Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Robin Zegers & Elie Mounzer

On double quantum affinization: 1. Type $\mathfrak{a}_1$

Article à paraître, mis en ligne le 5 février 2025, 64 p.

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
Article à paraître
Mis en ligne le 5 février 2025.

ON DOUBLE QUANTUM AFFINIZATION: 1. TYPE $\mathfrak{a}_1$

by Robin ZEGERS & Elie MOUNZER

Abstract. — We define the double quantum affinization $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ of type $\mathfrak{a}_1$ as a topological Hopf algebra. We prove that it admits a subalgebra $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$ whose completion is (bicontinuously) isomorphic to the completion of the quantum toroidal algebra $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$, defined as the (simple) quantum affinization of the untwisted affine Kač–Moody Lie algebra $\dot{\mathfrak{sl}}_2$ of type $\dot{\mathfrak{a}}_1$, equipped with a certain topology inherited from its natural $\mathbb{Z}$-grading. The isomorphism is constructed by means of a bicontinuous action by automorphisms of an affinized version $\ddot{\mathfrak{B}}$ – technically a split extension $\ddot{\mathfrak{B}} \cong \dot{\mathfrak{B}} \ltimes P^\vee$ by the coweight lattice $P^\vee$ – of the affine braid group $\dot{\mathfrak{B}}$ of type $\dot{\mathfrak{a}}_1$ on that completion of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$. It can be regarded as an affinized version of the Damiani–Beck isomorphism, familiar from the quantum affine setting. We eventually prove the corresponding triangular decomposition of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ and briefly discuss the consequences regarding the representation theory of quantum toroidal algebras.

Résumé. — Nous définissons la double affinisation quantique $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ de type $\mathfrak{a}_1$ comme une algèbre de Hopf topologique. Nous démontrons qu'elle admet une sous-algèbre $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$ dont la complétion est (bicontinûment) isomorphe à la complétion de l'algèbre quantique toroïdale $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$, elle-même définie comme l'affinisaton quantique (simple) de l'algèbre de Kač–Moody affine non-torsionnée $\dot{\mathfrak{sl}}_2$ de type $\dot{\mathfrak{a}}_1$, munie d'une certaine topologie héritée de sa $\mathbb{Z}$-graduation naturelle. L'isomorphisme est construit au moyen d'une action bicontinue par automorphismes d'une version affinisée $\ddot{\mathfrak{B}}$ – techniquement une extension scindée $\ddot{\mathfrak{B}} \cong \dot{\mathfrak{B}} \ltimes P^\vee$ par le réseau des co-poids $P^\vee$ – du groupe des tresses affine $\dot{\mathfrak{B}}$ de type $\dot{\mathfrak{a}}_1$ sur cette complétion de $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$. Il peut être vu comme une version affinisée de l'isomorphisme de Damiani–Beck, bien connu dans le cadre des algèbres quantiques affines. Nous prouvons finalement la décomposition triangulaire correspondante de $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ et discutons brièvement les conséquences sur la théorie des représentations des algèbres quantiques toroïdales.

1. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra and denote by $\dot{\mathfrak{g}}$ the corresponding untwisted affine Kač–Moody algebra. Starting from $\mathfrak{g}$ and $\dot{\mathfrak{g}}$ or from their

Keywords: Quantum Affine Algebras, Quantum Toroidal Algebras, Representation Theory.
2020 *Mathematics Subject Classification:* 17B37, 17B67.

respective root systems, one can construct two a priori different algebras: on one hand, the quantum affine algebra $\mathrm{U}_q(\dot{\mathfrak{g}})$ is the standard Drinfel'd–Jimbo algebra associated with $\dot{\mathfrak{g}}$; whereas on the other hand, the quantum affinization $\dot{\mathrm{U}}_q(\mathfrak{g})$ of $\mathfrak{g}$, which we define as $\mathrm{U}_q(\dot{\mathfrak{g}})$ in its Drinfel'd current presentation, is associated with the simple finite root system of $\mathfrak{g}$. Now $\dot{\mathrm{U}}_q(\mathfrak{g})$ and $\mathrm{U}_q(\dot{\mathfrak{g}})$ are isomorphic by virtue of a theorem established by Damiani and Beck, [2, 4], which can be regarded as a quantum version of the classic result that each affine Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ is isomorphic to the corresponding untwisted affine Kač–Moody Lie algebra $\dot{\mathfrak{g}}$. The situation can be summarized by the following diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\text{Classical Affinization}} & \dot{\mathfrak{g}} \\ {\scriptstyle \text{Quantum Affinization}}\big\downarrow & & \big\downarrow{\scriptstyle \text{Quantization}} \\ \dot{\mathrm{U}}_q(\mathfrak{g}) & \xrightarrow[\text{Damiani–Beck isom.}]{\sim} & \mathrm{U}_q(\dot{\mathfrak{g}}) \end{array} \tag{1.1}$$

It turns out that quantum affinization still makes sense for the already affine Lie algebra $\dot{\mathfrak{g}}$, thus yielding a doubly affine quantum algebra known as the *quantum toroidal algebra* $\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})$. These originally appeared in type $\mathfrak{a}_n$ in the work of Ginzburg, Kapranov and Vasserot, [12]. Quantum toroidal algebras have received a lot of attention in the past (see [15] for a review) and are presently the subject of a renewed interest due to their relevance for integrable systems (see e.g. [9, 10, 11]) and for 5 dimensional supersymmetric Yang–Mills theory and related AGT like correspondence (see [1]). From a more mathematical perspective, it is well known (see [24]) that they are the Frobenius–Schur duals of Cherednik's Doubly Affine Hecke Algebras (DAHA), (see [3, 18] for classic references on the latter).

The purpose of the present work is to reconsider quantum toroidal algebras as topological Hopf algebras. On the one hand, this is only natural since the existence of an algebraic comultiplication for quantum toroidal algebras is still an essentially open question to this date, (although see [13] for recent results on algebraic comultiplications for affine Yangians that may suggest the existence of similar results for quantum toroidal algebras), and only a topological coalgebra structure is provided by the so-called Drinfel'd current coproduct, (see [8] or [5] and [6] for the cases of $\dot{\mathrm{U}}_q(\mathfrak{a}_{n\geqslant 1})$, as well as [15] and references therein). On the other hand, the existence of a braid group action by bicontinuous algebra automorphisms, generalizing those in [7], provides us with a topological version of the Lusztig symmetries that prove pivotal in both Damiani's and Beck's proofs of Drinfel'd's current presentation. We may therefore expect, in that context, the existence of an alternative presentation for quantum toroidal algebras, in terms of double current generators. In the same spirit as Drinfel'd's current presentation, such a presentation could be regarded as defining a new quantum algebra, namely the *double quantum affinization* $\ddot{\mathrm{U}}_q(\mathfrak{g})$ of $\mathfrak{g}$ (see Section 3), and (a subalgebra $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$ of) $\ddot{\mathrm{U}}_q(\mathfrak{g})$ should be isomorphic to (the completion of) $\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})$ through an affinized version of the Damiani–Beck isomorphism (see Section 4). We therefore expect a diagram of the form

$$
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\text{Classical Affinization}} & \dot{\mathfrak{g}} \\
{\scriptstyle \text{Double Quantum Affinization}}\big\downarrow & & \big\downarrow{\scriptstyle \text{Quantum Affinization}} \\
\ddot{\mathrm{U}}'_q(\mathfrak{g}) & & \dot{\mathrm{U}}_q(\dot{\mathfrak{g}}) \\
{\scriptstyle \text{Completion}}\big\downarrow & & \big\downarrow{\scriptstyle \text{Completion}} \\
\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{g})} & \xrightarrow[\text{Affine Damiani–Beck isom.}]{\sim} & \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{g}})}
\end{array}
\tag{1.2}
$$

in which the last line, in the same way as that of diagram (1.1) in the previous paragraph, should be thought of as a quantum version of the classical isomorphism

$$\ddot{\mathfrak{g}} = \dot{\mathfrak{g}} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}C \cong \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}, z, z^{-1}] \oplus \mathbb{C}[z, z^{-1}]c \oplus \mathbb{C}C\,.$$

In the present paper we prove such results in the particular case where $\mathfrak{g}$ is of type $\mathfrak{a}_1$. It is fairly natural to conjecture that similar results hold for higher rank root systems, thus yielding

Conjecture 1.1. — *Every simple Lie algebra $\mathfrak{g}$ admits a (unique up to isomorphisms) double quantum affinization $\ddot{\mathrm{U}}_q(\mathfrak{g})$ and diagram 1.2 holds.*

and

Conjecture 1.2. — *Every untwisted affine Kač–Moody Lie algebra $\dot{\mathfrak{g}}$ admits a (unique up to isomorphisms) double quantum affinization $\ddot{\mathrm{U}}_q(\dot{\mathfrak{g}})$.*

Note that the latter would naturally provide a definition for the so far elusive triply affine quantum algebras. The latter are believed to play an important role in mathematical physics, as the conformal block side of an AGT type correspondence with 6-dimensional super Yang–Mills theories, [1].

In any case, $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ (and presumably other double quantum affinizations if any) admits a triangular decomposition $(\ddot{\mathrm{U}}^-_q(\mathfrak{a}_1), \ddot{\mathrm{U}}^0_q(\mathfrak{a}_1), \ddot{\mathrm{U}}^+_q(\mathfrak{a}_1))$. The latter naturally leads to an alternative notion of weight and highest weight modules that we shall refer to as t-weight and highest t-weight modules.

Natural analogues of the finite dimensional modules over quantum affine algebras also appear, that we refer to as *weight-finite* modules, see Section 3 for definitions. We actually expect that it will be possible to classify simple weight-finite modules over $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$, by essentially classifying those simple $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-modules that appear as their highest t-weight spaces, see Section 3 for the corresponding discussion. This is the subject of ongoing work.

Quite remarkably, there exists an algebra homomorphism $f : \mathcal{E}_{q^{-4},q^2,q^2} \to \ddot{\mathrm{U}}_q^{0^+}(\mathfrak{a}_1)$, where $\ddot{\mathrm{U}}_q^{0^+}(\mathfrak{a}_1)$ is a closed subalgebra of $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$ and, for every q_1, q_2, q_3 such that $q_1q_2q_3 = 1$, $\mathcal{E}_{q_1,q_2,q_3}$ is the corresponding elliptic Hall algebra, see Section 3. The latter was first defined by Miki in [20] as a (q,γ)-analogue of the $W_{1+\infty}$ algebra. It reappeared later in [9], as the quantum continuous $\mathfrak{gl}_\infty$ algebra. Schiffmann then identified it with the Hall algebra of the category of coherent sheaves on some elliptic curve whose Weil numbers are related to q_1, q_2, q_3, [23]. More recently, it also appeared in [10] and in subsequent works by Feigin et al. as the quantum toroidal algebra associated with $\mathfrak{gl}_1$. As we shall see, it appears natural to make the following

Conjecture 1.3. — *$\ddot{\mathrm{U}}_q^{0^+}(\mathfrak{a}_1)$ is isomorphic to the completion of $\mathcal{E}_{q^{-4},q^2,q^2}$.*

If it held true, the above conjecture would have many interesting implications. On one hand, in view of Schiffmann's results, it seems reasonable to expect that the double quantum affinization $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ admits a K-theoretic realization, in the spirit of Nakajima's quiver varieties realization of quantum affine algebras [21], wherein the generators outside of the elliptic Hall algebras would be realized as correspondences. At the level of representation theory on the other hand, Conjecture 1.3 would imply that the classification of the simple $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-modules that appear as highest t-weight spaces of simple weight-finite $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules would almost entirely reduce to a classification of the corresponding subclass of simple modules over the elliptic Hall algebra. Again, we leave these questions for future work (see [25]).

The paper is organized as follows. In Section 2, we briefly review some well known facts about quantum toroidal algebras, including their definition and natural gradings. We endow them with a topology and construct the corresponding completion. On the latter, we construct a set of automorphisms, including affinized versions of Lusztig's symmetry. Analogues of these for simply laced untwisted affine $\dot{\mathfrak{a}}_{n\geqslant 2}$-types appeared in the work of Ding and Khoroshkin [7]. The $\dot{\mathfrak{a}}_1$ version we give here plays a crucial role in Section 4 where we prove the main result of this paper. In Section 3, we define a new quantum algebra called the *double quantum affinization of type* $\mathfrak{a}_1$ and denoted $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$. We prove that there exists an algebra homomorphism from the elliptic Hall algebra $\mathcal{E}_{q_1,q_2,q_3}$ to its subalgebra $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$. We also elaborate on the consequences at the level of representation theory and introduce the notions of (highest) t-weights and of weight-finiteness. Finally, in Section 4, we construct the affinized version of the Damiani–Beck isomorphism

$$\widehat{\Psi} : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \xrightarrow{\sim} \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)} .$$

The appendix contains a short review of formal distributions as relevant to the present work. This is already covered in the literature (see e.g. [17]), however, since our conventions slightly differ from the standard ones, we included it for the sake of clarity.

Notations and conventions

We let $\mathbb{N} = \{0, 1, \dots\}$ be the set of natural integers including 0. We denote by $\mathbb{N}^\times$ the set $\mathbb{N}-\{0\}$. For every $m \leqslant n \in \mathbb{N}$, we denote by $[\![m, n]\!] = \{m, m+1, \dots, n\}$. We also let $[\![n]\!] = [\![1, n]\!]$ for every $n \in \mathbb{N}$. For every finite subset $\Sigma \subset \mathbb{N}$ with $\operatorname{card}\Sigma = N$, any $n \leqslant N$ and any $m_1, \dots, m_n \in \mathbb{N}$ such that $m_1 + \dots + m_n = N$, we let $\mathsf{P}_\Sigma^{(m_1,\dots,m_n)}$ denote the set of ordered $(m_1, \dots, m_n)$ set n-partitions, i.e. any $\mathbf{A} = (A^{(1)}, \dots, A^{(n)}) \in \mathsf{P}_\Sigma^{(m_1,\dots,m_n)}$ is such that

(i) for every $p \in [\![n]\!]$, $\operatorname{card} A^{(p)} = m_p$;
(ii) for every $p \in [\![n]\!]$, $A^{(p)} = \{A_1^{(p)}, \dots, A_{m_p}^{(p)}\} \subset \Sigma$, with $A_1^{(p)} < \dots < A_{m_p}^{(p)}$;
(iii) $A^{(1)} \sqcup \dots \sqcup A^{(n)} = \Sigma$.

We let $\operatorname{sign} : \mathbb{Z} \to \{-1, 0, 1\}$ be defined by setting, for any $n \in \mathbb{Z}$,

$$\operatorname{sign}(n) = \begin{cases} -1 & \text{if } n < 0, \\ 0 & \text{if } n = 0, \\ 1 & \text{if } n > 0. \end{cases}$$

We assume throughout that $\mathbb{K}$ is a field of characteristic 0 and we let $\mathbb{F} = \mathbb{K}(q)$ denote the field of rational functions over $\mathbb{K}$ in the formal variable q. As usual, we let $\mathbb{K}^\times = \mathbb{K}-\{0\}$ and $\mathbb{F}^\times = \mathbb{F}-\{0\}$. We make $\mathbb{F}$ a topological field by endowing it with the discrete topology.

For every $m, n \in \mathbb{N}$, we define the following elements of $\mathbb{F}$

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}\,, \tag{1.3}$$

$$[n]_q^! = \begin{cases} [n]_q[n-1]_q \cdots [1]_q & \text{if } n \in \mathbb{N}^\times, \\ 1 & \text{if } n = 0, \end{cases} \tag{1.4}$$

$$\binom{n}{m}_q = \frac{[n]_q^!}{[m]_q^! [n-m]_q^!}\,. \tag{1.5}$$

We shall let

$$_a[A, B]_b = aAB - bBA\,,$$

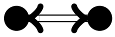
for any symbols a, b, A and B provided the r.h.s of the above equations makes sense. At some point we may need the following obvious identities

$$[[A, B]_a, C]_b = [[A, C]_b, B]_a + [A, [B, C]]_{ab}\,, \tag{1.6}$$

$$[_a[A, B], C]_b = {}_a[[A, C]_b, B] + {}_a[A, [B, C]]_b\,. \tag{1.7}$$

We refer to the Appendix for conventions and more details on formal distributions.

The Dynkin diagrams and correponding Cartan matrices in type $\mathfrak{a}_1$ and $\dot{\mathfrak{a}}_1$ are reminded in the following table.

Type	Dynkin diagram	Simple roots	Cartan matrix
$\mathfrak{a}_1$	1 	$\Phi = \{\alpha_1\}$	(2)
$\dot{\mathfrak{a}}_1$	0 1 	$\dot{\Phi} = \{\alpha_0, \alpha_1\}$	$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

2. The quantum toroidal algebra of type $\mathfrak{a}_1$ and its completion

2.1. Definition

Let $\dot{I} = \{0, 1\}$ be the above labeling of the nodes of the Dynkin diagram of type $\dot{\mathfrak{a}}_1$ and let $\dot{\Phi} = \{\alpha_0, \alpha_1\}$ be a choice of simple roots for the corresponding root system. We denote by $(c_{ij})_{i,j=0,1}$ the entries of the associated Cartan matrix. Let $\dot{Q}^\pm = \mathbb{Z}^\pm\alpha_0 \oplus \mathbb{Z}^\pm\alpha_1$ and let $\dot{Q} = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1$ be the type $\dot{\mathfrak{a}}_1$ root lattice.

DEFINITION 2.1. — *The* quantum toroidal algebra $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ *is the associative* $\mathbb{F}$*-algebra generated by the generators*

$$\left\{D, D^{-1}, C^{1/2}, C^{-1/2}, k^+_{i,n}, k^-_{i,-n}, x^+_{i,m}, x^-_{i,m} : i \in \dot{I}, m \in \mathbb{Z}, n \in \mathbb{N}\right\}$$

subject to the following relations

$$C^{\pm 1/2} \text{ is central,} \qquad C^{\pm 1/2}C^{\mp 1/2} = 1\,, \qquad D^{\pm 1}D^{\mp 1} = 1\,, \tag{2.1}$$

$$D\mathbf{k}^\pm_i(z)D^{-1} = \mathbf{k}^\pm_i(zq^{-1})\,, \qquad D\mathbf{x}^\pm_i(z)D^{-1} = \mathbf{x}^\pm_i(zq^{-1})\,, \tag{2.2}$$

$$\operatorname*{res}_{z_1,z_2} \frac{1}{z_1 z_2}\mathbf{k}^\pm_i(z_1)\mathbf{k}^\mp_i(z_2) = 1\,, \tag{2.3}$$

$$\mathbf{k}^\pm_i(z_1)\mathbf{k}^\pm_j(z_2) = \mathbf{k}^\pm_j(z_2)\mathbf{k}^\pm_i(z_1)\,, \tag{2.4}$$

$$\mathbf{k}^-_i(z_1)\mathbf{k}^+_j(z_2) = G^-_{ij}(C^{-1}z_1/z_2)G^+_{ij}(Cz_1/z_2)\mathbf{k}^+_j(z_2)\mathbf{k}^-_i(z_1)\,, \tag{2.5}$$

$$G^\mp_{ij}(C^{\mp 1/2}z_2/z_1)\mathbf{k}^+_i(z_1)\mathbf{x}^\pm_j(z_2) = \mathbf{x}^\pm_j(z_2)\mathbf{k}^+_i(z_1)\,, \tag{2.6}$$

$$\mathbf{k}^-_i(z_1)\mathbf{x}^\pm_j(z_2) = G^\mp_{ij}(C^{\mp 1/2}z_1/z_2)\mathbf{x}^\pm_j(z_2)\mathbf{k}^-_i(z_1)\,, \tag{2.7}$$

$$(z_1 - q^{\pm c_{ij}}z_2)\mathbf{x}^\pm_i(z_1)\mathbf{x}^\pm_j(z_2) = (z_1 q^{\pm c_{ij}} - z_2)\mathbf{x}^\pm_j(z_2)\mathbf{x}^\pm_i(z_1)\,, \tag{2.8}$$

$$\begin{aligned}&[\mathbf{x}^+_i(z_1), \mathbf{x}^-_j(z_2)]\\ &\quad= \frac{\delta_{ij}}{q-q^{-1}}\left[\delta\left(\frac{z_1}{Cz_2}\right)\mathbf{k}^+_i(z_1C^{-1/2}) - \delta\left(\frac{z_1 C}{z_2}\right)\mathbf{k}^-_i(z_2C^{-1/2})\right]\,,\end{aligned} \tag{2.9}$$

$$\begin{aligned}\sum_{\sigma\in S_{1-c_{ij}}}\sum_{k=0}^{1-c_{ij}}(-1)^k\binom{1-c_{ij}}{k}_q\mathbf{x}^\pm_i(z_{\sigma(1)})\cdots\mathbf{x}^\pm_i(z_{\sigma(k)})\mathbf{x}^\pm_j(z)&\\ \times\,\mathbf{x}^\pm_i(z_{\sigma(k+1)})\cdots\mathbf{x}^\pm_i(z_{\sigma(1-c_{ij})}) = 0\,,&\end{aligned} \tag{2.10}$$

where, for every $i \in \dot{I}$*, we define the following* $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$*-valued formal distributions*

$$\mathbf{x}^\pm_i(z) = \sum_{m\in\mathbb{Z}} x^\pm_{i,m}z^{-m} \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)[[z, z^{-1}]]\,, \tag{2.11}$$

$$\mathbf{k}^\pm_i(z) = \sum_{n\in\mathbb{N}} k^\pm_{i,\pm n}z^{\mp n} \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)[[z^{\mp 1}]]\,, \tag{2.12}$$

for every $i,j \in \dot{I}$, we define the following $\mathbb{F}$-valued formal power series

$$G^{\pm}_{ij}(z) = q^{\pm c_{ij}} + (q-q^{-1})[\pm c_{ij}]_q \sum_{m\in\mathbb{N}^\times} q^{\pm m c_{ij}} z^m \in \mathbb{F}[[z]] \tag{2.13}$$

and

$$\delta(z) = \sum_{m\in\mathbb{Z}} z^m \in \mathbb{F}[[z,z^{-1}]] \tag{2.14}$$

is an $\mathbb{F}$-valued formal distribution.

Note that $G^{\pm}_{ij}(z)$ is invertible in $\mathbb{F}[[z]]$ with inverse $G^{\mp}_{ij}(z)$, i.e.

$$G^{\pm}_{ij}(z)G^{\mp}_{ij}(z) = 1\,, \tag{2.15}$$

and that it can be viewed as the power series expansion of a rational function of $(z_1,z_2)\in\mathbb{C}^2$ as $|z_2| \gg |z_1|$, which we shall denote as follows

$$G^{\pm}_{ij}(z_1/z_2) = \left(\frac{z_1 q^{\mp c_{ij}} - z_2}{z_1 - q^{\mp c_{ij}} z_2}\right)_{|z_2|\gg|z_1|}. \tag{2.16}$$

Observe furthermore that we have the following useful identity in $\mathbb{F}[[z,z^{-1}]]$

$$\frac{G^{\pm}_{ij}(z) - G^{\mp}_{ij}(z^{-1})}{q-q^{-1}} = [\pm c_{ij}]_q \delta\left(z q^{\pm c_{ij}}\right)\,. \tag{2.17}$$

Remark 2.2. — In type $\mathfrak{a}_1$, $\dot{I} = \{0,1\}$, $c_{ij} = 4\delta_{ij} - 2$ and we have an additional identity, namely $G^{\pm}_{10}(z) = G^{\mp}_{11}(z)$. We refer to Section A.3 of the Appendix for more identities involving the formal power series $G^{\pm}_{ij}(z)$.

$\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ is obviously a $\mathbb{Z}$-graded algebra, i.e. we have

$$\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) = \bigoplus_{n\in\mathbb{Z}} \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)_n\,, \tag{2.18}$$

where, for all $n\in\mathbb{Z}$, $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)_n = \{x \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) : DxD^{-1} = q^n x\}$.

It was proven in [14] to admit a triangular decomposition

$$(\dot{\mathrm{U}}^-_q(\dot{\mathfrak{a}}_1), \dot{\mathrm{U}}^0_q(\dot{\mathfrak{a}}_1), \dot{\mathrm{U}}^+_q(\dot{\mathfrak{a}}_1))\,,$$

where $\dot{\mathrm{U}}^{\pm}_q(\dot{\mathfrak{a}}_1)$ and $\dot{\mathrm{U}}^0_q(\dot{\mathfrak{a}}_1)$ are the subalgebras of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ respectively generated by $\left\{x^{\pm}_{i,m} : i\in\dot{I}, m\in\mathbb{Z}\right\}$ and

$$\left\{C^{1/2}, C^{-1/2}, D, D^{-1}, k^+_{i,m}, k^-_{i,m} : i\in\dot{I}, m\in\mathbb{Z}\right\}\,.$$

Observe that $\dot{\mathrm{U}}_q^{\pm}(\dot{\mathfrak{a}}_1)$ admits a natural gradation over $\dot{Q}^{\pm}$ that we shall denote by

$$\dot{\mathrm{U}}_q^{\pm}(\dot{\mathfrak{a}}_1) = \bigoplus_{\alpha \in \dot{Q}^{\pm}} \dot{\mathrm{U}}_q^{\pm}(\dot{\mathfrak{a}}_1)_{\alpha} \,. \tag{2.19}$$

Of course $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ is graded over the root lattice $\dot{Q}$. We finally remark that the two Dynkin diagram subalgebras $\dot{\mathrm{U}}_q(\mathfrak{a}_1)^{(0)}$ and $\dot{\mathrm{U}}_q(\mathfrak{a}_1)^{(1)}$ of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ generated by

$$\left\{D, D^{-1}, C^{1/2}, C^{-1/2}, k^+_{i,n}, k^-_{i,-n}, x^+_{i,m}, x^-_{i,m} : m \in \mathbb{Z}, n \in \mathbb{N}\right\} ,$$

with $i = 0$ and $i = 1$ respectively, are both isomorphic to $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ and that the corresponding inclusion maps thus yield two injective algebra homomorphisms $\iota^{(i)} : \dot{\mathrm{U}}_q(\mathfrak{a}_1) \cong \dot{\mathrm{U}}_q(\mathfrak{a}_1)^{(i)} \hookrightarrow \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$.

2.2. Automorphisms of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$

Proposition 2.3.

(i) *For every Dynkin diagram automorphism $\pi : \dot{I} \xrightarrow{\sim} \dot{I}$, there exists a unique $\mathbb{F}$-algebra automorphism $T_\pi \in \mathrm{Aut}(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1))$ such that*

$$\begin{aligned} &T_\pi(C^{1/2}) = C^{1/2} \,, && T_\pi(D) = D \,, \\ &T_\pi(\mathbf{x}_i^{\pm}(z)) = \mathbf{x}^{\pm}_{\pi(i)}(z) \,, && T_\pi(\mathbf{k}_i^{\pm}(z)) = \mathbf{k}^{\pm}_{\pi(i)}(z) \,. \end{aligned} \tag{2.20}$$

(ii) *For every $i \in \dot{I}$, there exists a unique $\mathbb{F}$-algebra automorphism $T_{\omega_i^\vee} \in \mathrm{Aut}(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1))$ such that*

$$\begin{aligned} &T_{\omega_i^\vee}(C^{1/2}) = C^{1/2} \,, && T_{\omega_i^\vee}(D) = D \,, \\ &T_{\omega_i^\vee}(\mathbf{x}_j^{\pm}(z)) = z^{\pm\delta_{ij}}\mathbf{x}_j^{\pm}(z) \,, && T_{\omega_i^\vee}(\mathbf{k}_j^{\pm}(z)) = C^{\mp\delta_{ij}}\mathbf{k}_j^{\pm}(z) \,. \end{aligned} \tag{2.21}$$

(iii) *There exists a unique involutive $\mathbb{F}$-algebra anti-homomorphism $\eta \in \mathrm{Aut}(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1))$ such that*

$$\begin{aligned} &\eta(C^{1/2}) = C^{1/2} \,, && \eta(D) = D \,, \\ &\eta(\mathbf{x}_i^{\pm}(z)) = \mathbf{x}_i^{\pm}(1/z) \,, && \eta(\mathbf{k}_i^{\pm}(z)) = \mathbf{k}_i^{\mp}(1/z) \,. \end{aligned} \tag{2.22}$$

(iv) *There exists a unique involutive $\mathbb{K}$-algebra anti-homomorphism φ such that $\varphi(q) = q^{-1}$ and*

$$\begin{aligned} &\varphi(C^{1/2}) = C^{-1/2} \,, && \varphi(D) = D^{-1} \,, \\ &\varphi(\mathbf{x}_i^{\pm}(z)) = \mathbf{x}_i^{\mp}(1/z) \,, && \varphi(\mathbf{k}_i^{\pm}(z)) = \mathbf{k}_i^{\mp}(1/z) \,. \end{aligned} \tag{2.23}$$

Proof. — This is easily checked to be compatible with the defining relations (2.1)–(2.10) of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$. □

Remark 2.4. — In the present case, the Dynkin diagram being that of type $\dot{\mathfrak{a}}_1$, $\dot{I} = \{0, 1\}$ and the only nontrivial diagram automorphism is defined by setting $\pi(0) = 1$ and $\pi(1) = 0$.

Remark 2.5. — Note that φ restricts as a non-trivial automorphism of the field $\mathbb{F}$ and that, as such, it yields e.g.

$$\varphi(G^{\pm}_{ij}(z)) = G^{\mp}_{ij}(z)\,. \tag{2.24}$$

2.3. The completions $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ and $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\hat{\otimes} m \geqslant 2}$

Let, for every $n \in \mathbb{N}$,

$$\Omega_n = \bigoplus_{\substack{r \geqslant n \\ s \geqslant n}} \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) \cdot \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)_{-r} \cdot \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) \cdot \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)_s \cdot \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)\,.$$

Proposition 2.6. — *The following hold true:*

(i) *For every $n \in \mathbb{N}$, Ω_n is a two-sided ideal of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$;*
(ii) *For every $n \in \mathbb{N}$, $\Omega_n \supseteq \Omega_{n+1}$;*
(iii) $\Omega_0 = \bigcup_{n\in\mathbb{N}} \Omega_n = \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$;
(iv) $\bigcap_{n\in\mathbb{N}} \Omega_n = \{0\}$;
(v) *For every $m, n \in \mathbb{N}$, $\Omega_m + \Omega_n \subseteq \Omega_{\min(m,n)}$;*
(vi) *For every $m, n \in \mathbb{N}$, $\Omega_m \cdot \Omega_n \subseteq \Omega_{\max(m,n)}$.*

Proof. — Points (i) and (ii) are obvious. As sets, it is clear that $\Omega_0 \subseteq \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$. Now, $1 \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)_0$ and for every $x \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$, we can write $x = 1 \cdot x \cdot 1$ thus proving that $x \in \Omega_0$. Point (iii)follows. Point (v) is an easy consequence of point (ii). Point (vi) is obvious given (i). So let us finally prove point (iv). In order to do so, it suffices to prove that for every $x \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) - \{0\}$, there exists a largest integer $\nu_x \in \mathbb{N}$ such that $x \in \Omega_{\nu_x}$; for then indeed $x \notin \Omega_{\nu_x+1}$, whereas obviously $0 \in \Omega_n$, for every $n \in \mathbb{N}$. Relations (2.5)–(2.9) respectively imply that, for every $i, j \in \dot{I}$, every $m \in \mathbb{N}$

and every $n \in \mathbb{N}^\times$,

$$k^+_{i,m} k^-_{j,-n} = k^-_{j,-n} k^+_{i,m} - (q^{c_{ij}} - q^{-c_{ij}})(C - C^{-1}) \times \sum_{p=1}^{\min(m,n)} \frac{q^{-pc_{ij}} C^p - q^{pc_{ij}} C^{-p}}{q^{-c_{ij}} C - q^{c_{ij}} C^{-1}} k^-_{j,-n+p} k^+_{i,m-p} \,,$$

$$k^+_{i,m} x^\pm_{j,-n} = q^{\pm c_{ij}} x^\pm_{j,-n} k^+_{i,m} + (q^{\pm c_{ij}} - q^{\mp c_{ij}}) \sum_{p=1}^{m} C^{\mp p/2} q^{\pm p c_{ij}} x^\pm_{j,-n+p} k^+_{i,m-p} \,,$$

$$x^\pm_{i,m} k^-_{j,-n} = q^{\pm c_{ij}} k^-_{j,-n} x^\pm_{i,m} + (q^{\pm c_{ij}} - q^{\mp c_{ij}}) \sum_{p=1}^{n} C^{\mp p/2} q^{\pm p c_{ij}} k^-_{j,-n+p} x^\pm_{i,m-p} \,,$$

$$\begin{aligned} x^\pm_{i,m} x^\pm_{j,-n} = q^{\pm c_{ij}} x^\pm_{j,-n} x^\pm_{i,m} + (q^{\pm c_{ij}} - q^{\mp c_{ij}}) & \sum_{p=1}^{\min(m,n)-1} q^{\pm p c_{ij}} x^\pm_{j,-n+p} x^\pm_{i,m-p} \\ & - q^{\pm(\min(m,n)-1)c_{ij}} x^\pm_{j,\min(m,n)-n} x^\pm_{i,m-\min(m,n)} \\ & + q^{\pm \min(m,n) c_{ij}} x^\pm_{i,m-\min(m,n)} x^\pm_{j,\min(m,n)-n} \,, \end{aligned}$$

$$x^\pm_{i,m} x^\mp_{j,-n} = x^\mp_{j,-n} x^\pm_{i,m} \pm \frac{\delta_{ij}}{q - q^{-1}} \begin{cases} C^{\pm \frac{m+n}{2}} k^+_{i,m-n} & \text{if } m > n; \\ -C^{\mp \frac{m+n}{2}} k^-_{i,n-m} & \text{if } m < n; \\ \left[C^{\pm m} k^+_{i,0} - C^{\mp m} k^-_{i,0}\right] & \text{if } m = n. \end{cases}$$

Now let

$$B = \left\{ b_{\mathbf{a},\mathbf{m}} = \overrightarrow{\prod_{p \in [\![n]\!]}} \xi_{a_p, m_p} : n \in \mathbb{N}, \quad \begin{array}{c} \mathbf{a} = (a_1, \dots, a_n) \in (\dot{\Phi} \sqcup -\dot{\Phi} \sqcup \dot{I})^n, \\ \mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n \end{array} \right\},$$

where, for every $(a, m) \in (\dot{\Phi} \sqcup -\dot{\Phi} \sqcup \dot{I}) \times \mathbb{Z}$,

$$\xi_{a,m} = \begin{cases} x^\pm_{i,m} & \text{if } a = \pm\alpha_i \in \pm\dot{\Phi},\ i \in \dot{I}, \\ k^\pm_{i,m} & \text{if } a = i \in \dot{I} \text{ and } m \in \mathbb{Z}^\pm. \end{cases}$$

If we omit $C^{\pm 1/2}$ and $D^{\pm 1}$ which are clearly irrelevant for the present discussion, B is obviously a spanning set for $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$. Making repeated use of the above relations, one then easily shows that, for every $n \in \mathbb{N}$, every $\mathbf{a} \in (\dot{\Phi} \sqcup -\dot{\Phi} \sqcup \dot{I})^n$ and every $\mathbf{m} \in \mathbb{Z}^n$,

$$b_{\mathbf{a},\mathbf{m}} - c_{\mathbf{a},\mathbf{m}} \overrightarrow{\prod_{\substack{p \in [\![n]\!] \\ m_p < 0}}} \xi_{a_p, m_p} \overrightarrow{\prod_{\substack{p \in [\![n]\!] \\ m_p \geqslant 0}}} \xi_{a_p, m_p} \in \Omega_{N(\mathbf{m})-1} - \Omega_{N(\mathbf{m})} \,,$$

where $c_{\mathbf{a},\mathbf{m}} \in \mathbb{F}^{\times}$ and

$$N(\mathbf{m}) = \min\left(-\sum_{\substack{p\in[\![n]\!]\\ m_p<0}} m_p, \sum_{\substack{p\in[\![n]\!]\\ m_p\geqslant 0}} m_p\right).$$

As a consequence, $\nu_{b_{\mathbf{a},\mathbf{m}}} \leqslant N(\mathbf{m})$, which concludes the proof. □

Similarly, making use of the natural $\mathbb{Z}$-grading of the tensor algebras $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\otimes m}$, $m \in \mathbb{N}^{\times}$, we let, for every $n \in \mathbb{N}$,

$$\Omega_n^{(m)} = \bigoplus_{\substack{r\geqslant n\\ s\geqslant n}} \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\otimes m} \cdot \left(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\otimes m}\right)_{-r} \cdot \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\otimes m} \cdot \left(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\otimes m}\right)_{s} \cdot \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\otimes m}.$$

One easily checks that for every $m \in \mathbb{N}^{\times}$, $\{\Omega_n^{(m)} : n \in \mathbb{N}\}$ has the same properties as the ones established in Proposition 2.6 for $\{\Omega_n = \Omega_n^{(1)} : n \in \mathbb{N}\}$.

Definition-Proposition 2.7. — *We endow $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ with the topology τ whose open sets are either $\emptyset$ or nonempty subsets $\mathcal{O} \subseteq \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ such that for every $x \in \mathcal{O}$, $x + \Omega_n \subseteq \mathcal{O}$ for some $n \in \mathbb{N}$. These turn $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ into a (separated) topological algebra. We then let $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ denote its completion and we extend by continuity to $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ all the (anti)-automorphisms defined over $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ in the previous section. Similarly, we endow each tensor power $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\otimes m\geqslant 2}$ with the topology induced by $\{\Omega_n^{(m)} : n \in \mathbb{N}\}$ and denote by $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\hat{\otimes} m\geqslant 2}$ the corresponding completion. Note that the latter admits a topology induced by $\left\{\widehat{\Omega}_n^{(m)} : n \in \mathbb{N}\right\}$, where, for every $n \in \mathbb{N}$, $\widehat{\Omega}_n^{(m)}$ denotes the closure of $\Omega_n^{(m)}$ in $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\hat{\otimes} m\geqslant 2}$, and such that $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\otimes m\geqslant 2}$ injects densely in $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)^{\hat{\otimes} m\geqslant 2}$ with an induced topology equivalent to its above defined original topology.*

Proof. — The addition is automatically continuous in the above defined topology of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$. The continuity of the multiplication follows from point (vi) of Proposition 2.6. Point (vi), in turn, implies that $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$, as a topological space, is Hausdorff. The continuity of the unit map $\eta : \mathbb{F} \to \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ is easily checked – remember that $\mathbb{F}$ is given the discrete topology. □

Remark 2.8. — It is worth noting that the above topology is actually ultrametrizable. In the notations of the previous proof, let indeed, for every $x \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$,

$$\|x\| = \begin{cases} \exp(-\nu_x) & \text{if } x \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) - \{0\}, \\ 0 & \text{if } x = 0. \end{cases}$$

Since obviously $\nu_{x+y} \geqslant \min(\nu_x, \nu_y)$ for every $x, y \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$, the ultrametric inequality for the metric defined by $d(x,y) = \|x - y\|$ follows immediately as a consequence of the inequality $\|x + y\| \leqslant \max(\|x\|, \|y\|)$.

2.4. Continuous Lusztig automorphisms

Following [18] we make the following

Definition 2.9. — *The affine braid group $\dot{\mathfrak{B}}$ of type $\dot{\mathfrak{a}}_1$ is generated by t and y subject to the relation $ty^{-1}t = y$.*

The coweight lattice $P^\vee$ of $\dot{\mathfrak{a}}_1$ is an abelian group whose generators we shall denote as x_λ for every $\lambda \in P^\vee$. In particular, we shall write

$$x_\lambda x_\mu = x_\mu x_\lambda = x_{\lambda+\mu}\,, \tag{2.25}$$

assuming that $x_0 = 1$. There exists a unique group homomorphism $\dot{\mathfrak{B}} \to \mathrm{Aut}(P^\vee)$ defined by letting

$$t(x_\lambda) = x_{s_{\alpha_1}(\lambda)}\,, \qquad y(x_\lambda) = x_\lambda\,, \tag{2.26}$$

where s_{α_1} denotes the reflection in the simple root α_1, i.e. $s_{\alpha_1}(\lambda) = \lambda - (\alpha_1^\vee, \lambda)\alpha_1$. This action allows us to make the following

Definition 2.10. — *We let $\ddot{\mathfrak{B}} = \dot{\mathfrak{B}} \ltimes P^\vee$, i.e. $\ddot{\mathfrak{B}}$ is isomorphic to the group with generators t, y and $(x_\lambda)_{\lambda\in P^\vee}$ obeying the relations*

$$ty^{-1}t = y\,, \qquad tx_\lambda t^{-1} = x_{s_{\alpha_1}(\lambda)}\,, \qquad x_\lambda y = yx_\lambda\,, \tag{2.27}$$

for every $\lambda \in P^\vee$.

We now define an action of $\ddot{\mathfrak{B}}$ on $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ by bicontinuous algebra automorphisms, i.e. we construct a group homomorphism $\ddot{\mathfrak{B}} \to \mathrm{Aut}(\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)})$. In order to do so, we first describe the image of the latter, following [7].

Proposition 2.11. — *There exists a unique bicontinuous algebra automorphism $T \in \mathrm{Aut}(\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)})$ such that*

$$T(C^{1/2}) = C^{1/2}\,, \qquad T(D) = D\,, \tag{2.28}$$

$$T(\mathbf{k}_0^\pm(z)) = \mathbf{k}_0^\pm(zq^2)\mathbf{k}_1^\pm(z)\mathbf{k}_1^\pm(zq^2)\,, \qquad T(\mathbf{k}_1^\pm(z)) = \mathbf{k}_1^\pm(z)^{-1}\,, \tag{2.29}$$

$$T(\mathbf{x}_0^+(z)) = \frac{1}{[2]_q} \operatorname*{res}_{z_1,z_2} z_1^{-1} z_2^{-1} \Big[\mathbf{x}_1^+(z_1), \big[\mathbf{x}_1^+(z_2), \mathbf{x}_0^+(zq^2)\big]_{G_{10}^-(z_2/zq^2)}\Big]_{G_{11}^-(z_1/z_2)G_{10}^-(z_1/zq^2)}\,, \tag{2.30}$$

$$T(\mathbf{x}_0^-(z)) = \frac{1}{[2]_q} \operatorname*{res}_{z_1,z_2} z_1^{-1}z_2^{-1} \left[\left[\mathbf{x}_0^-(zq^2), \mathbf{x}_1^-(z_1)\right]_{G_{10}^+(zq^2/z_1)}, \mathbf{x}_1^-(z_2)\right]_{G_{11}^+(z_1/z_2)G_{10}^+(zq^2/z_2)}, \tag{2.31}$$

$$T(\mathbf{x}_1^+(z)) = -\mathbf{x}_1^-(C^{-1}z)\mathbf{k}_1^+(C^{-1/2}z)^{-1}, \tag{2.32}$$

$$T(\mathbf{x}_1^-(z)) = -\mathbf{k}_1^-(C^{-1/2}z)^{-1}\mathbf{x}_1^+(C^{-1}z). \tag{2.33}$$

Proof. — It suffices to check all the relations, which is cumbersome but straightforward, and to observe that T being $\mathbb{Z}$-graded, we have $T(\Omega_n) \subseteq \widehat{\Omega}_n$ for every $n \in \mathbb{N}$, from which it follows that T is continuous. The inverse automorphism is given by

$$T^{-1}(C^{1/2}) = C^{1/2}, \qquad T^{-1}(D) = D, \tag{2.34}$$

$$T^{-1}(\mathbf{k}_0^\pm(z)) = \mathbf{k}_0^\pm(zq^{-2})\mathbf{k}_1^\pm(z)\mathbf{k}_1^\pm(zq^{-2}), \tag{2.35}$$

$$T^{-1}(\mathbf{k}_1^\pm(z)) = \mathbf{k}_1^\pm(z)^{-1}, \tag{2.36}$$

$$T^{-1}(\mathbf{x}_0^+(z)) = \frac{1}{[2]_q} \operatorname*{res}_{z_1,z_2} z_1^{-1}z_2^{-1} \left[\left[\mathbf{x}_0^+(zq^{-2}), \mathbf{x}_1^+(z_1)\right]_{G_{10}^-(zq^{-2}/z_1)}, \mathbf{x}_1^+(z_2)\right]_{G_{11}^-(z_1/z_2)G_{10}^-(zq^{-2}/z_2)}, \tag{2.37}$$

$$T^{-1}(\mathbf{x}_0^-(z)) = \frac{1}{[2]_q} \operatorname*{res}_{z_1,z_2} z_1^{-1}z_2^{-1} \left[\mathbf{x}_1^-(z_1), \left[\mathbf{x}_1^-(z_2), \mathbf{x}_0^-(zq^{-2})\right]_{G_{10}^+(z_2/zq^{-2})}\right]_{G_{11}^+(z_1/z_2)G_{10}^+(z_1/zq^{-2})}, \tag{2.38}$$

$$T^{-1}(\mathbf{x}_1^+(z)) = -\mathbf{k}_1^-(C^{1/2}z)^{-1}\mathbf{x}_1^-(Cz), \tag{2.39}$$

$$T^{-1}(\mathbf{x}_1^-(z)) = -\mathbf{x}_1^+(Cz)\mathbf{k}_1^+(C^{1/2}z)^{-1}. \tag{2.40}$$

The continuity of T^{-1} is proven in the same way as that of T. □

Remark 2.12. — Making use of the defining relations of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$, one easily shows that indeed

$$\left[\mathbf{x}_1^+(z_1), \left[\mathbf{x}_1^+(z_2), \mathbf{x}_0^+(zq^2)\right]_{G_{10}^-(z_2/zq^2)}\right]_{G_{11}^-(z_1/z_2)G_{10}^-(z_1/zq^2)} = [2]_q\, \delta\left(\frac{z_1}{q^2 z_2}\right) \delta\left(\frac{z_2}{z}\right) T\left(\mathbf{x}_0^+(z)\right), \tag{2.41}$$

$$\left[\left[\mathbf{x}_0^-(zq^2), \mathbf{x}_1^-(z_1)\right]_{G_{10}^+(zq^2/z_1)}, \mathbf{x}_1^-(z_2)\right]_{G_{11}^+(z_1/z_2)G_{10}^+(zq^2/z_2)} = [2]_q\,\delta\left(\frac{z_1q^2}{z_2}\right)\delta\left(\frac{z_1}{z}\right)T(\mathbf{x}_0^-(z))\,. \tag{2.42}$$

The following is straightforward but will be useful.

Proposition 2.13. — *We have*

(i) $\varphi \circ T_\pi = T_\pi \circ \varphi$;
(ii) $\varphi \circ T = T \circ \varphi$;
(iii) $T^{-1} = \eta \circ T \circ \eta$.

We have finally,

Theorem 2.14. — *The assignement*

$$t \longmapsto T \qquad y \longmapsto Y = T_\pi \circ T \qquad x_{\omega_i^\vee} \longmapsto T_{\omega_i^\vee} \tag{2.43}$$

extends to a group homomorphism $\ddot{\mathfrak{B}} \to \mathrm{Aut}(\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)})$.

Proof. — This is a cumbersome but straightforward exercise that we leave to the reader. □

Remark 2.15. — In [19], Miki constructed an algebraic action by automorphisms of the extended elliptic braid group on $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ which should not be confused with the topological action of $\ddot{\mathfrak{B}}$ on $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ provided by the above theorem.

2.5. Topological Hopf algebra structure on $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$

Definition 2.16. — *We endow the topological* $\mathbb{F}$*-algebra* $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ *with:*

(i) *the comultiplication* $\Delta : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \to \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)\widehat{\otimes}\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ *defined by*

$$\Delta(C^{\pm 1/2}) = C^{\pm 1/2} \otimes C^{\pm 1/2}\,, \qquad \Delta(D^{\pm 1}) = D^{\pm 1} \otimes D^{\pm 1}\,, \tag{2.44}$$

$$\Delta(\mathbf{k}_i^\pm(z)) = \mathbf{k}_i^\pm(zC_{(2)}^{\pm 1/2}) \otimes \mathbf{k}_i^\pm(zC_{(1)}^{\mp 1/2})\,, \tag{2.45}$$

$$\Delta(\mathbf{x}_i^+(z)) = \mathbf{x}_i^+(z) \otimes 1 + \mathbf{k}_i^-(zC_{(1)}^{1/2})\widehat{\otimes}\mathbf{x}_i^+(zC_{(1)})\,, \tag{2.46}$$

$$\Delta(\mathbf{x}_i^-(z)) = \mathbf{x}_i^-(zC_{(2)})\widehat{\otimes}\mathbf{k}_i^+(zC_{(2)}^{1/2}) + 1 \otimes \mathbf{x}_i^-(z)\,, \tag{2.47}$$

where $C_{(1)}^{\pm 1/2} = C^{\pm 1/2} \otimes 1$ *and* $C_{(2)}^{\pm 1/2} = 1 \otimes C^{\pm 1/2}$*;*

(ii) *the counit* $\varepsilon : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \to \mathbb{F}$*, defined by* $\varepsilon(D^{\pm 1}) = \varepsilon(C^{\pm 1/2}) = \varepsilon(\mathbf{k}_i^{\pm}(z)) = 1$, $\varepsilon(\mathbf{x}_i^{\pm}(z)) = 0$ *and;*

(iii) *the antipode* $S : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \to \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$*, defined by* $S(D^{\pm 1}) = D^{\mp 1}$, $S(C^{\pm 1/2}) = C^{\mp 1/2}$ *and*

$$S(\mathbf{k}_i^{\pm}(z)) = \mathbf{k}_i^{\pm}(z)^{-1}\,,$$

$$S(\mathbf{x}_i^{+}(z)) = -\mathbf{k}_i^{-}(zC^{-1/2})^{-1}\mathbf{x}_i^{+}(zC^{-1})\,,$$

$$S(\mathbf{x}_i^{-}(z)) = -\mathbf{x}_i^{-}(zC^{-1})\mathbf{k}_i^{+}(zC^{-1/2})^{-1}\,.$$

With these operations so defined and the topologies defined in Section 2.3, $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ *is a topological Hopf algebra. Indeed, it is a Hopf algebra, (see [8] or [5] and [6] for the cases of* $\dot{\mathrm{U}}_q(\mathfrak{a}_{n\geqslant 1})$*), and since the maps are all* $\mathbb{Z}$*-graded, we have e.g.* $\Delta(\Omega_n) \subseteq \widehat{\Omega}_n^{(2)}$ *for every* $n \in \mathbb{N}$*, making* Δ *continuous.*

2.6. Non-degenerate Hopf algebra pairing

Define $\dot{\mathrm{U}}_q^{\geqslant}(\dot{\mathfrak{a}}_1)$ (resp. $\dot{\mathrm{U}}_q^{\leqslant}(\dot{\mathfrak{a}}_1)$) as the subalgebra of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ generated by $\{k_{i,-m}^-, x_{i,n}^+ : i \in I, m \in \mathbb{N}, n \in \mathbb{Z}\}$ (resp. $\{k_{i,m}^+, x_{i,n}^- : i \in I, m \in \mathbb{N}, n \in \mathbb{Z}\}$). In view of the triangular decompositon of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ (see [14]) and of its defining relations, it is clear that $\dot{\mathrm{U}}_q^{\geqslant}(\dot{\mathfrak{a}}_1)$ (resp. $\dot{\mathrm{U}}_q^{\leqslant}(\dot{\mathfrak{a}}_1)$), as an $\mathbb{F}$-vector space, is spanned by

$$\begin{aligned}\{x_{i_1,r_1}^+ \cdots x_{i_m,r_m}^+ k_{j_1,-s_1}^- \cdots k_{j_n,-s_n}^- : m,n \in \mathbb{N},\\ ((i_1,r_1),\dots,(i_m,r_m)) \in (\dot{I}\times\mathbb{Z})^m,\\ ((j_1,s_1),\dots,(j_n,s_n)) \in (\dot{I}\times\mathbb{N})^n\}\,,\end{aligned} \tag{2.48}$$

$$\begin{aligned}\big(\text{resp. } \{x_{i_1,r_1}^- \cdots x_{i_m,r_m}^- k_{j_1,s_1}^+ \cdots k_{j_n,s_n}^+ : m,n \in \mathbb{N},\\ ((i_1,r_1),\dots,(i_m,r_m)) \in (\dot{I}\times\mathbb{Z})^m,\\ ((j_1,s_1),\dots,(j_n,s_n)) \in (\dot{I}\times\mathbb{N})^n\}\big)\,.\end{aligned} \tag{2.49}$$

Proposition 2.17. — *There exists a unique non-degenerate Hopf algebra pairing* $\langle\,,\rangle : \dot{\mathrm{U}}_q^{\geqslant}(\dot{\mathfrak{a}}_1) \times \dot{\mathrm{U}}_q^{\leqslant}(\dot{\mathfrak{a}}_1) \to \mathbb{F}$*, defined by setting*

$$\left\langle \mathbf{x}_i^+(z), \mathbf{x}_j^-(v)\right\rangle = \frac{\delta_{ij}}{q-q^{-1}}\delta\left(\frac{z}{v}\right)\,, \tag{2.50}$$

$$\left\langle \mathbf{k}_i^-(z), \mathbf{k}_j^+(v)\right\rangle = G_{ij}^-\left(\frac{z}{v}\right)\,, \tag{2.51}$$

$$\left\langle \mathbf{k}_i^-(z), \mathbf{x}_j^-(v)\right\rangle = \left\langle \mathbf{x}_i^+(z), \mathbf{k}_j^+(v)\right\rangle = 0\,. \tag{2.52}$$

By definition, it is such that, for every $a, b \in \dot{\mathrm{U}}_q^{\geqslant}(\dot{\mathfrak{a}}_1)$ *and every* $x, y \in \dot{\mathrm{U}}_q^{\leqslant}(\dot{\mathfrak{a}}_1)$,

$$\langle a, xy\rangle = \sum \left\langle a_{(1)}, x\right\rangle \left\langle a_{(2)}, y\right\rangle \,,$$
$$\langle ab, x\rangle = \sum \left\langle a, x_{(2)}\right\rangle \left\langle b, x_{(1)}\right\rangle \,,$$
$$\langle a, 1\rangle = \varepsilon_{\geqslant}(a) \qquad \langle 1, x\rangle = \varepsilon_{\leqslant}(x)\,,$$

where we have set $\varepsilon_{\leqslant} = \varepsilon_{|\dot{\mathrm{U}}_q^{\leqslant}(\dot{\mathfrak{a}}_1)}$, $\varepsilon_{\geqslant} = \varepsilon_{|\dot{\mathrm{U}}_q^{\geqslant}(\dot{\mathfrak{a}}_1)}$ *and we have made use of Sweedler's notation for the comultiplication*

$$\Delta(x) = \sum x_{(1)} \widehat{\otimes} x_{(2)}\,.$$

Proof. — See Proposition 2.16 in [22]. □

Before we can establish the continuity of the above defined pairing, we need the following

Lemma 2.18. — *For every* $m_+, m_-, n_+, n_- \in \mathbb{N}$, $(i_1^\pm, \dots, i_{m_\pm}^\pm) \in \dot{I}^{m_\pm}$ *and every* $(j_1^\pm, \dots, j_{n_\pm}^\pm) \in \dot{I}^{n_\pm}$, *we have*

$$\begin{aligned}(2.53)\quad & \Big\langle \mathbf{x}^+_{i_1^+}(u_1)\cdots\mathbf{x}^+_{i_{m_+}^+}(u_{m_+})\mathbf{k}^-_{j_1^+}(v_1)\cdots\mathbf{k}^-_{j_{n_+}^+}(v_{n_+}), \\ &\qquad \mathbf{x}^-_{i_1^-}(w_1)\cdots\mathbf{x}^-_{i_{m_-}^-}(w_{m_-})\mathbf{k}^+_{j_1^-}(z_1)\cdots\mathbf{k}^+_{j_{n_-}^-}(z_{n_-})\Big\rangle \\ = \;& \delta_{m_+,m_-}\left(\prod_{\substack{r\in[\![n_+]\!]\\ s\in[\![n_-]\!]}} G^-_{j_r^+,j_s^-}\left(\frac{v_r}{z_s}\right)\right) \\ &\times \sum_{\sigma\in S_{m_+}}\left(\prod_{\substack{1\leqslant r<s\leqslant m_+\\ \sigma(r)>\sigma(s)}} G^-_{i_r^+,i_s^+}\left(\frac{u_r}{u_s}\right)\right)\prod_{t\in[\![m_+]\!]}\frac{\delta_{i_t^+,i_{\sigma(t)}^-}}{q-q^{-1}}\,\delta\left(\frac{w_{\sigma(t)}}{u_t}\right)\,.\end{aligned}$$

Proof. — One easily proves by recursion the results for $n_+ = n_- = 0$ and $m_+ = m_- = 0$, respectively. The general case then follows by a straightforward calculation. □

It follows that (remember $\mathbb{F}$ is given the discrete topology)

Corollary 2.19. — *The Hopf algebra pairing* $\langle\,,\rangle$ *is (separately) continuous.*

Proof. — It suffices to prove that for every $x \in \dot{\mathrm{U}}_q^{\geqslant}(\dot{\mathfrak{a}}_1)$ there exists an $m \in \mathbb{N}$ such that, for every $n \geqslant m$

$$\left\langle x, \Omega_n \cap \dot{\mathrm{U}}_q^{\leqslant}(\dot{\mathfrak{a}}_1)\right\rangle = \{0\}\,.$$

In order to prove the latter, it suffices to prove it over the spanning sets of (2.48) and (2.49). Now this easily follows by inspection, making use of Lemma 2.18 and of the fact that, for any $y \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) - \{0\}$, there exists $\nu_y \in \mathbb{N}$ such that $y \notin \Omega_{\nu_y+1}$ (see proof of Proposition 2.6). ☐

We can now extend $\langle\, ,\rangle$ from $\dot{\mathrm{U}}_q^{\geqslant}(\dot{\mathfrak{a}}_1) \times \dot{\mathrm{U}}_q^{\leqslant}(\dot{\mathfrak{a}}_1)$ to $\dot{\mathrm{U}}_q^{\geqslant}(\dot{\mathfrak{a}}_1) \times \widehat{\dot{\mathrm{U}}_q^{\leqslant}(\dot{\mathfrak{a}}_1)}$ by continuity. Importantly, we have

Proposition 2.20. — *The extended pairing $\langle\, ,\rangle : \dot{\mathrm{U}}_q^{\geqslant}(\dot{\mathfrak{a}}_1)\times\widehat{\dot{\mathrm{U}}_q^{\leqslant}(\dot{\mathfrak{a}}_1)} \to \mathbb{F}$ is non-degenerate in the sense that, if for every $x \in \dot{\mathrm{U}}_q^{\geqslant}(\dot{\mathfrak{a}}_1)$, $\langle x, y\rangle = 0$ for some $y \in \widehat{\dot{\mathrm{U}}_q^{\leqslant}(\dot{\mathfrak{a}}_1)}$, then $y = 0$.*

Proof. — Let $\{\mathcal{O}_n : n \in \mathbb{N}\}$ be any neighbourhood basis at $0 \in \mathbb{F}$ for the discrete topology on $\mathbb{F}$. Then, let for every $n \in \mathbb{N}$,

$$A_n = \left\langle \dot{\mathrm{U}}_q^{\geqslant}(\dot{\mathfrak{a}}_1), - \right\rangle^{-1}(\mathcal{O}_n) = \left\{ y \in \dot{\mathrm{U}}_q^{\leqslant}(\dot{\mathfrak{a}}_1) : \forall x \in \dot{\mathrm{U}}_q^{\geqslant}(\dot{\mathfrak{a}}_1) \quad \langle x, y\rangle \in \mathcal{O}_n \right\}.$$

We clearly have, for every $n \in \mathbb{N}$, $\{0\} \subseteq A_n \subseteq \dot{\mathrm{U}}_q^{\leqslant}(\dot{\mathfrak{a}}_1)$ and $A_n \supseteq A_{n+1}$. The non-degeneracy of the pairing further implies that

$$\bigcap_{n\in\mathbb{N}} A_n = \{0\}\,.$$

As a consequence, for every $n \in \mathbb{N}$ and every $y \in A_n - \{0\}$, there exists an $N \in \mathbb{N}$ such that for every $m \geqslant N$, $y \notin A_m$. Now, given $n_1 \in \mathbb{N}$, let $\mu(n_1) \in \mathbb{N}$ be the largest integer such that $A_{n_1} \subseteq \Omega_{\mu(n_1)}$. By the previous discussion, for every point $y \in A_{n_1} - \Omega_{\mu(n_1)+1}$, there exists (a smallest) $n_2 \in \mathbb{N}$ such that for every $m \geqslant n_2$, $y \notin A_m$. Hence, for every $m \geqslant n_2$, $A_m \subseteq \Omega_{\mu(n_1)+1}$ and we conclude that $\mu(n) = \mu(n_1)$ for every $n \in [\![n_1, n_2 - 1]\!]$, whereas $\mu(n_2) = \mu(n_1) + 1$. By induction, it follows that $\mu : \mathbb{N} \to \mathbb{N}$ so defined is increasing and that, as a consequence, $\lim_{n\to+\infty} \mu(n) = +\infty$. We have therefore proven that, for every $n \in \mathbb{N}$,

$$\forall x \in \dot{\mathrm{U}}_q^{\geqslant}(\dot{\mathfrak{a}}_1) \qquad \langle x, y\rangle \in \mathcal{O}_n \Longrightarrow y \in \Omega_{\mu(n)}\,. \tag{2.54}$$

If we finally let $(y_n)_{n\in\mathbb{N}} \in \dot{\mathrm{U}}_q^{\leqslant}(\dot{\mathfrak{a}}_1)^{\mathbb{N}}$ be any Cauchy sequence that does not converge to 0, the proposition is obviously equivalent to claiming that there exists an $x \in \dot{\mathrm{U}}_q^{\geqslant}(\dot{\mathfrak{a}}_1)$ such that

$$\lim_{n\to+\infty} \langle x, y_n\rangle \neq 0\,.$$

Indeed, since $(y_n)_{n\in\mathbb{N}}$ does not converge to 0, there exist $m \in \mathbb{N}$ such that for every $N \in \mathbb{N}$, $y_n \notin \Omega_m$ for some $n \geqslant N$. We can therefore extract a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ such that $y_{n_k} \notin \Omega_m$ for every $k \in \mathbb{N}$. The contrapositive of (2.54) then implies that there exists $(x_k)_{k\in\mathbb{N}} \in \dot{\mathrm{U}}_q^{\geqslant}(\dot{\mathfrak{a}}_1)^{\mathbb{N}}$ such that, for every $k \in \mathbb{N}$,

$$\langle x_k, y_{n_k} \rangle \notin \mathcal{O}_{\nu(m)}$$

where $\nu(m) = \min\{n \in \mathbb{N} : \mu(n) = m\}$. But since $(y_n)_{n\in\mathbb{N}}$ is Cauchy, so is $(y_{n_k})_{k\in\mathbb{N}}$ and, upon taking $k, l \in \mathbb{N}$ large enough, we can make $\langle x_k, y_{n_l} - y_{n_k} \rangle$ arbitrary small. This eventually concludes the proof. □

3. Double quantum affinization in type $\mathfrak{a}_1$

We now define and study the main object of interest in this paper; the double quantum affinization in type $\mathfrak{a}_1$, $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$. We let $I = \{1\}$ be the labeling of the unique node of the type $\mathfrak{a}_1$ Dynkin diagram and we let $Q^\pm = \mathbb{Z}^\pm \alpha_1$. We denote by $Q = \mathbb{Z}\alpha_1$ the type $\mathfrak{a}_1$ root lattice.

3.1. Definition of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$

Definition 3.1. — *The* double quantum affinization $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ *of type* $\mathfrak{a}_1$ *is defined as the* $\mathbb{F}$*-algebra generated by*

$$\big\{ \mathsf{D}_1, \mathsf{D}_1^{-1}, \mathsf{D}_2, \ \mathsf{D}_2^{-1}, \mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}^+_m, \mathsf{c}^-_{-m}, \mathsf{K}^+_{1,0,m}, \mathsf{K}^-_{1,0,-m}, \mathsf{K}^+_{1,n,r}, \mathsf{K}^-_{1,-n,r}, \mathsf{X}^+_{1,r,s}, \mathsf{X}^-_{1,r,s} : m \in \mathbb{N}, n \in \mathbb{N}^\times, r, s \in \mathbb{Z} \big\} \,,$$

subject to the relations

$$\mathsf{C}^{\pm 1/2} \text{ and } \boldsymbol{c}^\pm(z) \text{ are central} \tag{3.1}$$

$$\operatorname*{res}_{v,w} \frac{1}{vw} \boldsymbol{c}^\pm(v) \boldsymbol{c}^\mp(w) = 1 \,, \tag{3.2}$$

$$\mathsf{D}_1^{\pm 1} \mathsf{D}_1^{\mp 1} = 1 \qquad \mathsf{D}_2^{\pm 1} \mathsf{D}_2^{\mp 1} = 1 \qquad \mathsf{D}_1 \mathsf{D}_2 = \mathsf{D}_2 \mathsf{D}_1 \tag{3.3}$$

$$\mathsf{D}_1 \boldsymbol{K}^\pm_{1,\pm m}(z) \mathsf{D}_1^{-1} = q^{\pm m} \boldsymbol{K}^\pm_{1,\pm m}(z) \qquad \mathsf{D}_1 \boldsymbol{X}^\pm_{1,r}(z) \mathsf{D}_1^{-1} = q^r \boldsymbol{X}^\pm_{1,r}(z) \,, \tag{3.4}$$

$$\mathsf{D}_2 \boldsymbol{K}^\pm_{1,\pm m}(z) \mathsf{D}_2^{-1} = \boldsymbol{K}^\pm_{1,\pm m}(zq^{-1}) \qquad \mathsf{D}_2 \boldsymbol{X}^\pm_{1,r}(z) \mathsf{D}_2^{-1} = \boldsymbol{X}^\pm_{1,r}(zq^{-1}) \,, \tag{3.5}$$

$$\operatorname*{res}_{v,w} \frac{1}{vw} \boldsymbol{K}^\pm_{1,0}(v) \boldsymbol{K}^\mp_{1,0}(w) = 1 \,, \tag{3.6}$$

$$\begin{aligned} &(v - q^{\pm 2} z)(v - q^{2(m-n\mp 1)} z) \boldsymbol{K}^\pm_{1,\pm m}(v) \boldsymbol{K}^\pm_{1,\pm n}(z) \\ &\qquad = (vq^{\pm 2} - z)(vq^{\mp 2} - q^{2(m-n)} z) \boldsymbol{K}^\pm_{1,\pm n}(z) \boldsymbol{K}^\pm_{1,\pm m}(v) \,, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &(\mathsf{C} q^{2(1-m)} v - w)(q^{2(n-1)} v - \mathsf{C} w) \boldsymbol{K}^+_{1,m}(v) \boldsymbol{K}^-_{1,-n}(w) \\ &\qquad = (\mathsf{C} q^{-2m} v - q^2 w)(q^{2n} v - \mathsf{C} q^{-2} w) \boldsymbol{K}^-_{1,-n}(w) \boldsymbol{K}^+_{1,m}(v) \,, \end{aligned} \tag{3.8}$$

$$(v - q^{\pm 2} z)\boldsymbol{K}^{\pm}_{1,\pm m}(v)\boldsymbol{X}^{\pm}_{1,r}(z) = (q^{\pm 2} v - z)\boldsymbol{X}^{\pm}_{1,r}(z)\boldsymbol{K}^{\pm}_{1,\pm m}(v)\,, \tag{3.9}$$

$$(Cv - q^{2(m\mp 1)} z)\boldsymbol{K}^{\pm}_{1,\pm m}(v)\boldsymbol{X}^{\mp}_{1,r}(z) = (Cq^{\mp 2} v - q^{2m} z)\boldsymbol{X}^{\mp}_{1,r}(z)\boldsymbol{K}^{\pm}_{1,\pm m}(v)\,, \tag{3.10}$$

$$(v - q^{\pm 2} w)\boldsymbol{X}^{\pm}_{1,r}(v)\boldsymbol{X}^{\pm}_{1,s}(w) = (vq^{\pm 2} - w)\boldsymbol{X}^{\pm}_{1,s}(w)\boldsymbol{X}^{\pm}_{1,r}(v)\,, \tag{3.11}$$

$$\left[\boldsymbol{X}^{+}_{1,r}(v), \boldsymbol{X}^{-}_{1,s}(z)\right] = \frac{1}{q-q^{-1}} \times \left\{ \delta\left(\frac{Cv}{q^{2(r+s)}z}\right) \prod_{p=1}^{|s|} \boldsymbol{c}^{-}\left(C^{-1/2} q^{(2p-1)\mathrm{sign}(s)-1} z\right)^{-\mathrm{sign}(s)} \boldsymbol{K}^{+}_{1,r+s}(v) - \delta\left(\frac{C^{-1}v}{q^{2(r+s)}z}\right) \prod_{p=1}^{|r|} \boldsymbol{c}^{+}\left(C^{-1/2} q^{(1-2p)\mathrm{sign}(r)-1} v\right)^{\mathrm{sign}(r)} \boldsymbol{K}^{-}_{1,r+s}(z) \right\}, \tag{3.12}$$

where $m, n \in \mathbb{N}$, $r, s \in \mathbb{Z}$ *and we have set*

$$\boldsymbol{c}^{\pm}(z) = \sum_{m\in\mathbb{N}} \mathsf{c}^{\pm}_{\pm m} z^{\mp m}\,, \tag{3.13}$$

$$\boldsymbol{K}^{\pm}_{1,0}(z) = \sum_{m\in\mathbb{N}} \mathsf{K}^{\pm}_{1,0,\pm m} z^{\pm m}\,, \tag{3.14}$$

and, for every $m \in \mathbb{N}^{\times}$ *and* $r \in \mathbb{Z}$,

$$\boldsymbol{K}^{\pm}_{1,\pm m}(z) = \sum_{s\in\mathbb{Z}} \mathsf{K}^{\pm}_{1,\pm m,s} z^{-s}\,, \tag{3.15}$$

$$\boldsymbol{X}^{\pm}_{1,r}(z) = \sum_{s\in\mathbb{Z}} \mathsf{X}^{\pm}_{1,r,s} z^{-s}\,. \tag{3.16}$$

In (3.12), *we further assume that* $\boldsymbol{K}^{\pm}_{1,\mp m}(z) = 0$ *for every* $m \in \mathbb{N}^{\times}$.

Definition 3.2. — *We define* $\ddot{\mathrm{U}}^{0}_{q}(\mathfrak{a}_1)$ *as the subalgebra of* $\ddot{\mathrm{U}}_{q}(\mathfrak{a}_1)$ *generated by*

$$\left\{ C^{1/2}, C^{-1/2}, \mathsf{c}^{+}_{m}, \mathsf{c}^{-}_{-m}, \mathsf{K}^{+}_{1,0,m}, \mathsf{K}^{-}_{1,0,-m}, \mathsf{K}^{+}_{1,n,r}, \mathsf{K}^{-}_{1,-n,r} : m \in \mathbb{N}, n \in \mathbb{N}^{\times}, r \in \mathbb{Z} \right\}.$$

We define similarly $\ddot{\mathrm{U}}^{\pm}_{q}(\mathfrak{a}_1)$ *as the subalgebra of* $\ddot{\mathrm{U}}_{q}(\mathfrak{a}_1)$ *generated by*

$$\left\{ \mathsf{X}^{\pm}_{1,r,s} : r, s \in \mathbb{Z} \right\}.$$

Remark 3.3. — Obviously, $\ddot{\mathrm{U}}_q^{\pm}(\mathfrak{a}_1)$ is graded over $Q^{\pm}$ whereas $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ is graded over the root lattice Q of $\mathfrak{a}_1$. $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ is also graded over $\mathbb{Z}^2 = \mathbb{Z}_{(1)} \times \mathbb{Z}_{(2)}$;

$$\ddot{\mathrm{U}}_q(\mathfrak{a}_1) = \bigoplus_{(n_1,n_2)\in\mathbb{Z}^2} \ddot{\mathrm{U}}_q(\mathfrak{a}_1)_{(n_1,n_2)}\,,$$

where, for every $(n_1, n_2) \in \mathbb{Z}^2$, we let

$$\ddot{\mathrm{U}}_q(\mathfrak{a}_1)_{(n_1,n_2)} = \left\{x \in \ddot{\mathrm{U}}_q(\mathfrak{a}_1) : \mathsf{D}_1 x \mathsf{D}_1^{-1} = q^{n_1} x, \quad \mathsf{D}_2 x \mathsf{D}_2^{-1} = q^{n_2} x\right\}\,.$$

Remark 3.4. — It is worth emphasizing that, were it not for relation (3.12), the above definition of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ would be purely algebraic. However, the r.h.s. of (3.12) involves two infinite series and we may equip $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ with a topology, along the lines of what was done in Section 2.3 for $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$, making use of the $\mathbb{Z}_{(2)}$-grading in order to construct a basis $\left\{\dot{\Omega}_n : n \in \mathbb{N}\right\}$ of open neighbourhoods of 0. In that case, both series are convergent in the corresponding completion $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$ and we shall further require that the subalgebras $\ddot{\mathrm{U}}_q^-(\mathfrak{a}_1)$, $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$ and $\ddot{\mathrm{U}}_q^+(\mathfrak{a}_1)$ be defined as closed subalgebras of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$. We shall eventually denote with a hat their respective completions. An alternative point of view on this question, which might actually prove more useful when it comes to studying representation theory, consists in observing that $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ is *proalgebraic*. Indeed, for every $N \in \mathbb{N}$, let $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}$ be the $\mathbb{F}$-algebra generated by

$$\Big\{\mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}_n^+, \mathsf{c}_{-n}^-, \mathsf{K}_{1,0,m}^+, \mathsf{K}_{1,0,-m}^-,$$
$$\mathsf{K}_{1,p,r}^+, \mathsf{K}_{1,-p,r}^-, \mathsf{X}_{1,r,s}^+, \mathsf{X}_{1,r,s}^- : m \in \mathbb{N}, n \in [\![0, N]\!], p \in \mathbb{N}^\times, r, s \in \mathbb{Z}\Big\}\,,$$

subject to relations (3.1)–(3.12), where, this time,

$$\mathbf{c}^{\pm}(z) = \sum_{m=0}^{N} \mathsf{c}_{\pm m}^{\pm} z^{\mp m}\,. \tag{3.17}$$

Now clearly, each $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}$ is algebraic since the sums on the r.h.s. of (3.12) are both finite (whenever $\mathbf{c}^{\pm}(z)^{-1}$ is involved, just multiply through by $\mathbf{c}^{\pm}(z)$ to get an equivalent algebraic relation). Moreover, letting $\mathcal{I}_N$ be the two-sided ideal of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}$ generated by $\{\mathsf{c}_N^+, \mathsf{c}_{-N}^-\}$ (resp. $\{\mathsf{c}_0^+ - 1, \mathsf{c}_0^- - 1\}$) for every $N > 1$ (resp. for $N = 0$), we obviously have a surjective algebra homomorphism

$$\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)} \longrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N-1)} \cong \frac{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}}{\mathcal{I}_N} \tag{3.18}$$

and we can define $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ as the inverse limit

$$\ddot{\mathrm{U}}_q(\mathfrak{a}_1) = \varprojlim \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}$$

of the system of algebras

$$\cdots \longrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)} \longrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N-1)} \longrightarrow \cdots \longrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(0)} \longrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(-1)}.$$

We shall refer to the quotient of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(-1)}$ by the two-sided ideal generated by $\{\mathsf{C}^{1/2}-1\}$ as the *double quantum loop algebra* of type $\mathfrak{a}_1$.

Definition 3.5. — *In $\widehat{\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)}$, we define*

$$\boldsymbol{p}^\pm(z) = \sum_{m\in\mathbb{N}} \mathsf{p}^\pm_{\pm m} z^{\mp m} = \boldsymbol{c}^\pm(z)\boldsymbol{K}^\mp_{1,0}(C^{-1/2}z)^{-1}\boldsymbol{K}^\mp_{1,0}(C^{-1/2}zq^2)$$

and for every $m \in \mathbb{N}^\times$,

$$\boldsymbol{t}^+_{1,m}(z) = \sum_{n\in\mathbb{N}} \mathsf{t}^+_{1,m,n} z^{-n} = -\frac{1}{q-q^{-1}}\boldsymbol{K}^+_{1,0}(zq^{-2m})^{-1}\boldsymbol{K}^+_{1,m}(z),$$
$$\boldsymbol{t}^-_{1,-m}(z) = \sum_{n\in\mathbb{N}} \mathsf{t}^-_{1,-m,n} z^{n} = \frac{1}{q-q^{-1}}\boldsymbol{K}^-_{1,-m}(z)\boldsymbol{K}^-_{1,0}(zq^{-2m})^{-1}.$$

Then, we let $\ddot{\mathrm{U}}^{0^+}_q(\mathfrak{a}_1)$ be the closed subalgebra of $\widehat{\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)}$ generated by

$$\{C^{1/2}, C^{-1/2}, \mathsf{p}^+_m, \mathsf{p}^-_{-m}, \mathsf{t}^+_{1,p,n}, \mathsf{t}^-_{1,-p,n} : m\in\mathbb{N}, n\in\mathbb{Z}, p\in\mathbb{N}^\times\}.$$

Definition 3.6. — *We denote by $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$ the subalgebra of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ generated by*

$$\Big\{D_2, D_2^{-1}, C^{1/2}, C^{-1/2}, \mathsf{c}^+_m, \mathsf{c}^-_{-m}, \mathsf{K}^+_{1,0,m}, \mathsf{K}^-_{1,0,-m},$$
$$\mathsf{K}^+_{1,n,r}, \mathsf{K}^-_{1,-n,r}, \mathsf{X}^+_{1,r,s}, \mathsf{X}^-_{1,r,s} : m\in\mathbb{N}, n\in\mathbb{N}^\times, r,s\in\mathbb{Z}\Big\},$$

i.e. the subalgebra generated by all the generators of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ except D_1 and D_1^{-1}. We shall denote by

$$\jmath : \ddot{\mathrm{U}}'_q(\mathfrak{a}_1) \hookrightarrow \ddot{\mathrm{U}}_q(\mathfrak{a}_1)$$

the natural injective algebra homomorphism. We extend it by continuity into

$$\widehat{\jmath} : \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)} \hookrightarrow \widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}.$$

The main result of the present paper is the following

THEOREM 3.7. — *There exists a bicontinuous* $\mathbb{F}$*-algebra isomorphism* $\widehat{\Psi} : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \xrightarrow{\sim} \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$.

Proof. — Relations (3.7)–(3.10) respectively imply

$$\mathsf{K}^{\pm}_{1,0}(v)\mathsf{K}^{\pm}_{1,0}(z) = \mathsf{K}^{\pm}_{1,0}(z)\mathsf{K}^{\pm}_{1,0}(v)\,, \tag{3.19}$$

$$\mathsf{K}^{+}_{1,0}(v)\mathsf{K}^{-}_{1,0}(w) = G^{+}_{11}(\mathsf{C}v/w)G^{-}_{11}(\mathsf{C}^{-1}v/w)\mathsf{K}^{-}_{1,0}(w)\mathsf{K}^{+}_{1,0}(v) \tag{3.20}$$

$$\mathsf{K}^{\pm}_{1,0}(v)\mathsf{X}^{\pm}_{1,r}(z) = G^{\mp}_{11}(v/z)\mathsf{X}^{\pm}_{1,r}(z)\mathsf{K}^{\pm}_{1,0}(v)\,, \tag{3.21}$$

$$\mathsf{K}^{\pm}_{1,0}(v)\mathsf{X}^{\mp}_{1,r}(z) = G^{\pm}_{11}(\mathsf{C}v/z)\mathsf{X}^{\mp}_{1,r}(z)\mathsf{K}^{\pm}_{1,0}(v)\,, \tag{3.22}$$

since $\mathsf{K}^{\pm}_{1,0}(z) \in \ddot{\mathrm{U}}'_q(\mathfrak{a}_1)[[z^{\pm 1}]]$. It also easily follows from relation (3.11) that

$$\left[\mathsf{X}^{+}_{1,0}(v), \mathsf{X}^{+}_{1,-1}(w)\right]_{G^{-}_{11}(v/w)} = \delta\left(\frac{vq^{-2}}{w}\right)\Upsilon^{+}(w)\,, \tag{3.23}$$

$$\left[\mathsf{X}^{-}_{1,1}(v), \mathsf{X}^{-}_{1,0}(w)\right]_{G^{+}_{11}(v/w)} = \delta\left(\frac{vq^{2}}{w}\right)\Upsilon^{-}(w)\,, \tag{3.24}$$

for some $\Upsilon^{\pm}(w) \in \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}[[w, w^{-1}]]$. Hence, the only possible obstructions to setting

$$\Psi(D^{\pm 1}) = \mathsf{D}_2^{\pm 1} \qquad \Psi(C^{\pm 1/2}) = \mathsf{C}^{\pm 1/2}\,,$$
$$\Psi(\mathbf{k}^{\pm}_0(z)) = -\mathsf{c}^{\pm}(z)\mathsf{K}^{\mp}_{1,0}(\mathsf{C}^{-1/2}z)^{-1}\,, \qquad \Psi(\mathbf{k}^{\pm}_1(z)) = -\mathsf{K}^{\mp}_{1,0}(\mathsf{C}^{-1/2}z)\,,$$
$$\Psi(\mathbf{x}^{+}_0(z)) = -\mathsf{c}^{-}(\mathsf{C}^{1/2}z)\mathsf{K}^{+}_{1,0}(z)^{-1}\mathsf{X}^{-}_{1,1}(\mathsf{C}z)\,,$$
$$\Psi(\mathbf{x}^{-}_0(z)) = -\mathsf{X}^{+}_{1,-1}(\mathsf{C}z)\mathsf{c}^{+}(\mathsf{C}^{1/2}z)\mathsf{K}^{-}_{1,0}(z)^{-1}$$
$$\Psi(\mathbf{x}^{\pm}_1(z)) = \mathsf{X}^{\pm}_{1,0}(z)\,,$$

and to extending it as an algebra homomorphism $\Psi : \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) \to \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ are $\Upsilon^{\pm}(w)$ and the images under Ψ of the l.h.s. of the quantum Serre relations (2.10). We shall see in Section 4 that both obstructions actually vanish. We also postpone until Section 4 the construction of the continuous algebra homomorphism $\Psi^{-1} : \ddot{\mathrm{U}}'_q(\mathfrak{a}_1) \to \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$. □

3.2. The subalgebra $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$ and the elliptic Hall algebra

Another remarkable feature of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ and, more particularly of its subalgebra $\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1)$, is the existence of an algebra homomorphism onto it, from the elliptic Hall algebra that we now define.

Definition 3.8. — *Let q_1, q_2, q_3 be three (dependent) formal variables such that $q_1q_2q_3 = 1$. The* elliptic Hall algebra $\mathcal{E}_{q_1,q_2,q_3}$ *is the $\mathbb{Q}(q_1, q_2, q_3)$-algebra generated by $\left\{C^{1/2}, C^{-1/2}, \psi^+_m, \psi^-_{-m}, e^+_n, e^-_n : m \in \mathbb{N}, n \in \mathbb{Z}\right\}$, with $\psi^\pm_0$ invertible, subject to the relations*

$$C^{\pm 1/2} \text{ is central}, \tag{3.25}$$

$$\boldsymbol{\psi}^\pm(z)\boldsymbol{\psi}^\pm(w) = \boldsymbol{\psi}^\pm(w)\boldsymbol{\psi}^\pm(z)\,, \tag{3.26}$$

$$g(Cz, w)g(Cw, z)\boldsymbol{\psi}^+(z)\boldsymbol{\psi}^-(w) = g(z, Cw)g(w, Cz)\boldsymbol{\psi}^-(w)\boldsymbol{\psi}^+(z)\,, \tag{3.27}$$

$$g(C^{\frac{1\pm 1}{2}} z, w)\boldsymbol{\psi}^\pm(z)\mathbf{e}^+(w) = -g(w, C^{\frac{1\pm 1}{2}} z)\mathbf{e}^+(w)\boldsymbol{\psi}^\pm(z)\,, \tag{3.28}$$

$$g(w, C^{\frac{1\mp 1}{2}} z)\boldsymbol{\psi}^\pm(z)\mathbf{e}^-(w) = -g(C^{\frac{1\mp 1}{2}} z, w)\mathbf{e}^-(w)\boldsymbol{\psi}^\pm(z)\,, \tag{3.29}$$

$$[\mathbf{e}^+(z), \mathbf{e}^-(w)] = \frac{1}{g(1,1)}\left[\delta\left(\frac{Cw}{z}\right)\boldsymbol{\psi}^+(w) - \delta\left(\frac{w}{Cz}\right)\boldsymbol{\psi}^-(z)\right]\,, \tag{3.30}$$

$$g(z, w)\mathbf{e}^+(z)\mathbf{e}^+(w) = -g(w, z)\mathbf{e}^+(w)\mathbf{e}^+(z)\,, \tag{3.31}$$

$$g(w, z)\mathbf{e}^-(z)\mathbf{e}^-(w) = -g(z, w)\mathbf{e}^-(w)\mathbf{e}^-(z)\,, \tag{3.32}$$

$$\operatorname*{res}_{v,w,z}(vwz)^m(v + z)(w^2 - vz)\mathbf{e}^\pm(v)\mathbf{e}^\pm(w)\mathbf{e}^\pm(z) = 0\,, \tag{3.33}$$

where $m \in \mathbb{Z}$ and we have introduced

$$g(z, w) = (z - q_1w)(z - q_2w)(z - q_3w)\,, \tag{3.34}$$

$$\boldsymbol{\psi}^\pm(z) = \sum_{m\in\mathbb{N}} \psi^\pm_{\pm m} z^{\mp m}\,, \tag{3.35}$$

$$\mathbf{e}^\pm(z) = \sum_{m\in\mathbb{Z}} e^\pm_m z^{-m}\,. \tag{3.36}$$

Remark 3.9. — The elliptic Hall algebra $\mathcal{E}_{q_1,q_2,q_3}$ is $\mathbb{Z}$-graded and can be equipped with a natural topology along the lines of what we did for $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ in Section 2.3. It then becomes a topological algebra and we denote by $\widehat{\mathcal{E}_{q_1,q_2,q_3}}$ its completion.

Proposition 3.10. — *There exists a unique continuous $\mathbb{F}$-algebra homomorphism $f : \widehat{\mathcal{E}_{q^{-4},q^2,q^2}} \to \ddot{\mathrm{U}}^{0^+}_q(\mathfrak{a}_1)$ such that*

$$f(C^{1/2}) = \mathcal{C}^{1/2}\,, \tag{3.37}$$

$$f(\boldsymbol{\psi}^\pm(z)) = (q^2 - q^{-2})^2\, \boldsymbol{p}^\pm(\mathcal{C}^{1/2}zq^{-2})\,, \tag{3.38}$$

$$f(\mathbf{e}^\pm(z)) = \boldsymbol{t}^\pm_{1,\pm 1}(z)\,. \tag{3.39}$$

Proof. — We prove that, starting from (3.37)–(3.39), we can extend f as an algebra homomorphism. For that purpose, it suffices to check the relations in $\mathcal{E}_{q^{-4},q^2,q^2}$, observing that, in addition to (3.19) and (3.20), we also have

$$\mathsf{K}^{\pm}_{1,0}(v)\mathsf{K}^{\pm}_{1,\pm1}(z) = G^{\mp}_{11}(v/z)G^{\pm}_{11}(vq^2/z)\mathsf{K}^{\pm}_{1,\pm1}(z)\mathsf{K}^{\pm}_{1,0}(v)\,, \tag{3.40}$$

$$\mathsf{K}^{\mp}_{1,0}(v)\mathsf{K}^{\pm}_{1,\pm1}(w) = G^{\mp}_{11}(\mathsf{C}v/w)G^{\pm}_{11}(\mathsf{C}^{-1}q^2v/w)\mathsf{K}^{\pm}_{1,\pm1}(w)\mathsf{K}^{\mp}_{1,0}(v)\,, \tag{3.41}$$

as direct consequences of (3.7) and (3.8) respectively, since, by definition, $\mathsf{K}^{\pm}_{1,0}(z) \in \ddot{\mathrm{U}}'_q(\mathfrak{a}_1)[[z^{\pm1}]]$. One then easily obtains (3.26)–(3.29) and (3.31)–(3.32). For example, we have

$$\begin{aligned} &g(v,z)f(\mathbf{e}^+(v))f(\mathbf{e}^+(z)) \\ &= \frac{1}{(q-q^{-1})^2}g(v,z)G^{+}_{11}(z/v)G^{-}_{11}(zq^{-2}/v) \\ &\qquad\qquad \times \mathsf{K}^{+}_{1,0}(vq^{-2})\mathsf{K}^{+}_{1,0}(q^{-2}z)\mathsf{K}^{+}_{1,1}(v)\mathsf{K}^{+}_{1,1}(z) \\ &= \frac{v-z}{(q-q^{-1})^2}(v-q^2z)(v-q^{-2}z)\mathsf{K}^{+}_{1,0}(vq^{-2})\mathsf{K}^{+}_{1,0}(q^{-2}z)\mathsf{K}^{+}_{1,1}(v)\mathsf{K}^{+}_{1,1}(z) \\ &= \frac{v-z}{(q-q^{-1})^2}(vq^2-z)(vq^{-2}-z)\mathsf{K}^{+}_{1,0}(vq^{-2})\mathsf{K}^{+}_{1,0}(q^{-2}z)\mathsf{K}^{+}_{1,1}(z)\mathsf{K}^{+}_{1,1}(v) \\ &= \frac{v-z}{(q-q^{-1})^2}(vq^2-z)(vq^{-2}-z)G^{+}_{11}(vq^{-2}/z)G^{-}_{11}(v/z) \\ &\qquad\qquad \times \mathsf{K}^{+}_{1,0}(q^{-2}z)\mathsf{K}^{+}_{1,1}(z)\mathsf{K}^{+}_{1,0}(vq^{-2})\mathsf{K}^{+}_{1,1}(v) \\ &= -g(z,v)f(\mathbf{e}^+(z))f(\mathbf{e}^+(v))\,. \end{aligned} \tag{3.42}$$

Considering (3.30), we observe that (3.8) implies that there exist $\theta^{\pm}(z) \in \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}[[z,z^{-1}]]$ such that

$$\left[\mathsf{K}^{+}_{1,1}(v),\mathsf{K}^{-}_{1,-1}(w)\right]_{G^{+}_{11}(\mathsf{C}vq^{-2}/w)G^{-}_{11}(\mathsf{C}^{-1}vq^2/w)} = \delta\left(\frac{\mathsf{C}v}{w}\right)\theta^-(v) + \delta\left(\frac{v}{\mathsf{C}w}\right)\theta^+(w)$$

and one easily sees that

$$\begin{aligned} &\left[f(\mathbf{e}^+(v)),f(\mathbf{e}^-(w))\right] \\ &= -\frac{1}{(q-q^{-1})^2}\mathsf{K}^{+}_{1,0}(vq^{-2})^{-1} \\ &\qquad \times \left[\mathsf{K}^{+}_{1,1}(v),\mathsf{K}^{-}_{1,-1}(w)\right]_{G^{+}_{11}(\mathsf{C}vq^{-2}/w)G^{-}_{11}(\mathsf{C}^{-1}vq^2/w)}\mathsf{K}^{-}_{1,0}(wq^{-2})^{-1} \\ &= -\frac{1}{(q-q^{-1})^2}\mathsf{K}^{+}_{1,0}(vq^{-2})^{-1} \\ &\qquad \times \left\{\delta\left(\frac{\mathsf{C}v}{w}\right)\theta^-(v) + \delta\left(\frac{v}{\mathsf{C}w}\right)\theta^+(w)\right\}\mathsf{K}^{-}_{1,0}(wq^{-2})^{-1}\,. \end{aligned} \tag{3.43}$$

Therefore, it suffices to prove that

$$\text{(3.44)}\qquad \frac{1}{(q-q^{-1})^2}\mathsf{K}^+_{1,0}(\mathsf{C}wq^{-2})^{-1}\theta^+(w)\mathsf{K}^-_{1,0}(wq^{-2})^{-1} = \frac{(q^2-q^{-2})^2}{g(1,1)}\mathbf{p}^+(\mathsf{C}^{1/2}q^{-2}w)\,,$$

$$\text{(3.45)}\qquad -\frac{1}{(q-q^{-1})^2}\mathsf{K}^+_{1,0}(vq^{-2})^{-1}\theta^-(v)\mathsf{K}^-_{1,0}(\mathsf{C}vq^{-2})^{-1} = -\frac{(q^2-q^{-2})^2}{g(1,1)}\mathbf{p}^-(\mathsf{C}^{1/2}q^{-2}v)\,.$$

We postpone the proof of (3.44)–(3.45), as well as that of

$$\text{(3.46)}\qquad \operatorname*{res}_{v,w,z}(vwz)^m(v+z)(w^2-vz)f(\mathbf{e}^\pm(v))f(\mathbf{e}^\pm(w))f(\mathbf{e}^\pm(z))=0\,,$$

until Section 4. □

We now naturally make the following

Conjecture 3.11. — *$f:\widehat{\mathcal{E}_{q^{-4},q^2,q^2}}\to\ddot{\mathrm{U}}_q^{0^+}(\mathfrak{a}_1)$ is a bicontinuous $\mathbb{F}$-algebra isomorphism.*

Remark 3.12. — It is worth mentioning that the above conjecture is supported by the fact that, in view of (3.31)–(3.32), there clearly exists $\mathbf{e}^\pm_{\pm2}(z)\in\widehat{\mathcal{E}_{q_1,q_2,q_3}}[[z,z^{-1}]]$ such that

$$G^\mp_{01}(q^{\mp2}v/w)G^\mp_{11}(v/w)\big[\mathbf{e}^\pm(w),\mathbf{e}^\pm(v)\big]_{G^\mp_{01}(q^{\mp2}w/v)G^\mp_{11}(w/v)} = \pm[2]_q\left\{\delta\left(\frac{q^2v}{w}\right)\mathbf{e}^\pm_{\pm2}(w)-\delta\left(\frac{wq^2}{v}\right)\mathbf{e}^\pm_{\pm2}(v)\right\}$$

and that we can therefore set

$$f^{-1}(\mathbf{t}^\pm_{1,\pm2}(v))=\mathbf{e}^\pm_{\pm2}(v)\,.$$

In order to complete the proof, one would similarly need to construct $f^{-1}(\mathbf{t}^\pm_{1,\pm m}(v))$ for any $m>2$.

3.3. $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ subalgebras of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$

Interestingly, $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ admits countably many embeddings of the quantum affine algebra $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$. This is the content of the following

Proposition 3.13. — *For every $m \in \mathbb{Z}$, there exists a unique injective algebra homomorphism* $\iota_m : \dot{\mathrm{U}}_q(\mathfrak{a}_1) \hookrightarrow \widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$ *such that*

$$\iota_m(C^{\pm 1/2}) = \mathcal{C}^{\pm 1/2} \qquad \iota_m(D^{\pm 1}) = \mathcal{D}_2^{\pm 1} \tag{3.47}$$

$$\iota_m(\mathbf{k}_1^{\pm}(z)) = -\prod_{p=1}^{|m|} \boldsymbol{c}^{\pm}\left(q^{(1-2p)\mathrm{sign}(m)-1} z\right)^{\mathrm{sign}(m)} \boldsymbol{K}_{1,0}^{\mp}(\mathcal{C}^{-1/2} z)\,, \tag{3.48}$$

$$\iota_m(\mathbf{x}_1^{\pm}(z)) = \boldsymbol{X}_{1,\pm m}^{\pm}(z)\,. \tag{3.49}$$

Proof. — Let $\iota^{(1)} : \dot{\mathrm{U}}_q(\mathfrak{a}_1) \hookrightarrow \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ be the injective algebra homomorphism mapping $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ to the Dynkin diagram subalgebra of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ associated with the vertex labeled $1 \in \dot{I}$ (see Section 2.1). It naturally extends to an injective algebra homomorphism $\widehat{\iota}^{(1)} : \dot{\mathrm{U}}_q(\mathfrak{a}_1) \hookrightarrow \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$. Then, let for every $m \in \mathbb{Z}$, ι_m be the composite

$$\iota_m : \dot{\mathrm{U}}_q(\mathfrak{a}_1) \underset{\widehat{\iota}^{(1)}}{\hookrightarrow} \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \xrightarrow[Y^{-m}]{\sim} \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \xrightarrow[\widehat{\Psi}]{\sim} \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)} \underset{\hat{\jmath}}{\hookrightarrow} \widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}\,.$$

Thus, ι_m is clearly injective. Moreover, one easily checks (3.47)–(3.49) (see next section). ☐

3.4. Automorphisms of $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$

$\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ naturally inherits, through $\widehat{\Psi}$, the automorphisms defined over $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ in the previous section.

Proposition 3.14. — *Conjugation by $\widehat{\Psi}$ clearly provides a group isomorphism*

$$\mathrm{Aut}(\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}) \cong \mathrm{Aut}(\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}).$$

In particular, for every $f \in \mathrm{Aut}(\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)})$, we let $\dot{f} = \widehat{\Psi} \circ f \circ \widehat{\Psi}^{-1} \in \mathrm{Aut}(\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)})$.

3.5. Triangular decomposition of $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$

Definition 3.15. — *Let A be a complete topological algebra with closed subalgebras $A^{\pm}$ and A^0. We shall say that (A^-, A^0, A^+) is a* triangular decomposition *of A if the multiplication induces a bicontinuous isomorphism of vector spaces $A^- \widehat{\otimes} A^0 \widehat{\otimes} A^+ \xrightarrow{\sim} A$.*

In order to prove the triangular decomposition of $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$, we shall make use of the following classic

Lemma 3.16. — *Let A be a complete topological algebra with a triangular decomposition (A^-, A^0, A^+). Let $\mathcal{I}^\pm$ be a closed two-sided ideal of $A^\pm$ such that $\mathcal{I}^+.A \subseteq A.\mathcal{I}^+$ and $A.\mathcal{I}^- \subseteq \mathcal{I}^-.A$. Then the quotient algebra $B = A/(A.(\mathcal{I}^+ + \mathcal{I}^-).A)$ admits a triangular decomposition (B^-, A^0, B^+) where $B^\pm$ is the set of equivalence classes of $A^\pm$ in B. Moreover, there exists a bicontinuous algebra isomorphism $B^\pm \cong A^\pm/\mathcal{I}^\pm$.*

Proof. — See e.g. Section 4.21 in [16]. □

Recalling the definitions of $\ddot{\mathrm{U}}_q^\pm(\mathfrak{a}_1)$ and $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$ from Definition 3.1, we have

Proposition 3.17. — $(\ddot{\mathrm{U}}_q^-(\mathfrak{a}_1), \ddot{\mathrm{U}}_q^0(\mathfrak{a}_1), \ddot{\mathrm{U}}_q^+(\mathfrak{a}_1))$ *is a triangular decomposition of $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$ and $\ddot{\mathrm{U}}_q^\pm(\mathfrak{a}_1)$ is bicontinuously isomorphic to the algebra generated by $\{\mathsf{X}^\pm_{1,r,s} : r, s \in \mathbb{Z}\}$ subject to relation* (3.11).

Proof. — Let A be the $\mathbb{F}$-algebra generated by

$$\Big\{\mathsf{D}_1, \mathsf{D}_1^{-1}, \mathsf{D}_2, \mathsf{D}_2^{-1}, \mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}^+_m, \mathsf{c}^-_{-m}, \mathsf{K}^+_{1,0,m}, \mathsf{K}^-_{1,0,-m}, \\ \mathsf{K}^+_{1,n,r}, \mathsf{K}^-_{1,-n,r}, \mathsf{X}^+_{1,r,s}, \mathsf{X}^-_{1,r,s} : m \in \mathbb{N}, n \in \mathbb{N}^\times, r, s \in \mathbb{Z}\Big\} ,$$

subject to the relations (3.2)–(3.10) and (3.12), i.e. all the defining relations of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$ but relation (3.11). Endow A with a topology along the lines of what was done in Section 2.3 for $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$, making use of its $\mathbb{Z}_{(2)}$-grading. This yields a basis $\{\dot{\Omega}_n : n \in \mathbb{N}\}$ of open neighbourhoods of 0. Let furthermore A^0 be the closed subalgebra of A generated by

$$\Big\{\mathsf{D}_1, \mathsf{D}_1^{-1}, \mathsf{D}_2, \mathsf{D}_2^{-1}, \mathsf{C}^{1/2}, \mathsf{C}^{-1/2}, \mathsf{c}^+_m, \mathsf{c}^-_{-m}, \\ \mathsf{K}^+_{1,0,m}, \mathsf{K}^-_{1,0,-m}, \mathsf{K}^+_{1,n,r}, \mathsf{K}^-_{1,-n,r} : m \in \mathbb{N}, n \in \mathbb{N}^\times, r \in \mathbb{Z}\Big\}$$

and $A^\pm$ be the closed subalgebra of A generated by $\{\mathsf{X}^\pm_{1,r,s} : r, s \in \mathbb{Z}\}$. An easy recursion proves that relations (3.9) and (3.10) imply that, for every $N \in \mathbb{N}$ and every $m \in \mathbb{N}$, $l, r, s \in \mathbb{Z}$,

$$\mathsf{X}^+_{1,r,s}\mathsf{K}^+_{1,m,l} - q^2\mathsf{K}^+_{1,m,l}\mathsf{X}^+_{1,r,s} - (q^2 - q^{-2})\sum_{p=1}^{N} q^{2p}\mathsf{K}^+_{1,m,l+p}\mathsf{X}^+_{1,r,s-p} \\ + q^{2N}\mathsf{K}^+_{1,m,l+N+1}\mathsf{X}^+_{1,r,s-N-1} \in \dot{\Omega}_{\nu^+_{s,l}(N)}$$

$$\mathsf{K}^-_{1,-m,l}\mathsf{X}^-_{1,r,s} - q^{-2}\mathsf{X}^-_{1,r,s}\mathsf{K}^-_{1,-m,l} + (q^2-q^{-2})\sum_{p=1}^{N} q^{-2p}\mathsf{X}^-_{1,r,s+p}\mathsf{K}^-_{1,-m,l-p} + q^{2N}\mathsf{X}^-_{1,r,s+N+1}\mathsf{K}^-_{1,-m,l-N-1} \in \dot{\Omega}_{\nu^-_{s,l}(N)}$$

$$\mathsf{K}^+_{1,m,l}\mathsf{X}^-_{1,r,s} - q^{-2}\mathsf{X}^-_{1,r,s}\mathsf{K}^+_{1,m,l} + (q^2-q^{-2})\sum_{p=1}^{N} \mathsf{C}^{-p}q^{2p(m-1)}\mathsf{X}^-_{1,r,s+p}\mathsf{K}^+_{1,m,l-p} + \mathsf{C}^{-(N+1)}q^{2(N+1)(m-1)+2}\mathsf{X}^-_{1,r,s+N+1}\mathsf{K}^+_{1,m,l-N-1} \in \dot{\Omega}_{\nu^-_{s,l}(N)}$$

$$\mathsf{X}^+_{1,r,s}\mathsf{K}^-_{1,-m,l} - q^{2}\mathsf{K}^-_{1,-m,l}\mathsf{X}^+_{1,r,s} - (q^2-q^{-2})\sum_{p=1}^{N} \mathsf{C}^{p}q^{2p(1-m)}\mathsf{K}^-_{1,-m,l+p}\mathsf{X}^+_{1,r,s-p} + \mathsf{C}^{N+1}q^{2(N+1)(1-m)}\mathsf{K}^-_{1,-m,l+N+1}\mathsf{X}^+_{1,r,s-N-1} \in \dot{\Omega}_{\nu^+_{s,l}(N)}$$

where $\nu^{\pm}_{s,l}(N) = \min(\pm l, \mp s)+N+1$. It obviously follows that (A^-, A^0, A^+) is a triangular decomposition of A. Now let $\mathcal{I}^{\pm}$ be the closed two-sided ideal of $A^{\pm}$ generated by

$$\{\mathsf{X}^{\pm}_{1,r,m+1}\mathsf{X}^{\pm}_{1,s,n} - q^{\pm 2}\mathsf{X}^{\pm}_{1,r,m}\mathsf{X}^{\pm}_{1,s,n+1} - q^{\pm 2}\mathsf{X}^{\pm}_{1,s,n}\mathsf{X}^{\pm}_{1,r,m+1} + \mathsf{X}^{\pm}_{1,s,n+1}\mathsf{X}^{\pm}_{1,r,m} : r,s,m,n \in \mathbb{Z}\} \,.$$

Clearly $\ddot{\mathrm{U}}_q(\mathfrak{a}_1) \cong A/(A.(\mathcal{I}^+ + \mathcal{I}^-).A)$. In view of the above rewritings of (3.9) and (3.10), it is clear that $\mathcal{I}^+.A^0 \subseteq A^0.\mathcal{I}^+$ and $A^0.\mathcal{I}^- \subseteq \mathcal{I}^-.A^0$. Moreover, relations (3.9), (3.10) and (3.12) are easily shown to imply that, for every $r, s, t \in \mathbb{Z}$,

$$\left[(v - q^{\pm 2}w)\mathbf{X}^{\pm}_{1,r}(v)\mathbf{X}^{\pm}_{1,s}(w) - (vq^{\pm 2} - w)\mathbf{X}^{\pm}_{1,s}(w)\mathbf{X}^{\pm}_{1,r}(v), \mathbf{X}^{\mp}_{1,t}(u)\right] = 0\,,$$

hence proving that $\mathcal{I}^+.A^- \subseteq A.\mathcal{I}^+$ and $A^+.\mathcal{I}^- \subseteq \mathcal{I}^-.A$. The claim eventually follows as a consequence of Lemma 3.16 □

3.6. Weight-finite highest t-weight modules

DEFINITION 3.18. — *For every $N \in \mathbb{N}^{\times}$, we shall say that a (topological) module M over $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$ is of* type $(1, N)$ *if:*

(i) $C^{\pm 1/2}$ *acts as* id *on M;*
(ii) $\mathsf{c}^{\pm}_{\pm m}$ *acts by multiplication by* 0 *on M, for every $m \geqslant N$.*

We shall say that M is of type $(1,0)$ *if points (i) and (ii) above hold for every $m>0$ and, in addition, $\mathsf{c}_0^{\pm}$ acts as* id *on M.*

Remark 3.19. — Let $N \in \mathbb{N}^{\times}$. Then the $\ddot{\mathrm{U}}_q'(\mathfrak{a}_1)$-modules of type $(1,N)$ are in one-to-one correspondence with the $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N-1)}/(\mathsf{C}^{1/2}-1)$-modules (see Remark 3.4 for a definition of $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(N)}$). Similarly, $\ddot{\mathrm{U}}_q'(\mathfrak{a}_1)$-modules of type $(1,0)$ descend to modules over the double quantum loop algebra of type $\mathfrak{a}_1$, $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)^{(-1)}/(\mathsf{C}^{1/2}-1)$.

In view of the triangular decomposition $(\ddot{\mathrm{U}}_q^-(\mathfrak{a}_1), \ddot{\mathrm{U}}_q^0(\mathfrak{a}_1), \ddot{\mathrm{U}}_q^+(\mathfrak{a}_1))$ of $\widehat{\ddot{\mathrm{U}}_q'(\mathfrak{a}_1)}$ (see Proposition 3.17), we naturally expect that a new, adapted notion of highest weight modules exists, in which $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$, although non-abelian, plays the role usually played by the Cartan subalgebra. Thus, we restrict our attention to modules over $\ddot{\mathrm{U}}_q'(\mathfrak{a}_1)$ which, regarded as $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-modules, split as direct sums of indecomposable modules over $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$. We refer to those summands as *t-weight spaces*. Moreover, the injective algebra homomorphism ι_0 of Proposition 3.13 restricts to an injective algebra homomorphism $\dot{\mathrm{U}}_q^0(\mathfrak{a}_1) \to \ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$ from the quantum Heisenberg subalgebra $\dot{\mathrm{U}}_q^0(\mathfrak{a}_1)$ of $\dot{\mathrm{U}}_q(\mathfrak{a}_1)$ to $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$. Therefore, considering any $\widehat{\ddot{\mathrm{U}}_q'(\mathfrak{a}_1)}$-module M of type $(1,0)$, we get an action of the infinite-dimensional abelian algebra $\dot{\mathrm{U}}_q^0(\mathfrak{a}_1)/(C^{1/2}-1)$ on all the t-weight spaces of M. Whenever the latter decompose into direct sums of generalized eigenspaces of the commuting family of linear operators $\{\mathsf{K}_{1,0,m}^+, \mathsf{K}_{1,0,-m}^- : m \in \mathbb{N}\}$, we shall say that the t-weight-spaces are ℓ-weight. In the latter case, we let $\mathrm{Sp}(M)$ denote the set of all the eigenvalues of $\mathsf{K}_{1,0,0}^+$ over M.

Definition 3.20. — *We shall say that a (topological) $\ddot{\mathrm{U}}_q'(\mathfrak{a}_1)$-module M is a t-*weight *module if there exists a countable set $\{M_\alpha : \alpha \in A\}$ of indecomposable ℓ-weight $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-modules, called the t-*weight spaces *of M, such that, as $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-modules,*

$$M \cong \bigoplus_{\alpha \in A} M_\alpha \,. \tag{3.50}$$

We shall say that M is weight-finite *if, in addition, $\mathrm{Sp}(M)$ is finite. A vector $v \in M - \{0\}$ is a* highest t-weight vector *of M if $v \in M_\alpha$ for some $\alpha \in A$ and, for every $r, s \in \mathbb{Z}$,*

$$\mathsf{X}_{1,r,s}^+ . v = 0 \,. \tag{3.51}$$

We shall say that M is highest t-weight *if $M \cong \ddot{\mathrm{U}}_q'(\mathfrak{a}_1).v$ for some highest t-weight vector $v \in M - \{0\}$.*

It is reasonably clear that, owing to the triangular decomposition

$$(\ddot{\mathrm{U}}_q^-(\mathfrak{a}_1), \ddot{\mathrm{U}}_q^0(\mathfrak{a}_1), \ddot{\mathrm{U}}_q^+(\mathfrak{a}_1))$$

of $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$, for every highest t-weight $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-module M and every highest t-weight vector $v \in M - \{0\}$, we have

$$M \cong \ddot{\mathrm{U}}_q^-(\mathfrak{a}_1).\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1).v\,. \tag{3.52}$$

Remark 3.21. — In view of (3.52), simple highest t-weight $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-modules, including simple weight-finite $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$-modules, are entirely determined as $M \cong \ddot{\mathrm{U}}_q^-(\mathfrak{a}_1).M_0$, by the data of their unique highest t-weight space $M_0 \cong \ddot{\mathrm{U}}_q^0(\mathfrak{a}_1).v$. Classifying simple weight-finite $\ddot{\mathrm{U}}_q(\mathfrak{a}_1)$-modules therefore amounts to classifying those simple $\ddot{\mathrm{U}}_q^0(\mathfrak{a}_1)$-modules that appear as their highest t-weight spaces. We intend to undertake that classification in a future work, see [25].

Remark 3.22. — (3.52) induces a partial ordering of the t-weight spaces through the Q^--grading of $\ddot{\mathrm{U}}_q^-(\mathfrak{a}_1)$.

3.7. Topological Hopf algebra structure on $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$

Definition-Proposition 3.23. — *We define*

$$\dot{\Delta} = \left(\widehat{\Psi}\widehat{\otimes}\widehat{\Psi}\right) \circ \Delta \circ \widehat{\Psi}^{-1}\,, \tag{3.53}$$

$$\dot{S} = \widehat{\Psi} \circ S \circ \widehat{\Psi}^{-1}\,, \tag{3.54}$$

$$\dot{\varepsilon} = \varepsilon \circ \widehat{\Psi}^{-1}\,. \tag{3.55}$$

Equipped with the above comultiplication, antipode and counit, $\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ *is a topological Hopf algebra. The latter is easily extended into a topological Hopf algebraic structure on* $\widehat{\ddot{\mathrm{U}}_q(\mathfrak{a}_1)}$ *by setting, in addition,*

$$\dot{\Delta}(D_1^{\pm 1}) = D_1^{\pm 1} \otimes D_1^{\pm 1}\,, \qquad \dot{S}(D_1^{\pm 1}) = D_1^{\mp 1} \qquad \textit{and} \qquad \dot{\varepsilon}(D_1^{\pm 1}) = 1\,.$$

4. Doubly Affine Damiani–Beck isomorphism

In this last section, we complete the proof of Theorem 3.7 by constructing $\Psi^{-1} : \ddot{\mathrm{U}}'_q(\mathfrak{a}_1) \to \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$; i.e. by constructing a realization of the generators of $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$ in $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$.

4.1. Double loop generators

DEFINITION 4.1. — *For every $m \in \mathbb{Z}$, we set $\mathbf{X}^{\pm}_{1,m}(z) = Y^{\mp m}(\mathbf{x}^{\pm}_1(z))$.*

It is clear that

PROPOSITION 4.2. — *For every $m \in \mathbb{Z}$, we have*

$$\varphi\left(\mathbf{X}^{\pm}_{1,m}(z)\right) = \mathbf{X}^{\mp}_{1,-m}(1/z)\,. \tag{4.1}$$

DEFINITION-PROPOSITION 4.3.

(i) *There exists a unique $\boldsymbol{\psi}^{+}_{1,1}(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z,z^{-1}]]$ such that*

$$\left[Y\left(\mathbf{k}^{-}_1(w)^{-1}\mathbf{x}^{-}_1(C^{1/2}w)\right), \mathbf{x}^{+}_1(z)\right]_{G^{-}_{10}(C^{-1/2}w/z)} = -\delta\left(\frac{C^{-1/2}q^2w}{z}\right)\boldsymbol{\psi}^{+}_{1,1}(z)\,. \tag{4.2}$$

(ii) *Set $\boldsymbol{\psi}^{-}_{1,-1}(z) = \varphi\left(\boldsymbol{\psi}^{+}_{1,1}(1/z)\right)$. Then, we have*

$$\left[\mathbf{x}^{-}_1(z), Y\left(\mathbf{x}^{+}_1(C^{1/2}w)\mathbf{k}^{+}_1(w)^{-1}\right),\right]_{G^{+}_{10}(C^{1/2}z/w)} = -\delta\left(\frac{C^{-1/2}q^2w}{z}\right)\boldsymbol{\psi}^{-}_{1,-1}(z)\,. \tag{4.3}$$

Proof. — The proof of (i) is immediate from the definitions. (ii) then follows by applying φ to (4.2). □

Remark 4.4. — It is worth noting that $\boldsymbol{\psi}^{\pm}_{1,\pm 1}(z) \notin \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)[[z,z^{-1}]]$.

COROLLARY 4.5. — *For every $i \in \dot{I}$, we have*

(i) $\mathbf{k}^{-}_i(v)\boldsymbol{\psi}^{\pm}_{1,\pm 1}(z) = G^{\mp}_{i,0}(C^{\mp 1/2}q^2v/z)G^{\mp}_{i,1}(C^{\mp 1/2}v/z)\boldsymbol{\psi}^{\pm}_{1,\pm 1}(z)\mathbf{k}^{-}_i(v)$;

(ii) $\boldsymbol{\psi}^{\pm}_{1,\pm 1}(z)\mathbf{k}^{+}_i(v) = G^{\mp}_{i,0}(C^{\mp 1/2}q^{-2}z/v)G^{\mp}_{i,1}(C^{\mp 1/2}z/v)\mathbf{k}^{+}_i(v)\boldsymbol{\psi}^{\pm}_{1,\pm 1}(z)$;

Proof. — (ii) follows by applying φ to (i) and (i) is a direct consequence of (4.2) and (4.3) on one hand and of (2.6) and (2.7) on the other hand. □

Let us then define the following $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$-valued formal power series

$$\boldsymbol{\Gamma}^{\pm}_0(z) = \mathbf{k}^{\pm}_0(z)\mathbf{k}^{\pm}_1(z) \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)[[z^{\mp 1}]]\,. \tag{4.4}$$

Denoting by $\mathcal{Z}(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1))$ the center of $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$, it is straightforward to check that indeed

Proposition 4.6. — $\mathbf{\Gamma}_0^{\pm}(z) \in \mathcal{Z}(\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1))[[z^{\mp 1}]]$.

Similarly, define

$$\wp^{\pm}(z) = \mathbf{k}_0^{\pm}(z)\mathbf{k}_1^{\pm}(zq^2) \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)[[z^{\mp 1}]]\,. \tag{4.5}$$

Then we establish an important result.

Proposition 4.7. — *We have the following fixed points of Y;*

$$Y\left(\wp^{\pm}(z)\right) = \wp^{\pm}(z)\,, \tag{4.6}$$

$$Y\left(\boldsymbol{\psi}_{1,\pm 1}^{\pm}(z)\right) = \boldsymbol{\psi}_{1,\pm 1}^{\pm}(z)\,. \tag{4.7}$$

Moreover

$$Y\left(\mathbf{\Gamma}_0^{\pm}(z)\right) = \mathbf{\Gamma}_0^{\pm}(zq^2)\,, \tag{4.8}$$

Proof. — (4.6) and (4.8) are obvious. We prove (4.7) for the upper choice of signs. In order to do so, we first rewrite (4.2) as

$$\left[\mathbf{x}_0^+(w), \mathbf{x}_1^+(z)\right]_{G_{10}^-(w/z)} = \delta\left(\frac{q^2 w}{z}\right)\boldsymbol{\psi}_{1,1}^+(z)\,.$$

Now, (2.41) and the definition of Y imply that, on one hand,

$$\begin{aligned}
&\left[\left[\mathbf{x}_0^+(z_1), \left[\mathbf{x}_0^+(z_2), \mathbf{x}_1^+(wq^2)\right]_{G_{10}^-(z_2/wq^2)}\right]_{G_{11}^-(z_1/z_2)G_{10}^-(z_1/wq^2)},\right.\\
&\hspace{14em}\left.\mathbf{x}_0^-(C^{-1}z)\mathbf{k}_0^+(C^{-1/2}z)^{-1}\right]_{G_{10}^-(w/z)}\\
&\quad= -\left[2\right]_q \delta\left(\frac{z_1}{z_2 q^2}\right)\delta\left(\frac{z_2}{w}\right)\left[Y\left(\mathbf{x}_0^+(w)\right), Y\left(\mathbf{x}_1^+(z)\right)\right]_{G_{10}^-(w/z)}\\
&\quad= -\left[2\right]_q \delta\left(\frac{z_1}{z_2 q^2}\right)\delta\left(\frac{z_2}{w}\right) Y\left(\left[\mathbf{x}_0^+(w), \mathbf{x}_1^+(z)\right]_{G_{10}^-(w/z)}\right)\\
&\quad= -\left[2\right]_q \delta\left(\frac{z_1}{z_2 q^2}\right)\delta\left(\frac{z_2}{w}\right)\delta\left(\frac{wq^2}{z}\right) Y\left(\boldsymbol{\psi}_{1,1}^+(z)\right)\,,
\end{aligned}$$

whereas, on the other hand, (2.41), (2.6), (2.7) and (2.9), as well as Corollary 4.5, imply that

$$\Big[\Big[\mathbf{x}_0^+(z_1), \big[\mathbf{x}_0^+(z_2), \mathbf{x}_1^+(wq^2)\big]_{G_{10}^-(z_2/wq^2)}\Big]_{G_{11}^-(z_1/z_2)G_{10}^-(z_1/wq^2)}, \\ \mathbf{x}_0^-(C^{-1}z)\mathbf{k}_0^+(C^{-1/2}z)^{-1}\Big]_{G_{10}^-(w/z)}$$

$$= \Big[\Big[\mathbf{x}_0^+(z_1), \big[\mathbf{x}_0^+(z_2), \mathbf{x}_1^+(wq^2)\big]_{G_{10}^-(z_2/wq^2)}\Big]_{G_{11}^-(z_1/z_2)G_{10}^-(z_1/wq^2)}, \\ \mathbf{x}_0^-(C^{-1}z)\Big]\,\mathbf{k}_0^+(C^{-1/2}z)^{-1}$$

$$= \Big\{\delta\left(\frac{z_1}{z}\right)\delta\left(\frac{z_2}{w}\right)\Big[\mathbf{k}_0^+(z_1C^{-1/2}), \boldsymbol{\psi}_{1,1}^+(wq^2)\Big]_{G_{11}^-(z_1/z_2)G_{10}^-(z_1/wq^2)} \\ + \delta\left(\frac{z_2}{z}\right)\Big[\mathbf{x}_0^+(z_1), \Big[\mathbf{k}_0^+(z_2C^{-1/2}), \mathbf{x}_1^+(wq^2)\Big]_{G_{10}^-(z_2/wq^2)}\Big]_{G_{11}^-(z_1/z_2)G_{10}^-(z_1/wq^2)}\Big\}$$

$$= \frac{1}{q-q^{-1}}\delta\left(\frac{z_1}{z}\right)\delta\left(\frac{z_2}{w}\right)\big[G_{00}^+(w/z_1)G_{01}^+(q^2w/z_1) \\ -G_{11}^-(z_1/w)G_{10}^-(z_1/wq^2)\big]\,\boldsymbol{\psi}_{1,1}^+(wq^2) \\ + \delta\left(\frac{z_2}{z}\right)\delta\left(\frac{z_1}{w}\right)\frac{G_{01}^+(q^2w/z_2) - G_{10}^-(z_2/wq^2)}{q-q^{-1}}\boldsymbol{\psi}_{1,1}^+(wq^2)\,.$$

Making use of (2.17) and (A.5), (for the latter, see Appendix), we eventually get

$$\Big[\Big[\mathbf{x}_0^+(z_1), \big[\mathbf{x}_0^+(z_2), \mathbf{x}_1^+(wq^2)\big]_{G_{10}^-(z_2/wq^2)}\Big]_{G_{11}^-(z_1/z_2)G_{10}^-(z_1/wq^2)}, \\ \mathbf{x}_0^-(C^{-1}z)\mathbf{k}_0^+(C^{-1/2}z)^{-1}\Big]_{G_{10}^-(w/z)}$$

$$= [2]_q\,\delta\left(\frac{z_1}{z}\right)\delta\left(\frac{z_2}{w}\right)\left[\delta\left(\frac{w}{z_1}\right) - \delta\left(\frac{wq^2}{z_1}\right)\right]\boldsymbol{\psi}_{1,1}^+(wq^2) \\ - [2]_q\,\delta\left(\frac{z_2}{z}\right)\delta\left(\frac{z_1}{w}\right)\delta\left(\frac{w}{z_2}\right)\boldsymbol{\psi}_{1,1}^+(wq^2)$$

$$= -[2]_q\,\delta\left(\frac{z_1}{z}\right)\delta\left(\frac{z_2}{w}\right)\delta\left(\frac{wq^2}{z_1}\right)\boldsymbol{\psi}_{1,1}^+(z)\,,$$

thus proving the result. The case with lower choice of signs follows by applying φ. □

PROPOSITION 4.8. — *For every $m \in \mathbb{Z}$, we have*

(i) $\left[\boldsymbol{\psi}^+_{1,1}(z), \mathbf{X}^-_{1,m}(v)\right] = -[2]_q \delta\left(\frac{Cz}{v}\right) \wp^-(C^{1/2}q^{-2}z)\mathbf{X}^-_{1,m+1}(Cq^{-2}z)$;

(ii) $\left[\boldsymbol{\psi}^+_{1,1}(z), \mathbf{X}^+_{1,m}(v)\right]_{G^-_{10}(z/vq^2)G^-_{11}(z/v)} = [2]_q \delta\left(\frac{z}{vq^2}\right) \mathbf{X}^+_{1,m+1}(z)$;

(iii) $\left[\boldsymbol{\psi}^-_{1,-1}(z), \mathbf{X}^+_{1,-m}(v)\right] = [2]_q \delta\left(\frac{Cz}{v}\right) \mathbf{X}^+_{1,-(m+1)}(Cq^{-2}z) \wp^+(C^{1/2}q^{-2}z)$;

(iv) ${}_{G^+_{10}(vq^2/z)G^+_{11}(v/z)}\left[\boldsymbol{\psi}^-_{1,-1}(z), \mathbf{X}^-_{1,-m}(v)\right] = -[2]_q \delta\left(\frac{z}{vq^2}\right) \mathbf{X}^-_{1,-(m+1)}(z)$;

(v)

$$\left[\boldsymbol{\psi}^+_{1,1}(z), \boldsymbol{\psi}^-_{1,-1}(v)\right] = \frac{[2]_q}{q-q^{-1}} \left[\delta\left(\frac{z}{Cv}\right) \wp^+(C^{-1/2}q^{-2}z) - \delta\left(\frac{Cz}{v}\right) \wp^-(C^{-1/2}q^{-2}v)\right].$$

Proof. — (i) and (ii) are readily checked for $m = 0$. Then, assuming they hold for some $m \in \mathbb{Z}$ and applying $Y^{\pm 1}$, it follows from propositon 4.7 that they also hold for $m \pm 1$. (iii) and (iv) are obtained by applying φ to (i) and (ii) respectively. Finally (v) is obtained by direct calculation from the definitions of $\boldsymbol{\psi}^+_{1,1}(z)$ and $\boldsymbol{\psi}^-_{1,-1}(v)$, i.e.

$$\begin{aligned}
&\delta\left(\frac{C^{-1/2}q^2 w}{z}\right) \delta\left(\frac{C^{1/2}q^{-2}u}{v}\right) \left[\boldsymbol{\psi}^+_{1,1}(z), \boldsymbol{\psi}^-_{1,-1}(u)\right] \\
&= \left[\left[\mathbf{x}^+_0(C^{-1/2}w), \mathbf{x}^+_1(z)\right]_{G^-_{10}(C^{-1/2}w/z)}, \left[\mathbf{x}^-_1(u), \mathbf{x}^-_0(C^{-1/2}v)\right]_{G^+_{10}(C^{1/2}u/v)}\right] \\
&= [2]_q \delta\left(\frac{C^{-1/2}v}{uq^{-2}}\right) \Bigg\{ \delta\left(\frac{z}{Cu}\right) \Big[\mathbf{x}^+_0(C^{-1/2}w), \\
&\qquad \mathbf{x}^-_0(C^{-1/2}v)\mathbf{k}^+_1(C^{-1/2}z)\Big]_{G^-_{10}(C^{-1/2}w/z)} \\
&\qquad\qquad -\delta\left(\frac{Cw}{v}\right) \left[\mathbf{k}^-_0(C^{-1}v)\mathbf{x}^-_1(u), \mathbf{x}^+_1(z)\right]_{G^-_{10}(C^{-1/2}w/z)} \Bigg\} \\
&= \frac{[2]_q}{q-q^{-1}} \delta\left(\frac{C^{1/2}u}{vq^2}\right) \delta\left(\frac{C^{-1/2}w}{zq^{-2}}\right) \Bigg\{ \delta\left(\frac{z}{Cu}\right) \mathbf{k}^+_0(C^{-1}w)\mathbf{k}^+_1(C^{-1/2}z) \\
&\qquad\qquad -\delta\left(\frac{Cz}{u}\right) \mathbf{k}^-_0(C^{-1}v)\mathbf{k}^-_1(C^{-1/2}u) \Bigg\}.
\end{aligned}$$

Compare with (4.5) to conclude the proof. □

DEFINITION-PROPOSITION 4.9. — *For every $m \in \mathbb{N}^\times$ there exist*

$$\boldsymbol{\psi}^+_{1,m}(z), \boldsymbol{\Gamma}^+_m(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z, z^{-1}]],$$

such that

$$\boldsymbol{\Gamma}^+_1(v) = 0 \tag{4.9}$$

and, for every $m, n \in \mathbb{N}^\times$,

$$(4.10)\quad \left[Y^m\left(\mathbf{k}_1^-(z)^{-1}\mathbf{x}_1^-(C^{1/2}z)\right), \mathbf{x}_1^+(v)\right]_{G_{01}^-\left(z/C^{1/2}v\right)} = -\delta\left(\frac{z}{C^{1/2}v}\right)\boldsymbol{\Gamma}_m^+(v) + (q-q^{-1})\sum_{k=1}^{m-2}\delta\left(\frac{q^{2k}z}{C^{1/2}v}\right)\boldsymbol{\psi}_{1,k}^+(v)\boldsymbol{\Gamma}_{m-k}^+(v) - \delta\left(\frac{q^{2m}z}{C^{1/2}v}\right)\boldsymbol{\psi}_{1,m}^+(v)\,,$$

$$(4.11)\quad Y\left(\boldsymbol{\psi}_{1,m}^+(v)\right) = \boldsymbol{\psi}_{1,m}^+(v)\,,$$

$$(4.12)\quad Y\left(\boldsymbol{\Gamma}_m^+(v)\right) = \boldsymbol{\Gamma}_m^+(vq^2)\,,$$

$$(4.13)\quad {}_{G_{01}^-(q^{-2m}v/w)G_{11}^-(q^{2(1-m)}v/w)}\left[\boldsymbol{\psi}_{1,1}^+(w), \boldsymbol{\psi}_{1,m}^+(v)\right]_{G_{01}^-(w/vq^2)G_{11}^-(w/v)} = [2]_q\delta\left(\frac{w}{vq^2}\right)\boldsymbol{\psi}_{1,m+1}^+(q^2v) - [2]_q\delta\left(\frac{q^{2m}w}{v}\right)\boldsymbol{\psi}_{1,m+1}^+(v)\,,$$

$$(4.14)\quad [\boldsymbol{\psi}_{1,n}^+(w), \boldsymbol{\Gamma}_m^+(v)] = 0\,.$$

Proof. — It suffices to prove the proposition with $n = 1$ since the general case follows by an easy recursion on n once we have (4.13). The proof for $n = 1$ is by recursion on m. For $m = 1$, (4.9) and (4.10) are definition-Proposition (i), whereas (4.11) is Proposition 4.7. (4.12) and (4.14), with $n = 1$, automatically follow from (4.9). Making use of Proposition 4.8, it is straightforward to prove that, for every $m \in \mathbb{N}^\times$,

$$(4.15)\quad [2]_q\delta\left(\frac{z}{uq^2}\right)Y^{-1}\left(\left[Y^{m+1}\left(\mathbf{k}_1^-(C^{-1/2}v)^{-1}\mathbf{x}_1^-(v)\right), \mathbf{x}_1^+(uq^2)\right]_{G_{01}^-(C^{-1}q^{-2}v/u)}\right) - [2]_q\delta\left(\frac{Cz}{v}\right)\left[Y^{m+1}\left(\mathbf{k}_1^-(C^{1/2}q^{-2}z)^{-1}\mathbf{x}_1^-(Cq^{-2}z)\right), \mathbf{x}_1^+(u)\right]_{G_{01}^-(z/uq^2)} = {}_{G_{10}^-(v/Cz)G_{11}^-(vq^2/Cz)}\left[\boldsymbol{\psi}_{1,1}^+(z), \left[Y^m\left(\mathbf{k}_1^-(C^{-1/2}v)^{-1}\mathbf{x}_1^-(v)\right), \mathbf{x}_1^+(u)\right]_{G_{01}^-(C^{-1}v/u)}\right]_{G_{10}^-(z/uq^2)G_{11}^-(z/u)}\,.$$

If $m=1$, (4.13) is an easy consequence of the above equation. Now assume that the proposition holds up to some $m \in \mathbb{N}^\times$. Then (4.15) reads, for that m,

$$
\begin{aligned}
(4.16)\quad & [2]_q \delta\left(\frac{z}{uq^2}\right) Y^{-1}\left(\left[Y^{m+1}\left(\mathbf{k}_1^-(C^{-1/2}v)^{-1}\mathbf{x}_1^-(v)\right),\right.\right. \\
&\qquad\qquad \left.\left.\mathbf{x}_1^+(uq^2)\right]_{G_{01}^-(C^{-1}q^{-2}v/u)}\right) \\
&- [2]_q \delta\left(\frac{Cz}{v}\right)\left[Y^{m+1}\left(\mathbf{k}_1^-(C^{1/2}q^{-2}z)^{-1}\mathbf{x}_1^-(Cq^{-2}z)\right), \mathbf{x}_1^+(u)\right]_{G_{01}^-(z/uq^2)} \\
= & -\delta\left(\frac{v}{Cu}\right)_{G_{10}^-(v/Cz)G_{11}^-(vq^2/Cz)}\left[\boldsymbol{\psi}_{1,1}^+(z), \boldsymbol{\Gamma}_m^+(u)\right]_{G_{10}^-(z/uq^2)G_{11}^-(z/u)} \\
&+ (q-q^{-1})\sum_{k=1}^{m-2}\delta\left(\frac{q^{2k}v}{Cu}\right)_{G_{10}^-(v/Cz)G_{11}^-(vq^2/Cz)}\left[\boldsymbol{\psi}_{1,1}^+(z),\right. \\
&\qquad\qquad \left.\boldsymbol{\psi}_{1,k}^+(u)\right]_{G_{10}^-(z/uq^2)G_{11}^-(z/u)}\boldsymbol{\Gamma}_{m-k}^+(u) \\
&- \delta\left(\frac{q^{2m}v}{Cu}\right)_{G_{10}^-(v/Cz)G_{11}^-(vq^2/Cz)}\left[\boldsymbol{\psi}_{1,1}^+(z), \boldsymbol{\psi}_{1,m}^+(u)\right]_{G_{10}^-(z/uq^2)G_{11}^-(z/u)} \\
= & -[2]_q(q-q^{-1})\delta\left(\frac{v}{Cu}\right)\left\{\delta\left(\frac{v}{Cz}\right) - \delta\left(\frac{vq^2}{Cz}\right)\right\}\boldsymbol{\psi}_{1,1}^+(z)\boldsymbol{\Gamma}_m^+(u) \\
&+ [2]_q(q-q^{-1})\sum_{k=1}^{m-2}\delta\left(\frac{q^{2k}v}{Cu}\right)\left\{\delta\left(\frac{z}{uq^2}\right)\boldsymbol{\psi}_{1,k+1}^+(uq^2)\right. \\
&\qquad\qquad \left. -\delta\left(\frac{zq^{2k}}{u}\right)\boldsymbol{\psi}_{k+1}^+(u)\right\}\boldsymbol{\Gamma}_{m-k}^+(u) \\
&- [2]_q\delta\left(\frac{q^{2m}v}{Cu}\right)\left\{\delta\left(\frac{z}{uq^2}\right)\boldsymbol{\psi}_{1,m+1}^+(uq^2) - \delta\left(\frac{zq^{2m}}{u}\right)\boldsymbol{\psi}_{1,m+1}^+(u)\right\}.
\end{aligned}
$$

It immediately follows that (4.10) holds at rank $m+1$, for some $\boldsymbol{\Gamma}_{m+1}^+(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z,z^{-1}]]$ satisfying (4.12). Considering (4.15) at rank $m+1$, and substituting the above results, we get

$$[2]_q\delta\left(\frac{z}{uq^2}\right)Y^{-1}\left(\Big[Y^{m+2}\left(\mathbf{k}_1^-(C^{-1/2}v)^{-1}\mathbf{x}_1^-(v)\right),\right.$$
$$\left.\mathbf{x}_1^+(uq^2)\Big]_{G_{01}^-(C^{-1}q^{-2}v/u)}\right)$$
$$-[2]_q\delta\left(\frac{Cz}{v}\right)\Big[Y^{m+2}\left(\mathbf{k}_1^-(C^{1/2}q^{-2}z)^{-1}\mathbf{x}_1^-(Cq^{-2}z)\right),\mathbf{x}_1^+(u)\Big]_{G_{01}^-(z/uq^2)}$$
$$=-\delta\left(\frac{v}{Cu}\right)_{G_{10}^-(v/Cz)G_{11}^-(vq^2/Cz)}\big[\boldsymbol{\psi}_{1,1}^+(z),\boldsymbol{\Gamma}_{m+1}^+(u)\big]_{G_{10}^-(z/uq^2)G_{11}^-(z/u)}$$
$$+[2]_q(q-q^{-1})\sum_{k=1}^{m-1}\delta\left(\frac{q^{2k}v}{Cu}\right)\left\{\delta\left(\frac{z}{uq^2}\right)\boldsymbol{\psi}_{1,k+1}^+(uq^2)\right.$$
$$\left.-\delta\left(\frac{zq^{2k}}{u}\right)\boldsymbol{\psi}_{k+1}^+(u)\right\}\boldsymbol{\Gamma}_{m+1-k}^+(u)$$
$$-\delta\left(\frac{q^{2(m+1)}v}{Cu}\right)_{G_{10}^-(v/Cz)G_{11}^-(vq^2/Cz)}\big[\boldsymbol{\psi}_{1,1}^+(z),$$
$$\boldsymbol{\psi}_{1,m+1}^+(u)\big]_{G_{10}^-(z/uq^2)G_{11}^-(z/u)}.$$

It readily follows that, on one hand, there exists some

$$\boldsymbol{\psi}_{1,m+2}^+(v)\in\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[v,v^{-1}]]$$

such that (4.13) holds for $m+1$ and that, on the other hand,

$$(uq^2-z)(u-z)\left[\boldsymbol{\psi}_{1,1}^+(z),\boldsymbol{\Gamma}_{m+1}^+(u)\right]=0\,.$$

Since $Y(\boldsymbol{\Gamma}_{m+1}^+(u))=\boldsymbol{\Gamma}_{m+1}^+(uq^2)$, we have that

$$(uq^{2(p+1)}-z)(uq^{2p}-z)\left[\boldsymbol{\psi}_{1,1}^+(z),\boldsymbol{\Gamma}_{m+1}^+(u)\right]=0$$

for every $p\in\mathbb{Z}$ and, as a consequence, (4.14) holds for $m+1$. Finally, (4.11) for $m+1$ follows from the corresponding case of (4.13), which concludes the proof. □

Remark 4.10. — Note that since $[\boldsymbol{\psi}_{1,n}^+(z),\boldsymbol{\Gamma}_m^+(v)]=0$ for every $m,n\in\mathbb{N}^\times$, we have that

$$\boldsymbol{\psi}_{1,n,k}^+\boldsymbol{\Gamma}_{m,l}^+=\boldsymbol{\Gamma}_{m,l}^+\boldsymbol{\psi}_{1,n,k}^+\in\Omega_{l-k}\cap\Omega_{k-l}\,,\tag{4.17}$$

guaranteeing the convergence in $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$ of each of the terms of the series $\boldsymbol{\psi}_{1,k}^+(z)\boldsymbol{\Gamma}_{m-k}^+(z)$ on the r.h.s of (4.10).

Definition 4.11. — *For every $m\in\mathbb{N}^\times$, let*

$$\boldsymbol{\Gamma}_{-m}^-(z)=\varphi(\boldsymbol{\Gamma}_m^+(1/z))\qquad\textit{and}\qquad\boldsymbol{\psi}_{1,-m}^-(z)=\varphi(\boldsymbol{\psi}_{1,m}^+(1/z))\,.\tag{4.18}$$

Then,

Corollary 4.12. — *We have*

$$\boldsymbol{\Gamma}^-_{-1}(v) = 0 \tag{4.19}$$

and, for every $m, n \in \mathbb{N}^\times$,

$$\begin{aligned}[t] &\Big[\mathbf{x}_1^-(v), Y^m\left(\mathbf{x}_1^+(C^{1/2}z)\mathbf{k}_1^+(z)^{-1}\right)\Big]_{G^+_{01}(C^{1/2}v/z)} \\ &\quad = -\delta\left(\frac{z}{C^{1/2}v}\right)\boldsymbol{\Gamma}^-_{-m}(v) \\ &\qquad - (q-q^{-1})\sum_{k=1}^{m-2}\delta\left(\frac{q^{2k}z}{C^{1/2}v}\right)\boldsymbol{\Gamma}^-_{-(m-k)}(v)\boldsymbol{\psi}^-_{1,-k}(v) \\ &\qquad - \delta\left(\frac{q^{2m}z}{C^{1/2}v}\right)\boldsymbol{\psi}^-_{1,-m}(v)\,, \end{aligned} \tag{4.20}$$

$$Y\left(\boldsymbol{\psi}^-_{1,-m}(v)\right) = \boldsymbol{\psi}^-_{1,-m}(v)\,, \tag{4.21}$$

$$Y\left(\boldsymbol{\Gamma}^-_{-m}(v)\right) = \boldsymbol{\Gamma}^-_{-m}(vq^2)\,, \tag{4.22}$$

$$\begin{aligned}[t] &{}_{G^+_{01}(q^{2m}w/v)G^+_{11}(q^{2(m-1)}w/v)}\Big[\boldsymbol{\psi}^-_{1,-m}(v), \boldsymbol{\psi}^-_{1,-1}(w)\Big]_{G^+_{01}(vq^2/w)G^+_{11}(v/w)} \\ &\quad = [2]_q\delta\left(\frac{w}{vq^2}\right)\boldsymbol{\psi}^-_{1,-(m+1)}(q^2v) - [2]_q\delta\left(\frac{q^{2m}w}{v}\right)\boldsymbol{\psi}^-_{1,-(m+1)}(v)\,, \end{aligned} \tag{4.23}$$

$$[\boldsymbol{\psi}^-_{1,-n}(w), \boldsymbol{\Gamma}^-_{-m}(v)] = 0\,. \tag{4.24}$$

Proof. — It suffice to apply φ to the results of the previous proposition. □

Proposition 4.13. — *For every* $i \in \dot{I}$ *and for every* $m \in \mathbb{N}^\times$, *we have*

(i) $\mathbf{k}_i^-(v)\boldsymbol{\psi}^\pm_{1,\pm m}(z)$

$$= G^\mp_{i,0}(C^{\mp 1/2}q^{2m}v/z)G^\mp_{i,1}(C^{\mp 1/2}v/z)\boldsymbol{\psi}^\pm_{1,\pm m}(z)\mathbf{k}_i^-(v)\,;$$

(ii) $\boldsymbol{\psi}^\pm_{1,\pm m}(z)\mathbf{k}_i^+(v)$

$$= G^\mp_{i,0}(C^{\mp 1/2}q^{-2m}z/v)G^\mp_{i,1}(C^{\mp 1/2}z/v)\mathbf{k}_i^+(v)\boldsymbol{\psi}^\pm_{1,\pm m}(z)\,.$$

Proof. — Clearly (ii) follows by applying φ to (i). We prove (ii) by induction on $m \in \mathbb{N}^\times$. The case $m = 1$ is Corollary 4.5(i). Now, assuming that (i) holds for some $m \in \mathbb{N}^\times$, we can make use of (4.13) and (4.23) to show that

$$\begin{aligned}
&\mathbf{k}_i^-(v)\boldsymbol{\psi}^\pm_{1,\pm(m+1)}(z)\\
&\qquad = G^\mp_{i,0}(C^{\mp 1/2}q^{2(m+1)}v/z)G^\mp_{i,1}(C^{\mp 1/2}q^{2m}v/z)\\
&\qquad\qquad \times G^\mp_{i,0}(C^{\mp 1/2}q^{2m}v/z)G^\mp_{i,1}(C^{\mp 1/2}z/v)\boldsymbol{\psi}^\pm_{1,\pm m}(z)\mathbf{k}_i^-(v)\\
&\qquad = G^\mp_{i,0}(C^{\mp 1/2}q^{2(m+1)}v/z)G^\mp_{i,1}(C^{\mp 1/2}z/v)\boldsymbol{\psi}^\pm_{1,\pm m}(z)\mathbf{k}_i^-(v)
\end{aligned}$$

which completes the recursion. □

The above proposition has the obvious

Corollary 4.14. — *For every $m \in \mathbb{N}^\times$, we have*

$$\begin{aligned}
(4.25)\quad &\wp^-(v)\boldsymbol{\psi}^\pm_{1,\pm m}(z) = G^\mp_{00}(C^{\mp 1/2}q^{2m}v/z)G^\mp_{01}(C^{\mp 1/2}v/z)\\
&\qquad \times G^\mp_{01}(C^{\mp 1/2}q^{2(m+1)}v/z)G^\mp_{11}(C^{\mp 1/2}q^{2}v/z)\boldsymbol{\psi}^\pm_{1,\pm m}(z)\wp^-(v)\,;\\
(4.26)\quad &\boldsymbol{\psi}^\pm_{1,\pm m}(z)\wp^+(v) = G^\mp_{00}(C^{\mp 1/2}q^{-2m}z/v)G^\mp_{01}(C^{\mp 1/2}z/v)\\
&\qquad \times G^\mp_{01}(C^{\mp 1/2}q^{-2(m+1)}z/v)G^\mp_{11}(C^{\mp 1/2}q^{-2}z/v)\wp^+(v)\boldsymbol{\psi}^\pm_{1,\pm m}(z)\,.
\end{aligned}$$

Proposition 4.15. — *For every $m, n \in \mathbb{N}^\times$, we have*

$$\begin{aligned}
&\left[\boldsymbol{\psi}^+_{1,m}(v), \boldsymbol{\psi}^-_{1,-n}(w)\right]\\
&\quad = [2]_q\,(q-q^{-1})\left\{\delta\left(\frac{Cv}{wq^{2(m-1)}}\right)\wp^-(C^{-1/2}q^{-2m}v)\right.\\
&\qquad\qquad\qquad\qquad \times\boldsymbol{\psi}^-_{1,-(n-1)}(wq^{-2})\boldsymbol{\psi}^+_{1,m-1}(v)\\
&\qquad\quad \left.-\delta\left(\frac{q^{2(n-1)}v}{Cw}\right)\boldsymbol{\psi}^-_{1,-(n-1)}(w)\boldsymbol{\psi}^+_{1,m-1}(vq^{-2})\wp^+(C^{1/2}q^{-2}v)\right\}\,,
\end{aligned}$$

where we assume that

$$\boldsymbol{\psi}^\pm_{1,0}(z) = \frac{1}{q-q^{-1}}\,. \tag{4.27}$$

Proof. — The case $m = n = 1$ follows immediately by Proposition 4.8(v). Now, applying $a \mapsto [a, \boldsymbol{\psi}^-_{1,-n}(w)]$ and $a \mapsto [\boldsymbol{\psi}^+_{1,n}(w), a]$ to (4.13) and (4.23) respectively and making use of Corollary 4.14, one easily completes the recursion. □

4.2. Exchange relations

Proposition 4.16. — *For every $m \in \mathbb{N}$, there exists some $\xi_m(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z, z^{-1}]]$ such that, for every $n \in \mathbb{Z}$,*

$$[\mathbf{X}^-_{1,m+n+1}(w), \mathbf{X}^-_{1,n}(z)]_{G^-_{01}(w/z)} = -[\mathbf{X}^-_{1,n+1}(w), \mathbf{X}^-_{1,m+n}(z)]_{G^-_{01}(w/z)} = \delta\left(\frac{wq^2}{z}\right) Y^n\left(\xi_m(z)\right) . \tag{4.28}$$

Proof. — Assume first that $n = 0$. The case $m = 0$ then follows immediately from the definition of $\mathbf{X}^-_{1,1}(w)$ and relations (2.7) and (2.9), leading to $\xi_0(z) = 0$, as it should. Taking the commutator of the case $m = 0$ with $\boldsymbol{\psi}^+_{1,1}(v)$, we get

$$\begin{aligned}
0 &= [[\mathbf{X}^-_{1,1}(w), \mathbf{X}^-_{1,0}(z)]_{G^-_{01}(w/z)}, \boldsymbol{\psi}^+_{1,1}(v)]\\
&= [[\mathbf{X}^-_{1,1}(w), \boldsymbol{\psi}^+_{1,1}(v)], \mathbf{X}^-_{1,0}(z)]_{G^-_{01}(w/z)}\\
&\qquad + [\mathbf{X}^-_{1,1}(w), [\mathbf{X}^-_{1,0}(z), \boldsymbol{\psi}^+_{1,1}(v)]]_{G^-_{01}(w/z)}\\
&= [2]_q \wp^-(v) \Bigg\{ \delta\left(\frac{C^{1/2}q^2 v}{w}\right) [\mathbf{X}^-_{1,2}(wq^{-2}), \mathbf{X}^-_{1,0}(z)]_{G^-_{01}(wq^{-2}/z)}\\
&\qquad + \delta\left(\frac{C^{1/2}q^2 v}{z}\right) {}_{G^-_{01}(zq^{-2}/w) G^-_{11}(z/w)}[\mathbf{X}^-_{1,1}(w), \mathbf{X}^-_{1,1}(zq^{-2})]_{G^-_{01}(w/z)} \Bigg\} .
\end{aligned}$$

The latter implies that

$$[\mathbf{X}^-_{1,2}(wq^{-2}), \mathbf{X}^-_{1,0}(z)]_{G^-_{01}(wq^{-2}/z)} = \delta\left(\frac{w}{z}\right) \xi_1(z) , \tag{4.29}$$

$${}_{G^-_{01}(zq^{-2}/w) G^-_{11}(z/w)}[\mathbf{X}^-_{1,1}(w), \mathbf{X}^-_{1,1}(zq^{-2})]_{G^-_{01}(w/z)} = -\delta\left(\frac{w}{z}\right) \xi_1(z) , \tag{4.30}$$

for some $\xi_1(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z, z^{-1}]]$. Multiplying (4.30) by $(zq^{-2} - w)$ and subsequently factoring $(z - q^{-2}w)$, we get that

$${}_{G^-_{01}(zq^{-2}/w)}[\mathbf{X}^-_{1,1}(w), \mathbf{X}^-_{1,1}(zq^{-2})] = \delta\left(\frac{w}{z}\right) \xi_1(z) + \delta\left(\frac{w}{zq^2}\right) \eta_0(z) , \tag{4.31}$$

for some $\eta_0(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z, z^{-1}]]$. Multiplying the above equation by $q^{-2}(z - w)$, we get

$$(zq^{-4} - w)\mathbf{X}^-_{1,1}(w)\mathbf{X}^-_{1,1}(zq^{-2}) - q^{-2}(z - w)\mathbf{X}^-_{1,1}(zq^{-2})\mathbf{X}^-_{1,1}(w) = z(1 - q^2)\delta\left(\frac{w}{zq^2}\right) \eta_0(z) . \tag{4.32}$$

But, on the other hand,

$$(zq^{-4}-w)\mathbf{X}^-_{1,1}(w)\mathbf{X}^-_{1,1}(zq^{-2})-q^{-2}(z-w)\mathbf{X}^-_{1,1}(zq^{-2})\mathbf{X}^-_{1,1}(w)$$
$$=Y\left((zq^{-4}-w)\mathbf{x}^-_1(w)\mathbf{x}^-_1(zq^{-2})-q^{-2}(z-w)\mathbf{x}^-_1(zq^{-2})\mathbf{x}^-_1(w)\right)=0$$

by relation (2.8). Substituting back into (4.32) proves that $\eta_0(z)=0$ and that (4.31) eventually reads

$$(4.33)\qquad {}_{G^-_{01}(zq^{-2}/w)}[\mathbf{X}^-_{1,1}(w),\mathbf{X}^-_{1,1}(zq^{-2})]=\delta\left(\frac{w}{z}\right)\xi_1(z)\,.$$

Combining (4.29) and (4.33), we get the case $m=1$. Now assume that the result holds for all nonnegative integer less than $m\in\mathbb{N}$. Taking the commutator of (4.28) with $\boldsymbol{\psi}^+_{1,1}(v)$ yields

$$[2]_q\wp^-(v)\left\{\delta\left(\frac{C^{1/2}q^2v}{w}\right)[\mathbf{X}^-_{1,m+2}(wq^{-2}),\mathbf{X}^-_{1,0}(z)]_{G^-_{01}(wq^{-2}/z)}\right.$$
$$\left.+\delta\left(\frac{C^{1/2}q^2v}{z}\right){}_{G^-_{01}(zq^{-2}/w)G^-_{11}(z/w)}[\mathbf{X}^-_{1,m+1}(w),\mathbf{X}^-_{1,1}(zq^{-2})]_{G^-_{01}(w/z)}\right\}$$
$$=-[2]_q\wp^-(v)\left\{\delta\left(\frac{C^{1/2}q^2v}{w}\right)[\mathbf{X}^-_{1,2}(wq^{-2}),\mathbf{X}^-_{1,m}(z)]_{G^-_{01}(wq^{-2}/z)}\right.$$
$$\left.+\delta\left(\frac{C^{1/2}q^2v}{z}\right){}_{G^-_{01}(zq^{-2}/w)G^-_{11}(z/w)}[\mathbf{X}^-_{1,1}(w),\mathbf{X}^-_{1,m+1}(zq^{-2})]_{G^-_{01}(w/z)}\right\}$$
$$=\delta\left(\frac{wq^2}{z}\right)[\xi_m(z),\boldsymbol{\psi}^+_{1,1}(v)]$$

The latter implies that

$$(4.34)\quad [\mathbf{X}^-_{1,m+2}(wq^{-2}),\mathbf{X}^-_{1,0}(z)]_{G^-_{01}(wq^{-2}/z)}=\delta\left(\frac{w}{z}\right)\xi_{m+1}(z)+\delta\left(\frac{wq^2}{z}\right)\eta_1(z)\,,$$

$$(4.35)\quad {}_{G^-_{01}(zq^{-2}/w)G^-_{11}(z/w)}[\mathbf{X}^-_{1,m+1}(w),\mathbf{X}^-_{1,1}(zq^{-2})]_{G^-_{01}(w/z)}=-\delta\left(\frac{w}{z}\right)\xi_{m+1}(z)+\delta\left(\frac{wq^2}{z}\right)\eta_1(z)\,,$$

$$(4.36)\quad [\mathbf{X}^-_{1,2}(wq^{-2}),\mathbf{X}^-_{1,m}(z)]_{G^-_{01}(wq^{-2}/z)}=\delta\left(\frac{w}{z}\right)\eta_3(z)-\delta\left(\frac{wq^2}{z}\right)\eta_1(z)\,,$$

$$(4.37)\quad {}_{G^-_{01}(zq^{-2}/w)G^-_{11}(z/w)}[\mathbf{X}^-_{1,1}(w),\mathbf{X}^-_{1,m+1}(zq^{-2})]_{G^-_{01}(w/z)}=-\delta\left(\frac{w}{z}\right)\eta_3(z)-\delta\left(\frac{wq^2}{z}\right)\eta_2(z)\,,$$

for some $\xi_{m+1}(z), \eta_1(z), \eta_2(z), \eta_3(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z, z^{-1}]]$. Multiplying (4.37) by $(z - wq^2)$ and subsequently factoring $(zq^2 - w)$, we get that

$$(4.38)\quad [\mathbf{X}^-_{1,m+1}(z), \mathbf{X}^-_{1,1}(w)]_{G^-_{01}(z/w)} = -\delta\left(\frac{w}{zq^2}\right)\eta_3(w) + \delta\left(\frac{w}{zq^4}\right)\eta_4(z)\,,$$

for some $\eta_4(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z, z^{-1}]]$. But, by the recursion hypothesis,

$$\begin{aligned}[\mathbf{X}^-_{1,m+1}(z), \mathbf{X}^-_{1,1}(w)]_{G^-_{01}(z/w)} &= Y\left([\mathbf{X}^-_{1,m}(z), \mathbf{X}^-_{1,0}(w)]_{G^-_{01}(z/w)}\right)\\ &= \delta\left(\frac{w}{zq^2}\right) Y\left(\xi_{m-1}(w)\right)\,.\end{aligned}$$

Comparing with (4.38), it follows that

$$\eta_3(w) = -Y\left(\xi_{m-1}(w)\right) \qquad \text{and} \qquad \eta_4(z) = 0\,.$$

By the recursion hypothesis, we also have

$$\begin{aligned}&[\mathbf{X}^-_{1,2}(wq^{-2}), \mathbf{X}^-_{1,m}(z)]_{G^-_{01}(wq^{-2}/z)}\\ &\qquad = Y\left([\mathbf{X}^-_{1,1}(wq^{-2}), \mathbf{X}^-_{1,m-1}(z)]_{G^-_{01}(wq^{-2}/z)}\right)\\ &\qquad = -\delta\left(\frac{w}{z}\right) Y\left(\xi_{m-1}(z)\right) = \delta\left(\frac{w}{z}\right)\eta_3(z)\end{aligned}$$

Comparing the above result with (4.36), we conclude that $\eta_1(z) = 0$. As a consequence, (4.34) now reads

$$(4.39)\qquad [\mathbf{X}^-_{1,m+2}(w), \mathbf{X}^-_{1,0}(z)]_{G^-_{01}(w/z)} = \delta\left(\frac{wq^2}{z}\right)\xi_{m+1}(z)\,.$$

On the other hand, multiplying (4.35) by $(z - wq^2)$ and subsequently factoring $(zq^2 - w)$, we get that

$$\begin{aligned}(4.40)\quad & {}_{G^-_{01}(zq^{-2}/w)}[\mathbf{X}^-_{1,m+1}(w), \mathbf{X}^-_{1,1}(zq^{-2})]\\ &\qquad = \delta\left(\frac{w}{z}\right)\xi_{m+1}(z) + \delta\left(\frac{w}{zq^2}\right)\eta_5(z)\,,\end{aligned}$$

for some $\eta_5(z) \in \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}[[z, z^{-1}]]$. Multiplying the above equation by $(z-w)$ yields

$$\begin{aligned}(4.41)\quad & Y\left((zq^{-2} - wq^2)\mathbf{X}^-_{1,m}(w)\mathbf{X}^-_{1,0}(zq^{-2}) - (z-w)\mathbf{X}^-_{1,0}(zq^{-2})\mathbf{X}^-_{1,m}(w)\right)\\ &\qquad = z(1-q^2)\delta\left(\frac{zq^2}{w}\right)\eta_5(z)\,.\end{aligned}$$

But the recursion hypothesis

$$(4.42)\qquad [\mathbf{X}^-_{1,m}(w), \mathbf{X}^-_{1,0}(zq^{-2})]_{G^-_{01}(wq^2/z)} = \delta\left(\frac{wq^4}{z}\right)\xi_{m-1}(z)$$

implies, upon multiplication by $(zq^{-2} - wq^2)$, that

$$(zq^{-2}-wq^2)\mathbf{X}^-_{1,m}(w)\mathbf{X}^-_{1,0}(zq^{-2})-(z-w)\mathbf{X}^-_{1,0}(zq^{-2})\mathbf{X}^-_{1,m}(w)=0\,. \tag{4.43}$$

Substituting back into (4.41) proves that $\eta_5(z)=0$ and that (4.40) eventually reads

$$_{G^-_{01}(w/z)}[\mathbf{X}^-_{1,m+1}(z),\mathbf{X}^-_{1,1}(w)] = \delta\left(\frac{wq^2}{z}\right)\xi_{m+1}(z)\,. \tag{4.44}$$

Combining (4.39) and (4.44) completes the recursion and the result holds for any $m\in\mathbb{N}$, assuming $n=0$. The cases $n\in\mathbb{Z}^\times$ are then obtained by applying Y^n to the case $n=0$. □

Corollary 4.17. — *For every $m\in\mathbb{N}$ and every $n\in\mathbb{Z}$, we have*

$$\begin{aligned}(4.45)\quad [\mathbf{X}^+_{1,m+n+1}(z),\mathbf{X}^+_{1,n}(w)]_{G^+_{01}(z/w)} &\\ = -[\mathbf{X}^+_{1,n+1}(z),\mathbf{X}^+_{1,m+n}(w)]_{G^+_{01}(z/w)}&\\ = \delta\left(\frac{wq^2}{z}\right)\varphi\circ Y^{-m-n-1}\left(\xi_m(1/z)\right)&\end{aligned}$$

Proof. — It suffices to apply $\varphi\circ Y^{-m-n-1}$ to (4.28). □

We now return to the proof of Theorem 3.7 and to the map $\Psi:\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)\to\widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$.

Corollary 4.18. — *We have*

(i) $\Upsilon^\pm(w)=0$;

(ii) *and for every $i\neq j$,*

$$\sum_{\sigma\in S_3}\sum_{k=0}^{3}(-1)^k\binom{3}{k}_q\Psi(\mathbf{x}^\pm_i(z_{\sigma(1)}))\cdots\Psi(\mathbf{x}^\pm_i(z_{\sigma(k)}))\Psi(\mathbf{x}^\pm_j(z))\times\Psi(\mathbf{x}^\pm_i(z_{\sigma(k+1)}))\cdots\Psi(\mathbf{x}^\pm_i(z_{\sigma(3)}))=0\,.$$

Proof. — The proof of Proposition 4.16 makes it clear that the relations (2.9) with $i\neq j$ there, both follow from the relations

$$\left[\mathbf{X}^+_{1,0}(v),\mathbf{X}^+_{1,-1}(w)\right]_{G^-_{11}(v/w)}=0 \tag{4.46}$$

and

$$\left[\mathbf{X}^-_{1,1}(v),\mathbf{X}^-_{1,0}(w)\right]_{G^+_{11}(v/w)}=0 \tag{4.47}$$

in the completion $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$. A tedious but straightforward calculation shows that the quantum Serre relations (2.10) similarly follow from

$$\left[\mathbf{X}^+_{1,-1}(v),\mathbf{X}^+_{1,-2}(w)\right]_{G^-_{11}(v/w)}=0 \tag{4.48}$$

and

$$\left[\mathbf{X}^-_{1,2}(v), \mathbf{X}^-_{1,1}(w)\right]_{G^+_{11}(v/w)} = 0\,, \tag{4.49}$$

which in turn are a consequence of (4.46)–(4.47), (just apply Y there). We can therefore extend $\Psi : \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1) \to \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ by continuity [1] into $\widehat{\Psi} : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \to \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$ and it suffices to check point (i). Since by construction $\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)$ is dense in $\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$, there exists a sequence $(u_n(v,w))_{n\in\mathbb{N}} \in \dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)[[v,v^{-1},w,w^{-1}]]^{\mathbb{N}}$ such that

$$\lim_{n\to+\infty} u_n(v,w) = 0\,, \tag{4.50}$$

whereas, on the other hand,

$$\lim_{n\to+\infty} \widehat{\Psi}(u_n(v,w)) = \delta\left(\frac{vq^{\mp 2}}{w}\right)\Upsilon^{\pm}(w)\,. \tag{4.51}$$

Take for example the partial sum of the series involved on the l.h.s. of equations (4.46)–(4.47) above. The result now follows by the continuity of $\widehat{\Psi}$. □

Remark 4.19. — We have therefore completed the proof of that part of Theorem 3.7 that claims the existence of a continuous algebra homomorphism $\widehat{\Psi} : \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)} \to \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)}$. We still have to construct the inverse continuous algebra homomorphism $\widehat{\Psi}^{-1} : \widehat{\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)} \to \widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)}$. This shall be done at the end of the present section.

4.3. Weight grading relations

The results of the previous subsection have the following

COROLLARY 4.20. — *For every $m \in \mathbb{N}^\times$ and every $n \in \mathbb{Z}$, we have:*

(i) $[\boldsymbol{\Gamma}^+_{m+1}(u), \mathbf{X}^-_{1,n}(z)] = 0\,;$

(ii) $[\boldsymbol{\psi}^+_{1,m+1}(u), \mathbf{X}^-_{1,n}(z)]$

$$= -\wp^-(C^{1/2}uq^{-2(m+1)})_{G^+_{01}(Cuq^{2(1-m)}/z)}[\mathbf{X}^-_{1,n+1}(zq^{-2}), \boldsymbol{\psi}^+_{1,m}(u)]_{G^-_{01}(z/Cuq^{2(1-m)})}$$

$$\propto \delta\left(\frac{Cu}{zq^2 m}\right)\,;$$

[1] Ψ is obviously $\mathbb{Z}_{(2)}$-graded, hence continuous.

(iii) $[\mathbf{\Gamma}^+_{m+1}(u), \mathbf{X}^+_{1,n}(z)] = 0\,;$

(iv) $[\boldsymbol{\psi}^+_{1,m+1}(v), \mathbf{X}^+_{1,n}(z)]_{G^+_{01}(v/z)G^+_{11}(v/zq^{2(m+1)})}$

$$= - {}_{G^-_{01}(z/vq^{2m})}[\mathbf{X}^+_{1,n+1}(v), \boldsymbol{\psi}^+_{1,m}(z)]_{G^+_{01}(v/z)}$$

$$\propto \delta\left(\frac{zq^2}{v}\right).$$

Proof. — It suffices to prove the proposition for $n = 0$ as the general case then follows by applying Y^n for any $n \in \mathbb{Z}$. Assuming that $n = 0$ in (i) and (ii), it then suffices to take the commutator of (4.28), for $n = 1$ there, with $\mathbf{x}^+_1(z)$. □

Remark 4.21. — It turns out that, for every $m \in \mathbb{N}^\times$,

$$\mathbf{\Gamma}^+_m(z) \in \mathcal{Z}(\widehat{\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)})[[z, z^{-1}]]\,.$$

Indeed, in the next section we actually establish that these central elements consistently vanish.

4.4. The central elements $\mathbf{\Gamma}^\pm_{m>2}(z)$

Before we can actually establish that these central elements vanish, we need to establish a few lemmas. In what follows, we let $\dot{\mathrm{U}}^<_q(\dot{\mathfrak{a}}_1) = \dot{\mathrm{U}}^\leqslant_q(\dot{\mathfrak{a}}_1) - \dot{\mathrm{U}}^\leqslant_q(\dot{\mathfrak{a}}_1) \cap \dot{\mathrm{U}}^0_q(\dot{\mathfrak{a}}_1)$.

Lemma 4.22. — *For every $p \in \mathbb{N}^\times$,*

(i) $\Delta(\boldsymbol{\psi}^-_{1,-p}(v)) = 1 \otimes \boldsymbol{\psi}^-_{1,-p}(v) \mod \dot{\mathrm{U}}^<_q(\dot{\mathfrak{a}}_1)\widehat{\otimes}\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)\,;$

(ii) $$\Delta(\mathbf{X}^+_{1,-p}(v)) = \prod_{\ell=1}^{p-1} \mathbf{\Gamma}^+_0(C^{-1/2}q^{2\ell}v)^{-1}\mathbf{k}^+_0(C^{-1/2}v)^{-1}\widehat{\otimes}\mathbf{X}^+_{1,-p}(v) \mod \dot{\mathrm{U}}^<_q(\dot{\mathfrak{a}}_1)\widehat{\otimes}\dot{\mathrm{U}}_q(\dot{\mathfrak{a}}_1)\,.$$

Proof. — First one easily checks that

$$\Delta(\boldsymbol{\psi}^-_{1,-1}(z)) = 1 \otimes \boldsymbol{\psi}^-_{1,-1}(z) + [2]_q\,(q - q^{-1})\mathbf{x}^-_1(z)\widehat{\otimes}\mathbf{x}^-_0(q^{-2}z)\mathbf{k}^+_1(z) + \boldsymbol{\psi}^-_{1,-1}(z)\widehat{\otimes}\boldsymbol{\wp}^+(q^{-2}z)\,,$$

which proves (i) for $p = 1$. Assuming (i) holds for some $p \in \mathbb{N}$, the result for $p + 1$ easily holds making use of (4.23) and of the recursion hypothesis.

Similarly, one easily checks that

$$\Delta(\mathbf{X}^+_{1,-1}(v)) = \mathbf{X}^+_{1,-1}(v) \otimes 1 + \mathbf{k}^+_0(C^{-1/2}v)^{-1}\widehat{\otimes}\mathbf{X}^+_{1,-1}(v)\,,$$

which proves (ii) in the case $p = 1$. Assuming the result holds for some $p \in \mathbb{N}$, the result for $p+1$ easily follows making use of Proposition 4.8(iii) and of the recusrion hypothesis. □

For every $N \in \mathbb{N}^\times$, we let

$$S^<_{2N-1} = \left\{ \sigma \in S_{2N-1} : \begin{array}{c} \sigma(1) = 1 \\ \forall p \in [\![N-1]\!] \quad \sigma(2p) < \sigma(2p+1) \\ \sigma(2N-4) < \sigma(2N-1) \end{array} \right\}$$

Define $\varpi : \mathbb{Z} \to \dot{I} = \{0, 1\}$ by setting, for every $n \in \mathbb{Z}$,

$$\varpi(n) = \begin{cases} 0 & \text{if } n \text{ is even;} \\ 1 & \text{if } n \text{ is odd.} \end{cases} \tag{4.52}$$

Lemma 4.23. — *For every $r \in \mathbb{N}$ and every $i_1, \dots, i_{2r-1} \in \dot{I}$, there exists $(\beta_{r,\sigma})_{\sigma \in S^<_{2r-1}} \in \mathbb{F}^{S^<_{2r-1}}$ such that*

$$\begin{aligned} &\left\langle \mathbf{x}^+_{i_1}(z_1) \dots \mathbf{x}^+_{i_{2r-1}}(z_{2r-1}), \mathbf{X}^+_{1,-r}(v) \right\rangle \\ &\qquad = -\frac{[2]_q^{r-1}}{q - q^{-1}} \sum_{\sigma \in S^<_{2r-1}} \beta_{r,\sigma} \prod_{n=1}^{2r-1} \delta_{i_{\sigma(n)}, \pi(n)} \delta\left(\frac{z_{\sigma(n)} q^{\nu_r(n)}}{v} \right), \end{aligned} \tag{4.53}$$

where we have defined $\pi : \mathbb{N} \to \dot{I}$ and $\nu_r : \mathbb{N} \to \mathbb{Z}$ by setting, for every $n \in \mathbb{N}$,

$$\pi(n) = \begin{cases} 0 & \text{if } n = 1; \\ \varpi(n) & \text{if } n > 1 \end{cases} \tag{4.54}$$

and

$$\nu_r(n) = \begin{cases} 2(1-r) & \text{if } n = 1; \\ 2(1-r) + n - 3\varpi(n) & \text{if } n > 1. \end{cases} \tag{4.55}$$

Proof. — The case $r = 0$ holds by definition of the pairing. Assume that (4.53) holds for some $r \in \mathbb{N}$. Then, making use of the previous lemma, one easily shows that, for every $i_1, \dots, i_{2r+1} \in \dot{I}$

$$\left\langle \mathbf{x}^+_{i_1}(z_1) \dots \mathbf{x}^+_{i_{2r+1}}(z_{2r+1}), \left[\boldsymbol{\psi}^-_{1,-1}(z), \mathbf{X}^+_{1,-r}(v)\right] \right\rangle$$
$$= \frac{[2]_q}{q-q^{-1}} \sum_{\mathbf{A}\in \mathsf{P}^{(2,2r-1)}_{[\![2r+1]\!]}} \prod_{m\in[\![2]\!]} \delta_{i_{A^{(1)}_m},1-\varpi(m)} \delta\left(\frac{z_{A^{(1)}_m} q^{2\varpi(m)}}{z}\right)$$
$$\times \left\langle \mathbf{x}^+_{i_{A^{(2)}_1}}(z_{A^{(2)}_1}) \dots \mathbf{x}^+_{i_{A^{(2)}_{2r-1}}}(z_{A^{(2)}_{2r-1}}), \mathbf{X}^+_{1,-r}(v) \right\rangle$$
$$\times \left\{ R^<_{\mathbf{A}}(z_{\mathbf{A}}) - G^+_{i_{A^{(1)}_1},0}(C^{-1/2} z_{A^{(1)}_1}/v) G^+_{i_{A^{(1)}_2},0}(C^{-1/2} z_{A^{(1)}_2}/v) R^>_{\mathbf{A}}(z^{-1}_{\mathbf{A}}) \right\},$$

where

$$R^<_{\mathbf{A}}(z_{\mathbf{A}}) = \prod_{\substack{m\in[\![2]\!] \\ n\in[\![2r-1]\!] \\ A^{(2)}_n < A^{(1)}_m}} G^-_{i_{A^{(2)}_n}, i_{A^{(1)}_m}}(C^{-1/2} z_{A^{(2)}_n}/z_{A^{(1)}_m});$$

$$R^>_{\mathbf{A}}(z^{-1}_{\mathbf{A}}) = \prod_{\substack{m\in[\![2]\!] \\ n\in[\![2r-1]\!] \\ A^{(2)}_n > A^{(1)}_m}} G^-_{i_{A^{(2)}_n}, i_{A^{(1)}_m}}(C^{1/2} z_{A^{(1)}_m}/z_{A^{(2)}_n}).$$

Making use of Proposition 4.8(iii) on the l.h.s. and of the recursion hypothesis on the r.h.s., we get

$$(4.56)\quad [2]_q \, \delta\left(\frac{Cz}{v}\right) \left\langle \mathbf{x}^+_{i_1}(z_1) \dots \mathbf{x}^+_{i_{2r+1}}(z_{2r+1}), \mathbf{X}^+_{1,-(r+1)}(vq^{-2}) \right\rangle$$
$$= -\frac{[2]^r_q}{(q-q^{-1})^2} \sum_{\substack{\mathbf{A}\in \mathsf{P}^{(2,2r-1)}_{[\![2r+1]\!]} \\ \sigma\in S^<_{2r-1}}} \beta_{r,\sigma} \prod_{m\in[\![2]\!]} \delta_{i_{A^{(1)}_m},1-\varpi(m)} \delta\left(\frac{z_{A^{(1)}_m} q^{2\varpi(m)}}{z}\right)$$
$$\times \prod_{n\in[\![2r-1]\!]} \delta_{i_{A^{(2)}_{\sigma(n)}},\pi(n)} \delta\left(\frac{z_{A^{(2)}_{\sigma(n)}} q^{\nu_r(n)}}{v}\right)$$
$$\times \left\{ Q^<_{\sigma,\mathbf{A}}(v/z) - G^+_{0,0}(C^{-1/2} z q^{-2}/v) G^+_{1,0}(C^{-1/2} z/v) Q^>_{\sigma,\mathbf{A}}(z/v) \right\},$$

where

$$Q^<_{\sigma,\mathbf{A}}(v/z) = \prod_{\substack{m\in[\![2]\!] \\ n\in[\![2r-1]\!] \\ A^{(2)}_{\sigma(n)} < A^{(1)}_m}} G^-_{\pi(n),1-\varpi(m)}(C^{-1/2} v q^{\lambda_r(m,n)}/z);$$

and

$$Q^{>}_{\sigma,\mathbf{A}}(z/v) = \prod_{\substack{m\in[\![2]\!]\\ n\in[\![2r-1]\!]\\ A^{(2)}_{\sigma(n)}>A^{(1)}_m}} G^{-}_{\pi(n),1-\varpi(m)}(C^{1/2}z/vq^{\lambda_r(m,n)})\,;$$

where $\lambda_r(m,n) = 2\varpi(m) - \nu_r(n)$. In view of the $\delta(Cz/v)$ factor on the l.h.s of (4.56), it is clear that the relevant factors in $Q^{<}_{\sigma,\mathbf{A}}(v/z)$ and $Q^{>}_{\sigma,\mathbf{A}}(z/v)$ are the ones contributing to a pole at $Cz = v$, i.e. the ones for which $\lambda_r(m,n) = c_{\pi(n),1-\varpi(m)}$ or $\lambda_r(m,n) = -c_{\pi(n),1-\varpi(m)}$ respectively. We thus let

$$L^{\pm}_r = \left\{(m,n) \in [\![2]\!] \times [\![2r-1]\!] : \lambda_r(m,n) = \pm c_{\pi(n),1-\varpi(m)}\right\}$$

and determine, by inspection, that, for every $r \geqslant 3$,

$$L^{+}_r = \{(1,2r-2),(2,2r-3)\}\,, \qquad \text{whereas} \qquad L^{-}_r = \{(2,2r-4)\}\,.$$

Since we cannot have $A^{(2)}_{\sigma(2r-4)} > A^{(1)}_2$ while $A^{(2)}_{\sigma(2r-3)} < A^{(1)}_2$ for $\sigma \in S^{<}_{2r-1}$, we see that the relevant pole is necessarily a simple pole; as one might have expected, given the absence of a $\delta'(Cz/v)$ factor on the l.h.s of (4.56). It easily follows that

$$\left\{Q^{<}_{\sigma,\mathbf{A}}(v/z) - G^{+}_{0,0}(C^{-1/2}zq^{-2}/v)G^{+}_{1,0}(C^{-1/2}z/v)Q^{>}_{\sigma,\mathbf{A}}(z/v)\right\} \\ = [2]_q\,(q-q^{-1})\gamma_{\sigma,\mathbf{A}}\delta\left(\frac{Cz}{v}\right)$$

for every $(\sigma,\mathbf{A}) \in S^{<}_{2r-1} \times \mathsf{P}^{(2,2r-1)}_{[\![2r+1]\!]}$ such that $A^{(2)}_{\sigma(2r-2)} < A^{(1)}_1$ and either:

- $A^{(2)}_{\sigma(2r-4)} > A^{(1)}_2$ (and then necessarily, $A^{(2)}_{\sigma(2r-3)} > A^{(1)}_2$); or
- $A^{(2)}_{\sigma(2r-4)} < A^{(1)}_2$ and $A^{(2)}_{\sigma(2r-3)} < A^{(1)}_2$;

and, for each such pair $(\sigma,\mathbf{A})$, $\gamma_{\sigma,\mathbf{A}} \in \mathbb{F}$. Note that the above conditions impose that $A^{(2)}_{\sigma(1)=1} < A^{(1)}_1$ and hence $A^{(2)}_1 = 1$. Now, for each pair $(\sigma,\mathbf{A})$ as above, define

$$\sigma' = \begin{pmatrix} 1 & 2 & \dots & 2r-1 & 2r & 2r+1 \\ 1 & A^{(2)}_{\sigma(2)} & \dots & A^{(2)}_{\sigma(2r-1)} & A^{(1)}_1 & A^{(1)}_2 \end{pmatrix}.$$

It is obvious that $\sigma' \in S^{<}_{2r+1}$. Actually, setting $(\sigma,\mathbf{A}) \mapsto \sigma'$ defines a map $S^{<}_{2r-1} \times \mathsf{P}^{(2,2r-1)}_{[\![2r+1]\!]} \to S^{<}_{2r+1}$ which is easily seen to be a bijection. Observing furthermore that $\nu_r - 2 = \nu_{r+1}$ and setting $\beta_{r+1,\sigma'} = \beta_{r,\sigma}\gamma_{\sigma,\mathbf{A}}$, we can rewrite (4.56) as

$$\left\langle \mathbf{x}^+_{i_1}(z_1)\dots\mathbf{x}^+_{i_{2r+1}}(z_{2r+1}), \mathbf{X}^+_{1,-(r+1)}(v)\right\rangle$$
$$= -\frac{[2]_q^r}{q-q^{-1}} \sum_{\sigma'\in S^<_{2r+1}} \beta_{r+1,\sigma'} \prod_{n=1}^{2r+1} \delta_{i_{\sigma'(n)},\pi(n)}\delta\left(\frac{z_{\sigma'(n)}q^{\nu_{r+1}(n)}}{v}\right),$$

which completes the recursion. □

Proposition 4.24. — *For every $m\in\mathbb{N}^\times$, we actually have $\mathbf{\Gamma}^+_m(v) = \mathbf{\Gamma}^-_{-m}(v) = 0$.*

Proof. — It suffices to prove that, say $\mathbf{\Gamma}^-_{-m}(z) = 0$ for every $m\in\mathbb{N}^\times$ and to apply φ^{-1} to get the result for $\mathbf{\Gamma}^+_m(z)$. Considering the root space decomposition, it is obvious that having

$$\left\langle \mathbf{x}^+_{i_1}(z_1)\cdots\mathbf{x}^+_{i_{2m}}(z_{2m}), \mathbf{\Gamma}^-_{-m}(z)\right\rangle = 0\,,$$

for every $i_1,\dots,i_{2m}\in I$, is a sufficient condition. Now, making use of the previous lemma, one easily shows that

$$\left\langle \mathbf{x}^+_{i_1}(z_1)\cdots\mathbf{x}^+_{i_{2m}}(z_{2m}), \left[\mathbf{X}^+_{1,-m}(v), \mathbf{x}^-_1(z)\right]\right\rangle$$
$$= -\frac{[2]_q}{(q-q^{-1})^2} \sum_{\substack{\mathbf{A}\in\mathsf{P}^{(1,2m-1)}_{[\![2m]\!]}\\ \sigma\in S^<_{2m-1}}} \beta_{m,\sigma}\delta_{i_{A^{(1)}_1},1}\delta\left(\frac{z_{A^{(1)}_1}}{z}\right)$$
$$\times \prod_{n\in[\![2m-1]\!]} \delta_{i_{A^{(2)}_n},\pi(n)}\delta\left(\frac{z_{A^{(2)}_{\sigma(n)}}q^{\nu_m(n)}}{v}\right)$$
$$\times\left\{G^+_{i_{A^{(1)}_1},0}(C^{1/2}z_{A^{(1)}_1}/v)R^<_{\mathbf{A}}(z_{\mathbf{A}}) - R^>_{\mathbf{A}}(z^{-1}_{\mathbf{A}})\right\},$$

where

$$R^<_{\mathbf{A}}(z_{\mathbf{A}}) = \prod_{\substack{n\in[\![2m-1]\!]\\ A^{(2)}_n > A^{(1)}_1}} G^-_{i_{A^{(1)}_1},i_{A^{(2)}_n}}(C^{-1/2}z_{A^{(1)}_1}/z_{A^{(2)}_n}),$$
$$R^>_{\mathbf{A}}(z^{-1}_{\mathbf{A}}) = \prod_{\substack{n\in[\![2m-1]\!]\\ A^{(2)}_n < A^{(1)}_1}} G^-_{i_{A^{(1)}_1},i_{A^{(2)}_n}}(C^{1/2}z_{A^{(2)}_n}/z_{A^{(1)}_1}).$$

Hence, upon rewriting, we get

$$\begin{aligned}&\left\langle \mathbf{x}^+_{i_1}(z_1)\cdots \mathbf{x}^+_{i_{2m}}(z_{2m}), \left[\mathbf{X}^+_{1,-m}(v), \mathbf{x}^-_1(z)\right]\right\rangle \\ &\qquad = -\frac{[2]_q}{(q-q^{-1})^2} \sum_{\substack{\mathbf{A}\in\mathsf{P}^{(1,2m-1)}_{[\![2m]\!]}\\ \sigma\in S^<_{2m-1}}} \beta_{m,\sigma}\delta_{i_{A^{(1)}_1},1}\delta\left(\frac{z_{A^{(1)}_1}}{z}\right) \\ &\qquad\qquad \times \prod_{n\in[\![2m-1]\!]} \delta_{i_{A^{(2)}_n},\pi(n)}\delta\left(\frac{z_{A^{(2)}_{\sigma(n)}}q^{\nu_m(n)}}{v}\right) \\ &\qquad\qquad \times \left\{G^+_{1,0}(C^{1/2}z/v)Q^<_{\sigma,\mathbf{A}}(z/v) - Q^>_{\sigma,\mathbf{A}}(v/z)\right\},\end{aligned}$$

where

$$\begin{aligned} Q^<_{\sigma,\mathbf{A}}(z/v) &= \prod_{\substack{n\in[\![2m-1]\!]\\ A^{(2)}_{\sigma(n)}>A^{(1)}_1}} G^-_{1,\pi(n)}(C^{-1/2}zq^{\nu_m(n)}/v)\,,\\ Q^>_{\sigma,\mathbf{A}}(v/z) &= \prod_{\substack{n\in[\![2m-1]\!]\\ A^{(2)}_{\sigma(n)}<A^{(1)}_1}} G^-_{1,\pi(n)}(C^{1/2}vq^{-\nu_m(n)}/z)\,.\end{aligned}$$

In view of (4.20), the contributions to $\left\langle \mathbf{x}^+_{i_1}(z_1)\cdots\mathbf{x}^+_{i_{2m}}(z_{2m}), \mathbf{\Gamma}^-_{-m}(z)\right\rangle$ in the above expression must come from terms with a pole at $z=C^{1/2}v$. The latter happen for factors in $Q^<_{\sigma,\mathbf{A}}(z/v)$ or $Q^>_{\sigma,\mathbf{A}}(v/z)$ such that $\nu_m(n) = c_{1,\pi(n)}$ or $\nu_m(n) = -c_{1,\pi(n)}$ respectively. We thus let

$$M^\pm_m = \{n\in[\![2m-1]\!] : \nu_m(n) = \pm c_{1,\pi(n)}\}\,.$$

Upon inspection, one easily sees that

$$M^+_m = \{2m-4\}\,, \qquad \text{whereas} \qquad M^-_m = \{2m-1\}\,.$$

Now, for $\sigma\in S^<_{2m-1}$, we have $\sigma(2m-4)<\sigma(2m-1)$ and no term has a pole at $z=C^{1/2}v$. We conclude that $\left\langle \mathbf{x}^+_{i_1}(z_1)\cdots\mathbf{x}^+_{i_{2m}}(z_{2m}), \mathbf{\Gamma}^-_{-m}(z)\right\rangle = 0$. □

4.5. Relations in $\Psi^{-1}(\ddot{\mathrm{U}}^0_q(\mathfrak{a}_1))$

Definition 4.25. — *We set* $\mathbf{K}^+_{1,0}(v) = -\mathbf{k}^-_1(C^{1/2}v)$ *and, for every* $m\in\mathbb{N}^\times$,

$$\mathbf{K}^+_{1,m}(v) = (q-q^{-1})\mathbf{k}^-_1(C^{1/2}vq^{-2m})\boldsymbol{\psi}^+_{1,m}(v)\,.$$

We then let

$$\mathbf{K}^-_{1,0}(v) = \varphi\left(\mathbf{K}^+_{1,0}(1/v)\right) = -\mathbf{k}^+_1(C^{1/2}v)$$

and, for every $m \in \mathbb{N}^\times$,

$$\mathbf{K}^-_{1,-m}(v) = \varphi\left(\mathbf{K}^+_{1,m}(1/v)\right) = -(q-q^{-1})\boldsymbol{\psi}^-_{1,-m}(v)\mathbf{k}^+_1(C^{1/2}vq^{-2m})\,.$$

It is straigthforward to establish that

$$\mathbf{k}^-_1(C^{1/2}w)\boldsymbol{\psi}^+_{1,m}(v) = G^+_{11}\left(\frac{wq^{2m}}{v}\right)G^-_{11}\left(\frac{w}{v}\right)\boldsymbol{\psi}^+_{1,m}(v)\mathbf{k}^-_1(C^{1/2}w)\,. \tag{4.57}$$

By making repeated use of the above relation, one readily checks that, in terms of $(\mathbf{K}^+_{1,m}(v))_{m\in\mathbb{N}^\times}$, the relations (4.10) and (4.13), as well as the relations in Corollary 4.20(ii) and (iv) of the previous subsections respectively read

$$\begin{aligned}[]&[\mathbf{x}^+_1(v), \mathbf{X}^-_{1,n}(z)] \\ &\qquad = \frac{1}{q-q^{-1}}\delta\left(\frac{zq^{2n}}{Cv}\right)\left(\prod_{p=0}^{n-1}\boldsymbol{\Gamma}^-_0(C^{-1/2}zq^{2p})^{-1}\right)\mathbf{K}^+_{1,n}(v)\,,\end{aligned} \tag{4.58}$$

$$\begin{aligned}[]&[\mathbf{K}^+_{1,1}(w), \mathbf{K}^+_{1,m}(v)]_{G^-_{11}(w/v)G^+_{11}(wq^{2(m-1)}/v)} \\ &\qquad = [2]_q\left\{\delta\left(\frac{wq^{2m}}{v}\right)\mathbf{K}^+_{1,0}(vq^{-2m})\mathbf{K}^+_{1,m+1}(v)\right. \\ &\qquad\qquad \left. -\delta\left(\frac{w}{vq^2}\right)\mathbf{K}^+_{1,0}(v)\mathbf{K}^+_{1,m+1}(vq^2)\right\}\,,\end{aligned} \tag{4.59}$$

$$\begin{aligned}[]&[\mathbf{K}^+_{1,m+1}(v), \mathbf{X}^-_{1,n}(z)]_{G^+_{11}(Cv/zq^{2(m+1)})} \\ &= -\boldsymbol{\Gamma}^-_0(C^{1/2}vq^{-2(m+1)})[\mathbf{X}^-_{1,n+1}(zq^{-2}), \mathbf{K}^+_{1,m}(v)]_{G^+_{11}(zq^{2(m-1)}/Cv)} \\ &\propto \delta\left(\frac{zq^{2m}}{Cv}\right)\,,\end{aligned} \tag{4.60}$$

$$\begin{aligned}[]&[\mathbf{K}^+_{1,m+1}(v), \mathbf{X}^+_{1,n}(z)]_{G^-_{11}(v/z)} = -[\mathbf{X}^+_{1,n+1}(v), \mathbf{K}^+_{1,m}(z)]_{G^-_{11}(v/z)} \\ &\qquad \propto \delta\left(\frac{zq^2}{v}\right)\,.\end{aligned} \tag{4.61}$$

Proposition 4.26. — *For every* $m, n \in \mathbb{N}$, *we have*

$$\begin{aligned}&(v-q^{\pm 2}z)(v-q^{2(m-n\mp 1)}z)\mathbf{K}^\pm_{1,\pm m}(v)\mathbf{K}^\pm_{1,\pm n}(z) \\ &\qquad = (vq^{\pm 2}-z)(vq^{\mp 2}-q^{2(m-n)}z)\mathbf{K}^\pm_{1,\pm n}(z)\mathbf{K}^\pm_{1,\pm m}(v)\,,\end{aligned} \tag{4.62}$$

Proof. — We apply the map $a \mapsto [a, \mathbf{X}^-_{1,n}(u)]_{G^+_{11}(Cv/uq^{2(m+1)})}$ to the relation (4.61) with $n=0$ there. Making use of identity (1.6) on the left hand side, we get

$$\underbrace{[[\mathbf{K}^+_{1,m+1}(v), \mathbf{X}^-_{1,n}(u)]_{G^+_{11}(Cq^{-2(m+1)}v/u)}}_{\propto\delta\left(\frac{C^{-1}uq^{2m}}{v}\right)}, \mathbf{x}^+_1(z)]_{G^+_{10}(v/z)}$$

$$+ \left[\mathbf{K}^+_{1,m+1}(v), \left[\mathbf{x}^+_1(z), \mathbf{X}^-_{1,n}(u)\right]\right]_{G^+_{10}(v/z)G^+_{11}(Cq^{-2(m+1)}v/u)} \propto \delta\left(\frac{zq^2}{v}\right)$$

Multiplying through by $\left(C^{-1}uq^{2m} - v\right)\left(zq^2 - v\right)$ and making use of (4.58), it follows that

$$\begin{aligned} 0 &= \left(C^{-1}uq^{2m} - v\right)\left(zq^2 - v\right)\delta\left(\frac{uq^{2n}}{Cz}\right) \\ &\qquad\times \left[\mathbf{K}^+_{1,m+1}(v), \mathbf{K}^+_{1,n}(z)\right]_{G^+_{10}(v/z)G^+_{11}(Cq^{-2(m+1)}v/u)} \\ &= \left(zq^{2(m-n)} - v\right)\left(zq^2 - v\right)\delta\left(\frac{uq^{2n}}{Cz}\right) \\ &\qquad\times \left[\mathbf{K}^+_{1,m+1}(v), \mathbf{K}^+_{1,n}(z)\right]_{G^+_{10}(v/z)G^+_{11}(q^{2(n-m-1)}v/z)} \end{aligned}$$

Hence the result for the upper choice of signs in (4.62). The case with lower choice of signs follows by applying φ to the above equation. □

At this point it should be clear that we have obtained Ψ^{-1}. Indeed, it suffices to let, for every $m \in \mathbb{N}$ and every $n \in \mathbb{Z}$,

$$\Psi^{-1}(\mathsf{D}_2^{\pm 1}) = D^{\pm 1} \tag{4.63}$$

$$\Psi^{-1}(\mathsf{C}^{\pm 1/2}) = C^{\pm 1/2} \tag{4.64}$$

$$\Psi^{-1}(\mathsf{c}^{\pm}(z)) = \mathbf{\Gamma}^{\pm}_0(z) \tag{4.65}$$

$$\Psi^{-1}(\mathsf{K}^{\pm}_{1,\pm m}(z)) = \mathbf{K}^{\pm}_{1,\pm m}(z) \tag{4.66}$$

$$\Psi^{-1}(\mathsf{X}^{\pm}_{1,n}(z)) = \mathbf{X}^{\pm}_{1,n}(z) \tag{4.67}$$

The relations in $\ddot{\mathrm{U}}'_q(\mathfrak{a}_1)$ are obviously all the relations we have derived in the present section. Ψ^{-1} therefore extends as an algebra homomorphism. This concludes the proof of Theorem 3.7.

Returning to the proof of Proposition 3.10, it is also clear that

$$f(\psi^{\pm}(z)) = (q^2 - q^{-2})^2 \Psi(\wp^{\pm}(C^{1/2}zq^{-2})) \tag{4.68}$$

$$f(\mathbf{e}^{\pm}(z)) = \Psi(\boldsymbol{\psi}^{\pm}_{1,\pm 1}(z)) \tag{4.69}$$

Therefore (3.44)–(3.45) follow from Proposition 4.8(v). In order to complete the proof of Proposition 3.10, we still have to prove the compatibility of f with the Serre relations (3.33) of $\mathcal{E}_{q_1,q_2,q_3}$. This is the purpose of the next section.

4.6. The Serre relations of the elliptic Hall algebra

By the compatibility of f with (3.33), we actually mean that we should have, for every $m \in \mathbb{Z}$,

$$\operatorname*{res}_{v,w,z} (vwz)^m (v+z)(w^2 - vz) f(\mathbf{e}^{\pm}(v)) f(\mathbf{e}^{\pm}(w)) f(\mathbf{e}^{\pm}(z)) = 0\,. \tag{4.70}$$

Now we have already identified $f(\mathbf{e}^{\pm}(z))$ with $\Psi(\boldsymbol{\psi}^{\pm}_{1,\pm 1}(z))$ in (4.69) above. The latter means that proving (4.70) is equivalent to proving

Proposition 4.27. — *For every $m \in \mathbb{Z}$, we have*

$$\operatorname*{res}_{v_1,v_2,v_3} (v_1v_2v_3)^m (v_1+v_3)(v_2^2 - v_1v_3) \boldsymbol{\psi}^{\pm}_{1,\pm 1}(v_1) \boldsymbol{\psi}^{\pm}_{1,\pm 1}(v_2) \boldsymbol{\psi}^{\pm}_{1,\pm 1}(v_3) = 0\,. \tag{4.71}$$

Proof. — The upper choice of signs immediately follows from the lower one upon applying φ. Moreover, considering the root space decomposition, it is clear that having

$$\operatorname*{res}_{v_1,v_2,v_3} (v_1v_2v_3)^m (v_1+v_3)(v_2^2 - v_1v_3) \left\langle \mathbf{x}^+_{i_1}(z_1) \dots \mathbf{x}^+_{i_6}(z_6), \boldsymbol{\psi}^-_{1,-1}(v_1) \boldsymbol{\psi}^-_{1,-1}(v_2) \boldsymbol{\psi}^-_{1,-1}(v_3) \right\rangle = 0$$

for every $i_1, \dots, i_6 \in \dot{I}$ is a sufficient condition for the result to hold. Now, making use of Lemma 4.22, one easily obtains that

$$\begin{aligned} &\left\langle \mathbf{x}^+_{i_1}(z_1) \dots \mathbf{x}^+_{i_6}(z_6), \boldsymbol{\psi}^-_{1,-1}(v_1) \boldsymbol{\psi}^-_{1,-1}(v_2) \boldsymbol{\psi}^-_{1,-1}(v_3) \right\rangle \\ &\quad = \left(\frac{[2]_q}{q-q^{-1}} \right)^3 \sum_{\mathbf{A} \in \mathsf{P}^{(2,2,2)}_{[\![6]\!]}} \prod_{p=1}^{3} \prod_{\substack{m \in A^{(p+1)} \sqcup \dots \sqcup A^{(3)} \\ n \in A^{(p)} \\ n > m}} \prod_{k=1}^{2} \delta_{i_{A^{(p)}_k}, \varpi(k)} \\ &\qquad \times \delta\left(\frac{z_{A^{(p)}_k} q^{\varpi(k)}}{v_k} \right) G^-_{i_m,i_n}(C^{-1/2} z_m/z_n)\,. \end{aligned}$$

There is obviously an action of S_3 on $\mathsf{P}^{(2,2,2)}_{[\![6]\!]}$ given by setting $\sigma(\mathbf{A}) = (A^{(\sigma(1))}, A^{(\sigma(2))}, A^{(\sigma(3))})$ for every $\sigma \in S_3$ and every $\mathbf{A} \in \mathsf{P}^{(2,2,2)}_{[\![6]\!]}$. It is also quite clear that

$$\frac{\mathsf{P}^{(2,2,2)}_{[\![6]\!]}}{S_3} \cong \mathsf{T}^{(2,2,2)}_{[\![6]\!]}\,,$$

where

$$\mathsf{T}^{(2,2,2)}_{[\![6]\!]} = \left\{ \mathbf{A} \in \mathsf{P}^{(2,2,2)}_{[\![6]\!]} : A^{(1)}_1 < A^{(2)}_1 < A^{(3)}_1 \right\}\,.$$

For every triple $\mathbf{n} = \{n_1, n_2, n_3\} \subset [\![6]\!]$, we further let

$$\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\mathbf{n}) = \left\{ \mathbf{A} \in \mathsf{T}^{(2,2,2)}_{[\![6]\!]} : \left\{ A_2^{(p)} : p \in [\![3]\!] \right\} = \mathbf{n} \right\} .$$

With these notations in place, we can now write

$$\begin{aligned}\operatorname*{res}_{v_1,v_2,v_3} (v_1v_2v_3)^m (v_1+v_3)(v_2^2 - v_1v_3) \left\langle \mathbf{x}^+_{i_1}(z_1) \dots \mathbf{x}^+_{i_6}(z_6), \right.& \\ \left. \boldsymbol{\psi}^-_{1,-1}(v_1)\boldsymbol{\psi}^-_{1,-1}(v_2)\boldsymbol{\psi}^-_{1,-1}(v_3) \right\rangle& \\ = \left(\frac{[2]_q}{q-q^{-1}} \right)^3 \sum_{\substack{\mathbf{n} \subset [\![6]\!] \\ \operatorname{card} \mathbf{n} = 3}} z_{\mathbf{n}}^m \delta_{i_{[\![6]\!]-\mathbf{n}},1} \delta_{i_{\mathbf{n}},0} \sum_{\mathbf{A} \in \mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\mathbf{n})} \prod_{p=1}^{3} \delta \left(\frac{z_{A_1^{(p)}} q^2}{z_{A_2^{(p)}}} \right) c_{\mathbf{A}} \, ,&\end{aligned}$$

where, by definition,

$$z_{\mathbf{n}}^m = \prod_{i=1}^{3} z_{n_i}^m \, , \qquad \delta_{i_{\mathbf{n}},0} = \prod_{j=1}^{3} \delta_{i_{n_j},0} \, , \qquad \delta_{i_{[\![6]\!]-\mathbf{n}},1} = \prod_{m \in [\![6]\!]-\mathbf{n}} \delta_{i_m,1} \tag{4.72}$$

$$c_{\mathbf{A}} = \sum_{\sigma \in S_3} F\left(z_{A_2^{(\sigma(1))}}, z_{A_2^{(\sigma(2))}}, z_{A_2^{(\sigma(3))}}\right) \prod_{1 \leqslant p' < p \leqslant 3} H_{\mathbf{A},\sigma,p,p'}\left(z_{A_2^{(\sigma(p))}} / z_{A_2^{(\sigma(p'))}}\right) \tag{4.73}$$

$$F(x,y,z) = (x+z)(y^2 - xz) \tag{4.74}$$

$$H_{\mathbf{A},\sigma,p,p'} \left(\frac{z_{A_2^{(\sigma(p))}}}{z_{A_2^{(\sigma(p'))}}} \right) = \prod_{k,k'=1}^{2} G^-_{\varpi(k),\varpi(k')} \left(\frac{C^{-1/2} z_{A_2^{(\sigma(p))}}}{z_{A_2^{(\sigma(p'))}} q^{2(k'-k)}} \right)^{\epsilon(\mathbf{A},\sigma,p,p',k,k')} \tag{4.75}$$

$$\epsilon(\mathbf{A},\sigma,p,p',k,k') = \begin{cases} 1 & \text{if } A_k^{(\sigma(p))} < A_{k'}^{(\sigma(p'))}; \\ 0 & \text{otherwise.} \end{cases} \tag{4.76}$$

Denoting each $\mathbf{A} \in \mathsf{T}^{(2,2,2)}_{[\![6]\!]}$ as the tableau

$A_1^{(1)}$	$A_1^{(2)}$	$A_1^{(3)}$
$A_2^{(1)}$	$A_2^{(2)}$	$A_2^{(3)}$

,

one easily checks that, actually,

$$\begin{aligned}\mathsf{T}^{(2,2,2)}_{[\![6]\!]} =& \mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{2,4,6\}) \sqcup \mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{2,5,6\}) \sqcup \mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{3,4,6\}) \\ &\sqcup \mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{3,5,6\}) \sqcup \mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{4,5,6\}) ,\end{aligned}$$

with

$$\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{2,4,6\}) = \left\{ \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array} \right\},$$

$$\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{2,5,6\}) = \left\{ \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&6&5\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array} \right\},$$

$$\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{3,4,6\}) = \left\{ \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 4&3&6\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array} \right\},$$

$$\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{3,5,6\}) = \left\{ \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 6&3&5\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&6&5\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 5&3&6\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array} \right\},$$

$$\begin{aligned}\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{4,5,6\}) = \Bigg\{ & \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 6&5&4\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 5&6&4\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 6&4&5\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&6&5\\\hline\end{array}, \\ & \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 5&4&6\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array} \Bigg\} .\end{aligned}$$

A tedious but straightforward calculation (see Appendix for useful identities) shows that, e.g.

$$\begin{aligned} c_{\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 6&3&5\\\hline\end{array}} =& (q^2-q^{-2})^2(1+q^2)(1-q^2)z_3^3\left[q^2\delta\left(\frac{z_3q^2}{z_6}\right)\delta\left(\frac{z_6}{z_5}\right) \right. \\ & \left. -\delta\left(\frac{z_3}{z_6}\right)\delta\left(\frac{z_6q^2}{z_5}\right)\right] \\ & + q^{-2}(q^2-q^{-2})(1+q^2)^2(1-q^2)^6 \\ & \times H_1(z_3/z_5)\left[z_5\delta\left(\frac{z_6}{z_5}\right) - z_3\delta\left(\frac{z_3}{z_6}\right)\right] \end{aligned}$$

$$\begin{aligned} c_{\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&6&5\\\hline\end{array}} =& (q^2-q^{-2})^2(1+q^2)(1-q^2)z_3^3\delta\left(\frac{z_3}{z_6}\right)\delta\left(\frac{z_6q^2}{z_5}\right) \\ & + q^{-2}(q^2-q^{-2})(1+q^2)^2(1-q^2)^6H_1(z_3/z_5)z_3\delta\left(\frac{z_3}{z_6}\right) \\ & + q^{-2}(q^2-q^{-2})(1+q^2)^2(1-q^2)^6H_2(z_3/z_5)z_5\delta\left(\frac{z_6}{z_5}\right) \end{aligned}$$

$$
\begin{aligned}
c_{\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 5&3&6\\\hline\end{array}} &= -q^2(q^2-q^{-2})^2(1+q^2)(1-q^2)z_3^3\delta\left(\frac{z_5}{z_6}\right)\delta\left(\frac{z_3q^2}{z_5}\right)\\
&\qquad - q^{-2}(q^2-q^{-2})(1+q^2)(1-q^2)^6 H_1(z_3/z_5)z_5\delta\left(\frac{z_5}{z_6}\right)\\
&\qquad - q^{-2}(q^2-q^{-2})(1+q^2)(1-q^2)^6 H_2(z_3/z_6)z_3\delta\left(\frac{z_3}{z_5}\right)
\end{aligned}
$$

$$
c_{\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}} = q^{-2}(q^2-q^{-2})(1+q^2)(1-q^2)^6 H_2(z_3/z_6)\left[z_3\delta\left(\frac{z_3}{z_5}\right)-z_6\delta\left(\frac{z_5}{z_6}\right)\right].
$$

where we have set

$$
\begin{aligned}
H_1(z_3/z_5) &= \left(\frac{z_3^2z_5^2(z_3+z_5)^3}{(z_3q^2-z_5)(q^4z_3-z_5)(z_3-q^2z_5)^3}\right)_{|z_5|\gg|z_3|},\\
H_2(z_3/z_5) &= \left(\frac{z_3^2z_5^2(z_3+z_5)^3}{(z_3q^4-z_5)(z_3-q^2z_5)^4}\right)_{|z_5|\gg|z_3|}.
\end{aligned}
$$

It easily follows that

$$
\sum_{\mathbf{A}\in\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\{3,5,6\})}\prod_{p=1}^{3}\delta\left(\frac{z_{A_1^{(p)}}q^2}{z_{A_2^{(p)}}}\right)c_{\mathbf{A}}=0\,. \tag{4.77}
$$

Similar calculations show that, eventually, for every $\mathbf{n}\subset[\![6]\!]$ such that $\operatorname{card}\mathbf{n}=3$ and $\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\mathbf{n})\neq\emptyset$, we have

$$
\sum_{\mathbf{A}\in\mathsf{T}^{(2,2,2)}_{[\![6]\!]}(\mathbf{n})}\prod_{p=1}^{3}\delta\left(\frac{z_{A_1^{(p)}}q^2}{z_{A_2^{(p)}}}\right)c_{\mathbf{A}}=0\,, \tag{4.78}
$$

thus proving the result. □

Appendix A. Formal distributions

A.1. Definitions and main properties

Let $\mathbb{K}$ be a field of characteristic 0. For any $\mathbb{K}$-vector space V, we let $V[z,z^{-1}]$ denote the ring of V-valued Laurent polynomials. Writing

$$
v(z)=\sum_{n\in\mathbb{Z}}v_nz^n\,,
$$

where the sum runs over finitely many terms, for any $v(z) \in V[z, z^{-1}]$, we can define

$$\operatorname{supp}(v(z)) = \{n \in \mathbb{Z} : v_n \neq 0\} ,$$

and set

$$V_n[z, z^{-1}] = \left\{v(z) \in V[z, z^{-1}] : \operatorname{supp}(v(z)) \subseteq [\![-n, n]\!]\right\} .$$

It is clear that, for every $n \in \mathbb{N}$, $V_n[z, z^{-1}] \cong V^{2n+1}$ as $\mathbb{K}$-vector spaces. Now, if in addition V is a topological vector space with topology τ_1, making use of that isomorphism, we can endow $V_n[z, z^{-1}]$ with the box topology of V^{2n+1}, for every $n \in \mathbb{N}$. Denote by τ_n that topology.

The obvious inclusions $V_n[z, z^{-1}] \hookrightarrow V_{n+1}[z, z^{-1}]$ are clearly continuous and we define a topology τ on $V[z, z^{-1}]$ as the inductive limit

$$\tau = \varinjlim \tau_n .$$

We now assume that $\mathbb{K}$ is a topological field.

Definition A.1. — *The space* $V[[z, z^{-1}]]$ *of* V*-valued* formal distributions *is the* $\mathbb{K}$*-vector space of continuous* V*-valued linear functions over the ring of* $\mathbb{K}$*-valued Laurent polynomials* $\mathbb{K}[z, z^{-1}]$*, the latter being endowed with the final topology induced as above from the topology of* $\mathbb{K}$*.*

Proposition A.2. — *Any V-valued formal distribution* $v(z) \in V[[z, z^{-1}]]$ *reads*

$$v(z) = \sum_{n \in \mathbb{Z}} v_n z^n ,$$

for some $(v_n)_{n\in\mathbb{Z}} \in V^{\mathbb{Z}}$ *and the action of* $v(z)$ *on any Laurent polynomial* $f(z) \in \mathbb{K}[z, z^{-1}]$ *is given by*

$$\langle v(z), f(z)\rangle = \operatorname*{res}_z \left(v(z)f(z)z^{-1}\right) ,$$

where we let

$$\operatorname*{res}_z a(z) = \operatorname*{res}_z \left(\sum_{n\in\mathbb{Z}} a_n z^n\right) = a_{-1} ,$$

for any $a(z) \in V[[z, z^{-1}]]$*.* $V[[z, z^{-1}]]$ *is given the weak* $*$*-topology. It is actually a module over the ring* $\mathbb{K}[z, z^{-1}]$ *of* $\mathbb{K}$*-valued Laurent polynomials.*

Proof. — It is clear that, due to its linearity, any $v(z) \in V[[z, z^{-1}]]$ is entirely characterized by the data, for every $n \in \mathbb{N}$, of

$$v_n = \left\langle v(z), z^{-n}\right\rangle \in V .$$

Now, writing $v(z) = \sum_{n\in\mathbb{Z}} v_n z^n$, we also have

$$v_n = \left\langle v(z), z^{-n}\right\rangle = \operatorname*{res}_z v(z) z^{-n-1} ,$$

for every $n \in \mathbb{N}$ and the claim follows. $\square$

Let A be a topological $\mathbb{K}$-algebra. Then $A[[z, z^{-1}]]$ is the space of A-valued formal distributions, i.e. of A-valued linear functions over $A[z, z^{-1}]$. In that case, the action of $a(z) \in A[[z, z^{-1}]]$ on $b(z) \in A[z, z^{-1}]$ is given by

$$\langle a(z), b(z)\rangle = \operatorname*{res}_z a(z)b(z)z^{-1} \,. \tag{A.1}$$

Clearly, $A[[z, z^{-1}]]$ is a module over the ring $A[z, z^{-1}]$ of A-valued Laurent polynomials. It is generally impossible to consistently extend that structure into a full-fledged product over $A[[z, z^{-1}]]$. However, since A is a topological algebra, we can set

$$a(z)b(z) = \sum_{p\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} a_m b_{p-m} \right) z^p \,,$$

whenever the series

$$\sum_{m\in\mathbb{Z}} a_m b_{p-m}$$

is convergent for every $p \in \mathbb{Z}$. If A is complete as a topological algebra, it suffices that the above series be Cauchy.

We let similarly $V[[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]]$ denote the space of V-valued formal distributions in $n \in \mathbb{N}$ variables, so that any V-valued formal distribution $v(z_1, \dots, z_n)$ in n variable reads

$$v(z_1, \dots, z_n) = \sum_{p_1,\dots,p_n\in\mathbb{Z}} v_{p_1,\dots,p_n} z_1^{p_1} \cdots z_n^{p_n} \,,$$

for some $(v_{p_1\dots,p_n})_{p_1,\dots,p_n\in\mathbb{Z}} \in V^{\mathbb{Z}^n}$. For every $i = 1, \dots, n$, we define

$$\operatorname*{res}_{z_i} : V[[z_1, z_1^{-1}, \dots z_n, z_n^{-1}]] \longrightarrow V[[z_1, z_1^{-1}, \dots, \widehat{z_i}, \widehat{z_i^{-1}} \dots, z_n, z_n^{-1}]] \,,$$

where a hat over a variable indicates omission of that variable, by setting

$$\begin{aligned} \operatorname*{res}_{z_i} v(z_1, \dots, z_n) &= \operatorname*{res}_{z_i} \sum_{p_1,\dots,p_n\in\mathbb{Z}} v_{p_1,\dots,p_n} z_1^{p_1} \cdots z_n^{p_n} \\ &= \sum_{p_1,\dots,\hat{p}_i,\dots,p_n\in\mathbb{Z}} v_{p_1,\dots,p_{i-1},-1,p_{i+1},\dots,p_n} z_1^{p_1} \cdots \widehat{z_i^{-1}} \cdots z_n^{p_n} \,. \end{aligned}$$

For every $i = 1, \dots, n$, we define

$$\partial_i : V[[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]] \longrightarrow V[[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]]$$

by setting

$$\partial_i v(z_1, \dots, z_n) = \sum_{p_1,\dots,p_n\in\mathbb{Z}} p_i v_{p_1,\dots,p_n} z_1^{p_1} \cdots z_i^{p_i-1} \cdots z_n^{p_n} \,.$$

If A is a topological $\mathbb{K}$-algebra, then the multiplication in A naturally extends to bilinear maps

$$A[[z_1, z_1^{-1}, \dots, z_m, z_m^{-1}]] \times A[[z_{m+1}, z_{m+1}^{-1}, \dots, z_{m+n}, z_{m+n}^{-1}]] \longrightarrow A[[z_1, z_1^{-1}, \dots, z_{m+n}, z_{m+n}^{-1}]]$$

by setting

$$a(z_1, \dots, z_m)b(z_{m+1}, \dots, z_{m+n}) = \sum_{p_1,\dots,p_{m+n}\in\mathbb{Z}} a_{p_1,\dots,p_m} b_{p_{m+1},\dots,p_{m+n}} z_1^{p_1} \cdots z_{m+n}^{p_{m+n}} .$$

Let $a(z_1, \dots, z_n) \in A[[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]]$ be an A-valued formal distribution in n variables. Since A is a topological $\mathbb{K}$-agebra, we can define the *localization* $a_{|z_{n-1}=z_n}(z_1, \dots, z_{n-1}) \in A[[z_1, z_1^{-1}, \dots, z_{n-1}, z_{n-1}^{-1}]]$ of $a(z_1, \dots, z_n)$ at $z_{n-1} = z_n$, by setting

$$a_{|z_{n-1}=z_n}(z_1, \dots, z_{n-1}) = \sum_{p_1,\dots,p_{n-1}\in\mathbb{Z}} \left(\sum_{p\in\mathbb{Z}} a_{p_1,\dots,p_{n-2},p,p_{n-1}-p} \right) z_1^{p_1} \dots z_{n-1}^{p_{n-1}} ,$$

whenever

$$\sum_{p\in\mathbb{Z}} a_{p_1,\dots,p_{n-2},p,p_{n-1}-p}$$

is convergent. If A is complete as a topological algebra, it suffices that the above series be Cauchy.

A.2. Laurent expansion and the Dirac formal distribution

One way to obtain formal power series is to take the Laurent expansion of some holomorphic function $f : \mathbb{C} \to \mathbb{C}$. We shall usually write $f(z)_{|z|\ll 1}$ to denote the Laurent expansion around 0. Similarly, we shall denote by $f(z)_{|z|\gg 1}$ the Laurent expansion around ∞.

Let

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n .$$

Lemma A.3. — *For every $n \in \mathbb{N}^\times$, we have*

$$\left(\frac{1}{1-z}\right)^n_{|z|\ll 1} - \left(\frac{1}{1-z}\right)^n_{|z|\gg 1} = \frac{\delta^{(n-1)}(z)}{(n-1)!} .$$

Proof. — It is straightforward to check that the result holds for $n = 1$. Assuming it holds for some n, it follows, upon differentiation, that

$$n\left[\left(\frac{1}{1-z}\right)^{n+1}_{|z|\ll 1} - \left(\frac{1}{1-z}\right)^{n+1}_{|z|\gg 1}\right] = \frac{\delta^{(n)}(z)}{(n-1)!}\,,$$

which completes te recursion. □

Lemma A.4. — *For any $n \in \mathbb{N}$ and any A-valued Laurent polynomial $f(z) \in A[z, z^{-1}]$, we have*

$$f(z)\delta^{(n)}(z) = \sum_{p=0}^{n}(-1)^{n-p}\binom{n}{p} f^{(n-p)}(1)\delta^{(p)}(z)\,.$$

Proof. — The case $n = 1$ is straightforward. Assuming the results holds for some $n \in \mathbb{N}$, we have, upon differentiation,

$$f'(z)\delta^{(n)}(z) + f(z)\delta^{(n+1)}(z) = \sum_{p=0}^{n}(-1)^{n-p}\binom{n}{p} f^{(n-p)}(1)\delta^{(p+1)}(z)\,,$$

which completes the recursion. □

Example A.5. — In particular, for any A-valued formal distribution $f(z_1, z_2) \in A[[z_1, z_1^{-1}, z_2, z_2^{-1}]]$ with a well-defined localization $f_{|z_1=z_2}(z_1)$ (see previous subsection for a definition), we have

$$f(z_1, z_2)\delta\left(\frac{z_1}{z_2}\right) = f_{|z_1=z_2}(z_1)\delta\left(\frac{z_1}{z_2}\right)\,,$$

Assuming that $\mathbb{K}$ is an algebraically closed field, we have

Lemma A.6. — *Let $P(z) \in \mathbb{K}[z]$ be a polynomial of degree N, with roots $\{\lambda_i : i \in [\![n]\!]\}$ and respective multiplicities $\{m_i : i \in [\![n]\!]\}$. If $a(z) \in \mathbb{K}[[z, z^{-1}]]$ is a $\mathbb{K}$-valued formal distribution, then*

$$P(z)a(z) = 0 \Longleftrightarrow a(z) = \sum_{i=1}^{n}\sum_{p_i=0}^{m_i-1}\alpha_{i,p}\delta^{(p_i)}\left(\frac{z}{\lambda_i}\right)\,,$$

for some $\alpha_{i,p} \in \mathbb{K}$.

Proof. — The if part is easily checked making use of the previous lemma. The only if part follows by an easy recursion, after writing that $P(z) = \prod_{i\in[\![n]\!]}(z - \lambda_i)^{m_i}$. □

Lemma A.7. — *Let $P(z), Q(z) \in \mathbb{K}[z]$ be two coprime polynomials. Let $\{\lambda_i : i \in [\![n]\!]\}$ be the set of roots of $Q(z)$ and let $\{m_i : i \in [\![n]\!]\}$ be their respective multiplicities. Then, in $\mathbb{K}[[z, z^{-1}]]$,*

$$\text{(A.2)}\quad \left(\frac{P(z)}{Q(z)}\right)_{|z|\ll 1} - \left(\frac{P(z)}{Q(z)}\right)_{|z|\gg 1} = \sum_{i=1}^{n}\sum_{p_i=0}^{m_i-1} \frac{(-1)^{p_i+1}\alpha_{i,p_i+1}}{(p_i)!\lambda_i^{p_i+1}}\delta^{(p_i)}\left(\frac{z}{\lambda_i}\right),$$

where, for every $i \in [\![n]\!]$ and every $p_i \in [\![m_i]\!]$, α_{i,p_i} is obtained from the partial fraction decomposition

$$\text{(A.3)}\quad \frac{P(z)}{Q(z)} = A(z) + \sum_{i=1}^{n}\sum_{p_i=1}^{m_i} \frac{\alpha_{i,p_i}}{(z-\lambda_i)^{p_i}},$$

in which $A(z) \in \mathbb{K}[z]$ is a polynomial of degree $\deg(P) - \deg(Q)$.

Proof. — Given the partial fraction decomposition (A.3), we can write

$$\begin{aligned}\text{(A.4)}\quad &\left(\frac{P(z)}{Q(z)}\right)_{|z|\ll 1} - \left(\frac{P(z)}{Q(z)}\right)_{|z|\gg 1} \\ &= \sum_{i=1}^{n}\sum_{p_i=1}^{m_i} \alpha_{i,p_i}\left[\left(\frac{1}{(z-\lambda_i)^{p_i}}\right)_{|z|\ll 1} - \left(\frac{1}{(z-\lambda_i)^{p_i}}\right)_{|z|\gg 1}\right] \\ &= \sum_{i=1}^{n}\sum_{p_i=1}^{m_i} \frac{(-1)^{p_i}\alpha_{i,p_i}}{(p_i-1)!\lambda_i^{p_i}}\delta^{(p_i-1)}\left(\frac{z}{\lambda_i}\right)\end{aligned}$$

where we have used Lemma A.3 to derive the last equality. The claim obviously follows. □

A.3. The structure power series $G^{\pm}_{ij}(z)$

In this last subsection, we derive identities involving the structure power series $G^{\pm}_{ij}(z)$ by applying Lemma A.7. Remember (see Remark 2.2) that in type $\dot{\mathfrak{a}}_1$, we have $G^{\pm}_{10}(z) = G^{\mp}_{11}(z)$.

Proposition A.8. — *The following hold true in $\mathbb{F}[[z, z^{-1}]]$.*

(i) *For every $p \in \mathbb{Z} - \{2\}$,*

$$\text{(A.5)}\quad \frac{G^{+}_{10}(zq^{p})G^{+}_{11}(zq^{-p}) - G^{-}_{10}(z^{-1}q^{-p})G^{-}_{11}(z^{-1}q^{p})}{q-q^{-1}} = \frac{[2]_q[p]_q}{[p-2]_q}\left[\delta\left(zq^{2-p}\right) - \delta\left(zq^{p-2}\right)\right].$$

In particular, when $p = 1$, we have

$$\text{(A.6)} \quad \frac{G^+_{10}(zq)G^+_{11}(zq^{-1}) - G^-_{10}(z^{-1}q^{-1})G^-_{11}(z^{-1}q)}{q-q^{-1}} = [2]_q \left[\delta\left(zq^{-1}\right) - \delta\left(zq\right)\right] .$$

If $p = 2$, we have instead

$$\text{(A.7)} \quad \frac{G^+_{10}(zq^2)G^+_{11}(zq^{-2}) - G^-_{10}(z^{-1}q^{-2})G^-_{11}(z^{-1}q^2)}{(q-q^{-1})^2} = [2]_q^2 \left[\delta\left(z\right) - \delta'\left(z\right)\right] .$$

(ii) *Similarly,*

$$\text{(A.8)} \quad \frac{G^+_{11}(zq^{-2})^2 - G^-_{11}(z^{-1}q^2)^2}{(q-q^{-1})^2} = \frac{2q^{-2}[2]_q}{q-q^{-1}}\delta\left(z\right) + [2]_q^2\delta'\left(z\right) .$$

Proof. — In each case, it suffices to determine the partial fraction decomposition of the l.h.s and to apply Lemma A.7 to get the desired result. ☐

BIBLIOGRAPHY

[1] H. Awata, H. Kanno, A. Mironov, A. Morozov, A. Morozov, Y. Ohkubo & Y. Zenkevich, "Generalized Knizhnik–Zamolodchikov equation for Ding–Iohara–Miki algebra", *Phys. Rev. D* **96** (2017), no. 2, article no. 026021 (19 pages).

[2] J. Beck, "Braid group action and quantum affine algebras", *Commun. Math. Phys.* **165** (1994), no. 3, p. 555-568.

[3] I. Cherednik, *Double affine Hecke algebras*, London Mathematical Society Lecture Note Series, vol. 319, Cambridge University Press, 2005, xii+434 pages.

[4] I. Damiani, "A basis of type Poincaré–Birkhoff–Witt for the quantum algebra of $\widehat{\mathrm{sl}}(2)$", *J. Algebra* **161** (1993), no. 2, p. 291-310.

[5] J. Ding & I. B. Frenkel, "Isomorphism of two realizations of quantum affine algebra $U_q(\mathfrak{gl}(n))$", *Commun. Math. Phys.* **156** (1993), no. 2, p. 277-300.

[6] J. Ding & K. Iohara, "Generalization of Drinfeld quantum affine algebras", *Lett. Math. Phys.* **41** (1997), no. 2, p. 181-193.

[7] J. Ding & S. Khoroshkin, "Weyl group extension of quantized current algebras", *Transform. Groups* **5** (2000), no. 1, p. 35-59.

[8] V. G. Drinfeld, unpublished note.

[9] B. Feigin, E. Feigin, M. Jimbo, T. Miwa & E. Mukhin, "Quantum continuous $\mathfrak{gl}_\infty$: semiinfinite construction of representations", *Kyoto J. Math.* **51** (2011), no. 2, p. 337-364.

[10] B. Feigin, M. Jimbo, T. Miwa & E. Mukhin, "Quantum toroidal $\mathfrak{gl}_1$-algebra: plane partitions", *Kyoto J. Math.* **52** (2012), no. 3, p. 621-659.

[11] ———, "Quantum toroidal $\mathfrak{gl}_1$ and Bethe ansatz", *J. Phys. A. Math. Theor.* **48** (2015), no. 24, article no. 244001 (27 pages).

[12] V. Ginzburg, M. Kapranov & É. Vasserot, "Langlands reciprocity for algebraic surfaces", *Math. Res. Lett.* **2** (1995), no. 2, p. 147-160.

[13] N. Guay, H. Nakajima & C. Wendlandt, "Coproduct for Yangians of affine Kac-Moody algebras", *Adv. Math.* **338** (2018), p. 865-911.

[14] D. Hernandez, "Representations of quantum affinizations and fusion product", *Transform. Groups* **10** (2005), no. 2, p. 163-200.

[15] ———, "Quantum toroidal algebras and their representations", *Sel. Math., New Ser.* **14** (2009), no. 3-4, p. 701-725.

[16] J. C. Jantzen, *Lectures on quantum groups*, Graduate Studies in Mathematics, vol. 6, American Mathematical Society, 1996, viii+266 pages.

[17] V. Kac, *Vertex algebras for beginners*, second ed., University Lecture Series, vol. 10, American Mathematical Society, 1998, vi+201 pages.

[18] I. G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Cambridge Tracts in Mathematics, vol. 157, Cambridge University Press, 2003, x+175 pages.

[19] K. Miki, "Toroidal braid group action and an automorphism of toroidal algebra $U_q(\mathrm{sl}_{n+1,\mathrm{tor}})(n \geqslant 2)$", *Lett. Math. Phys.* **47** (1999), no. 4, p. 365-378.

[20] ———, "A (q,γ) analog of the $W_{1+\infty}$ algebra", *J. Math. Phys.* **48** (2007), no. 12, article no. 123520 (35 pages).

[21] H. Nakajima, "Quiver varieties and finite-dimensional representations of quantum affine algebras", *J. Am. Math. Soc.* **14** (2001), no. 1, p. 145-238.

[22] A. Negut, "Quantum toroidal and shuffle algebras, R-matrices and a conjecture of Kuznetsov", 2013, `https://arxiv.org/abs/1302.6202v2`.

[23] O. Schiffmann, "Drinfeld realization of the elliptic Hall algebra", *J. Algebr. Comb.* **35** (2012), no. 2, p. 237-262.

[24] M. Varagnolo & É. Vasserot, "Schur duality in the toroidal setting", *Commun. Math. Phys.* **182** (1996), no. 2, p. 469-483.

[25] R. Zegers & E. Mounzer, "Weight-finite modules over the quantum affine and double quantum affine algebras of type $\mathfrak{a}_1$", *Algebr. Represent. Theory* **25** (2022), no. 6, p. 1631-1684.

Manuscrit reçu le 27 octobre 2020,
accepté le 2 octobre 2023.

Robin ZEGERS
Université Paris-Saclay, CNRS, IJCLab
91405, Orsay (France)
robin.zegers@universite-paris-saclay.fr

Elie MOUNZER
Université Paris-Saclay, CNRS, IJCLab
91405, Orsay (France)