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NEVANLINNA CLASS, DIRICHLET SERIES AND SZEGŐ'S PROBLEM

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ABSTRACT. — This paper is associated with Nevanlinna class, Dirichlet series and Szegő's problem in infinitely many variables. As we will see, there is a natural connection between these topics. The paper first introduces the Nevanlinna class and the Smirnov class in this context, and generalizes the classical theory in finitely many variables to the infinite-variable setting. These results applied to Szegő's problem on Hardy spaces in infinitely many variables. Moreover, this paper is also devoted to the study of the correspondence between the Nevanlinna functions and Dirichlet series.

RÉSUMÉ. — Cet article est associé à la classe de Nevanlinna, aux séries de Dirichlet et au problème de Szegő en un nombre infini de variables. Comme nous le verrons, il existe une connexion naturelle entre ces sujets. L'article introduit d'abord la classe de Nevanlinna et la classe de Smirnov dans ce contexte, et généralise la théorie classique en un nombre fini de variables au cadre des variables infinies. Ces résultats sont ensuite appliqués au problème de Szegő dans les espaces de Hardy en un nombre infini de variables. De plus, cet article est également consacré à l'étude de la correspondance entre les fonctions de Nevanlinna et les séries de Dirichlet.

1. Introduction

This paper is of three purposes. The first is to study the function theory of the Nevanlinna class and the Smirnov class in infinitely many variables, the second is to consider Szegő's problem in infinitely many variables, and the third is to discuss the relationship between Dirichlet series and these functions by the Bohr correspondence.

The function theory in infinitely many variables has received attention in recent years, see [3, 14, 17, 33, 34]. We begin with the familiar Hardy

spaces. Let $\mathbb{T}^\infty = \mathbb{T} \times \mathbb{T} \times \cdots$ denote the cartesian product of countably infinitely many unit circles $\mathbb{T}$, equipped with the product topology. Then $\mathbb{T}^\infty$ is a compact group with the Haar measure $dm_\infty = \frac{d\theta_1}{2\pi} \times \frac{d\theta_2}{2\pi} \times \cdots$. By a polynomial we mean that it is an analytic polynomial only depending on finitely many complex variables. Let $\mathcal{P}_\infty$ denote the ring consisting of all polynomials. For $0 < p < \infty$, the Hardy space $H^p(\mathbb{T}^\infty)$ is defined to be the closure of $\mathcal{P}_\infty$ in $L^p(\mathbb{T}^\infty)$. Therefore, when $1 \leqslant p < \infty$, $H^p(\mathbb{T}^\infty)$ is a Banach space with the norm of $L^p(\mathbb{T}^\infty)$, and when $0 < p < 1$, $H^p(\mathbb{T}^\infty)$ is complete in the metric $d_p(f, g) = \int_{\mathbb{T}^\infty} |f - g|^p dm_\infty$. It is clear that the Hardy space $H^p(\mathbb{T}^n)$ over the n-torus $\mathbb{T}^n$ can be viewed as a closed subspace of $H^p(\mathbb{T}^\infty)$.

Assume $0 < p < \infty$. In finite-variable setting, it is known that $H^p(\mathbb{T}^n)$ is canonically isometrically isomorphic to the Hardy space $H^p(\mathbb{D}^n)$ over the polydisk $\mathbb{D}^n$. We now turn to the infinite-variable setting. Let $\mathbb{D}^\infty = \mathbb{D} \times \mathbb{D} \times \cdots$ be the cartesian product of countably infinitely many open unit disks $\mathbb{D}$. Cole and Gamelin [17] showed that every function $f \in H^p(\mathbb{T}^\infty)$, by evaluation functional, can be extended to a function $\widetilde{f}$ holomorphic on $\mathbb{D}_2^\infty = \ell^2 \cap \mathbb{D}^\infty$, a domain in the Hilbert space ℓ^2 of all square-summable sequences. In particular, for $1 \leqslant p < \infty$, this holomorphic function can be represented by taking Poisson integrals [17]. On the other hand, the Hardy space $H^p(\mathbb{D}_2^\infty)$ over $\mathbb{D}_2^\infty$ is defined as follows:

$$H^p(\mathbb{D}_2^\infty)$$
$$= \left\{ F \text{ is holomorphic on } \mathbb{D}_2^\infty : \|F\|_p^p = \sup_{0 < r < 1} \int_{\mathbb{T}^\infty} |F_{[r]}|^p dm_\infty < \infty \right\},$$

where

$$F_{[r]}(w) = F(rw_1, \ldots, r^n w_n, \ldots), \quad w = (w_1, w_2, \cdots) \in \mathbb{T}^\infty,$$

see [14, 18]. As the same in the finite-variable setting, for every nonzero function $F \in H^p(\mathbb{D}_2^\infty)$, the radial limit $F^*(w) = \lim_{r \to 1} F_{[r]}(w)$ exists for almost every $w \in \mathbb{T}^\infty$, and $\log|F^*| \in L^1(\mathbb{T}^\infty)$ [3, 15]. Furthermore, the map $F \mapsto F^*$ gives a canonical isometric isomorphism from $H^p(\mathbb{D}_2^\infty)$ onto $H^p(\mathbb{T}^\infty)$, and its inverse is given by $f \mapsto \widetilde{f}$, $f \in H^p(\mathbb{T}^\infty)$, see [3, 14, 18, 33, 34].

Hardy spaces in infinitely many variables are also closely related to spaces formed by Dirichlet series. Let $\mathcal{P}_D$ be the set of all Dirichlet polynomials $Q(s) = \sum_{n=1}^N a_n n^{-s}$. For $0 < p < \infty$ and $Q \in \mathcal{P}_D$, it follows from the almost periodicity of the function $t \mapsto |Q(it)|^p$ that

$$\|Q\|_p^p = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T |Q(it)|^p dt$$

exists, see [10], or [37, Theorem 1.5.6]. The Hardy–Dirichlet space $\mathcal{H}^p$ is defined to be the completion of $\mathcal{P}_D$ in the metric $\| \cdot \|_p$ [7]. Bohr's vision below [13] allows us to investigate $\mathcal{H}^p$ via the Hardy space $H^p(\mathbb{T}^\infty)$. Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of positive integers and p_j the jth prime number. With each $n \in \mathbb{N}$ is associated a unique prime factorization $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, and set $\alpha(n) = (\alpha_1, \ldots, \alpha_k, 0, \ldots)$. For a sequence of complex numbers $\zeta = (\zeta_1, \zeta_2, \ldots)$, write $\zeta^{\alpha(n)} = \zeta_1^{\alpha_1} \cdots \zeta_k^{\alpha_k}$. The Bohr correspondence

$$\mathcal{B} : \sum_{n=1}^{N} a_n n^{-s} \mapsto \sum_{n=1}^{N} a_n \zeta^{\alpha(n)}$$

is an algebraic isomorphism from $\mathcal{P}_D$ onto $\mathcal{P}_\infty$. Then by Birkhoff–Oxtoby theorem [37, Theorem 6.5.1], for every $Q \in \mathcal{P}_D$, $\|Q\|_p^p = \int_{\mathbb{T}^\infty} |\mathcal{B}Q|^p dm_\infty$, and hence the Bohr correspondence can be extended to an isometric isomorphism from $\mathcal{H}^p$ onto $H^p(\mathbb{T}^\infty)$.

When $p = \infty$, let $H^\infty(\mathbb{T}^\infty)$ be the weak*-closure of $\mathcal{P}_\infty$ in $L^\infty(\mathbb{T}^\infty)$. As done in [5, 17, 27], there is a canonical isometric isomorphism from $H^\infty(\mathbb{T}^\infty)$ onto $H^\infty(\mathbb{D}_2^\infty)$, the Banach algebra consisting of all bounded holomorphic functions on $\mathbb{D}_2^\infty$, by taking Poisson integrals. In addition, the Hardy space $H^\infty(\mathbb{T}^\infty)$ can be identified with the Hardy–Dirichlet space $\mathcal{H}^\infty$ by the Bohr correspondence, see works of Hedenmalm, Lindqvist and Seip [26]. For some recent works on the Hardy–Dirichlet spaces $\mathcal{H}^p$ ($0 < p \leqslant \infty$), we refer the reader to [2, 8, 9, 14, 16, 36].

The above statements briefly sketch some background material of both H^p and $\mathcal{H}^p$ in the case $0 < p \leqslant \infty$. This paper is intended as an attempt to develop the theory of limit function spaces in two cases of both H^p and $\mathcal{H}^p$ as $p \to 0^+$. We first consider the case H^p as $p \to 0^+$, which is parallel to the finite-variable setting [38]. For $0 < p < \infty$ and $f \in L^p(\mathbb{T}^\infty)$, write

$$\|f\|_p = \left(\int_{\mathbb{T}^\infty} |f|^p dm_\infty \right)^{\frac{1}{p}},$$

then when $0 < p < q \leqslant \infty$, $L^q(\mathbb{T}^\infty) \subset L^p(\mathbb{T}^\infty)$. As well known, if $f \in L^r(\mathbb{T}^\infty)$ for some $0 < r \leqslant \infty$, then $\|f\|_p$ tends to $\exp\left(\int_{\mathbb{T}^\infty} \log|f| dm_\infty\right)$ as $p \to 0^+$, where $\exp(-\infty)$ is defined to be zero. An observation is that if f is a complex measurable function on $\mathbb{T}^\infty$, then $\exp\left(\int_{\mathbb{T}^\infty} \log|f| dm_\infty\right)$ is finite if and only if $\exp\left(\int_{\mathbb{T}^\infty} \log(1 + |f|) dm_\infty\right)$ is finite, and the latter is equivalent to

$$\|f\|_0 = \int_{\mathbb{T}^\infty} \log(1 + |f|) dm_\infty < \infty.$$

So the limit space of $L^p(\mathbb{T}^\infty)$ as $p \to 0^+$, denoted by $L^0(\mathbb{T}^\infty)$, is defined to be the set of all complex measurable functions f on $\mathbb{T}^\infty$ for which $\|f\|_0$ is finite. Then $L^0(\mathbb{T}^\infty)$ is a topological vector space with the complete metric $d_0(f,g) = \|f - g\|_0$. The limit space of $H^p(\mathbb{T}^\infty)$ as $p \to 0^+$, denoted by $N_*(\mathbb{T}^\infty)$, is defined to be the closure of $\mathcal{P}_\infty$ in $L^0(\mathbb{T}^\infty)$, so-called the Smirnov class over $\mathbb{T}^\infty$. It is shown that each function in $N_*(\mathbb{T}^\infty)$ can be uniquely analytically extended to a domain $\mathbb{D}_1^\infty$ of ℓ^1, where $\mathbb{D}_1^\infty = \ell^1 \cap \mathbb{D}^\infty$ is a domain in the Banach space ℓ^1 of summable sequences. This leads to bring in the Smirnov class $N_*(\mathbb{D}_1^\infty)$ for holomorphic functions on $\mathbb{D}_1^\infty$. Section 2 will be concerned with the Smirnov class $N_*(\mathbb{D}_1^\infty)$ and the Nevanlinna class $N(\mathbb{D}_1^\infty)$, a larger class than $N_*(\mathbb{D}_1^\infty)$. It is shown that there is a natural correspondence between the class $N_*(\mathbb{T}^\infty)$ and the class $N_*(\mathbb{D}_1^\infty)$. For functions in the class $N(\mathbb{D}_1^\infty)$, there exists an analogue of the classical Fatou's theorem, that is, for every function $F \in N(\mathbb{D}_1^\infty)$, the radial limit $F^*(w) = \lim_{r\to 1} F(rw_1, \ldots, r^n w_n, \ldots)$ exists for almost every $w \in \mathbb{T}^\infty$. Furthermore, if $F \neq 0$, then $\log|F^*| \in L^1(\mathbb{T}^\infty)$.

In Section 3, we apply the preceding results to Szegő's problem in infinitely many variables. Let us first recall Szegő's theorem in one variable case [23, 28, 43]. Assume that K is a nonnegative function in $L^1(\mathbb{T})$ with $\log K \in L^1(\mathbb{T})$, and m_1 is the normalized Lebesgue measure on $\mathbb{T}$. Write $\mathbb{C}[z]$ for the ring of all one-variable analytic polynomials, and $\mathbb{C}_0[z]$ for the set of polynomials $q \in \mathbb{C}[z]$ with $q(0) = 0$. Szegő's theorem states that when $1 < p < \infty$, the following equality holds:

$$\inf_{q \in \mathbb{C}_0[z]} \int_{\mathbb{T}} |1 - q|^p K dm_1 = \exp\left(\int_{\mathbb{T}} \log K dm_1\right).$$

In fact, this holds because such K is exactly the modulus of an outer function [28]. Szegő's theorem has a profound influence in many areas, especially in the theory of orthogonal polynomials on the unit circle [11, 42, 43], and it can be regarded as the cornerstone of the development of the invariant subspace theory [23]. For more works relating to Szegő's theorem, we refer the reader to [1, 6, 24, 25, 30, 40]. When $p = 2$, Nakazi gave an analogous version of Szegő's theorem in two-variable setting [31]. Whereas in the case of infinitely many variables, things become much more complicated. Let K be a nonnegative function in $L^1(\mathbb{T}^\infty)$ with $\log K \in L^1(\mathbb{T}^\infty)$. Without loss of generality, we assume that $\|K\|_1 = 1$. Write

$$S(K) = \inf_{q \in \mathcal{P}_0} \int_{\mathbb{T}^\infty} |1 - q|^p K dm_\infty,$$

where $\mathcal{P}_0$ denotes the set of polynomials $q \in \mathcal{P}_\infty$ with $q(0) = 0$. It is easy to verify that $S(K)$ falls into the closed interval $[\exp(\int_{\mathbb{T}^\infty} \log K dm_\infty), 1]$. Naturally, the problem arises which is called Szegő's problem: For which function K, $S(K)$ attains the lower bound $\exp(\int_{\mathbb{T}^\infty} \log K dm_\infty)$ or the upper bound 1? We will give a complete answer to this problem in Theorem 3.1 and Theorem 3.4 by applying the previous results. It is worth pointing out this paper provides a general method which also applies to the finite-variable setting for Szegő's problem.

As in cases of the Nevanlinna class and the Smirnov class in the infinite-variable setting, there exist analogies for Dirichlet series, that is, we are concerned with the limit Hardy–Dirichlet space in the situation $p \to 0^+$. In Section 4, we introduce the Smirnov–Dirichlet class $\mathcal{N}_*$, which is defined to be the completion of $\mathcal{P}_D$ in the metric

$$\|Q\|_0 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \log(1 + |Q(it)|)dt, \quad Q \in \mathcal{P}_D.$$

Then $\mathcal{N}_*$ can be viewed as the limit space of $\mathcal{H}^p$ when $p \to 0^+$. Moreover, we draw a conclusion that there exists a canonical isometric algebra isomorphism from $\mathcal{N}_*$ onto the Smirnov class $N_*(\mathbb{D}_1^\infty)$ by the Bohr correspondence. In order to study composition operators on spaces of Dirichlet series, Brevig and Perfekt [15] defined the class $\mathcal{N}_u$ of Dirichlet series f with the abscissa of uniform convergence $\sigma_u(f) \leqslant 0$ and

$$\limsup_{\sigma \to 0^+} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \log^+ |f(\sigma + it)| dt < \infty,$$

where $\log^+ x = \max\{0, \log x\}$ for $x > 0$. This leads to an introduction of the Nevanlinna–Dirichlet class $\mathcal{N}$, the completion of $\mathcal{N}_u$ in the metric

$$\|f\|_0 = \limsup_{\sigma \to 0^+} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \log(1 + |f(\sigma + it)|)dt, \quad f \in \mathcal{N}_u.$$

We will prove that the Nevanlinna–Dirichlet class $\mathcal{N}$ can be isometrically embedded into the Nevanlinna class $N(\mathbb{D}_1^\infty)$ by the Bohr correspondence.

Acknowledgments

2. The Nevanlinna class and the Smirnov class

This section is devoted to introducing the Nevanlinna class and the Smirnov class in infinitely many variables.

2.1. The Nevanlinna class and the Smirnov class over $\mathbb{D}_1^\infty$

We begin with ∞-subharmonic functions on $\mathbb{D}_1^\infty$. Motivated by [38], an upper semicontinuous function $u : \mathbb{D}_1^\infty \to [-\infty, \infty)$ is called ∞-subharmonic, if u is subharmonic in each variable separately. As the same in single variable, if $u : \mathbb{D}_1^\infty \to [-\infty, \infty)$ is ∞-subharmonic, and φ is a nondecreasing convex function on the real line $\mathbb{R}$, then $\varphi \circ u$ is ∞-subharmonic (setting $\varphi(-\infty) = \lim_{t \to -\infty} \varphi(t)$).

For each function F on $\mathbb{D}_1^\infty$ and $0 < r < 1$, the function $F_{[r]}$ over the infinite torus $\mathbb{T}^\infty$ is defined by

$$F_{[r]}(w) = F\left(rw_1, \ldots, r^n w_n, \ldots\right), \quad w \in \mathbb{T}^\infty.$$

The following result will be used frequently in this paper.

PROPOSITION 2.1. — *If u is ∞-subharmonic on $\mathbb{D}_1^\infty$, then integrals*

$$I_r = \int_{\mathbb{T}^\infty} u_{[r]} dm_\infty \quad (0 \leqslant r < 1)$$

increase with r. Therefore, if $\{r_n\}_{n=1}^\infty$ is an increasing sequence in $(0,1)$ with $r_n \to 1$ as $n \to \infty$, then $\sup_{0 < r < 1} I_r = \sup_{n \in \mathbb{N}} I_{r_n}$.

Proof. — Given $0 \leqslant r \leqslant s < 1$, it suffices to prove that $I_r \leqslant I_s$. For each $n \in \mathbb{N}$, write

$$u_{[r,s,n]}(w) = u\left(sw_1, \ldots, s^n w_n, r^{n+1} w_{n+1}, r^{n+2} w_{n+2}, \ldots\right), \quad w \in \mathbb{T}^\infty.$$

Using subharmonicity successively in the first n variables, we obtain

$$I_r \leqslant \int_{\mathbb{T}^\infty} u_{[r,s,n]} dm_\infty.$$

Note that $s\overline{\mathbb{D}} \times \cdots \times s^n \overline{\mathbb{D}} \times \cdots$ is compact in ℓ^1, where $\overline{\mathbb{D}}$ denotes the closed unit disk. The upper semicontinuity of u implies that $\{u_{[r,s,n]}\}_{n=1}^\infty$ is uniformly bounded above on $\mathbb{T}^\infty$. Therefore, it follows from Fatou's lemma that

$$I_r \leqslant \limsup_{n \to \infty} \int_{\mathbb{T}^\infty} u_{[r,s,n]} dm_\infty \leqslant \int_{\mathbb{T}^\infty} \limsup_{n \to \infty} u_{[r,s,n]} dm_\infty \leqslant I_s,$$

as desired.   □

Let $\overline{\mathbb{D}}^\infty = \overline{\mathbb{D}} \times \overline{\mathbb{D}} \times \cdots$ denote the cartesian product of countably infinitely many closed unit disks $\overline{\mathbb{D}}$, then it is compact with respect to the product topology. The set of all continuous functions on $\overline{\mathbb{D}}^\infty$, denoted by $C(\overline{\mathbb{D}}^\infty)$, is a Banach algebra with the uniform norm. The following corollary is useful.

COROLLARY 2.2. — *Let* $u \in C(\overline{\mathbb{D}}^\infty)$. *If* u *is* ∞-*subharmonic on* $\mathbb{D}_1^\infty$, *then for every* $0 < r < 1$,

$$u(0) \leqslant \int_{\mathbb{T}^\infty} u_{[r]} dm_\infty \leqslant \int_{\mathbb{T}^\infty} u \, dm_\infty.$$

Proof. — By Proposition 2.1, for each $r < s < 1$, we have

$$u(0) \leqslant \int_{\mathbb{T}^\infty} u_{[r]} dm_\infty \leqslant \int_{\mathbb{T}^\infty} u_{[s]} dm_\infty.$$

The desired corollary follows from Lebesgue's dominated convergence theorem by letting $s \to 1$. $\qquad\square$

Recall that a complex-valued function F defined on an open subset V of a Banach space X is called holomorphic [22], if F satisfies the following two conditions: (i) F is locally bounded. (ii) For each $x_0 \in V$ and $x \in X$, the function $F(x_0 + zx)$ is holomorphic in parameter z for $x_0 + zx \in V$. One easily checks that holomorphic functions are continuous.

In this paper, we mainly concern with holomorphic functions on the domain $\mathbb{D}_1^\infty \subset \ell^1$. For every $0 < r < 1$ and $M > 0$, set

$$(2.1) \qquad V_{r,M} = \left\{ \zeta \in \ell^1 : \|\zeta\|_1 < M \text{ and for each } n \in \mathbb{N}, \ |\zeta_n| < r \right\},$$

then $V_{r,M}$ are domains in ℓ^1 which increase to $\mathbb{D}_1^\infty$ as $r \uparrow 1$ and $M \uparrow \infty$. As we will see later, when discussing holomorphic functions on $\mathbb{D}_1^\infty$, the role of the domains $V_{r,M}$ in $\mathbb{D}_1^\infty$ is similar to $r\mathbb{D}$ in $\mathbb{D}$.

It is also worth mentioning that every function F holomorphic on $\mathbb{D}_1^\infty$ has a unique monomial expansion

$$(2.2) \qquad\qquad F(\zeta) = \sum_{n=1}^\infty c_n \zeta^{\alpha(n)},$$

and the series converges uniformly and absolutely on compact subsets of $\mathbb{D}_1^\infty$ for which we refer readers to [21].

Now we introduce the Nevanlinna class over $\mathbb{D}_1^\infty$. Let F be a holomorphic function on $\mathbb{D}_1^\infty$. Since $\varphi(x) = \max\{0, x\}$ is a non-decreasing convex function on $\mathbb{R}$, $\varphi(\log |F|) = \log^+ |F|$ is ∞-subharmonic. Hence by Proposition 2.1,

$$(2.3) \qquad \sup_{0<r<1} \int_{\mathbb{T}^\infty} \log^+ |F_{[r]}| dm_\infty = \lim_{r \to 1} \int_{\mathbb{T}^\infty} \log^+ |F_{[r]}| dm_\infty.$$

The Nevanlinna class $N(\mathbb{D}_1^\infty)$ over $\mathbb{D}_1^\infty$ is defined to be the class of all holomorphic functions F on $\mathbb{D}_1^\infty$ such that the value of (2.3) is finite. The following proposition is needed in the sequel.

PROPOSITION 2.3. — *Let F be a nonzero holomorphic function on $\mathbb{D}_1^\infty$. Then the following statements are equivalent:*

(1) $F \in N(\mathbb{D}_1^\infty)$.
(2) $\sup_{0<r<1} \|F_{[r]}\|_0 < \infty$.
(3) $\limsup_{r\to 1} \| \log |F_{[r]}| \|_1 < \infty$.

To prove Proposition 2.3, we need to introduce the infinite polydisk algebra [17]. The infinite polydisk algebra $A(\mathbb{D}^\infty)$ is defined to be the closure of $\mathcal{P}_\infty$ in $C(\overline{\mathbb{D}}^\infty)$. It is worth mentioning that if F is a nonzero holomorphic function on $\mathbb{D}_1^\infty$, then for every $0 < r < 1$, $F_{[r]} \in A(\mathbb{D}^\infty)$, and hence $\log |F_{[r]}| \in L^1(\mathbb{T}^\infty)$ [3, 15]. On the other hand, we can also understand the infinite polydisk algebra by "boundary functions". Let $A(\mathbb{T}^\infty)$ be the norm-closure of $\mathcal{P}_\infty$ in $C(\mathbb{T}^\infty)$, the Banach space of all continuous functions on $\mathbb{T}^\infty$. Since the infinite polydisk algebra $A(\mathbb{D}^\infty)$ is of the Shilov boundary $\mathbb{T}^\infty$, one naturally identifies $A(\mathbb{D}^\infty)$ with $A(\mathbb{T}^\infty)$ by the restriction map.

Proof of Proposition 2.3. (1)⇔(2): By inequalities

$$(2.4) \qquad \log^+ x \leqslant \log(1 + x) \leqslant \log^+ x + \log 2, \quad x \geqslant 0,$$

we see that (1) is equivalent to (2).

(1)⇔(3): It suffices to show (1) implies (3). Given $0 < r_0 < 1$, since F is a nonzero function, $\log |F_{[r_0]}| \in L^1(\mathbb{T}^\infty)$. Then for every $r_0 < r < 1$,

$$\big\| \log \big|F_{[r]}\big| \big\|_1 = 2 \int_{\mathbb{T}^\infty} \log^+ \big|F_{[r]}\big| \, dm_\infty - \int_{\mathbb{T}^\infty} \log \big|F_{[r]}\big| \, dm_\infty$$

$$\leqslant 2 \int_{\mathbb{T}^\infty} \log^+ \big|F_{[r]}\big| \, dm_\infty - \int_{\mathbb{T}^\infty} \log \big|F_{[r_0]}\big| \, dm_\infty,$$

and hence

$$\limsup_{r\to 1} \big\| \log \big|F_{[r]}\big| \big\|_1$$

$$\leqslant 2 \lim_{r\to 1} \int_{\mathbb{T}^\infty} \log^+ \big|F_{[r]}\big| \, dm_\infty - \int_{\mathbb{T}^\infty} \log \big|F_{[r_0]}\big| \, dm_\infty < \infty,$$

as desired.   □

Let F be a function on $\mathbb{D}_1^\infty$. For each $n \in \mathbb{N}$, we denote by $A_n F$ Bohr's $n$te Abschnitt of F, which is defined by

$$(A_n F)(\zeta) = F(\zeta_1, \ldots, \zeta_n, 0, \ldots), \quad \zeta \in \mathbb{D}_1^\infty.$$

Since $A_n F$ does not depend on variables ζ_j $(j > n)$, we may consider it as a function defined on the polydisk $\mathbb{D}^n$. For each $w \in \mathbb{T}^\infty$, the slice function of F at w is defined by

$$F_w(z) = F\left(zw_1, \ldots, z^n w_n, \ldots\right), \quad z \in \mathbb{D}.$$

If F is holomorphic on $\mathbb{D}_1^\infty$, and $w \in \mathbb{T}^\infty$, then for every $n \in \mathbb{N}$,

$$(A_n F)_w(z) = F\left(zw_1, \ldots, z^n w_n, 0, \ldots\right), \quad z \in \mathbb{D}$$

is holomorphic on $\mathbb{D}$. Note that $\{(A_n F)_w\}_{n=1}^\infty$ converges to F_w uniformly on each compact subset of $\mathbb{D}$. It implies that F_w is holomorphic on $\mathbb{D}$.

Let $N(\mathbb{D})$ be the Nevanlinna class over the open unit disk, then for every $F \in N(\mathbb{D})$, the radial limit $F^*(\lambda) = \lim_{r \to 1} F(r\lambda)$ exists for almost every $\lambda \in \mathbb{T}$, see [39, p. 346]. Now we give the following theorem for which its second part is an analogue of the classical Fatou's theorem.

THEOREM 2.4. — *Let F be a nonzero function in $N(\mathbb{D}_1^\infty)$, then for almost every $w \in \mathbb{T}^\infty$, $F_w \in N(\mathbb{D})$. Furthermore, for almost every $w \in \mathbb{T}^\infty$, the radial limit*

$$F^*(w) = \lim_{r \to 1} F_{[r]}(w)$$

exists, and $\log|F^*| \in L^1(\mathbb{T}^\infty)$.

The next corollary implies that every function $F \in N(\mathbb{D}_1^\infty)$ can be uniquely determined by its "boundary function" F^*.

COROLLARY 2.5. — *Let $F \in N(\mathbb{D}_1^\infty)$. If F^* vanishes on some subset of $\mathbb{T}^\infty$ with positive measure, then $F = 0$.*

For every $z \in \overline{\mathbb{D}}$ and $\zeta \in \overline{\mathbb{D}}^\infty$, set

$$(2.5) \qquad\qquad z \star \zeta = (z\zeta_1, \ldots, z^n \zeta_n, \ldots) \in \overline{\mathbb{D}}^\infty.$$

To prove Theorem 2.4, we need the following lemma, an analogue of [38, Lemma 3.3.2].

LEMMA 2.6. — *Let f be a nonnegative measurable function on $\mathbb{T}^\infty$. Then*

$$\int_{\mathbb{T}^\infty} f(w) dm_\infty(w) = \int_{\mathbb{T}^\infty} \int_{\mathbb{T}} f(\lambda \star w) dm_1(\lambda) dm_\infty(w).$$

Proof. — Since the Haar measure m_∞ is rotation-invariant, for every $\lambda \in \mathbb{T}$,

$$\int_{\mathbb{T}^\infty} f(w) dm_\infty(w) = \int_{\mathbb{T}^\infty} f(\lambda \star w) dm_\infty(w).$$

Integrating with respect to λ over $\mathbb{T}$ and applying Fubini's theorem yield that

$$\int_{\mathbb{T}^\infty} f(w)dm_\infty(w) = \int_{\mathbb{T}} \int_{\mathbb{T}^\infty} f(\lambda \star w)dm_\infty(w)dm_1(\lambda)$$

$$= \int_{\mathbb{T}^\infty} \int_{\mathbb{T}} f(\lambda \star w)dm_1(\lambda)dm_\infty(w),$$

which completes the proof.   □

Proof of Theorem 2.4. We use ideas from [38, Theorem 3.3.3] to complete the proof. For every fixed $w \in \mathbb{T}^\infty$, $\log^+ |F_w|$ is subharmonic on $\mathbb{D}$, and hence the integrals

$$I_{w,r} = \int_{\mathbb{T}} \log^+ |F_w(r\lambda)|dm_1(\lambda) \quad (0 < r < 1)$$

increase with r. Then the monotone convergence theorem gives

$$(2.6) \quad \int_{\mathbb{T}^\infty} \left(\sup_{0<r<1} I_{w,r} \right) dm_\infty(w)$$

$$= \sup_{0<r<1} \int_{\mathbb{T}^\infty} \int_{\mathbb{T}} \log^+ |F_w(r\lambda)| \, dm_1(\lambda)dm_\infty(w)$$

$$= \sup_{0<r<1} \int_{\mathbb{T}^\infty} \int_{\mathbb{T}} \log^+ |F_{[r]}(\lambda \star w)| \, dm_1(\lambda)dm_\infty(w).$$

Applying Lemma 2.6 to $\log^+ |F_{[r]}|$, we have

$$\sup_{0<r<1} \int_{\mathbb{T}^\infty} \int_{\mathbb{T}} \log^+ |F_{[r]}(\lambda \star w)| \, dm_1(\lambda)dm_\infty(w)$$

$$= \sup_{0<r<1} \int_{\mathbb{T}^\infty} \log^+ |F_{[r]}(w)| \, dm_\infty(w) < \infty.$$

And hence by (2.6), for almost every $w \in \mathbb{T}^\infty$, $\sup_{0<r<1} I_{w,r}$ is finite, which implies that $F_w \in N(\mathbb{D})$ for such w.

Let E be the set of points $w \in \mathbb{T}^\infty$ for which $F^*(w) = \lim_{r \to 1} F_{[r]}(w)$ exists. Then E is a measurable set. Put

$$S = \left\{ w \in \mathbb{T}^\infty : \lim_{r \to 1} F_w(r\lambda) \text{ exists for almost every } \lambda \in \mathbb{T} \right\}.$$

Since for almost every $w \in \mathbb{T}^\infty$, $F_w \in N(\mathbb{D})$, we see that S is a measurable set, and $m_\infty(S) = 1$. Furthermore, for every fixed $w \in S$, $\lambda \star w \in E$ for almost every $\lambda \in \mathbb{T}$, and thus

$$\int_{\mathbb{T}} \chi_E(\lambda \star w)dm_1(\lambda) = 1,$$

where χ_E denotes the characteristic function of E. Moreover, applying Lemma 2.6 to χ_E gives

$$
\begin{aligned}
m_\infty(E) &= \int_{\mathbb{T}^\infty} \chi_E(w) dm_\infty(w) \\
&= \int_{\mathbb{T}^\infty} \int_{\mathbb{T}} \chi_E(\lambda \star w) dm_1(\lambda) dm_\infty(w) \\
&\geqslant \int_S \int_{\mathbb{T}} \chi_E(\lambda \star w) dm_1(\lambda) dm_\infty(w) \\
&= \int_S 1 dm_\infty(w) \\
&= 1,
\end{aligned}
$$

forcing $m_\infty(E) = 1$. Then for almost every $w \in \mathbb{T}^\infty$,

$$
F^*(w) = \lim_{r \to 1} F_{[r]}(w)
$$

exists, and it follows from Proposition 2.3 and Fatou's lemma that

$$
\left\| \log |F^*| \right\|_1 \leqslant \liminf_{r \to 1} \left\| \log |F_{[r]}| \right\|_1 < \infty,
$$

which implies that $\log |F^*| \in L^1(\mathbb{T}^\infty)$. $\qquad\square$

As done in the finite-variable setting, we characterize Nevanlinna functions via the ∞-harmonic majorant. Motivated by [38], a continuous function $u : \mathbb{D}_1^\infty \to \mathbb{C}$ is called ∞-harmonic, if u is harmonic in each variable separately. The harmonic Hardy space $h^1(\mathbb{D}_1^\infty)$ over $\mathbb{D}_1^\infty$ is defined to be

$$
\begin{aligned}
& h^1(\mathbb{D}_1^\infty) \\
& = \left\{ F \text{ is } \infty\text{-harmonic on } \mathbb{D}_1^\infty : \|F\|_h = \sup_{0 < r < 1} \int_{\mathbb{T}^\infty} |F_{[r]}| dm_\infty < \infty \right\}.
\end{aligned}
$$

As in finite-variable cases, each function in $h^1(\mathbb{D}_1^\infty)$ can be represented by the Poisson integral of a complex regular Borel measure on $\mathbb{T}^\infty$. To show this, we recall some notions of Poisson kernels in infinitely many variables, see [17]. For each $\zeta \in \mathbb{D}_1^\infty$, the Poisson kernel at ζ is defined to be

$$
\mathbf{P}_\zeta(w) = \prod_{n=1}^\infty \mathrm{P}_{\zeta_n}(w_n), \quad w \in \mathbb{T}^\infty,
$$

where

$$
\mathrm{P}_{\zeta_n}(w_n) = \frac{1 - |\zeta_n|^2}{|\zeta_n - w_n|^2}, \quad n \in \mathbb{N}
$$

are Poisson kernels for the unit disk. It is easy to prove that when $\zeta \in \mathbb{D}_1^\infty$, $\mathbf{P}_\zeta$ is continuous on $\mathbb{T}^\infty$. Furthermore, if $u \in C(\overline{\mathbb{D}}^\infty)$ is ∞-harmonic on $\mathbb{D}_1^\infty$, then for every $\zeta \in \mathbb{D}_1^\infty$,

$$(2.7) \qquad u(\zeta) = \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta u \, dm_\infty.$$

Let $M(\mathbb{T}^\infty)$ denote the Banach space of all complex regular Borel measures μ on $\mathbb{T}^\infty$ with the norm $\|\mu\|_M = |\mu|(\mathbb{T}^\infty)$. For every $\mu \in M(\mathbb{T}^\infty)$, set

$$\mathrm{P}[d\mu](\zeta) = \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta d\mu, \quad \zeta \in \mathbb{D}_1^\infty.$$

Then $\mathrm{P}[d\mu]$ is ∞-harmonic on $\mathbb{D}_1^\infty$, and by Fubini's theorem, $\|\mathrm{P}[d\mu]\|_h \leqslant \|\mu\|_M < \infty$, and hence $\mathrm{P}[d\mu] \in h^1(\mathbb{D}_1^\infty)$. Moreover, $\mathrm{P}[d\mu] = 0$ implies $\mu = 0$.

PROPOSITION 2.7. — *The Poisson integral $\mu \mapsto \mathrm{P}[d\mu]$ establishes an isometric isomorphism from $M(\mathbb{T}^\infty)$ onto $h^1(\mathbb{D}_1^\infty)$, and hence $h^1(\mathbb{D}_1^\infty)$ is a Banach space.*

Proof. — By the arguments above, the map $\mu \mapsto \mathrm{P}[d\mu]$ is a one-to-one contraction. On the other hand, for every $F \in h^1(\mathbb{D}_1^\infty)$ and $0 < r < 1$, write $d\mu_r = F_{[r]} dm_\infty$. Then it follows from Banach–Alaoglu theorem that there exist a sequence $r_n \to 1$ $(n \to \infty)$ and a complex Borel measure μ_F on $\mathbb{T}^\infty$ such that $\{\mu_{r_n}\}_{n=1}^\infty$ converges to μ_F in the weak*-topology, in the dual space of $C(\mathbb{T}^\infty)$. Combining this fact with (2.7) shows that for every $\zeta \in \mathbb{D}_1^\infty$,

$$\mathrm{P}[d\mu_F](\zeta) = \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta d\mu_F = \lim_{n \to \infty} \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta F_{[r_n]} dm_\infty = \lim_{n \to \infty} F_{[r_n]}(\zeta) = F(\zeta),$$

which implies that $F = \mathrm{P}[d\mu_F]$, and thus the map $\mu \mapsto \mathrm{P}[d\mu]$ is onto. Furthermore, we have

$$\|\mu_F\|_M \leqslant \liminf_{n \to \infty} \|\mu_{r_n}\|_M \leqslant \sup_{0 < r < 1} \int_{\mathbb{T}^\infty} |F_{[r]}| \, dm_\infty = \|F\|_h,$$

and hence the map $\mu \mapsto \mathrm{P}[d\mu]$ is an isometry. $\square$

Recall that $\mathbf{P}_\zeta dm_\infty$ is a Jensen measure for $\zeta \in \mathbb{D}_1^\infty$ with respect to $A(\mathbb{D}^\infty)$, that is,

$$(2.8) \qquad \log |F(\zeta)| \leqslant \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log |F| dm_\infty, \quad F \in A(\mathbb{D}^\infty),$$

see [17]. This says that

$$G(\zeta) = \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log |F| dm_\infty, \quad \zeta \in \mathbb{D}_1^\infty$$

is an ∞-harmonic majorant of $\log|F|$. The next proposition gives a characterization of functions in $N(\mathbb{D}_1^\infty)$ via ∞-harmonic majorant.

PROPOSITION 2.8. — *Let F be a holomorphic function on $\mathbb{D}_1^\infty$. Then the following statements are equivalent:*

(1) $F \in N(\mathbb{D}_1^\infty)$.
(2) $\log|F|$ *has an ∞-harmonic majorant in $h^1(\mathbb{D}_1^\infty)$. That is, there is a function $H \in h^1(\mathbb{D}_1^\infty)$ such that $\log|F| \leqslant H$ on $\mathbb{D}_1^\infty$.*

Proof. — (1)$\Rightarrow$(2): For every $0 < r < 1$, set $d\nu_r = \log|F_{[r]}|dm_\infty$. Then it follows from (3) of Proposition 2.3 and Banach–Alaoglu theorem that there is a sequence $r_n \to 1$ ($n \to \infty$) and a complex Borel measure ν on $\mathbb{T}^\infty$, such that $\{\nu_{r_n}\}_{n=1}^\infty$ converges to ν in the weak*-topology, in the dual space of $C(\mathbb{T}^\infty)$. Since for every $n \in \mathbb{N}$, $F_{[r_n]} \in A(\mathbb{D}^\infty)$, we see from (2.8) that

$$\log\left|F_{[r_n]}(\zeta)\right| \leqslant \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log\left|F_{[r_n]}\right| dm_\infty, \quad \zeta \in \mathbb{D}_1^\infty.$$

Letting $n \to \infty$ yields that

$$\log|F(\zeta)| \leqslant \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta d\nu, \quad \zeta \in \mathbb{D}_1^\infty,$$

and the right side of this inequality gives an ∞-harmonic majorant of $\log|F|$ in $h^1(\mathbb{D}_1^\infty)$.

(2)$\Rightarrow$(1): By assumptions, we have

$$\sup_{0<r<1} \int_{\mathbb{T}^\infty} \log^+\left|F_{[r]}\right| dm_\infty \leqslant \sup_{0<r<1} \int_{\mathbb{T}^\infty} H_{[r]}^+ dm_\infty$$

$$\leqslant \sup_{0<r<1} \int_{\mathbb{T}^\infty} \left|H_{[r]}\right| dm_\infty < \infty,$$

which gives $F \in N(\mathbb{D}_1^\infty)$. □

We next consider the metric in the class $N(\mathbb{D}_1^\infty)$. Writing $\varphi(x) = \log(1 + e^x)$, then it is a non-decreasing convex function on $\mathbb{R}$. Therefore, for every $F \in N(\mathbb{D}_1^\infty)$, $\varphi(\log|F|) = \log(1 + |F|)$ is ∞-subharmonic. Hence by Proposition 2.1, integrals

$$\int_{\mathbb{T}^\infty} \log\left(1 + \left|F_{[r]}\right|\right) dm_\infty \quad (0 < r < 1)$$

increase with r. As done in [19, 41, 44] for finite-variable cases, we define

$$\|F\|_0 = \sup_{0<r<1} \int_{\mathbb{T}^\infty} \log\left(1 + \left|F_{[r]}\right|\right) dm_\infty = \lim_{r\to 1} \int_{\mathbb{T}^\infty} \log\left(1 + \left|F_{[r]}\right|\right) dm_\infty$$

for $F \in N(\mathbb{D}_1^\infty)$ and

$$d_0(F, G) = \|F - G\|_0, \quad F, G \in N(\mathbb{D}_1^\infty),$$

then d_0 is a translation-invariant metric on the space $N(\mathbb{D}_1^\infty)$. In fact, we have

PROPOSITION 2.9. — $N(\mathbb{D}_1^\infty)$ is complete with respect to the metric d_0.

To prove Proposition 2.9, we need Lemma 2.10, which implies that the evaluation functional at $\zeta \in \mathbb{D}_1^\infty$ on $N(\mathbb{D}_1^\infty)$ is continuous, and will be used frequently in the sequel.

Recall that for any $\zeta \in \mathbb{D}^\infty$, the Möbius map over the infinite-dimensional polydisk is defined as

$$\Phi_\zeta(w) = \left(\frac{\zeta_1 - w_1}{1 - \overline{\zeta}_1 w_1}, \frac{\zeta_2 - w_2}{1 - \overline{\zeta}_2 w_2}, \dots \right), \quad w \in \overline{\mathbb{D}}^\infty.$$

Then Φ_ζ maps $\mathbb{D}^\infty$ and $\mathbb{T}^\infty$ onto themselves, respectively. Moreover, a direct verification shows that Φ_ζ maps $\mathbb{D}_1^\infty$ onto itself if and only if $\zeta \in \mathbb{D}_1^\infty$. When $\zeta \in \mathbb{D}_1^\infty$, an observation is that for every nonnegative continuous function f on $\mathbb{T}^\infty$,

$$(2.9) \qquad \int_{\mathbb{T}^\infty} f \circ \Phi_\zeta \, dm_\infty = \int_{\mathbb{T}^\infty} f \mathbf{P}_\zeta \, dm_\infty \leqslant \|\mathbf{P}_\zeta\|_\infty \int_{\mathbb{T}^\infty} f \, dm_\infty.$$

LEMMA 2.10. — If $\zeta \in \mathbb{D}_1^\infty$, then for every $F \in N(\mathbb{D}_1^\infty)$,

$$\log(1 + |F(\zeta)|) \leqslant \|\mathbf{P}_\zeta\|_\infty \|F\|_0.$$

Proof. — For each $0 < r < 1$, $\log(1 + |F_{[r]} \circ \Phi_\zeta|) \in C(\overline{\mathbb{D}}^\infty)$ is ∞-subharmonic on $\mathbb{D}_1^\infty$. By Corollary 2.2,

$$\log\left(1 + \left|F_{[r]}(\zeta)\right|\right) = \log\left(1 + \left|\left(F_{[r]} \circ \Phi_\zeta\right)(0)\right|\right)$$

$$(2.10) \qquad\qquad\qquad \leqslant \int_{\mathbb{T}^\infty} \log\left(1 + \left|F_{[r]} \circ \Phi_\zeta\right|\right) dm_\infty.$$

Applying the inequality (2.9) to $\log(1 + |F_{[r]}|)$, we have

$$\int_{\mathbb{T}^\infty} \log\left(1 + \left|F_{[r]} \circ \Phi_\zeta\right|\right) dm_\infty \leqslant \|\mathbf{P}_\zeta\|_\infty \int_{\mathbb{T}^\infty} \log\left(1 + \left|F_{[r]}\right|\right) dm_\infty.$$

Combining this with (2.10) yields that

$$\log\left(1 + \left|F_{[r]}(\zeta)\right|\right) \leqslant \|\mathbf{P}_\zeta\|_\infty \int_{\mathbb{T}^\infty} \log\left(1 + \left|F_{[r]}\right|\right) dm_\infty.$$

Letting $r \to 1$ gives the desired conclusion.   □

As defined earlier, for every $0 < r < 1$ and $M > 0$, the domain

$$V_{r,M} = \left\{ \zeta \in \ell^1 : \|\zeta\|_1 < M \text{ and for each } n \in \mathbb{N}, \ |\zeta_n| < r \right\}.$$

One can easily check that $\{\mathbf{P}_\zeta\}_{\zeta \in V_{r,M}}$ is a bounded set in $L^\infty(\mathbb{T}^\infty)$. We now present the proof of Proposition 2.9.

Proof of Proposition 2.9. Suppose that $\{F_n\}_{n=1}^\infty$ is a Cauchy sequence in $N(\mathbb{D}_1^\infty)$. By Lemma 2.10, for each $\zeta \in \mathbb{D}_1^\infty$ and $k, l \in \mathbb{N}$,

$$\log\left(1 + |(F_k - F_l)(\zeta)|\right) \leqslant \|\mathbf{P}_\zeta\|_\infty \|F_k - F_l\|_0.$$

Note that for each $0 < r < 1$ and $M > 0$, $\{\mathbf{P}_\zeta\}_{\zeta \in V_{r,M}}$ is bounded in $L^\infty(\mathbb{T}^\infty)$. It follows that $\{F_n\}_{n=1}^\infty$ converges uniformly to a holomorphic function on each $V_{r,M}$, and hence there exists a holomorphic function F on $\mathbb{D}_1^\infty$, such that $\{F_n\}_{n=1}^\infty$ converges to F uniformly on each $V_{r,M}$. It remains to prove $F \in N(\mathbb{D}_1^\infty)$ and $\|F_k - F\|_0 \to 0$ as $k \to \infty$. Given $\varepsilon > 0$, choose N large enough such that for $k, l > N$, $\|F_k - F_l\|_0 < \varepsilon$. Then for each $0 < r < 1$,

$$\int_{\mathbb{T}^\infty} \log\left(1 + \left|(F_k - F_l)_{[r]}\right|\right) dm_\infty < \varepsilon.$$

Note that $\{F_l\}_{l=1}^\infty$ converges to F uniformly on $r\overline{\mathbb{D}} \times \cdots \times r^n \overline{\mathbb{D}} \times \cdots$. Letting $l \to \infty$ yields that when $k > N$,

$$(2.11) \qquad \int_{\mathbb{T}^\infty} \log\left(1 + \left|(F_k - F)_{[r]}\right|\right) dm_\infty \leqslant \varepsilon,$$

and hence for all $0 < r < 1$,

$$\int_{\mathbb{T}^\infty} \log\left(1 + \left|F_{[r]}\right|\right) dm_\infty \leqslant \varepsilon + \int_{\mathbb{T}^\infty} \log\left(1 + \left|(F_k)_{[r]}\right|\right) dm_\infty$$
$$\leqslant \varepsilon + \|F_k\|_0 < \infty.$$

This shows $F \in N(\mathbb{D}_1^\infty)$. Furthermore, by (2.11), we have $\|F_k - F\|_0 \leqslant \varepsilon$, which means that $N(\mathbb{D}_1^\infty)$ is complete with respect to d_0. $\qquad\square$

An important subclass of $N(\mathbb{D}_1^\infty)$ is the Smirnov class $N_*(\mathbb{D}_1^\infty)$, which consists of all functions $F \in N(\mathbb{D}_1^\infty)$ for which $\{\log^+ |F_{[r]}|\}_{0 < r < 1}$ forms a uniformly integrable family. When $F \in N(\mathbb{D}_1^\infty)$, (2.4) implies that $F \in N_*(\mathbb{D}_1^\infty)$ if and only if $\{\log(1+|F_{[r]}|)\}_{0<r<1}$ is uniformly integrable over $\mathbb{T}^\infty$.

By Fatou's lemma, for every function $F \in N(\mathbb{D}_1^\infty)$, using the metric in $L^0(\mathbb{T}^\infty)$,

$$\|F^*\|_0 \leqslant \lim_{r \to 1} \|F_{[r]}\|_0.$$

This inequality inspires the following characterization of functions in the Smirnov class $N_*(\mathbb{D}_1^\infty)$. For finite-variable cases, see [19, 41, 44].

PROPOSITION 2.11. — *Let F be a function in $N(\mathbb{D}_1^\infty)$. Then the following statements are equivalent:*

 (1) $F \in N_*(\mathbb{D}_1^\infty)$.
 (2) $\|F^*\|_0 = \lim_{r \to 1} \|F_{[r]}\|_0$.
 (3) $\lim_{r \to 1} \|F_{[r]} - F^*\|_0 = 0$.
 (4) $\lim_{r \to 1} \int_{\mathbb{T}^\infty} \log^+ |F_{[r]}| dm_\infty = \int_{\mathbb{T}^\infty} \log^+ |F^*| dm_\infty$.

It is worth mentioning that when F only depends on single variable, the equality (2) in Proposition 2.11 is another common definition for the Smirnov class $N_*(\mathbb{D})$.

To prove Proposition 2.11, the the general Lebesgue's dominated convergence theorem [12, Theorem 2.8.8] is needed.

LEMMA 2.12 (General Lebesgue's dominated convergence theorem). —
Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ two sequences of measurable functions on X that converge to f, g almost everywhere on X, respectively. Assume that for every $n \in \mathbb{N}$, $g_n \geqslant 0$, and $|f_n| \leqslant g_n$ on X. If $g \in L^1(X, \mu)$ and

$$\lim_{n \to \infty} \int_X g_n d\mu = \int_X g d\mu,$$

then

$$\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu.$$

Furthermore, this conclusion remain valid if "convergence almost everywhere" is replaced by "convergence in measure".

The following lemma immediately follows.

LEMMA 2.13. — Let $0 \leqslant p < \infty$ and $\{h_n\}_{n=1}^{\infty}$ be a sequence in $L^p(\mathbb{T}^{\infty})$ that converges to $h \in L^p(\mathbb{T}^{\infty})$ almost everywhere on $\mathbb{T}^{\infty}$. Then $\|h_n - h\|_p \to 0$ if and only if $\|h_n\|_p \to \|h\|_p$ as $n \to \infty$.

Proof. — We prove this lemma by using the general Lebesgue's dominated convergence theorem. It suffices to show that $\|h_n\|_p \to \|h\|_p$ implies $\|h_n - h\|_p \to 0$. Write

$$f_n = \begin{cases} \log\left(1 + |h_n - h|\right), & p = 0, \\ |h_n - h|^p, & p > 0, \end{cases}$$

and

$$g_n = \begin{cases} \log\left(1 + |h_n|\right) + \log(1 + |h|), & p = 0, \\ 2^p\left(|h_n|^p + |h|^p\right), & p > 0. \end{cases}$$

For every $n \in \mathbb{N}$, it is clear that $0 \leqslant f_n \leqslant g_n$ on $\mathbb{T}^{\infty}$. Applying Lemma 2.12 to sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ shows that $\|h_n - h\|_p \to 0$ as $n \to \infty$. $\square$

It is worth mentioning that when $0 < p < \infty$, [39, pp. 73, Exercise 17] presents two proofs for Lemma 2.13 using Egoroff's theorem and Fatou's lemma, respectively.

Recall that a subset Λ of $L^1(\mathbb{T}^\infty)$ is uniformly integrable if and only if there is a non-decreasing convex function $\varphi : \mathbb{R} \to [0, \infty)$ satisfying $\frac{\varphi(t)}{t} \to \infty$ as $t \to +\infty$, called strongly convex function, such that $\{\varphi \circ |f|\}_{f \in \Lambda}$ is bounded in $L^1(\mathbb{T}^\infty)$, see [12, Theorem 4.5.9].

We now present the proof of Proposition 2.11.

Proof of Proposition 2.11. $(1) \Rightarrow (2)$: When $F \in N_*(\mathbb{D}_1^\infty)$, the family of functions $\{\log(1 + |F_{[r]}|)\}_{0 < r < 1}$ is uniformly integrable over $\mathbb{T}^\infty$. Applying Lebesgue–Vitali's theorem [12, Theorem 4.5.4], we conclude that $\|F_{[r]}\|_0 \to \|F^*\|_0$ as $r \to 1$.

$(2) \Rightarrow (3)$: It immediately follows from Lemma 2.13.

$(3) \Rightarrow (4)$: By the inequality

$$(2.12) \qquad \left|\log^+ x - \log^+ y\right| \leqslant \log(1 + |x - y|), \quad x, y \geqslant 0,$$

we have

$$\left| \int_{\mathbb{T}^\infty} \log^+ |F_{[r]}| \, dm_\infty - \int_{\mathbb{T}^\infty} \log^+ |F^*| \, dm_\infty \right| \leqslant \left\|F_{[r]} - F^*\right\|_0 \to 0 \quad (r \to 1),$$

which implies (4).

$(4) \Rightarrow (1)$: Let $\{r_n\}_{n=1}^\infty$ be an increasing sequence in $(0, 1)$ with $r_n \to 1$ as $n \to \infty$. By (4) and Lemma 2.13, the sequence $\{\log^+ |F_{[r_n]}|\}_{n=1}^\infty$ converges to $\log^+ |F^*|$ in $L^1(\mathbb{T}^\infty)$. Then it follows from Lebesgue–Vitali's theorem that $\{\log^+ |F_{[r_n]}|\}_{n=1}^\infty$ is uniformly integrable over $\mathbb{T}^\infty$. Therefore, there is a strongly convex function φ such that $\{\varphi(\log^+ |F_{[r_n]}|)\}_{n=1}^\infty$ is bounded in $L^1(\mathbb{T}^\infty)$. By the ∞–subharmonicity of $\varphi(\log^+ |F|)$, applying Proposition 2.1 shows that $\{\varphi(\log^+ |F_{[r]}|)\}_{0 < r < 1}$ is also bounded in $L^1(\mathbb{T}^\infty)$. Therefore, the family $\{\log^+ |F_{[r]}|\}_{0 < r < 1}$ is uniformly integrable, which implies $F \in N_*(\mathbb{D}_1^\infty)$. $\qquad\square$

The following Proposition immediately follows from Proposition 2.11.

PROPOSITION 2.14. — $N_*(\mathbb{D}_1^\infty)$ is a closed subclass of $N(\mathbb{D}_1^\infty)$.

Proof. — Assume that $\{G_n\}_{n=1}^\infty$ is a sequence in $N_*(\mathbb{D}_1^\infty)$, and $\|G_n - G\|_0 \to 0$ for some $G \in N(\mathbb{D}_1^\infty)$ as $n \to \infty$. To prove $G \in N_*(\mathbb{D}_1^\infty)$, let $0 < r < 1$ and $n \in \mathbb{N}$, then

$$\left\|G_{[r]} - G^*\right\|_0 \leqslant \left\|G_{[r]} - (G_n)_{[r]}\right\|_0 + \left\|(G_n)_{[r]} - G_n^*\right\|_0 + \left\|G_n^* - G^*\right\|_0$$
$$\leqslant 2\|G_n - G\|_0 + \left\|(G_n)_{[r]} - G_n^*\right\|_0.$$

Since $G_n \in N_*(\mathbb{D}_1^\infty)$, combining the above inequality with Proposition 2.11 shows that

$$\limsup_{r \to 1} \left\|G_{[r]} - G^*\right\|_0 \leqslant 2\|G_n - G\|_0.$$

Letting $n \to \infty$, we see that $\|G_{[r]} - G^*\|_0 \to 0$ as $r \to 1$. Again by Proposition 2.11, $G \in N_*(\mathbb{D}_1^\infty)$. □

Applying the inequality

$$\log(1 + xy) \leqslant \log(1 + x) + \log(1 + y), \quad x, y \geqslant 0$$

yields that the Nevanlinna class $N(\mathbb{D}_1^\infty)$ is an algebra, and the Smirnov class $N_*(\mathbb{D}_1^\infty)$ is its subalgebra. Unfortunately, as shown in [19, 41] for one-variable case, the Nevanlinna class $N(\mathbb{D}_1^\infty)$ is not a topological linear space since the scalar multiplication is not continuous. However, the following proposition shows that the Smirnov class $N_*(\mathbb{D}_1^\infty)$ is in fact a topological algebra.

PROPOSITION 2.15. — *If* $\{F_n\}_{n=1}^\infty$ *and* $\{G_n\}_{n=1}^\infty$ *are two sequences in* $N_*(\mathbb{D}_1^\infty)$ *that converge to* F, G *in* $N_*(\mathbb{D}_1^\infty)$, *respectively, then* $\{F_n G_n\}_{n=1}^\infty$ *converges to* FG *in* $N_*(\mathbb{D}_1^\infty)$.

Proof. — From Proposition 2.11, we have $\|F_n^* - F^*\|_0 \to 0$, and $\|G_n^* - G^*\|_0 \to 0$ as $n \to \infty$. Passing to subsequences, assume that $\{F_n^*\}_{n=1}^\infty$ and $\{G_n^*\}_{n=1}^\infty$ converge almost everywhere to F^* and G^* on $\mathbb{T}^\infty$, respectively. Write

$$f_n = \log\left(1 + |F_n^* G_n^* - F^* G^*|\right)$$

and

$$g_n = \log(1 + |F_n^*|) + \log(1 + |G_n^* - G^*|) + \log(1 + |F_n^* - F^*|) + \log(1 + |G^*|).$$

It is easy to verify that for every $n \in \mathbb{N}$, $0 \leqslant f_n \leqslant g_n$ on $\mathbb{T}^\infty$. Applying Lemma 2.12 to sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ shows that $\|F_n^* G_n^* - F^* G^*\|_0 \to 0$ as $n \to \infty$. Then by Proposition 2.11, $\|F_n G_n - FG\|_0 \to 0$ as $n \to \infty$, which completes the proof. □

By Proposition 2.8, for every $F \in N(\mathbb{D}_1^\infty)$, $\log |F|$ has an ∞-harmonic majorant in $h^1(\mathbb{D}_1^\infty)$. The following proposition shows that if, furthermore, $F \in N_*(\mathbb{D}_1^\infty)$, the ∞-harmonic majorant can be taken as $\mathrm{P}[\log |F^*|dm_\infty]$. The finite-variable version of this proposition is presented in [38].

PROPOSITION 2.16. — *Suppose that* $F \in N(\mathbb{D}_1^\infty)$. *Then* $F \in N_*(\mathbb{D}_1^\infty)$ *if and only if for every* $\zeta \in \mathbb{D}_1^\infty$,

$$(2.13) \qquad \log |F(\zeta)| \leqslant \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log |F^*| dm_\infty.$$

Proof. — We first assume that $F \in N_*(\mathbb{D}_1^\infty)$. Then for each $0 < r < 1$, $F_{[r]} \in A(\mathbb{D}^\infty)$, and (2.8) implies that for $\zeta \in \mathbb{D}_1^\infty$,

$$(2.14) \qquad \log \left|F_{[r]}(\zeta)\right| \leqslant \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log \left|F_{[r]}\right| dm_\infty.$$

Since $F \in N_*(\mathbb{D}_1^\infty)$ and $\mathbf{P}_\zeta \in C(\mathbb{T}^\infty)$, the family $\{\mathbf{P}_\zeta \log^+ |F_{[r]}|\}_{0<r<1}$ is uniformly integrable. By Lebesgue–Vitali's theorem [12, Theorem 4.5.4],

$$(2.15) \qquad \lim_{r\to 1} \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log^+ |F_{[r]}|\, dm_\infty = \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log^+ |F^*|\, dm_\infty.$$

On the other hand, it follows from Fatou's lemma that

$$(2.16) \qquad \liminf_{r\to 1} \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log^- |F_{[r]}|\, dm_\infty \geqslant \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log^- |F^*|\, dm_\infty,$$

where $\log^- x = -\min\{0, \log x\}$ for $x > 0$. By (2.15) and (2.16) we see that

$$\limsup_{r\to 1} \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log |F_{[r]}|\, dm_\infty \leqslant \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log |F^*|\, dm_\infty.$$

Letting $r \to 1$ in (2.14) shows that

$$\log |F(\zeta)| \leqslant \limsup_{r\to 1} \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log |F_{[r]}|\, dm_\infty \leqslant \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log |F^*| dm_\infty.$$

Conversely, suppose that $F \neq 0$ and (2.13) holds for every $\zeta \in \mathbb{D}_1^\infty$, then for each $0 < r < 1$ and $w \in \mathbb{T}^\infty$,

$$\log |F_{[r]}(w)| \leqslant \int_{\mathbb{T}^\infty} \mathbf{P}_{r\star w} \log |F^*| dm_\infty,$$

where $r \star w \in \mathbb{D}_1^\infty$ by definition (2.5). Since $\log |F^*| \in L^1(\mathbb{T}^\infty)$, there is a strongly convex function φ such that $\varphi(\log |F^*|) \in L^1(\mathbb{T}^\infty)$. As done in finite-variable cases by Rudin [38], applying the convexity of φ and Fubini's theorem, we obtain that for $0 < r < 1$,

$$(2.17) \quad \int_{\mathbb{T}^\infty} \varphi \left(\log |F_{[r]}| \right) dm_\infty$$

$$\leqslant \int_{\mathbb{T}^\infty} \varphi \left(\int_{\mathbb{T}^\infty} \mathbf{P}_{r\star w}(\xi) \log |F^*(\xi)| dm_\infty(\xi) \right) dm_\infty(w)$$

$$\leqslant \int_{\mathbb{T}^\infty} \int_{\mathbb{T}^\infty} \mathbf{P}_{r\star w}(\xi) \varphi(\log |F^*(\xi)|) dm_\infty(\xi) dm_\infty(w)$$

$$= \int_{\mathbb{T}^\infty} \left(\int_{\mathbb{T}^\infty} \mathbf{P}_{r\star w}(\xi) dm_\infty(w) \right) \varphi(\log |F^*(\xi)|) dm_\infty(\xi)$$

$$= \int_{\mathbb{T}^\infty} \varphi(\log |F^*|) dm_\infty.$$

Let E_r be the set of points $w \in \mathbb{T}^\infty$ such that $|F_{[r]}(w)| < 1$, and E_r^c the complement of E_r with respect to $\mathbb{T}^\infty$. Then (2.17) implies that

$$\int_{\mathbb{T}^\infty} \varphi \left(\log^+ |F_{[r]}| \right) dm_\infty = \int_{E_r} \varphi(0) dm_\infty + \int_{E_r^c} \varphi \left(\log |F_{[r]}| \right) dm_\infty$$

(2.18)
$$\leqslant \varphi(0) + \int_{\mathbb{T}^\infty} \varphi \left(\log |F_{[r]}| \right) dm_\infty$$

$$\leqslant \varphi(0) + \int_{\mathbb{T}^\infty} \varphi(\log |F^*|) dm_\infty.$$

Therefore, the family $\{\log^+ |F_{[r]}|\}_{0<r<1}$ is uniformly integrable over $\mathbb{T}^\infty$, which implies $F \in N_*(\mathbb{D}_1^\infty)$. $\qquad\square$

Motivated by Proposition 2.16, we ask when the inequality in this proposition attains an equality at some point, and such a nonzero function is called an outer function.

PROPOSITION 2.17. — *If F is a nonzero function in $N_*(\mathbb{D}_1^\infty)$, then the following statements are equivalent:*

(1) *F is outer.*
(2) *For every $\zeta \in \mathbb{D}_1^\infty$, $\log |F(\zeta)| = \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log |F^*| dm_\infty$.*
(3) *$\log |F(0)| = \int_{\mathbb{T}^\infty} \log |F^*| dm_\infty$.*

To prove Proposition 2.17, the maximum principle of ∞-subharmonic functions is needed.

LEMMA 2.18 (The maximum principle). — *Let u be an ∞-subharmonic function on $\mathbb{D}_1^\infty$, and $a \in \mathbb{R}$. If $u \leqslant a$ on $\mathbb{D}_1^\infty$ and $u(\eta) = a$ for some $\eta \in \mathbb{D}_1^\infty$, then $u \equiv a$ on $\mathbb{D}_1^\infty$.*

Proof. — For every $n \in \mathbb{N}$, set

$$u_n(z) = u(z_1, \ldots, z_n, \eta_{n+1}, \eta_{n+2}, \ldots), \quad z = (z_1, \ldots, z_n) \in \mathbb{D}^n.$$

Then $u_n \leqslant a$ is subharmonic in variables $z_1, \ldots, z_n$ separately, and

$$u_n(\eta_1 \ldots, \eta_n) = a.$$

By using the subharmonicity successively in each variable, we see that $u_n \equiv a$ on $\mathbb{D}^n$. Now it follows from the upper semicontinuity of u that for every $\zeta \in \mathbb{D}_1^\infty$,

$$u(\zeta) \geqslant \limsup_{n \to \infty} u(\zeta_1, \ldots, \zeta_n, \eta_{n+1}, \eta_{n+2}, \ldots) = \limsup_{n \to \infty} u_n(\zeta_1, \ldots, \zeta_n) = a,$$

which leads to the desired conclusion. $\qquad\square$

Proof of Proposition 2.17. It suffices to show that (1) implies (2). Put

$$\rho(\zeta) = \log |F(\zeta)| - \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log |F^*| dm_\infty, \quad \zeta \in \mathbb{D}_1^\infty.$$

Since $\log |F^*| \in L^1(\mathbb{T}^\infty)$, ρ is ∞-subharmonic on $\mathbb{D}_1^\infty$. By Proposition 2.16, $\rho \leqslant 0$ on $\mathbb{D}_1^\infty$. Noticing that $\rho(\xi) = 0$ for some $\xi \in \mathbb{D}_1^\infty$, it follows from Lemma 2.18 that ρ is identically zero on $\mathbb{D}_1^\infty$, as desired. $\qquad\square$

For every nonzero function $F \in N_*(\mathbb{D}_1^\infty)$ and $\zeta \in \mathbb{D}_1^\infty$, $\mathbf{P}_\zeta \log |F^*| \in L^1(\mathbb{T}^\infty)$. A combination of this fact and Proposition 2.17 immediately gives the following result.

COROLLARY 2.19. — *If $F \in N_*(\mathbb{D}_1^\infty)$ is outer, then F is zero-free in $\mathbb{D}_1^\infty$.*

For $0 < p \leqslant \infty$, the proof of Proposition 2.16 implies that $H^p(\mathbb{D}_2^\infty) \subset N_*(\mathbb{D}_1^\infty)$. Therefore, when $F \in H^p(\mathbb{D}_2^\infty)$ is outer, F is zero-free in $\mathbb{D}_1^\infty$. In fact, it is also zero-free in $\mathbb{D}_2^\infty$.

COROLLARY 2.20. — *Let $0 < p \leqslant \infty$. If $F \in H^p(\mathbb{D}_2^\infty)$ is outer, then F is zero-free in $\mathbb{D}_2^\infty$.*

Proof. — Assume that there exists $\zeta \in \mathbb{D}_2^\infty$ such that $F(\zeta) = 0$. Choose $\eta \in \mathbb{D}$ and an integer N such that

$$\left(\eta^{-1}\zeta_1, \ldots, \eta^{-1}\zeta_N, 2\zeta_{N+1}, 2\zeta_{N+2}, \ldots\right) \in \mathbb{D}_2^\infty.$$

For each $n \in \mathbb{N}$ and $z = (z_1, z_2) \in \mathbb{D}^2$, set

$$\phi_n(z) = F\left(z_1\eta^{-1}\zeta_1, \ldots, z_1\eta^{-1}\zeta_N, 2z_2\zeta_{N+1}, \ldots, 2z_2\zeta_{N+n}, 0, \ldots\right),$$

and

$$\phi(z) = F\left(z_1\eta^{-1}\zeta_1, \ldots, z_1\eta^{-1}\zeta_N, 2z_2\zeta_{N+1}, 2z_2\zeta_{N+2}, \ldots\right).$$

Then ϕ_n is holomorphic on $\mathbb{D}^2$, and $\{\phi_n\}_{n=1}^\infty$ converges uniformly to ϕ on each compact subset of $\mathbb{D}^2$. Therefore, ϕ is holomorphic on $\mathbb{D}^2$. Since F is zero-free in $\mathbb{D}_1^\infty$, ϕ_n is zero-free in $\mathbb{D}^2$. Noticing that $\phi(\eta, \frac{1}{2}) = 0$, Hurwitz's theorem [29, pp. 310, Exercise 3] implies that ϕ is identically zero on $\mathbb{D}^2$, and hence

$$F(\zeta_1, \ldots, \zeta_N, 0, \ldots) = \phi(\eta, 0) = 0,$$

a contradiction to that F is zero-free in $\mathbb{D}_1^\infty$. $\qquad\square$

The following corollary will be used in Section 3.

COROLLARY 2.21. — *If $F \in N_*(\mathbb{D}_1^\infty)$ is outer, then $\frac{1}{F} \in N_*(\mathbb{D}_1^\infty)$ is outer.*

Proof. — Since F is outer, it follows from Corollary 2.19 that F is zero-free in $\mathbb{D}_1^\infty$. Hence by Proposition 2.3, $\frac{1}{F} \in N(\mathbb{D}_1^\infty)$. On the other hand, for every $\zeta \in \mathbb{D}_1^\infty$,

$$\log |F(\zeta)| = \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log |F^*| dm_\infty,$$

and hence

$$\log \left| \frac{1}{F(\zeta)} \right| = \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log \left| \frac{1}{F^*} \right| dm_\infty.$$

Combining this equality and Proposition 2.16 gives $\frac{1}{F} \in N_*(\mathbb{D}_1^\infty)$ is outer, as desired. □

In what follows we consider cyclic vectors. A closed subspace $S \subset N_*$ $(\mathbb{D}_1^\infty)$ is said to be invariant, if for every $F \in S$ and $q \in \mathcal{P}_\infty$, $qF \in S$. We say that a function $F \in N_*(\mathbb{D}_1^\infty)$ is cyclic, if the invariant subspace generated by F is exactly $N_*(\mathbb{D}_1^\infty)$. For $0 < p < \infty$, cyclic vectors in the Hardy space $H^p(\mathbb{D}_2^\infty)$ can be defined similarly.

The following theorem gives a quantitative description of cyclic vectors in $N_*(\mathbb{D}_1^\infty)$. Although its proof is similar as in [38, Theorem 4.4.6], we present it here for completeness.

THEOREM 2.22. — *Each cyclic vector in $N_*(\mathbb{D}_1^\infty)$ is outer.*

Proof. — For a nonzero function $F \in N_*(\mathbb{D}_1^\infty)$, set

$$\Gamma F = \int_{\mathbb{T}^\infty} \log |F^*| dm_\infty - \log |F(0)|.$$

Then Proposition 2.17 yields that F is outer if and only if $\Gamma F = 0$. We claim that Γ is upper semicontinuous on $N_*(\mathbb{D}_1^\infty)\backslash\{0\}$. Indeed, let $\{G_n\}_{n=1}^\infty$ be a sequence in $N_*(\mathbb{D}_1^\infty)\backslash\{0\}$ and $G \in N_*(\mathbb{D}_1^\infty)\backslash\{0\}$ such that for all $n \in \mathbb{N}$, $\Gamma G_n \geqslant c \geqslant 0$, and $\|G_n - G\|_0 \to 0$ as $n \to \infty$. Hence by (2.12) and Proposition 2.11,

$$\left| \int_{\mathbb{T}^\infty} \log^+ |G_n^*| \, dm_\infty - \int_{\mathbb{T}^\infty} \log^+ |G^*| \, dm_\infty \right| \leqslant \|G_n^* - G^*\|_0 = \|G_n - G\|_0,$$

and thus

$$(2.19) \qquad \lim_{n \to \infty} \int_{\mathbb{T}^\infty} \log^+ |G_n^*| dm_\infty = \int_{\mathbb{T}^\infty} \log^+ |G^*| dm_\infty.$$

On the other hand, by the convergence of $\{G_n^*\}_{n=1}^\infty$ in $L^0(\mathbb{T}^\infty)$, there exists a subsequence $\{G_{n_k}^*\}_{k=1}^\infty$ converging to G^* almost everywhere on $\mathbb{T}^\infty$. Then it follows from Fatou's lemma that

$$(2.20) \qquad \int_{\mathbb{T}^\infty} \log^- |G^*| \, dm_\infty \leqslant \liminf_{k \to \infty} \int_{\mathbb{T}^\infty} \log^- \left| G_{n_k}^* \right| dm_\infty.$$

A combination of (2.19) and (2.20) shows that

$$\int_{\mathbb{T}^\infty} \log |G^*|\, dm_\infty \geqslant \limsup_{k\to\infty} \int_{\mathbb{T}^\infty} \log \left|G^*_{n_k}\right| dm_\infty.$$

Noticing that $\|G_n - G\|_0 \to 0$ as $n \to \infty$, Lemma 2.10 gives $G_n(0) \to G(0)$ as $n \to \infty$. Therefore, $\Gamma G \geqslant \limsup_{k\to\infty} \Gamma G_{n_k} \geqslant c$, and hence the claim holds.

Now we assume that F is cyclic in $N_*(\mathbb{D}_1^\infty)$, then there is a sequence of polynomials $\{q_n\}_{n=1}^\infty$ in $\mathcal{P}_\infty$ such that $\|q_n F - 1\|_0 \to 0$ $(n \to \infty)$. Hence

$$0 = \Gamma 1 \geqslant \limsup_{n\to\infty} \Gamma(q_n F) \geqslant \limsup_{n\to\infty} \Gamma q_n + \Gamma F \geqslant \Gamma F,$$

forcing $\Gamma F = 0$, and thus F is outer. $\qquad\square$

We mention that for every $0 < p < \infty$, there is a constant $C_p > 0$ such that

$$(2.21) \qquad \|f\|_0 \leqslant C_p \|f\|_p^{\min\{p,1\}}, \quad f \in L^p(\mathbb{T}^\infty).$$

Hence for every $F \in H^p(\mathbb{D}_2^\infty) \subset N_*(\mathbb{D}_1^\infty)$,

$$\|F\|_0 = \sup_{0<r<1} \left\|F_{[r]}\right\|_0 \leqslant C_p \sup_{0<r<1} \left\|F_{[r]}\right\|_p = C_p \|F\|_p.$$

A combination of this fact and Theorem 2.22 gives the following corollary.

COROLLARY 2.23. — *If $0 < p < \infty$, then each cyclic vector in $H^p(\mathbb{D}_2^\infty)$ is outer.*

However, not every outer function is cyclic in $H^p(\mathbb{D}_2^\infty)$, see [38, pp. 75-78] for a counterexample. We also conjecture that not every outer function is cyclic in $N_*(\mathbb{D}_1^\infty)$, but we still cannot construct a counterexample, even in finite-variable setting.

2.2. Correspondence between $N_*(\mathbb{T}^\infty)$ and $N_*(\mathbb{D}_1^\infty)$

The purpose of this subsection is to show that there is a canonical isometric isomorphism between $N_*(\mathbb{T}^\infty)$ and $N_*(\mathbb{D}_1^\infty)$. We refer readers to [3, 18] for cases of the Hardy spaces in infinitely many variables.

Recall that the Smirnov class $N_*(\mathbb{T}^\infty)$ over the infinite torus is defined to be the closure of $\mathcal{P}_\infty$ in $L^0(\mathbb{T}^\infty)$. A similar argument as in Proposition 2.15 shows that $N_*(\mathbb{T}^\infty)$ is a topological algebra. By (2.21), for every $0 < p < \infty$, the space $H^p(\mathbb{T}^\infty)$ is contained in $N_*(\mathbb{T}^\infty)$. For each point $\zeta \in \mathbb{D}_1^\infty$ and $q \in \mathcal{P}_\infty$, it follows from Lemma 2.10 that

$$\log(1 + |q(\zeta)|) \leqslant \|\mathbf{P}_\zeta\|_\infty \int_{\mathbb{T}^\infty} \log(1 + |q|) dm_\infty.$$

This means that the evaluation functional $E_\zeta : q \mapsto q(\zeta)$ is continuous on the dense subspace $\mathcal{P}_\infty$ of $N_*(\mathbb{T}^\infty)$, and hence E_ζ can be continuously extended to the whole space $N_*(\mathbb{T}^\infty)$, still denoted by E_ζ. For every $f \in N_*(\mathbb{T}^\infty)$ and $\zeta \in \mathbb{D}_1^\infty$, set $\widetilde{f}(\zeta) = E_\zeta f$. Then $\widetilde{f}$ defines a function on $\mathbb{D}_1^\infty$, and

$$(2.22) \qquad \log\left(1 + \left|\widetilde{f}(\zeta)\right|\right) \leqslant \|\mathbf{P}_\zeta\|_\infty \int_{\mathbb{T}^\infty} \log(1 + |f|)dm_\infty.$$

When $q \in \mathcal{P}_\infty$, $\widetilde{q} = q$ on $\mathbb{D}_1^\infty$, so we will no longer distinguish between q and $\widetilde{q}$.

For $0 < r < 1$ and $M > 0$, the domain $V_{r,M}$ in ℓ^1 is defined by (2.1). The following lemma is obvious by using (2.22). It will be used not only in the proof of Propostion 2.25, but also in Section 3.

LEMMA 2.24. — *Suppose that $\{f_n\}_{n=1}^\infty$ is a sequence in $N_*(\mathbb{T}^\infty)$. If $\|f_n - f\|_0 \to 0$ for some $f \in N_*(\mathbb{T}^\infty)$ as $n \to \infty$, then $\{\widetilde{f}_n\}_{n=1}^\infty$ converges to $\widetilde{f}$ uniformly on each $V_{r,M}$.*

We now establish the correspondence between $N_*(\mathbb{T}^\infty)$ and $N_*(\mathbb{D}_1^\infty)$.

PROPOSITION 2.25. — *If $f \in N_*(\mathbb{T}^\infty)$, then $\widetilde{f} \in N_*(\mathbb{D}_1^\infty)$.*

Proof. — We first show that $\widetilde{f}$ is holomorphic on $\mathbb{D}_1^\infty$. Choose a sequence of polynomials $\{q_n\}_{n=1}^\infty$ such that $\|q_n - f\|_0 \to 0$ as $n \to \infty$. By Lemma 2.24, $\{q_n\}_{n=1}^\infty$ converges uniformly to $\widetilde{f}$ on each $V_{r,M}$, and thus $\widetilde{f}$ is holomorphic on $\mathbb{D}_1^\infty$. It remains to show $\{\log(1 + |\widetilde{f}_{[r]}|)\}_{0<r<1}$ forms a uniformly integrable family. Since $\|q_n - f\|_0 \to 0$ as $n \to \infty$, applying Lebesgue-Vitali's theorem [12, Theorem 4.5.4] shows that $\{\log(1 + |q_n|)\}_{n=1}^\infty$ is uniformly integrable. Therefore, there exists a strongly convex function φ such that

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{T}^\infty} \varphi\big(\log(1 + |q_n|)\big)dm_\infty < \infty.$$

By Fatou's lemma, we have

$$(2.23) \quad \int_{\mathbb{T}^\infty} \varphi\left(\log\left(1 + \left|\widetilde{f}_{[r]}\right|\right)\right)dm_\infty$$

$$\leqslant \liminf_{n \to \infty} \int_{\mathbb{T}^\infty} \varphi\left(\log\left(1 + \left|(q_n)_{[r]}\right|\right)\right)dm_\infty.$$

For every $n \in \mathbb{N}$, since $\varphi(\log(1 + |q_n|))$ is continuous on $\overline{\mathbb{D}}^\infty$, and ∞-subharmonic on $\mathbb{D}_1^\infty$, applying Corollary 2.2 gives that for every $0 < r < 1$,

$$\int_{\mathbb{T}^\infty} \varphi\left(\log\left(1 + \left|(q_n)_{[r]}\right|\right)\right)dm_\infty \leqslant \int_{\mathbb{T}^\infty} \varphi(\log(1 + |q_n|))dm_\infty.$$

Taking this inequality back into (2.23) yields

$$\sup_{0<r<1} \int_{\mathbb{T}^\infty} \varphi\left(\log\left(1+\left|\widetilde{f}_{[r]}\right|\right)\right) dm_\infty \leqslant \sup_{n\in\mathbb{N}} \int_{\mathbb{T}^\infty} \varphi(\log(1+|q_n|))dm_\infty < \infty,$$

which ensures the uniform integrability of $\{\log(1+|\widetilde{f}_{[r]}|)\}_{0<r<1}$.   □

Recall that for every $F \in N_*(\mathbb{D}_1^\infty)$, F's "boundary function"

$$F^*(w) = \lim_{r\to 1} F_{[r]}(w)$$

exists for almost every $w \in \mathbb{T}^\infty$. In fact, this "boundary function" belongs to the Smirnov class $N_*(\mathbb{T}^\infty)$ over the infinite torus.

PROPOSITION 2.26. — *If $F \in N_*(\mathbb{D}_1^\infty)$, then $F^* \in N_*(\mathbb{T}^\infty)$ and $\|F\|_0 = \|F^*\|_0$.*

Proof. — The equality $\|F\|_0 = \|F^*\|_0$ immediately follows from Proposition 2.11. Given $\varepsilon > 0$, again by Proposition 2.11, there exists $0 < r < 1$ such that $\|F_{[r]} - F^*\|_0 < \varepsilon$. Since $F_{[r]} \in A(\mathbb{T}^\infty)$, we can find a polynomial $q \in \mathcal{P}_\infty$ satisfying $\|q - F_{[r]}\|_\infty < e^\varepsilon - 1$, and hence $\|q - F_{[r]}\|_0 < \varepsilon$. Therefore,

$$\|q - F^*\|_0 \leqslant \|q - F_{[r]}\|_0 + \|F_{[r]} - F^*\|_0 < 2\varepsilon,$$

which implies $F^* \in N_*(\mathbb{T}^\infty)$.   □

By Proposition 2.26, the algebra homomorphism defined by

$$\Lambda : N_*(\mathbb{D}_1^\infty) \to N_*(\mathbb{T}^\infty), \quad F \mapsto F^*,$$

is a linear isometry. The main theorem of this subsection is stated as follows.

THEOREM 2.27 (Generalized Fatou's theorem). — *The map*

$$\Lambda : N_*(\mathbb{D}_1^\infty) \to N_*(\mathbb{T}^\infty)$$

is an isometric algebra isomorphism, and its inverse map

$$\Lambda^{-1} : N_*(\mathbb{T}^\infty) \to N_*(\mathbb{D}_1^\infty)$$

is given by $f \mapsto \widetilde{f}$.

Proof. — It suffices to show that for every $f \in N_*(\mathbb{T}^\infty)$, $\Lambda\widetilde{f} = f$. Choosing a sequence of polynomials $\{q_n\}_{n=1}^\infty$ satisfying

$$(2.24) \qquad\qquad \|q_n - f\|_0 \to 0 \quad (n \to \infty)$$

and applying Lemma 2.24 lead that $\{q_n\}_{n=1}^\infty$ converges to $\widetilde{f}$ pointwise on $\mathbb{D}_1^\infty$. On the other hand, (2.24) also implies that $\{q_n\}_{n=1}^\infty$ is a Cauchy sequence in $N_*(\mathbb{D}_1^\infty)$, and hence $\|q_n - G\|_0 \to 0$ for some $G \in N_*(\mathbb{D}_1^\infty)$ as

$n \to \infty$. By using Lemma 2.10, $\{q_n\}_{n=1}^\infty$ converges to G pointwise on $\mathbb{D}_1^\infty$, therefore $G = \widetilde{f}$. We conclude from Proposition 2.26 that

$$\left\| q_n - \Lambda \widetilde{f} \right\|_0 = \|q_n - \Lambda G\|_0 = \|q_n - G\|_0 \to 0 \quad (n \to \infty).$$

From this and (2.24), $\Lambda \widetilde{f} = f$, which yields the desired conclusion. □

For $0 < p < \infty$, every function in $H^p(\mathbb{T}^\infty)$, by evaluation functional, can be extended to a holomorphic function on $\mathbb{D}_2^\infty$. Comparison of this fact and Theorem 2.27 leads us to ask the following question.

QUESTION 2.28. — *Does there exist a function in $N_*(\mathbb{T}^\infty)$ which can not be extended to a function holomorphic on $\mathbb{D}_2^\infty$?*

As an application of the generalized Fatou's theorem, we obtain Corollary 2.29, which will be used in Section 3.

COROLLARY 2.29. — *If $0 < p \leqslant \infty$, then $N_*(\mathbb{T}^\infty) \cap L^p(\mathbb{T}^\infty) = H^p(\mathbb{T}^\infty)$.*

Proof. — Clearly, $H^p(\mathbb{T}^\infty) \subset N_*(\mathbb{T}^\infty) \cap L^p(\mathbb{T}^\infty)$. Conversely, for every $f \in N_*(\mathbb{T}^\infty) \cap L^p(\mathbb{T}^\infty)$, we will show that $f \in H^p(\mathbb{T}^\infty)$. Write $F = \Lambda^{-1} f$, then $F \in N_*(\mathbb{D}_1^\infty)$, and $F^* = f$. By Proposition 2.16, for each $\zeta \in \mathbb{D}_1^\infty$,

$$(2.25) \qquad \log |F(\zeta)| \leqslant \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log |F^*| dm_\infty.$$

When $0 < p < \infty$, by (2.25), taking $\varphi(t) = \exp(pt)$ and applying the reasoning as in (2.17) show that

$$\lim_{r \to 1} \int_{\mathbb{T}^\infty} \left| F_{[r]} \right|^p dm_\infty \leqslant \int_{\mathbb{T}^\infty} |F^*|^p dm_\infty.$$

On the other hand, we infer from Fatou's lemma that

$$\int_{\mathbb{T}^\infty} |F^*|^p dm_\infty \leqslant \lim_{r \to 1} \int_{\mathbb{T}^\infty} \left| F_{[r]} \right|^p dm_\infty,$$

and hence $\|F^*\|_p = \lim_{r \to 1} \|F_{[r]}\|_p$. Then by Lemma 2.13, $\|F_{[r]} - F^*\|_p \to 0$ as $r \to 1$. Noticing that for $0 < r < 1$, $F_{[r]} \in A(\mathbb{T}^\infty)$, we see $f = F^* \in H^p(\mathbb{T}^\infty)$.

When $p = \infty$, by (2.25) and Jensen's inequality, for every $\zeta \in \mathbb{D}_1^\infty$,

$$|F(\zeta)| \leqslant \exp \left(\int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log |F^*| dm_\infty \right) \leqslant \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta |F^*| dm_\infty \leqslant \|F^*\|_\infty < \infty,$$

which implies that F is a bounded holomorphic function on $\mathbb{D}_1^\infty$. By the argument in [26, pp. 7-8], F can be extended to a bounded holomorphic function on $\mathbb{D}_2^\infty$, still denoted by F. Then $F \in H^\infty(\mathbb{D}_2^\infty)$, and thus $f = F^* \in H^\infty(\mathbb{T}^\infty)$. □

3. Szegő's problem in infinitely many variables

In this section, we will apply function theory of the Smirnov class to discuss Szegő's problem in infinitely many variables. Write $\mathcal{P}_0$ for the set of polynomials $q \in \mathcal{P}_\infty$ for which $q(0) = 0$. In what follows we assume that $1 < p < \infty$, and $K \in L^1(\mathbb{T}^\infty)$ is a nonnegative function with $\log K \in L^1(\mathbb{T}^\infty)$. Without loss of generality, suppose that $\|K\|_1 = 1$. Motivated by Szegő's theorem mentioned in Introduction, one is naturally concerned with the following quantity:

$$S(K) = \inf_{q \in \mathcal{P}_0} \int_{\mathbb{T}^\infty} |1 - q|^p K \, dm_\infty.$$

Obviously, $S(K) \leqslant 1$. On the other hand, it follows from Jensen's inequality and (2.8) that for every $q \in \mathcal{P}_0$,

$$(3.1) \quad \int_{\mathbb{T}^\infty} |1 - q|^p K \, dm_\infty$$

$$\geqslant \exp \left(\int_{\mathbb{T}^\infty} \log |1 - q|^p \, dm_\infty + \int_{\mathbb{T}^\infty} \log K \, dm_\infty \right)$$

$$\geqslant \exp \left(\log |1 - q(0)|^p + \int_{\mathbb{T}^\infty} \log K \, dm_\infty \right)$$

$$= \exp \left(\int_{\mathbb{T}^\infty} \log K \, dm_\infty \right).$$

Therefore, $S(K)$ falls into the closed interval $\left[\exp \left(\int_{\mathbb{T}^\infty} \log K \, dm_\infty \right), 1 \right]$. Then it is natural to ask for which K, $S(K)$ attains the lower bound $\exp \left(\int_{\mathbb{T}^\infty} \log K \, dm_\infty \right)$ or the upper bound 1, and this is called Szegő's problem.

When $p = 2$, Nakazi gave an answer to Szegő's problem in two-variable setting [31]. His proof heavily depends on the fact that if $K \in L^1(\mathbb{T}^2)$ is a nonnegative function with $\log K$ in the RP class over $\mathbb{T}^2$, then there exists an outer function $h \in H^2(\mathbb{T}^2)$ such that $K = |h|^2$, see [38, p. 77]. In what follows, we will give a unifying treatment for Szegő's problem as an application of function theory developed in the previous section, which works for both finite-variable and infinite-variable setting.

We first study when $S(K)$ attains the lower bound $\exp \left(\int_{\mathbb{T}^\infty} \log K \, dm_\infty \right)$. The next theorem gives the answer.

THEOREM 3.1. — *The equality*

$$(3.2) \quad S(K) = \exp \left(\int_{\mathbb{T}^\infty} \log K \, dm_\infty \right)$$

holds if and only if $K = |h|^p$ for some cyclic vector $h \in H^p(\mathbb{T}^\infty)$.

Before proving Theorem 3.1, we need the following proposition, which states that the weighted Hardy space $H^p(K dm_\infty)$, the closure of $\mathcal{P}_\infty$ in $L^p(K dm_\infty)$, is a subset of the Smirnov class $N_*(\mathbb{T}^\infty)$.

PROPOSITION 3.2. — $H^p(K dm_\infty) \subset N_*(\mathbb{T}^\infty)$.

Proof. — For every $h \in H^p(K dm_\infty)$, there is a sequence of polynomials $\{q_n\}_{n=1}^\infty$ for which

$$(3.3) \qquad \int_{\mathbb{T}^\infty} |q_n - h|^p K dm_\infty \to 0 \quad (n \to \infty).$$

Then there is a subsequence $\{q_{n_k}\}_{k=1}^\infty$ that converges to h almost everywhere on $\mathbb{T}^\infty$. Moreover, by the inequality $\log(1+x) \leqslant x$ $(x \geqslant 0)$ and (3.3), we have

$$\int_{\mathbb{T}^\infty} \log\left(1 + |q_{n_k} - h|^p K\right) dm_\infty \to 0 \quad (k \to \infty).$$

Write

$$f_{n_k} = \log\left(1 + |q_{n_k} - h|^p\right),$$
$$g_{n_k} = \log\left(1 + |q_{n_k} - h|^p K\right) + \log(1 + K) - \log K.$$

For every $k \in \mathbb{N}$, it is easy to verify that $0 \leqslant f_{n_k} \leqslant g_{n_k}$ on $\mathbb{T}^\infty$. Applying Lemma 2.12 to sequences $\{f_{n_k}\}_{k=1}^\infty$ and $\{g_{n_k}\}_{k=1}^\infty$, we obtain

$$(3.4) \qquad \int_{\mathbb{T}^\infty} \log\left(1 + |q_{n_k} - h|^p\right) dm_\infty \to 0 \quad (k \to \infty).$$

Let E_k be the set of points $w \in \mathbb{T}^\infty$ such that $|q_{n_k}(w) - h(w)| \leqslant 1$, and E_k^c the complement of E_k with respect to $\mathbb{T}^\infty$. Write χ_{E_k} for the characteristic function of E_k. Then for every $k \in \mathbb{N}$, $0 \leqslant \chi_{E_k}|q_{n_k} - h| \leqslant 1$ on $\mathbb{T}^\infty$. It follows from Lebesgue's dominated convergence theorem that

$$(3.5) \int_{E_k} \log\left(1 + |q_{n_k} - h|\right) dm_\infty = \int_{\mathbb{T}^\infty} \log\left(1 + \chi_{E_k}|q_{n_k} - h|\right) dm_\infty \to 0$$

as $k \to \infty$. On the other hand, by inequalities

$$\int_{E_k^c} \log\left(1 + |q_{n_k} - h|\right) dm_\infty \leqslant \int_{E_k^c} \log\left(1 + |q_{n_k} - h|^p\right) dm_\infty$$
$$\leqslant \int_{\mathbb{T}^\infty} \log\left(1 + |q_{n_k} - h|^p\right) dm_\infty,$$

we conclude from (3.4) that

$$\int_{E_k^c} \log\left(1 + |q_{n_k} - h|\right) dm_\infty \to 0 \quad (k \to \infty).$$

Combining this with (3.5) gives $\|q_{n_k} - h\|_0 \to 0$ as $k \to \infty$, and thus $h \in N_*(\mathbb{T}^\infty)$, as desired. □

Recall that to every $f \in N_*(\mathbb{T}^\infty)$ corresponds a holomorphic function $\widetilde{f} \in N_*(\mathbb{D}_1^\infty)$ whose radial limit is f. We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. We first assume $K = |h|^p$ for some cyclic vector $h \in H^p(\mathbb{T}^\infty)$. To prove (3.2), it suffices to show that

$$(3.6) \qquad S(K) \leqslant \exp\left(\int_{\mathbb{T}^\infty} \log K \, dm_\infty\right).$$

By the cyclicity of h, it is not difficult to verify that $h - \widetilde{h}(0)$ belongs to the closure of $\{qh \in H^p(\mathbb{T}^\infty) : q \in \mathcal{P}_0\}$ in $H^p(\mathbb{T}^\infty)$. Therefore,

$$(3.7) \qquad \begin{aligned} S(K) &= \inf_{q \in \mathcal{P}_0} \int_{\mathbb{T}^\infty} |h - qh|^p \, dm_\infty \\ &\leqslant \int_{\mathbb{T}^\infty} \left| h - \left(h - \widetilde{h}(0)\right) \right|^p dm_\infty = \left| \widetilde{h}(0) \right|^p. \end{aligned}$$

Since $h \in H^p(\mathbb{T}^\infty)$ is outer, that is,

$$|\widetilde{h}(0)|^p = \exp\left(\int_{\mathbb{T}^\infty} \log |h|^p dm_\infty\right) = \exp\left(\int_{\mathbb{T}^\infty} \log K \, dm_\infty\right),$$

we deduce from this and (3.7) that K satisfies (3.6).

Conversely, we assume that (3.2) holds. Since $H^p(K dm_\infty)$ is reflexive, there exists a function φ belonging to the closure of $\mathcal{P}_0$ in $H^p(K dm_\infty)$ such that

$$(3.8) \qquad S(K) = \int_{\mathbb{T}^\infty} |1 - \varphi|^p K dm_\infty.$$

Choose a sequence of polynomials $\{q_n\}_{n=1}^\infty$ in $\mathcal{P}_0$ satisfying

$$(3.9) \qquad \int_{\mathbb{T}^\infty} |q_n - \varphi|^p K dm_\infty \to 0 \quad (n \to \infty).$$

It follows from Proposition 3.2 that $\varphi \in N_*(\mathbb{T}^\infty)$, and the proof of this proposition also implies that there is a subsequence $\{q_{n_k}\}_{k=1}^\infty$ for which $\|q_{n_k} - \varphi\|_0 \to 0$ as $k \to \infty$. Hence by Lemma 2.24, $\{q_{n_k}\}_{k=1}^\infty$ converges to $\widetilde{\varphi}$ pointwise on $\mathbb{D}_1^\infty$, and thus $\widetilde{\varphi}(0) = 0$. We see from Proposition 2.16 that

$$(3.10) \qquad \int_{\mathbb{T}^\infty} \log |1 - \varphi|^p dm_\infty \geqslant \log |1 - \widetilde{\varphi}(0)|^p = 0.$$

By Jensen's inequality,

$$(3.11) \qquad \int_{\mathbb{T}^\infty} |1 - \varphi|^p K dm_\infty \geqslant \exp\left(\int_{\mathbb{T}^\infty} \log |1 - \varphi|^p + \log K dm_\infty\right),$$

and the equality holds if and only if $|1 - \varphi|^p K$ is a constant. Combining (3.10) with (3.11) shows that

$$\int_{\mathbb{T}^\infty} |1 - \varphi|^p K dm_\infty \geqslant \exp\left(\int_{\mathbb{T}^\infty} \log K dm_\infty\right).$$

By (3.2) and (3.8), this inequality is actually an equality, forcing (3.10) and (3.11) to be equalities. This means that $1 - \varphi \in N_*(\mathbb{T}^\infty)$ is outer, and there is a constant $C > 0$ such that $|1 - \varphi|^p K = C^p$.

We claim that $\frac{1}{1-\varphi} \in H^p(\mathbb{T}^\infty)$. Indeed, since $1 - \varphi \in N_*(\mathbb{T}^\infty)$ is outer, by Corollary 2.21 and the generalized Fatou's theorem, $\frac{1}{1-\varphi} \in N_*(\mathbb{T}^\infty)$, and it is outer. On the other hand, since $\frac{1}{|1-\varphi|^p} = \frac{K}{C^p}$, we have $\frac{1}{1-\varphi} \in L^p(\mathbb{T}^\infty)$. Then by Corollary 2.29, $\frac{1}{1-\varphi} \in H^p(\mathbb{T}^\infty)$, and the claim holds.

Write $h = \frac{C}{1-\varphi} \in H^p(\mathbb{T}^\infty)$, then $K = |h|^p$. We proceed to prove that h is cyclic by (3.9). Noticing that

$$|q_n - \varphi|^p K = |(1 - q_n) - (1 - \varphi)|^p |h|^p = |(1 - q_n)h - C|^p,$$

hence by (3.9),

$$\int_{\mathbb{T}^\infty} |(1 - q_n)h - C|^p \, dm_\infty = \int_{\mathbb{T}^\infty} |q_n - \varphi|^p K dm_\infty \to 0 \quad (n \to \infty).$$

The above reasoning shows that the constant function C is in the invariant subspace of $H^p(\mathbb{T}^\infty)$ generated by h. Therefore, h is cyclic, and the proof of Theorem 3.1 is complete. $\qquad\square$

As it is clear, Szegő's problem is closely related to cyclic vectors in Hardy spaces. In fact, its connection with cyclicity is more than the external presentation. We will show this via a special case $K = |f|^p$, where $1 < p < \infty$ and $f \in H^p(\mathbb{T}^\infty)$. By Theorem 2.4, $\log K = p \log |f| \in L^1(\mathbb{T}^\infty)$. Write

$$M(|f|^p) = \exp\left(\int_{\mathbb{T}^\infty} \log |f|^p dm_\infty\right).$$

Then it follows from (3.1) and Proposition 2.16 that

$$S(|f|^p) \geqslant M(|f|^p) \geqslant \left|\widetilde{f}(0)\right|^p,$$

where $\widetilde{f} \in H^p(\mathbb{D}_2^\infty)$ denotes the holomorphic function corresponding to f. It seems interesting to ask when two among these three quantities are equal. The complete answer will be given in Proposition 3.3 below. For $p = 2$, Nakazi considered when $S(|f|^2) = M(|f|^2)$ in the case of two variables, see [32]. In the same setting, Guo and Zhou [24] solved the problem when $S(|f|^2) = |\widetilde{f}(0)|^2$, but their methods do not apply for $p \neq 2$, since $H^p(\mathbb{T}^2)$ is not a Hilbert space.

A function $\eta \in H^\infty(\mathbb{T}^\infty)$ is said to be inner, if $|\eta| = 1$ almost everywhere on $\mathbb{T}^\infty$ [18]. Since there is a canonical isometric isomorphism from $H^\infty(\mathbb{T}^\infty)$ onto $H^\infty(\mathbb{D}_2^\infty)$, we see that if η is inner, then $|\widetilde{\eta}(\zeta)| \leqslant 1$ for all $\zeta \in \mathbb{D}_2^\infty$. By Corollary 2.29, it is easy to verify that a function η is inner if $\eta \in N_*(\mathbb{T}^\infty)$ and is of modulus 1 almost everywhere on $\mathbb{T}^\infty$.

PROPOSITION 3.3. — *Suppose that $1 < p < \infty$ and $f \in H^p(\mathbb{T}^\infty)$, then:*
  (1) $M(|f|^p) = |\widetilde{f}(0)|^p$ *if and only if f is outer.*
  (2) $S(|f|^p) = M(|f|^p)$ *if and only if $f = \eta h$, where η is inner and h is a cyclic vector in $H^p(\mathbb{T}^\infty)$.*
  (3) $S(|f|^p) = |\widetilde{f}(0)|^p$ *if and only if f is cyclic.*

Proof. — (1) This is a direct conclusion of Proposition 2.17.

(2) If $f = \eta h$, then it follows from Theorem 3.1 that $S(|f|^p) = M(|f|^p)$. Conversely, suppose that $S(|f|^p) = M(|f|^p)$. By Theorem 3.1, there is a cyclic vector $h \in H^p(\mathbb{T}^\infty)$ such that $|f|^p = |h|^p$. Since $h \in H^p(\mathbb{T}^\infty)$ is cyclic, Corollary 2.23 implies that h is outer, and hence

$$(3.12) \qquad \log\left|\widetilde{h}(\zeta)\right| = \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log|h| dm_\infty, \quad \zeta \in \mathbb{D}_1^\infty$$

by Proposition 2.17. On the other hand, Proposition 2.16 states that

$$(3.13) \qquad \log\left|\widetilde{f}(\zeta)\right| \leqslant \int_{\mathbb{T}^\infty} \mathbf{P}_\zeta \log|f| dm_\infty, \quad \zeta \in \mathbb{D}_1^\infty.$$

Combining (3.12), (3.13) and the fact $|f| = |h|$ yields that for all $\zeta \in \mathbb{D}_1^\infty$, $|\widetilde{f}(\zeta)| \leqslant |\widetilde{h}(\zeta)|$. By the continuity of $\widetilde{f}$ and $\widetilde{h}$ on $\mathbb{D}_2^\infty$, this inequality remains valid for $\zeta \in \mathbb{D}_2^\infty$. It follows from Corollary 2.20 that $\widetilde{h}$ is zero-free in $\mathbb{D}_2^\infty$, so $F = \frac{\widetilde{f}}{\widetilde{h}} \in H^\infty(\mathbb{D}_2^\infty)$. Therefore, $F^* = \frac{f}{h} \in H^\infty(\mathbb{T}^\infty)$ is of modulus 1 almost everywhere on $\mathbb{T}^\infty$, which implies that F^* is inner and $f = F^* h$.

(3) By (1) and (2), it is clear that $S(|f|^p) = |\widetilde{f}(0)|^p$ if and only if f is an outer function with factorization $f = \eta h$, where η is inner and h is a cyclic vector in $H^p(\mathbb{T}^\infty)$. In this case, it will be shown that η is a constant function, and hence f is cyclic. Since f is outer, it follows from Corollary 2.21 that $\frac{1}{f} \in N_*(\mathbb{T}^\infty)$. As mentioned before, $N_*(\mathbb{T}^\infty)$ is a topological algebra, so $\frac{1}{\eta} = \frac{1}{f} \cdot h \in N_*(\mathbb{T}^\infty)$. Noticing that η is inner, $\frac{1}{\eta}$ is of modulus 1 almost everywhere on $\mathbb{T}^\infty$, which implies that $\frac{1}{\eta}$ is also inner. Then

$$|\widetilde{\eta}(\zeta)| \leqslant 1, \quad \frac{1}{|\widetilde{\eta}(\zeta)|} \leqslant 1, \quad \zeta \in \mathbb{D}_2^\infty.$$

Therefore, $|\widetilde{\eta}(\zeta)| = 1$ for all $\zeta \in \mathbb{D}_2^\infty$, and thus $\widetilde{\eta}$ is a constant function, as desired. $\qquad\square$

We continue to discuss when $S(K)$ attains the upper bound 1. Let $\mathbb{Z}_0^\infty$ be the set of all finitely supported sequences of integers. Recall that for every $\alpha \in \mathbb{Z}_0^\infty$, the Fourier coefficient $\widehat{K}(\alpha)$ of K is defined to be

$$\widehat{K}(\alpha) = \int_{\mathbb{T}^\infty} K\bar{e}_\alpha dm_\infty,$$

where

$$e_\alpha(w) = w_1^{\alpha_1} w_2^{\alpha_2} \cdots, \quad w \in \mathbb{T}^\infty.$$

THEOREM 3.4. — $S(K) = 1$ if and only if

(3.14) $\widehat{K}(\alpha) = \widehat{K}(-\alpha) = 0, \quad 0 \neq \alpha \in \mathbb{N}_0^\infty,$

where $\mathbb{N}_0^\infty$ denotes the set of all finitely supported sequences of nonnegative integers.

Proof. — Let $H_0^p(Kdm_\infty)$ denote the closure of $\mathcal{P}_0$ in $L^p(Kdm_\infty)$. Applying Hahn–Banach theorem shows that the quotient space

$$\mathcal{Q} = L^p(Kdm_\infty)/H_0^p(Kdm_\infty)$$

is of the dual space $\mathcal{Q}^*$ as follows:

$$\mathcal{Q}^* = \left\{ g \in L^q(Kdm_\infty) : \int_{\mathbb{T}^\infty} fgKdm_\infty = 0 \text{ for } f \in H_0^p(Kdm_\infty) \right\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Therefore,

(3.15)
$$S(K) = \inf_{f \in H_0^p(Kdm_\infty)} \int_{\mathbb{T}^\infty} |1 - f|^p Kdm_\infty$$
$$= \|\pi(1)\|_{\mathcal{Q}}^p = \sup_{g \in \mathbf{B} \cap \mathcal{Q}^*} \left| \int_{\mathbb{T}^\infty} gKdm_\infty \right|^p,$$

where π is the quotient map of $L^p(Kdm_\infty)$ onto $\mathcal{Q}$, and $\mathbf{B}$ is the closed unit ball of $L^q(Kdm_\infty)$.

When (3.14) holds, it is easy to verify that $1 \in \mathbf{B} \cap \mathcal{Q}^*$. Then by (3.15),

$$S(K) = \sup_{g \in \mathbf{B} \cap \mathcal{Q}^*} \left| \int_{\mathbb{T}^\infty} gKdm_\infty \right|^p \geqslant \left| \int_{\mathbb{T}^\infty} Kdm_\infty \right|^p = 1,$$

forcing $S(K) = 1$.

Conversely, we assume that $S(K) = 1$. Since $L^q(Kdm_\infty)$ is reflexive, $\mathbf{B} \cap \mathcal{Q}^*$ is weakly compact. Hence by (3.15) there exists a $\psi \in \mathbf{B} \cap \mathcal{Q}^*$ such that

$$1 = S(K) = \left| \int_{\mathbb{T}^\infty} \psi Kdm_\infty \right|^p.$$

Noticing that $\|\psi\|_{L^q(Kdm_\infty)} \leqslant 1$, Hölder's inequality gives that $\psi = 1$, and hence $1 \in \mathcal{Q}^*$. For each $0 \neq \alpha \in \mathbb{N}_0^\infty$, it follows from $e_\alpha \in H_0^p(Kdm_\infty)$ that

$$(3.16) \qquad \widehat{K}(-\alpha) = \int_{\mathbb{T}^\infty} Ke_\alpha dm_\infty = \int_{\mathbb{T}^\infty} 1 \cdot e_\alpha Kdm_\infty = 0.$$

Since K is nonnegative, the conjugate of (3.16) gives that

$$\widehat{K}(\alpha) = \int_{\mathbb{T}^\infty} K\bar{e}_\alpha dm_\infty = 0,$$

and the proof of Theorem 3.4 is complete. $\hfill\square$

4. Nevanlinna functions and Dirichlet series

It is well known that there exists a fascinating connection between functions in infinitely many variables and Dirichlet series via Bohr correspondence. As in cases of the Nevanlinna class and the Smirnov class in the infinite-variable setting, there exist analogies for Dirichlet series. In this section, we will develop the Nevanlinna–Dirichlet class $\mathcal{N}$ and the Smirnov-Dirichlet class $\mathcal{N}_*$ for Dirichlet series. Moreover, the relationships between $\mathcal{N}$ and $N(\mathbb{D}_1^\infty)$, $\mathcal{N}_*$ and $N_*(\mathbb{D}_1^\infty)$ will be established, respectively.

First, we briefly recall some elements from the theory of Dirichlet series. A Dirichlet series is a series of the following form:

$$f(s) = \sum_{n=1}^\infty a_n n^{-s},$$

where s is the complex variable. Let $\mathbb{R}$ be the real line and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ the extended real line. When $\sigma \in \mathbb{R}$, set $\mathbb{C}_\sigma = \{z \in \mathbb{C} : \text{Re } z > \sigma\}$. For each Dirichlet series $f(s) = \sum_{n=1}^\infty a_n n^{-s}$, its abscissa of uniform convergence $\sigma_u(f)$ is defined as

$$\sigma_u(f) = \inf\left\{\sigma \in \mathbb{R} : \sum_{n=1}^\infty a_n n^{-s} \text{ converges uniformly on } \mathbb{C}_\sigma\right\} \in \overline{\mathbb{R}},$$

see [20, p. 10]. When $\sigma \in \mathbb{R}$, write

$$f_\sigma(s) = \sum_{n=1}^\infty a_n n^{-(s+\sigma)},$$

then $\sigma_u(f_\sigma) = \sigma_u(f) - \sigma$.

Recall that a subset S of $\mathbb{R}$ is relatively dense in $\mathbb{R}$, if there exists a constant $L > 0$ such that each closed interval of length L intersects S.

A function $f : \mathbb{R} \to \mathbb{C}$ is called almost periodic, if for each $\varepsilon > 0$, there is a relatively dense set $E_\varepsilon \subset \mathbb{R}$ satisfying

$$\sup_{t \in \mathbb{R}} |f(t + \tau) - f(t)| < \varepsilon, \quad \forall \, \tau \in E_\varepsilon.$$

We mention that when f is almost periodic, the limit $\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t) dt$ exists. Let f be a Dirichlet series with $\sigma_u(f) < 0$, then $t \mapsto f(it)$ is almost periodic. Furthermore, if h is a non-decreasing continuous function on the half real line $\mathbb{R}^+ = [0, \infty)$, then $h(|f(it)|)$ is almost periodic, see [10, 37].

Let p_j be the jth prime number. For every $n \in \mathbb{N}$, there exists a unique prime factorization $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and set $\alpha(n) = (\alpha_1, \ldots, \alpha_k, 0, \ldots)$. When $\zeta = (\zeta_1, \zeta_2, \ldots)$ is a sequence of complex numbers, we write $\zeta^{\alpha(n)} = \zeta_1^{\alpha_1} \cdots \zeta_k^{\alpha_k}$. From Bohr's point of view [13], each Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ can be associated with a formal power series in infinitely many variables as follows:

$$(\mathcal{B}f)(\zeta) = \sum_{n=1}^{\infty} a_n \zeta^{\alpha(n)}.$$

It is easy to verify that if $\sigma_u(f) < 0$, then the partial sums of $\mathcal{B}f$ converge uniformly on $\mathbb{T}^\infty$, and $\mathcal{B}f \in A(\mathbb{T}^\infty)$, see [26, 37].

This section contains two parts. In the first part, we define the Nevanlinna–Dirichlet class $\mathcal{N}$, and discuss the relationship between this class and the Nevanlinna class $N(\mathbb{D}_1^\infty)$. The main result is Theorem 4.1. The second part is devoted to proving this theorem.

4.1. The Nevanlinna–Dirichlet class $\mathcal{N}$

Before defining the Nevanlinna–Dirichlet class $\mathcal{N}$, let us start with a smaller class, the Smirnov–Dirichlet class $\mathcal{N}_*$.

Let f be a Dirichlet series with $\sigma_u(f) < 0$, then $t \mapsto \log(1 + |f(it)|)$ is almost periodic, and hence the limit

$$\|f\|_0 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \log(1 + |f(it)|) dt$$

exists. Note that $\mathcal{B}f \in A(\mathbb{T}^\infty)$. Applying Birkhoff–Oxtoby theorem [37, Theorem 6.5.1] yields

(4.1) $$\|f\|_0 = \int_{\mathbb{T}^\infty} \log(1 + |\mathcal{B}f|) dm_\infty = \|\mathcal{B}f\|_0.$$

Let $\mathcal{P}_D$ denote the set of all Dirichlet polynimials, then for every $Q \in \mathcal{P}_D$, $\sigma_u(Q) = -\infty$. Hence $\|Q\|_0$ is well defined and $\|Q\|_0 = \|\mathcal{B}Q\|_0$. We define the

Smirnov–Dirichlet class $\mathcal{N}_*$ to be the completion of $\mathcal{P}_D$ in the metric $\|\cdot\|_0$. Then there exists a natural algebraic structure on $\mathcal{N}_*$. When $0 < p < \infty$, there is a constant $C_p > 0$ such that

$$\|Q\|_0 \leqslant C_p \|Q\|_p^{\min\{p,1\}}, \quad Q \in \mathcal{P}_D.$$

Hence all Hardy–Dirichlet spaces $\mathcal{H}^p$ ($0 < p \leqslant \infty$) are contained in $\mathcal{N}_*$. As defined in Introduction, the Smirnov class $N_*(\mathbb{T}^\infty)$ is the closure of $\mathcal{P}_\infty$ in $L^0(\mathbb{T}^\infty)$. Hence the Bohr transform $\mathcal{B} : \mathcal{P}_D \to \mathcal{P}_\infty$ can be extended to an isometric isomorphism from $\mathcal{N}_*$ onto $N_*(\mathbb{T}^\infty)$, still denoted by $\mathcal{B}$. Furthermore, $\mathcal{B}$ is also an algebra isomorphism.

For a fixed $k \in \mathbb{N}$, let Ξ_k be the multiplicative subsemigroup of $\mathbb{N}$ generated by the first k prime numbers $p_1, \ldots, p_k$. When f is a Dirichlet series of the form $f(s) = \sum_{n \in \Xi_k} a_n n^{-s}$, then $(\mathcal{B}f)(\zeta) = \sum_{n \in \Xi_k} a_n \zeta^{\alpha(n)}$ only depends on the first k variables $\zeta_1, \ldots, \zeta_k$. If $\mathcal{B}f$ is holomorphic on $\mathbb{D}^k$, then $f(s) = (\mathcal{B}f)(p_1^{-s}, \ldots, p_k^{-s})$ converges uniformly on each $\mathbb{C}_\sigma$ ($\sigma > 0$), and thus $\sigma_u(f) \leqslant 0$. This inspires us to consider the class $\mathcal{N}_u$ of Dirichlet series f with $\sigma_u(f) \leqslant 0$ and $\limsup_{\sigma \to 0+} \|f_\sigma\|_0 < \infty$. By (2.4), this coincides with Brevig and Perfekt's definition [15] mentioned in Introduction. A similar argument as in the proof of Proposition 2.1 gives that if f is a Dirichlet series with $\sigma_u(f) \leqslant 0$, then $\|f_\sigma\|_0$ defines a non-increasing function of σ, which is also mentioned in [15]. Thus we have

$$\limsup_{\sigma \to 0^+} \|f_\sigma\|_0 = \sup_{\sigma > 0} \|f_\sigma\|_0 .$$

It is clear that $\mathcal{N}_u$ is a linear space. We define

$$\|f\|_0 = \sup_{\sigma > 0} \|f_\sigma\|_0 , \quad f \in \mathcal{N}_u.$$

Then

$$d_0(f, g) = \|f - g\|_0, \quad f, g \in \mathcal{N}_u$$

defines a metric on $\mathcal{N}_u$. Unfortunately, the class $\mathcal{N}_u$ does not contain the familiar Hardy–Dirichlet space $\mathcal{H}^2$. For example, put $a_n = (\sqrt{n} \log n)^{-1}$, then the Dirichlet series $f_{\boldsymbol{a}}(s) = \sum_{n=2}^\infty a_n n^{-s}$ belongs to $\mathcal{H}^2$, and $\sigma_u(f_{\boldsymbol{a}}) = \frac{1}{2}$. Hence $f_{\boldsymbol{a}} \notin \mathcal{N}_u$. So we define the Nevanlinna–Dirichlet class $\mathcal{N}$ to be the completion of $\mathcal{N}_u$ in the metric $\|\cdot\|_0$. Since $\mathcal{P}_D$ is contained in $\mathcal{N}_u$, we see that $\mathcal{N}_*$ is contained in $\mathcal{N}$. In conclusion, we have

$$\mathcal{H}^\infty \subset \mathcal{H}^q \subset \mathcal{H}^p \subset \mathcal{N}_* \subset \mathcal{N} \quad (0 < p < q < \infty).$$

The following is the main result of this section, which will be proved in Subsection 4.2.

THEOREM 4.1. — *Let* $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ *be a Dirichlet series in* $\mathcal{N}_u$. *Then the formal power series* $(\mathcal{B}f)(\zeta) = \sum_{n=1}^{\infty} a_n \zeta^{\alpha(n)}$ *converges in* $\mathbb{D}_1^{\infty}$. *Moreover, we have* $\mathcal{B}f \in N(\mathbb{D}_1^{\infty})$, *and* $\|\mathcal{B}f\|_0 = \|f\|_0$. *Therefore, the Bohr transform* $\mathcal{B}$ *can be extended to an isometry from* $\mathcal{N}$ *to* $N(\mathbb{D}_1^{\infty})$, *still denoted by* $\mathcal{B}$.

For a general Dirichlet series f with $\sigma_u(f) \leqslant 0$, the conclusion of Theorem 4.1 fails. Here is an example.

Example 4.2. — Let p_k be the kth prime number and $\{p_{k_j}\}_{j=1}^{\infty}$ a subsequence of $\{p_k\}_{k=1}^{\infty}$ satisfying $p_{k_{j+1}} > 2p_{k_j}$ for all $j \in \mathbb{N}$. Write

$$f(s) = \sum_{j=1}^{\infty} \frac{\log p_{k_j}}{p_{k_j}^s}.$$

Let $\sigma > 0$, then there is a constant $M_\sigma > 0$ such that for each $j \in \mathbb{N}$, $\log p_{k_j} \leqslant M_\sigma p_{k_j}^\sigma$. Therefore,

$$\sum_{j=1}^{\infty} \frac{\log p_{k_j}}{p_{k_j}^{2\sigma}} \leqslant M_\sigma \sum_{j=1}^{\infty} p_{k_j}^{-\sigma} \leqslant M_\sigma p_{k_1}^{-\sigma} \sum_{j=1}^{\infty} 2^{-(j-1)\sigma} < \infty.$$

Since σ is arbitrary, we see that $\sigma_u(f) \leqslant 0$. However, $\{\log p_{k_j}\}_{j=1}^{\infty}$ is unbounded, which implies that $(\mathcal{B}f)(\zeta) = \sum_{j=1}^{\infty} (\log p_{k_j}) \zeta_{k_j}$ diverges for some $\zeta \in \mathbb{D}_1^{\infty}$.

The following example implies that $\mathcal{N}_*$ is a proper subclass of $\mathcal{N}$.

Example 4.3. — Write $\varphi(z) = \exp(\frac{1+z}{1-z})$, $z \in \mathbb{D}$. It is easy to see that $\varphi \in N(\mathbb{D})$, but $\varphi \notin N_*(\mathbb{D})$. Let a_n denote the nth Taylor's coefficient of φ, and put $f(s) = \sum_{n=0}^{\infty} a_n 2^{-ns}$. Then for every $\zeta \in \mathbb{D}_1^{\infty}$, we have $(\mathcal{B}f)(\zeta) = \varphi(\zeta_1)$, and hence $\mathcal{B}f \notin N_*(\mathbb{D}_1^{\infty})$. Note that $\mathcal{B}f$ is a holomorphic function on $\mathbb{D}$. As we mentioned before, $\sigma_u(f) \leqslant 0$. On the other hand, it follows from (4.1) that

$$\sup_{\sigma>0} \|f_\sigma\|_0 = \sup_{\sigma>0} \|\mathcal{B}f_\sigma\|_0 = \sup_{\sigma>0} \int_{\mathbb{T}} \log\left(1 + \left|\varphi(2^{-\sigma}\lambda)\right|\right) dm_1(\lambda) < \infty,$$

which implies $f \in \mathcal{N}_u$. Now we claim that $f \notin \mathcal{N}_*$. Indeed, if $f \in \mathcal{N}_*$, then there exists a sequence of Dirichlet polynomials $\{Q_n\}_{n=1}^{\infty}$ such that $\|Q_n - f\|_0 \to 0$ as $n \to \infty$. By Theorem 4.1, $\|\mathcal{B}Q_n - \mathcal{B}f\|_0 \to 0$ as $n \to \infty$, which gives that $\mathcal{B}f \in N_*(\mathbb{D}_1^{\infty})$, a contradiction.

Let $0 < p < q < \infty$. We draw up the following figure to show relations between all spaces, where "$\simeq$" denotes that the corresponding spaces

are canonically isometrically isomorphic, and "$\hookrightarrow$" denotes that the former space can be isometrically embedded into the latter. It remains to be clarified whether $\mathcal{B} : \mathcal{N} \to N(\mathbb{D}_1^\infty)$ is surjective.

$$
\begin{array}{ccccccccc}
\mathcal{H}^\infty & \subset & \mathcal{H}^q & \subset & \mathcal{H}^p & \subset & \mathcal{N}_* & \subset & \mathcal{N} \\
\wr\wr & & \wr\wr & & \wr\wr & & \wr\wr & & \downarrow \\
H^\infty(\mathbb{D}_2^\infty) & \subset & H^q(\mathbb{D}_2^\infty) & \subset & H^p(\mathbb{D}_2^\infty) & \subset & N_*(\mathbb{D}_1^\infty) & \subset & N(\mathbb{D}_1^\infty)
\end{array}
$$

4.2. Proof of Theorem 4.1

This subsection is mainly dedicated to proving Theorem 4.1. To prove this theorem, a series of preparations is needed.

Let F be a holomorphic function on the polydisk $\mathbb{D}^k$. For each $\sigma > 0$, write

$$
F_{\{\sigma\}}(w) = F\left(p_1^{-\sigma} w_1, \ldots, p_k^{-\sigma} w_k\right), \quad w \in \mathbb{T}^k,
$$

where p_j denotes the jth prime number. Then by the subharmonicity of $\log(1 + |F|)$ in each variable separately,

$$
(4.2) \qquad \sup_{\sigma > 0} \int_{\mathbb{T}^k} \log\left(1 + \left|F_{\{\sigma\}}\right|\right) dm_k = \sup_{0 < r < 1} \int_{\mathbb{T}^k} \log\left(1 + \left|F_{[r]}\right|\right) dm_k,
$$

where m_k denotes the normalized Lebesgue measure on $\mathbb{T}^k$. Moreover, integrals in the left side of this equality decrease with σ.

Suppose that $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ is a Dirichlet series in $\mathcal{N}_u$. We first consider Bohr's $k$te Abschnitt

$$
(\mathcal{B}_k f)(\zeta) = \sum_{n \in \Xi_k} a_n \zeta^{\alpha(n)}
$$

of $\mathcal{B}f$. It is a formal power series only depending on the first k variables $\zeta_1, \ldots, \zeta_k$. We have the following proposition.

PROPOSITION 4.4. — *If $f \in \mathcal{N}_u$, then for every $k \in \mathbb{N}$, the formal power series $\mathcal{B}_k f$ converges in $\mathbb{D}^k$, and it defines a function in the Nevanlinna class $N(\mathbb{D}^k)$ over $\mathbb{D}^k$.*

Proof. — Let $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ be a Dirichlet series in $\mathcal{N}_u$, then for every $\sigma > 0$, $\sigma_u(f_\sigma) < 0$, and thus $\mathcal{B}f_\sigma \in A(\mathbb{T}^\infty)$. Applying Lebesgue's dominated convergence theorem gives

$$
\begin{aligned}
(4.3) \qquad \lim_{k \to \infty} \int_{\mathbb{T}^\infty} \log\left(1 + |\mathcal{B}_k f_\sigma|\right) dm_\infty &= \int_{\mathbb{T}^\infty} \log\left(1 + |\mathcal{B}f_\sigma|\right) dm_\infty \\
&= \|\mathcal{B}f_\sigma\|_0 .
\end{aligned}
$$

It follows from Corollary 2.2 that the integrals in the left side of (4.3) increase with k. Therefore we have

$$\sup_{k \in \mathbb{N}} \sup_{\sigma>0} \int_{\mathbb{T}^\infty} \log\left(1 + |\mathcal{B}_k f_\sigma|\right) dm_\infty$$

$$= \sup_{\sigma>0} \sup_{k \in \mathbb{N}} \int_{\mathbb{T}^\infty} \log\left(1 + |\mathcal{B}_k f_\sigma|\right) dm_\infty = \sup_{\sigma>0} \|\mathcal{B} f_\sigma\|_0.$$

By (4.1), for every $\sigma > 0$, $\|\mathcal{B} f_\sigma\|_0 = \|f_\sigma\|_0$, and hence

$$(4.4) \quad \begin{aligned} &\sup_{k \in \mathbb{N}} \sup_{\sigma>0} \int_{\mathbb{T}^\infty} \log\left(1 + |\mathcal{B}_k f_\sigma|\right) dm_\infty \\ &\qquad = \sup_{\sigma>0} \|\mathcal{B} f_\sigma\|_0 = \sup_{\sigma>0} \|f_\sigma\|_0 = \|f\|_0 < \infty. \end{aligned}$$

For a fixed $k \in \mathbb{N}$, the family $\{\mathcal{B}_k f_\sigma\}_{\sigma>0}$ is contained in the polydisk algebra $A(\mathbb{D}^k)$. It follows from Lemma 2.10 that for each $\zeta \in \mathbb{D}^k$,

$$\log\left(1 + |(\mathcal{B}_k f_\sigma)(\zeta)|\right) \leqslant \|\mathbf{P}_\zeta\|_\infty \int_{\mathbb{T}^\infty} \log\left(1 + |\mathcal{B}_k f_\sigma|\right) dm_\infty \leqslant \|\mathbf{P}_\zeta\|_\infty \|f\|_0.$$

This implies that $\{\mathcal{B}_k f_\sigma\}_{\sigma>0}$ is uniformly bounded on each compact subset of $\mathbb{D}^k$. By Montel's theorem [35, Theorem 1.5], there is a sequence $\sigma_m \to 0^+$ $(m \to \infty)$ and a function G_k holomorphic on $\mathbb{D}^k$ such that $\{\mathcal{B}_k f_{\sigma_m}\}_{m=1}^\infty$ converges to G_k uniformly on each compact subset of $\mathbb{D}^k$. Let

$$G_k(\zeta) = \sum_{n \in \Xi_k} b_n \zeta^{\alpha(n)}, \quad \zeta \in \mathbb{D}^k$$

be the Taylor expansion of G_k. Since for every $m \in \mathbb{N}$,

$$\left(\mathcal{B}_k f_{\sigma_m}\right)(\zeta) = \sum_{n \in \Xi_k} a_n n^{-\sigma_m} \zeta^{\alpha(n)}, \quad \zeta \in \mathbb{D}^k,$$

the uniform convergence implies that

$$b_n = \lim_{m \to \infty} a_n n^{-\sigma_m} = a_n, \quad n \in \Xi_k,$$

and hence $(\mathcal{B}_k f)(\zeta) = \sum_{n \in \Xi_k} a_n \zeta^{\alpha(n)}$ converges in $\mathbb{D}^k$. Noticing that for every $\sigma > 0$, $\mathcal{B}_k f_\sigma = (\mathcal{B}_k f)_{\{\sigma\}}$, and combining (4.2) with (4.4) show that

$$(4.5) \quad \sup_{0<r<1} \int_{\mathbb{T}^\infty} \log\left(1 + |(\mathcal{B}_k f)_{[r]}|\right) dm_\infty$$

$$= \sup_{\sigma>0} \int_{\mathbb{T}^\infty} \log\left(1 + |\mathcal{B}_k f_\sigma|\right) dm_\infty \leqslant \|f\|_0 < \infty,$$

which implies $\mathcal{B}_k f \in N(\mathbb{D}^k)$. $\qquad\square$

For every $k \in \mathbb{N}$, $N(\mathbb{D}^k)$ can be regarded as a subset of $N(\mathbb{D}_1^\infty)$. Hence by Proposition 4.4, when $f \in \mathcal{N}_u$, $\{\mathcal{B}_k f\}_{k=1}^\infty$ is a sequence in $N(\mathbb{D}_1^\infty)$. Then it follows from Lemma 2.10 and (4.5) that for each $\zeta \in \mathbb{D}_1^\infty$,

$$\log \left(1 + |(\mathcal{B}_k f)(\zeta)|\right) \leqslant \|\mathbf{P}_\zeta\|_\infty \|\mathcal{B}_k f\|_0 \leqslant \|\mathbf{P}_\zeta\|_\infty \|f\|_0.$$

Therefore we have

COROLLARY 4.5. — *If $f \in \mathcal{N}_u$, then the sequence $\{\mathcal{B}_k f\}_{k=1}^\infty$ is uniformly bounded on each domain $V_{r,M}$ $(0 < r < 1, M > 0)$ defined by (2.1).*

We need the following result.

PROPOSITION 4.6. — *If $f \in \mathcal{N}_u$, then the sequence $\{\mathcal{B}_k f\}_{k=1}^\infty$ converges pointwise on $\mathbb{D}_1^\infty$, and*

$$(\widehat{\mathcal{B}} f)(\zeta) = \lim_{k \to \infty} (\mathcal{B}_k f)(\zeta), \quad \zeta \in \mathbb{D}_1^\infty$$

defines a holomorphic function on $\mathbb{D}_1^\infty$.

The following lemma is used in the proof of Proposition 4.6.

LEMMA 4.7. — *If $\{a_n\}_{n=1}^\infty \in \ell^1$, then there exist $\{b_n\}_{n=1}^\infty \in \ell^1$ and $\{c_n\}_{n=1}^\infty \in c_0$ such that $a_n = b_n c_n$, $n = 1, 2, \ldots$, where c_0 denotes the Banach space of null sequences.*

Proof. — Without loss of generality, we may assume that $\{a_n\}_{n=1}^\infty$ has infinitely many nonzero entries. It suffices to show that there is a sequence $\{\lambda_n\}_{n=1}^\infty$ such that $\lambda_n \to \infty$ as $n \to \infty$, and $\sum_{n=1}^\infty |a_n \lambda_n| < \infty$. Write $A = \sum_{n=1}^\infty |a_n|$ and

$$k_j = \min \left\{ k \in \mathbb{N} : \sum_{n=1}^k |a_n| \geqslant \frac{6A}{\pi^2} \sum_{n=1}^j \frac{1}{n^2} \right\}, \quad j \in \mathbb{N}.$$

Then there is a subsequence $\{k_{s_j}\}_{j=1}^\infty$ of $\{k_j\}_{j=1}^\infty$ such that $k_{s_1} < k_{s_2} < \cdots$, and for each $j \in \mathbb{N}$, we have

$$\sum_{n=1}^{k_{s_j}} |a_n| \geqslant \frac{6A}{\pi^2} \sum_{n=1}^{s_j} \frac{1}{n^2}, \quad \sum_{n=1}^{k_{s_{j+1}}} |a_n| \leqslant \sum_{n=1}^{k_{s_{j+2}}-1} |a_n| \leqslant \frac{6A}{\pi^2} \sum_{n=1}^{s_{j+2}} \frac{1}{n^2},$$

and hence

(4.6)
$$\sum_{n=k_{s_j}+1}^{k_{s_{j+1}}} |a_n| \leqslant \frac{6A}{\pi^2} \sum_{n=s_j+1}^{s_{j+2}} \frac{1}{n^2}.$$

If $k_{s_j} + 1 \leqslant n \leqslant k_{s_{j+1}}$, set $\lambda_n = \sqrt{s_j + 1}$. Then $\lambda_n \to \infty$ as $n \to \infty$, and by (4.6),

$$\sum_{n=k_{s_1}+1}^{\infty} |a_n \lambda_n| = \sum_{j=1}^{\infty} \sum_{n=k_{s_j}+1}^{k_{s_{j+1}}} |a_n \lambda_n|$$

$$= \sum_{j=1}^{\infty} \sqrt{s_j + 1} \sum_{n=k_{s_j}+1}^{k_{s_{j+1}}} |a_n|$$

$$\leqslant \frac{6A}{\pi^2} \sum_{j=1}^{\infty} \sqrt{s_j + 1} \sum_{n=s_j+1}^{s_{j+2}} \frac{1}{n^2}$$

$$\leqslant \frac{6A}{\pi^2} \sum_{j=1}^{\infty} \sum_{n=s_j+1}^{s_{j+2}} \frac{1}{n\sqrt{n}}$$

$$\leqslant \frac{12A}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$$

$$< \infty,$$

which completes the proof of Lemma 4.7. $\qquad \square$

We now present the proof of Proposition 4.6.

Proof of Proposition 4.6. We first show that $\{\mathcal{B}_k f\}_{k=1}^{\infty}$ converges pointwise on $\mathbb{D}_1^{\infty}$. For a fixed $\zeta \in \mathbb{D}_1^{\infty}$, it follows from Lemma 4.7 that there exist $\alpha \in \mathbb{D}_1^{\infty}$ and $\beta \in \mathbb{B}_0$ such that $\zeta_n = \alpha_n \beta_n$, $n = 1, 2, \ldots$, where $\mathbb{B}_0$ denotes the open unit ball of c_0. By Corollary 4.5, $\{\mathcal{B}_k f\}_{k=1}^{\infty}$ is uniformly bounded on $\Delta_\alpha = \alpha_1 \overline{\mathbb{D}} \times \cdots \times \alpha_n \overline{\mathbb{D}} \times \cdots$. Set

$$C_\alpha = \sup_{k \in \mathbb{N},\, v \in \Delta_\alpha} |(\mathcal{B}_k f)(v)| + 1.$$

For any $\varepsilon > 0$, there exists an index K for which if $k > K$, then $|\beta_k| < \frac{\varepsilon}{2C_\alpha}$. By a similar argument as in [26, 27], we conclude that $\{(\mathcal{B}_k f)(\zeta)\}_{k=1}^{\infty}$ is Cauchy. Indeed, for $k > l > K$ and $z = (z_1, \ldots, z_{k-l}) \in \mathbb{D}^{k-l}$, set

$$F(z) = (\mathcal{B}_k f)\left(\alpha_1 \beta_1, \ldots, \alpha_l \beta_l, \alpha_{l+1} z_1, \ldots, \alpha_k z_{k-l} \right).$$

We see that F is holomorphic on $\mathbb{D}^{k-l}$, and $\sup_{z \in \mathbb{D}^{k-l}} |F(z)| \leqslant C_\alpha$. By Schwarz's lemma [35, Theorem 1.9], we have

$$|(\mathcal{B}_k f)(\zeta) - (\mathcal{B}_l f)(\zeta)| = |F(\beta_{l+1}, \ldots, \beta_k) - F(0)| \leqslant 2C_\alpha \max_{l+1 \leqslant j \leqslant k} |\beta_j| < \varepsilon,$$

which implies that $\{(\mathcal{B}_k f)(\zeta)\}_{k=1}^{\infty}$ is Cauchy, and hence $\{\mathcal{B}_k f\}_{k=1}^{\infty}$ converges pointwise on $\mathbb{D}_1^{\infty}$. Write

(4.7) $$(\widehat{\mathcal{B}} f)(\zeta) = \lim_{k \to \infty} (\mathcal{B}_k f)(\zeta), \quad \zeta \in \mathbb{D}_1^{\infty}.$$

Then Corollary 4.5 implies that $\widehat{\mathcal{B}}f$ is locally bounded. In what follows we show that $\widehat{\mathcal{B}}f$ is holomorphic on $\mathbb{D}_1^\infty$. For fixed $\zeta \in \mathbb{D}_1^\infty$ and $\xi \in \ell^1$, there exists an open domain $\Omega \subset \mathbb{C}$ such that $0 \in \Omega$ and $\{\zeta + \lambda\xi : \lambda \in \Omega\} \subset V_{r,M}$ for some $0 < r < 1$, $M > 0$. For each $k \in \mathbb{N}$, let

$$G_k(\lambda) = (\mathcal{B}_k f)(\zeta + \lambda\xi), \quad \lambda \in \Omega.$$

Then $\{G_k\}_{k=1}^\infty$ is sequence of holomorphic functions on Ω, and by Corollary 4.5, it is uniformly bounded on Ω. Applying Montel's theorem, there is a subsequence that converges uniformly on each compact subset of Ω. By combining this with (4.7), $(\widehat{\mathcal{B}}f)(\zeta + \lambda\xi)$ is holomorphic in parameter $\lambda \in \Omega$, and hence $\widehat{\mathcal{B}}f$ is holomorphic on $\mathbb{D}_1^\infty$. $\square$

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. We first prove that $(\mathcal{B}f)(\zeta) = \sum_{n=1}^\infty a_n \zeta^{\alpha(n)}$ converges to a holomorphic function on $\mathbb{D}_1^\infty$. By Proposition 4.6, the sequence $\{\mathcal{B}_k f\}_{k=1}^\infty$ converges to a holomorphic function $\widehat{\mathcal{B}}f$. Let

$$(4.8) \qquad (\widehat{\mathcal{B}}f)(\zeta) = \sum_{n=1}^\infty c_n \zeta^{\alpha(n)}, \quad \zeta \in \mathbb{D}_1^\infty$$

be the monomial expansion of $\widehat{\mathcal{B}}f$. For a fixed $k \in \mathbb{N}$, Bohr's $k$te Abschnitt $A_k(\widehat{\mathcal{B}}f)$ is holomorphic on $\mathbb{D}^k$, and

$$A_k(\widehat{\mathcal{B}}f)(\zeta) = \sum_{n \in \Xi_k} c_n \zeta^{\alpha(n)}, \quad \zeta \in \mathbb{D}^k.$$

On the other hand, by the definition (4.7) of $\widehat{\mathcal{B}}f$, we have

$$A_k(\widehat{\mathcal{B}}f)(\zeta) = (\mathcal{B}_k f)(\zeta) = \sum_{n \in \Xi_k} a_n \zeta^{\alpha(n)}, \quad \zeta \in \mathbb{D}^k.$$

It follows from the uniqueness of Taylor expansion that for all $n \in \Xi_k$, $a_n = c_n$, and hence $a_n = c_n$ for all $n \in \mathbb{N}$ by the arbitrariness of k. Comparison of this and (4.8) shows that

$$(\mathcal{B}f)(\zeta) = \sum_{n=1}^\infty a_n \zeta^{\alpha(n)} = \sum_{n=1}^\infty c_n \zeta^{\alpha(n)}$$

converges in $\mathbb{D}_1^\infty$, and $\mathcal{B}f = \widehat{\mathcal{B}}f$.

We next show that $\mathcal{B}f \in N(\mathbb{D}_1^\infty)$ and $\|\mathcal{B}f\|_0 = \|f\|_0$. Since $\mathcal{B}f$ is holomorphic on $\mathbb{D}_1^\infty$, $(\mathcal{B}f)_{[r]} \in A(\mathbb{T}^\infty)$ for all $0 < r < 1$, and thus

$$\|\mathcal{B}f\|_0 = \sup_{0<r<1} \int_{\mathbb{T}^\infty} \log\left(1 + \left|(\mathcal{B}f)_{[r]}\right|\right) dm_\infty$$

$$= \sup_{0<r<1} \sup_{k \in \mathbb{N}} \int_{\mathbb{T}^\infty} \log\left(1 + \left|(\mathcal{B}_k f)_{[r]}\right|\right) dm_\infty.$$

By interchanging the order of taking supremum for r and k, we have

$$(4.9) \qquad \|\mathcal{B}f\|_0 = \sup_{k \in \mathbb{N}} \sup_{0<r<1} \int_{\mathbb{T}^\infty} \log\left(1 + \left|(\mathcal{B}_k f)_{[r]}\right|\right) dm_\infty.$$

Since for $k \in \mathbb{N}$, $\mathcal{B}_k f$ is holomorphic on $\mathbb{D}^k$, it follows from (4.2) that

$$\sup_{0<r<1} \int_{\mathbb{T}^\infty} \log\left(1 + \left|(\mathcal{B}_k f)_{[r]}\right|\right) dm_\infty = \sup_{\sigma>0} \int_{\mathbb{T}^\infty} \log\left(1 + |\mathcal{B}_k f_\sigma|\right) dm_\infty.$$

Substituting this into (4.9) yields

$$(4.10) \qquad \begin{aligned} \|\mathcal{B}f\|_0 &= \sup_{k \in \mathbb{N}} \sup_{\sigma>0} \int_{\mathbb{T}^\infty} \log\left(1 + |\mathcal{B}_k f_\sigma|\right) dm_\infty \\ &= \sup_{\sigma>0} \sup_{k \in \mathbb{N}} \int_{\mathbb{T}^\infty} \log\left(1 + |\mathcal{B}_k f_\sigma|\right) dm_\infty. \end{aligned}$$

Note that for every $\sigma > 0$, $\mathcal{B}f_\sigma \in A(\mathbb{T}^\infty)$. Then

$$\sup_{k \in \mathbb{N}} \int_{\mathbb{T}^\infty} \log\left(1 + |\mathcal{B}_k f_\sigma|\right) dm_\infty = \int_{\mathbb{T}^\infty} \log\left(1 + |\mathcal{B}f_\sigma|\right) dm_\infty = \|\mathcal{B}f_\sigma\|_0.$$

We conclude from this equality, (4.1) and (4.10) that

$$\|\mathcal{B}f\|_0 = \sup_{\sigma>0} \|\mathcal{B}f_\sigma\|_0 = \sup_{\sigma>0} \|f_\sigma\|_0 = \|f\|_0 < \infty.$$

The proof of Theorem 4.1 is complete. $\qquad\qquad\qquad\qquad\qquad\square$

Let $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ be a Dirichlet series with $\sigma_u(f) \leqslant 0$. Example 4.2 shows that $(\mathcal{B}f)(\zeta) = \sum_{n=1}^\infty a_n \zeta^{\alpha(n)}$ does not need to converge at all points in $\mathbb{D}_1^\infty$. However, for every $0 < r < 1$, the power series

$$(\mathcal{B}_r f)(\zeta) = \sum_{n=1}^\infty a_n \left(r\zeta_1, \ldots, r^k\zeta_k, \ldots\right)^{\alpha(n)}$$

converges in $\overline{\mathbb{D}}^\infty$, and $\mathcal{B}_r f \in A(\mathbb{D}^\infty)$. Indeed, let p_n be the nth prime number, then the prime number theorem implies that $\frac{p_n}{n \log n} \to 1$ as $n \to \infty$ [4, Theorem 4.5]. Therefore, for a fixed $0 < r < 1$, there exists $\sigma > 0$ such that for all $n \in \mathbb{N}$, $r^n \leqslant p_n^{-\sigma}$, and hence

$$(4.11) \qquad r\overline{\mathbb{D}} \times \cdots \times r^n\overline{\mathbb{D}} \times \cdots \subset p_1^{-\sigma}\overline{\mathbb{D}} \times \cdots \times p_n^{-\sigma}\overline{\mathbb{D}} \times \cdots.$$

As mentioned in [26], the partial sums of $\mathcal{B}f_\sigma$ converge uniformly on $\overline{\mathbb{D}}^\infty$. That is, for any $\varepsilon > 0$, there is an index N for which if $M_2 > M_1 > N$, then

$$\left\| \sum_{n=M_1}^{M_2} a_n \left(p_1^{-\sigma} \zeta_1, \ldots, p_k^{-\sigma} \zeta_k, \ldots \right)^{\alpha(n)} \right\|_\infty < \varepsilon,$$

where $\| \cdot \|_\infty$ denotes the uniform norm in Banach algebra $C(\overline{\mathbb{D}}^\infty)$. From (4.11), we have

$$\left\| \sum_{n=M_1}^{M_2} a_n \left(r\zeta_1, \ldots, r^k \zeta_k, \ldots \right)^{\alpha(n)} \right\|_\infty$$

$$\leqslant \left\| \sum_{n=M_1}^{M_2} a_n \left(p_1^{-\sigma} \zeta_1, \ldots, p_k^{-\sigma} \zeta_k, \ldots \right)^{\alpha(n)} \right\|_\infty < \varepsilon,$$

and thus the partial sums of $\mathcal{B}_r f$ converge uniformly on $\overline{\mathbb{D}}^\infty$. This gives $\mathcal{B}_r f \in A(\mathbb{D}^\infty)$ as desired. By this fact and a similar argument as in the proof of Theorem 4.1, we have the following conclusion.

PROPOSITION 4.8. — *Let f be a Dirichlet series with $\sigma_u(f) \leqslant 0$. Then*

$$\sup_{\sigma > 0} \| f_\sigma \|_0 = \sup_{\sigma > 0} \int_{\mathbb{T}^\infty} \log \left(1 + |\mathcal{B}f_\sigma| \right) dm_\infty$$

$$= \sup_{0 < r < 1} \int_{\mathbb{T}^\infty} \log \left(1 + |\mathcal{B}_r f| \right) dm_\infty.$$

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