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THE MOSER ISOTOPY FOR HOLOMORPHIC SYMPLECTIC AND C-SYMPLECTIC STRUCTURES

by Andrey SOLDATENKOV & Misha VERBITSKY (*)

ABSTRACT. — A \mathbb{C} -symplectic structure is a complex-valued 2-form which is holomorphically symplectic for an appropriate complex structure. We prove an analogue of Moser's isotopy theorem for families of \mathbb{C} -symplectic structures and list several applications of this result. We prove that the degenerate twistorial deformation associated to a holomorphic Lagrangian fibration is locally trivial over the base of this fibration. This is used to extend several theorems about Lagrangian fibrations, known for projective hyperkähler manifolds, to the non-projective case. We also exhibit new examples of non-compact complex manifolds with infinitely many pairwise non-birational algebraic compactifications.

RÉSUMÉ. — Une structure \mathbb{C} -symplectique est une 2-forme à valeurs complexes qui est holomorphiquement symplectique pour une structure complexe appropriée. Nous prouvons un analogue du théorème d'isotopie de Moser pour les familles de structures \mathbb{C} -symplectiques et donnons plusieurs applications de ce résultat. Nous prouvons que la déformation twistorielle dégénérée associée à une fibration lagrangienne holomorphe est localement triviale sur la base de cette fibration. Ceci est utilisé pour étendre plusieurs théorèmes sur les fibrations lagrangiennes, connus pour les variétés hyperkähler projectives, au cas non projectif. Nous présentons également de nouveaux exemples de variétés complexes non compactes avec une infinité de compactifications algébriques non birationnelles deux-à-deux.

1. Introduction

Let X be a complex manifold. Recall that a holomorphic 2-form $\Omega \in H^0(X, \Omega_X^2)$ is symplectic if it is closed and non-degenerate at every point of X . Viewing Ω as a section of the bundle $\Lambda^{2,0}X$, we note that one can

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uniquely recover the complex structure of X from the form Ω . Namely, the subbundle $T^{0,1}X$ is the kernel of Ω and $T^{1,0}X$ is its complex conjugate. We see that the complex 2-form Ω alone can be used to encode both the complex structure and the symplectic structure of X . Abstracting the properties of Ω that are necessary to reconstruct the complex structure, we arrive at the notion of a C-symplectic form, see Definition 2.1.

A similar point of view on holomorphic symplectic structures appeared in the work of Hitchin [11] and was further explored in [30] and [6]. One advantage of this point of view is that we can describe deformations of both the complex and symplectic structures on X simultaneously.

Our aim is to investigate the properties of C-symplectic structures further. Recall that the ordinary real symplectic structures admit a simple local description. Namely, by Darboux theorem, any real symplectic manifold M is locally symplectomorphic to an open subset in a vector space with the standard symplectic form. Moreover, any compact Lagrangian submanifold $N \subset M$ admits a neighbourhood that is symplectomorphic to a neighbourhood of the zero section in T^*N with its canonical symplectic form, see e.g. [23, Section 3]. To prove these statements one can use an idea that goes back to Moser [24] and construct an isotopy that maps one symplectic form to another.

In the present paper, we prove some analogues of Moser's isotopy theorem for C-symplectic structures (Section 2.3) and discuss their applications (Section 3). It turns out that it is useful to apply Moser's isotopy to holomorphic Lagrangian fibrations. Given such a fibration $\pi: X \rightarrow S$ over a projective base S , we consider a degenerate twistorial deformation of X (introduced in [30]), which is a family of holomorphic symplectic manifolds parametrized by \mathbb{C} (see Section 2.2 for details). Such families have been classically studied in the case when X is a K3 surface, see e.g. the discussion in [13], where they are called Brauer families. In [18] these families are studied in the case when X is of K3^[n]-type.

Our main observation is the following. Given a hyperplane section $D \subset S$, we consider its open complement $U \subset S$. If \mathcal{X}_t is the fibre of the degenerate twistorial deformation of X over the point $t \in \mathbb{C}$, it still admits a holomorphic Lagrangian fibration over S . Let $\mathcal{X}_{U,t}$ be the preimage of U in \mathcal{X}_t . We prove in Theorem 3.1 that $\mathcal{X}_{U,t}$ are isomorphic as complex manifolds for all $t \in \mathbb{C}$. This has several implications.

First, we construct examples of complex manifolds that admit infinitely many structures of a quasi-projective variety that are pairwise non-birational, see Corollary 3.4. Examples of complex manifolds that admit

several non-isomorphic algebraic structures are well known, see e.g. [10, Chapter VI, Section 3] and [14]. One may also compare our examples with those arising from non-abelian Hodge theory of Hitchin–Simpson. Recall that de Rham and Betti moduli spaces are isomorphic as complex analytic varieties (the Riemann–Hilbert correspondence), but not as algebraic varieties, see e.g. [28, Proposition 9].

Next, we observe that any fibre of a holomorphic Lagrangian fibration admits a Zariski-open neighbourhood that has a structure of a quasi-projective variety, see Corollary 3.5. In particular, any Lagrangian fibration is a Zariski-locally projective morphism. This strengthens a result of Campana [7] who proved analogous statement in the analytic topology. Using the same idea, we show in Section 3.4 that any hyperkähler manifold admitting a Lagrangian fibration contains an open dense subset that is Stein. This is related to a question of Bogomolov about the existence of “Stein cells” in compact complex manifolds, see [5]. We use quasi-projective neighbourhoods to generalize to the non-projective setting the results of Matsushita [21] about higher direct images of the structure sheaf, see Corollary 3.10. This shows how to obtain Matshushita’s theorem in full generality and in elementary way, without using more complicated techniques of Saito [27].

In Section 3.6 we consider a holomorphic symplectic manifold that admits a bimeromorphic contraction such that the contraction centre is mapped to a point. We use Moser’s isotopies to show that such contraction centres are rigid. More precisely, in a family of such contractions the contraction centres of nearby fibres are isomorphic, and moreover the isomorphisms are induced by symplectomorphisms of open neighbourhoods of the contraction centres, see Theorem 3.11.

The Moser isotopy method which we describe can be considered as an analytic version of the work [16], which deals with formal deformations of holomorphic symplectic manifolds, not necessarily compact. In [16] it was shown that any deformation of a holomorphic symplectic manifold with $H^{>0}(X, \mathcal{O}_X) = 0$ is trivial in the formal category, as long as the de Rham cohomology class of its holomorphic symplectic form does not change. This theorem, which has many useful applications, is in many aspects deficient. Indeed, a formal deformation of a Stein manifold or an affine manifold is always trivial. Using the Moser isotopy for \mathbb{C} -symplectic structures, we can prove a local analytic version of this formal deformation theorem, Corollary 2.6.

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2. C-symplectic structures

2.1. Definition and main properties

We recall the definition of a C-symplectic structure from [6].

DEFINITION 2.1. — *Let X be a real $4n$ -dimensional C^∞ -manifold. A C-symplectic form on X is a smooth section Ω of the complexified bundle of 2-forms $\Lambda_{\mathbb{C}}^2 X$ that satisfies three conditions:*

- (1) $d\Omega = 0$,
- (2) $\Omega^{n+1} = 0$,
- (3) $\Omega^n \wedge \bar{\Omega}^n \neq 0$ pointwise on X .

Decompose a C-symplectic form into its real and imaginary parts: $\Omega = \omega_1 + i\omega_2$. Recall the following properties (see [6]).

- (1) Both ω_1 and ω_2 are real symplectic forms.
- (2) The kernel of Ω has rank $2n$ at all points of X . Let the subbundle $T^{0,1}X \subset T_{\mathbb{C}}X$ be the kernel of Ω and $T^{1,0}X$ its complex conjugate. Then $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$.
- (3) The subbundle $T^{1,0}X$ is closed under the Lie bracket. This defines an integrable almost-complex structure on X . Therefore any manifold with a C-symplectic form admits an intrinsically defined complex structure.
- (4) The complex structure operator $I \in \text{End}(TX)$ can be expressed as follows: $I = \omega_2^{-1} \circ \omega_1$. Here we view the 2-forms ω_j as morphisms $TX \rightarrow \Lambda^1 X$ defined by contracting ω_j with the tangent vectors.
- (5) With respect to the above complex structure, the form Ω is holomorphically symplectic.

2.2. Lagrangian fibrations

We will be interested in certain families of C-symplectic structures that arise from holomorphic Lagrangian fibrations. Let X be a complex manifold of dimension $2n$ and S a normal complex analytic variety of dimension

n . Let $\pi: X \rightarrow S$ be a proper surjective holomorphic morphism with connected fibres. Let $S^\circ \subset S$ be the maximal open subset over which the morphism π is smooth. Recall that by Sard's theorem S° is non-empty and dense in S . We denote by π° the restriction of π to $X^\circ = \pi^{-1}(S^\circ)$.

Assume that $\Omega \in \Lambda^{2,0}X$ is a holomorphic symplectic form. We will call π as above a *holomorphic Lagrangian fibration* if for any fibre F of π° we have $\Omega|_F = 0$. Since the fibres are n -dimensional, the latter condition means that F is Lagrangian. It is known that F is a complex torus, see e.g. [8, Theorem 2.6].

Remark 2.2. — The base S is smooth in all known examples of Lagrangian fibrations that are of interest for us. However this condition is not really necessary, so we do not include it in the definition. We will need to use C^∞ differential forms on S , and in the case of singular S we define them as in [29].

Let $\eta \in \Lambda^{2,0}S \oplus \Lambda^{1,1}S$ be a closed 2-form. For any $t \in \mathbb{C}$ define $\Omega_t = \Omega + t\pi^*\eta$.

THEOREM 2.3 ([6]). — *In the above setting, the 2-forms Ω_t have the following properties.*

- (1) Ω_t is a C -symplectic form on X for any $t \in \mathbb{C}$.
- (2) If I_t denotes the complex structure corresponding to Ω_t , then π is a holomorphic Lagrangian fibration from (X, I_t) to S for any $t \in \mathbb{C}$.

Proof.

Part (1). — It is clear that Ω_t is closed. We will show that $\Omega_t^{n+1} = 0$ and $\Omega_t^n \wedge \bar{\Omega}_t^n = \Omega^n \wedge \bar{\Omega}^n$ for any $t \in \mathbb{C}$. It is enough to check both equalities on the dense open subset X° .

The tangent space at any point $x \in X^\circ$ can be decomposed into a direct sum of two Lagrangian subspaces: $T_x^{1,0}X \simeq T_x^{1,0}F \oplus W$, where F is the fibre of π through x . We can choose bases in the two Lagrangian subspaces that are dual with respect to Ω : $T_x^{1,0}F = \langle e_1, \dots, e_n \rangle$, $W = \langle f_1, \dots, f_n \rangle$, so that $\Omega(x) = \sum_j e_j^* \wedge f_j^*$. Assume that $\alpha \in \Lambda^{p,q}S$. Then $(\pi^*\alpha)(x) \in \Lambda^p W^* \otimes \Lambda^q \bar{W}^*$ and from the formula for Ω it is clear that

$$(2.1) \quad \Omega^k \wedge \pi^*\alpha = 0 \quad \text{if } p + k > n.$$

The equality (2.1) clearly implies that $\Omega_t^{n+1} = 0$. We write $\eta = \eta_1 + \eta_2$ with $\eta_1 \in \Lambda^{2,0}S$ and $\eta_2 \in \Lambda^{1,1}S$ and consider the form $\Omega_t^n \wedge \bar{\Omega}_t^n = (\Omega + t\pi^*\eta)^n \wedge (\bar{\Omega} + \bar{t}\pi^*\bar{\eta})^n$. Opening the brackets, we obtain a sum, where the summands equal up to a constant $\Omega^{k_1} \wedge \bar{\Omega}^{k_2} \wedge \pi^*\alpha$ with $\alpha = \eta_1^{l_1} \wedge \eta_2^{m_1} \wedge \bar{\eta}_1^{l_2} \wedge \bar{\eta}_2^{m_2}$ and $k_1 + l_1 + m_1 = k_2 + l_2 + m_2 = n$. The form α has Hodge type

$(2l_1 + m_1 + m_2, 2l_2 + m_1 + m_2)$. Then the formula (2.1) and its complex conjugate imply that our summand is zero unless $l_1 = m_1 = l_2 = m_2 = 0$. This proves that $\Omega_t^n \wedge \bar{\Omega}_t^n = \Omega^n \wedge \bar{\Omega}^n \neq 0$ pointwise on X and shows that Ω_t is a \mathbb{C} -symplectic form.

Part (2). — To prove that π is a holomorphic map from (X, I_t) to S it is again enough to restrict to the dense open subset X° . Using the splitting of the tangent space at arbitrary $x \in X^\circ$ as above, we write

$$(\pi^*\eta)(x) = \sum_{j,k} (a_{jk} f_j^* \wedge f_k^* + b_{jk} f_j^* \wedge \bar{f}_k^*).$$

Then it is clear that the kernel of $\Omega_t(x) = \sum_j e_j^* \wedge f_j^* + t(\pi^*\eta)(x)$ is spanned by the vectors

$$\bar{e}_j, j = 1 \dots n \quad \text{and} \quad \bar{f}_j + t \sum_{l=1}^n b_{lj} e_l, j = 1 \dots n.$$

Since $e_l \in \ker d\pi$, we see that $d\pi(T_{I_t}^{0,1} X) \subset T^{0,1} S$, hence $d\pi$ is complex-linear and π is holomorphic. To prove that it is a Lagrangian fibration note that for any fibre $F \subset X^\circ$ we clearly have $\Omega_t|_F = 0$. □

In the case when X is a compact hyperkähler manifold, it has been shown in [30] that the complex manifolds (X, I_t) form a smooth complex analytic family over the complex line \mathbb{C} which we denote by \mathcal{X} . This family is called *degenerate twistorial deformation* of X . The terminology can be explained as follows. Inside the period domain \mathcal{D} the usual twistor conic can degenerate into the union of a pair of projective lines. Removing the point of intersection of those lines, we get a disjoint union of two affine lines. The periods of the fibres of the degenerate twistorial deformation form one of such affine lines. The two lines are exchanged by complex conjugation, i.e. one obtains the family parametrized by the second affine line if one replaces the holomorphic form Ω with $\bar{\Omega}$ in the construction described above.

2.3. Moser’s isotopy

Recall the idea of Moser’s isotopy theorem: given a family of real symplectic forms ω_t parametrized by $t \in [0, 1]$ such that the cohomology classes $[\omega_t] \in H^2(X, \mathbb{R})$ do not depend on t , one constructs a flow of diffeomorphisms φ_t such that $\omega_0 = \varphi_t^* \omega_t$, see [24, Theorem 2]. Our aim is to prove a similar statement for \mathbb{C} -symplectic forms. However, no strict analogy is apparent, because certain additional cohomological obstructions arise, and one needs to introduce some further conditions to construct the isotopy.

We work in the following setting: X is a manifold of real dimension $4n$, $\Omega_t \in \Lambda_{\mathbb{C}}^2 X$ is a family of \mathbb{C} -symplectic forms for $t \in [0, 1]$. When talking about families of tensor fields, we always assume that the dependence on t is continuous and, if necessary, sufficiently differentiable. We denote by I_t the complex structure corresponding to Ω_t and by $\mathcal{X}_t = (X, I_t)$ the corresponding complex manifold. If we use the (p, q) -decomposition with respect to I_t , the corresponding bundles have subscript I_t , e.g. $\Lambda_{I_t}^{p,q} X$. We denote by \mathcal{L}_V the Lie derivative along a vector field V .

LEMMA 2.4. — *In the above setting assume that there exists a family of complex 1-forms $\alpha_t \in \Lambda_{\mathbb{C}}^1 X$ such that for all $t \in [0, 1]$ we have*

- (1) $\alpha_t \in \Lambda_{I_t}^{1,0} X$,
- (2) $\frac{d}{dt} \Omega_t = d\alpha_t$.

Then there exists a family of real vector fields $V_t \in TX$ such that $\frac{d}{dt} \Omega_t = -\mathcal{L}_{V_t} \Omega_t$.

Proof. — Since the forms Ω_t are closed, we have $\mathcal{L}_{V_t} \Omega_t = d(V_t \lrcorner \Omega_t)$ for any real vector field V_t . Decompose $V_t = V_t^{1,0} + V_t^{0,1}$, where $V_t^{1,0} \in T_{I_t}^{1,0} X$ and $V_t^{0,1} = \overline{V_t^{1,0}}$, and note that $V_t \lrcorner \Omega_t = V_t^{1,0} \lrcorner \Omega_t \in \Lambda_{I_t}^{1,0} X$. Since the 2-form Ω_t is symplectic for any t , the map $W \mapsto W \lrcorner \Omega_t$ defines an isomorphism $T_{I_t}^{1,0} X \xrightarrow{\sim} \Lambda_{I_t}^{1,0} X$. Applying the inverse of this isomorphism to $-\alpha_t$, we get the vector field $V_t^{1,0}$, and its real part is the desired real vector field. □

In the setting of the above lemma, it remains to integrate the family of vector fields V_t in order to obtain isotopies that deform Ω_0 into Ω_t .

THEOREM 2.5 (Moser’s isotopy, version I). — *In the above setting, assume additionally that $H^1(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 0$ and $[\Omega_t] = [\Omega_0] \in H^2(X, \mathbb{C})$ for all $t \in [0, 1]$. In this case:*

- (1) *There exists a family of vector fields V_t such that $\frac{d}{dt} \Omega_t = -\mathcal{L}_{V_t} \Omega_t$;*
- (2) *If X is compact, then there exists a family of diffeomorphisms φ_t such that $\varphi_0 = \text{id}$ and $\varphi_t^* \Omega_t = \Omega_0$ for $t \in [0, 1]$.*

Proof.

(1). — We need to find a family of 1-forms α_t that satisfies the conditions of Lemma 2.4. By our assumption $[\Omega_t] = [\Omega_0]$ for all $t \in [0, 1]$, so $\Omega_{t+\varepsilon} - \Omega_t$ is exact. It follows that there exists a family of 1-forms $\beta_t \in \Lambda_{\mathbb{C}}^1 X$ such that $\frac{d}{dt} \Omega_t = d\beta_t$. Differentiating the equation $\Omega_t^{n+1} = 0$ we see that $\Omega_t^n \wedge \frac{d}{dt} \Omega_t = 0$. Since Ω_t^n is a holomorphic volume form, we deduce that $\frac{d}{dt} \Omega_t \in \Lambda_{I_t}^{2,0} \oplus \Lambda_{I_t}^{1,1}$. Decomposing $\beta_t = \beta_t^{1,0} + \beta_t^{0,1}$, we see that $\bar{\partial} \beta_t^{0,1} = 0$. By our assumption $H^1(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 0$, therefore $\beta_t^{0,1} = \bar{\partial} f_t$ for some f_t . Define

$\alpha_t = \beta_t^{1,0} - \partial f_t$. Then $d\alpha_t = d\beta_t^{1,0} - \bar{\partial}\partial f_t = d\beta_t^{1,0} + \partial\bar{\partial}f_t = d\beta_t$ and α_t is of type $(1, 0)$ as required. By Lemma 2.4 we obtain a family of vector fields V_t .

(2). — Since X is compact, we can integrate V_t to a flow of diffeomorphisms $\varphi_t, t \in [0, 1]$ with $\varphi_0 = \text{id}$. We compute:

$$\frac{d}{dt}(\varphi_t^*\Omega_t) = \varphi_t^* \left(\mathcal{L}_{V_t}\Omega_t + \frac{d}{dt}\Omega_t \right) = 0$$

by the definition of V_t in part (1). Since $\varphi_0^*\Omega_0 = \Omega_0$, we conclude that $\varphi_t^*\Omega_t = \Omega_0$ for all $t \in [0, 1]$. □

This result can be applied to compact hyperkähler manifolds, which satisfy $H^1(X, \mathcal{O}_X) = 0$, but in that case it easily follows from Bogomolov’s version of Kuranishi–Kodaira–Spencer theory, [4]. On non-compact manifolds, the assumption $H^1(X, \mathcal{O}_X) = 0$ is often quite restrictive. Moreover, on non-compact manifolds one needs some additional information to guarantee that the flow φ_t exists globally on X . Without such information, integrating the vector field V_t in a neighbourhood of a compact subset, one can deduce from Theorem 2.5 the following local statement.

COROLLARY 2.6. — *Let $\pi: \mathcal{X} \rightarrow \Delta$ be a smooth family of holomorphic symplectic manifolds (not necessarily compact) over the unit disc, trivial as a family of C^∞ manifolds. Denote by $\mathcal{X}_t = \pi^{-1}(t)$ a fibre, and by $\Omega_t \in H^0(\mathcal{X}_t, \Omega_{\mathcal{X}_t}^2)$ its holomorphic symplectic form. Using the C^∞ trivialization to identify cohomology groups of the fibres, assume that the cohomology class of Ω_t does not depend on $t \in \Delta$, and $H^1(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 0$. Let $K \subset \mathcal{X}_{t_0}$ be a compact subset. Then there exists an open neighbourhood $U \subset \Delta$ of $t_0 \in \Delta$, and an open subset $\tilde{U} \subset \pi^{-1}(U)$, with $K \subset \tilde{U}$, with the following property. The set \tilde{U} is locally trivially fibred over U , with all fibres $\tilde{U} \cap \pi^{-1}(t), t \in U$ isomorphic as holomorphic symplectic manifolds.*

It turns out that one can prove a useful version of Moser’s theorem in the situation when the family Ω_t comes from a Lagrangian fibration.

THEOREM 2.7 (Moser’s isotopy, version II). — *Let $\pi: X \rightarrow S$ be a holomorphic Lagrangian fibration in the sense of Section 2.2. Denote by Ω the holomorphic symplectic form on X and assume that $\alpha \in \Lambda^{1,0}S$. Let $\Omega_t = \Omega + t\pi^*(d\alpha)$ for $t \in [0, 1]$ be the family of C -symplectic forms on X . Then there exists a family of diffeomorphisms φ_t of X such that $\varphi_0 = \text{id}$ and $\varphi_t^*\Omega_t = \Omega_0$ for $t \in [0, 1]$. ⁽¹⁾*

(1) A version of this theorem was independently obtained by Abasheva and Rogov in [1].

Proof. — As we recalled in Section 2.2, the 2-forms Ω_t are C-symplectic and π is a holomorphic Lagrangian fibration on (X, I_t) . It follows that $\pi^*\alpha$ is in $\Lambda_{I_t}^{1,0}X$ for any $t \in [0, 1]$. Hence we can apply Lemma 2.4 with $\alpha_t = \pi^*\alpha$ (independent on t) and obtain a family of vector fields V_t .

We need to integrate V_t to a flow of diffeomorphisms. Recall that by the construction in the proof of Lemma 2.4 we have $V_t \lrcorner \Omega_t = -\alpha_t$. Let F be a smooth fibre of π and $W \in T^{1,0}F$ some locally defined vector field. Then $\Omega_t(V_t, W) = -\alpha_t(W) = -(\pi^*\alpha)(W) = 0$. Since F is Lagrangian and W arbitrary, the last equality implies that V_t is tangent to F . So $d\pi(V_t) = 0$ at all points of smooth fibres, hence everywhere on X , because smooth fibres are dense.

We conclude that integral curves of V_t are contained in the fibres of π . Let now F be an arbitrary fibre. By our assumptions about Lagrangian fibrations, F is compact. This implies that any integral curve of V_t that starts at a point of F can be extended for all $t \in [0, 1]$. Since this is true for all fibres, we get a well-defined flow of diffeomorphisms $\varphi_t, t \in [0, 1]$ and the rest of the proof goes as in Theorem 2.5. □

3. Applications

We apply Theorem 2.7 to Lagrangian fibrations on compact hyperkähler manifolds of maximal holonomy, that is, compact simply connected Kähler manifolds X with $H^0(X, \Omega_X^2)$ spanned by a symplectic form Ω . Further on, we shall call these manifolds just “hyperkähler manifolds”. We let $2n = \dim_{\mathbb{C}} X$. Assume that $\pi: X \rightarrow S$ is a holomorphic Lagrangian fibration in the sense explained in Section 2.2. For the discussion of basic properties of such fibrations we refer to [12, 19] and [20]. It is explained in [2, footnote on page 53] that in this setting the base S is always projective.

3.1. A degenerate twistorial deformation is Zariski locally trivial over the base

We fix a projective embedding $\iota: S \hookrightarrow \mathbb{P}^m$, let $\omega_{FS} \in \Lambda^{1,1}\mathbb{P}^m$ be the Fubini–Study form and $\eta = \iota^*\omega_{FS}$. Setting $\Omega_t = \Omega + t\pi^*\eta, t \in \mathbb{C}$, we get a family of C-symplectic structures that form a degenerate twistorial deformation of X . We use the notation $\mathcal{X}_t = (X, I_t)$ as above.

Let $H \subset \mathbb{P}^m$ be an arbitrary hyperplane and $U = S \setminus H$ its open complement in S . We denote by $\mathcal{X}_{U,t}$ the preimage of U in \mathcal{X}_t and $X_U = \mathcal{X}_{U,0}$.

THEOREM 3.1. — *For any $t_1, t_2 \in \mathbb{C}$ the complex manifolds \mathcal{X}_{U,t_1} and \mathcal{X}_{U,t_2} are isomorphic.*

Proof. — It is enough to prove that \mathcal{X}_{U,t_1} is isomorphic to $\mathcal{X}_{U,0}$. For this we shall use Moser’s isotopy theorem 2.7.

We may choose local coordinates $(x_0 : \dots : x_m)$ on \mathbb{P}^m such that H is given by $x_0 = 0$. In the open affine chart $x_0 \neq 0$ the Fubini–Study form is given by

$$\omega_{FS}|_{\mathbb{P}^m \setminus H} = i\partial\bar{\partial} \log \left(1 + \sum_{j=1}^m |x_j|^2 \right)$$

Note that U is the intersection of $\mathbb{P}^m \setminus H$ with S . We let α be the restriction of the 1-form $-it_1\partial \log(1 + \sum_{j=1}^m |x_j|^2)$ to U . Then $\alpha \in \Lambda^{1,0}U$ and $Hu1\alpha = t_1\omega_{FS}|_U$. We can apply Theorem 2.7, obtaining a flow of diffeomorphisms $\varphi_t, t \in [0, 1]$, such that $\varphi_1^*\Omega_{t_1}|_{\pi^{-1}(U)} = \Omega|_{\pi^{-1}(U)}$. Then φ_1 defines an isomorphism of complex manifolds $\mathcal{X}_{U,0} \xrightarrow{\sim} \mathcal{X}_{U,t_1}$. \square

Remark 3.2. — The restriction of the Fubini–Study form to S represents the cohomology class of a hyperplane section, so this restriction is not an exact form. Therefore to apply Moser’s lemma it is necessary to have $H \neq \emptyset$ in the above theorem. This condition is not redundant, because the fibres \mathcal{X}_t are typically not pairwise isomorphic to each other.

Denote by $\mathcal{A}ut^s(X/S)$ the following sheaf of abelian groups in the Zariski topology on S . For an open subset $U \subset S$ the sections of $\mathcal{A}ut^s(X/S)$ over U are complex analytic automorphisms of the complex manifold X_U that commute with π and preserve the symplectic form $\Omega|_{X_U}$. Since a general fibre of π is a complex torus, such automorphisms form an abelian group.

COROLLARY 3.3. — *For any $t \in \mathbb{C}$ the complex manifold \mathcal{X}_t can be obtained as the twist of X by a 1-cocycle with values in $\mathcal{A}ut^s(X/S)$.*

Proof. — Keeping the notation as above, we cover S by $m + 1$ affine charts U_i and denote by $\varphi_i: X_{U_i} \xrightarrow{\sim} \mathcal{X}_{U_i,t}$ the isomorphisms constructed in Theorem 3.1. Then $\psi_{ij} = \varphi_j^{-1} \circ \varphi_i \in \mathcal{A}ut^s(X/S)(U_i \cap U_j)$ define the 1-cocycle in question. \square

Below we consider several applications of the above theorem.

3.2. A complex manifold with several algebraic structures

Recall that small deformations of compact Kähler manifolds remain Kähler ([17]). This implies that in the above setting for t sufficiently close to

zero the manifolds \mathcal{X}_t are hyperkähler. Using the projectivity criterion of Huybrechts ([12, Proposition 26.13]), we can determine for which t the manifolds \mathcal{X}_t are projective. Assume that two projective fibres \mathcal{X}_{t_1} and \mathcal{X}_{t_2} are not birational. Note that \mathcal{X}_{U,t_1} and \mathcal{X}_{U,t_2} are both quasi-projective. Theorem 3.1 shows that \mathcal{X}_{U,t_1} and \mathcal{X}_{U,t_2} are isomorphic as complex manifolds. But they can not be isomorphic (or even birational) as algebraic varieties, otherwise \mathcal{X}_{t_1} and \mathcal{X}_{t_2} would be birational, contrary to our choice.

Starting from a K3 surface X one can produce a sufficiently general degenerate twistorial line in the moduli space of X , and make sure that infinitely many fibres \mathcal{X}_{t_i} are projective and pairwise non-birational. This gives us the following statement.

COROLLARY 3.4. — *There exists a complex K3 surface X , a Zariski-open subset $\mathcal{X} \subset X$ and infinitely many complex quasi-projective varieties \mathcal{Y}_j that are pairwise non-birational and such that $\mathcal{Y}_j^{\text{an}}$ are isomorphic to \mathcal{X} as complex manifolds.*

Proof. — For a complex K3 surface X , let $V_R = H^2(X, R)$ for $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and \mathbb{C} . Let q be the intersection form on $V_{\mathbb{Z}}$. Let $\mathcal{D} = \{x \in \mathbb{P}V_{\mathbb{C}} \mid q(x) = 0, q(x, \bar{x}) > 0\}$ be the period domain. We assume that $\pi: X \rightarrow \mathbb{P}^1$ is a Lagrangian fibration and denote by $\ell \in V_{\mathbb{Z}}$ the primitive cohomology class $[\pi^* \omega_{FS}]$.

Step 1. — We first choose sufficiently general degenerate twistorial line. Let $\mathcal{D}_{\ell} = \mathcal{D} \cap \mathbb{P}(\ell^{\perp})$. The period domain has dimension 20, and \mathcal{D}_{ℓ} is a hypersurface in it. The variety \mathcal{D}_{ℓ} is an \mathbb{A}^1 -bundle over $\mathcal{D}' = \{x \in \mathbb{P}(\ell^{\perp}/\langle \ell \rangle) \mid q(x) = 0, q(x, \bar{x}) > 0\}$, the fibres being the degenerate twistorial lines.

Let $W \subset V_{\mathbb{Q}}$ be a 3-dimensional subspace that contains ℓ , and denote by \mathcal{S} the set of all such subspaces. This set is countably infinite. Let $\mathcal{D}_W = \mathcal{D} \cap \mathbb{P}(W^{\perp} \otimes \mathbb{C})$ and \mathcal{D}_W^+ be the union of all degenerate twistorial lines in \mathcal{D}_{ℓ} that pass through the points of \mathcal{D}_W . Since $q|_{W^{\perp}}$ is non-zero, the dimension of \mathcal{D}_W is 17, and the dimension of \mathcal{D}_W^+ is not bigger than 18. Hence \mathcal{D}_W^+ is of positive codimension in \mathcal{D}_{ℓ} , and $\mathcal{D}_{\ell}^{\circ} = \mathcal{D}_{\ell} \setminus \cup_{W \in \mathcal{S}} \mathcal{D}_W^+$ is non-empty. Deforming X we may assume that its period $p \in \mathcal{D}_{\ell}^{\circ}$.

Step 2. — We consider the degenerate twistorial line through $[\Omega]$ and ℓ . Cohomology classes of C-symplectic forms on this line are of the form $[\Omega_t] = [\Omega] + t\ell$. Recall that $V_{\mathbb{Z}}$ is the even unimodular lattice of signature (3, 19). Denote by $\mathcal{H}(d)$ the rank two hyperbolic plane with intersection form multiplied by d . By [25, Appendix to Section 6, Theorem 1], there exists a primitive embedding of lattices $\mathcal{H}(d) \hookrightarrow V_{\mathbb{Z}}$, unique up to an automorphism of $V_{\mathbb{Z}}$. It is well known that the automorphism group of the

lattice $V_{\mathbb{Z}}$ acts transitively on the set of primitive isotropic vectors in $V_{\mathbb{Z}}$, see e.g. [9, Lemma 3.5]; note that in our case the lattice is unimodular. Therefore, conjugating our embedding $\mathcal{H}(d) \hookrightarrow V_{\mathbb{Z}}$ by an automorphism of $V_{\mathbb{Z}}$, we find for arbitrary d a primitive sublattice $\Lambda_d \subset V_{\mathbb{Z}}$ such that $\Lambda_d \simeq \mathcal{H}(d)$ and $\ell \in \Lambda_d$. Let t_d be such that $[\Omega_{t_d}] \in \Lambda_d^{\perp}$.

Step 3. — We consider the fibres \mathcal{X}_{t_d} of the degenerate twistorial family. These fibres are complex K3 surfaces, in particular they are Kähler. By our construction, $\text{NS}(\mathcal{X}_{t_d})$ contains Λ_d . The rank of $\text{NS}(\mathcal{X}_{t_d})$ cannot be bigger than 2 by our choice of p made in Step 1. Hence $\text{NS}(\mathcal{X}_{t_d}) \simeq \Lambda_d$. Since Λ_d contains positive elements, \mathcal{X}_{t_d} are projective. The discriminants of Λ_d are different for different d , hence \mathcal{X}_{t_d} are pairwise non-birational. Now it remains to set $\mathcal{X} = \mathcal{X}_{U,0}$, $\mathcal{Y}_j = \mathcal{X}_{U,t_j}$, where $U \simeq \mathbb{A}^1 \subset \mathbb{P}^1$, and apply Theorem 3.1. □

3.3. Local projectivity of Lagrangian fibrations

We will sharpen the result of Campana [7] about local projectivity of Lagrangian fibrations. Let $\pi: X \rightarrow S$ be a Lagrangian fibration with X compact hyperkähler and S projective. It was shown in [7] that any fibre F of π admits an open analytic neighbourhood \mathcal{V} , such that $\pi|_{\mathcal{V}}$ is a projective morphism.

COROLLARY 3.5. — *In the above setting, F admits a Zariski-open neighbourhood \mathcal{V} that is a quasi-projective variety and such that $\pi|_{\mathcal{V}}$ is a projective morphism.*

Proof. — Since S is projective, we can find a Kähler form $\eta \in \Lambda^{1,1}S$ whose cohomology class is rational. Let $\ell = [\pi^*\eta] \in H^2(X, \mathbb{Q})$. As before, we denote by Ω the holomorphic symplectic form on X .

Consider the degenerate twistorial deformation of X determined by η . Using the density of $H^2(X, \mathbb{Q})$ in $H^2(X, \mathbb{R})$, we can find a class $x \in H^2(X, \mathbb{Q})$ with $q(x) > 0$ and $t = -q([\Omega], x)/q(\ell, x)$ arbitrary close to zero. Then for the fibre \mathcal{X}_t of the degenerate twistorial family we have $[\Omega_t] = [\Omega] + t\ell$, $x \in [\Omega_t]^{\perp}$, hence $\text{NS}(\mathcal{X}_t)$ contains an element with positive BBF square. Since $|t|$ is small, \mathcal{X}_t is Kähler, and hence projective by Huybrechts’s criterion [12, Proposition 26.13]. Pick a hyperplane section H of S that does not contain $\pi(F)$, let $U = S \setminus H$ and $\mathcal{V} = \pi^{-1}(U)$. Then apply Theorem 3.1. □

Remark 3.6. — This corollary gives, in particular, an alternative proof of the well known fact that all fibres of π are projective varieties.

Remark 3.7. — If X is not projective, it can not be Moishezon, because being Kähler and Moishezon implies being projective. Using the open subsets \mathcal{V} from Corollary 3.5, we get an example of a compact complex manifold that admits a covering by Zariski-open quasi-projective subsets, but is not Moishezon.

3.4. Stein cells in hyperkähler manifolds with Lagrangian fibrations

It was asked by Bogomolov (see [5]) whether any compact complex manifold X admits a “Stein cell”. By this one means an open subset $\mathcal{V} \subset X$ that is a Stein manifold, and such that the complement $X \setminus \mathcal{V}$ is “small”. Ideally the complement of a Stein cell should be a finite CW complex whose dimension is strictly smaller than the dimension of X . If X is projective, one can take for \mathcal{V} the complement of an ample divisor, which is an affine variety, hence Stein. For non-algebraic varieties the question is more subtle, and no general construction is known, even conjecturally. The condition on the complement of a Stein cell stated above is quite restrictive, and it is debatable how realistic it is. We observe here that if one weakens this condition, then Theorem 3.1 provides a way to construct Stein cells in arbitrary (possibly non-algebraic) hyperkähler manifolds admitting a Lagrangian fibration.

COROLLARY 3.8. — *Let $\pi: X \rightarrow S$ be a Lagrangian fibration with X compact hyperkähler and S projective. There exists an open subset $\mathcal{V} \subset X$ that is Stein and dense in the analytic topology.*

Proof. — Consider the degenerate twistorial deformation of X as in the proof of Corollary 3.5 and let \mathcal{X}_t be a projective fibre. Let $H \subset S$ be a hyperplane section, $U = S \setminus H$ and $\varphi: X_U \rightarrow \mathcal{X}_{U,t}$ the isomorphism of complex manifolds constructed in Theorem 3.1. Let $Z = \pi^{-1}(H) \subset \mathcal{X}_t$. For any ample divisor $A \subset \mathcal{X}_t$ the divisor $mA + Z$ is ample for m big enough. Hence $\mathcal{V}' = \mathcal{X}_{U,t} \setminus A$ is an affine variety. Let $\mathcal{V} = \varphi^{-1}(\mathcal{V}')$, then \mathcal{V} is an open dense Stein subset of X . \square

Remark 3.9. — In the notation of the proof above, note that the closure of $\varphi^{-1}(A \cap \mathcal{X}_{U,t}) \subset X_U$ may not be a complex-analytic subset of X . So the complement of \mathcal{V} constructed above may be more complicated than a CW complex.

3.5. Higher direct images of the structure sheaf

Let $\pi: X \rightarrow S$ be a Lagrangian fibration with X compact hyperkähler and S a smooth projective variety. In the case when X is projective, Matsushita has shown in [21] that $R^j \pi_* \mathcal{O}_X \simeq \Omega_S^j$. One can deduce the same result for non-projective X using the theory of Saito [27]. We show how to deduce the non-projective case directly from [21] without using the complicated techniques from [27].

COROLLARY 3.10. — *In the above setting $R^j \pi_* \mathcal{O}_X \simeq \Omega_S^j$ for all j .*

Proof. — We let η and ℓ be as in the proof of Corollary 3.5, and we pick a projective fibre \mathcal{X}_t in the degenerate twistorial deformation of X as in that proof. We let $\omega \in \Lambda^{1,1} X$ be a Kähler form and $L \in \text{Pic}(\mathcal{X}_t)$ be an ample line bundle.

Let $\pi^\circ: X^\circ \rightarrow S^\circ$ be the smooth part of the fibration π , as in Section 2.2. We recall from [21, Section 2] the construction of the isomorphism $R^1 \pi_*^\circ \mathcal{O}_{X^\circ} \simeq \Omega_{S^\circ}^1$. Consider the isomorphism $T_{X^\circ} \xrightarrow{\sim} \Omega_{X^\circ}^1$ induced by the symplectic form Ω . Compose this isomorphism with the natural surjection $\Omega_{X^\circ}^1 \rightarrow \Omega_{X^\circ/S^\circ}^1$ and note that this composition vanishes on the subbundle T_{X°/S° , because the fibres of π° are Lagrangian. It therefore induces an isomorphism $(\pi^\circ)^* T_{S^\circ} \xrightarrow{\sim} \Omega_{X^\circ/S^\circ}^1$. Projecting it to S we get an isomorphism $T_{S^\circ} \xrightarrow{\sim} \pi_*^\circ \Omega_{X^\circ/S^\circ}^1$. We have an exact triple

$$0 \longrightarrow \pi_*^\circ \Omega_{X^\circ/S^\circ}^1 \longrightarrow (R^1 \pi_*^\circ \mathbb{C}) \otimes \mathcal{O}_{S^\circ} \longrightarrow R^1 \pi_*^\circ \mathcal{O}_{X^\circ} \longrightarrow 0,$$

and the bundle in the middle carries a polarized variation of Hodge structures, the polarization being induced by the restriction of the Kähler class $[\omega]$ to the fibres of π° . Using the polarization, we get an isomorphism $R^1 \pi_*^\circ \mathcal{O}_{X^\circ} \simeq (\pi_*^\circ \Omega_{X^\circ/S^\circ}^1)^\vee$ and hence

$$(3.1) \quad R^1 \pi_*^\circ \mathcal{O}_{X^\circ} \simeq \Omega_{S^\circ}^1.$$

From the construction we see that the isomorphism (3.1) depends on the class of the polarization $[\omega] \in H^2(X, \mathbb{R})$. Recall, however, that for any fibre F of π° the restriction map $H^2(X, \mathbb{R}) \rightarrow H^2(F, \mathbb{R})$ has rank one and its kernel is the orthogonal complement of ℓ , see [22, Lemma 2.2]. The polarization on the fibres therefore depends only on the value of $q(\ell, [\omega])$, and not on $[\omega]$ itself. We can rescale ω so that $q(\ell, [\omega]) = q(\ell, c_1(L))$, where L is the ample bundle on \mathcal{X}_t chosen above. Then $[\omega]$ and $c_1(L)$ induce the same polarization on the VHS $(R^1 \pi_*^\circ \mathbb{C}) \otimes \mathcal{O}_{S^\circ}$ and thus the same isomorphism (3.1).

We need to show that (3.1) extends over the discriminant locus of π . We choose an arbitrary hyperplane section of S in some projective embedding and denote by $U \subset S$ its open complement. We let $U^\circ = U \cap S^\circ$ and $X_U = \pi^{-1}(U)$. Theorem 3.1 implies that the complex manifolds X_U and $\mathcal{X}_{U,t}$ are isomorphic. Applying the results of [21] to \mathcal{X}_t and using the line bundle L to define the polarization, we get an isomorphism $R^1\pi_*\mathcal{O}_{X_U} \simeq R^1\pi_*\mathcal{O}_{\mathcal{X}_{U,t}} \simeq \Omega_U^1$. When restricted to U° , this isomorphism coincides with the restriction of (3.1), because of the remark about polarizations in the previous paragraph. Therefore we can extend (3.1) to an isomorphism over U . Since U was arbitrary, we conclude that (3.1) extends to an isomorphism $R^1\pi_*\mathcal{O}_X \simeq \Omega_S^1$.

To finish the proof, we use the isomorphisms $\Lambda^j(R^1\pi_*\mathcal{O}_X) \simeq R^j\pi_*\mathcal{O}_X$ from [21]. These isomorphisms may be checked locally on S , so we again replace X_U by $\mathcal{X}_{U,t}$ as above and apply the results of [21]. \square

3.6. Deformation invariance of contraction centres

Consider a smooth family $\pi: \mathcal{X} \rightarrow \Delta$ of holomorphic symplectic manifolds over the unit disc. Assume that $\mathcal{Z} \subset \mathcal{X}$ is a subvariety that is flat over Δ and such that $\pi|_{\mathcal{Z}}$ is proper, i.e. the fibres \mathcal{Z}_t are compact subvarieties of \mathcal{X}_t . Assume that for every $t \in \Delta$ the subvariety \mathcal{Z}_t can be contracted to a point. More precisely, there exists a flat family $\pi': \mathcal{Y} \rightarrow \Delta$ and a proper morphism $\rho: \mathcal{X} \rightarrow \mathcal{Y}$ such that $\pi' \circ \rho = \pi$ and $\rho_t: \mathcal{X}_t \rightarrow \mathcal{Y}_t$ is a bimeromorphic morphism with exceptional set \mathcal{Z}_t , and such that $\rho_t(\mathcal{Z}_t)$ is a point in \mathcal{Y}_t . We call \mathcal{Z}_t the contraction centre of ρ_t .

We would like to use Moser's isotopy to show that all \mathcal{Z}_t are isomorphic, the isomorphisms being induced by symplectomorphisms of some open neighbourhoods of \mathcal{Z}_t . Every \mathcal{Z}_t is a deformation retract of its open neighbourhood in \mathcal{X}_t , see e.g. [26, Proposition 3.5]. By shrinking \mathcal{X} we may assume that every fibre \mathcal{Z}_t is a deformation retract of \mathcal{X}_t . Moreover, since every point in \mathcal{Y}_t admits a Stein neighbourhood, by shrinking \mathcal{Y} we may assume that all \mathcal{Y}_t are Stein spaces. Therefore we are reduced to the following statement.

THEOREM 3.11. — *Let X be a manifold with a family of C -symplectic structures $\Omega_t \in \Lambda_{\mathbb{C}}^2 X$, $t \in [0, 1]$ and corresponding complex structures I_t . Denote $\mathcal{X}_t = (X, I_t)$ and assume that $\mathcal{Z}_t \subset \mathcal{X}_t$ is a flat family of compact complex subvarieties such that:*

- (1) \mathcal{Z}_t is a deformation retract of X ;

- (2) \mathcal{X}_t admits a bimeromorphic morphism ρ_t onto a Stein space \mathcal{Y}_t such that the exceptional set of ρ_t is \mathcal{Z}_t .

Then there exist open neighbourhoods U_0 of \mathcal{Z}_0 , U_1 of \mathcal{Z}_1 and a diffeomorphism $\varphi: U_0 \rightarrow U_1$ such that $\varphi^*(\Omega_1|_{U_1}) = \Omega_0|_{U_0}$ and $\varphi(\mathcal{Z}_0) = \mathcal{Z}_1$.

Proof. — Since \mathcal{X}_t are holomorphic symplectic manifolds, \mathcal{Y}_t are holomorphic symplectic varieties in the sense of Beauville [3]. It is known [3, Proposition 1.3] that the singularities of \mathcal{Y}_t are rational, and since \mathcal{Y}_t are Stein, $H^j(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 0$ for all $j > 0$.

As before, we denote by $[\Omega_t]$ the cohomology class of Ω_t in $H^2(X, \mathbb{C})$. We know from [15, Corollary 2.8] that $[\Omega_t]|_{\mathcal{Z}_t} = 0$. Since \mathcal{Z}_t is a deformation retract of X , we have $[\Omega_t] = 0$ for all t . Hence we can apply part (1) of Theorem 2.5 and obtain a family of vector fields V_t .

Since X is not compact, it is not possible, in general, to integrate V_t to a flow of diffeomorphisms. However, since \mathcal{Z}_0 is compact, we can integrate V_t in an open neighbourhood U_0 of \mathcal{Z}_0 for small t . More precisely, there exist diffeomorphisms φ_t of U_0 onto some open subsets of X integrating V_t and defined for t sufficiently close to zero. Note that $\varphi_t^*\Omega_t = \Omega_0|_{U_0}$ and $\varphi_t(\mathcal{Z}_0)$ is a compact complex subvariety of \mathcal{X}_t . By the assumption that the family of subvarieties \mathcal{Z}_t is flat, the dimension of \mathcal{Z}_t is a constant that we denote by d . Since \mathcal{X}_t is contractible onto a Stein space \mathcal{Y}_t with contraction centre \mathcal{Z}_t , the only d -dimensional compact subvariety of \mathcal{X}_t is \mathcal{Z}_t . Hence $\varphi_t(\mathcal{Z}_0) = \mathcal{Z}_t$, and since all \mathcal{Z}_t are compact, we can extend the flow for all $t \in [0, 1]$, after possibly shrinking U_0 . Then we define $U_1 = \varphi_1(U_0)$ and $\varphi = \varphi_1$. \square

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