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# EIGENVALUE ASYMPTOTICS AND UNIQUE CONTINUATION OF EIGENFUNCTIONS ON PLANAR GRAPHS

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ABSTRACT. — We study planar graphs with large negative curvature outside of a finite set and the spectral theory of Schrödinger operators on these graphs. We obtain estimates on the first and second order term of the eigenvalue asymptotics. Moreover, we prove a unique continuation result for eigenfunctions and decay properties of general eigenfunctions. The proofs rely on a detailed analysis of the geometry which employs a Copy-and-Paste procedure based on the Gauß–Bonnet theorem.

RÉSUMÉ. — Dans ce travail, nous étudions les graphes planaires à courbure négative grande en dehors d'un ensemble fini et plus précisément la théorie spectrale d'opérateurs de Schrödinger sur de tels graphes. Nous obtenons des estimées du premier et du second ordre pour l'asymptotique des valeurs propres. Nous prouvons de plus un résultat de continuation unique ainsi que des propriétés de décroissance des fonctions propres. Les preuves sont basées sur une analyse fine de la géométrie et utilisent une procédure de "copier-coller" qui repose elle-même sur le théorème de Gauß–Bonnet.

## 1. Introduction

In recent years consequences of curvature bounds on the geometry and spectral theory of graphs have been intensively studied. For planar graphs a notion of curvature was introduced by Stone [26] going back to ideas to Alexandrov and even Descartes. Recently, the study of this curvature gained some momentum. For positive and non-negative curvature geometric consequences and harmonic functions were studied in [6, 7, 14, 15, 26,

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*Keywords:* planar graph, Gauß–Bonnet Theorem, Schrödinger operator, eigenvalues asymptotics, unique continuation.

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29]. On the other hand, the geometry of negative and non-positive curvature was investigated in [2, 3, 13, 17, 20, 24, 27] as well as for spectral consequences see [1, 16, 17, 23]. For more recent work on sectional curvature of polygonal complexes see [21].

The subject of this paper are planar graphs with large negative curvature outside a finite set and we are interested in the spectral theory of the Laplacian or more general that of Schrödinger operators. Especially, we study the asymptotics of eigenvalues, existence of eigenfunctions of compact support and decay properties of eigenfunctions in general.

Let us discuss the results of the paper in the light of the existing literature. In [16] it was proven that if the curvature tends to negative infinity uniformly then the spectrum of the Laplacian is purely discrete. The first order term of the eigenvalue asymptotics was obtained [5, 10] for so called sparse graphs which include planar graphs. Here we get a hold on the second order term of the asymptotics of the eigenvalues in the case of planar graphs with uniformly decreasing curvature, see Theorem 1.6 and Corollary 1.8.

Next, we turn to eigenfunctions. In [17, 23] unique continuation results for graphs under non-positive curvature were shown. However, the relevant curvature notion is the one of corner curvature, see Formula (C) below, and these results are rather delicate. For example they fail to hold for the Kagome lattice which has non-positive vertex curvature only, see [23]. Moreover, we also present an example showing that the failure of the curvature assumption on a finite set can lead to the existence of infinitely many compactly supported eigenfunctions, see Section 3.7. On the other hand, we show that if the curvature is sufficiently negative outside a finite set, then compactly supported eigenfunctions can occur in a finite region only, see Theorem 1.9.

Finally, we prove Agmon estimates as they were recently obtained in [22] to give decay results on general eigenfunctions, Theorem 1.11.

To prove these results we carefully study the geometry of graphs with large degree outside a finite set. The underlying philosophy (which is made precise later in the paper) is that we can continue such a planar graph to a tessellation with non-positive corner curvature after generously removing the set of positive curvature.

While the geometric results are mainly phrased without mentioning curvature the proofs make use of the Gauß–Bonnet theorem – which essentially involves curvature. Firstly, the geometric results include statements about

the sphere structure of the graph sufficiently far outside. These considerations yield immediately a Cartan–Hadamard type result about continuation of geodesics, see Theorem 1.1 for these results. While we also recover some of the results of [2, 3] our approach is independent of theirs. Secondly we investigate the existence of spanning trees that are in some sense close to the original graphs. In particular, we show that there exist spanning trees that are bounded perturbations of the original graph, Theorem 1.2.

These geometric results are then applied to the study of the spectral theory of Schrödinger operators in the following.

The paper is structured as follows. In the next subsection we introduce the basic notions and in the following two subsections we present the geometric and spectral results. The proof of the geometric result relies heavily on a so called Copy-and-Paste procedure presented in Section 2. Next, we closely study the case of triangulations in Section 3. Then the geometric results follow rather directly from considering triangulation supergraphs. A result about spanning trees is proven in Subsection 4.1 and the result about continuing a graph with negative curvature outside a finite set to a non-positively curved tessellation is shown in Section 4. The unique continuation result is also proven in this section. Finally, in Section 5 we prove the results about discrete spectrum, the asymptotics of eigenvalues and the decay of eigenfunctions.

### 1.1. Set up and definitions

Let an infinite connected simple graph  $G = (V, E)$  be given. The *degree*  $\text{deg}(v)$  of a vertex  $v \in V$  is the number of adjacent vertices. We assume  $2 \leq \text{deg}(v) < \infty$  for all  $v \in V$ . We call a sequence of vertices  $(v_0, \dots, v_n)$  a *walk* of length  $n$  if  $v_0 \sim \dots \sim v_n$ , where  $v \sim w$  denotes that  $v$  and  $w$  are adjacent.

We denote by  $d$  the *natural graph distance* on  $G$  which is the length of the shortest walk between two vertices.

We fix a vertex  $o \in V$  which we call the *root*. For  $r \geq 0$ , we define the *sphere* with respect the natural graph distance by

$$S_r := S_r(o) := \{v \in V \mid d(o, v) = r\}.$$

The distance *balls* are defined as

$$B_r := B_r(o) := \{v \in V \mid d(o, v) \leq r\}.$$

For a vertex  $v \in S_r$ ,  $r \geq 0$ , we call  $w \in S_{r\pm 1}$ ,  $v \sim w$  a *forward/backward neighbor* and denote

$$\text{deg}_\pm(v) := |\{w \in S_{r\pm 1} \mid w \sim v\}| \quad \text{and} \quad \text{deg}_0(v) := |\{w \in S_r \mid w \sim v\}|,$$

where  $|A|$  denotes the number of elements of a set  $A \subseteq V$ .

We assume that  $G$  is a planar graph which is embedded into an orientable topological surface  $\mathcal{S}$  homeomorphic to  $\mathbb{R}^2$  or the sphere  $\mathbb{S}^2$ . We assume that the embedding of  $G$  is *locally finite*, that is for every point in  $\mathcal{S}$  there is a neighborhood which intersects only finitely many edges.

From now on, when we speak about *planar graphs* we always assume to have an infinite connected simple planar graph which admits a locally finite embedding.

We associate to  $G$  the set of *faces*  $F$  whose elements are defined as the closures of the connected components of  $\mathcal{S} \setminus \bigcup E$ , i.e., the connected components of  $\mathcal{S}$  after removing the edge segments. The *boundary* of a face  $f$  is defined as the elements of  $V$  whose image belongs to  $f$ . A *boundary walk* of  $f$  is a closed walk which visits every vertex of  $f$ . The length of the shortest boundary walk is called the *degree*  $\text{deg}(f)$  of the face  $f \in F$  and if no closed boundary walk exists we say that  $f$  has infinite degree. In what follows we do not distinguish between the graph and its embedding.

The set of *corners*  $C(G)$  is given as the set of pairs  $(v, f) \in V \times F$  such that  $v$  is contained in  $f$ . The *degree*  $|(v, f)|$  of a corner  $(v, f)$  is the minimal number of times the vertex  $v$  is met by a boundary walk of  $f$ . The *corner curvature*  $\kappa_C : C(G) \rightarrow \mathbb{R}$  is given by

$$(C) \quad \kappa_C(v, f) := \frac{1}{\text{deg}(v)} - \frac{1}{2} + \frac{1}{\text{deg}(f)}.$$

This quantity was first introduced in [2, 3] for tessellations and in [17] for general planar graphs. Summing over all corners of a vertex gives the *vertex curvature*  $\kappa : V \rightarrow \mathbb{R}$

$$\kappa(v) := \sum_{(v, f) \in C(G)} |(v, f)| \kappa_C(v, f).$$

This quantity was first defined in [26] for tessellations following ideas of Alexandrov and for general planar graphs in [17]. In [17, Proposition 1] (or [2] for tessellations) a Gauß–Bonnet formula for this curvature is shown which reads for a connected finite planar graph embedded in  $\mathbb{S}^2$  as

$$(GB) \quad \sum_{v \in V} \kappa(v) = \chi(\mathbb{S}^2) = 2,$$

where the  $\chi(\mathbb{S}^2)$  is the Euler characteristic of  $\mathbb{S}^2$ . Moreover, it is shown in [17], that since  $\deg(v) = \sum_{f \ni v} |(v, f)|$ , for all  $v \in V$ , one has

$$\kappa(v) = 1 - \frac{\deg(v)}{2} + \sum_{f \in F, f \ni v} |(v, f)| \frac{1}{\deg(f)}.$$

The most interesting examples are tessellations which are discussed in a slightly more general form in Section 3.7.

We continue by introducing some more notation needed for the paper. For two walks  $p = (v_0, \dots, v_n)$  and  $q = (w_0, \dots, w_m)$  with  $v_n = w_0$  or  $v_0 = w_m$ , we denote by  $p + q$  the walk  $(v_0, \dots, v_n, w_1, \dots, w_m)$  if  $v_n = w_0$  or  $(v_1, \dots, v_n, w_{m-1}, \dots, w_0)$  if  $v_n = w_m$ . A walk  $(v_0, \dots, v_n)$  is called a *path* if the vertices in a walk are pairwise different except for possibly  $v_0 = v_n$ . We say that  $n$  is the length of the path. Moreover, a walk  $(v_n)$  is called a *geodesic* if  $d(v_0, v_n) = n$  for all  $n$ . For a walk  $p = (v_0, \dots, v_n)$ , we denote its vertex set by  $V(p) = \{v_0, \dots, v_n\}$  and call it the *trace* of  $p$ . We call the vertices  $v_1, \dots, v_{n-1}$  the *inner vertices* and  $v_0, v_n$  the *outer vertices* of the walk  $p$ . We call  $p$  *closed* if  $v_0 = v_n$ . Note that the definition of a path does not allow repetition of vertices apart from the beginning and the ending vertex. To stress this we sometimes refer to closed paths also as simply closed paths.

Each simply closed path  $p$  in the graph induces a simply closed curve with image  $\gamma(p)$  in the surface  $\mathcal{S}$  where the graph is embedded. By Jordan's curve theorem, this induces a partition of  $\mathcal{S}$  as

$$\mathcal{S} = \mathcal{B}(p) \cup \gamma(p) \cup \mathcal{U}(p),$$

where  $\mathcal{B}(p)$  and  $\mathcal{U}(p)$  are respectively the bounded and the unbounded connected component of  $\mathcal{S} \setminus \gamma(p)$ .

For a subset  $W \subseteq V$ , let  $G_W$  be the *induced subgraph*  $(W, E_W)$ , where  $E_W \subseteq E$  is the set of edges with beginning and end vertex in  $W$ . We say that  $G_W$  has a *closed boundary path* if there is a closed path  $p$  within the graph  $G_W$  such that  $\mathcal{B}(p) \cap V = W$ . Every vertex in  $W$  not contained in a boundary path is called an *interior vertex* of  $G_W$ . Indeed, boundary walks are unique up to enumeration.

When we consider two planar graphs  $G$  and  $G'$  at the same time we denote the degree on  $G'$  by  $\deg'$  or  $\deg^{(G')}$ , the curvatures by  $\kappa'_C$  or  $\kappa_C^{(G')}$ ,  $\kappa'$  or  $\kappa^{(G')}$  and the natural graph distance by  $d'$  or  $d^{(G')}$ .

## 1.2. Geometric results

In this work we first show that planar graphs with vertex degree large outside a finite set are in some sense really close to tree graphs. We shall consider two situations. First, we consider  $\deg \geq 6$  for all vertices except possibly for the root and secondly  $\deg \geq 7$  outside a finite set.

The first theorem is a Cartan–Hadarmard type theorem. This says that (sufficiently long) geodesics can be continued indefinitely which is equivalent to the function  $d(o, \cdot)$  not having local maxima (outside a finite set). In the literature this is also referred to as absence or emptiness of the cut-locus (which is the set where  $d(o, \cdot)$  attains its local maxima), [2, 3].

Furthermore the theorem includes a remarkable structural statement about distance spheres. To this end, we say a subset  $W$  of a planar graph  $G$  can be *cyclically ordered* if there is a planar supergraph  $G'$  of  $G$  such that  $W$  is the trace of a simply closed path of  $G'$ .

**THEOREM 1.1.** — *Let  $G = (V, E)$  be a planar graph, such that one of the following conditions hold:*

- (a)  $\deg \geq 6$  outside the root  $o$ .
- (b)  $\deg \geq 7$  outside some finite set.

*Then, there exists a finite set  $K \subseteq V$  (which can be chosen to be  $K = \{o\}$  in case (a)) such that for all  $v \in V \setminus K$*

$$\deg_0(v) \leq 2 \quad \text{and} \quad 1 \leq \deg_-(v) \leq 2.$$

*In particular, any geodesic reaching  $V \setminus K$  from  $o$  can be continued indefinitely. Furthermore, all the spheres outside  $V \setminus K$  can be cyclically ordered.*

Observe that (a) is already included in [2, 3] since  $\deg \geq 6$  implies  $\kappa_C \leq 0$  but our proof follows a completely different strategy. However, our techniques also allow us to change our graphs by replacing a finite set with a vertex such that they become graphs with  $\kappa_C \leq 0$  everywhere. This is discussed in detail in Section 3.7.

The second result is that planar graphs with large vertex degree are close to some of their spanning trees in the sense that it is a bounded perturbation. Two graphs over the same vertex set have the *same sphere structure* if for a fixed root the distance spheres are the same sets for both graphs.

**THEOREM 1.2.** — *Let  $G$  be a planar graph, such that one of the following conditions hold:*

- (a)  $\deg \geq 6$  outside the root  $o$ .

(b)  $\text{deg} \geq 7$  outside some finite set.

Then, there exists a spanning tree  $T$  of  $G$  such that outside a finite set the vertex degrees of  $T$  and  $G$  differ at most by 4, where the finite set is empty in case (a). Furthermore,  $T$  and  $G$  have the same sphere structure.

*Remark 1.3.* — The existence of spanning trees with certain properties is also treated in [4, 8].

*Remark 1.4.* — Our techniques of proof allow us to quantify the finite set in the theorems, Theorem 1.1 and Theorem 1.2. In fact, given the radius of the ball out of which the degree is larger than 7, one can give an estimate on the radius of the ball such that outside this ball the statements hold.

*Remark 1.5.* — It would be interesting to study whether the criteria  $\text{deg} \geq 6$  outside the root or  $\text{deg} \geq 7$  outside a finite set can be replaced by a weaker curvature type assumption. In the case of triangulations  $\text{deg} \geq 6$  is equivalent to  $\kappa_C \leq 0$  and  $\text{deg} \geq 7$  is equivalent to  $\kappa_C < 0$ . It remains an open question which of the results still hold for  $\text{deg} \geq 6$  outside a finite set.

The proof of the geometrical results above for general planar graphs is given in Section 4. It will follow from the case of planar triangulations by an embedding into a triangulation supergraph. The case of triangulation is investigated in Section 3. It uses a Copy-and-Paste procedure given in Section 2 and a fine study of an adapted new sphere structure.

### 1.3. Spectral consequences

In this section, we turn to some spectral consequences for the Laplacian on  $\ell^2(V)$ . We introduce some notation first.

Denote the space of square summable real valued functions by  $\ell^2(V)$ , the corresponding scalar product by  $\langle \cdot, \cdot \rangle$  and the norm by  $\| \cdot \|$ .

We consider the Laplace operator  $\Delta = \Delta_G$  defined as

$$D(\Delta) := \left\{ \varphi \in \ell^2(V) \mid \left( v \mapsto \sum_{w \sim v} (\varphi(v) - \varphi(w)) \right) \in \ell^2(V) \right\}$$

$$\Delta\varphi(v) := \sum_{w \sim v} (\varphi(v) - \varphi(w)).$$

The operator is positive and selfadjoint (confer [28, Theorem 1.3.1.]).

For a function  $g : V \rightarrow \mathbb{R}$ , we denote with slight abuse of notation the operator of multiplication again by  $g$ , as well as the corresponding quadratic form

$$g(\varphi) := \langle g\varphi, \varphi \rangle = \sum_{v \in V} g(v)\varphi(v)^2$$

and for  $\varphi \in C_c(V)$  (which are the real valued functions of compact support).

For two self-adjoint operators  $A$  and  $A'$  on  $\ell^2(V)$  and a subspace  $D_0 \subseteq D(A) \cap D(A')$  we write  $A \leq A'$  on  $D_0$  if  $\langle A\varphi, \varphi \rangle \leq \langle A'\varphi, \varphi \rangle$  for  $\varphi \in D_0$ .

For a function  $q$ , we let the positive and negative part be given by  $q_{\pm} = \max\{\pm q, 0\}$ . We denote by  $K_{\alpha}$ ,  $\alpha \in (0, 1]$ , the class of potentials  $q : V \rightarrow \mathbb{R}$  such that there is  $C \geq 0$  such that

$$q_- \leq \alpha(\Delta + q_+) + C, \quad \text{on } C_c(V).$$

As the operator  $(\Delta + q)|_{C_c(V)}$  is symmetric and bounded from below for  $q$  in  $K_{\alpha}$ ,  $\alpha \in (0, 1]$ , it admits a Friedrich's extension which we also denote by  $\Delta + q$ .

For a self-adjoint operator  $A$  which is bounded from below, we denote the discrete eigenvalues below the bottom of the essential spectrum by  $\lambda_n(A)$  in increasing order counted with multiplicity for all  $n \geq 0$  for which they exist. We denote

$$d_n = \lambda_n(\text{deg} + q), \quad n \geq 0.$$

Furthermore, we use the Landau notation  $o(a_n)$  for a sequences  $(b_n)$  such that  $b_n/a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

The following theorem is the main result about the asymptotics of eigenvalues.

**THEOREM 1.6.** — *Let  $G$  be a planar graph and  $q \in K_{\alpha}$ ,  $\alpha \in (0, 1)$ . Then the spectrum of  $\Delta + q$  is purely discrete if and only if*

$$\sup_{K \subset V \text{ finite}} \inf_{v \in V \setminus K} (-\kappa(v) + q(v)) = \infty.$$

*In this case and if  $q \geq 0$*

$$d_n - 2\sqrt{d_n} - o\left(\sqrt{d_n}\right) \leq \lambda_n(\Delta + q) \leq d_n + 2\sqrt{d_n} + o\left(\sqrt{d_n}\right).$$

*Remark 1.7.*

- (a) The first part of the theorem above was announced in [18] and is a unification of [16, Theorem 3] and [19, Corollary 21] for Schrödinger operators on planar graphs. The second part improves the considerations of [5, 10] by giving the second order term on the eigenvalue asymptotics.

(b) For potentials  $q$  in  $\bigcap_{\alpha \in (0,1)} K_\alpha$  instead of  $q \geq 0$ , one has still the same first term of the eigenvalue asymptotics, see [5].

In the case of planar graphs with constant face degree we can even prove bounds with an even more geometric flavor. For  $k \geq 3$ , denote

$$\gamma(k) := 2\pi \frac{2(k-2)}{k},$$

that is, if a face  $f \in F$  with  $\deg(f) = k$  is a regular  $k$ -gon, then  $\gamma(k)$  is the inner angle of  $f$ . Moreover, denote

$$\kappa_n = -\lambda_n(-\kappa), \quad n \geq 0,$$

and in case there are infinitely many  $\kappa_n$  (which are a decreasing sequence) we let

$$\kappa_\infty = \lim_{n \rightarrow \infty} \kappa_n.$$

**COROLLARY 1.8.** — *Let  $G$  be a planar graph whose face degree is constantly  $k$  outside some finite set. The operator  $\Delta$  has purely discrete spectrum if and only if*

$$\kappa_\infty = -\infty.$$

In this case,  $\kappa_n \leq 0$  for large  $n$  and

$$\begin{aligned} -\frac{2\pi}{\gamma(k)}\kappa_n - 2\sqrt{-\frac{2\pi}{\gamma(k)}\kappa_n} - o(\sqrt{\kappa_n}) \\ \leq \lambda_n(\Delta) \leq -\frac{2\pi}{\gamma(k)}\kappa_n + 2\sqrt{-\frac{2\pi}{\gamma(k)}\kappa_n} + o(\sqrt{\kappa_n}). \end{aligned}$$

The proofs of the preceding theorems and corollaries are given in Section 5.

From the results above we learn that in the case of uniformly unbounded curvature the spectrum consists of discrete eigenvalues. The following corollary is a unique continuation result telling us that outside a compact set eigenfunctions have unbounded support.

**THEOREM 1.9.** — *Let  $G$  be a planar graph. Assume*

$$\kappa_\infty = -\infty.$$

*Then, outside a finite set there are no eigenfunctions of compact support of  $\Delta + q$  for all  $q \in K_1$ . In particular, there are at most finitely many linearly independent eigenfunctions of compact support.*

*Remark 1.10.*

- (a) In [23, Theorem 1], [17, Theorem 9] it is proven for tessellations and locally tessellating graphs that  $\kappa_C \leq 0$  implies absence of compactly supported eigenfunctions for the operator  $\Delta + q$ . We emphasize that Theorem 1.9 is not a simple perturbation result of [17, 23]. Indeed, unique continuation is a rather subtle issue on discrete spaces. For example there are spaces with  $\kappa \leq 0$  and which admit compactly supported eigenfunctions see e.g. [2]. See also Example 3.18 in Section 3.6.
- (b) Validity of the theorem does not depend on the particular choice of  $\Delta + q$  but holds for arbitrary nearest neighbor operators (i.e., with arbitrary complex coefficients for the edges and arbitrary complex potentials), see Theorem 3.15 in Section 3.6.
- (c) From the proof of Theorem 1.9, we can deduce an explicit estimate on the size of the set where compactly supported eigenfunctions can be supported, see Theorem 4.3.

For the other eigenfunctions we obtain a result on the decay which is based on Agmon type estimates as they are developed in [22]. In Section 5, we give a simplified proof which is adapted to the situation of planar graphs.

**THEOREM 1.11.** — *Let  $G$  be a planar graph. Assume*

$$\kappa_\infty = -\infty.$$

*Then, any eigenfunction  $u \in D(\Delta)$  of  $\Delta$  satisfies*

$$\alpha^{d(o,\cdot)} u \in \ell^2(X, \text{deg}),$$

*for all  $0 < \alpha < 1 + \sqrt{2}$ .*

## 2. Copy-and-Paste Lemma

In this section we prove a lemma that shows that certain subgraphs imply the presence of positive curvature. The proof works by making several copies of the subgraph and pasting them along the boundary path. The resulting graph can be embedded in the two dimensional sphere. We then apply a discrete Gauß–Bonnet theorem. Similar ideas can be found in [17].

**LEMMA 2.1** (Copy-and-Paste Lemma). — *Let  $G' = (V', E')$  be a subgraph of a planar graph  $G = (V, E)$  with a simply closed boundary path  $p$  in  $G'$  such that there are (at most) three vertices  $v_0, v_1, v_2 \in V(p)$  such that*

- (a)  $\text{deg}'(v) \geq 4$  for all  $v \in V(p) \setminus \{v_0, v_1, v_2\}$ ,
- (b)  $\text{deg}'(v_0) \geq 3$ ,
- (c)  $\text{deg}'(v_1), \text{deg}'(v_2) \geq 2$ .

Then, there is  $v \in V' \setminus V(p)$  such that

$$\kappa(v) = \kappa'(v) > 0.$$

*Proof.* — The proof uses a copy and paste procedure applied to  $G'$  with boundary path  $p = p_0 + p_1 + p_2$  which is illustrated in Figure 2.1.

We denote the subpath of  $p$  from  $v_1$  to  $v_2$  by  $p_0$ , the subpath of  $p$  from  $v_2$  to  $v_0$  by  $p_1$  the one from  $v_0$  to  $v_1$  by  $p_2$ . We make two copies  $G^{(1)}, G^{(2)}$  of  $G'$  and denote the corresponding copies of  $p_j$  by  $p_j^{(1)}$  and  $p_j^{(2)}$ ,  $j = 0, 1, 2$ . We paste  $G^{(1)}$  and  $G^{(2)}$  along  $p_0^{(1)}$  and  $p_0^{(2)}$  (after reflecting  $G^{(2)}$  — where reflecting always means with respect to the path along the graphs are pasted). We denote the resulting graph by  $G_1$ . We denote the boundary path of  $G_1$  in the following way: Denote  $p_1^{(1)}$  by  $q_4$ ,  $p_1^{(2)}$  by  $q_3$ ,  $p_2^{(2)}$  by  $q_2$  and  $p_2^{(1)}$  by  $q_1$ .

Let us keep track of the vertex degrees in  $G_1$ :

- All vertices in the glued path of  $p_0^{(1)}$  and  $p_0^{(2)}$  have degree at least 6 in  $G_1$ .
- The copies of the vertices  $v_1$  and  $v_2$  in  $G^{(1)}$  and  $G^{(2)}$  that are merged have now vertex degree at least 3 in  $G_1$ .
- The two copies of  $v_0$  in  $G_1$  have still vertex degree at least 3.
- All other vertices in the boundary path have degree at least 4.

Next, we make seven copies  $G_1^{(j)}$ ,  $j = 0, \dots, 6$ , of  $G_1$  and we denote the corresponding subpaths of the boundary by  $q_1^{(j)}, \dots, q_4^{(j)}$ ,  $j = 0, \dots, 6$ .

We paste:

- $G_1^{(0)}$  to  $G_1^{(1)}$  along  $q_1^{(0)}$  and  $q_1^{(1)}$  (after reflecting  $G_1^{(1)}$ ),
- $G_1^{(0)}$  to  $G_1^{(2)}$  along  $q_2^{(0)}$  and  $q_2^{(2)}$  (after reflecting  $G_1^{(2)}$ ),
- $G_1^{(1)}$  to  $G_1^{(2)}$  along  $q_2^{(1)}$  and  $q_1^{(2)}$  (which is possible as  $q_2^{(1)}$  and  $q_1^{(2)}$  both originate from  $p_1$ ),
- $G_1^{(2)}$  to  $G_1^{(3)}$  along  $q_3^{(2)}$  and  $q_3^{(3)}$  (after rotating  $G_1^{(3)}$ ),
- $G_1^{(0)}$  to  $G_1^{(4)}$  along  $q_3^{(0)}$  and  $q_3^{(4)}$  (after reflecting  $G_1^{(4)}$ ),
- $G_1^{(3)}$  to  $G_1^{(4)}$  along  $q_2^{(3)}$  and  $q_2^{(4)}$ ,
- $G_1^{(0)}$  to  $G_1^{(5)}$  along  $q_4^{(0)}$  and  $q_4^{(5)}$  (after reflecting  $G_1^{(5)}$ ),
- $G_1^{(4)}$  to  $G_1^{(5)}$  along  $q_4^{(4)}$  and  $q_3^{(5)}$  (which is possible as  $q_4^{(4)}$  and  $q_3^{(5)}$  both originate from  $p_2$ ),
- $G_1^{(5)}$  to  $G_1^{(6)}$  along  $q_1^{(5)}$  and  $q_1^{(6)}$  (after rotating  $G_1^{(6)}$ ),
- $G_1^{(1)}$  to  $G_1^{(6)}$  along  $q_4^{(1)}$  and  $q_4^{(6)}$ .

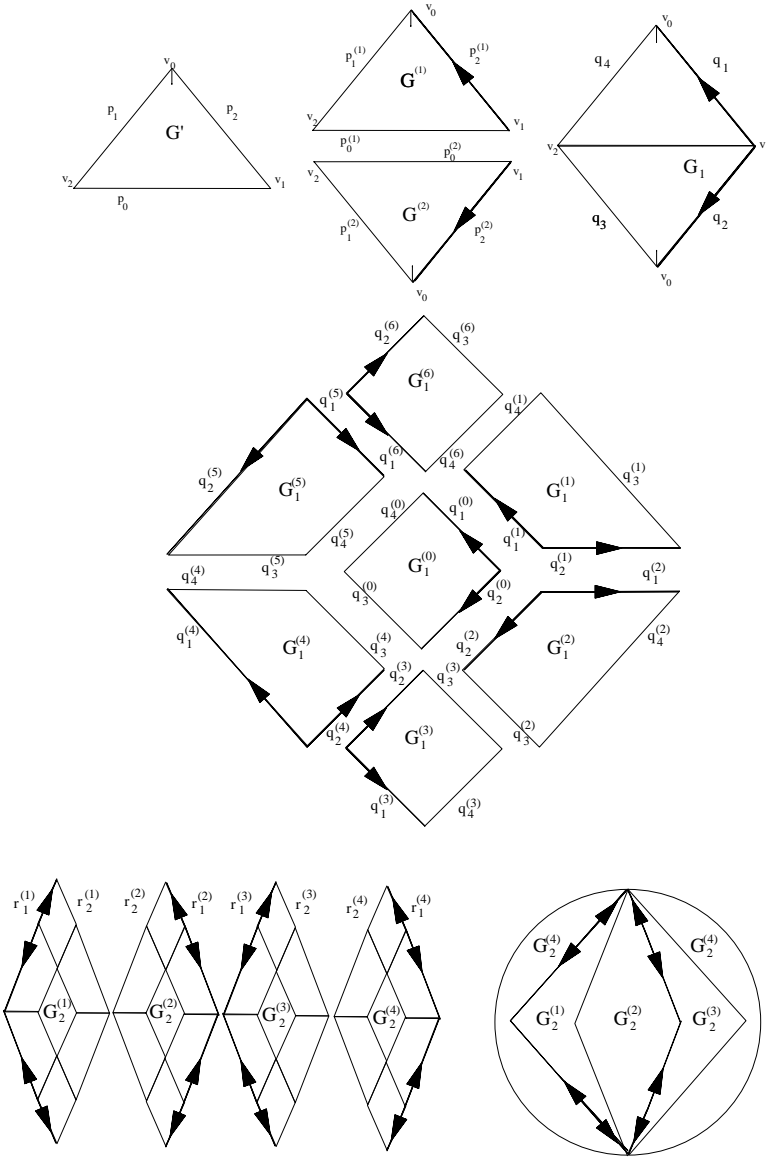


Figure 2.1. An illustration of the copy and paste procedure. (The arrows only indicate the orientation of the subpaths.)

We denote the resulting graph by  $G_2$  and the boundary path  $q_1^{(3)} + q_1^{(4)} + q_2^{(5)} + q_2^{(6)}$  by  $r_1$  and  $q_3^{(6)} + q_3^{(1)} + q_4^{(2)} + q_4^{(3)}$  by  $r_2$ . We summarize some facts about the vertex degrees in  $G_2$ :

- All vertices in  $G_2$  that are not in the boundary path of  $G_2$  but were in the boundary paths of  $G_1^{(0)}, \dots, G_1^{(6)}$  have vertex degree at least 6. (The inner vertices of  $q_1^{(j)}, \dots, q_4^{(j)}, j = 0, \dots, 6$ , had vertex degree at least 4 before being pasted. The outer vertices of  $q_1^{(j)}, \dots, q_4^{(j)}$ , which are originating from the vertices  $u, u'$  and  $v$ , had vertex degree at least 3 before and each is pasted to at least three copies.)
- The vertex in the boundary path in the intersection of  $q_1^{(3)}$  and  $q_4^{(3)}$  from  $G_1^{(3)}$  (which originated from a copy of  $v$ ) has vertex degree at least 3. The same applies to the vertex in the intersection of  $q_2^{(6)}$  and  $q_3^{(6)}$  from  $G_1^{(6)}$ .
- All other vertices in the boundary path of  $G_2$  have vertex degree at least 4.

Next, we make four copies  $G_2^{(1)}, \dots, G_2^{(4)}$  of  $G_2$ . We paste:

- $G_2^{(1)}$  to  $G_2^{(2)}$  along  $r_2^{(1)}$  and  $r_2^{(2)}$  (after reflecting  $G_2^{(2)}$ ),
- $G_2^{(2)}$  to  $G_2^{(3)}$  along  $r_1^{(2)}$  and  $r_1^{(3)}$ ,
- $G_2^{(3)}$  to  $G_2^{(4)}$  along  $r_2^{(3)}$  and  $r_2^{(4)}$  (after reflecting  $G_2^{(4)}$ )
- after embedding the resulting graph into the two dimensional sphere  $\mathbb{S}^2$  we paste  $G_2^{(1)}$  to  $G_2^{(4)}$  along  $r_1^{(1)}$  and  $r_1^{(4)}$ .

The resulting graph  $G_3 = (V_3, E_3)$  is a planar graph that can be embedded in the sphere  $\mathbb{S}^2$ . By the Gauß–Bonnet formula, (GB) which is taken from [17, Proposition 1] (or [2] for tessellations),

$$\sum_{v \in V_3} \kappa^{(G_3)}(v) = 2.$$

Denote the vertices in  $G_3$  that result from copies of vertices in  $V(p)$  of the original subgraph  $G'$  by  $V_3^{(p)}$ . By our construction they have degree at least 6 in  $G_3$ . Thus,  $\kappa^{(G_3)}(v) \leq 0$  for any such vertex  $v \in V_3 \setminus V_3^{(p)}$ . (Note that the minimal face degree is 3 due to triangles and, therefore,  $\kappa_C^{(G_3)}(v, f) \leq 1/6 - 1/2 + 1/3 = 0$  which implies  $\kappa^{(G_3)}(v) \leq 0$ ). On the other hand, for every vertex  $v' \in V' \setminus V(p)$  there are  $56 = 2 \cdot 7 \cdot 4$  copies in  $V_3 \setminus V_3^{(p)}$  and for any such copy  $v$  of  $v'$  we have  $\kappa^{(G_3)}(v) = \kappa^{(G')}(v) = \kappa^{(G)}(v)$ . In conclusion, we have

$$2 = \sum_{v \in V_3} \kappa^{(G_3)}(v) \leq \sum_{v \in V_3 \setminus V_3^{(p)}} \kappa^{(G_3)}(v) = 56 \sum_{v \in V' \setminus V(p)} \kappa^{(G')}(v),$$

which implies the statement.  $\square$

There is an immediate corollary which will play a major role in the considerations below.

**COROLLARY 2.2.** — *Let  $G' = (V', E')$  be a subgraph of a planar graph  $G = (V, E)$  with a simply closed boundary path such that every interior vertex has degree larger or equal to 6 in  $G'$  and all but two vertices in the boundary path have degree at least 3. Then, there are at least four vertices in the boundary path that have vertex degree at most 3 in  $G'$ .*

*Proof.* — Suppose all but three vertices in the boundary path have degree larger or equal to 4. Then, the assumptions of the lemma above are fulfilled and, therefore, there exists a vertex in the interior with positive curvature. This however is impossible by the assumption that the vertex degrees of the interior vertices are larger or equal to 6 as this implies  $\kappa \leq 0$  by the definition of the curvature, see Formula (C) and the formulas below.  $\square$

### 3. Triangulations

In this section we begin by proving Theorem 1.1 and its consequences in the simpler case of triangulations. The case of general planar graphs will be investigated in Section 4.

To phrase the result in the special case of triangulations we need some notation first. We denote by  $\mathcal{B}_r$  the embedding of the vertices and the edges of the distance balls  $B_r$ , confer Section 1.1, into  $\mathcal{S}$ . Since  $\mathcal{B}_r$  is a compact set,  $\mathcal{S} \setminus \mathcal{B}_r$  admits only one unbounded connected component that we denote by  $\mathcal{U}_r$ . We also denote

$$U_r := V \cap \mathcal{U}_r.$$

Observe that in general  $V$  only strictly includes the union  $B_r \cup U_r$  as there can be vertices in  $\mathcal{B}_r$  which distance larger than  $r$  from the root which are however enclosed by vertices in  $B_r$ .

**THEOREM 3.1.** — *Let  $G = (V, E)$  be a planar triangulation such that one of the following assumptions holds:*

- (a)  $\deg \geq 6$  outside the root  $o$ .

(b)  $\text{deg} \geq 7$  outside  $B_r$  for some  $r \geq 0$ .

Let  $v \in S_R \cap U_r$  with  $R > r + \log_2 |S_r|$ , (where  $r = 0$  in case (a)). Then,

$$\text{deg}_0(v) = 2 \quad \text{and} \quad 1 \leq \text{deg}_-(v) \leq 2.$$

In particular, any geodesic reaching such a vertex  $v$  from  $o$  can be continued indefinitely. Furthermore, all spheres  $S_R \cap U_r$  are given by a cyclic path.

*Remark 3.2.* — In Section 3.4 we show that we can extend our techniques so that the results also hold for all  $v \in S_R$  instead of  $S_R \cap U_r$ . However, the result as stated above is enough to prove our main results.

To prove the theorem we employ the copy and paste procedures above. But before we have to introduce a new sphere structure that reflects the fact that geodesics might not be continued to infinity.

### 3.1. A new sphere structure

We first introduce some notation. Recall that each simply closed path  $p$  in the graph induces a simply closed curve with image  $\gamma(p)$  in the surface  $\mathcal{S}$  where the graph is embedded. Furthermore, recall that Jordan’s curve theorem induces a partition of  $\mathcal{S}$  as

$$\mathcal{S} = \mathcal{B}(p) \cup \gamma(p) \cup \mathcal{U}(p),$$

with  $\mathcal{B}(p)$  and  $\mathcal{U}(p)$  being respectively the bounded and the unbounded connected component of  $\mathcal{S} \setminus \gamma(p)$ .

We also denote

$$B(p) := V \cap \overline{\mathcal{B}(p)} = V \cap (\mathcal{B}(p) \cup \gamma(p)) \quad \text{and} \quad U(p) := V \cap \mathcal{U}(p).$$

Furthermore, recall that we denoted  $V(p) := V \cap \gamma(p)$ .

**LEMMA 3.3.** — *Let  $G$  be a planar triangulation and  $r \geq 0$ . The subgraph induced by  $U_r$  is connected and infinite. Moreover if  $w \sim w'$  with  $w \in U_r$  and  $w' \notin B_r$ , then  $w' \in U_r$ .*

*Proof.* — First since  $\mathcal{U}_r$  is unbounded, the fact that the subgraph induced by  $G$  on  $U_r$  is infinite is clear. Let  $v, w \in U_r$ . First, let  $\gamma$  be a simply closed curve in  $\mathcal{S}$  such that  $v, w$  are on  $\gamma$  and  $B_r$  lies in the open bounded region enclosed by  $\gamma$ . Let  $f_1, \dots, f_n$  be a path of faces which  $\gamma$  passes through from  $v$  to  $w$ , i.e., two subsequent faces intersect in exactly one edge which is crossed by  $\gamma$ . These edges have at least one vertex outside  $B_r$  which we denote by  $v_1, \dots, v_n$ . Two subsequent vertices  $v_j$  and  $v_{j+1}$

are connected by a path of boundary vertices of  $f_j$  which are not included in  $B_r, j = 1, \dots, n - 1$ . This induces a path in the graph between  $v$  and  $w$  in  $U_r$ . The last statement is easy: indeed if  $w \sim w', w \in U_r$  and  $w' \notin B_r$ , the edges joining  $w$  to  $w'$  gives a curve in  $\mathcal{S} \setminus B_r$ . Thus  $w$  and  $w'$  are in the same connected component of  $\mathcal{S} \setminus B_r$  and the statement follows.  $\square$

Note that since the other connected components of  $\mathcal{S} \setminus B_r$  are bounded, the other induced graphs are finite.

The following definition turns to be an important object in our study.

$$V_r := \{v \in B_r \mid \text{there is } w \in U_r \text{ such that } v \sim w\}.$$

LEMMA 3.4. — *Let  $G$  be a planar triangulation. For every  $r \geq 1$ , there exists a simply closed path of vertices  $p_r$  such that  $V_r = V(p_r)$ . Moreover, one has*

$$B_r \subseteq B(p_r), \quad \text{and} \quad U_r = U(p_r) \subseteq V \setminus B_r.$$

Furthermore, one also has

$$\begin{aligned} V_r &= \{v \in S_r \mid \text{there is } w \in U_r \text{ such that } v \sim w\} \\ &= \{v \in B(p_r) \mid \text{there is } w \in U_r \text{ such that } v \sim w\}. \end{aligned}$$

*Proof.* — We show  $\emptyset \neq V_r \subseteq S_r$ : Since the graph is connected,  $V_r$  is not empty. Moreover, by construction,  $U_r \subseteq V \setminus B_r$ . So, for a vertex in  $V_r$  to be connected to  $U_r$  it cannot be in  $B_{r-1}$ . Thus, we have  $V_r \subseteq (V \setminus B_{r-1}) \cap B_r = S_r$ .

*Existence of a simply closed path  $p_r$  with  $V(p_r) \subseteq V_r$ :* We now claim that for every vertex in  $v \in V_r$ , there are (at least) two distinct adjacent vertices in  $V_r$ . This easily follows by considering the combinatorial neighborhood of  $v \in S_r$  which includes a neighbor  $v_-$  in  $S_{r-1}$  and  $v_+$  in  $S_{r+1} \cap U_r$  and using that  $G$  is a triangulation. Since  $S_r$  is finite, there necessarily exists a simply closed path  $p_r$  of vertices in  $V_r$ .

*We show that the root  $o$  is in  $B(p_r)$ :* By construction of the simply closed path  $p_r$ , and its image  $\gamma(p_r)$  which is a simply closed curve in  $\mathcal{S}$ , there exist two vertices  $v_- \in S_{r-1}$  and  $v_+ \in S_{r+1} \cap U_r$  which do not belong to the same connected component of  $\mathcal{S} \setminus \gamma(p_r)$ . Since  $V(p_r) \subseteq S_r \subseteq B_r$ , one has  $U_r \subseteq U(p_r)$ . Thus,  $v_+ \in U(p_r)$  and  $v_- \in B(p_r)$ . Furthermore, all the vertices on the geodesic between the root  $o$  and  $v_-$  belong to the same connected component and, therefore, belong to  $B(p_r)$ .

*We show  $U(p_r) \subseteq V \setminus B_r$  and  $B_r \subseteq B(p_r)$ :* The two statements are equivalent by taking the complement. Let us prove the first statement. Let  $v \in U(p_r)$  and consider a geodesic from  $o$  to  $v$ . By connectedness, it has

to cross the simply closed curve  $\gamma(p_r)$  arising from  $p_r$  and must contain a vertex in  $V(p_r)$ , necessarily different from  $v$ . Since  $V(p_r) \subseteq S_r$ , one has  $d(o, v) \geq r + 1$ . Thus,  $v$  is not in  $B_r$ .

We show  $V_r = V(p_r)$ : Since we constructed  $p_r$  such that  $V(p_r) \subseteq V_r$ , we are left to show  $V_r \subseteq V(p_r)$ . Let  $v \in V_r$  and  $w \in U_r$  such that  $v \sim w$ . Consider a geodesic  $p$  from  $o$  to  $v$ . Thus, adding  $w$  to the end of  $p$  is a geodesic between  $o$  and  $w$  which, by connectedness, must cross  $V(p_r)$ , since  $o \in B(p_r)$  and  $w \in U_r \subseteq U(p_r)$ . Recalling  $V(p_r) \subseteq S_r$ , this implies  $v \in V(p_r)$ .

We show  $U_r = U(p_r)$ : We already noticed that  $U_r \subseteq U(p_r)$ . The reverse inclusion follows from the following general connectedness result: Let  $A \subseteq B \subseteq E$  in a topological space  $E$  and let  $O_A$  be an arc-connected component of  $E \setminus A$  such that  $O_A \subseteq E \setminus B$ . Then,  $O_A$  is also an arc-connected component of  $E \setminus B$ .

Finally, we turn to the last two equalities concerning  $V_r$ . The first equality is clear, since we already know  $V_r \subseteq S_r$ . We turn to the second equality. Let  $v \in B(p_r)$  and  $w \in U_r = U(p_r)$  such that  $v \sim w$ . Let  $p$  be any path from  $o$  to  $v$ . Since  $o \in B(p_r)$ ,  $w \in U(p_r)$  and  $v \sim w$ , there is  $u \in V(p_r)$  such that  $u \in V(p_r)$ . Consider the last vertex  $u$  in the path  $p_r$  with this property. Then this vertex and all the following vertices including  $v$  must be in  $V(p_r) \cup U(p_r)$ . Since  $v \notin U(p_r)$ , we conclude  $v \in V(p_r) = V_r$ .  $\square$

We define inductively  $\Sigma_0 = S_0 = \{o\}$  and

$$\Sigma_r := B(p_r) \setminus \Sigma_{r-1} \quad \text{and} \quad \partial \Sigma_r := V(p_r), \quad r \geq 1.$$

This gives a decomposition of  $V$  into a “new sphere structure”. This new sphere structure is also called a “1-dimensional decomposition” in the literature. We denote

$$B_r^{(\Sigma)} := \bigcup_{k=0}^r \Sigma_k$$

for  $r \geq 0$ . Note that  $B_r^{(\Sigma)} = B(p_r)$  for  $r \geq 1$ .

LEMMA 3.5. — *Let  $G$  be a planar triangulation. Let  $r, r' \geq 0$  and  $v \in \Sigma_r, w \in \Sigma_{r'}$  such that  $v \sim w$ , then*

$$|r' - r| \leq 1.$$

*Proof.* — Let  $r \geq 0, l \geq 1$  and  $v \in \Sigma_r, w \in \Sigma_{r+l}$  such that  $v \sim w$ . Since by construction  $B(p_r) = B_r^{(\Sigma)}$ , we deduce  $w \in U(p_r) = U_r$  (where  $p_r$  is taken from Lemma 3.4). By definition of  $V_r$  and  $v \sim w$ , we obtain  $v \in$

$V_r = \partial\Sigma_r \subseteq S_r$  and, therefore,  $w \in S_{r+1}$ . Hence,  $w \in B_{r+1}^{(\Sigma)}$  as  $B_{r+1} \subseteq V \cap B(p_{r+1}) = B_{r+1}^{(\Sigma)}$ . Thus,  $w \in \Sigma_{r+1}$  that is  $l = 1$  and the result follows.  $\square$

Next, we analyze this new sphere structure more closely. To this end, we denote by  $\text{deg}_{\pm}^{(\Sigma)}(v)$  (respectively  $\text{deg}_0^{(\Sigma)}(v)$ ) the number of neighbors of a vertex  $v \in \Sigma_r$  in  $\Sigma_{r\pm 1}$  (respectively in  $\Sigma_r$ ).

LEMMA 3.6. — *In a planar triangulation we have*

$$\partial\Sigma_r = V_r,$$

for all  $r \geq 1$ . Moreover, on  $\partial\Sigma_r$ ,

$$\text{deg}_+ \geq \text{deg}_+^{(\Sigma)} \geq 1, \quad \text{deg}_- = \text{deg}_-^{(\Sigma)} \geq 1 \quad \text{and} \quad 2 \leq \text{deg}_0 \leq \text{deg}_0^{(\Sigma)}$$

and, on  $\Sigma_r \setminus \partial\Sigma_r$ ,

$$\text{deg}_+^{(\Sigma)} = 0.$$

*Proof.* — The first statement  $\partial\Sigma_r = V_r$  follows directly from Lemma 3.4 and the definition of  $\partial\Sigma_r$ . We first consider the case of  $v \in \partial\Sigma_r = V_r$ .

We show  $\text{deg}_+(v) \geq \text{deg}_+^{(\Sigma)}(v) \geq 1$  for  $v \in \partial\Sigma_r$ : We first claim

$$\{w \in S_{r+1} \mid w \sim v\} \supseteq \{w \in \Sigma_{r+1} \mid w \sim v\} \neq \emptyset.$$

Vertices in  $V_r$  have (at least) one neighbor in  $U_r = U(p_r) = V \setminus B(p_r)$ . By definition of  $\Sigma_{r+1}$  these neighbors are exactly the neighbors of  $v$  in  $\Sigma_{r+1}$  and we have already seen in Lemma 3.4 that they must belong to  $S_{r+1}$ . Thus, we obtain  $\text{deg}_+(v) \geq \text{deg}_+^{(\Sigma)}(v) \geq 1$ .

We show  $\text{deg}_-(v) = \text{deg}_-^{(\Sigma)}(v) \geq 1$  for  $v \in \partial\Sigma_r$ : To this end, we claim

$$\{w \in \Sigma_{r-1} \mid w \sim v\} = \{w \in \partial\Sigma_{r-1} \mid w \sim v\} = \{w \in S_{r-1} \mid w \sim v\} \neq \emptyset.$$

To see this, we first note that by connectedness  $\partial\Sigma_r \subseteq U_{r-1}$ . Thus, if  $w \in S_{r-1}$  is a neighbor of  $v \in \partial\Sigma_r$ , then  $w \in V_{r-1} = \partial\Sigma_{r-1}$  by definition and, therefore,  $w \in \Sigma_{r-1}$ . On the other hand, if  $w \in \Sigma_{r-1}$  is a neighbor of  $v \in \partial\Sigma_r$ , then it must belong to  $\partial\Sigma_{r-1}$  and as noticed before to  $S_{r-1}$ .

We show  $2 \leq \text{deg}_0 \leq \text{deg}_-^{(\Sigma)}$   $v \in \partial\Sigma_r$ : The second inequality follows from the first two inequalities. Furthermore,  $v \in \partial\Sigma_r = V_r$  has two neighbors in  $V_r = V(p_r) \subseteq S_r$ .

We show  $\text{deg}_+^{(\Sigma)}(v) = 0$  for  $v \in \Sigma_r \setminus \partial\Sigma_r$ : If  $v \in \Sigma_r \setminus \partial\Sigma_r$ , then  $v$  does not admit any neighbor in  $U_r$ . Thus,  $v$  has no neighbors in  $\Sigma_{r+1}$  since  $\Sigma_{r+1} \subseteq U_r$  from which  $\text{deg}_+^{(\Sigma)}(v) = 0$  follows.

This finishes the proof.  $\square$

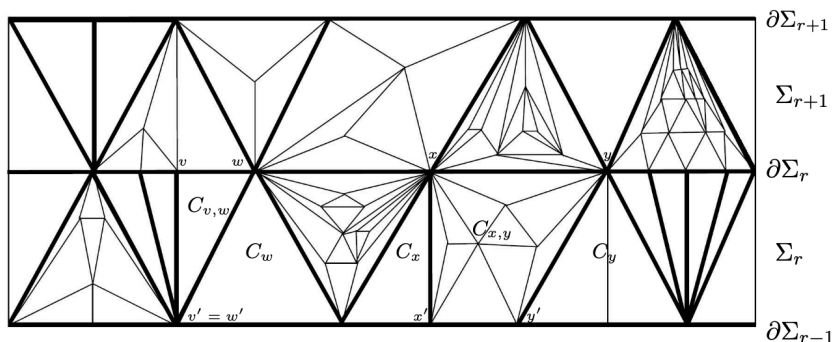


Figure 3.1. An example of a part of  $\Sigma_r$  and  $\Sigma_{r+1}$ , where the horizontal lines indicate the boundary edges connecting  $\partial\Sigma_{r-1}$ ,  $\partial\Sigma_r$  and  $\partial\Sigma_{r+1}$ . The thick lines enclose the elementary cells and different types of elementary cells for the vertices  $v, w, x, y$  are illustrated.

### 3.2. Elementary cells

In this section we define elementary cells associated to the new sphere structure introduced above. More precisely, we define the elementary cells  $C_{v,w}$  and  $C_v$  associated to vertices  $v, w \in \partial\Sigma_r$  with  $v \sim w$ .

Let  $E_r$  be the set of edges  $e_1, \dots, e_N$  connecting vertices in  $\partial\Sigma_r$  to vertices  $\partial\Sigma_{r-1}$ , where the enumeration is in cyclic order with respect to the boundary path  $p_r$  from Lemma 3.4. In particular, each vertex in  $\partial\Sigma_r$  is contained in at least one of these edges by the definition of  $V_r$  which equals  $V(p_r) = \partial\Sigma_r$  by the lemmas above. Now, the subgraph  $(\Sigma_r \setminus \Sigma_{r-1}) \cup \partial\Sigma_{r-1}$  can be decomposed into  $N$  subgraphs  $W_1, \dots, W_N$  that have a closed boundary path with edges of  $\partial\Sigma_r$ ,  $\partial\Sigma_{r-1}$  and  $E_r$  and such that  $W_j$  and  $W_{j+1}$  intersect precisely in  $e_j$  for  $j = 1, \dots, N$  (modulo  $N$ ). Note that each  $W_j$  contains exactly one or two vertices of  $\partial\Sigma_r$ .

If there are two vertices  $v, w \in \partial\Sigma_r$  contained in  $W_j$  we denote  $C_{v,w} := W_j$  and call  $C_{v,w}$  an elementary cell of type (EC1). We denote the neighbors of  $v$  and  $w$  in  $\partial\Sigma_{r-1}$  by the edges  $e_j$  and  $e_{j+1}$  by  $v'$  and  $w'$  (while it might very well happen that  $v' = w'$ ).

On the other hand, if there is only one vertex  $v \in \partial\Sigma_r$  contained in  $W_j$ , then  $v$  is contained in more than one edge of  $E_r$ , say  $e_i, \dots, e_{i+n}$ . We denote the union of  $W_{i+1}, \dots, W_{i+n}$  by  $C_v$  and call  $C_v$  an elementary cell type (EC2).

See Figure 3.1 for an illustration of the definition.

### 3.3. The induction and the proof for triangulations

In this section, we give the proof of Theorem 3.1 which deals with the case of triangulations. Our strategy is to show that if a triangulation has large vertex degree (outside a finite set) then all the elementary cells (at least outside some other finite set) must have empty interior.

This will yield the results on the degree and the Cartan–Hadamard type result. Moreover, this will also give that the graph has a nice structure since (at least outside some finite set) the sphere are composed exactly by a cyclic closed path.

We introduce the set  $Z_R \subseteq \partial\Sigma_R \subseteq S_R$  by

$$Z_R := \left\{ v \in \partial\Sigma_R \mid \deg_+^{(\Sigma)}(v) = 1 \right\}, \quad R \geq 0.$$

Our strategy is to show that if  $Z_R$  is non-empty for some  $R$  then  $Z_R$  grows as  $R$  decays. The underlying idea is that vertices in  $Z_R$  have a lot of “backward” neighbors due to the large vertex degree which then inductively yields vertices in  $Z_{R-1}$ .

We first consider the case  $\deg \geq 6$  outside of a finite set. The case  $\deg \geq 7$  outside a finite set will need somewhat more effort.

#### 3.3.1. The case $\deg \geq 6$ .

LEMMA 3.7. — *Let  $G = (V, E)$  be a triangulation such that  $\deg \geq 6$  outside  $\Sigma_r$  for some  $r \geq 0$ . If there are  $v, w \in \partial\Sigma_R$ ,  $v \sim w$ ,  $R > r$ , such that one of the induced elementary cells  $C_v$  and  $C_{v,w}$  does not have an empty interior, then this elementary cell contains points in  $Z_{R-1}$  and*

$$Z_{R-1} \neq \emptyset.$$

*In particular, if there is  $R > r$  such that  $\Sigma_R \setminus \partial\Sigma_R \neq \emptyset$ , then  $Z_{R-1} \neq \emptyset$ .*

*Proof.* — Let  $v, w \in \partial\Sigma_R$ ,  $v \sim w$ , and consider the elementary cell  $C_{v,w}$ . Assume the interior of the elementary cell  $C_{v,w}$  is non-empty. Since we are in a triangulation, non-emptiness of  $C_{v,w}$  implies that each of the vertices  $v, w, w', v'$  has vertex degree at least 3 in  $C_{v,w}$ . Moreover, by Lemma 3.6 any other boundary vertex  $u$  of  $C_{v,w}$  (in  $\partial\Sigma_{R-1}$ ) has  $\deg_+^{(\Sigma)}(u) \geq 1$ . So, every vertex in the boundary path of  $C_{v,w}$  has degree at least 3 in  $C_{v,w}$ . We distinguish two cases:

*Case 1: One of the vertices  $v, w, w', v'$  has degree at least 4 in  $C_{v,w}$ .* Since every vertex in the boundary of  $C_{v,w}$  has degree at least 3 and every interior vertex of  $C_{v,w}$  has degree at least 6 by assumption, we can apply

Corollary 2.2. This yields that there are at least four vertices in the boundary with degree at most 3. Since at least one of the vertices  $v, w, w', v'$  has degree 4 there is another boundary vertex with degree at most 3. Since the only vertices in  $C_{v,w} \cap \partial\Sigma_R$  are  $v, w$  this vertex is in  $\Sigma_{R-1}$  and, therefore, this vertex is in  $Z_{R-1}$ .

*Case 2: The vertices  $v, w, w', v'$  have all degree 3 in  $C_{v,w}$ .* Since we are in a triangulation, the vertices share a unique common neighbor which we denote by  $u$ . The subgraph  $G'$  induced by  $C_{v,w} \setminus \{v, w\}$  has the boundary  $p' = (w', u, v') + q$ , where  $q$  is the subpath of the boundary path  $p$  of  $C_{v,w}$  such that  $p = (w', w, v, v') + q$ . Note that by the assumption  $\text{deg} \geq 6$ , we infer that  $u$  has degree at least 4 in  $G'$ . If every inner vertex in the subpath  $q$  has vertex degree at least 4 in  $C_{v,w}$ , then  $G'$  satisfies the assumption of the Copy-and-Paste Lemma, Lemma 2.1. Hence, there is an interior vertex with positive curvature in  $G'$ . This is, however, impossible by the assumption  $\text{deg} \geq 6$ . Thus, there is at least one inner vertex in the boundary path  $q$  that has vertex degree strictly less than 4 in  $C_{v,w}$ . Since all vertices in  $q$  are in  $\partial\Sigma_{R-1}$ , they have vertex degree exactly 3 and, thus, this vertex is in  $Z_{R-1}$ .

Consider now the elementary cell  $C_v$ . If the interior is not empty, then  $v$  has vertex degree at least 3 with in  $C_v$ . By the Copy-and-Paste Lemma, Lemma 2.1 and a similar argument as above this implies that there is a vertex in  $Z_{R-1}$ .

The “in particular” is clear since if  $v \in \Sigma_R \setminus \partial\Sigma_R \neq \emptyset$ , then  $v$  must belong to the interior of some elementary cell. □

LEMMA 3.8 (The base case). — *Let  $G = (V, E)$  be a triangulation with  $\text{deg} \geq 6$  outside  $S_r$  for  $r \geq 0$ . If there is a vertex  $v \in \partial\Sigma_R$ ,  $R > r$ , such that*

$$\text{deg}_-^{(\Sigma)}(v) + \text{deg}_0^{(\Sigma)}(v) \geq 5,$$

then

$$Z_{R-1} \neq \emptyset.$$

In particular, if  $Z_R \neq \emptyset$ , then  $Z_{R-1} \neq \emptyset$ .

*Proof.* — The assumption on  $v$  implies  $\text{deg}_-^{(\Sigma)}(v) \geq 3$  or  $\text{deg}_0^{(\Sigma)}(v) \geq 3$ .

First assume  $\text{deg}_0^{(\Sigma)}(v) \geq 3$ . As  $\partial\Sigma_R = V(p_R)$  by definition and  $p_R$  is a simply closed path by Lemma 3.4, the vertex  $v$  has at most two neighbors in  $\partial\Sigma_R$ . Thus, there is another neighbor of  $v$  in  $\Sigma_R \setminus \partial\Sigma_R$ . By Lemma 3.7 we infer  $Z_{R-1} \neq \emptyset$ .

Now, if  $\text{deg}_-^{(\Sigma)}(v) \geq 3$ , then the elementary cell  $C_v$  which has three vertices in  $\partial\Sigma_{R-1}$  in its boundary and at least one of them (i.e., the ones in the middle) belong to  $Z_{R-1}$ .

The “in particular” is clear since for  $v \in Z_R$  we have by definition  $\text{deg}_+^{(\Sigma)}(v) = 1$  and, therefore,  $\text{deg}_-^{(\Sigma)}(v) + \text{deg}_0^{(\Sigma)}(v) \geq 5$ . Thus,  $Z_{R-1} \neq \emptyset$ . □

The proof of Theorem 3.1 now follows directly from the following lemma in which we show that emptiness of  $Z_R$  implies the statement of Theorem 3.1.

LEMMA 3.9. — *Let  $G = (V, E)$  be a triangulation such  $\text{deg} \geq 6$  outside  $B_r$  and  $Z_r = \emptyset$  for some  $r \geq 0$ . Then, for all  $R > r$*

$$\Sigma_R = \partial\Sigma_R = S_R \cap U_r.$$

Furthermore, the following statements hold:

(a) For all  $v \in U_r$

$$\text{deg}_0(v) = 2 \quad \text{and} \quad 1 \leq \text{deg}_-(v) \leq 2.$$

(b)  $S_R \cap U_r$  is given by a cyclic path for each  $R > r$ .

*Proof.* — We first show that  $\Sigma_R = \partial\Sigma_R$  for all  $R > r$ . Assume by contradiction that there is  $R > r$  such that  $\Sigma_R \setminus \partial\Sigma_R \neq \emptyset$ . Then, by Lemma 3.7, this implies that  $Z_{R-1} \neq \emptyset$ . By induction we infer  $Z_r \neq \emptyset$  which contradicts the assumption. Thus,  $\Sigma_R = \partial\Sigma_R$ .

We now introduce the two following partitions of  $U_r$

$$U_r = \bigcup_{R \geq r+1} \Sigma_R \quad \text{and} \quad U_r = \bigcup_{R \geq r+1} S_R \cap U_r.$$

The first one follows since, by construction,  $B_r^{(\Sigma)} = B(p_r)$  and the second one since  $U_r \subseteq V \setminus B_r$ . Thus, using  $\Sigma_R = \partial\Sigma_R$  and Lemma 3.6, we obtain

$$\Sigma_R = \partial\Sigma_R = V_R \subseteq S_R \cap U_r.$$

Necessarily, as the disjoint union over  $R$  on both sides gives  $U_r$ , we infer the equality  $\Sigma_R = \partial\Sigma_R = V_R = S_R \cap U_r$ .

Statement (b) follows directly since  $V_R$  is a cyclic path by Lemma 3.6.

We now turn to statement (a). Let  $v \in U_r$ . By the considerations above, there is  $R > r$  such that  $v \in \Sigma_R$ . Since  $\Sigma_R = \partial\Sigma_R = V(p_R)$  for some closed path  $p_R$ , the vertex  $v$  has exactly two neighbors in  $\Sigma_R$  and  $\text{deg}_0^{(\Sigma)}(v) = 2$ . So, we infer by Lemma 3.6

$$2 = \text{deg}_0^{(\Sigma)}(v) \geq \text{deg}_0(v) \geq 2.$$

The inequality  $\deg_-(v) \geq 1$  is obvious. Now, we prove by contradiction that  $\deg_-(v) \leq 2$ . Indeed, if  $\deg_-(v) \geq 3$ , we have by Lemma 3.6

$$\deg_0^{(\Sigma)}(v) + \deg_-^{(\Sigma)}(v) = \deg_0(v) + \deg_-(v) \geq 5.$$

Lemma 3.8 therefore implies  $Z_{R-1} \neq \emptyset$  and by induction  $Z_r \neq \emptyset$ . This contradicts the assumption and we infer  $\deg_-(v) \leq 2$ .  $\square$

We can now proceed to deduce the first geometrical results in the case of triangulations from the two lemmas above.

*Proof of Theorem 3.1(a).* — Assume  $\deg \geq 6$  outside the root. Obviously,  $Z_0 = \emptyset$  since  $\deg_+(o) = \deg(o) \geq 3$  in a triangulation for the root vertex  $o$ . Thus, the “in particular” of Lemma 3.8 implies

$$Z_R = \emptyset$$

by induction for all  $R \geq 1$ . Hence, the statements of Theorem 3.1(a) follow directly from Lemma 3.9(a) and (b) as well as the observation that  $U_0 = V \setminus \{o\}$ .  $\square$

### 3.3.2. The case $\deg \geq 7$

In the case  $\deg \geq 7$ , we estimate how the size of  $Z_r$  increases exponentially as  $r$  decays.

LEMMA 3.10 (The induction step). — *Let  $G = (V, E)$  be a triangulation such  $\deg \geq 7$  outside  $\Sigma_r$ ,  $r \geq 0$ . Then, for  $R > r$*

$$|Z_R| \geq 2|Z_{R+1}|.$$

*Proof.* — Assume  $Z_{R+1} \neq \emptyset$ . We show that any vertex in  $v \in Z_{R+1}$  induces two vertices in  $Z_R$ . To this end consider the two distinct neighbors  $w, w' \in \partial\Sigma_{R+1}$  such that  $w \sim v \sim w'$  which exist as  $\partial\Sigma_{R+1} = V(p_{R+1})$ , confer Lemma 3.4. Say  $w$  is to the left and  $w'$  is to the right of  $v$ . We now construct paths from  $v$  in  $C_{v,w}$  and  $C_{v,w'}$  to vertices in  $\partial\Sigma_R$ . Since we are in a triangulation  $v$  and  $w$  are contained in a triangle in  $C_{v,w}$  which is induced by a common neighbor  $w_1$  in  $\Sigma_{R+1} \cup \partial\Sigma_R$ . If  $w_1 \in \partial\Sigma_R$  we set  $u = w_1$  and denote the path  $p = (v, u)$ . Otherwise, since  $\deg(w_1) \geq 7$ , there are at least 5 neighbors of  $w_1$  in  $C_{v,w}$ . Starting from  $v$  and counting the vertices along the edges around  $w_1$  clockwise, we pick the third vertex  $w_2$  (such that there are two more edges between the edge  $(w_1, v)$  and  $(w_1, w_2)$ ). If  $w_2$  is not in  $\partial\Sigma_R$ , then we continue inductively by choosing vertices  $w_1, w_2, \dots, w_m$  in the same manner until we reach  $\partial\Sigma_R$ . That is having chosen  $w_j$  we pick  $w_{j+1}$  to be the third neighbor of  $w_j$  starting from  $w_{j-1}$  and counting

clockwise. We then set  $p = (v, w_1, \dots, w_m)$  and  $u = w_m$ . Analogously, we choose a path  $w'_1, \dots, w'_n$  in  $C_{v,w'}$  with difference of choosing  $w'_{j+1}$  to be the third neighbor of  $w'_j$  counter clockwise (instead of clockwise) and set  $w'_n = u'$ . Finally we pick the path between  $u$  and  $u'$  in  $\partial\Sigma_R$  within  $C_{v,w} \cup C_{v,w'}$  and denote it by  $q$ . Thus, the paths  $p, p'$  and  $q$  enclose a finite subgraph  $G_v$  within  $C_{v,w} \cup C_{v,w'}$  such that all interior vertices within the paths  $p$  and  $p'$  have degree at least 4 in  $G_v$  by construction. Moreover, also  $v$  has degree at least 4 within  $G_v$  and any vertex of  $G_v$  which is not in the boundary path has degree at least 7 within  $G_v$ . Thus, by Corollary 2.2 there are at least 4 vertices in the boundary of  $G_v$  with degree 3 or less in  $G_v$ . By the consideration about the degrees above, these vertices must be in  $q$ . Thus, other than  $u, u'$  there are at least two more vertices  $x, y \in V(q)$  with degree 3 or less in  $G_v$ . Since  $x, y \in \partial\Sigma_R$  they must have degree 3 and, therefore,  $x, y \in Z_R$ .

To finish the proof we observe that for two distinct vertices  $v, \tilde{v} \in Z_{R+1}$  the subgraphs  $G_v$  and  $G_{\tilde{v}}$  intersect at most in the boundary paths  $p, p'$  and  $\tilde{p}, \tilde{p}'$  and, therefore, the vertices  $x, y \in Z_R$  and  $\tilde{x}, \tilde{y} \in Z_R$  are distinct.

Thus, any vertex in  $Z_{R+1}$  induces at least two vertices in  $Z_R$  and, therefore,  $|Z_R| \geq 2|Z_{R+1}|$  follows. □

The proof of Theorem 3.1 for the case  $\text{deg} \geq 7$  outside a finite set uses the same idea as the proof in the case  $\text{deg} \geq 6$ .

*Proof of Theorem 3.1(b).* — Assume  $\text{deg} \geq 7$  outside  $B_r$ . By definition we have  $Z_r \subseteq \partial\Sigma_r \subseteq S_r$ . Then, we obtain for any  $R > r + \log_2 |S_r|$  by Lemma 3.10 and a direct induction

$$|Z_R| \leq \frac{|Z_r|}{2^{R-r}} < 1.$$

Thus,

$$Z_R = \emptyset,$$

for  $R > r + \log_2 |S_r|$ . Now, the statements of Theorem 3.1 (b) follow directly from Lemma 3.9 (a) and (b). □

### 3.4. Other connected components of $V \setminus B_r$

In this section we show how to extend the statement of Theorem 3.1(b) to all  $v \in S_R$  with  $R > r + \log_2 |S_r|$  if  $\text{deg} \geq 7$  outside  $B_r$ . Previously, we did this only for vertices in the unbounded connected component  $U_r$ . Now, we look at the other connected components of  $V \setminus B_r$  and discuss how the same arguments as above apply.

Here, we discuss briefly the other finitely many finite connected components. We show that  $B_R \cap \mathcal{B}_r = \emptyset$  for  $R > r + \log_2 |S_r|$ .

Let  $r_0 \geq 0$ . Observe that  $V \setminus B_{r_0}$  has finitely many connected components which are finite. We fix one such connected component and denote it by  $C_{r_0}$ . Inductively, we choose a decreasing sequence  $C = (C_r)_{r \geq r_0}$

$$C_{r+1} \subseteq C_r \subseteq C_{r_0}$$

of finite connected components of  $V \setminus B_r$ ,  $r \geq r_0$ . By finiteness of  $C_{r_0}$ , we have  $C_r = \emptyset$  for  $r$  large enough.

For  $r \geq r_0$ , we introduce

$$V_r^{(C)} := \{v \in B_r \mid \text{there is } w \in C_r \text{ such that } v \sim w\}.$$

Observe that  $V_r^{(C)} \subseteq C_{r_0}$  for  $r > r_0$ .

Indeed, with the same analysis as in Lemma 3.4 we can show that each  $V_r^{(C)}$  is induced by a simple cyclic path  $p_r^{(C)}$  such that

$$V_r^{(C)} = V \left( p_r^{(C)} \right).$$

However, a fundamental difference is that the unbounded component  $\mathcal{U}(p_r^{(C)})$  of  $\mathcal{S} \setminus \gamma(p_r^{(C)})$  now includes  $o$  and, moreover,

$$\begin{aligned} B_r \cup U_r &\subseteq V \cap \overline{\mathcal{U} \left( p_r^{(C)} \right)} =: \bar{U} \left( p_r^{(C)} \right) \\ \text{and } C_r &= V \cap \mathcal{B} \left( p_r^{(C)} \right) =: \dot{B} \left( p_r^{(C)} \right) \end{aligned}$$

for  $R \geq 1$ . One can now define a similar new sphere structure  $\Sigma_r^{(C)}$ ,  $r \geq r_0$ , on  $C_{r_0}$  by letting

$$\Sigma_{r_0}^{(C)} = V \setminus C_{r_0} = \bar{U} \left( p_{r_0}^{(C)} \right), \quad \partial \Sigma_{r_0}^{(C)} = V \left( p_{r_0}^{(C)} \right)$$

and

$$\Sigma_r^{(C)} = (B_r \cap C_{r_0}) \setminus \Sigma_{r-1}^{(C)}, \quad \partial \Sigma_r^{(C)} = V \left( p_r^{(C)} \right),$$

for  $r > r_0$ . As in Lemma 3.5 we can show that, for  $r, r' \geq r_0$  and  $v \in \Sigma_r^{(C)}$  and  $w \in \Sigma_{r'}^{(C)}$  with  $v \sim w$ , we have

$$|r - r'| \leq 1.$$

Furthermore, to obtain a similar result as in Lemma 3.6 we denote by  $\text{deg}_{\pm}^{(C)}(v)$  (respectively  $\text{deg}_0^{(C)}(v)$ ) the number of neighbors of  $v \in \Sigma_r^{(C)}$  in  $\Sigma_{r \pm 1}^{(C)}$  (respectively in  $\Sigma_r^{(C)}$ ). Then, following the arguments given in the proof of Lemma 3.6 we obtain for  $r > r_0$

$$\partial \Sigma_r^{(C)} = V_r^{(C)}$$

and on  $\partial\Sigma_r^{(C)}$

$$\deg_+ \geq \deg_+^{(C)} \geq 1, \quad \deg_- = \deg_-^{(C)} \geq 1 \quad \text{and} \quad 2 \leq \deg_0 \leq \deg_0^{(C)}$$

and on  $\Sigma_r^{(C)} \setminus \partial\Sigma_r^{(C)}$ ,

$$\deg_+^{(C)} = 0.$$

This does not stand in contradiction to the finiteness of  $C$  since  $\partial\Sigma_r^{(C)} = V(p_r^{(C)}) = \emptyset$  for large  $r$ .

On  $\Sigma_r^{(C)}$ , we can also define the elementary cells  $C_v^{(C)}$  and  $C_{v,w}^{(C)}$  for  $v, w \in \partial\Sigma_r^{(C)}$  with  $v \sim w$  as above in Section 3.2. Finally, one defines the set  $Z_r^{(C)}$  of vertices  $v$  in  $\partial\Sigma_r^{(C)}$  with  $\deg_+^{(C)} = 1$ . In the case, where  $\deg \geq 7$  outside  $B_{r_0}$  we show as in Lemma 3.10

$$\left| Z_R^{(C)} \right| \geq 2 \left| Z_{R+1}^{(C)} \right|$$

for  $R \geq r_0$  and, therefore,

$$Z_R^{(C)} = \emptyset$$

for  $R > r_0 + \log_2 |S_{r_0}|$ . As in Lemma 3.8, one sees that  $\deg_0^{(C)}(v) + \deg_-^{(C)}(v) \geq 5$  for  $v \in \Sigma_{R+1}$  implies  $Z_R^{(C)} \neq \emptyset$ . So there cannot be vertices of degree larger  $\deg \geq 7$  in  $B_R \cap C$  for  $R > r_0 + \log_2 |S_{r_0}|$ . In other words

$$B_R \cap C_{r_0} = \emptyset$$

for  $R > r_0 + \log_2 |S_{r_0}|$ .

Thus, we have proven  $S_R = S_R \cap U_r$  for  $R > r_0 + \log_2 |S_{r_0}|$  and, therefore, we deduce the following generalization of Theorem 3.1 directly from these theorems.

**THEOREM 3.11.** — *Let  $G = (V, E)$  be a planar triangulation. If  $\deg \geq 7$  outside  $B_{r_0}$  for some  $r_0 \geq 0$ . Then, for all  $v \in S_R$  such that  $R > r_0 + \log_2 |S_{r_0}|$ ,*

$$\deg_0(v) = 2 \quad \text{and} \quad 1 \leq \deg_-(v) \leq 2$$

and  $S_R$  is a simple cyclic path.

*Remark 3.12.* — In order to deduce a conclusion similar to Theorem 3.1 (b) in the case  $\deg \geq 6$  outside a finite ball  $B_r$ , it is sufficient to prove that there is  $R \geq r$  such that  $Z_R = \emptyset$ . Whether it is possible to prove this remains an open question. In Section 3.7 we present an example that satisfies  $\deg \geq 6$  outside  $B_1$  and  $Z_1 = \emptyset$ . Thus, if we continue this example outside  $B_2$  in any graph  $H$ , such that it is still a planar triangulation with  $\deg \geq 6$  outside  $B_1$ , then  $H$  will satisfies the conclusion of Theorem 3.1 (a).

### 3.5. Spanning trees for triangulations

In this section, we prove Theorem 1.2 on spanning trees for planar triangulations.

Recall the new sphere structure  $\Sigma_R$  introduced in Section 3.1. Given a vertex  $v \in \Sigma_R$  and  $w \in V$  such that  $v \sim w$ , we say that  $w$  is a *forward* (respectively *horizontal and backward*) neighbor if  $w \in \Sigma_{R+1}$  (respectively  $\Sigma_R$  and  $\Sigma_{R-1}$ ).

**THEOREM 3.13.** — *Let  $G$  be a planar triangulation. Assume one of the following assumptions:*

- (a)  $\text{deg} \geq 6$  outside the root.
- (b)  $\text{deg} \geq 7$  outside a finite set.

*Then, then there exists a spanning tree  $T$  of  $G$  such the vertex degrees of  $T$  and  $G$  differ at most by 4 outside a finite set, where the finite set is empty in case (a). Furthermore,  $T$  and  $G$  have the same sphere structure.*

*Proof.* — Let  $r = 0$  in the case (a) and let  $r$  be such that  $\text{deg} \geq 7$  outside  $B_r$  in the case (b).

We construct the spanning tree inductively and assume we already have chosen a spanning tree of  $B_R$  in  $G$  for  $R > \log_2 |S_r| + r$  (which means  $R > 0$  in case (a)).

By Theorem 3.11, we know that every sphere  $S_R \cap U_r$  is a simple cyclic path for  $R > r + \log_2 |S_r|$  and by choosing  $R$  even larger we have  $S_R = S_R \cap U_r$ . (Alternatively one can also apply Theorem 3.11 so one does not have increase  $R$  further.)

This implies that every vertex has two neighbors in the same sphere. For these we remove the connecting edges. Since  $\mathcal{S}$  is oriented, for some fixed  $v \in \Sigma_R$ , we can identify the most left and the most right forward neighbor in the next sphere  $\Sigma_{R+1}$ . By planarity, only the most left and the most right forward of these neighbors  $\Sigma_{R+1}$  have more than one (and thus two) backward neighbors. We remove the edge between  $v$  and the most right forward neighbor. See Figure 3.2 for an illustration.

For any given vertex we therefore remove at most two horizontal edges, one edge to a forward neighbor and one edge to a backward neighbor. Moreover after this procedure, every vertex has exactly one backward neighbor and no horizontal edges, meaning that the graph obtained by removing these edges is a tree. □

*Remark 3.14.* — Using Theorem 3.11 for which we above sketched the proof allows us to quantify the finite set which is excluded as the ball  $B_R$  with  $R = r + \log_2 |S_r|$ , where  $r$  is such that  $\text{deg} \geq 7$  outside  $B_r$ .

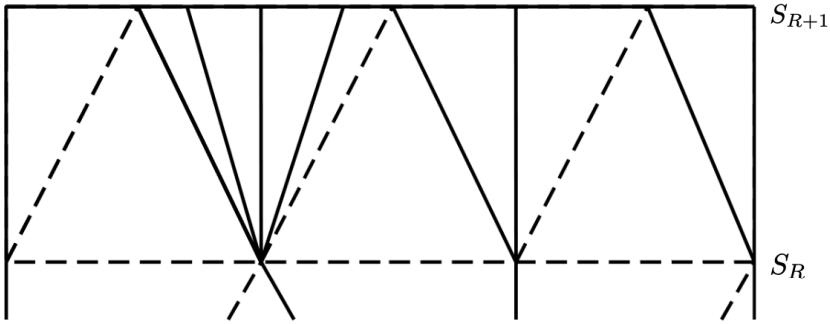


Figure 3.2. The dashed edges are removed from the vertices in the  $R^{\text{th}}$  step of the procedure.

### 3.6. Unique continuation of eigenfunctions for triangulations

In this section, we study the unique continuation of eigenfunctions. Our result is not limited to the Laplacian but holds for general nearest neighbor operators. We start with a definition.

Let a locally finite graph  $G = (V, E)$  be given. A linear operator  $A$  defined on a subspace of  $C(V) = \mathbb{C}^V$  is called a *nearest neighbor operator associated to  $G$* , if it has a matrix representation with respect to the standard basis which is given by some  $a : V \times V \rightarrow \mathbb{C}$  such that for  $v \neq w$ ,

$$a(v, w) \neq 0 \quad \text{if and only if} \quad v \sim w.$$

In this case,  $A$  acts as

$$(A\varphi)(v) = \sum_{w \in V} a(v, w)\varphi(w) = a(v, v)\varphi(v) + \sum_{v \sim w} a(v, w)\varphi(w)$$

where the sum is finite due to local finiteness of the graph. Moreover, the compactly supported functions are included in the domain of definition of  $A$  as  $A$  allows a matrix representation with respect to the standard basis.

The following theorem is a unique continuation result for eigenfunctions in the case of triangulations.

**THEOREM 3.15.** — *Let  $G$  be a planar triangulation such that  $\deg \geq 7$  outside a finite set and let  $A$  be a nearest neighbor operator. Then, there are only finitely many linearly independent eigenfunctions of compact support.*

**Remark 3.16.** — By the use of Theorem 3.11, one can even quantify the set where the compactly supported eigenfunctions are supported. Indeed,

if  $\text{deg} \geq 7$  outside  $B_r$ , then all compactly supported eigenfunctions are supported in  $B_R$  with  $R = r + \log_2 |S_r|$ .

To prove Theorem 3.15 we introduce the polar coordinate representation of nearest neighbor operators as it is used in [9, 17]. For a function  $\varphi \in C(V)$ , let  $\varphi_r$  be the restriction of  $\varphi$  to  $C(S_r)$  and let  $s_r = |S_r|$ ,  $r \geq 0$ . For a nearest neighbor operator  $A$ , let the matrices  $E_r \in \mathbb{C}^{s_{r+1} \times s_r}$ ,  $D_r \in \mathbb{C}^{s_r \times s_r}$ ,  $E_r^+ \in \mathbb{C}^{s_r \times s_{r+1}}$  be given such that

$$(A\varphi)_r = -E_{r-1}\varphi_{r-1} + D_r\varphi_r - E_r^+\varphi_{r+1},$$

for all  $\varphi$  in the domain of  $A$ . In particular, the matrix  $D_r$  is the restriction of  $A$  to  $C(S_r)$ .

The key point is to prove the matrices  $E_r$  are injective. This comes from the following geometric lemma.

LEMMA 3.17. — *Let  $G$  be a planar triangulation. Assume that  $\Sigma_R = \partial\Sigma_R$  for all  $R > r$  and that  $\text{deg} \geq 7$  outside of  $B_r^{(\Sigma)}$ . Then for all  $R > r$  and all  $v \in \Sigma_R$ , there exists  $w \in \Sigma_{R+1}$  with  $v \sim w$  and*

$$\text{deg}_-(w) = 1.$$

*In particular, for any nearest neighbor operator associated to  $G$  and all  $R \geq r$ , the matrices  $E_R$  are injective.*

*Proof.* — Note that the assumption implies that  $\text{deg}_\pm^{(\Sigma)} = \text{deg}_\pm$  and  $\text{deg}_0^{(\Sigma)} = \text{deg}_0$  outside  $B_r^{(\Sigma)}$ . Let  $v \in \Sigma_R$  for some  $R > r$ . By Theorem 3.1, we have  $\text{deg}_-(v) + \text{deg}_0(v) \leq 4$ . Hence,  $\text{deg}_+^{(\Sigma)}(v) = \text{deg}_+(v) \geq 3$  by the assumption  $\text{deg} \geq 7$  outside  $B_{R_0}^{(\Sigma)}$ . Since all the elementary cells are empty, this means that  $v \in \partial\Sigma_R$  has at least 3 neighbors in  $\Sigma_{R+1}$ . Recall here that  $\Sigma_{R+1} = \partial\Sigma_{R+1}$  is a cyclic closed path. Since  $G$  is a triangulation, for the vertex  $v \in \partial\Sigma_R$  the forward neighbors induce a path in  $\partial\Sigma_{R+1}$  of length  $\geq 2$ . By planarity, the inner vertices in the path cannot have another neighbor in  $\partial\Sigma_R$  different from  $v$ .

As a consequence, the matrix  $E_{R-1}$  is injective. Indeed, let  $\varphi_{R-1}$  be a non trivial function on  $\partial\Sigma_{R-1}$ . That is, there exists  $v \in \partial\Sigma_{R-1}$  such that  $\varphi_{R-1}(v) \neq 0$ . As a consequence, with  $w \in \Sigma_R$ ,  $w \sim v$  and  $\text{deg}_-(w) = 1$  as above

$$E_{R-1}\varphi_{R-1}(w) = a(v, w)\varphi_{R-1}(v) \neq 0,$$

and the conclusion follows. □

With the help of the lemma above the proof of Theorem 3.15 is along the lines of the proof of [17, Theorem 9].

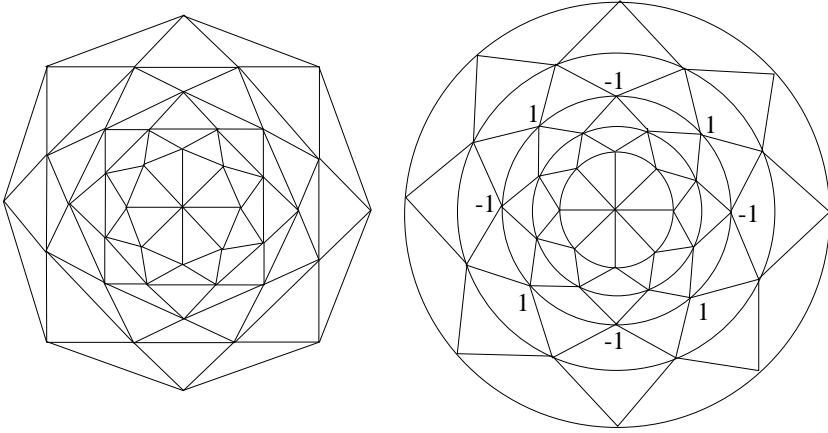


Figure 3.3. Two illustrations of the first 6 distance spheres of the same tessellation allowing for infinitely many linearly independent compactly supported eigenfunctions.

*Proof of Theorem 3.15.* — Let  $G$  be a triangulation which satisfies  $\text{deg} \geq 7$  outside a ball and let  $R_0 \geq r_0 + \log_2 |S_{r_0}|$ . By Lemma 3.17 the matrices  $E_R$  are injective for  $R \geq R_0$ . Let  $R \geq R_0 + 1$ . Suppose there is an eigenfunction  $\varphi$  of  $A$  such that  $\varphi_{R-1} \not\equiv 0$ . Rewriting the eigenvalue equation  $(A\varphi)_R = \lambda\varphi_R$  on the  $R^{\text{th}}$  sphere, one gets

$$E_{R-1}\varphi_{R-1} = (D_R - \lambda)\varphi_R - E_R^+\varphi_{R+1}.$$

Since  $E_{R-1}$  is injective, either  $\varphi_R$  or  $\varphi_{R+1}$  must be non-zero. Hence, if an eigenfunction is supported on a sphere  $S_R$  with  $R > R_0$ , then it has infinite support.

Since the space of functions supported on a ball is finite dimensional, there can be at most finitely many linearly independent eigenfunctions of compact support. □

### 3.7. Counter-example and continuation to tessellations

In this section, we give an example showing, that  $\text{deg} \geq 6$  or even  $\kappa_C \leq 0$  outside a finite set is not enough to exclude the existence of infinitely many compactly supported eigenfunctions such as it was shown in Theorem 3.15.

*Example 3.18.* — The planar graph whose ball  $B_6$  are pictured in Figure 3.3 is given such that the root vertex in the middle is adjacent to

8 triangles, the vertices in the first sphere are adjacent to 5 triangles and all further vertices are adjacent to exactly 6 triangles. Hence,  $\text{deg} = 6$  and  $\kappa_C = 0$  outside  $B_1$ . However, there are eigenfunctions to the eigenvalue 8 for  $\Delta$  supported on any sphere  $S_r$ ,  $r \geq 1$ . One eigenfunction is illustrated on the right hand side of Figure 3.3 in the third sphere.

In [23] it is shown that  $\kappa_C \leq 0$  everywhere implies the absence of compact supported eigenfunction. The example above underscores clearly that our results are not merely simple perturbation results of [23] but the issue in question is much more subtle.

Furthermore, we conclude that graphs as the one above cannot be changed on a finite set such that one obtains a graph with  $\kappa_C \leq 0$ . We finish this section to show that such a procedure is indeed possible if one has  $\text{deg} \geq 7$  outside a finite set.

To this end, we recall the notion of strictly locally tessellating graphs from [17] which slightly more general than tessellations as they allow for unbounded faces. We call a planar locally finite graph  $G$  *strictly locally tessellating* if the following three assumptions are satisfied:

- (T1) Every edge is included in two faces.
- (T2) Every two faces are either disjoint or intersect in one vertex or one edge.
- (T3) Every face is homeomorphic to a closed disc or to the half space.

In a strictly locally tessellating graph the vertex curvature can be seen to be equal to (confer [17, Lemma 3])

$$\kappa(v) = 1 - \frac{\text{deg}(v)}{2} + \sum_{f \in F, f \ni v} \frac{1}{\text{deg}(f)}.$$

In [2, 3] tessellations are considered, that are strictly locally tessellating graphs with the following stronger assumption replacing (T3)

- (T3\*) Every face is homeomorphic to a closed disc.

We show that a planar graph with  $\text{deg} \geq 7$  outside a finite set can be embedded modified on a finite set such that one obtains a strictly locally tessellating graph with non-positive corner curvature. This is useful as for non-positively corner curved graphs various subtle structural results are known, see e.g. [2, 3, 17].

**THEOREM 3.19.** — *Let  $G = (V, E)$  be a planar graph such  $\text{deg} \geq 7$  outside  $B_r$  for some  $r \geq 0$ . Then, there is a strictly locally tessellating graph  $G' = (V', E')$  and  $R \geq \log_2 |S_r| + r + 1$  such that*

- (a)  $\kappa'_C \leq 0$ .

- (b)  $V' = (V \setminus B_R) \cup \{o\}$  and  $G'_{V' \setminus \{o\}} = G_{V \setminus B_R}$ .
- (c)  $d'(\cdot, o) + R = d(\cdot, o)$ ,

Moreover, there is a tessellation  $G'' = (V', E'')$  and  $R \geq r$  such that (a), (c) and  $E' \subseteq E''$ .

The theorem above directly follows from the next lemma (valid for any planar graphs) as  $\text{deg}' \geq 6$  implies  $\kappa'_C \leq 0$ . The underlying idea is to replace a ball around the set with small degree by a single vertex. Since outside this ball the backward degrees are at most two one changes the overall degree at most by one at these vertices.

LEMMA 3.20. — *Let  $N \geq 7$  and a planar graph  $G = (V, E)$  be given such that  $\text{deg} \geq N$  and  $\text{deg}_- + \text{deg}_0 \leq 4$  outside a ball  $B_r$ . Then, there exists  $r + 1 \leq R \leq r + 1 + \lceil \log_2(N - 1) \rceil$  and a strictly locally tessellating graph  $G' = (V', E')$  such that*

- (a)  $\text{deg}' \geq N - 1$ .
- (b)  $V' = (V \setminus B_R) \cup \{o\}$  and  $G'_{V' \setminus \{o\}} = G_{V \setminus B_R}$ .
- (c)  $d(\cdot, o) = d'(\cdot, o) + R$ .

Moreover, there is a tessellation  $G'' = (V', E'')$  with  $E' \subseteq E''$  and  $R \geq r$  such that (a), (c) hold.

*Proof.* — We may assume  $|S_{r+1}| \geq N - 1$  and let  $R = r$ . (Otherwise, we observe that  $|S_{r+2}| = 2|S_{r+1}|$  since  $\text{deg}_+ = \text{deg} - (\text{deg}_- + \text{deg}_0) \geq N - 4 \geq 3$ . Therefore, we can continue with  $R = r + \lceil \log_2(N - 1) \rceil$ .)

Now, we remove all vertices in  $B_R(o)$  except for  $o$ , all edges starting and ending in  $B_R(o)$  and connect  $o$  to all vertices in  $S_{R+1}$  by an edge. We denote this graph by  $G' = (V', E')$ . Then  $V \setminus B_R(o) = V' \setminus \{o\}$ . Notice that  $d'(o, \cdot) + R = d(o, \cdot)$  on this set. The vertex degrees agree on  $V \setminus B_{R+1}(o)$  and  $V' \setminus B'_1(o)$ , so,  $\text{deg}' \geq N$  on  $V' \setminus B'_R(o)$ . Moreover,  $\text{deg}'(v) \geq (\text{deg}(v) - 2) + 1 \geq N - 1$  for  $v \in S'_1(o)$  and  $\text{deg}'(o) \geq N$ . This shows that  $\text{deg}' \geq N - 1$  on  $G'$ . Since  $N - 1 \geq 6$ , we have  $\kappa_C \leq 0$ . By [17, Theorem 1] we conclude that  $G'$  is strictly locally tessellating. This shows the first part of the theorem.

For the “moreover” part, we note that a strictly locally tessellating graph can be easily continued to a tessellation by closing unbounded faces by horizontal edges (i.e., edges connecting vertices in the same sphere). □

### 4. General planar graphs

In this section we show how to carry over the result of Section 3 from triangulations to general planar graphs.

### 4.1. Triangulation supergraphs

The results of this section are based on next lemma. It says that every planar graphs has a triangulating supergraph with the same sphere structure.

LEMMA 4.1. — *Let  $G = (V, E)$  be a locally finite, connected, planar graph and  $o \in V$ . There is a locally finite, planar triangulation  $G' = (V, E')$  with  $E \subseteq E'$  and the same sphere structure, i.e.,  $d(v, o) = d'(v, o)$ ,  $v \in V$ .*

*Proof.* — Adding an edge between the vertices  $v, w \in V$ ,  $v \neq w$  and  $v \not\sim w$  changes the sphere structure of a graph with respect to  $o$  if and only if  $|d(v, o) - d(w, o)| \geq 2$ . Thus, the result can be deduced from the following claim.

CLAIM 4.2. — *For every face  $f$  in a planar graph which is not a triangle there exist vertices  $v_0, v_1 \in V \cap f$  with  $v_0 \not\sim v_1$  and  $|d(v_0, o) - d(v_1, o)| \leq 1$ . In the case of an unbounded face there are infinitely many such pairs of distinct vertices.*

*Proof of the Claim.* — For a face  $f$  which is not a triangle let  $v \in V \cap f$  be such that  $d(v, o) = \min\{d(w, o) \mid w \in V \cap f\}$ . Let  $v_0, v_1 \in V \cap f$  be adjacent to  $v$ . Then,  $d(v, o) \leq d(v_i, o) \leq d(v, o) + 1$ ,  $i = 0, 1$  and, thus,  $v_0, v_1$  satisfy the assertion. In the case of an infinite face  $f$ , there is a two sided infinite sequence where succeeding vertices are adjacent and every vertex of  $f$  is visited. We say that this the boundary walk of  $f$ . By the local finiteness in each sphere there are at most finitely many vertices of this boundary walk. Hence, for a given  $n > \min\{d(w, o) \mid w \in V \cap f\}$  there is at least one vertex in  $S_n(o)$  on each side of the boundary walk and these two are therefore also not adjacent. This proves the claim. □

### 4.2. Proofs of the geometric results for general planar graphs

By Lemma 4.1, a planar graph  $G$  with  $\text{deg} \geq 6$  or  $\text{deg} \geq 7$  has a triangulation supergraph  $G'$  that satisfies  $\text{deg}' \geq 6$  or  $\text{deg}' \geq 7$ .

We first to turn to the proof of Cartan–Hadamard type result, Theorem 1.1, which we conclude from Theorem 3.1.

*Proof of Theorem 1.1.* — Given a planar graph  $G$ , we consider the triangulation supergraph  $G'$  given by Lemma 4.1 above which has the same vertex set  $V$ . Let  $r = 0$  for (a) and  $r$  being the radius such that  $\text{deg} \geq 7$  outside  $B_r$ . For this triangulation  $G'$ , we have  $1 \leq \text{deg}'_0 \leq 2$  and  $1 \leq \text{deg}'_- \leq 2$

on  $S_R \cap U_r$  with  $R > r + \log |S_r|$ . Now, we choose  $R$  even larger such that  $S_R \cap U_r = S_R$  which is possible since  $V \setminus U_r$  is a finite set and let  $K = V \setminus B_R$ . Since the supergraph  $G'$  has the same sphere structure as  $G$ , we have  $\deg_- \leq \deg'_-$  and  $\deg_0 \leq \deg'_0$ . Note also that since the graph is connected,  $\deg_- \geq 1$  on  $V \setminus \{o\}$ . Thus, the statement  $\deg_0 \leq 2$  and  $1 \leq \deg_- \leq 2$  follows on  $V \setminus K$ . This readily gives that geodesics can be continued as  $\deg_+ = \deg - \deg_0 - \deg_- \geq 3$  outside  $K$ . Finally, spheres outside  $K$  in the triangulation supergraph  $G'$  are given by cyclic path which gives that the spheres in  $G$  are cyclically ordered.  $\square$

Next we turn to the proof of Theorem 1.2 which says that one obtains spanning tree by changing the vertex degree at most by 4 outside a finite set.

*Proof of Theorem 1.2.* — The proof follows along the lines for the corresponding proof for triangulations. Let the finite set  $K$  be chosen as in Theorem 1.1. Then spheres are cyclically ordered outside  $K$ . We first remove the vertices within the spheres which by Theorem 1.1 changes the vertex degree at most by 2. Now the cyclic ordering of the spheres allows us to speak of the most right and the most left forward neighbor of a vertex  $v$  in a sphere  $S_R$  for large  $R$ . By planarity only these two vertices can have more than one backward neighbor. If the most right forward neighbor has more than one backward neighbor, call it  $w$ , we remove this edge, confer Figure 3.2 above. On the other hand, any vertex  $w$  in  $S_{R'}$  with backward degree more than 1 is a most right forward of some vertex in  $S_{R'-1}$ . By Theorem 1.1, such a vertex  $w$  satisfies  $\deg_-(w) \leq 2$ . By this procedure we remove all cycles in this way without changing the sphere structure of the graph. In summary for each vertex we have removed at most two edges within the same sphere, one edge to a forward and one edge to a backward neighbor which makes at most 4. This proves the statement.  $\square$

We next come to the unique continuation statement for eigenfunctions on general planar graphs. To this end we recall the definition of a nearest neighbor operator from Section 3.6.

**THEOREM 4.3.** — *Let  $G$  be a planar graph such that  $\deg \geq 7$  outside a finite set and let  $A$  be a nearest neighbor operator. Then, there are only finitely many linearly independent eigenfunctions of compact support.*

*Proof.* — Consider the triangulation supergraph  $G'$  of  $G$  given by Lemma 4.1. Now, for a vertex  $v \in S_R$  for sufficiently large  $R$  there is a forward neighbor  $w$  such that  $\deg'_-(w) = 1$  by Lemma 3.17. Since  $v$  is

the only backward neighbor of  $w$  in  $G'$ , it must also be a backward neighbor in  $G$  (otherwise  $d(o, w) < d'(o, w)$  contradicting Lemma 4.1). Thus, following the argument as in the proof of Theorem 3.15 we conclude the statement.  $\square$

From this theorem we can immediately deduce that the operator  $\Delta + q$  admits at most finitely many linearly independent eigenfunctions whenever  $\kappa_\infty = -\infty$ .

*Proof of Theorem 1.9.* — Obviously  $\Delta + q$  is a nearest neighbor operator and  $\kappa_\infty = -\infty$  implies  $\text{deg} \geq 7$  outside a finite set. Hence, the statement follows from Theorem 4.3.  $\square$

### 5. Discrete spectrum, eigenvalue asymptotics and decay of eigenfunctions

In this section we prove Theorem 1.6, Corollary 1.8 and Theorem 1.11. To this end we extend the inequalities presented in [5, 10] for planar graphs. Here, we use that Laplacians of planar graphs with large vertex degree outside a finite set are bounded perturbations of a Laplacian of a tree. In particular, an immediate consequence of Theorem 1.2 is the following.

**COROLLARY 5.1.** — *Let  $G = (V, E)$  be a planar graph such  $\text{deg} \geq 7$  outside  $B_r$  for some  $r \geq 0$ . Then, there is a tree  $T = (V, E')$  with  $E' \subseteq E$  such that  $\Delta_T$  is a bounded perturbation of  $\Delta_G$ .*

From Corollary 5.1, we derive the following inequality which improves the considerations of [5] for planar graphs. These inequalities might be of interest in their own rights.

**THEOREM 5.2.** — *Let  $G$  be a planar graph such that  $\text{deg} \geq 6$  outside the root or  $\text{deg} \geq 7$  outside a finite set and  $q : V \rightarrow [0, \infty)$ . Then, there is  $C \geq 0$  such that*

(a) for all  $\varepsilon > 0$

$$(1 - \varepsilon)(\text{deg} + q) - \frac{1}{\varepsilon} - C \leq \Delta + q \leq (1 + \varepsilon)(\text{deg} + q) + \frac{1}{\varepsilon} + C,$$

(b) for all  $\varphi \in C_c(V)$ ,  $\|\varphi\| = 1$ , we additionally have

$$\begin{aligned} \langle \varphi, (\text{deg} + q)\varphi \rangle - 2\sqrt{\langle \varphi, (\text{deg} + q)\varphi \rangle} - C &\leq \langle \varphi, (\Delta + q)\varphi \rangle \\ &\leq \langle \varphi, (\text{deg} + q)\varphi \rangle + 2\sqrt{\langle \varphi, (\text{deg} + q)\varphi \rangle} + C. \end{aligned}$$

*Remark 5.3.*

- (a) The constant  $C$  in the theorem above depends only on the norm of the Laplacian on a neighborhood of the finite set outside which we have  $\deg \geq 7$ . In particular, if  $\deg \geq 7$  everywhere the constant can be chosen  $C = 0$ .
- (b) The considerations of [5] yield an estimate that has  $\pm 3/\varepsilon$  instead of the constants  $\pm(1/\varepsilon + C)$  in (a).

The essential step in the proof of Theorem 5.2 is to combine Theorem 1.2 with techniques developed in [5, 10]. Our rather special situation allows for a very transparent and non-technical treatment. For sake of being self-contained and to illustrate the core of the techniques of both [10] and [5] we obtain (a) by the Hardy inequality techniques of [10] and (b) by the isoperimetric techniques of [5]. Observe that one could also derive (a) from (b) using some technical estimates of [5].

*Proof of Theorem 5.2.* — By Corollary 5.1 there is a tree  $T = (V, E')$  such that for the Laplacian  $\Delta_T$  on the tree there is  $C \geq 0$  such that  $\Delta_T - C \leq \Delta \leq \Delta_T + C$ . Denote the vertex degree in  $T$  by  $\deg_T$  and observe that by Theorem 1.2 we have  $\deg_T \leq \deg \leq \deg_T + 4$ .

- (a) For a positive function  $m : V \rightarrow (0, \infty)$  let  $q_m : V \rightarrow \mathbb{R}$  be given by

$$q_m(v) = \deg(v) - \sum_{w \sim v} \frac{m(w)}{m(v)}.$$

By direct calculation, which is sometimes referred to as the ground state representation, (confer [10, Proposition 1.1] or [11, Proposition 3.2]) we obtain for  $\varphi \in C_c(V)$

$$\begin{aligned} \langle \varphi, \Delta_T \varphi \rangle &= \frac{1}{2} \sum_{v \sim w} (\varphi(v) - \varphi(w))^2 \\ &= \frac{1}{2} \sum_{v \sim w} m(v)m(w) \left( \frac{\varphi(v)}{m(v)} - \frac{\varphi(w)}{m(w)} \right)^2 + \sum_{v \in V} q_m(v) \varphi(v)^2 \end{aligned}$$

and, therefore,  $\Delta_T \geq q_m$  on  $C_c(V)$ . Now, for  $\varepsilon > 0$ , we choose  $m(v) = \varepsilon^{d(v,o)}$  and observe  $q_m = (1 - \varepsilon) \deg_T - 1/\varepsilon$ . Thus,

$$(1 - \varepsilon) \deg_T - \frac{1}{\varepsilon} \leq \Delta_T$$

on  $C_c(V)$ . Since  $T$  is a tree, the operator  $\Delta_T$  is unitarily equivalent to the operator  $2 \deg - \Delta_T$  (the unitary operator is multiplication by  $(-1)^{d(\cdot, o)}$ ).

Hence, we conclude

$$\Delta_T \leq 2 \deg_T - \Delta_T \leq (1 + \varepsilon) \deg_T + \frac{1}{\varepsilon}$$

on  $C_c(V)$ . Statement (a) follows now from  $\Delta_T - C \leq \Delta \leq \Delta_T + C$  and  $\deg_T \leq \deg \leq \deg_T + 4$  discussed in the beginning of the proof and the assumption  $q \geq 0$ .

(b) By Theorem 3.19 the tree  $T$  has degree larger than 2 outside a finite set  $K$ . Since  $T$  is a spanning tree and, hence, connected it has degree greater or equal to 1 everywhere. We define

$$d_T = \deg_T + q' \quad \text{with} \quad q' = q + 1_K.$$

By the discussion above  $d_T \geq \deg_T + 1_K \geq 2$ . We further notice that on a tree any subgraph  $T_W = (W, E'_W)$  of  $T = (V, E')$  induced by a finite set  $W \subseteq V$  satisfies  $|E'_W| \leq |W|$  (confer [5, Lemma 6.2]). This implies

$$d_T(1_W) = 2|E'_W| + |\partial W| + q'(1_W) \leq 2|W| + |\partial W| + q'(1_W),$$

where  $\partial W = \{(v, w) \in W \times V \setminus W \mid v \sim w\}$  and  $q'(\varphi) = \sum_{v \in V} \varphi(v)^2 q'(v)$ ,  $\varphi \in C_c(V)$ . Let  $\varphi \in C_c(V)$ ,  $\|\varphi\| = 1$ . Using an area and a co-area formula (cf. [19, Theorem 12 and Theorem 13]) with  $\Omega_t := \{v \in V \mid |\varphi(v)|^2 > t\}$ , and the discussion above, we obtain

$$\begin{aligned} \langle \varphi, (d_T - 2)\varphi \rangle &= \int_0^\infty \left( d_T(1_{\Omega_t}) - 2|\Omega_t| \right) dt \leq \int_0^\infty |\partial \Omega_t| + q'(1_{\Omega_t}) dt \\ &= \frac{1}{2} \sum_{v \sim w} |\varphi(v)^2 - \varphi(w)^2| + q'(\varphi) \\ &= \frac{1}{2} \sum_{v \sim w} |(\varphi(v) - \varphi(w))(\varphi(v) + \varphi(w))| + q'(\varphi) \\ &\leq \frac{1}{2} \left( \sum_{v \sim w} |\varphi(v) - \varphi(w)|^2 + 2q'(\varphi) \right)^{1/2} \left( \sum_{v \sim w} |\varphi(v) + \varphi(w)|^2 + 2q'(\varphi) \right)^{1/2} \\ &= \langle \varphi, (\Delta_T + q')\varphi \rangle^{\frac{1}{2}} \left( 2\langle \varphi, d_T \varphi \rangle - \langle \varphi, (\Delta_T + q')\varphi \rangle \right)^{\frac{1}{2}}. \end{aligned}$$

Since  $d_T \geq 2$ , we have  $\langle \varphi, (d_T - 2)\varphi \rangle \geq 0$  and, thus, we can square both sides of the inequality to obtain after reordering the terms,

$$\langle \varphi, (\Delta_T + q')\varphi \rangle^2 - 2\langle \varphi, d_T \varphi \rangle \langle \varphi, (\Delta_T + q')\varphi \rangle + \langle \varphi, (d_T - 2)\varphi \rangle^2 \leq 0.$$

Resolving the inequality and using  $\|\varphi\| = 1$  yields

$$\begin{aligned} \langle \varphi, d_T \varphi \rangle - 2\sqrt{\langle \varphi, d_T \varphi \rangle - 1} &\leq \langle \varphi, (\Delta_T + q')\varphi \rangle \\ &\leq \langle d_T \varphi, \varphi \rangle + 2\sqrt{\langle \varphi, d_T \varphi \rangle - 1}. \end{aligned}$$

Now, we further observe that

$$\sqrt{\langle \varphi, d_T \varphi \rangle - 1} = \sqrt{\langle \varphi, (\deg_T + q + 1_K) \varphi \rangle - 1} \leq \sqrt{\langle \varphi, (\deg_T + q) \varphi \rangle}.$$

Thus, (b) follows from the inequalities subtracting  $\langle \varphi, 1_K \varphi \rangle$  from the inequalities and using  $\Delta_T - C \leq \Delta \leq \Delta_T + C$  and  $d_T \leq \deg + q \leq d_T + C$ .  $\square$

Next, we turn to the proof of Theorem 1.6.

*Proof of Theorem 1.6.* — For a potential  $q \in K_\alpha$ ,  $\alpha \in (0, 1)$ , there is  $C_\alpha$  such that  $q_- \leq \alpha(\Delta + q_+) + C_\alpha$ . We deduce from Theorem 5.2(a) (confer [5, Lemma A.3])

$$\frac{(1 - \alpha)(1 - \varepsilon)}{(1 - \alpha(1 - \varepsilon))}(\deg + q) - \frac{(1 - \alpha)(1/\varepsilon + C) + \varepsilon C_\alpha}{(1 - \alpha(1 - \varepsilon))} \leq \Delta + q, \quad \text{on } C_c(X)$$

for all  $\varepsilon > 0$ .

By an application of the Min-Max-Principle, Theorem A.1, the spectrum of  $\Delta + q$  is purely discrete if the spectrum of  $\deg + q$  is purely discrete. On the other hand, if there are vertices  $v_n$  such that  $(\deg + q)(v_n) \leq C$  for some  $C$ , then  $\langle \Delta 1_{\{v_n\}}, 1_{\{v_n\}} \rangle = (\deg + q)(v_n) \leq C$ ,  $n \geq 0$ . By a Persson-type theorem, [12, Proposition 2.1] we conclude that the bottom of the essential spectrum of  $\Delta + q$  is bounded from above by  $C$ . Hence, the essential spectrum of  $\Delta + q$  is non-empty. We summarize that the spectrum of  $\Delta + q$  is purely discrete if and only if  $\sup_{K \subset V \text{ finite}} \inf_{v \in V} (\deg(v) + q(v)) = \infty$ . Since

$$-\frac{\deg(v)}{2} \leq \kappa(v) \leq 1 - \frac{\deg(v)}{6}$$

(due to  $\deg(f) \geq 3$ ), this in turn is equivalent to

$$\sup_{K \subset V \text{ finite}} \inf_{v \in V} (-\kappa(v) + q(v)) = \infty.$$

Next, we assume  $q \geq 0$ . The eigenvalue asymptotics follow directly from Theorem 5.2(b) and the Min-Max-Principle, Theorem A.1 as  $x \mapsto x - 2\sqrt{x}$  is continuous and monotone increasing on  $[1, \infty)$  and  $\lambda_0(\deg + q) \geq 1$ .  $\square$

Now, we turn to the proof of Corollary 1.8.

*Proof of Corollary 1.8.* — If the face degree is constantly  $k$  outside a compact set  $K \subseteq X$ , then

$$\kappa(v) = 1 - \frac{k - 2}{2k} \deg(v),$$

for  $v \in V \setminus K$ . The eigenvalue asymptotics follow now from Theorem 1.6.  $\square$

Finally, we prove Theorem 1.11 on the decay of eigenfunctions. The proof we give here is similar to the techniques developed of [22]. However, for the

convenience of the reader we include a short proof. Indeed, in our situation of planar graphs with large degree, the proof simplifies even substantially.

*Proof of Theorem 1.11.* — A direct computation, often referred to as the ground state representation as used in the proof of Theorem 5.2 above (confer [10, Proposition 1.1] or [11, Proposition 3.2]), gives for  $\varphi$  with compact support,  $u \in D(\Delta)$  and the basic estimate  $2ab \leq a^2 + b^2$

$$\begin{aligned} \frac{1}{2} \sum_{x,y, y \sim x} u(x)^2(\varphi(x) - \varphi(y))^2 &\geq \frac{1}{2} \sum_{x,y, x \sim y} u(x)u(y)(\varphi(x) - \varphi(y))^2 \\ &= \langle \varphi u, \Delta(\varphi u) \rangle - \langle \varphi^2 u, \Delta u \rangle. \end{aligned}$$

If  $u \in D(\Delta)$  is an eigenvector of  $\Delta$  with eigenvalue  $\lambda$ , we estimate together with the form bound from Theorem 5.2(a)

$$\frac{1}{2} \sum_{x,y, y \sim x} u(x)^2(\varphi(x) - \varphi(y))^2 \geq \langle ((1 - \varepsilon) \deg - C_\varepsilon - \lambda)\varphi u, \varphi u \rangle$$

for  $0 < \varepsilon < 1$  and with  $C_\varepsilon = 1/\varepsilon + C$  some  $C \geq 0$ . We define for  $N \geq 0$

$$\varphi_N = 1_{B_N} \alpha^{d(\cdot, o)} + 1_{B_{2N} \setminus B_N} \alpha^{2N - d(\cdot, o)}$$

with  $\alpha = \sqrt{2}(1 - \varepsilon) + 1$  and observe that for  $\varepsilon > 0$  small enough,

$$\frac{1}{2} \sum_{y, y \sim \cdot} (\varphi_N(\cdot) - \varphi_N(y))^2 \leq (1 - \varepsilon)^2 \varphi_N^2 \deg + \frac{1}{2} 1_{S_{2N+1}} \deg_- ,$$

where we used that  $\varphi_N(x) - \varphi_N(y) = 1$  for  $x \in S_{2N}$  and  $y \in S_{2N+1}$ . Combining this with the estimate above, we obtain after reordering the terms

$$\frac{1}{2} \sum_{x \in S_{2N+1}} \deg_-(x) u(x)^2 \geq \langle (\varepsilon(1 - \varepsilon) \deg - C_\varepsilon - \lambda)\varphi_N u, \varphi_N u \rangle.$$

Since we assumed that  $\deg$  becomes arbitrarily large outside finite sets, there is a finite set  $K$  and a constant  $c_\varepsilon > 0$  such that  $\varepsilon(1 - \varepsilon) \deg - C_\varepsilon - \lambda \geq c_\varepsilon \deg$  outside  $K$ . Furthermore, by Theorem 1.1,  $\deg_-$  is a bounded function. Thus, there is  $C' = C'_\varepsilon$  such that for all  $N$

$$C' \|u\|^2 \geq \sum_{x \in X} \deg(x) \varphi_N^2(x) u^2(x) \geq \sum_{x \in B_N} \deg(x) \alpha^{2d(o,x)} u^2(x).$$

By monotone convergence, we conclude

$$C' \|u\|^2 \geq \sum_{x \in X} \deg(x) \alpha^{2d(o,x)} u^2(x).$$

This finishes the proof. □

### Appendix A. Applications of the Min-Max-Principle

In this appendix we shortly discuss an application of the Min-Max-Principle to (non-linear) functions of operators.

Let  $H$  be a Hilbert space with norm  $\|\cdot\|$ . For a quadratic form  $Q$ , denote the form norm by  $\|\cdot\|_Q := \sqrt{Q(\cdot) + \|\cdot\|^2}$ . For a selfadjoint operator  $A$  which is bounded from below, we denote the bottom of the essential spectrum by  $\lambda_0^{\text{ess}}(A)$ . Let  $n(A) \in \mathbb{N}_0 \cup \{\infty\}$  be the dimension of the range of the spectral projection of the interval  $(-\infty, \lambda_0^{\text{ess}}(A))$ . Whenever  $\lambda_0(A) < \lambda_0^{\text{ess}}(A)$ , we denote the eigenvalues below  $\lambda_0^{\text{ess}}(A)$  by  $\lambda_n(A)$ , for  $0 \leq n \leq n(A)$ , in increasing order counted with multiplicity.

**THEOREM A.1.** — *Let  $(Q_1, D(Q_1))$  and  $(Q_2, D(Q_2))$  be closed symmetric non-negative quadratic forms with a common form core  $D_0$  and let the corresponding selfadjoint operators be denoted by  $A_1$  and  $A_2$ . Assume there are continuous monotone increasing functions  $f_1, f_2 : [\lambda_0(A_2), \infty) \rightarrow \mathbb{R}$  such for all  $\varphi \in D_0$  with  $\|\varphi\| = 1$*

$$f_1(Q_2(\varphi)) \leq Q_1 \leq f_2(Q_2(\varphi)).$$

Then, for  $0 \leq n \leq \min(n(A_1), n(A_2))$ ,

$$f_1(\lambda_n(A_2)) \leq \lambda_n(A_1) \leq f_2(\lambda_n(A_2)).$$

Moreover, if  $\lim_{r \rightarrow \infty} f_1(r) = \lim_{r \rightarrow \infty} f_2(r) = \infty$ , then  $\sigma_{\text{ess}}(A_1) = \emptyset$  if and only if  $\sigma_{\text{ess}}(A_2) = \emptyset$ .

*Proof.* — Letting

$$\mu_n(A) := \sup_{\varphi_1, \dots, \varphi_n \in H} \inf_{\psi \in \{\varphi_1, \dots, \varphi_n\}^\perp \cap D_0, \|\psi\|=1} Q(\psi),$$

for a selfadjoint operator  $A$  with form  $Q$  and  $D_0 \subseteq D(Q)$ , we have by the Min-Max-Principle [25, Chapter XIII.1]  $\mu_n(A) = \lambda_n(A)$  if  $\mu_n(A) < \lambda_0^{\text{ess}}(A)$  and  $\mu_n(A) = \lambda_0^{\text{ess}}(A)$  otherwise,  $n \geq 0$ . Now, observe that for a continuous monotone increasing function  $f : [0, \infty) \rightarrow \mathbb{R}$  and a function  $g : X \rightarrow [0, \infty)$  defined on an arbitrary set  $X$  we have

$$\inf_{x \in X} f(g(x)) = f\left(\inf_{x \in X} g(x)\right).$$

Now, assume  $n \leq \min\{n(A_1), n(A_2)\}$  and let  $\varphi_0^{(j)}, \dots, \varphi_n^{(j)}$  be the eigenfunctions of  $A_j$  to  $\lambda_0(A_j), \dots, \lambda_n(A_j)$  and denote

$$U_j^{(n)} := \left\{ \varphi_1^{(j)}, \dots, \varphi_n^{(j)} \right\}^\perp \cap \{ \psi \in D_0 \mid \|\psi\| = 1 \}, \quad j = 1, 2.$$

We apply the discussion above with  $f = f_1$  and

$$g = g_1 : U_2^{(n)} \rightarrow [0, \infty), \quad \psi \mapsto Q_2(\psi),$$

first and  $f = f_2$  and

$$g = g_2 : U_1^{(n)} \rightarrow [0, \infty), \quad \psi \mapsto Q_2(\psi),$$

later on, to obtain

$$\begin{aligned} f_1(\lambda_n(A_2)) &= f_1 \left( \inf_{\psi \in U_2^{(n)}} Q_2(\psi) \right) = \inf_{\psi \in U_2^{(n)}} f_1(Q_2(\psi)) \leq \inf_{\psi \in U_2^{(n)}} Q_1(\psi) \\ &\leq \mu_n(A_1) = \lambda_n(A_1) = \inf_{\psi \in U_2^{(n)}} Q_1(\psi) \leq \inf_{\psi \in U_2^{(n)}} f_2(Q_2(\psi)) \\ &= f_2 \left( \inf_{\psi \in U_2^{(n)}} Q_2(\psi) \right) \leq f_2(\mu_n(A_2)) = f_2(\lambda_n(A_2)). \end{aligned}$$

This directly implies the first statement. Assuming now  $\lambda_0^{\text{ess}}(A_2) = \infty$  implies  $n(A_2) = \infty$  and  $\lim_{n \rightarrow \infty} \lambda_n(A_2) = \infty$  and, therefore, we get  $\lim_{n \rightarrow \infty} f_1(\lambda_n(A_2)) = \infty$ , by the assumption on  $f_1$ . Hence, by the above we get  $\lambda_0^{\text{ess}}(A_1) = \infty$ . The other implication follows analogously.  $\square$

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