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A CAPORASO–HARRIS TYPE FORMULA FOR RELATIVE REFINED INVARIANTS

by Thomas BLOMME (*)

Abstract. — In 2015, G. Mikhalkin introduced a refined count for real rational curves in a toric surface which pass through some points on the toric boundary of the surface. The refinement is provided by the value of a so-called quantum index. Moreover, he proved that the result of this refined count does not depend on the choice of the points. The correspondence theorem allows one to compute these invariants using the tropical geometry approach and the refined Block–Göttsche multiplicities. In this paper we give a recursive formula for these invariants, that leads to an algorithm to compute them.

Résumé. — En 2015, G. Mikhalkin a introduit un compte raffiné des courbes rationnelles réelles dans les surfcaes toriques passant par certains points situés sur le bord de cette dernière. Le raffinement est apporté par la valeur d'un certain indice quantique. Le compte introduit s'avère ne pas dépendre de du choix des points, donnant lieu à un invariant. Il est alors possible de les calculer à la limite tropicale en utilisant le théorème de correspondance et la multiplicité de Block–Göttsche. Dans ce papier on donne une formule récursive qui permet de calculer ces invariants, ce qui donne un algorithme de calcul.

1. Introduction

1.1. Setting

1.1.1. Rational curves in toric surfaces

The paper deals with enumerative problems involving rational curves in toric surfaces, for which we quickly give our notations. Let N be a 2-dimensional lattice and let $\Delta = (n_j) \subset N$ be a multiset of m lattice vectors,

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whose total sum is zero. The vectors of Δ define a fan Σ_Δ in $N_\mathbb{R} = N \otimes \mathbb{R}$, yielding a toric surface $\mathbb{C}\Delta$ whose dense complex torus is $N_{\mathbb{C}^*} = N \otimes \mathbb{C}^*$. The dual lattice M of N is the set of monomial functions on $N_{\mathbb{C}^*}$. The toric divisors of $\mathbb{C}\Delta$ are in bijection with the rays of the fan Σ_Δ. The complex conjugation on $\mathbb{C}$ defines an involution on $N_{\mathbb{C}^*}$, which extends to $\mathbb{C}\Delta$, making it into a real surface. Its fixed locus, also called the real locus, is denoted by $\mathbb{R}\Delta$.

A parametrized rational curve $\mathbb{C}P^1 \to \mathbb{C}\Delta$ of degree Δ is a curve that admits a parametrization

$$\varphi : t \in \mathbb{C} \dashrightarrow \chi \prod_{i=1}^{m} (t - \alpha_i)^{n_i} \in N_{\mathbb{C}^*},$$

where the numbers α_j are some complex points inside $\mathbb{C}$, and $\chi \in N_{\mathbb{C}^*}$. The vectors n_j can also be recovered as the functions $\mathfrak{m} \mapsto \mathrm{val}_{\alpha_j}(\varphi^*\mathfrak{m})$, where $\mathfrak{m} \in M$ is a monomial function, and val_{α_j} is the order of vanishing at α_j. Such a parametrization is unique up to the automorphisms of $\mathbb{C}P^1$, therefore the space of rational curves $\mathbb{C}P^1 \to \mathbb{C}\Delta$ is of dimension $m-1$. A rational curve is real if it admits a parametrization invariant by conjugation. Equivalently, χ should belong to $N_{\mathbb{R}^*} \subset N_{\mathbb{C}^*}$, and the numbers α_i are either real, or come in pairs of complex conjugate points with the same exponent vector n_j.

1.1.2. Enumeration of real and complex rational curves

We choose a generic configuration $\mathcal{P}$ of $m-1$ points inside $\mathbb{C}\Delta$, and look for rational curves passing through this configuration. By dimension count and genericity, we have a finite number of complex solutions. The cardinal $|\mathcal{S}^\mathbb{C}(\mathcal{P})|$ of the set $\mathcal{S}^\mathbb{C}(\mathcal{P})$ of solutions is independent of the choice of the point configuration $\mathcal{P}$, and the value of this cardinal is denoted by N_Δ.

We now consider a configuration of points $\mathcal{P}$ which is invariant by the complex conjugation, i.e. it consists of real points and pairs of complex conjugate points. Such a configuration is called a *real configuration of points*. If all the points are real, it is a *totally real configuration of points*, otherwise we call it a *non-totally real configuration*. Contrarily to the complex case, the number $|\mathcal{S}^\mathbb{R}(\mathcal{P})|$ of real curves passing through the configuration $\mathcal{P}$ might depend on the chosen configuration. However, in [15] J-Y. Welschinger showed that if the toric surface $\mathbb{C}\Delta$ is del Pezzo, the vectors of Δ are primitive and the curves are counted with an appropriate sign, the signed number of solutions in $\mathcal{S}^\mathbb{R}(\mathcal{P})$ only depends on the number of pairs of complex conjugate points in the configuration. This invariant is called Welschinger invariant, often denoted by $W_{\Delta,s}$, where s is the number of pairs of complex conjugate points. This is for instance the case of $\mathbb{C}P^2$.

1.1.3. Tropical correspondence statement

In the case of $\Delta = \{(-1,0)^d, (0,-1)^d, (1,1)^d\} \subset \mathbb{Z}^2$, which corresponds to the case of degree d rational curves in $\mathbb{C}P^2$, the invariants are denoted by N_d and $W_{d,s}$. The values of the complex invariants N_d are computed by the Caporaso–Harris formula [4] or by Kontsevich's formula [9]. Roughly at the same time Welschinger invariants were introduced, Mikhalkin [11] proved a correspondence theorem that provided a way of computing both invariants N_d (in fact all N_Δ for any toric surface) and Welschinger invariants $W_{d,0}$ in the case of a totally real configuration of points, using the tropical geometry approach. Later E. Shustin [13] also computed the invariants N_d and $W_{d,0}$, and managed in [14] to make a tropical calculation of the $W_{d,s}$ for any s, i.e. not only in the case of a totally real configuration.

To compute the values of N_d and $W_{d,0}$, Mikhalkin counts tropical curves solution to a tropical enumerative problem with two specific choices of multiplicity. Following his computation, F. Block and L. Göttsche [1] proposed a way of combining these integer multiplicities into a Laurent polynomial multiplicity. The latter evaluated at ± 1 gives back the multiplicities used to compute the invariants N_d and $W_{d,0}$. Moreover, the counting of tropical curves of fixed degree passing through a generic configuration of points using refined multiplicity was proved in [8] to give a tropical invariant. This new choice of multiplicity seems to appear in more and more situations, while its meaning in classical geometry remains quite mysterious. Conjecturally, the refined invariant coincides with the refinement of Severi degrees by the χ_{-y}-genera proposed by L. Göttsche and V. Shende in [7]. This invariant bears also similarities with some Donaldson–Thomas wall-crossing invariants considered by M. Kontsevich and Y. Soibelman [10].

1.1.4. Curves with boundary conditions

In an attempt to find a classical analog to the tropical refined invariants, Mikhalkin introduced in [12] a quantum index for real type I curves. Not going into further details, this is the case of real rational curves. He then proved that a signed count of real rational curves having tangencies at fixed points of the toric boundary according to the value of their quantum index only depends on the number of pairs of complex conjugate points on each toric divisor. He finally proved that in the case of a totally real configuration of points (i.e. no pairs of complex conjugate points), this new invariant can be computed via tropical geometry using the same Block–Göttsche refined multiplicity. This provides another interpretation of these mysterious refined tropical invariants. The result was then generalized by the author in [2, 3] to the case of a non-totally real configuration.

The tropical counterpart to the enumerative problem consisting in putting all the point constraints on the toric boundary is as follows: finding rational tropical curves whose ends are contained in fixed lines. We refer to Section 2 for details on tropical curves and Section 3.1 for a precise statement of the enumerative problem. The count of the curves with the Block–Göttsche multiplicities does not depend on the chosen configuration of points and its value is denoted by $N_\Delta^{\partial,\mathrm{trop}}$.

In the case of $\mathbb{C}P^2$ and $s=(s_1,s_2,s_3)$ with $s_1,s_2,s_3\leqslant \frac{d}{2}$, we set

$$\Delta_d(s) = \left\{(-1,0)^{d-2s_1},(-2,0)^{s_1},(0,-1)^{d-2s_2},(0,-2)^{s_2},(1,1)^{d-2s_3},(2,2)^{s_3}\right\}.$$

The real refined invariant introduced in [12] obtained by counting real rational curves of degree $2d$ having tangencies at s_i pairs of complex conjugate points and $d-2s_i$ real points on each axis is denoted by $R_{d,s}$. We have the following correspondence results.

Theorem 1.1 ((Mikhalkin [12])). — *One has*

$$R_{d,0} = \left(q^{1/2}-q^{-1/2}\right)^{3d-2} N_{\Delta_d(0)}^{\partial,\mathrm{trop}}.$$

Theorem 1.2 ((B. [2])). — *Writing $|s|=s_1+s_2+s_3$, one has*

$$R_{d,s} = \frac{\left(q^{1/2}-q^{-1/2}\right)^{3d-2-|s|}}{\left(q-q^{-1}\right)^{|s|}} N_{\Delta_d(s)}^{\partial,\mathrm{trop}}.$$

Remark 1.3. — The normalization $(q^{1/2}-q^{-1/2})^{3d-2(-|s|)}$ amounts to clear the denominators of the Block–Göttsche multiplicities.

1.2. Result and organization of the paper

The results of [12] and [2, 3] reduce the computation of the invariants $R_{d,s}$ to a tropical one for the invariants $N_\Delta^{\partial,\mathrm{trop}}$. In this paper we prove a recursive formula for the refined invariants $N_\Delta^{\partial,\mathrm{trop}}$, mimicking the Caporaso–Harris formula. This formula allows one to compute them.

The paper is organized as follows. In the second section, we recall the standard definitions related to tropical curves. In the third section, we describe the tropical enumerative problem that defines $N_\Delta^{\partial,\mathrm{trop}}$, we prove its invariance and state the recursive formula. The fourth section is devoted to the proof of the formula. In the last section, we provide some examples of computations obtained using the formula.

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2. Tropical Curves

2.1. Abstract and parametrized tropical curves

2.1.1. Abstract rational tropical curves

Let $\overline{\Gamma}$ be finite graph without cycles and without bivalent vertices. The set of 1-valent vertices is denoted by Γ_∞, and its complement $\Gamma = \overline{\Gamma}\backslash\Gamma_\infty$ is endowed with a complete metric. Such a metric is defined up to isometry by declaring edges adjacent to a removed 1-valent vertex to be isometric to $[0;+\infty[$ (such edges are called *ends*), and the other edges to be isometric to some $[0;l]$ (such edges are called *bounded*), and $l>0$ is called the *length* of the edge.

The ends of Γ are indexed by the 1-valent vertices of $\overline{\Gamma}$. The homeomorphism type of Γ, which is obtained after forgetting the metric, is called the *combinatorial type* of Γ. For a fixed number of ends $m=|\Gamma_\infty|$, there are a finite number of combinatorial types, and the ends can be labelled by $[\![1;m]\!]$.

2.1.2. Parametrized rational tropical curves

We consider parametrized rational tropical curves inside $N_\mathbb{R}=N\otimes\mathbb{R}\simeq\mathbb{R}^2$. When the context is clear, we drop the parametrized and just speak about tropical curves.

Definition 2.1. — *A parametrized rational tropical curve in $N_\mathbb{R}$ is a map $h:\Gamma\to N_\mathbb{R}$ where:*

(i) *Γ is an abstract rational tropical curve,*
(ii) *h is affine with integer slope on each edge of Γ: if the slope of h along an oriented edge e is denoted by $\partial_e h$, it lies in $N\simeq\mathbb{Z}^2$,*
(iii) *h is balanced: at each vertex V of Γ, $\sum_{e\ni V}\partial_e h=0$, where the edges are oriented outside V.*

For any edge of Γ, the integral length of $\partial_e h\in N$ is called the weight w_e of e.

Remark 2.2. — We could assume that N and its dual M are both $\mathbb{Z}^2$. We prefer to see the lattice M as the set of integer linear functions on the space $N_\mathbb{R}$ where the tropical curves live, while N is the space of the slopes of the edges of a tropical curve. Moreover, notice that we deal with tropical curves in the affine space $N_\mathbb{R}$, identified with its tangent space at 0.

If $e\in\Gamma^1_\infty$ is an end of Γ, let $n_e\in N$ be the slope of h alongside e, oriented out of its unique adjacent vertex, i.e. toward infinity. The multiset

$$\Delta=\{n_e\in N\}_{e\in\Gamma^1_\infty}\subset N,$$

is called the *degree* of the curve. It is a multiset since an element may appear several times. Using the balancing condition, one can easily see that $\sum_{n_e\in\Delta}n_e=0$.

Example 2.3. — We say that a parametrized curve is of degree d if $\Delta_d=\{(-1,0)^d,(0,-1)^d,(1,1)^d\}$ where the d exponent means that the vector appears d times.

2.2. Moment of an edge

In this section, curves are not necessarily rational, which just means removing the assumption that the graphs are without cycles.

2.2.1. Tropical moment

Let ω be a generator of Λ^2M, i.e. a volume form in $N_\mathbb{R}$.

Definition 2.4. — *Let $h:\Gamma\to N_\mathbb{R}$ be a parametrized rational tropical curve. Let $e\in\Gamma^1_\infty$ be an end oriented toward infinity, directed by n_e. We define the moment of e as the scalar*

$$\mu_e=\omega(n_e,p)\in\mathbb{R},$$

where $p \in e$ is any point on the edge e. We similarly define the moment of a bounded edge if we specify its orientation. The moment of a bounded edge is reversed when its orientation is reversed.

The definition uses the identification between $N_{\mathbb{R}}$ and its tangent space. Intuitively, the moment of an unbounded end is just a way of measuring its position alongside a transversal axis. Therefore, fixing the moment of an end amounts to impose on the curve to pass through some point at infinity, or equivalently the corresponding end to be contained in a fixed line with the same slope.

Remark 2.5. — In a way, this allows us to do toric geometry in a compactification of $\mathbb{R}^2$ but staying in $\mathbb{R}^2$. It provides a coordinate on the components of the toric boundary without even having to introduce the concept of toric boundary in the tropical world.

2.2.2. Complex moment

The above tropical notion has also a meaning in toric geometry, where it corresponds to a power of the coordinate of the intersection point of the curve with the toric divisor. Let $\varphi : \mathbb{C}C \dashrightarrow N_{\mathbb{C}^*} \simeq (\mathbb{C}^*)^2$ be a parametrized curve, and $q \in \mathbb{C}C$ a point where φ is not defined, meaning φ has zeros or poles at q. The vanishing order of coordinates of φ at q yields a non-zero vector $n_q \in N$. Taking a fan that contains $\mathbb{R}_{\geqslant 0} n_q$ yields a toric variety in which we can extend φ at q, sending it to the toric divisor associated to the ray $\mathbb{R}_{\geqslant 0} n_q$. The coordinate along this toric divisor is measured by any primitive monomial that is orthogonal to n_q. For instance $\omega(n_q, -)$ if n_q is primitive. If n_q is not primitive, evaluating the monomial $\omega(n_q, -)$, which indeed extends on the toric divisor, amounts to take the $l(n_q)$-power of the coordinate of the intersection point, where $l(n_q)$ is the lattice length of n_q.

2.2.3. Menelaus relation

In the case of a complex curve, the Weil reciprocity law gives us the following relation between the moments :

$$\prod_{i=1}^{m} \mu_i = (-1)^m.$$

We could also prove the relation using Viète's formula. In the tropical world we have an analog called the tropical Menelaus theorem, which gives a relation between the moments of the unbounded ends of a parametrized tropical curve.

Proposition 2.6 (Tropical Menelaus [12]). — *For a parametrized tropical curve $h : \Gamma \to N_{\mathbb{R}}$ of degree Δ, we have*

$$\sum_{n_e \in \Delta} \mu_e = 0.$$

In the tropical case (resp. in the complex case), a configuration of m points on the toric divisors is said to satisfy the *Menelaus condition* if $\sum \mu_e = 0$ (resp. $\prod \mu_e = (-1)^m$).

2.3. Moduli space of tropical curves and refined multiplicity of a simple tropical curve

2.3.1. Simple rational tropical curves.

Definition 2.7. — *We say that a parametrized rational tropical curve $h : \Gamma \to N_{\mathbb{R}}$ is* simple *if every vertex is trivalent, and it has no* flat vertex*: a vertex all of whose adjacent edges lie in the same (classical) line.*

In particular, no edge can have a zero slope since it would imply that its extremities are flat vertices.

2.3.2. Moduli space of abstract rational tropical curves

Given a graph without cycle Γ, to make it into a rational tropical curve, one just needs to specify the lengths of the bounded edges. If the curve is trivalent and has m ends, there are $m-3$ bounded edges, otherwise the number of bounded edges is $m-3-\mathrm{ov}(\Gamma)$, where

$$\mathrm{ov}(\Gamma) = \sum_V \mathrm{val}(V) - 3,$$

is the *overvalence of the graph*. Therefore, the set of curves having the same combinatorial type is homeomorphic to $\mathbb{R}_{\geqslant 0}^{m-3-\mathrm{ov}(\Gamma)}$, and the coordinates are the lengths of the bounded edges. If Γ is an abstract tropical curve, we denote by $\mathrm{Comb}(\Gamma)$ the set of curves having the same combinatorial type as Γ.

For a given $\mathrm{Comb}(\Gamma)$, the boundary of $\mathbb{R}_{\geqslant 0}^{m-3-\mathrm{ov}(\Gamma)}$ corresponds to curves for which the length of an edge is zero, and therefore corresponds to a graph having a different combinatorial type. This graph is obtained by deleting the edge with zero length and merging its extremities. We can thus glue together all the cones of the finitely many combinatorial types and obtain the *moduli space* $\mathcal{M}_{0,m}$ *of rational tropical curves with* m *marked points*. It is a simplicial fan of pure dimension $m-3$, and the top-dimensional cones correspond to trivalent curves. The combinatorial types of codimension 1 are called *walls*.

2.3.3. Moduli space of parametrized curves

Given a tropical curve Γ, if we specify the slope of every end, and the position of a vertex, we can define uniquely a parametrized tropical curve $h:\Gamma\to N_{\mathbb{R}}$. Therefore, if $\Delta\subset N$ denotes the set of slopes of the ends, the *moduli space* $\mathcal{M}_0(\Delta,N_{\mathbb{R}})$ *of parametrized rational tropical curves of degree* Δ is isomorphic to $\mathcal{M}_{0,m}\times N_{\mathbb{R}}$ as a fan, where the $N_{\mathbb{R}}$ factor corresponds for instance to the position of the vertex adjacent to the first end.

2.3.4. Evaluation map

On $\mathcal{M}_0(\Delta,N_{\mathbb{R}})$ we have a well-defined evaluation map that associates to each parametrized curve the family of moments of its ends:

$$\begin{aligned}\mathrm{ev}:\mathcal{M}_0(\Delta,N_{\mathbb{R}}) &\longrightarrow \mathbb{R}^m\\ (\Gamma,h) &\longmapsto \mu=(\mu_i)_{1\leqslant i\leqslant m}\end{aligned}$$

By the tropical Menelaus theorem, the image lies in the hyperplane $H=\{\sum_1^m \mu_i=0\}$. Notice that the evaluation map is linear on every cone of $\mathcal{M}_0(\Delta,N_{\mathbb{R}})$. Moreover, restraining the codomain to H, both spaces have dimension equal to $m-1$. Thus, if Γ is a trivalent curve, the restriction of ev on $\mathrm{Comb}(\Gamma)\times N_{\mathbb{R}}$ has a determinant well-defined up to sign when H and $\mathrm{Comb}(\Gamma)\subset\mathcal{M}_{0,m}$ are both endowed with a basis of their underlying lattice, and $N_{\mathbb{R}}$ is endowed with a basis of N. The absolute value $m_\Gamma^{\mathbb{C}}$ is called the *complex multiplicity* of the curve, well-known to factor into the following product over the vertices of Γ:

$$m_\Gamma^{\mathbb{C}}=\prod_V m_V,$$

where $m_V=|\omega(a_V,b_V)|$ if a_V and b_V are the slopes of two adjacent edges of V. The balancing condition ensures that m_V does not depend on the chosen edges. This multiplicity is the one that appears in the correspondence theorem of Mikhalkin [11]. Notice that the simple parametrized tropical curves are precisely the ones with non-zero multiplicity. The refined multiplicity of a simple nodal tropical curve, sometimes also called the Block–Göttsche multiplicity is given by

$$m_\Gamma^q = \prod_V \left[m_V^{\mathbb{C}}\right]_q ,$$

where $[a]_q = \frac{q^{a/2}-q^{-a/2}}{q^{1/2}-q^{-1/2}}$ is the q-analog of a.

3. Definition of the invariants and recursive formula

3.1. Tropical enumerative problem

We now turn our focus into the tropical enumerative problem that provides the refined tropical invariants $N_\Delta^{\partial,\mathrm{trop}}$ used in [12]. This family of enumerative problems depends on the choice of the degree Δ, and the recursive formula proven in the present paper gives a relation between all these invariants.

Let $\Delta = \{n_1, \ldots, n_m\} \subset N$ be a tropical degree. We do not assume the vectors n_i to be primitive since the recursive formula almost always makes appear degrees with non-primitive vectors. For each $n_i \in \Delta$ we choose some scalar $\mu_i \in \mathbb{R}$ such that the Menelaus relation $\sum_1^m \mu_i = 0$ is satisfied. We now look at the set $\mathcal{S}(\mu)$ of parametrized rational tropical curves of degree Δ that have $\mu = (\mu_i)$ as family of moments, i.e. $\mathrm{ev}(\Gamma) = \mu$. This tropical enumerative problem is called the Δ-problem.

Due to the linear character of the evaluation map restricted to any combinatorial type, each one of them contributes at most one solution unless the map is non-injective. Let $\mathrm{Comb}(\Gamma)$ be a top-dimensional combinatorial type, for which the evaluation map is not injective. Since $\mathrm{Comb}(\Gamma)$ and $H \subset \mathbb{R}^m$ have the same dimension, the evaluation map is not surjective either. Similarly, the restriction of the evaluation map to non top-dimensional combinatorial types fails to be surjective for dimensional reasons. A family of moments μ is said to be generic if it is chosen outside the image of the evaluation map restricted to non top-dimensional combinatorial types, and top-dimensional types with non-injective evaluation map. Thus, if the configuration of moments μ is chosen generically, the set of solutions $\mathcal{S}(\mu)$ is finite, and the rational curves of $\mathcal{S}(\mu)$ solution to the problem are simple, and they have a well-defined refined tropical multiplicity m_Γ^q. Then, we set

$$N_\Delta^{\partial,\mathrm{trop}}(q,\mu) = \sum_{\Gamma \in \mathcal{S}(\mu)} m_\Gamma^q \in \mathbb{Z}\left[q^{\pm 1/2}\right].$$

Remark 3.1. — Let $\mathrm{Comb}(\Gamma)$ be a combinatorial type, and A the linear map associated to the restriction $\mathrm{ev}_{\mathrm{Comb}(\Gamma)}$ of ev to $\mathrm{Comb}(\Gamma)$. Let (l,V) denotes the coordinates on $\mathrm{Comb}(\Gamma)$ given by the length of the edges and the position of a specified vertex. To see if $\mathrm{Comb}(\Gamma)$ contributes a solution, one just needs to solve the system $A(l,V)=\mu$, leading to a formal solution $(l,V)=A^{-1}(\mu)\in\mathbb{R}^{m-3}\times N_{\mathbb{R}}$, and check that all its first $m-3$ coordinates are non-negative.

THEOREM 3.2. — *The value of $N_\Delta^{\partial,\mathrm{trop}}(q,\mu)$ does not depend on μ as long as μ is generic.*

The invariance statement can be seen as a corollary of the correspondence statement from [12] and [2]. Alternatively it can be seen as a particular case of the invariance statement from [8] restricting to the case where point constraints are forced to lie on the ends of the curves. To make the paper self-contained, we provide a proof in the next subsection.

3.2. Proof of tropical invariance

The proof of invariance goes in the same way as many tropical proofs of invariance by showing that we have a local invariance of the count around the walls of the tropical moduli space.

Proof of Theorem 3.2. — We choose two generic configurations $\mu(0)$ and $\mu(1)$, and choose a generic path $\mu(t)$ between them. Due to the genericity, we know that the set F of values of t where $\mu(t)$ meets the non-generic configurations is finite, and $N_\Delta^{\partial,\mathrm{trop}}(q,\mu(t))$ is constant on the connected components of the complement of this exceptional set F. We now need to check that the value is constant around these special values.

Let t^* be such a special value. Thanks to the genericity of the path, it means that at least one of the curves of $\mathcal{S}(\mu(t^*))$ has a unique four-valent vertex V. There are three ways to deform this curve into a trivalent one by choosing a splitting of the quadrivalent vertex, meaning there are three maximal cones adjacent to the wall. In some cases, one of the deformation leads to a flat vertex, i.e. a non-injective combinatorial type. Let e_1,e_2,e_3,e_4 be the adjacent edges directed by a_1,a_2,a_3,a_4, with ingoing orientations. Their index i is taken in $\mathbb{Z}/4\mathbb{Z}$. The splittings are denoted by $12//34$, $13//24$ and $14//23$ according to the pairing of vertices. Around a wall, one curve solution may divide in two solutions, or the other way around two solutions may merge into one solution.

We first assume that there are no parallel edges among the edges e_i. Let us prove that up to a relabelling we can assume:

- for each i we have $\omega(a_i, a_{i+1}) > 0$,
- we have $\omega(a_2, a_3) > \omega(a_1, a_2)$.

The first point essentially consists in finding some cyclic counterclockwise order on the adjacent vectors a_i oriented out of the vertex. Let us take such a cyclic order and prove that it satisfies these conditions : if we had $\omega(a_4, a_1) < 0$ (same for $\omega(a_1, a_2) < 0$ and other values), then because of the counterclockwise cyclic order all the vectors a_i are in a half-plane and their sum would not be zero, which is absurd. So we have $\omega(a_i, a_{i+1}) > 0$. For the second point, the assumption that there are no parallel vectors ensures that $\omega(a_i, a_{i+1} + a_{i-1}) \neq 0$ for any i, thus there are no consecutive equal values. Hence we can assume that $\omega(a_2, a_3) > \omega(a_1, a_2)$ up to a cyclic shift of the indices.

Let us notice that

$$\begin{aligned} \omega(a_4, a_1) &= \omega(-a_1 - a_2 - a_3, a_1) \quad \text{and } \omega(a_3, a_4) = \omega(a_3, -a_1 - a_2 - a_3) \\ &= \omega(a_1, a_2 + a_3) > 0, \qquad\qquad\qquad\quad = \omega(a_1 + a_2, a_3) > 0. \end{aligned}$$

To prove the local invariance, we need to know the repartition of the combinatorial types around the wall, that is, the adjacent combinatorial types providing a solution when $\mu(t)$ moves slightly. Using the correspondence theorem of Mikhalkin [11] or the tropical proof of invariance of the count with complex multiplicities given by A. Gathmann–H. Markwig in [6], this repartition is known to match the equality given between complex multiplicities $m_\Gamma = \prod_V m_V > 0$. All the vertices in the respective products for the three adjacent combinatorial types are the same, except the two vertices resulting from the splitting of the quadrivalent vertex. The desired relation is then

$$\begin{array}{ccccc} \underset{\text{for } 12//34}{\omega(a_1, a_2)\omega(a_1 + a_2, a_3)} & + & \underset{\text{for } 13//24}{\omega(a_1, a_3)\omega(a_2, a_1 + a_3)} & & \\ & + & \underset{\text{for } 14//23}{\omega(a_2, a_3)\omega(a_2 + a_3, a_1)} & = & 0, \end{array}$$

and the repartition of combinatorial types around the wall is given by the sign of each term. It means up to sign that one is positive and is on one side of the wall, and the two other ones are negative, on the other side of the wall. Hence, we just need to study the signs of each term to know which curve is on which side. We know that $\omega(a_1, a_2)$ and $\omega(a_1 + a_2, a_3)$ are positive, therefore their product, which is the term of $12//34$, is also positive.

We know that $\omega(a_2, a_3)$ is positive, but $\omega(a_2 + a_3, a_1)$ is negative, therefore their product is negative and 14//23 is on the other side of the wall. It means that the combinatorial types 12//34 and 14//23 are on opposite sides of the wall. We need to determine on which side the type 13//24 is, and that is given by the sign of the middle term. As by assumption $\omega(a_2, a_1 + a_3) = \omega(a_2, a_3) - \omega(a_1, a_2) > 0$, it is determined by the sign of $\omega(a_1, a_3)$.

* If $\omega(a_1, a_3) > 0$, then 12//34 and 13//24 are on the same side, and the invariance for refined multiplicities is dealt with the identity, obtained by just developing the products,

$$[\omega(a_2, a_3)]_q[\omega(a_1, a_2 + a_3)]_q = [\omega(a_1, a_2)]_q[\omega(a_1 + a_2, a_3)]_q + [\omega(a_1, a_3)]_q[\omega(a_2, a_1 + a_3)]_q.$$

* and if $\omega(a_1, a_3) < 0$, then 14//23 and 13//24 are on the same side and then the invariance for refined multiplicities is true since

$$[\omega(a_2, a_3)]_q[\omega(a_1, a_2 + a_3)]_q + [\omega(a_3, a_1)]_q[\omega(a_2, a_1 + a_3)]_q = [\omega(a_1, a_2)]_q[\omega(a_1 + a_2, a_3)]_q.$$

If we have some edges parallel among the vectors a_i, either two consecutive vectors are parallel, and then the invariance is straightforward, since there are only two adjacent combinatorial types with equal non-zero multiplicity, or we can choose a cyclic labelling such that a_1 and a_3 are parallel. We then have $\omega(a_1, a_3) = 0$. It means that one of the determinant multiplicities is zero, which is normal since the associated combinatorial type would have a flat vertex. Thus,

$$\underset{\text{for } 12//34}{\omega(a_1, a_2)\omega(a_1 + a_2, a_3)} + \underset{\text{for } 14//23}{\omega(a_2, a_3)\omega(a_2 + a_3, a_1)} = 0.$$

It means that the two terms are of opposite sign. Assume the first one is positive, and thus $\omega(a_1, a_2)$ and $\omega(a_1 + a_2, a_3) = \omega(a_2, a_3)$ have the same sign. The refined multiplicity is then

$$[\omega(a_1, a_2)]_q[\omega(a_2, a_3)]_q.$$

The second term being negative, it means that $\omega(a_2, a_3)$ and $\omega(a_1, a_2 + a_3) = \omega(a_1, a_2)$ have the same sign. The refined multiplicity is the the same and we have the desired local invariance. It is the same if the first term is negative. □

Remark 3.3. — The technicalities in the proof are just needed to find the repartition of the combinatorial types around the wall. This repartition could also be found by looking at the subdivisions of the Newton polygon, which are dual to the tropical curves. The quadrilateral dual to the quadrivalent vertex has three subdivisions (resp. two in the case of a flat vertex) matching the three splittings of the vertex: one using the big diagonal, one using the small diagonal, and one by using a parallelogram. Then, we can show that the repartition is given by putting the subdivision using the big diagonal alone on one side of the wall. See [8].

3.3. Recursive formula

Before stating the formula we need to introduce some notations. Let $\Delta = \{v_1, v_2, \ldots, v_m\} \subset N \simeq \mathbb{Z}^2$ be a degree for plane curves, i.e. a family of vectors that whose sum is zero. Let $\mu \in \mathbb{R}^m$ be the family of moments satisfying the tropical Menelaus condition, and let Γ be a parametrized tropical curve in $\mathcal{S}(\mu)$. As Γ is a tree, there is a unique shortest path between the edges directed by v_1 and v_m, which we call a *string*. The string has a natural orientation from the end v_1 to the end v_m. Once we remove the string, the curve Γ disconnects into several components $\Gamma_1, \ldots, \Gamma_p$, indexed in the order in which they meet the string. Let e_i be the edge in Γ_i adjacent to the string, and V_i the vertex in which both meet. All these notations along with the ones which are about to follow are depicted on Figure 3.1.

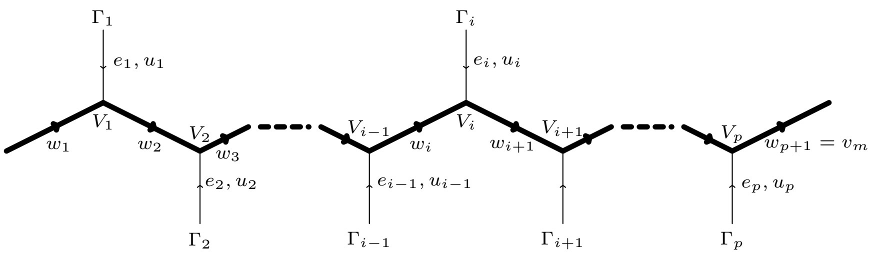

Figure 3.1. skeleton of the curve Γ with every notation. The string is in fat.

There are two possibilities for Γ_i :

* either e_i is an end, directed by some $v_i \in \Delta$, and we set $u_i = -v_i$,

* or e_i is bounded and Γ_i contains more than one end of Γ. We then denote by $\widetilde{\Delta_i} \subset \Delta$ the set of slopes of the ends of Γ that belong to Γ_i. Let $u_i = -\sum_{v \in \widetilde{\Delta_i}} v$ be the directing vector of e_i going toward the string, and $\Delta_i = \widetilde{\Delta_i} \sqcup \{u_i\}$ be the degree of the curve Γ_i' obtained by letting e_i going to infinity instead of stopping when meeting the string at V_i.

Finally, let w_i be the vector directing the edge of the string between V_{i-1} and V_i. This means that $w_1 = -v_1$ and $w_{i+1} = w_i + u_i$. Let also $\sigma_i = \omega(w_i, w_{i+1})$ be the signed multiplicity of the vertex V_i. We now can derive a recursive formula from this description.

Theorem 3.4. — *With the above notation, we have*

$$N_{\Delta}^{\partial,\mathrm{trop}}(q) = \sum_{\bullet} \prod_{i=1}^{p} [\sigma_i]_q N_{\Delta_i}^{\partial,\mathrm{trop}}(q),$$

where the sum $\bullet$ is over the ordered partitions of $\Delta - \{v_1, v_m\}$ into

$$\Delta - \{v_1, v_m\} = \bigsqcup_{i=1}^{p} \widetilde{\Delta_i},\ u_i = -\sum_{v \in \widetilde{\Delta_i}} v,$$

such that

$$\begin{cases} \sigma_i > 0 \Rightarrow \left|\widetilde{\Delta_i}\right| = 1 \\ \omega(\sigma_i u_i, \sigma_{i+1} u_{i+1}) \geqslant 0, \end{cases}$$

and up to a reordering of consecutive indices i having respective collinear vectors u_i.

Remark 3.5. — The term "ordered partition" means that the set is subdivided into several subsets, but we keep track of their order by labelling them. Ordered partitions in p subsets are thus in bijection with the surjective maps to $[\![1;p]\!]$. The reordering means that orders that differ by a sequence of permutations of consecutive indices i and $i+1$ such that u_i and u_{i+1} are collinear, are counted only once.

4. Proof of the recursive formula

To prove the recursive formula, we find a way to describe the curves solution to the problem for a specific value of μ. The idea is to choose an idealistic configuration of constraints and then a 1-parameter family $\mu(t)$ of moments getting closer and closer to this idealistic but unreachable configuration. Such a 1-parameter family $\mu(t)$ is called a deformation. Then, we describe the specific combinatorial types that continue to provide a solution through the deformation process toward this ideal configuration, which we call *surviving combinatorial types*.

Remark 4.1. — The same idea derives the tropical proof of the Caporaso–Harris formula in [5]: one deforms the constraints by making one of the marked points going to infinity on the left. The only combinatorial types that "survive" the deformation (see definition below) are those that either have the corresponding marked point on a horizontal edge of the curve, or that split into a *floor* containing the marked point and a curve of lower degree, joined to the floor by horizontal edges. We implement this heuristic in our setting to provide a way of computing the invariants $N_\Delta^{\partial,\mathrm{trop}}$.

First, recall the evaluation map $\mathrm{ev} : \mathcal{M}_0(\Delta, N_\mathbb{R}) \to H \subset \mathbb{R}^m$. If $\mu \in \mathbb{R}^m$ is a family of moments, we say that a parametrized tropical curve is solution to the Δ-problem with value μ if $\mathrm{ev}(\Gamma) = \mu$. In some cases we see μ as a function $\Delta - \{v_1\} \to \mathbb{R}$ that assigns to any end its moment. If $\widetilde{\Xi} \subset \Delta - \{v_1\}$ is a subset, then

$$\Xi = \left\{ -\sum_{v \in \widetilde{\Xi}} v \right\} \sqcup \widetilde{\Xi}$$

is still a tropical degree and this notation allows us to consider the Ξ-problem with value $\mu|_{\widetilde{\Xi}}$.

Definition 4.2. — *We call a* deformation vector *a lattice vector $\delta \in \mathbb{Z}^{m-1} \subset \mathbb{R}^{m-1}$. The associated* deformation *of an element $\mu \in \mathbb{R}^{m-1}$ is the half-line $\mu + \mathbb{R}_{\geqslant 0}\delta$, parametrized by $t \mapsto \mu + t\delta$.*

Definition 4.3. — *Let Γ be a tropical curve with non-zero multiplicity. On the orthant $\mathrm{Comb}(\Gamma) \times N_\mathbb{R}$ of curves having the same combinatorial type, the evaluation map is linear with matrix A in the canonical basis:*

$$A = \mathrm{ev}|_{\mathrm{Comb}(\Gamma)\times N_\mathbb{R}} : \mathbb{R}^{m-3}_{\geqslant 0} \times N_\mathbb{R} \to \mathbb{R}^{m-1}.$$

If δ is a deformation vector, we say that the combinatorial type of Γ survives *the deformation if the first $m-3$ coordinates of $A^{-1}\delta$ are non-negative.*

Remark 4.4. — As the multiplicity of the curve is non-zero, the matrix A is invertible. Hence, any small deformation of the image can be pulled back by the evaluation map to a small deformation of the curve, which means a variation of the length of the bounded edges and maybe a translation. The assumption that the first coordinates are non-negative means that going along the deformation δ, the length of the edges are non-decreasing, and the half-line $\mathrm{ev}|^{-1}_{\mathrm{Comb}(\Gamma)\times N_\mathbb{R}}(\mu) + \mathbb{R}_{\geqslant 0}A^{-1}\delta$ does not meet the boundary of the orthant, place where the deformation of the curve cannot go on since an edge has zero length and a quadri-valent vertex appears.

Let Γ be a tropical curve solution to the Δ-problem with value μ, with non-zero-multiplicity, and δ a deformation vector. Let f be some affine function defined on the orthant of the combinatorial type of Γ, with linear part $\overline{f}$. Then we write

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \overline{f}\left(A^{-1}\delta\right),$$

for the variation of f along the deformation. The functions of interest are the position of a vertex V, the length of a bounded edge E, and the moment of some edge E.

Example 4.5. — If $f = V : \mathrm{Comb}(\Gamma) \times N_{\mathbb{R}} \to N_{\mathbb{R}}$ is the position of a vertex of Γ, then $\frac{\mathrm{d}V}{\mathrm{d}t}$ is the direction in which V moves when the curve is deformed by making $\mathrm{ev}(\Gamma)$ go in direction δ.

From now on, we consider the deformation vector $\delta = (0, \ldots, 0, -1)$, which means that the moment of the edge directed by v_m goes to $-\infty$, and thus the moment of the edge directed by v_1 goes to $+\infty$. We look for combinatorial types that survive this deformation.

Proposition 4.6. — *For t large enough, the only combinatorial types that contribute a solution to $N_\Delta^{\partial,\mathrm{trop}}(q, \mu + t\delta)$ are surviving combinatorial types.*

Proof. — Each combinatorial type $\mathrm{Comb}(\Gamma)$ of tropical curves provides a formal solution to the problem, meaning that we can solve

$$\mathrm{ev}|_{\mathrm{Comb}(\Gamma)\times N_{\mathbb{R}}}(l, V)) = \mu$$

formally and find the lengths of the edges, but some of them might be negative. The formal solution is a true solution if the length of each edge is non-negative. If $\mathrm{Comb}(\Gamma)$ is not a surviving combinatorial type, the length of some edge strictly decreases when t increases, and it becomes negative if t is large enough, therefore the combinatorial type no longer provides a solution. □

Remark 4.7. — As the length of some edges might be constant through the deformation, the survival property is not enough to ensure a combinatorial type ultimately provide a true solution. More precisely, among the combinatorial types differing from one another by a permutation of consecutive indices i having collinear directing vectors u_i, exactly one ultimately provides a true solution. This is the place where the reordering appears.

Using the balancing condition, we see that the moment μ_{e_i} of e_i is constant equal to minus the sum of coordinates of $\mu|_{\widetilde{\Delta}_i}$. This means that the edge e_i is contained in a fixed line. The vertices of Γ_i are fixed because the moment of two of their incident edges are constant. Thus, if we change the moment of v_1 and v_m, the only way the edges e_i can move is by varying their length while the edges in each Γ_i different from e_i are fixed. Moreover, only their extremity V_i in which Γ_i meets the string can move, and these vertices move on the lines that respectively contain e_i.

For the combinatorial type of Γ to survive the deformation, we need to check that neither the length of the edges of the string nor the length of the bounded edges e_i go to 0. Recall $\sigma_i = \omega(w_i, w_{i+1})$ the signed multiplicity of the vertex V_i. If $\sigma_i < 0$, that means that at V_i, the string turns right, and if $\sigma_i > 0$, it means that the string turns left, as we can see on Figure 4.1.

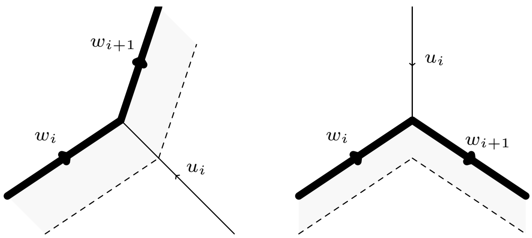

Figure 4.1. On the left, a vertex that turns left, i.e. $\sigma_i > 0$, and on the right a vertex that turns right,i.e. $\sigma_i < 0$.

Let $\tau_i = \omega(w_i, V_i) = \omega(w_i, V_{i-1})$ be the moment of the edge directed by w_i. The balancing condition ensures that $\tau_{i+1} = \tau_i + \mu_{e_i}$, hence all the moments τ_i only differ from one another by a constant, and thus all go to $-\infty$ through the deformation process, since τ_{p+1} is the moment of the edge directed by v_m that goes to $-\infty$ with velocity -1 by assumption. We then have $\frac{\mathrm{d}\tau_i}{\mathrm{d}t} = -1$.

We now write down some equations whose derivative allows us to obtain the variations of the lengths of the bounded edges of the curve. We separate the case of edges which are adjacent to the string from the edges which are part of the string. We denote by a dependence on t the fact that the functions (position of a vertex, moment of an edge, length of an edge) are taken on the curve of $\mathrm{Comb}(\Gamma) \times N_{\mathbb{R}}$ whose evaluation is $\mu + t\delta$, in case the orthant provides a true solution. As $t = 0$ provides a true solution, the formal solutions are true solutions at least for small values of t. The formal solution is true for any t if the combinatorial type is a surviving one.

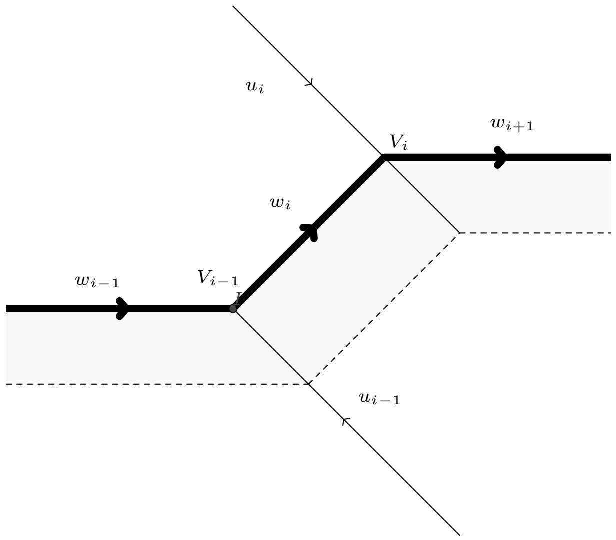

Figure 4.2. Deformation of an edge of the string.

- First, let P_i be a fixed point on e_i, which is the other extremity if e_i is bounded and any point otherwise. We have

$$V_i(t) = P_i + l_i(t)u_i = P_i + l_i(t)(w_{i+1} - w_i),$$

where $l_i(t)$ is the length of the edge between V_i and P_i. Hence,

$$\begin{aligned}\tau_i = \omega(w_i, V_i(t)) &= \omega(w_i, P_i) + l_i(t)\omega(w_i, w_{i+1} - w_i)\\ &= \omega(w_i, P_i) + l_i(t)\sigma_i.\end{aligned}$$

Thus, by derivating, we get $-1 = \frac{\mathrm{d}\tau_i}{\mathrm{d}t} = \sigma_i \frac{\mathrm{d}l_i}{\mathrm{d}t}$. This means that:
  - If at V_i the string turns left ($\sigma_i > 0$), then l_i decreases as t goes to $+\infty$, the vertex V_i goes up the edge e_i and will meet a vertex if there is one, which is the case if and only if e_i is bounded.
  - If at V_i the string turns right ($\sigma_i < 0$), then l_i increases as t goes to $+\infty$.

- We consider the edge between the vertices V_{i-1} and V_i. Let λ_i be its length so that we have

$$V_i(t) - V_{i-1}(t) = \lambda_i(t) w_i.$$

By derivating with respect to t and using previous notations, we get

$$\frac{\mathrm{d}l_i}{\mathrm{d}t}(w_{i+1} - w_i) - \frac{\mathrm{d}l_{i-1}}{\mathrm{d}t}(w_i - w_{i-1}) = \frac{\mathrm{d}\lambda_i}{\mathrm{d}t} w_i,$$

which is equivalent to

$$-\frac{w_{i+1} - w_i}{\sigma_i} + \frac{w_i - w_{i-1}}{\sigma_{i-1}} = \frac{\mathrm{d}\lambda_i}{\mathrm{d}t} w_i$$

since $\frac{\mathrm{d}l_i}{\mathrm{d}t} = -\frac{1}{\sigma_i}$. Multiplying by $\sigma_i \sigma_{i-1}$ we get

$$\sigma_{i-1}(w_{i+1} - w_i) - \sigma_i(w_i - w_{i-1}) = -\sigma_i \sigma_{i-1} \frac{\mathrm{d}\lambda_i}{\mathrm{d}t} w_i.$$

At this point we can check that $\sigma_{i-1} w_{i+1} + \sigma_i w_{i-1}$ is indeed collinear to w_i :

$$\begin{aligned} \omega(w_i, \sigma_{i-1} w_{i+1} + \sigma_i w_{i-1}) &= \sigma_{i-1} \omega(w_i, w_{i+1}) + \sigma_i \omega(w_i, w_{i-1}) \\ &= \sigma_{i-1}\sigma_i - \sigma_i \sigma_{i-1} \\ &= 0. \end{aligned}$$

In order to check the sign of $\frac{\mathrm{d}\lambda_i}{\mathrm{d}t}$, we can evaluate any linear form on this vector equation, for instance $\omega(w_{i-1}, -)$, which gives us

$$\sigma_{i-1}^2 \sigma_i \frac{\mathrm{d}\lambda_i}{\mathrm{d}t} = \sigma_{i-1}^2 + \sigma_i \sigma_{i-1} - \sigma_{i-1} \omega(w_{i-1}, w_{i+1}).$$

By noticing that

$$\begin{aligned} \omega(u_{i-1}, u_i) &= \omega(w_i - w_{i-1}, w_{i+1} - w_i) \\ &= \sigma_i + \sigma_{i-1} - \omega(w_{i-1}, w_{i+1}), \end{aligned}$$

after dividing by σ_{i-1}, we are left with

$$\sigma_{i-1} \sigma_i \frac{\mathrm{d}\lambda_i}{\mathrm{d}t} = \omega(u_{i-1}, u_i).$$

Hence, $\frac{\mathrm{d}\lambda_i}{\mathrm{d}t}$ is non-negative if and only if $\omega(\sigma_{i-1} u_{i-1}, \sigma_i u_i)$ is non-negative.

We now can describe the conditions for a combinatorial type to survive our deformation.

Proposition 4.8. — *Let* Γ *be a parametrized tropical curve. In the above notations, the combinatorial type of* Γ *survives the deformation* $t \to +\infty$ *if and only if*

- *the edge* e_i *is an unbounded edge whenever* $\sigma_i > 0$*;*
- *for each* i*, we have* $\omega(\sigma_{i-1}u_{i-1}, \sigma_i u_i) \geqslant 0$.

Proof. — The statement follows from the previous description: a combinatorial type survives the deformation if and only if the length of the bounded edges is non-decreasing along the deformation. All the cases have previously been studied:

- The length of the bounded edges of the string is non-decreasing, hence for each i we have $\omega(\sigma_{i-1}u_{i-1}, \sigma_i u_i) \geqslant 0$.
- The bounded edges inside some Γ_i but different from e_i are constant.
- The edges e_i have a non-decreasing length unless $\sigma_i > 0$, and then we need the edge to be unbounded. □

We can now prove the recursive formula.

Proof of Theorem 3.4. — Let $\mu \in \mathbb{R}^{m-1}$ be any value. Thanks to the previous proposition, up to a change of μ by $\mu + t\delta$ for a very large t, we can assume that the combinatorial types of the solutions to the Δ-problem with the value μ are surviving combinatorial types for our deformation. However, as noticed, the subtlety is that not all surviving combinatorial types provide a true solution. Nevertheless, each of the curves Γ'_i is solution to the Δ_i-problem with value $\mu|_{\widetilde{\Delta}_i}$. We use this to construct the solutions.

Let Γ be a solution to the Δ-problem with value $\mu+t\delta$ for t large enough. By assumption it has a surviving combinatorial type. Moreover, the Γ'_i provide solutions to the underlying Δ_i-problems with values $\mu|_{\widetilde{\Delta}_i}$, and the multiplicity of Γ factors in the following way:

$$m^q_\Gamma = \prod_1^p [\sigma_i]_q m^q_{\Gamma_i}.$$

Conversely, any combinatorial type can be described by the ordered partition $\Delta - \{v_1, v_m\} = \bigsqcup \Delta_i$ along with the combinatorial type of the curves Γ_i. Let (Γ_i) be a family of solutions to the respective Δ_i-problems with respective values $\mu|_{\widetilde{\Delta}_i}$, and we try to glue them into a global solution, for t large enough. The gluing is given by the order of the partition in which we glue $\Gamma_1, \dots, \Gamma_p$ on the string linking the unbounded end 1 to the unbounded end m. We have a formal solution obtained by resolving the length of the edges on the combinatorial type, and need to check that they indeed provide a true solution.

The lengths of the bounded edges inside the graphs Γ_i are constant. The only orders that have non-decreasing lengths for the edges of the strings and the edges e_i when t goes to $+\infty$ are the orders considered in the formula. If the length of some of these edges is negative but increases through the deformation, it becomes positive for t large enough. Conversely, if the order is not one of the considered, some edge length decreases along the deformation process and is ultimately negative, so the combinatorial type ceases to provide a solution. Finally, if the length of some edge of the string is constant through the deformation, meaning that consecutive incidents vectors to the string are collinear, there is a unique order between them that matches the order of their moments, i.e. the order in which a transversal oriented line would meet them, and provides positive lengths for these edges, hence the consideration of the order on $\Gamma_1, \ldots, \Gamma_p$ up to a reordering of consecutive collinear vectors u_i.

Finally, by putting together the contribution of the different possibilities of Γ_i, we get

$$\sum_{\Gamma_i \in \mathcal{S}\left(\mu|_{\widetilde{\Delta}_i}\right)} \prod_1^p [\sigma_i]_q m^q_{\Gamma_i} = \prod_1^p [\sigma_i]_q N^{\partial,\mathrm{trop}}_{\Delta_i},$$

and thus the desired formula. □

Provided that the deformation is big enough, the moments of all the unbounded edges except v_1 and v_m are really small regarding these two specific moments. It means that when we look at a solution to the Δ-problem with our value of μ, all the edges adjacent to the string seem to go through the origin of $\mathbb{R}^2$ (Although they do not, but they are not far from it if we look at them from far far away) while the string goes around the origin, changing its direction when meeting an adjacent edge in the right order.

This decomposition can be compared with the usual floor decomposition of tropical curves with an h-transverse degree coming from the tropical Caporaso–Harris formula of [5]. However, here the floors have a more complicated shape. For instance, even for degree d curves, and the two edges whose moments vary are directed by $(-1,0)$ and $(0,-1)$, the string may not be a usual floor and can make a loop around the origin as we can see on the left of Figure 4.3. The figure shows a quartic curve, with two ends going to infinity. The movement of these ends is depicted with an arrow. The region coloured in grey is the zone through which the curve travels

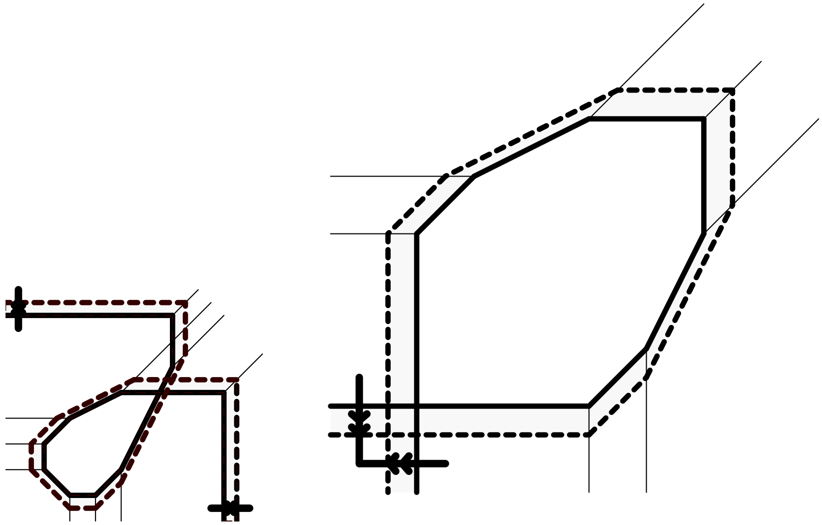

Figure 4.3. On the left the image of a quartic curve during the deformation, and on the right the image of a cubic.

through the deformation. We have a similar situation for a cubic depicted on the right.

5. Computations

We now provide a few values $N_\Delta^{\partial,\mathrm{trop}}$ for various families $\Delta \subset N$, computed using the recursive formula. Because of the exponential complexity of the algorithm, we only manage to make computations for small degrees. Concerning curves of degree d in $\mathbb{C}P^2$, computations can be done by hand up to degree 5 or 6. Degree 7 seems to be out of reach without computer assistance.

Let $d \geqslant 1$ be an integer, and let $\lambda \vdash d$ be a partition of d. We denote by $N_d^{\partial,\mathrm{trop}}(\lambda)$ the polynomial $N_\Delta^{\partial,\mathrm{trop}}$ when $\Delta = \{(-1,0)^d, (1,1)^d, (0,-\lambda_1), (0,-\lambda_2), \ldots\}$. These are the degrees that appear in the proof of the Caporaso–Harris formula in [5]. We have:

- $N_1^{\partial,\mathrm{trop}}(1) = N_2^{\partial,\mathrm{trop}}(1^2) = 1$,
- $N_2^{\partial,\mathrm{trop}}(2) = q^{1/2} + q^{-1/2}$,
- $N_3^{\partial,\mathrm{trop}}(1^3) = q + 7 + q^{-1}$,
- $N_3^{\partial,\mathrm{trop}}(2,1) = q^{3/2} + 6q^{1/2} + 6q^{-1/2} + q^{-3/2}$,

- $N_3^{\partial,\mathrm{trop}}(3) = q^2 + 5q + 6 + 5q^{-1} + q^{-2}$,
- $N_4^{\partial,\mathrm{trop}}(1^4) = q^3 + 10q^2 + 55q + 172 + 55q^{-1} + 10q^{-2} + q^{-3}$,
- $N_4^{\partial,\mathrm{trop}}(2,1^2) = q^{7/2}+9q^{5/2}+45q^{3/2}+133q^{1/2}+133q^{-1/2}+45q^{-3/2}+9q^{-5/2} + q^{-7/2}$,
- $N_4^{\partial,\mathrm{trop}}(3,1) = q^4+8q^3+36q^2+96q+117+96q^{-1}+36q^{-2}+8q^{-3}+q^{-4}$,
- $N_4^{\partial,\mathrm{trop}}(4) = q^{9/2} + 7q^{7/2} + 28q^{5/2} + 68q^{3/2} + 88q^{1/2} + 88q^{-1/2} + 68q^{-3/2} + 28q^{-5/2} + 8q^{-7/2} + q^{-9/2}$,
- $N_4^{\partial,\mathrm{trop}}(2^2) = q^4 + 8q^3 + 36q^2 + 104q + 150 + 104q^{-1} + 36q^{-2} + 8q^{-3} + q^{-4}$,
- $N_5^{\partial,\mathrm{trop}}(1^5) = q^6 + 13q^5 + 91q^4 + 455q^3 + 1695q^2 + 5023q + 11185 + 5023q^{-1} + 1695q^{-2} + 455q^{-3} + 91q^{-4} + 13q^{-5} + q^{-6}$.

The computation is handled with the formula: for each degree Δ one chooses two specific vectors and makes the associated unbounded edges going to infinity, reducing the computation of $N_\Delta^{\partial,\mathrm{trop}}$ to the computations of invariants with families of smaller size. We here show some of the computations for degree d curves, choosing unbounded ends directed by $(-1,0)$ and $(1,1)$. We explain only the main features, and draw the tropical curves resulting from the deformation. The shape of the tropical curves illustrate some of the involved phenomena.

Very low degrees. The values of $N_1^{\partial,\mathrm{trop}}(1), N_2^{\partial,\mathrm{trop}}(1^2)$ and $N_2^{\partial,\mathrm{trop}}(2)$ are easy to find since for each choice of boundary conditions, there is only one tropical curve matching the constraints. The only curve for $N_2(2)$ has a vertex of complex multiplicity 2, leading to the value $N_2^{\partial,\mathrm{trop}}(2) = q^{1/2} + q^{-1/2}$.

For curves of degree 3, the choice of unbounded edges going to infinity leads to tropical curves having a floor decomposition in the sense of [5], and the computation is reduced to the value of $N_1^{\partial,\mathrm{trop}}$, $N_2^{\partial,\mathrm{trop}}(1^2)$ and $N_2^{\partial,\mathrm{trop}}(2)$, which we already know. The appearance of a floor means that the string degenerates into a toric divisor, which is the coordinate axis $y = 0$ of $\mathbb{C}P^2$ in our case. The computation leads to the value $q + 7 + q^{-1}$.

Curves of degree 4 and 5. For curves of degree 4, we still get a contribution of the curves having a floor in the sense of [5]. Their contribution is

$$N_4^{\partial,\mathrm{trop}}(1^4) = q^3 + 10q^2 + 55q + 166 + 55q^{-1} + 10q^{-2} + q^{-3}.$$

However, it would be wrong to assume that all the contributing curves are of this form. There are in fact 6 additional curves having the shape of Figure 5.1 (a), where the string makes a loop around the origin. This means

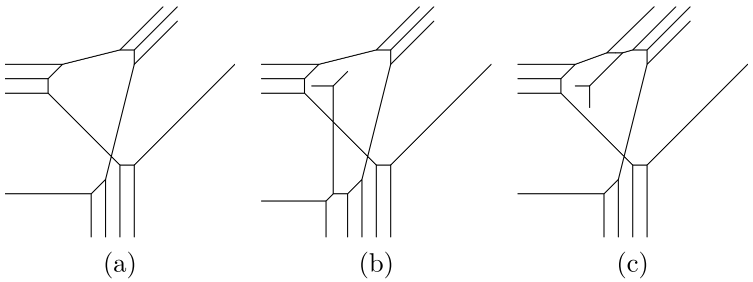

Figure 5.1. On (a) a quartic curve with a loop, and on (b) and (c) examples of quintics with a loop.

that the string degenerates to the union of all toric divisors, rather than going to only one, as it would happen if it was degenerating on a floor. The six curves come from the six possible repartitions of the bottom ends in two groups of two.

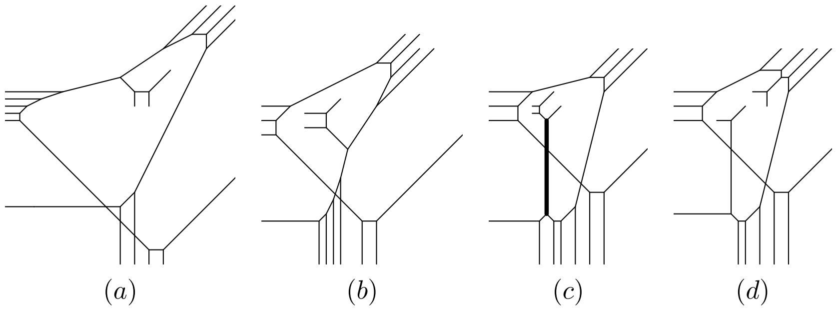

Figure 5.2. Different examples of sextics with a loop : in (a) and (b) the degree of the remaining curve is not a standard triangle, in (c) it is a degree 2 curve, in (d) it is two lines.

For curves of degree 5, the string can yet again degenerate into a floor, or just as in the degree 4, degenerate into a loop. Only this time, contrarily to the degree 4 case, there might still be ends leftover, leading to a line attached to the loop, as we can see on Figure 5.1 (b)(c).

Degree 6 and higher. Up to degree 5, thanks to the particular choice of ends going to infinity, the computations were reduced only to values of the form $N_d^{\partial,\mathrm{trop}}(\lambda)$ for some λ, as in the classical Caporaso–Harris formula. Once again, it would be wrong to assume that these values are sufficient to compute $N_d^{\partial,\mathrm{trop}}(\lambda)$, in the sense that some smart choice of formula reduces the computation of $N_d^{\partial,\mathrm{trop}}(\lambda)$ to the computation of some $N_l^{\partial,\mathrm{trop}}(\mu)$ for $l < d$. Starting from $d = 6$, the curves resulting from the eviction of the string (i.e. the curves Γ_i) may not be of degree l, meaning that the degree of the plane curve is not a standard triangle of size l, as we can see on Figure 5.2: the remaining curves might be of degree 1 or 2 as it is the case in (c) and (d), but can also have a more complicated shape, as we can see on (a) and (b).

For degree 7, the situation becomes even more complicated since the growing number of unbounded edges increases the number of possible degrees for the curves Γ_i, and the string can now make two loops. The number of loops that the string can makes goes higher with the degree.

Finally, the recursive formula involves every $N_\Delta^{\partial,\mathrm{trop}}$, associated with different toric surfaces, contrarily to the usual Caporaso–Harris formula for curves in $\mathbb{C}P^2$, which restricts to specific degrees. Furthermore, the formula applies for curves of any degree, while the Caporaso–Harris formula restricts to h-transverse polygons.

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