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# HIGHLY UNIFORM PRIME NUMBER THEOREMS

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ABSTRACT. — We prove a highly uniform version of the prime number theorem for a certain class of  $L$ -functions. The range of  $x$  depends polynomially on the analytic conductor, and the error term is expressed in terms of an optimization problem depending explicitly on the available zero-free region. The class contains the Rankin–Selberg  $L$ -function  $L(s, \pi \times \pi')$  associated to cuspidal automorphic representations  $\pi$  and  $\pi'$  of  $\mathrm{GL}_m$  and  $\mathrm{GL}_{m'}$ , respectively. Our main result implies the first nontrivial prime number theorem for such an  $L$ -function (with analytic conductor uniformity) in complete generality.

RÉSUMÉ. — Nous prouvons une version très uniforme du théorème des nombres premiers pour une certaine classe de fonctions  $L$ . L'étendue de  $x$  dépend polynomialement du conducteur analytique, et le terme d'erreur est exprimé en termes d'un problème d'optimisation dépendant explicitement de la région sans zéro disponible. La classe contient la fonction  $L$  de Rankin–Selberg  $L(s, \pi \times \pi')$  associée aux représentations automorphes cuspidales  $\pi$  et  $\pi'$  de  $\mathrm{GL}_m$  et  $\mathrm{GL}_{m'}$ , respectivement. Notre résultat principal implique le premier théorème non trivial des nombres premiers pour une telle fonction  $L$  (avec uniformité du conducteur analytique) en toute généralité.

## 1. Introduction and statement of the main result

We prove prime number theorems for a certain class of  $L$ -functions possessing a Dirichlet series, Euler product, analytic continuation, and functional equation of the usual type with strong uniformity in the analytic conductor. This problem has received attention before (see Iwaniec and Kowalski [6, Section 5.6]), but our work provides several new and substantial improvements. The class that we consider is slightly more restrictive

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*Keywords:* Prime number theorem, zero-free region, Rankin–Selberg  $L$ -function.

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than the class  $\mathcal{S}(m)$  considered by Soundararajan and Thorner [11, Sections 1.1–1.4]. Given an integer  $m \geq 1$ ,  $\mathcal{S}(m)$  denotes the set of  $L$ -functions satisfying the following four properties (A)–(D):

(A) (*Dirichlet series and Euler product.*) Let  $p$  run over the primes. The  $L$ -function  $L(s, \pi)$  is given by a Dirichlet series and an Euler product

$$(1.1) \quad L(s, \pi) = \sum_{n=1}^{\infty} \frac{\lambda_{\pi}(n)}{n^s} = \prod_p \prod_{j=1}^m \frac{1}{1 - \alpha_{j,\pi}(p)p^{-s}},$$

both converging absolutely for  $\operatorname{Re}(s) > 1$ . Let  $\Lambda(n)$  be the von Mangoldt function. We define the function  $a_{\pi}(n)$ , supported on prime powers, by the identity

$$-\frac{L'}{L}(s, \pi) = \sum_{n=1}^{\infty} \frac{a_{\pi}(n)\Lambda(n)}{n^s} = \sum_p \sum_{k=1}^{\infty} \frac{\sum_{j=1}^m \alpha_{j,\pi}(p)^k \log p}{p^{ks}}, \quad \operatorname{Re}(s) > 1.$$

(B) (*Analytic continuation and functional equation.*) There exist an integer  $q_{\pi} \geq 1$  attached to  $\pi$ , called the *conductor* of  $\pi$ , and complex numbers  $\mu_{\pi}(j)$  for  $1 \leq j \leq m$  such that if

$$r_{\pi} = -\operatorname{ord}_{s=1} L(s, \pi) \in [0, m] \quad \text{and} \quad L(s, \pi_{\infty}) = \pi^{-\frac{ms}{2}} \prod_{j=1}^m \Gamma\left(\frac{s + \mu_{\pi}(j)}{2}\right),$$

then

$$\Lambda(s, \pi) = (s(1-s))^{r_{\pi}} q_{\pi}^{s/2} L(s, \pi) L(s, \pi_{\infty})$$

is an entire function of order 1. Moreover, there exists a complex number  $\kappa_{\pi}$  of modulus 1 such that  $\Lambda(s, \pi) = \kappa_{\pi} \Lambda(1-s, \tilde{\pi})$ , where  $q_{\tilde{\pi}} = q_{\pi}$ ,  $\{\mu_{\tilde{\pi}}(j)\} = \{\overline{\mu_{\pi}(j)}\}$ ,  $\{\alpha_{\tilde{\pi},j}(p)\} = \{\overline{\alpha_{\pi,j}(p)}\}$ .

We define the: *analytic conductor*

$$(1.2) \quad C(\pi) = q_{\pi} \prod_{j=1}^m (|\mu_{\pi}(j)| + 3),$$

which serves as a key measure of “complexity” for  $L(s, \pi)$ . The zeros of  $\Lambda(s, \pi)$  are the *nontrivial* zeros of  $L(s, \pi)$ , and the poles of  $s^{r_{\pi}} L(s, \pi_{\infty})$  are the *trivial* zeros of  $L(s, \pi)$ . If  $p \nmid q_{\pi}$ , then for all  $1 \leq j \leq m$ , we have that  $\alpha_{j,\pi}(p) \neq 0$ . If  $p|q_{\pi}$ , then at least one of the  $\alpha_{j,\pi}(p)$  equals 0.

(C) (*Pointwise bounds on local parameters.*) If  $1 \leq j \leq m$  and  $p$  is prime, then

$$|\alpha_{j,\pi}(p)| \leq p^{1-\frac{1}{m}}, \quad \operatorname{Re}(\mu_{\pi}(j)) \geq -1 + \frac{1}{m}.$$

(D) (*ℓ<sup>1</sup> estimates.*) There exists a constant<sup>(1)</sup>  $c_1$  such that if  $\eta > 0$  and  $T \geq 1$ , then

$$\sum_{n=1}^{\infty} \frac{|a_{\pi}(n)|\Lambda(n)}{n^{1+\eta}} \leq \frac{m}{\eta} + m \log C(\pi) + O(m^2)$$

and

$$(1.3) \quad \sum_{x < n \leq xe^{1/T}} |a_{\pi}(n)|\Lambda(n) \ll m \frac{x}{T},$$

provided that  $x \geq c_1 m^{182m^4} (C(\pi)T)^{144m^3}$ .

*Remark.* — In the version of (D) in [11], it is only assumed that there exist certain unspecified constants  $c(m) > 0$  and  $c'(m) > 0$ , depending at most on  $m$ , such that

$$(1.4) \quad \sum_{x < n \leq xe^{1/T}} |a_{\pi}(n)|\Lambda(n) \leq c(m) \frac{x}{T}, \quad \text{provided that } x \geq c'(m)(C(\pi)T)^{144m^3}.$$

In (1.3), we assume that  $c(m)$  and  $c'(m)$  depend on  $m$  in a particular way.

The  $L$ -functions that we consider here satisfy two additional properties that are not part of the definition of  $\mathcal{S}(m)$  in [11].

(E) (*Nonvanishing on  $\text{Re}(s) = 1$ .*) If  $\text{Re}(s) \geq 1$ , then  $L(s, \pi) \neq 0$ . Consequently, there exists a function

$$\delta_{\pi} : [0, \infty) \rightarrow (0, \frac{1}{2})$$

such that if  $T > 0$ , then  $L(s, \pi) \neq 0$  in the region

$$\{s \in \mathbb{C} : \text{Re}(s) \geq 1 - \delta_{\pi}(T + 3), |\text{Im}(s)| \leq T\}$$

except for at most one real zero.

(F) (*Zero repulsion.*) Let  $\delta_{\pi}$  be as in (E). Define

$$\beta_0 = \max \left( \left\{ \beta > \max \left\{ \frac{3}{4}, 1 - \delta_{\pi}(3) \right\} : L(\beta, \pi) = 0 \right\} \cup \left\{ \frac{1}{2} \right\} \right).$$

If  $\beta_0 > \frac{1}{2}$ , then

- (i)  $\beta_0$  is a simple zero of  $L(s, \pi)$ ,
  - (ii) there exists a constant  $c_2 \geq 1$  such that  $\beta_0 \leq 1 - C(\pi)^{-c_2 m}$ ,
- and

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(1) The numbers  $c_1, c_2, c_3, \dots$  form a sequence of certain positive, absolute and effectively computable constants. The notation  $f \ll_{\nu} g$  or  $f = O_{\nu}(g)$  means that there exists an effectively computable constant  $c = c(\nu) > 0$ , depending at most on the parameter  $\nu$ , such that  $|f(z)| \leq c|g(z)|$  for all  $z$  in a range that is clear from context. If no parameter  $\nu$  is present, then  $c$  is absolute.

- (iii) there exist constants  $c_3$  and  $c_4$  such that if  $\rho = \beta + i\gamma \neq \beta_0$  is a nontrivial zero of  $L(s, \pi)$ , then

$$\beta \leq 1 - c_3 \frac{\log \left( \frac{c_4}{(1 - \beta_0) m \log(C(\pi) (|\gamma| + 3)^m)} \right)}{m \log(C(\pi) (|\gamma| + 3)^m)}.$$

*Remark.* — If (E) holds, then there are infinitely many choices of  $\delta_\pi$  such that  $\beta_0 = \frac{1}{2}$ . Property (F) asserts that there exists a  $\delta_\pi$  in (E) such if  $\beta_0 > \frac{1}{2}$ , then  $\beta_0$  is a simple zero.

*Remark.* — As in [11], one has some latitude in the formulation of (A)–(F). Our formulation is based on what we can prove when  $L(s, \pi)$  is the  $L$ -function of a cuspidal automorphic representation or the Rankin–Selberg  $L$ -function associated to a pair of such representations.

We define  $\mathfrak{S}(m)$  to be the set of  $L$ -functions  $L(s, \pi)$  that satisfy (A)–(F). Condition (E) is equivalent to the prime number theorem for  $L(s, \pi) \in \mathfrak{S}(m)$ , namely

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} a_\pi(n) \Lambda(n) = r_\pi.$$

We prove a highly uniform version of the prime number theorem for all  $L(s, \pi) \in \mathfrak{S}(m)$ .

**THEOREM 1.1.** — *There exist constants  $c_5 \geq 1$ ,  $c_6$ , and  $c_7 \geq 1$  such that the following is true. Let  $m \geq 1$ , and let  $L(s, \pi) \in \mathfrak{S}(m)$ . Let  $\delta_\pi(t)$  be given by (E) and  $\beta_0$  by (F), and define*

$$(1.5) \quad \eta_\pi(x) = \inf_{t \geq 3} (\delta_\pi(t) \log x + \log t).$$

*If  $A \geq 2$  and  $x \geq C(\pi)^{c_5 A^2 m^5}$ , then*

$$\sum_{n \leq x} a_\pi(n) \Lambda(n) = r_\pi x - \frac{x^{\beta_0}}{\beta_0} + O \left( \left( x - \frac{x^{\beta_0}}{\beta_0} \right) \left( m^5 x^{-c_6/m^4} + m^{c_7 m^3} A^2 e^{-(1-\frac{1}{A})\eta_\pi(x)} \right) \right).$$

It is natural to compare Theorem 1.1 with the following result of Iwaniec and Kowalski, which we present in our notation using properties (A)–(F).

**THEOREM 1.2** ([6, Theorem 5.13]). — *There exists a constant  $c_8$  such that the following is true. Let  $L(s, \pi)$  satisfy (A), (B), (E) with*

$$\delta_\pi(T) = \frac{c_8}{m^4 \log(C(\pi)T)},$$

and the  $\ell^2$  estimate

$$(1.6) \quad \sum_{n \leq x} |a_\pi(n)|^2 \Lambda(n)^2 \ll m^2 x (\log(C(\pi)x))^2, \quad x \geq 1.$$

Let  $\beta_0$  be as in (F). If  $x \geq 3$ , then

$$\sum_{n \leq x} a_\pi(n) \Lambda(n) = r_\pi x - \frac{x^{\beta_0}}{\beta_0} + O\left(m^4 (\log x C(\pi))^4 x \exp\left(-c_8 \frac{\log x}{m^4 (\log C(\pi) + \sqrt{\log x})}\right)\right).$$

The  $O$ -term is nontrivial when  $x \geq C(\pi)^{4c_8^{-1}m^4 \log(m \log C(\pi))}$ .

Theorem 1.1 has many advantages over Theorem 1.2. First, if one thinks of  $m$  as fixed (as is typical in many applications, but not all), then the range of  $x$  in Theorem 1.1 depends polynomially on  $C(\pi)$ , unlike Theorem 1.2. This is comparable with Linnik’s theorem [8, 9], which states that if  $q \geq 1$  and  $\gcd(a, q) = 1$ , then there exists a constant  $c_9$  such that the counting function  $\pi(x; q, a)$  for primes  $p \equiv a \pmod{q}$  is positive once  $x \geq q^{c_9}$ . Second, if  $\beta_0$  is especially close to  $s = 1$ , then the error term in Theorem 1.1 improves, unlike Theorem 1.2. This is a general extension of the zero repulsion phenomenon of Deuring and Heilbronn for Dirichlet  $L$ -functions, which served a crucial role in Linnik’s work [8, 9]. Until now, such a quantitative manifestation of this phenomenon has only been available when  $m = 1$  (see [12, Theorem 1.4]). Third, there are many important  $L$ -functions that are not yet known to satisfy the  $\ell^2$  bound (1.6) in Theorem 1.2, but the  $\ell^1$  bounds in (D) and the pointwise bounds in (C) are known quite generally. Fourth, Theorem 1.1 produces prime number theorems for  $L$ -functions having zero-free regions that are weaker than what Theorem 1.2 assumes.

Ultimately, Theorem 1.1 reduces the problem of establishing a prime number theorem for  $L(s, \pi) \in \mathfrak{S}(m)$  to the estimation of  $\eta_\pi(x)$ . This is a straightforward optimization calculation depending only on the available zero-free region. This feature, as well as the improved range of  $x$ , stems from our utilization of a log-free zero density estimate that follows from properties (A)–(D). In Section 2, we catalogue the most uniform versions to date of the prime number theorems that follow from Theorem 1.1 for the standard  $L$ -function  $L(s, \pi)$  and the Rankin–Selberg  $L$ -function  $L(s, \pi \times \pi')$  associated to cuspidal automorphic representations  $\pi$  of  $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$  and  $\pi'$  of  $\mathrm{GL}_{m'}(\mathbb{A}_\mathbb{Q})$ . When neither  $\pi$  nor  $\pi'$  is self-dual, our prime number theorem for  $L(s, \pi \times \pi')$  is completely new. Section 3 assembles various results on zeros of  $L$ -functions in  $\mathfrak{S}(m)$ , including a log-free zero density estimate that

improves as  $\beta_0$  worsens. In Section 4, we prove Theorem 1.1. The results in Section 2 are proved in Sections 5 and 6.

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## 2. Applications

Let  $\mathfrak{F}_m$  denote the family of cuspidal automorphic representations of  $GL_m(\mathbb{A}_{\mathbb{Q}})$  possessing unitary central character, normalized so that the central character is trivial on the diagonally embedded copy of the positive reals. Let  $\pi = \pi_{\infty} \otimes (\otimes_p \pi_p) \in \mathfrak{F}_m$  have arithmetic conductor  $q_{\pi} \geq 1$ , where  $\pi_p$  (resp.  $\pi_{\infty}$ ) is a smooth admissible representation of  $GL_m(\mathbb{Q}_p)$  for every prime  $p$  (resp.  $GL_m(\mathbb{R})$ ). The standard  $L$ -function  $L(s, \pi)$  associated to  $\pi$  can be expressed as a Dirichlet series and an Euler product, each absolutely convergent for  $\text{Re}(s) > 1$ :

$$L(s, \pi) = \prod_p L(s, \pi_p) = \sum_{n=1}^{\infty} \frac{\lambda_{\pi}(n)}{n^s}, \quad L(s, \pi_p) = \prod_{j=1}^m \frac{1}{1 - \alpha_{j,\pi}(p)p^{-s}}.$$

Here  $\lambda_{\pi}(n)$  is the  $n$ -th Hecke eigenvalue of  $\pi$ . When  $p \nmid q_{\pi}$ , the Satake isomorphism assigns to  $\pi_p$  the eigenvalues  $\{\alpha_{1,\pi}(p), \dots, \alpha_{m,\pi}(p)\}$  of a certain semisimple conjugacy class in  $GL_m(\mathbb{C})$ . If  $p \mid q_{\pi}$ , then some of the  $\alpha_{j,\pi}(p)$  might equal zero. We define the numbers  $a_{\pi}(n)$  by

$$\sum_{n=1}^{\infty} \frac{a_{\pi}(n)\Lambda(n)}{n^s} = -\frac{L'}{L}(s, \pi) = \sum_p \sum_{k=1}^{\infty} \frac{\sum_{j=1}^m \alpha_{j,\pi}(p)^k \log p}{p^{ks}}, \quad \text{Re}(s) > 1,$$

where  $\Lambda(n)$  denotes the usual von Mangoldt function. We define  $a_{\pi}(n) = 0$  when  $n$  is not a prime power. If  $p$  is prime, then  $a_{\pi}(p) = \lambda_{\pi}(p)$ . There are  $m$  Langlands parameters  $\mu_{\pi}(j)$ ,  $1 \leq j \leq m$ , from which we define

$$L(s, \pi_{\infty}) = \pi^{-\frac{ms}{2}} \prod_{j=1}^m \Gamma\left(\frac{s + \mu_{\pi}(j)}{2}\right).$$

If  $\tilde{\pi} \in \mathfrak{F}_m$  is the contragredient representation, then  $\tilde{\pi} \in \mathfrak{F}_m$  and

$$q_{\tilde{\pi}} = q_{\pi}, \quad \{\mu_{\tilde{\pi}}(j)\} = \{\overline{\mu_{\pi}(j)}\}, \quad \{\alpha_{\tilde{\pi},j}(p)\} = \{\overline{\alpha_{\pi,j}(p)}\}.$$

We denote by  $\mathbb{1} \in \mathfrak{F}_1$  the trivial representation, whose  $L$ -function is  $\zeta(s)$ .

Given  $\pi \in \mathfrak{F}_m$  with conductor  $q_\pi$  and  $\pi' \in \mathfrak{F}_{m'}$  with conductor  $q_{\pi'}$ , consider the Rankin–Selberg  $L$ -function

$$L(s, \pi \times \pi') = \prod_p L(s, \pi_p \times \pi'_p) = \sum_{n=1}^{\infty} \frac{\lambda_{\pi \times \pi'}(n)}{n^s},$$

absolutely convergent for  $\text{Re}(s) > 1$ , with

$$L(s, \pi_p \times \pi'_p) = \begin{cases} \prod_{j=1}^m \prod_{j'=1}^{m'} (1 - \alpha_{j,\pi}(p)\alpha_{j',\pi'}(p)p^{-s})^{-1} & \text{if } p \nmid q_\pi q_{\pi'}, \\ \prod_{j=1}^m \prod_{j'=1}^{m'} (1 - \alpha_{j,j',\pi \times \pi'}(p)p^{-s})^{-1} & \text{if } p | q_\pi q_{\pi'}. \end{cases}$$

See [11, Appendix] for a complete description of the numbers  $\alpha_{j,j',\pi \times \pi'}(p)$  when  $p | q_\pi q_{\pi'}$ . The conductor  $q_{\pi \times \pi'}$  divides  $q_\pi^{m'} q_{\pi'}^m$  [2]. The  $L$ -function  $L(s, \pi \times \pi')$  analytically continues to  $\mathbb{C}$ . By our normalization of the central characters,  $L(s, \pi \times \pi')$  is entire unless  $\pi' = \tilde{\pi}$ , in which case there is a pole of order 1 at  $s = 1$ . There are  $m'm$  Langlands parameters  $\mu_{\pi \times \pi'}(j, j')$ , with  $1 \leq j \leq m$  and  $1 \leq j' \leq m'$ , such that

$$L(s, \pi_\infty \times \pi'_\infty) = \pi^{-\frac{m'ms}{2}} \prod_{j=1}^m \prod_{j'=1}^{m'} \Gamma\left(\frac{s + \mu_{\pi,\pi'}(j, j')}{2}\right).$$

If  $\pi_\infty$  and  $\pi'_\infty$  are unramified, then

$$\{\mu_{\pi,\pi'}(j, j')\} = \{\mu_\pi(j) + \mu_{\pi'}(j')\}.$$

See [10, Section 3] for a complete description of the numbers  $\mu_{\pi \times \pi'}(j, j')$  when at least one of  $\pi_\infty$  and  $\pi'_\infty$  is ramified. We define the numbers  $a_{\pi \times \pi'}(n)$  by the identity

$$\sum_{n=1}^{\infty} \frac{a_{\pi \times \pi'}(n)\Lambda(n)}{n^s} = -\frac{L'}{L}(s, \pi \times \pi').$$

The sum converges absolutely for  $\text{Re}(s) > 1$ , and

$$a_{\pi \times \pi'}(p) = \lambda_{\pi \times \pi'}(p).$$

We require bounds for  $C(\pi \times \pi')$  in terms of  $C(\pi)$ ,  $C(\pi')$ ,  $C(\pi \times \tilde{\pi})$ , and  $C(\pi' \times \tilde{\pi}')$ .

LEMMA 2.1. — *If  $\pi \in \mathfrak{F}_m$  and  $\pi' \in \mathfrak{F}_{m'}$ , then*

$$C(\pi \times \tilde{\pi})^{\frac{m'}{4m}} C(\pi' \times \tilde{\pi}')^{\frac{m}{4m'}} \leq C(\pi \times \pi') \leq C(\pi)^{m'} C(\pi')^m.$$

*Proof.* — By combining [4, Lemma A.2] and [11, Lemma 2.1], we see that

$$C(\pi \times \tilde{\pi})^{(m')^2} C(\pi' \times \tilde{\pi}')^{m^2} \leq e^{O((m'm)^2)} C(\pi \times \pi')^{4m'm},$$

$$C(\pi \times \pi') \leq e^{O(m'm)} C(\pi)^{m'} C(\pi')^m.$$

In both of those proofs, the analytic conductor is defined to be

$$C(\pi) = q_\pi \prod_{j=1}^m (|\mu_\pi(j)| + 1), \quad C(\pi \times \pi') = q_{\pi \times \pi'} \prod_{j=1}^m \prod_{j'=1}^{m'} (|\mu_{\pi \times \pi'}(j, j')| + 1).$$

A careful inspection of the proofs shows that we can remove the factors  $e^{O((m'm)^2)}$  and  $e^{O(m'm)}$  when the shift of +1 is increased to +3, as in (1.2). Otherwise, the details are the same. □

The following result is [11, Proposition 2.5].

**PROPOSITION 2.2.** — *If  $\pi \in \mathfrak{F}_m$  and  $\pi' \in \mathfrak{F}_{m'}$ , then  $L(s, \pi) \in \mathcal{S}(m)$  and  $L(s, \pi \times \pi') \in \mathcal{S}(m'm)$ .*

We refine Proposition 2.2 as follows.

**PROPOSITION 2.3.** — *If  $\pi \in \mathfrak{F}_m$  and  $\pi' \in \mathfrak{F}_{m'}$ , then  $L(s, \pi) \in \mathfrak{S}(m)$  and  $L(s, \pi \times \pi') \in \mathfrak{S}(m'm)$ .*

*Proof.* — First, we confirm that  $L(s, \pi \times \pi') \in \mathfrak{S}(m'm)$ . Properties (A), (B), and (C) are true because  $L(s, \pi \times \pi') \in \mathcal{S}(m'm)$ , as proved in [11]. The first estimate in (D) is proved in [11, pp. 1241-1242]. The second estimate in (D) is proved by proceeding as in [11, Section 6], but with certain specific choices of test functions  $\Phi$  and  $\Phi_1$ . If  $\mathbf{1}_{(a,b)}(t)$  is the indicator function of the open interval  $(a, b)$  and one chooses

$$\Phi(t) = \exp\left(\frac{4}{3} + \frac{1}{(t - \frac{1}{2})^2 - 1}\right) \mathbf{1}_{(-\frac{1}{2}, \frac{3}{2})}(t),$$

$$\Phi_1(t) = \exp\left(1 + \frac{1}{(2t - 1)^2 - 1}\right) \mathbf{1}_{(0,1)}(t)$$

in the proof of [11, Theorem 2.4], then bounds for the Mellin transforms of  $\Phi$  and  $\Phi_1$  that follow from [1, Lemma 9] permit us to take

$$c(m) \ll m, \quad c'(m) = c_1 m^{182m^4}$$

in (1.4). Property (F) and a strong form of property (E) are given in Propositions 5.1, 5.2, and 5.3 below. We conclude that  $L(s, \pi \times \pi') \in \mathfrak{S}(m'm)$ . If  $\pi' = \mathbf{1}$ , then  $L(s, \pi) = L(s, \pi \times \pi') \in \mathfrak{S}(m)$ . □

Once we incorporate the best known zero-free regions for  $L(s, \pi)$  and  $L(s, \pi \times \pi')$ , we arrive at the most uniform versions of the prime number theorem for  $L(s, \pi)$  and  $L(s, \pi \times \pi')$  up to now. First, we apply Theorem 1.1 to the standard  $L$ -function  $L(s, \pi)$ .

THEOREM 2.4. — *Let  $\pi \in \mathfrak{F}_m - \{1\}$ . Let*

$$\beta_1 = \max \left( \left\{ \beta > \frac{3}{4} : L(\beta, \pi) = 0 \right\} \cup \left\{ \frac{1}{2} \right\} \right).$$

*There exist constants  $c_5$  and  $c_{10}$  such that if  $x \geq C(\pi)^{4c_5 m^8}$ , then*

$$\begin{aligned} & \sum_{n \leq x} a_\pi(n) \Lambda(n) \\ &= -\frac{x^{\beta_1}}{\beta_1} + O \left( \left( x - \frac{x^{\beta_1}}{\beta_1} \right) \exp \left( -c_{10} \frac{\log x}{m \log C(\pi) + \sqrt{m \log x}} \right) \right). \end{aligned}$$

For  $L(s, \pi \times \pi')$ , we consider two separate cases.

THEOREM 2.5. — *Let  $\pi \in \mathfrak{F}_m$  and  $\pi' \in \mathfrak{F}_{m'}$ . Let*

$$\beta_1 = \max \left( \left\{ \beta > \frac{3}{4} : L(\beta, \pi \times \pi') = 0 \right\} \cup \left\{ \frac{1}{2} \right\} \right).$$

*There exist constants  $c_5$  and  $c_{10}$  such that if*

$$(2.1) \quad \pi' \in \{ \tilde{\pi}, \tilde{\pi}' \}$$

*and  $x \geq (C(\pi)C(\pi'))^{4c_5(m'm)^8}$ , then*

$$\begin{aligned} \sum_{n \leq x} a_{\pi \times \pi'}(n) \Lambda(n) &= r_{\pi \times \pi'} x - \frac{x^{\beta_1}}{\beta_1} + O \left( \left( x - \frac{x^{\beta_1}}{\beta_1} \right) \right. \\ & \quad \left. \exp \left( -c_{10} \frac{\log x}{(m + m') \log(C(\pi)C(\pi')) + \sqrt{m(m + m') \log x}} \right) \right). \end{aligned}$$

All preceding prime number theorems for  $L(s, \pi \times \pi')$  with a nontrivial error term and a range of  $x$  with specified effective dependence on  $C(\pi)$  and  $C(\pi')$  have required an assumption of a “standard” zero-free region for  $L(s, \pi \times \pi')$ , which is known when (2.1) is true (Lemma 6.1 below). When (2.1) is true, Theorems 2.4 and 2.5 produce the strongest known error terms in ranges of  $x$  that are polynomial in the associated analytic conductors. When (2.1) is false, we only have Brumley’s narrow zero-free region (Lemma 6.2 below). For such  $\pi$  and  $\pi'$ , Theorem 1.1 and Lemma 6.2 together imply the first prime number theorem with a nontrivial error term of any sort, with an effective range of  $x$  in terms of  $C(\pi)$  and  $C(\pi')$ .

THEOREM 2.6. — *Let  $\pi \in \mathfrak{F}_m, \pi' \in \mathfrak{F}_{m'}$ . If  $\pi' \neq \tilde{\pi}$ , then there exists a constant  $c_5 > 0$  such that if*

$$x \geq \exp(c_5 (C(\pi)C(\pi'))^{2(m+m')^2}),$$

then

$$\sum_{n \leq x} a_{\pi \times \pi'}(n) \Lambda(n) \ll_{m,m'} x(\log x)^{-\frac{1}{m'}}.$$

Remark. — Note that if  $p \nmid q_\pi q_{\pi'}$ , then  $a_{\pi \times \pi'}(p^k) = a_\pi(p^k) a_{\pi'}(p^k)$ . Therefore, since (C) holds for  $L(s, \pi \times \pi')$  even when  $p|q_\pi q_{\pi'}$ , Theorems 2.5 and 2.6 remain the same if we sum  $a_\pi(n) a_{\pi'}(n) \Lambda(n)$  instead of  $a_{\pi \times \pi'}(n) \Lambda(n)$ .

### 3. Zeros of $L$ -functions in $\mathfrak{S}(m)$

Let  $m \geq 1$  be an integer, and let  $\pi \in \mathfrak{S}(m)$ . Since  $\Lambda(s, \pi)$  is entire of order 1 by (B), there exist constants  $a_\pi, b_\pi \in \mathbb{C}$  such that we have the Hadamard factorisation

$$(3.1) \quad \Lambda(s, \pi) = e^{a_\pi + b_\pi s} \prod_{\Lambda(\rho, \pi) = 0} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}.$$

LEMMA 3.1. — *If  $L(s, \pi) \in \mathfrak{S}(m), t \in \mathbb{R}$ , and  $0 < \eta \leq 2$ , then*

$$\#\{\rho: |\rho - (1 + it)| \leq \eta, L(\rho, \pi) = 0\} \ll \eta m \log(C(\pi)(2 + |t|)) + m^2,$$

where the zeros  $\rho$  are counted with multiplicity. In particular,

$$\#\{\rho = \beta + i\gamma : 0 < \beta < 1, |\gamma - t| \leq 1, L(\rho, \pi) = 0\} \ll m \log(C(\pi)(2 + |t|)).$$

Proof. — Since  $\mathfrak{S}(m) \subseteq \mathcal{S}(m)$ , this follows from [11, Lemma 3.1] when  $0 < \eta \leq 1$ . Otherwise, this follows from [6, Proposition 5.7].  $\square$

Next, we refine the  $m$ -dependence for the log-free zero density estimate in [11, Theorem 1.2].

THEOREM 3.2. — *Let  $L(s, \pi) \in \mathfrak{S}(m)$  and  $T \geq 1$ . For  $\sigma \geq 0$ , define*

$$N_\pi(\sigma, T) = \#\{\rho = \beta + i\gamma : L(\rho, \pi) = 0, \beta \geq \sigma, |\gamma| \leq T\},$$

where each  $\rho$  is counted with multiplicity. There exists a constant  $c_{11}$  such that

$$N_\pi(\sigma, T) \ll m^{c_{11} m^3} (C(\pi)T)^{10^7 m^3(1-\sigma)}.$$

*Proof.* — The proof proceeds as in [11, Section 4] with three small modifications. First, we use the bound (1.3) instead of the bound (1.4) (cf. [11, (1.10)]). This helps us to explicate the suppressed  $m$ -dependence in the implied constant in the third-to-last equation on [11, p. 1252]. Second, we require that  $\eta$  in [11, Proof of Theorem 1.2] satisfy

$$\frac{1}{200 \log(C(\pi)T)} < \eta \leq \frac{1}{200m} \quad \text{instead of} \quad \frac{1}{\log(C(\pi)T)} < \eta \leq \frac{1}{200m}.$$

When  $T = 1$ , this ensures that the interval containing  $\eta$  is always nonempty, even if  $C(\pi) < e^{200m}$ . (Since  $m$  was implicitly assumed to be fixed in [11], such considerations were inconsequential.) Third, one chooses

$$K = 10^5 m^3 \eta \log(C(\pi)T) + 300m^3 \log(em) + c_{12}m^2$$

in [11, (4.4)], where  $c_{12}$  is suitably large. This ensures that the range of  $x$  in (1.3) is compatible with the range of integration in the  $x$ -integral two equations below [11, (4.6)], even when  $m$  is not fixed. These modifications allow us to determine the dependence of the implied constant in [11, Theorem 1.2] on  $m$ . □

We use (F) to refine Theorem 3.2.

**COROLLARY 3.3.** — *Let  $L(s, \pi) \in \mathfrak{S}(m)$ . For  $\sigma \geq 0$  and  $T \geq 1$ , define*

$$N_\pi^*(\sigma, T) = \begin{cases} \#\{\rho = \beta + i\gamma \neq \beta_0 : L(\rho, \pi) = 0, \beta \geq \sigma, |\gamma| \leq T\} & \text{if } \beta_0 > \frac{1}{2} \\ N_\pi(\sigma, T) & \text{otherwise,} \end{cases}$$

where each  $\rho$  is counted with multiplicity. Let  $\beta_0$  be as in Theorem 1.1, and define

$$\nu_\pi(T) = \min\{1, (1 - \beta_0) \log(C(\pi)T)\}.$$

There exists a constant  $c_{13} \geq 1$  such that

$$N_\pi^*(\sigma, T) \ll \nu_\pi(T) m^{c_{13}m^3} (C(\pi)T)^{c_{13}m^3(1-\sigma)}.$$

*Proof.* — If  $\beta_0 = \frac{1}{2}$  or  $(1 - \beta_0)m \log(C(\pi)T^m) \geq \frac{c_4}{e}$ , then the result follows from Theorem 3.2. Now, suppose that

$$\beta_0 > \frac{1}{2}, \quad (1 - \beta_0)m \log(C(\pi)T^m) < \frac{c_4}{e}.$$

If

$$\sigma > 1 - c_3 \frac{\log\left(\frac{c_4}{(1 - \beta_0)m \log(C(\pi)(|\gamma| + 3)^m)}\right)}{m \log(C(\pi)(|\gamma| + 3)^m)}$$

then by (F), we have that  $N_\pi^*(\sigma, T) = 0$ . Otherwise,  $\sigma$  satisfies

$$(3.2) \quad \frac{c_4}{m^2} (C(\pi)(T+3)^m)^{-\frac{m}{c_3}(1-\sigma)} \leq \frac{1-\beta_0}{m} \log(C(\pi)(T+3)^m) \ll \nu_\pi(T).$$

It follows from Theorem 3.2 that  $N_\pi^*(\sigma, T) \leq N_\pi(\sigma, T)$  is

$$\ll m^{c_{11}m^3} (C(\pi)T)^{10^7 m^3(1-\sigma)} = \nu_\pi(T) m^{c_{11}m^3} (C(\pi)T)^{10^7 m^3(1-\sigma)} \nu_\pi(T)^{-1}.$$

Bounding  $\nu_\pi(T)^{-1}$  using (3.2), we obtain the corollary.  $\square$

## 4. Proof of Theorem 1.1

Let  $L(s, \pi) \in \mathfrak{S}(m)$ . We will prove Theorem 1.1 when  $\beta_0 > \frac{1}{2}$  in (E), in which case (F) states that  $\beta_0$  is a real simple zero of  $L(s, \pi)$ . If  $\beta_0 = \frac{1}{2}$ , then the proof is easier.

### 4.1. Preliminaries

We use the following smooth weight function.

LEMMA 4.1. — *Let  $x \geq 3$ ,  $\varepsilon \in (0, \frac{1}{4})$ , and an integer  $\ell \geq 2$ . Define  $B = \varepsilon/(2\ell \log x)$ . There exists a continuous function  $f(t) = f(t; x, \ell, \varepsilon)$  of a real variable  $t$  such that:*

- (i)  $0 \leq f(t) \leq 1$  for all  $t \in \mathbb{R}$ , and  $f(t) \equiv 1$  for  $\frac{1}{2} \leq t \leq 1$ .
- (ii) The support of  $f$  is contained in the interval  $[\frac{1}{2} - \frac{\varepsilon}{\log x}, 1 + \frac{\varepsilon}{\log x}]$ .
- (iii) Its Laplace transform  $F(z) = \int_{\mathbb{R}} f(t)e^{-zt} dt$  is entire and given by

$$F(z) = e^{-(1+2\ell B)z} \cdot \left( \frac{1 - e^{(\frac{1}{2}+2\ell B)z}}{-z} \right) \left( \frac{1 - e^{2Bz}}{-2Bz} \right)^\ell.$$

- (iv) Let  $s = \sigma + it$ ,  $\sigma > 0$ ,  $t \in \mathbb{R}$  and  $\alpha$  be any real number satisfying  $0 \leq \alpha \leq \ell$ . Then

$$|F(-s \log x)| \leq \frac{e^{\sigma\varepsilon} x^\sigma}{|s| \log x} \cdot (1 + x^{-\sigma/2}) \cdot \left( \frac{2\ell}{\varepsilon|s|} \right)^\alpha.$$

Moreover,  $|F(-s \log x)| \leq e^{\sigma\varepsilon} x^\sigma$  and  $1/2 < F(0) < 3/4$ .

- (v) If  $\frac{3}{4} < \sigma \leq 1$  and  $x \geq 10$ , then

$$F(-\log x) - F(-\sigma \log x) = \left( \frac{x}{\log x} - \frac{x^\sigma}{\sigma \log x} \right) (1 + O(\varepsilon)) + O\left( \frac{x^{1/2}}{\log x} \right).$$

*Proof.* — This is contained in the statement of [12, Lemma 2.2]. □

Using Lemma 4.1 and (C), we closely approximate

$$\sum_{n \leq x} a_\pi(n)\Lambda(n)$$

with a smoothed sum.

LEMMA 4.2. — *Let  $\pi \in \mathfrak{S}(m)$  and*

$$x \geq c_1^{145} m^{26390m^4} C(\pi)^{20880m^3}, \quad 0 < \varepsilon < \min \left\{ x^{-\frac{1}{145m^3}}, \frac{1}{4} \right\}.$$

*If  $f$  is given by Lemma 4.1, then*

$$\left| \sum_{n \leq x} a_\pi(n)\Lambda(n) - \sum_{n=1}^{\infty} a_\pi(n)\Lambda(n) f\left(\frac{\log n}{\log x}\right) \right| \ll mx^{1-\frac{1}{2m}} + \varepsilon x.$$

*Proof.* — By hypothesis, we have  $0 < \varepsilon < \frac{1}{4}$ . As such, Lemma 4.1 renders the equality

$$\sum_{n \leq x} a_\pi(n)\Lambda(n) = \sum_{n=1}^{\infty} a_\pi(n)\Lambda(n) f\left(\frac{\log n}{\log x}\right) + O\left(\sum_{\substack{1 \leq n \leq \sqrt{x} \\ x \leq n \leq xe^\varepsilon}} |a_\pi(n)|\Lambda(n)\right).$$

We apply (A), (C), and (D) with  $T = \varepsilon^{-1}$ , the prime number theorem  $\sum_{n \leq x} \Lambda(n) \sim x$ , and partial summation to obtain

$$\begin{aligned} \left( \sum_{n \leq \sqrt{x}} + \sum_{x < n \leq xe^\varepsilon} \right) |a_\pi(n)|\Lambda(n) &\ll m \sum_{n \leq \sqrt{x}} n^{1-\frac{1}{m}} \Lambda(n) + \varepsilon mx \\ &\ll mx^{1-\frac{1}{2m}} + \varepsilon mx. \end{aligned} \quad \square$$

We proceed to asymptotically evaluate the smoothed sum of  $a_\pi(n)\Lambda(n)$ . We let  $\rho = \beta + i\gamma$  run through the nontrivial zeros of  $L(s, \pi)$ , and  $\sum'_\rho$  denotes a sum over  $\rho \neq \beta_0$ , where each zero is counted with multiplicity.

LEMMA 4.3. — *If  $x \geq 3$  and  $\ell \geq m^3$ , then*

$$\begin{aligned} \frac{1}{\log x} \sum_{n=1}^{\infty} a_\pi(n)\Lambda(n) f\left(\frac{\log n}{\log x}\right) &= r_\pi F(-\log x) - F(-\beta_0 \log x) \\ &\quad - \sum'_{|\rho| > \frac{1}{4}} F(-\rho \log x) + O\left(\left(\frac{\ell}{\varepsilon} \frac{x^{1-\frac{1}{2m}}}{\log x} + mx^{\frac{1}{4}}\right) \log C(\pi)\right). \end{aligned}$$

*Proof.* — By Laplace inversion and (B), we obtain the identity

$$\begin{aligned}
 (4.1) \quad & \frac{1}{\log x} \sum_{n=1}^{\infty} a_{\pi}(n)\Lambda(n)f\left(\frac{\log n}{\log x}\right) \\
 &= \frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} -\frac{L'}{L}(s, \pi)F(-s \log x)ds \\
 &= \frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} \left(\frac{r_{\pi}}{s-1} + \frac{r_{\pi}}{s} + \frac{\log q_{\pi}}{2} + \frac{L'}{L}(s, \pi_{\infty}) - \frac{\Lambda'}{\Lambda}(s, \pi)\right)F(-s \log x)ds.
 \end{aligned}$$

By Lemma 4.1,  $F$  is entire and decays rapidly in vertical strips. By (C), we have that  $-\frac{L'}{L}(s, \pi_{\infty})$  is holomorphic for  $\text{Re}(s) > 1 - \frac{1}{m}$ . It follows that (4.1) equals

$$\begin{aligned}
 & r_{\pi}F(-\log x) - \frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} \frac{\Lambda'}{\Lambda}(s, \pi)F(-s \log x)ds \\
 & + \frac{1}{2\pi i} \int_{1-\frac{1}{2m}-i\infty}^{1-\frac{1}{2m}+i\infty} \left(\frac{r_{\pi}}{s-1} + \frac{r_{\pi}}{s} + \frac{\log q_{\pi}}{2} + \frac{L'}{L}(s, \pi_{\infty})\right)F(-s \log x)ds.
 \end{aligned}$$

By (A), we have that  $r_{\pi} \in [0, m]$ . Using Stirling’s formula and (C), it follows that

$$\begin{aligned}
 \left| \frac{r_{\pi}}{s-1} + \frac{r_{\pi}}{s} + \frac{\log q_{\pi}}{2} + \frac{L'}{L}(s, \pi_{\infty}) \right| &\ll m^2 + m \log(|\text{Im}(s)| + 3) + \log C(\pi), \\
 \text{Re}(s) &= 1 - \frac{1}{2m}.
 \end{aligned}$$

Therefore, by an application of Lemma 4.1 (iv) (with  $\alpha = 0$  when  $|\text{Im}(s)| \leq m$  and  $\alpha = 1$  when  $|\text{Im}(s)| > m$ ), we observe that

$$\begin{aligned}
 & \left| \frac{1}{2\pi i} \int_{1-\frac{1}{2m}-i\infty}^{1-\frac{1}{2m}+i\infty} \left(\frac{r_{\pi}}{s-1} + \frac{r_{\pi}}{s} + \frac{\log q_{\pi}}{2} + \frac{L'}{L}(s, \pi_{\infty})\right)F(-s \log x)ds \right| \\
 & \ll \frac{x^{1-\frac{1}{2m}}}{\log x} \int_{-m}^m (m^2 + m \log(|t| + 3) + \log C(\pi)) dt \\
 & + \frac{\ell x^{1-\frac{1}{2m}}}{\varepsilon \log x} \int_{|t|>m} (m^2 + m \log(|t| + 3) + \log C(\pi)) \frac{dt}{|t|^2} \\
 & \ll \frac{x^{1-\frac{1}{2m}}}{\log x} (m^3 + m \log C(\pi)) + \frac{\ell}{\varepsilon m} \frac{x^{1-\frac{1}{2m}}}{\log x} (m^2 + \log C(\pi)) \\
 & \ll \frac{\ell}{\varepsilon} \frac{x^{1-\frac{1}{2m}}}{\log x} \log C(\pi).
 \end{aligned}$$

Consequently, by the residue theorem, (4.1) equals

$$r_\pi F(-\log x) - F(-\beta_0 \log x) - \sum'_\rho F(-\rho \log x) + O\left(\frac{\ell x^{1-\frac{1}{2m}}}{\varepsilon \log x} \log C(\pi)\right).$$

For the zeros  $\rho$  such that  $|\rho| \leq \frac{1}{4}$ , Lemmata 3.1 and 4.1 (iv) imply that

$$\sum_{|\rho| \leq \frac{1}{4}} |F(-\rho \log x)| \ll x^{\frac{1}{4}} \#\{\rho: |\rho| < \frac{1}{4}\} \ll mx^{\frac{1}{4}} \log C(\pi).$$

The lemma follows once we combine the estimates above. □

### 4.2. Estimating the sum over zeros

We are in a position to evaluate the sum over nontrivial zeros  $\rho$  in Lemma 4.3 using the log-free zero density estimate in Corollary 3.3.

LEMMA 4.4. — *Let*

$$(4.2) \quad A \geq 2, \quad \ell = Ac_{13}m^3, \quad \varepsilon = \min\left\{\frac{1}{5}, 2A\ell x^{-1/(2A\ell)}\right\}.$$

*Let  $\delta_\pi$  be as in (E), and let  $\eta_\pi(x)$  be as in (1.5). Let  $\nu_\pi(T)$  be as in Corollary 3.3. If*

$$(4.3) \quad x \geq C(\pi)^{2A^2 c_{13} m^3},$$

*then*

$$\sum'_{|\rho| \geq \frac{1}{4}} |F(-\rho \log x)| \ll A^2 \nu_\pi(1) m^{c_{13} m^3} \frac{x}{\log x} e^{-(1-\frac{1}{A})\eta_\pi(x)}.$$

*Proof.* — Let  $T_0 = 0$ , and for  $j \geq 1$ , let  $T_j = 2^{j-1}$ . Consider the sum

$$(4.4) \quad Z_j = \frac{\log x}{x} \sum'_{\substack{|\rho| \geq \frac{1}{4} \\ T_{j-1} \leq |\gamma| \leq T_j}} |F(-\rho \log x)|.$$

First, we estimate the contribution of each zero  $\rho$  appearing in  $Z_j$ . Let  $\rho = \beta + i\gamma$  satisfy  $T_{j-1} \leq |\gamma| \leq T_j$  and  $|\rho| \geq \frac{1}{4}$ , so that  $|\rho| \geq \max(T_{j-1}, 1/4) \geq T_j/4$  and  $|\rho| \geq \frac{1}{13}(|\gamma| + 3)$ . Therefore, by Lemma 4.1(iv) with  $\alpha = \ell(1 - \beta)$  and our choice of  $\varepsilon$ , we have that

$$\begin{aligned} \frac{\log x}{x} |F(-\rho \log x)| &\ll \frac{x^{\beta-1}}{|\rho|} \left(\frac{2\ell}{\varepsilon|\rho|}\right)^{\ell(1-\beta)} \\ &\ll T_j^{-\frac{1}{A}} (|\gamma| + 3)^{-(1-\frac{1}{A})} x^{-(1-\beta)(1-\frac{1}{A})} (x^{\frac{1}{2A}} T_j^\ell)^{-(1-\beta)}. \end{aligned}$$

By (4.2) and (4.3), we have that

$$(4.5) \quad \frac{\log x}{x} |F(-\rho \log x)| \ll T_j^{-\frac{1}{A}} (|\gamma| + 3)^{\frac{1}{A}-1} x^{-(1-\beta)(1-\frac{1}{A})} (C(\pi)T_j)^{-Ac_{13}(1-\beta)m^3}.$$

From (E) and (1.5), one has

$$(4.6) \quad (|\gamma| + 3)^{\frac{1}{A}-1} x^{-(1-\beta)(1-\frac{1}{A})} \leq e^{-(1-\frac{1}{A})\eta_\pi(x)}.$$

Combining (4.4), (4.5), and (4.6), we derive

$$Z_j \ll e^{-(1-\frac{1}{A})\eta_\pi(x)} T_j^{-\frac{1}{A}} \sum'_{T_{j-1} \leq |\gamma| \leq T_j} (C(\pi)T_j)^{-Ac_{13}(1-\beta)m^3}.$$

By partial summation and Corollary 3.3, it follows that

$$\begin{aligned} \sum'_{T_{j-1} \leq |\gamma| \leq T_j} (C(\pi)T_j)^{-Ac_{13}(1-\beta)m^3} &\ll \int_0^1 (C(\pi)T_j)^{-Ac_{13}m^3\alpha} dN_\pi^*(1-\alpha, T_j) \\ &\ll m^{c_{13}m^3} \nu_\pi(T_j). \end{aligned}$$

Observe that

$$\nu_\pi(T_j) T_j^{-\frac{1}{2A}} \leq (1 - \beta_0) \sup_{t \geq 1} \left\{ t^{-\frac{1}{2A}} \log(C(\pi)t) \right\} \ll A \nu_\pi(1).$$

The lemma now follows from the bound

$$\begin{aligned} \sum_{j=1}^\infty Z_j &\ll A \nu_\pi(1) m^{c_{13}m^3} e^{-(1-\frac{1}{A})\eta_\pi(x)} \sum_{j=0}^\infty 2^{-\frac{j-1}{2A}} \\ &\ll A^2 \nu_\pi(1) m^{c_{13}m^3} e^{-(1-\frac{1}{A})\eta_\pi(x)}. \end{aligned} \quad \square$$

LEMMA 4.5. — *If  $x \geq C(\pi)^{1056c_2c_{13}m^5}$ , then*

$$x^{1-\frac{1}{1056c_{13}m^4}} \ll \nu_\pi(1)x \ll x - \frac{x^{\beta_0}}{\beta_0}.$$

*Proof.* — It suffices to prove the lemma when  $\nu_\pi(1) = (1 - \beta_0) \log C(\pi) < 1$ . We consider two cases. First, if  $(1 - \beta_0) \log x \geq 1$ , then

$$\nu_\pi(1)x \ll x \ll x(1 - 2e^{-1}) \leq x \left( 1 - \frac{x^{-(1-\beta_0)}}{\beta_0} \right) = x - \frac{x^{\beta_0}}{\beta_0}.$$

Second, assume that  $0 < (1 - \beta_0) \log x < 1$ . Our hypothesis on the range of  $x$  implies that  $x \geq e^4$ . We claim that

$$(4.7) \quad \frac{(1 - \beta_0) \log(x/e)}{1 - e^{-(1-\beta_0) \log x / \beta_0}} \leq \frac{e}{e - 1},$$

from which we deduce the desired bound

$$\begin{aligned} \nu_\pi(1)x &= (1 - \beta_0)x \log C(\pi) \ll (1 - \beta_0)x \log \frac{x}{e} \ll x \left(1 - \frac{e^{-(1-\beta_0)\log x}}{\beta_0}\right) \\ &= x - \frac{x^{\beta_0}}{\beta_0}. \end{aligned}$$

To finish the proof of the lemma, we observe that  $C(\pi)^{-c_2m} \ll \nu_\pi(1)$  by (F). Now, the lemma now follows from our range of  $x$ .

To prove the claimed bound in (4.7), we make the change of variables  $(1 - \beta_0)\log x = t$ , in which case the left hand side of (4.7) equals

$$f(x, t) = \frac{e^t t (\log x - t) (\log x - 1)}{(e^t (\log x - t) - \log x) \log x}.$$

We maximize  $f(x, t)$  when  $x \geq e^4$  and  $0 < t \leq 1$ . Observe that

$$\lim_{t \rightarrow 0^+} f(x, t) = 1 \leq \frac{e(\log x - 1)^2}{((e - 1)\log x - e)\log x} = \lim_{t \rightarrow 1^-} f(x, t),$$

and the sign of  $\frac{d}{dt}f(x, t)$  for  $t \in (0, 1]$  is the same as the sign of

$$\begin{aligned} (e^t - (t + 1)) (\log x)^2 - t(2e^t - (t + 2)) \log x + e^t t^2 \\ \geq ((e^t - (t + 1)) \log x - t(2e^t - (t + 2))) \log x \\ \geq (4(e^t - (t + 1)) - t(2e^t - (t + 2))) \log x \geq 0. \end{aligned}$$

Thus, as  $t$  monotonically increases from 0 to 1,  $f(x, t)$  monotonically increases from 1 to

$$\frac{e(\log x - 1)^2}{((e - 1)\log x - e)\log x}.$$

It follows that

$$\begin{aligned} \sup_{t \in (0, 1], x \geq e^4} f(x, t) &= \sup_{x \geq e^4} \frac{e(\log x - 1)^2}{((e - 1)\log x - e)\log x} \\ &= \lim_{x \rightarrow \infty} \frac{e(\log x - 1)^2}{((e - 1)\log x - e)\log x} \\ &= \frac{e}{e - 1}. \end{aligned}$$

□

**4.3. Proof of Theorem 1.1**

Without loss of generality, we may assume that  $\frac{3}{4} < \beta_0 < 1$ . Let  $c_5$  be suitably large, and let  $A \geq 4$ . If  $x \geq C(\pi)^{c_5 A^2 m^5}$ , then by Lemmata 4.2–4.4,

$$\sum_{n \leq x} a_\pi(n)\Lambda(n) = (r_\pi F(-\log x) - F(-\beta_0 \log x)) \log x + O\left(\nu_\pi(1)x\left(\frac{m}{\nu_\pi(1)x^{\frac{1}{2m}}} + \frac{\varepsilon}{\nu_\pi(1)} + \frac{\ell \log C(\pi)}{\varepsilon \nu_\pi(1)x^{\frac{1}{2m}}} + m^{c_{13}m^3} A^2 e^{-(1-\frac{1}{A})\eta_\pi(x)}\right)\right).$$

By Lemma 4.5 and the choices of  $\ell$  and  $\varepsilon$  in Lemma 4.4, the  $O$ -term is

$$\ll \nu_\pi(1)x\left(m^4 x^{-\frac{1}{33c_{13}m^4}} + m^{c_{13}m^3} A^2 e^{-(1-\frac{1}{A})\eta_\pi(x)}\right).$$

By Lemma 4.1 (iii), if  $\frac{3}{4} < \sigma \leq 1$ , then

$$F(-\sigma \log x) \log x = \frac{x^\sigma}{\sigma} \left(\frac{e^{\varepsilon\sigma/\ell} - 1}{\varepsilon\sigma/\ell}\right)^\ell + O\left(x^{\frac{\sigma}{2}}\right) = \frac{x^\sigma}{\sigma} (1 + O(\varepsilon\sigma)) + O\left(x^{\frac{\sigma}{2}}\right).$$

This bound, along with Lemma 4.1 (v), implies that

$$(r_\pi F(-\log x) - F(-\beta_0 \log x)) \log x = r_\pi x - \frac{x^{\beta_0}}{\beta_0} + O(m(\varepsilon x + \sqrt{x})).$$

Our choice of  $\varepsilon$  and the lower bound for  $\nu_\pi(1)x$  in Lemma 4.5 imply that

$$r_\pi(\varepsilon x + \sqrt{x}) \ll m\varepsilon x \ll m^5 \nu_\pi(1)x^{1-\frac{1}{33c_{13}m^4}},$$

from which we conclude that

$$\sum_{n \leq x} a_\pi(n)\Lambda(n) = r_\pi x - \frac{x^{\beta_0}}{\beta_0} + O\left(\nu_\pi(1)x\left(m^5 x^{-\frac{1}{33c_{13}m^4}} + m^{c_{13}m^3} A^2 e^{-(1-\frac{1}{A})\eta_\pi(x)}\right)\right).$$

To finish the proof, we invoke the upper bound for  $\nu_\pi(1)x$  in Lemma 4.5.

**5. Properties (E) and (F) for Rankin–Selberg  $L$ -functions**

Let  $\pi \in \mathfrak{F}_m$  and  $\pi' \in \mathfrak{F}_{m'}$ . We now compile the best known zero-free regions for  $L(s, \pi \times \pi')$ .

PROPOSITION 5.1. — *There exists a constant  $c_{14}$  such that if  $\pi \in \mathfrak{F}_m$  and  $\pi' \in \mathfrak{F}_{m'}$  satisfy (2.1), then  $L(s, \pi \times \pi') \neq 0$  in the region*

$$\operatorname{Re}(s) \geq 1 - \frac{c_{14}}{(m + m') \log(C(\pi)C(\pi')(|\operatorname{Im}(s)| + 3)^m)}$$

apart from at most one exceptional zero  $\beta_1 < 1$ . If  $\beta_1$  exists, then  $\beta_1$  is both real and simple, and

$$\pi = \tilde{\pi} \text{ and } \pi' = \tilde{\pi}', \quad \text{or} \quad \pi' = \tilde{\pi}.$$

*Remark.* — This implies a zero-free region for  $L(s, \pi) = L(s, \pi \times \mathbb{1})$ . If  $\beta_1$  exists, then  $\pi = \tilde{\pi}$ .

*Proof.* — When  $\pi' = \tilde{\pi}$ , this is [5, Theorem 2.1(1)]. When  $\pi' = \tilde{\pi}'$ , this is [4, Theorem A.1] with a small improvement in the dependence on  $m$  and  $m'$  stemming from the fact that if  $\Pi$  is the isobaric automorphic representation  $\pi \otimes |\det|^{i\gamma} \boxplus \tilde{\pi} \otimes |\det|^{-i\gamma} \boxplus \pi'$ , then the Dirichlet coefficients of  $\log L(s, \Pi \times \tilde{\Pi})$  are nonnegative [3, Lemma a]. This produces an improved degree dependence in [6, Lemma 5.9] that we insert into the proof of [4, Theorem A.1]. □

PROPOSITION 5.2. — Let  $\pi \in \mathfrak{F}_m$  and  $\pi' \in \mathfrak{F}_{m'}$ . Assume that  $\pi' \neq \tilde{\pi}$ . For all  $\varepsilon > 0$ , there exists an effectively computable constant  $c_{m,m',\varepsilon} > 0$  such that  $L(s, \pi \times \pi') \neq 0$  in the region

$$(5.1) \quad \text{Re}(s) \geq 1 - \frac{c_{m,m',\varepsilon}}{((C(\pi)C(\pi'))^{m+m'}(3 + |t|)^{m'm})^{1 - \frac{1}{m+m'} + \frac{\varepsilon}{2}}}.$$

*Proof.* — This follows from [7, Theorem A.1] and Lemma 2.1. □

Finally, property (F) for  $L(s, \pi)$  and  $L(s, \pi \times \pi')$  follows from the next result.

PROPOSITION 5.3. — Let  $\pi \in \mathfrak{F}_m$  and  $\pi' \in \mathfrak{F}_{m'}$ . If  $\beta_0 > \frac{1}{2}$  is a real simple zero of  $L(s, \pi \times \pi')$ , then  $\beta_0 \leq 1 - C(\pi \times \pi')^{-c_2 m'm}$ , and apart from  $s = \beta_0$ ,  $L(s, \pi \times \pi')$  is nonzero in the region

$$\text{Re}(s) \geq 1 - c_3 \frac{\log \left( \frac{c_4}{(1 - \beta_0)m'm \log(C(\pi \times \pi')(|\text{Im}(s)| + 3)^{m'm})} \right)}{m'm \log(C(\pi \times \pi')(|\text{Im}(s)| + 3)^{m'm})}.$$

*Proof.* — When  $\pi' = \tilde{\pi}$ , this was shown in [5, Proposition 5.3 and Corollary 5.4]. When  $\pi' \neq \tilde{\pi}$ , one applies the same ideas in [5, Proposition 5.3 and Corollary 5.4] to the  $L$ -function

$$D(s) = L(s, \pi \times \tilde{\pi})L(s, \pi' \times \tilde{\pi}')L(s, \pi \times \pi')L(s, \tilde{\pi} \times \tilde{\pi}')$$

instead of  $L(s, \pi \times \tilde{\pi})$ , which has nonnegative Dirichlet coefficients by [3, Lemma a]. The key observation is that while  $D(s)$  has a pole of order 2 at  $s = 1$ , if  $\rho$  is a nontrivial zero of  $L(s, \pi \times \pi')$ , then  $\bar{\rho}$  is a nontrivial zero of  $L(s, \tilde{\pi} \times \tilde{\pi}')$ . It remains to bound the analytic conductor of  $D(s)$  in terms of  $C(\pi \times \pi')$ , which is accomplished using Lemma 2.1. □

### 6. Proofs of prime number theorems for $L(s, \pi)$ and $L(s, \pi \times \pi')$

Let  $\pi \in \mathfrak{F}_m$  and  $\pi' \in \mathfrak{F}_{m'}$ . To prove Theorems 2.4, 2.5, and 2.6, it remains (after invoking Theorem 1.1) to bound  $e^{-\eta_\pi(x)}$  and  $e^{-\eta_{\pi \times \pi'}(x)}$  for  $x \geq 3$  using Propositions 5.1 and 5.2.

LEMMA 6.1. — *If  $\pi \in \mathfrak{F}_m$  and  $\pi' \in \mathfrak{F}_{m'}$  satisfy (2.1), then*

$$e^{-\eta_{\pi \times \pi'}(x)} \leq \exp \left( -c_{14} \frac{\log x}{(m + m') \log(C(\pi)C(\pi')) + \sqrt{m(m + m')}c_{14} \log x} \right).$$

*Proof.* — By (1.5) with the change of variables  $t \mapsto e^u$  and proposition 5.1, we have that

$$\eta_{\pi \times \pi'}(x) \geq \inf_{u \geq 0} \phi_x(u), \phi_x(u) = \frac{c_{14} \log x}{(m + m') \log(C(\pi)C(\pi')) + m(m + m')u} + u.$$

Note that  $\lim_{u \rightarrow \infty} \phi_x(u) = \infty$ . The equation  $\frac{d}{du} \phi_x(u) = 0$  has the unique positive solution

$$u = u_0 := \left( \frac{c_{14} \log x}{m(m + m')} \right)^{\frac{1}{2}} - \frac{\log(C(\pi)C(\pi'))}{m}.$$

We have that  $u_0 > 0$  if and only if  $x > \exp\left(\frac{m+m'}{c_{14}m}(\log(C(\pi)C(\pi')))^2\right)$ , so

$$\begin{aligned} \phi_x(u) &\geq \begin{cases} \phi_x(u_0) & \text{if } x > \exp\left(\frac{m + m'}{c_{14}m}(\log(C(\pi)C(\pi')))^2\right), \\ \phi_x(0) & \text{if } 3 \leq x \leq \exp\left(\frac{m + m'}{c_{14}m}(\log(C(\pi)C(\pi')))^2\right) \end{cases} \\ &= \begin{cases} 2 \left( \frac{c_{14} \log x}{m(m + m')} \right)^{\frac{1}{2}} - \frac{\log(C(\pi)C(\pi'))}{m} & \text{if } x > \exp\left(\frac{m + m'}{c_{14}m}(\log(C(\pi)C(\pi')))^2\right), \\ \frac{c_{14} \log x}{(m + m') \log(C(\pi)C(\pi'))} & \text{if } 3 \leq x \leq \exp\left(\frac{m + m'}{c_{14}m}(\log(C(\pi)C(\pi')))^2\right) \end{cases} \\ &\geq \min \left\{ \left( \frac{c_{14} \log x}{m(m + m')} \right)^{\frac{1}{2}}, \frac{c_{14} \log x}{(m + m') \log(C(\pi)C(\pi'))} \right\}. \end{aligned}$$

Since  $\exp(-\min\{a, b\}) \leq \exp(-\frac{ab}{a+b})$  when  $a > 0$  and  $b > 0$ , the lemma follows.  $\square$

*Proof of Theorems 2.4 and 2.5.* — Theorem 2.5 follows from Theorem 1.1 (with  $A = 2$ ) and Lemma 6.1. We restrict the range of  $x$  in order to absorb the factor of  $(m'm)^{c_7(m'm)^3}$  in the error term in Theorem 1.1. Theorem 2.4 follows from Theorem 2.5 by choosing  $\pi' = \mathbb{1}$ .  $\square$

We perform similar analysis using Brumley’s narrow zero-free region.

LEMMA 6.2. — Let  $\pi \in \mathfrak{F}_m$  and  $\pi' \in \mathfrak{F}_{m'}$  satisfy  $\pi' \neq \tilde{\pi}$ . Let  $0 < \varepsilon < 1$ , and let  $c_{m,m',\varepsilon}$  be as in (5.1). Let

$$(6.1) \quad \begin{aligned} \mathcal{A} &= c_{m,m',\varepsilon} / (C(\pi)C(\pi'))^{(m+m')(1+\frac{\varepsilon}{2})-1}, \\ \mathcal{B} &= m'm \left( 1 - \frac{1}{m+m'} + \frac{\varepsilon}{2} \right). \end{aligned}$$

If  $x > \exp(3^{\mathcal{B}}/(\mathcal{A}\mathcal{B}))$ , then

$$e^{-\eta_{\pi \times \pi'}(x)} \leq (\mathcal{A}\mathcal{B}e \log x)^{-1/\mathcal{B}}.$$

*Proof.* — Let  $\mathcal{A}$  and  $\mathcal{B}$  be given by (6.1). By Proposition 5.2 and (1.5), we have that

$$\eta_{\pi \times \pi'}(x) \geq \inf_{t \geq 3} \psi_x(t), \quad \psi_x(t) = t^{-\mathcal{B}} \mathcal{A} \log x + \log t.$$

Note that  $\lim_{t \rightarrow \infty} \psi_x(t) = \infty$ . The equation  $\frac{d}{dt} \psi_x(t) = 0$  has a unique positive solution  $t_0 = (\mathcal{A}\mathcal{B} \log x)^{1/\mathcal{B}}$ . We have that  $t_0 > 3$  if and only if  $x > \exp(3^{\mathcal{B}}/(\mathcal{A}\mathcal{B}))$ , in which case

$$\begin{aligned} \psi_x(t) &\geq \begin{cases} \phi_x(t_0) & \text{if } x > \exp(3^{\mathcal{B}}/(\mathcal{A}\mathcal{B})), \\ \phi_x(3) & \text{if } 3 \leq x \leq \exp(3^{\mathcal{B}}/(\mathcal{A}\mathcal{B})) \end{cases} \\ &= \begin{cases} \frac{1+\log(\mathcal{A}\mathcal{B} \log x)}{\mathcal{B}} & \text{if } x > \exp(3^{\mathcal{B}}/(\mathcal{A}\mathcal{B})), \\ \log 3 + \frac{\mathcal{A} \log x}{3^{\mathcal{B}}} & \text{if } 3 \leq x \leq \exp(3^{\mathcal{B}}/(\mathcal{A}\mathcal{B})). \end{cases} \end{aligned}$$

The lemma now follows.  $\square$

*Proof of Theorem 2.6.* — If  $A \geq 2$  and  $x \geq \exp(3^{\mathcal{B}}/(\mathcal{A}\mathcal{B}))$ , then

$$e^{-(1-\frac{1}{A})\eta_{\pi \times \pi'}(x)} \leq (\mathcal{A}\mathcal{B}e \log x)^{-(1-\frac{1}{A})/\mathcal{B}}$$

by Lemma 6.2. If  $A = 2(m + m')$ , then

$$\begin{aligned} &(\mathcal{A}\mathcal{B}e \log x)^{-(1-\frac{1}{A})/\mathcal{B}} \\ &\ll_{m,m',\varepsilon} (C(\pi)C(\pi'))^{\frac{1}{m} + \frac{1}{m'} - \frac{1}{2m'm}} (\log x)^{-\frac{2(m+m')-1}{m'm((2+\varepsilon)(m+m')-2)}}. \end{aligned}$$

If we let  $\varepsilon = (m + m')^{-2}$  and impose the condition

$$x \geq \exp \left( (C(\pi)C(\pi'))^{2(m+m')^2} \right),$$

then

$$(C(\pi)C(\pi'))^{\frac{1}{m} + \frac{1}{m'} - \frac{1}{2m'm}} (\log x)^{-\frac{2(m+m')-1}{m'm((2+\varepsilon)(m+m')-2)}} \ll_{m,m'} (\log x)^{-\frac{1}{m'm}}.$$

Theorem 2.6 follows from this estimate, Theorem 1.1, and Lemma 2.1.  $\square$

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