqslant v} F_{\mathfrak{p}}^u .$$

This defines an increasing filtration of V by subspaces $U_{\mathfrak{p}}^v$ generated by $\widehat{E}_{\mathfrak{p}}^v = D(E_{\mathfrak{p}}^v)$;

$$\{0\} = U_{\mathfrak{p}}^0 \subsetneq U_{\mathfrak{p}}^1 \subsetneq U_{\mathfrak{p}}^2 \subsetneq \cdots \subsetneq U_{\mathfrak{p}}^r = V$$

and a decreasing filtration by subspaces $V_{\mathfrak{p}}^v$ generated by $\Delta - E_{\mathfrak{p}}^v$:

$$V = V_{\mathfrak{p}}^0 \supsetneq V_{\mathfrak{p}}^1 \supsetneq V_{\mathfrak{p}}^2 \supsetneq \cdots \supsetneq V_{\mathfrak{p}}^r = \{0\}.$$

For any $v \in \{0, 1 \cdots, r\}$ we have an orthogonal decomposition: $V = U_{\mathfrak{p}}^v \oplus V_{\mathfrak{p}}^v$. Let $W_{\mathfrak{p}}^u$ be the orthogonal supplement to $U_{\mathfrak{p}}^{u-1}$ in $U_{\mathfrak{p}}^u$ or, equivalently to $V_{\mathfrak{p}}^u$ in $V_{\mathfrak{p}}^{u-1}$:

$$U_{\mathfrak{p}}^{u-1} \oplus W_{\mathfrak{p}}^u = U_{\mathfrak{p}}^u \qquad\qquad V_{\mathfrak{p}}^u \oplus W_{\mathfrak{p}}^u = V_{\mathfrak{p}}^{u-1} \ .$$

One has an orthogonal direct sum decomposition of V:

$$V = \bigoplus_{u=1}^{u=r} W_{\mathfrak{p}}^u \ .$$

For $\lambda \in F_{\mathfrak{p}}^u$ denote by $\lambda_{\mathfrak{p}}$ the orthogonal projection of λ on $W_{\mathfrak{p}}^u$; the $\lambda_{\mathfrak{p}}$ build a basis $\Delta_{\mathfrak{p}}^u$ of it. Similarly for $\mu \in \widehat{F}_{\mathfrak{p}}^u := D(F_{\mathfrak{p}}^u)$ denote by $\mu_{\mathfrak{p}}$ the orthogonal projection of μ on $W_{\mathfrak{p}}^u$, they build a basis $\widehat{\Delta}_{\mathfrak{p}}^u$. Since $U_{\mathfrak{p}}^u$ is orthogonal to $V_{\mathfrak{p}}^u$ then, for any $\lambda \in F_{\mathfrak{p}}^u$ and any $\mu \in \widehat{F}_{\mathfrak{p}}^u$ one has

$$\langle \mu, \lambda \rangle = \langle \mu, \lambda_{\mathfrak{p}} \rangle = \langle \mu_{\mathfrak{p}}, \lambda \rangle = \langle \mu_{\mathfrak{p}}, \lambda_{\mathfrak{p}} \rangle \ .$$

This shows that the sets $\Delta_{\mathfrak{p}}^u$ and $\widehat{\Delta}_{\mathfrak{p}}^u$ are dual basis. Denote by $\Delta_{\mathfrak{p}}$ the union of the $\Delta_{\mathfrak{p}}^u$ (resp. $\widehat{\Delta}_{\mathfrak{p}}$ the dual basis) and

$$D_{\mathfrak{p}} : \Delta_{\mathfrak{p}} \to \widehat{\Delta}_{\mathfrak{p}}$$

the natural bijection.

Functions $\phi_{\mathfrak{p}}^\Lambda$ and $\psi_{\mathfrak{p}}^\Lambda$

Let $H \mapsto \phi_{\mathfrak{p}}^\Lambda(H)$ be the characteristic function of the cone in V [1] defined by

$$\lambda_{\mathfrak{p}}(H) \leqslant 0 \ \text{ if } \ \mu_{\mathfrak{p}}(\Lambda) > 0 \ \text{ and } \ \lambda_{\mathfrak{p}}(H) > 0 \ \text{ otherwise}$$

for all $\lambda \in \Delta$ (or equivalently for all $\lambda_{\mathfrak{p}} \in \Delta_{\mathfrak{p}}$) and where $\mu_{\mathfrak{p}} = D_{\mathfrak{p}}(\lambda_{\mathfrak{p}})$. Now let $H \mapsto \psi_{\mathfrak{p}}^\Lambda(H)$ be the characteristic function of the cone $\lambda_{\mathfrak{p}}(H) > 0$ for $\lambda \in F_{\mathfrak{p}}^1$ and

$$\lambda_{\mathfrak{p}}(H) \leqslant 0 \ \text{ if } \ \mu_{\mathfrak{p}}(\Lambda) > 0 \ \text{ and } \ \lambda_{\mathfrak{p}}(H) > 0 \ \text{ if } \ \mu_{\mathfrak{p}}(\Lambda) \leqslant 0 \ \text{ for } \ \lambda \notin F_{\mathfrak{p}}^1$$

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[1] Although we have identified V with its dual we shall look at elements in $\Delta_{\mathfrak{p}}$ and $\widehat{\Delta}_{\mathfrak{p}}$ as linear forms rather than vectors. Langlands does not indentify V with V'; he considers H as an element of V and Λ as an element in V'. Here we emphasis a kind of symmetry between H and Λ.

where $\mu_{\mathfrak{p}} = D(\lambda_{\mathfrak{p}})$. Let $b_{\mathfrak{p}}^{\Lambda}$ be the number of $\mu \in \widehat{\Delta}$ such that $\mu_{\mathfrak{p}}(\Lambda) \leqslant 0$ and $c_{\mathfrak{p}}^{\Lambda}$ the number of those μ such that moreover $\mu \notin \widehat{F}_{\mathfrak{p}}^1$. Finally Langlands defines integers

$$\alpha_{\mathfrak{p}}^{\Lambda} = b_{\mathfrak{p}}^{\Lambda} + \sum_{u=1}^{u=r} \left(a_{\mathfrak{p}}^u + 1\right) \quad \text{and} \quad \beta_{\mathfrak{p}}^{\Lambda} = 1 + c_{\mathfrak{p}}^{\Lambda} + \sum_{u=2}^{u=r} \left(a_{\mathfrak{p}}^u + 1\right) \ .$$

An element $\Lambda \in V$ is said to be "in the positive chamber" (with respect to Δ) if $\lambda(\Lambda) > 0$ for all $\lambda \in \Delta$. Let $d_{\Delta}(\Lambda) = 1$ if Λ is in the positive chamber and zero otherwise. We may now state the Boulder lemma [6, Section 8][2].

PROPOSITION 2.1. — *Let $\mathcal{P}(\Delta)$ be the set of ordered partitions of Δ. Then on has the following identity*

$$\sum_{\mathfrak{p} \in \mathcal{P}(\Delta)} (-1)^{\alpha_{\mathfrak{p}}^{\Lambda}} \phi_{\mathfrak{p}}^{\Lambda}(H) = d_{\Delta}(\Lambda) + \sum_{\mathfrak{p} \in \mathcal{P}(\Delta)} (-1)^{\beta_{\mathfrak{p}}^{\Lambda}} \psi_{\mathfrak{p}}^{\Lambda}(H) \ .$$

Proof. — Langlands says that the proof is a "pleasant exercise". We observe that when $\Lambda = 0$ then $\phi_{\mathfrak{p}}^{\Lambda} = \psi_{\mathfrak{p}}^{\Lambda}$ and $\alpha_{\mathfrak{p}}^{\Lambda} = \beta_{\mathfrak{p}}^{\Lambda} + 2a_{\mathfrak{p}}^1$ for all $\mathfrak{p}$; now since $p \geqslant 1$ we have $d_{\Delta}(\Lambda) = 0$ and the proposition is trivially true. For arbitrary Λ a proof is given at the end of Section 4 as an immediate consequence of Corollaries 3.5 and 3.6. □

3. Matrix equations

A slightly more general setting

Consider a euclidean vector space V_0, a basis Δ_0 and two subsets $P \subset Q \subset \Delta_0$. Let Δ_P^Q be the projection of Q on the orthogonal of P and V_P^Q the subspace generated by Δ_P^Q. We denote by D_P^Q the bijection between Δ_P^Q and its dual basis $\widehat{\Delta}_P^Q$. For $H \in V$ we denote by H_P^Q the orthogonal projection of H on V_P^Q. With no loss of generality, we may assume that $\Delta = \Delta_P^Q$ and $V = V_P^Q$ for a pair of subsets $P \subset Q \subset \Delta_0$ for some Δ_0. This allows us to use Arthur's notation introduced in [1, 2] and [3] where Δ_0 is the set of simple roots attached to a minimal parabolic subgroup P_0 in a reductive group G. Using that standard parabolic subgroups are bijectively attached to subsets (that may be empty) of Δ_0 and that this bijection is

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[2] We observe that Langlands defines $\phi_{\mathfrak{p}}^{\Lambda}(H)$ and $\psi_{\mathfrak{p}}^{\Lambda}(H)$ only for H and Λ regular i.e. for H outside the walls defined by the $\lambda_{\mathfrak{p}}$ and Λ outside the walls defined by the $\mu_{\mathfrak{p}}$ for some $\mathfrak{p}$. Non regular elements are excluded in his formulation of the Boulder lemma. Moreover Langlands assumes Δ is obtuse.

compatible with inclusion then, by abuse of notation, using the same letter for a subset P of Δ_0 and the standard parabolic subgroup it defines, we recover Arthur's setting and notation. Hence one may forget about G, and consider P_0 as the empty subset in Δ_0. We have to give new proof of some of the classical results since we shall not assume Δ_0 obtuse.

$$\text{Functions } \theta^\Lambda_{P,Q} \text{ and } \widehat{\theta}^\Lambda_{P,Q}$$

For $P \subset Q$ let $\theta^\Lambda_{P,Q}$ be the characteristic function of the set of $H \in V_0$ such that for $\lambda \in \Delta^Q_P$ and $\mu = D^Q_P(\lambda)$

$$\lambda(H) \leqslant 0 \text{ if } \mu(\Lambda) > 0 \text{ and } \lambda(H) > 0 \text{ otherwise}$$

and let $\widehat{\theta}^\Lambda_{P,Q}$ be[3] the characteristic function of the set of H such that

$$\mu(H) \leqslant 0 \text{ if } \lambda(\Lambda) > 0 \text{ and } \mu(H) > 0 \text{ otherwise.}$$

By convention $\theta^\Lambda_{P,Q} = \widehat{\theta}^\Lambda_{P,Q} = 0$ when $P \not\subset Q$ and $\theta^\Lambda_{P,P} = \widehat{\theta}^\Lambda_{P,P} = 1$.

When $-\Lambda$ is in the closure of the positive chamber with respect to an obtuse basis $\widehat{\Delta}^Q_P$ (which implies $\mu(\Lambda) \leqslant 0$ for all $\mu \in \widehat{\Delta}^Q_P$) and in all cases when $\Lambda = 0$, we have

$$\theta^\Lambda_{P,Q} = \tau^Q_P \text{ and } \widehat{\theta}^\Lambda_{P,Q} = \widehat{\tau}^Q_P$$

where τ^Q_P and $\widehat{\tau}^Q_P$ are Arthur's functions. More generally, one can express the $\theta^\Lambda_{P,Q}$ and $\widehat{\theta}^\Lambda_{P,Q}$ in terms of the τ^Q_P and $\widehat{\tau}^Q_P$. To do this we need some notation. Let P_Λ (or P^Q_Λ if some confusion may arise) be the subset of Δ_0 such that $\widehat{\Delta}^Q_{P_\Lambda}$ is the set of $\mu \in \widehat{\Delta}^Q_P$ with $\mu(\Lambda) > 0$. Similarly, let Q^Λ (or Q^Λ_P if some confusion may arise) be the subset of Δ_0 such that $\Delta^{Q^\Lambda}_P$ is the set of $\lambda \in \Delta^Q_P$ with $\lambda(\Lambda) > 0$.

LEMMA 3.1. —

$$\theta^\Lambda_{P,Q} = \sum_{P_\Lambda \subset S \subset Q} (-1)^{a^S_{P_\Lambda}} \tau^S_P \qquad \text{and} \qquad \widehat{\theta}^\Lambda_{P,Q} = \sum_{P \subset S \subset Q^\Lambda} (-1)^{a^{Q^\Lambda}_S} \widehat{\tau}^Q_S$$

Proof. — This is an immediate consequence of the binomial identity. □

LEMMA 3.2. —

$$\sum_{P \subset Q \subset R} (-1)^{a^Q_P} \tau^Q_P \widehat{\tau}^R_Q = \delta_{P,R}$$

where $\delta_{P,R}$ is the Dirac symbol (it is zero if $P \neq R$ and 1 if $P = R$).

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[3] Our $\widehat{\theta}^\Lambda_{P,Q}(H)$ is denoted $\phi^Q_P(\Lambda, H)$ by Arthur in [2, p. 940]. Unfortunately Arthur's notation is in conflict with Langlands one in [6] and since we stick to Langlands' notation we had to introduce a different one.

When Δ_P^R is an obtuse basis this is [5, Proposition 1.7.2] which itself is a special case of [2, Lemma 6.1]. We now give a proof which does not use this assumption[4].

Proof. — Fix $H \in V_0$. As above, but with H in place of Λ, we introduce subsets $P_H \subset \Delta_0$ and $R^H \subset \Delta_0$. Then

$$\tau_P^Q(H)\widehat{\tau}_Q^R(H) = 1$$

if and only if $P_H \subset Q \subset R^H$ and hence

$$\sum_{P \subset Q \subset R} (-1)^{a_P^Q} \tau_P^Q(H)\widehat{\tau}_Q^R(H) = \sum_{P_H \subset Q \subset R^H} (-1)^{a_P^Q} .$$

The binomial identity tells us this alternating sum vanishes unless $P_H = R^H$. Assume it does not vanish, then $H_P^R \neq 0$ if $P \neq R$. Now consider $\lambda \in \Delta_P^R$ and $\mu \in \widehat{\Delta}_P^R$ such that $\mu = D_P^R(\lambda)$ then, since $P_H = R^H = Q$, we have

$$\lambda(H) > 0 \qquad \text{and} \qquad \mu(H) \leqslant 0 \quad \text{for} \quad \lambda \in \Delta_P^Q$$

while

$$\lambda(H) \leqslant 0 \qquad \text{and} \qquad \mu(H) > 0 \quad \text{for} \quad \mu \in \widehat{\Delta}_Q^R$$

and hence

$$\left\langle H_P^R, H_P^R \right\rangle = \sum_{\left\{\lambda \in \Delta_P^R \,|\, \mu = D_P^R(\lambda)\right\}} \lambda(H)\mu(H) \leqslant 0$$

which contradicts the non vanishing of H_P^R unless $P = R$. □

Matrices $\boldsymbol{\theta}^\Lambda$ and $\widehat{\boldsymbol{\theta}}^\Lambda$

We shall now consider matrices whose entries are indexed by pairs (P, Q) of subsets of Δ_0. Given matrices $\boldsymbol{A}$ with entries $\boldsymbol{A}_{P,Q}$ and $\boldsymbol{B}$ with entries $\boldsymbol{B}_{P,Q}$ the product $\boldsymbol{AB}$ is the matrix $\boldsymbol{C}$ with entries

$$\boldsymbol{C}_{P,Q} = \sum_{R \subset \Delta_0} \boldsymbol{A}_{P,R} \boldsymbol{B}_{R,Q} .$$

Let a_P^Q be the cardinal of Δ_P^Q (so that $a_{P_0}^P$ is the cardinal of P), $b_{P,Q}^\Lambda$ the cardinal of the set of $\mu \in \widehat{\Delta}_P^Q$ such that $\mu(\Lambda) \leqslant 0$ and $\widehat{b}_{P,Q}^\Lambda$ the cardinal of the set of $\lambda \in \Delta_P^Q$ such that $\lambda(\Lambda) \leqslant 0$. Let

$$\eta_{P,Q}^\Lambda = a_{P_0}^P + b_{P,Q}^\Lambda \quad \text{and} \quad \widehat{\eta}_{P,Q}^\Lambda = a_{P_0}^P + \widehat{b}_{P,Q}^\Lambda .$$

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[4] Casselman has already observed, some years ago, that the obtuse assumption is not necessary (private communication).

Consider matrices $\boldsymbol{\theta}^\Lambda$ and $\widehat{\boldsymbol{\theta}}^\Lambda$ with entries

$$\theta^\Lambda_{P,Q} = (-1)^{\eta^\Lambda_{P,Q}} \theta^\Lambda_{P,Q} \qquad \text{resp. } \widehat{\boldsymbol{\theta}}^\Lambda_{P,Q} = (-1)^{\widehat{\eta}^\Lambda_{P,Q}} \widehat{\theta}^\Lambda_{P,Q} .$$

We shall be interested in the products $\boldsymbol{A}^{\Lambda_1,\Lambda_2} = \boldsymbol{\theta}^{\Lambda_1}\widehat{\boldsymbol{\theta}}^{\Lambda_2}$ and $\boldsymbol{B}^{\Lambda_1,\Lambda_2} = \widehat{\boldsymbol{\theta}}^{\Lambda_2}\boldsymbol{\theta}^{\Lambda_1}$.

LEMMA 3.3. — *Assume either $\Lambda_i = 0$ or Δ_0 is an obtuse basis and the $-\Lambda_i$ are in the closure of the positive chamber. Then*

$$(1) \qquad\qquad \boldsymbol{A}^{\Lambda_1,\Lambda_2} = \boldsymbol{\theta}^{\Lambda_1}\widehat{\boldsymbol{\theta}}^{\Lambda_2} = \widehat{\boldsymbol{\theta}}^{\Lambda_2}\boldsymbol{\theta}^{\Lambda_1} = \boldsymbol{B}^{\Lambda_1,\Lambda_2} = \boldsymbol{1} .$$

In particular

$$(2) \qquad\qquad \sum_{P \subset Q \subset R} (-1)^{a^Q_P} \widehat{\tau}^Q_P \tau^R_Q = \delta_{P,R} .$$

Proof. — The assumptions on $\Lambda = \Lambda_i$ imply $\theta^\Lambda_{P,Q} = \tau^Q_P$ and $\widehat{\theta}^\Lambda_{P,Q} = \widehat{\tau}^Q_P$. Then, Lemma 3.2 yields

$$\boldsymbol{A}^{\Lambda_1,\Lambda_2}_{P,R} = \sum_{P \subset Q \subset R} (-1)^{\eta^{\Lambda_1}_{P,Q} - \widehat{\eta}^{\Lambda_2}_{Q,R}} \theta^{\Lambda_1}_{P,Q}\widehat{\theta}^{\Lambda_2}_{Q,R} = \sum_{P \subset Q \subset R} (-1)^{a^Q_P} \tau^Q_P \widehat{\tau}^R_Q = \delta_{P,R}$$

and hence $\boldsymbol{\theta}^{\Lambda_1}\widehat{\boldsymbol{\theta}}^{\Lambda_2} = \boldsymbol{1}$ which proves (1). Assertion (2) is an explicit form of equation $\widehat{\boldsymbol{\theta}}^{\Lambda_2}\boldsymbol{\theta}^{\Lambda_1} = \boldsymbol{1}$. $\qquad\square$

The key ingredient

The next proposition and its proof is a generalization of Arthur's [2, Lemma 6.3] (which appears below as Corollary 3.6).

PROPOSITION 3.4. —

$$\boldsymbol{B}^{\Lambda_1,\Lambda_2}_{P,R} = (-1)^{a^{P^{\Lambda_1}}_P} \delta_{P_{\Lambda_1},R^{\Lambda_2}} .$$

Proof. — By definition

$$\boldsymbol{B}^{\Lambda_1,\Lambda_2}_{P,R} = \sum_{P \subset Q \subset R} (-1)^{\widehat{\eta}^{\Lambda_2}_{P,Q} - \eta^{\Lambda_1}_{Q,R}} \widehat{\theta}^{\Lambda_2}_{P,Q}\theta^{\Lambda_1}_{Q,R} .$$

In view of Lemma 3.1 we have

$$\boldsymbol{B}^{\Lambda_1,\Lambda_2}_{P,R} = \sum_{P \subset Q \subset R} (-1)^{\widehat{\eta}^{\Lambda_2}_{P,Q} - \eta^{\Lambda_1}_{Q,R}} \sum_{P \subset S \subset Q^{\Lambda_2}_P} (-1)^{a^{Q^{\Lambda_2}_P}_S} \widehat{\tau}^Q_S \sum_{Q_{\Lambda_1} \subset S' \subset R} (-1)^{a^{S'}_{Q^R_{\Lambda_1}}} \tau^{S'}_Q$$

Since $a^{Q^R_\Lambda}_Q = b^\Lambda_{Q,R}$ and $a^Q_{Q^\Lambda_P} = \widehat{b}^\Lambda_{P,Q}$ we see that

$$\boldsymbol{B}^{\Lambda_1,\Lambda_2}_{P,R} = \sum (-1)^{a^Q_P}(-1)^{a^{S'}_S} \widehat{\tau}^Q_S \tau^{S'}_Q$$

where the sum runs over triples (S, Q, S') verifying the inclusions

$$(\star) \qquad P \subset S \subset Q_P^{\Lambda_2} \subset Q \subset Q_{\Lambda_1}^R \subset S' \subset R .$$

Observe that $Q_{\Lambda_1}^R \supset P_{\Lambda_1}^R$ and $Q_P^{\Lambda_2} \subset R_P^{\Lambda_2}$. This shows that given any pair (S, S') with $S \subset R_P^{\Lambda_2}$ and $S' \supset P_{\Lambda_1}^R$ we have the inclusions $(\star)$ for any Q with $S \subset Q \subset S'$. Then

$$\boldsymbol{B}_{P,R}^{\Lambda_1,\Lambda_2} = \sum_{\left\{ S,S' \,\middle|\, S \subset R_P^{\Lambda_2}, \, S' \supset P_{\Lambda_1}^R \right\}} (-1)^{a_P^{S'}} \sum_{\{Q \,|\, S \subset Q \subset S'\}} (-1)^{a_S^Q} \, \widehat{\tau}_S^Q \, \tau_Q^{S'} \ .$$

It follows from equation (2) in Lemma 3.3 that the sum over Q vanishes unless $S = S'$ and hence we have

$$\boldsymbol{B}_{P,R}^{\Lambda_1,\Lambda_2} = \sum_{\left\{ S \,\middle|\, P_{\Lambda_1}^R \subset S \subset R_P^{\Lambda_2} \right\}} (-1)^{a_P^S} \ .$$

This in turn, by the binomial identity, vanishes unless $P_{\Lambda_1} = R^{\Lambda_2}$. $\qquad\square$

COROLLARY 3.5. — *The matrices* $\boldsymbol{\theta}^\Lambda$ *and* $\widehat{\boldsymbol{\theta}}^\Lambda$ *are inverse of each other.*

Proof. — Proposition 3.4 shows that $\boldsymbol{B}_{P,R}^{\Lambda,\Lambda}$ vanishes unless $P_\Lambda = R^\Lambda$. But, as in the proof of Lemma 3.2 we see this condition implies $\langle \Lambda_P^R, \Lambda_P^R \rangle \leqslant 0$. Hence, if $\boldsymbol{B}_{P,R}^{\Lambda,\Lambda}$ does not vanish we have $\Lambda_P^R = 0$ and then $P = R^\Lambda$ and $P_\Lambda = R$ which yields $P = R$. $\qquad\square$

COROLLARY 3.6. —

$$\sum_{P \subset Q \subset R} (-1)^{\widehat{b}_{P,Q}^\Lambda} \, \widehat{\theta}_{P,Q}^\Lambda \, \tau_Q^R = d_{\Delta_P^R}(\Lambda) \ .$$

Proof. — Consider the particular case $\Lambda_1 = 0$ and $\Lambda_2 = \Lambda$. Then $P_{\Lambda_1} = R$ and Proposition 3.4 shows that $B_{P,R}^{0,\Lambda} = 0$ unless $R = R^\Lambda$ which holds if and only if Λ is in the positive chamber with respect to Δ_P^R so that

$$B_{P,R}^{0,\Lambda} = (-1)^{a_P^R} d_{\Delta_P^R}(\Lambda) \ .$$

It remains to observe that

$$\widehat{\eta}_{P,Q}^{\Lambda_2} - \eta_{Q,R}^{\Lambda_1} = a_{P_0}^P + \widehat{b}_{P,Q}^\Lambda - a_{P_0}^Q - \widehat{b}_{Q,R}^0 \equiv a_P^R + \widehat{b}_{P,Q}^\Lambda \quad (\text{modulo } 2) \ .$$

$\qquad\square$

4. Application to the Boulder lemma

Where ordered partitions disappear

Consider an ordered partition $\mathfrak{p}$ of Δ_P^R such that $F_\mathfrak{p}^1 = \Delta_P^R \backslash \Delta_P^Q$ with $P \subsetneq Q \subsetneq R$. Let $\mathfrak{q}$ be the ordered partition of Δ_P^Q defined by

$$F_\mathfrak{q}^i = F_\mathfrak{p}^{i+1} \ \text{ for } \ 1 \leqslant i \leqslant r(\mathfrak{q}) = r(\mathfrak{p}) - 1 \ .$$

Then, Langlands' characteristic functions $\phi_\mathfrak{p}^\Lambda$ and $\psi_\mathfrak{p}^\Lambda$ may be rewritten:

$(\star)$ $$\phi_\mathfrak{p}^\Lambda = \theta_{Q,R}^\Lambda \, \phi_\mathfrak{q}^\Lambda \ \text{ and } \ \psi_\mathfrak{p}^\Lambda = \tau_Q^R \, \phi_\mathfrak{q}^\Lambda \ .$$

Moreover

$(\star\star)$ $$b_\mathfrak{p}^\Lambda = b_{Q,R}^\Lambda + b_\mathfrak{q}^\Lambda \ \text{ and } \ \alpha_\mathfrak{p}^\Lambda = \alpha_\mathfrak{q}^\Lambda + a_P^Q + 1 + b_{Q,R}^\Lambda \ .$$

The next proposition shows how to replace, up to a sign, alternating sums over ordered partitions of functions $\phi_\mathfrak{p}^\Lambda$ by a single characteristic function.

PROPOSITION 4.1. — *Let $\widehat{b}_{P,R}^\Lambda$ be the number of $\lambda \in \Delta_P^R$ such that $\lambda(\Lambda) \leqslant 0$. Then,*

$$\sum_{\mathfrak{p} \in \mathcal{P}\left(\Delta_P^R\right)} (-1)^{\alpha_\mathfrak{p}^\Lambda} \phi_\mathfrak{p}^\Lambda = (-1)^{\widehat{b}_{P,R}^\Lambda} \widehat{\theta}_{P,R}^\Lambda \ .$$

Proof. — The above remarks $(\star)$ and $(\star\star)$ show that

$$\sum_{\mathfrak{p} \in \mathcal{P}\left(\Delta_P^R\right)} (-1)^{\alpha_\mathfrak{p}^\Lambda} \phi_\mathfrak{p}^\Lambda = \sum_{P \subset Q \subsetneq R} (-1)^{a_P^Q + 1 + b_{Q,R}^\Lambda} \theta_{Q,R}^\Lambda \sum_{\mathfrak{q} \in \mathcal{P}\left(\Delta_P^Q\right)} (-1)^{\alpha_\mathfrak{q}^\Lambda} \phi_\mathfrak{q}^\Lambda \ .$$

The equivalence of Proposition 4.1 and Corollary 3.5 now follows by induction on the cardinal of Δ_P^R.   □

Proof of the Boulder lemma

Proposition 2.1 claims an identity that in view of $(\star)$ can be written

$$\sum_{\mathfrak{p} \in \mathcal{P}\left(\Delta_P^R\right)} (-1)^{\alpha_\mathfrak{p}^\Lambda} \phi_\mathfrak{p}^\Lambda(H) = d_{\Delta_P^R}(\Lambda) - \sum_{P \subset Q \subsetneq R} \tau_Q^R(H) \sum_{\mathfrak{q} \in \mathcal{P}\left(\Delta_P^Q\right)} (-1)^{\alpha_\mathfrak{q}^\Lambda} \phi_\mathfrak{q}^\Lambda(H) \ .$$

To prove it we have to show that, thanks to Proposition 4.1,

$$(-1)^{\widehat{b}_{P,R}^\Lambda} \widehat{\theta}_{P,R}^\Lambda(H) = d_{\Delta_P^R}(\Lambda) - \sum_{P \subset Q \subsetneq R} (-1)^{\widehat{b}_{P,Q}^\Lambda} \tau_Q^R(H) \widehat{\theta}_{P,Q}^\Lambda(H)$$

but this is nothing than Corollary 3.6.

5. Truncated Eisenstein series

Series E'' versus truncated Eisenstein series

Here we borrow the notation from [5]. Let G be a reductive group over a number field F with minimal parabolic subgroup P_0. Let P be a standard parabolic subgroup with Levi M, S an associate standard parabolic subgroup and Φ a cuspidal form on $\mathbf{X}_P$ and T some parameter in $\mathfrak{a}_0$. Let $\Delta = \Delta_P^G$ and consider $\Lambda \in V \otimes C$ such that $\Re(\Lambda)$ is far enough inside the positive Weyl chamber (with respect to Δ). Langlands introduces in [6, § 9] functions

$$
\begin{aligned}
&F_S''(x, \Phi, \Lambda) \\
&= \sum_{s \in W(\mathfrak{a}_P, \mathfrak{a}_S)} \sum_{\mathfrak{p} \in \mathcal{P}(\Delta)} (-1)^{\alpha_\mathfrak{p}^{\Re(s\Lambda)}} \phi_\mathfrak{p}^{\Re(s\Lambda)}(H_S(x) - T_S)(M(s, \Lambda)\Phi)(x, s\Lambda)
\end{aligned}
$$

where, by definition,

$$
\Phi(x, \Lambda) = e^{\langle \Lambda + \rho_P, H_P(x) \rangle} \Phi(x) \ .
$$

The reader is warned that the dependence on T (denoted H_0 by Langlands in [6]) does not appear explicitly on the left hand side. Then, Langlands introduces

$$
E''(x, \Phi, \Lambda) = \sum_S \sum_{\xi \in S(F) \backslash G(F)} F_S''(\xi x, \Phi, \Lambda)
$$

where the sum over S runs over parabolic subgroups associated to P. On the other hand Arthur has defined in [3] a truncation operator denoted $\mathbf{\Lambda}^T$. When applied to the Eisenstein series

$$
E(x, \Phi, \Lambda) = \sum_{\gamma \in P(F) \backslash G(F)} \Phi(\gamma x, \Lambda)
$$

then, according to [5, Proposition 5.4.1], one obtains the following expression

$$
\begin{aligned}
\mathbf{\Lambda}^T E(x, \Phi, \Lambda) = \sum_S \sum_{s \in W(\mathfrak{a}_P, \mathfrak{a}_S)} \\
\sum_{\xi \in S(F) \backslash G(F)} (-1)^{a(s)} \phi_{M,s}\left(s^{-1}(H_S(\xi x) - T_S)\right)(M(s, \Lambda)\Phi)(\xi x, s\Lambda)
\end{aligned}
$$

where, by definition,

$$
\phi_{M,s}(H) = \widehat{\theta}_{P,G}^{\Re(s\Lambda)}(sH)
$$

and the integer $a(s)$ is the number of simple roots $\lambda \in \Delta_S$ such that $\lambda(s\Lambda) < 0$.

PROPOSITION 5.1. —

$$E^{''}(x, \Phi, \Lambda) = \mathbf{\Lambda}^T E(x, \Phi, \Lambda) \ .$$

Proof. — It suffices to observe that if $\Re(\Lambda)$ is in the positive chamber, then for $H \in \mathfrak{a}_P$ and $s \in W(\mathfrak{a}_P, \mathfrak{a}_S)$ one has

$$\sum_{\mathfrak{p} \in \mathcal{P}(\Delta)} (-1)^{\alpha_{\mathfrak{p}}^{\Re(s\Lambda)}} \phi_{\mathfrak{p}}^{\Re(s\Lambda)}(sH) = (-1)^{a(s)} \phi_{M,s}(H) \ .$$

This assertion is a particular case of Proposition 4.1 where Δ_0 is the set of simple roots of G with respect to the minimal parabolic subgroup P_0 and $\Delta = \Delta_P^G$. $\qquad\qquad\square$

On scalar products

A formula for the scalar product of two series E'' is stated by Langlands [6, § 9] without proof. Arthur proves a formula [3, Lemma 4.2] for the scalar product of two truncated Eisenstein series which is said (again without proof) to be equivalent to Langlands' one. As already observed this equivalence is not obvious since ordered partitions that show up in Langlands' formula never occur in Arthur's statement. Now, this follows from Proposition 5.1. This equivalence of the formulas for scalar products can also be proved directly using again Proposition 4.1. We leave it to the reader. We observe that a proof of the scalar product formula simpler than in [3] is given in [5, Chapter 5].

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