ANNALES DE L'INSTITUT FOURIER

Thomas Leistner & Stuart Teisseire

Conformal transformations of Cahen–Wallach spaces

CONFORMAL TRANSFORMATIONS OF CAHEN–WALLACH SPACES

by Thomas LEISTNER & Stuart TEISSEIRE (*)

————

ABSTRACT. — We study conformal transformations of indecomposable Lorentzian symmetric spaces of non-constant sectional curvature, the so-called Cahen–Wallach spaces. When a Cahen–Wallach space is conformally curved, its conformal transformations are homotheties. Using this we show that a conformal transformation of a conformally curved Cahen–Wallach space is essential if and only if it has a fixed point. Then we explore the possibility of properly discontinuous groups of conformal transformations acting with a compact orbit space on a conformally curved Cahen–Wallach space. We show that any such group cannot centralise an essential homothety and that for Cahen–Wallach spaces of imaginary type, such a group must be contained within the isometries.

RÉSUMÉ. — Nous étudions les transformations conformes d'espaces symétriques lorentziens indécomposables de courbure sectionnelle non constante, appelés espaces de Cahen–Wallach. Lorsqu'un espace de Cahen–Wallach n'est pas conformement plat, ses transformations conformes sont des homothéties. En utilisant cela, nous montrons qu'une transformation conforme d'un espace de Cahen–Wallach qui n'est pas conformement plat est essentielle si et seulement si il a un point fixe. Nous explorons ensuite la possibilité de groupes proprement discontinus de transformations conformes agissant avec un espace d'orbites compact sur un espace de Cahen–Wallach non conformement plat. Nous montrons qu'un tel groupe ne peut centraliser une homothétie essentielle et que pour les espaces de Cahen–Wallach de type imaginaire, un tel groupe doit être contenu dans les isométries.

1. Introduction

It is a remarkable feature of Lorentzian geometry that indecomposable Lorentzian symmetric spaces either have constant sectional curvature or are

universally covered by a Cahen–Wallach space, [4, 10]. A *Cahen–Wallach space* is a Lorentzian manifold $(\mathbb{R}^{n+2}, g_S)$ with $n \geqslant 1$ and Lorentzian metric

$$(1.1) \qquad \begin{aligned} g_S &= 2\,\mathrm{d}v\,\mathrm{d}t + S_{ij}x^i x^j\,(\mathrm{d}t)^2 + \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \\ &= 2\,\mathrm{d}v\,\mathrm{d}t + \boldsymbol{x}^\top S \boldsymbol{x}(\mathrm{d}t)^2 + \mathrm{d}\boldsymbol{x}^\top \mathrm{d}\boldsymbol{x}, \end{aligned}$$

where $(t, x^1, \ldots, x^n, v) = (t, \boldsymbol{x}, v)$ are coordinates on $\mathbb{R}^{n+2}$ and $S = (S_{ij})$ is a symmetric $(n \times n)$-matrix with non-zero determinant (using Einstein's summation convention). Whereas the constant sectional curvature spaces are Einstein and conformally flat, the Cahen–Wallach spaces in general are neither Einstein nor conformally flat. Motivated by the Clifford–Klein program, one may ask which compact manifolds arise as compact quotients of an indecomposable Lorentzian symmetric space by a group of isometries. For the constant curvature spaces, Calabi and Markus [11] have shown that a group that acts properly discontinuously on de Sitter space must be finite and hence cannot produce a compact quotient. Moreover, Kulkarni has shown [25] that the universal cover of anti-de Sitter space admits a compact quotient if and only if its dimension is odd. For Cahen–Wallach spaces this question only recently has been studied by Kath and Olbrich in [19], who gave arithmetic criteria for Cahen–Wallach spaces to admit a compact quotient by a properly discontinuously acting group of isometries. Together with the completeness results in [12, 20, 26, 27] this paves the way for a classification of Lorentzian compact locally symmetric spaces. Note that for Cahen–Wallach spaces examples of compact quotients by isometries exist and were given in [19], see also our Examples 4.11 and 4.12.

One may extend such questions to conformal structures, i.e. to manifolds equipped with an equivalence class of conformally equivalent metrics. For example, one may consider the conformal class of a Cahen–Wallach metric and ask for groups of conformal transformation that yield interesting and perhaps compact quotients. The resulting manifold would be equipped with a conformal structure that is locally conformally equivalent to a Cahen–Wallach metric. The local conformal geometry of Cahen–Wallach spaces, and more generally pseudo-Riemannian symmetric spaces, in particular their Killing vector fields, have been studied in papers by Cahen and Kerbrat [5, 6, 7, 8, 9]. Questions about the global conformal geometry, and in particular the existence of compact conformal quotients, to our knowledge have not been considered in the literature, and in this paper we are going to address some of these questions.

For our first result, we follow the convention in [19] and say that a Cahen–Wallach space is of *imaginary type* if S is negative definite.

THEOREM 1.1. — *Let $(\mathbb{R}^{n+2}, g_S)$ be a conformally curved Cahen–Wallach space of imaginary type and Γ a subgroup of its conformal group. If Γ acts properly discontinuously and with compact orbit space $M = \mathbb{R}^{n+2}/\Gamma$, then Γ is a group of isometries. Consequently, M with the metric induced from g_S is locally isometric to $(\mathbb{R}^{n+2}, g_S)$ and its conformal group is equal to its isometry group.*

Consequently, the question about compact conformal quotients of Cahen–Wallach spaces of imaginary type is reduced to the case of compact isometric quotients and the results in [19]. Our proof of Theorem 1.1 relies on an analysis of the group of conformal transformations of a Cahen–Wallach space. First, it is straightforward to show that a Cahen–Wallach metric g_S is conformally flat (i.e. with vanishing Weyl tensor) if and only if the matrix S is a scalar matrix, see Proposition 3.1. Since Cahen–Wallach spaces are locally symmetric, a result in [7, Proposition 2.1] implies that the conformal group of a conformally curved Cahen–Wallach space is equal to its homothety group. This implies in particular, that for compact quotients by a group of isometries, the conformal group is equal to the isometry group, see Corollary 2.4. The homothety group of a Cahen–Wallach space is equal to $\mathrm{Isom}(\mathbb{R}^{n+2}, g_S) \rtimes \mathbb{R}$, and the isometry group $\mathrm{Isom}(\mathbb{R}^{n+2}, g_S)$ is well-known [10, 19] to be isomorphic to

$$\mathbf{Hei}_n \rtimes \left(\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \right),$$

where $\mathbf{Hei}_n$ is the $2n + 1$-dimensional Heisenberg group, $\mathbf{Euc}(1) = \mathbb{R} \rtimes \mathbb{Z}_2$ is the Euclidean group in one dimension, and $C_{\mathbf{O}(n)}(S)$ is the centraliser of the matrix S in $\mathbf{O}(n)$. For details about the group structure and the result, see Section 3.2 and Corollary 3.10. Explicitly, each homothety ϕ of $(\mathbb{R}^{n+2}, g_S)$ is given by

$$(1.2) \qquad \begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \stackrel{\phi}{\longmapsto} \begin{pmatrix} \epsilon\, t + c \\ \mathrm{e}^s A \boldsymbol{x} + \beta(t) \\ \epsilon \left(\mathrm{e}^{2s} v + b - \langle \dot{\beta}(t), \mathrm{e}^s A \boldsymbol{x} + \frac{1}{2}\beta(t) \rangle \right) \end{pmatrix},$$

where $(c, \epsilon) \in \mathbb{R} \times \{\pm 1\} = \mathbf{Euc}(1)$, $A \in C_{\mathbf{O}(n)}(S)$, $b \in \mathbb{R}$, $\beta : \mathbb{R} \to \mathbb{R}^n$ is a solution to $\ddot{\beta} = S\beta$, and $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on $\mathbb{R}^n$. For Cahen–Wallach spaces of imaginary type the solutions to $\ddot{\beta} = S\beta$ are given by rotating the initial conditions. As a consequence, the boundedness of the solutions will yield the result in Theorem 1.1. For Cahen–Wallach spaces that are not of imaginary type (i.e. when S has at least one positive eigenvalue), the solutions to $\ddot{\beta} = S\beta$ are not necessarily bounded. Hence, the method of our proof does not immediately apply, and we are not able to answer the question of whether there are properly

discontinuous groups of homotheties acting with a compact orbit space. In Section 4.3 we illustrate the difficulties that arise when trying to construct an example of such group.

A motivation for studying conformal transformations of Cahen–Wallach spaces also comes from rigidity questions in conformal geometry, namely the question for which conformal manifolds the group of conformal transformations is *essential*, i.e. not contained in the isometry group of a metric in the conformal class. Of course, by definition, the group of conformal transformations of a semi-Riemannian manifold is larger than the group of isometries, however, examples of manifolds with *essential conformal transformations* are relatively rare. In fact, for Riemannian manifolds, Ferrand [29] and Obata [33] showed that a compact Riemannian manifold with essential conformal transformations must be conformally diffeomorphic to the round sphere. More surprisingly, any non-compact Riemannian manifold with essential conformal transformations must be conformally diffeomorphic to Euclidean space, [13]. These results confirmed the *Lichnerowicz conjecture*, [30]. The conjecture can be extended to conformal structures of indefinite metrics, however already in Lorentzian signature it turns out to be false: there are many non compact Lorentzian manifolds that are not conformally flat but which have essential conformal transformations [2, 21, 22, 37, 38], and Cahen–Wallach spaces are amongst them. In Section 3.3 we determine which conformal transformations of a conformally curved Cahen–Wallach space are essential.

THEOREM 1.2. — *Let ϕ be a homothety of a Cahen–Wallach space that is not an isometry. Then the following are equivalent:*

  (1) *ϕ is essential;*
  (2) *ϕ has a fixed point;*
  (3) *in equation (1.2) for ϕ it is $\epsilon = -1$ or $c = 0$.*

In particular, when the Cahen–Wallach space is not conformally flat, then every essential conformal transformation is given by a homothety with the above properties.

Returning to the compact case in the Lichnerowicz conjecture for indefinite metrics, in [14] Frances constructed examples of compact Lorentzian manifolds with essential conformal transformations that are not conformally diffeomorphic to the flat model of constant curvature, however, all of the examples constructed by Frances are conformally flat, i.e. have vanishing Weyl tensor. This leads to the *generalised pseudo-Riemannian Lichnerowicz conjecture:* any compact pseudo-Riemannian manifold with

essential conformal transformations must have vanishing Weyl tensor. Again Frances constructed counterexamples in all but Lorentzian signature [16], which leaves us with the *Lorentzian Lichnerowicz conjecture:* any compact Lorentzian manifold with essential conformal transformations is conformally flat. This conjecture remains unproven in general, although substantial progress has been made [17, 31, 35, 36] and it has recently has been proved for compact real analytic manifolds that are simply connected [32] or of dimension three [18].

The counterexamples constructed by Frances [16] in signatures beyond Lorentzian signature start with a locally symmetric space in signature $(2 + p, 2 + q)$ which admits a group of homotheties that acts with compact quotient and centralises an essential homothety, which then descends to the compact quotient manifold. These examples are a motivation for our results in Section 4.2. Here we study whether, for a given essential homothety ϕ of a Cahen–Wallach space, there is a group Γ of conformal transformations that acts properly discontinuously and cocompactly, and such that ϕ is in the centraliser of Γ. In this case, ϕ would descend to an essential conformal transformation of the compact conformal manifold M. We will show however, that this is not possible.

THEOREM 1.3. — *For a conformally curved Cahen–Wallach space, a group of conformal transformations that centralises an essential homothety cannot act properly discontinuously and cocompactly.*

This does not entirely exclude the possibility of compact conformal quotients of Cahen–Wallach spaces with essential conformal transformation. We believe however, that no such quotient exists.

The structure of the paper is as follows: in Section 2 we recall some basic notations and facts from conformal geometry including a short section about group actions; in Section 3 we give a criterion for conformal flatness, describe the isometries, homotheties and conformal transformations of Cahen–Wallach spaces and prove Theorem 1.2; Section 4 contains the proofs of the non-existence results in Theorems 1.1 and 1.3. The article concludes with a few examples that illustrate the obstacles for constructing compact conformal quotients.

Acknowledgements

2. Preliminaries on conformal geometry

2.1. Curvature conventions and conformal rescalings

Let (M, g) be a semi-Riemannian manifold of dimension m and ∇ the Levi-Civita connection. Our convention for the curvature tensors are as follows: the Riemannian curvature as a 2-form with values in $\mathfrak{so}(TM, g)$, or equivalently as a $(1, 3)$-tensor, is defined as

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

and the $(0, 4)$-curvature tensor as

$$R(X, Y, Z, V) = g(R(X, Y)V, Z).$$

The Ricci-tensor is the trace of the $(1, 3)$-curvature tensor

$$\operatorname{Ric}(Y, Z) = \operatorname{tr}(X \longmapsto R(X, Y)Z).$$

We denote the corresponding endomorphism also by Ric and and the scalar curvature scal as its trace. Moreover, we define the Schouten tensor P by

$$\operatorname{Ric} = (m - 2)\mathsf{P} + \frac{\operatorname{scal}}{2(m - 1)}g,$$

and the $(0, 4)$-Weyl tensor as

$$W(X, Y, Z, V) = R(X, Y, Z, V) - g \owedge \mathsf{P},$$

where $\owedge$ is the *Kulkarni–Nomizu product* of two symmetric bilinear forms defined as

$$\begin{aligned} A \owedge B(X, Y, Z, V) = {} & A(X, Z)B(Y, V) + B(X, Z)A(Y, V) \\ & - A(X, V)B(Y, Z) + B(X, V)A(Y, Z), \end{aligned}$$

that produces a $(0, 4)$-tensor with the same symmetries as the Riemannian curvature tensor. We define the $(1, 3)$-Weyl tensor by

$$g(W(X, Y)Z, V) = W(X, Y, Z, V).$$

We say that g is *Weyl-flat* if the Weyl tensor of g vanishes.

If $\widehat{g} = \mathrm{e}^{2f} g$ is a conformally equivalent metric to g, where f is a smooth function on M, then the Levi-Civita connection, the $(0,4)$-curvature tensor, and the Ricci and scalar curvatures change as follows, see [3, Section 1.J],

$$\widehat{\nabla}_X Y = \nabla_X Y + \mathrm{d}f(X)Y + \mathrm{d}f(Y)X - g(X,Y)\nabla f,$$

(2.1)
$$\mathrm{e}^{-2f}\widehat{R} = R - g \oslash \left(\nabla \mathrm{d}f - (\mathrm{d}f)^2 + \frac{1}{2}g(\nabla f, \nabla f)g \right),$$

$$\widehat{\mathrm{Ric}} = \mathrm{Ric} - (m-2)\left(\nabla \mathrm{d}f - (\mathrm{d}f)^2 \right) + (\Delta f - (m-2)g(\nabla f, \nabla f))\, g,$$

$$\mathrm{e}^{2f}\widehat{\mathrm{scal}} = \mathrm{scal} + (m-1)\left(2\Delta f - (m-2)g(\nabla f, \nabla f) \right),$$

whereas the $(1,3)$-Weyl tensor is conformally invariant. Here ∇f and Δf are the gradient and the Laplacian of f, both with respect to g. We observe the following useful relation.

LEMMA 2.1. — *If g and $\widehat{g} = \mathrm{e}^{2f} g$ both have vanishing scalar curvature, then*

$$\mathrm{e}^{-2f}\widehat{R} = R + \frac{1}{m-2}g \oslash \left(\widehat{\mathrm{Ric}} - \mathrm{Ric} \right).$$

2.2. Homotheties, conformal and essential conformal transformations

A *conformal diffeomorphism* between semi-Riemannian manifolds (M,g) and $(\widehat{M},\widehat{g})$ is a diffeomorphism $\phi : M \to \widehat{M}$ for which there is a smooth function f on M such that

$$\phi^*\widehat{g} = \mathrm{e}^{2f} g.$$

A conformal diffeomorphism for which f is constant is called a *homothety*. We call a homothety *strict* if it is not an isometry. We denote by $\mathrm{Conf}(M,g)$ the conformal transformations of (M,g), i.e. the conformal diffeomorphisms from (M,g) to itself, by $\mathrm{Homoth}(M,g)$ the homotheties of (M,g), and by $\mathrm{Isom}(M,g)$ the isometries of (M,g).

We say that (M,g) is *conformally flat* if each point admits a local conformal diffeomorphism from a neighbourhood into a flat semi-Riemannian manifold. If there is a global conformal diffeomorphism from M to a flat semi-Riemannian manifold, we say that (M,g) *globally conformally flat*. All manifolds of dimension 2 are conformally flat. In dimension 3, (M,g) is conformally flat if and only if the Cotton tensor A of g vanishes, which is defined as

$$A(X,Y,Z) = \nabla_Y \mathsf{P}(Z,X) - \nabla_Z \mathsf{P}(Y,X).$$

When $\dim(M) \geqslant 4$, (M, g) is conformally flat if and only if g is Weyl-flat. Moreover, if two metrics g and $\widehat{g}$ on M are *conformally equivalent*, i.e. $\widehat{g} = \mathrm{e}^{2f} g$, then the identity transformation is a conformal diffeomorphism between (M, g) and $(M, \widehat{g})$.

Let (M, g) be a semi-Riemannian manifold. The map $\mathrm{Homoth}(M, g) \to \mathbb{R}$ that sends a homothety ϕ with $\phi^* g = \mathrm{e}^{2s} g$ to s is a group homomorphism with kernel $\mathrm{Isom}(M, g)$. Hence we have that

$$\mathrm{Homoth}(M, g) = \mathrm{Isom}(M, g) \rtimes H,$$

for some subgroup H of $\mathbb{R}$. Further, if for each $s \in \mathbb{R}$, there is a h_s such that $h_s^* g = \mathrm{e}^{2s} g$, then $\mathrm{Homoth}(M, g) = \mathrm{Isom}(M, g) \rtimes \mathbb{R}$. An important result for our purposes is the following:

THEOREM 2.2 (Cahen & Kerbrat [7, Proposition 2.1]). — *Let (M, g) be a connected semi-Riemannian manifold of dimension $m \geqslant 4$ with parallel Weyl tensor, $\nabla W = 0$. Let $U \subset M$ be open and $\phi : U \to \phi(U) \subset M$ be a conformal diffeomorphism. Then ϕ is a homothety or the Weyl tensor is identically zero.*

Since $\nabla R = 0$ implies that $\nabla W = 0$, we obtain:

COROLLARY 2.3. — *Let (M, g) be a connected, locally symmetric, semi-Riemannian manifold of dimension $m \geqslant 4$. Let $U \subset M$ be open, and let $\phi : U \to \phi(U) \subset M$ be a conformal diffeomorphism. Then ϕ is a homothety, or the Weyl tensor is identically zero.*

If ϕ is a homothety with $\phi^* g = \mathrm{e}^{2s} g$, the volume form ν of a semi-Riemannian manifold satisfies that $\phi^* \nu = \mathrm{e}^{ms} \nu$. Hence, compact semi-Riemannian manifolds cannot have any strict homotheties (see for example [2]). This yields another result, which we could not find in the literature.

COROLLARY 2.4. — *Let (M, g) be a connected, compact and locally symmetric semi-Riemannian manifold of dimension $m \geqslant 4$. Then the conformal group is equal to its isometry group.*

A conformal transformation ϕ on a semi-Riemannian manifold (M, g) is called *essential* if there is no conformally equivalent metric on M for which ϕ is an isometry. Similarly, the conformal group $\mathrm{Conf}(M, g)$ is called *essential* if there is no conformally equivalent metric $\widehat{g}$ on M for which $\mathrm{Conf}(M, g)$ is contained in the isometries of $(M, \widehat{g})$. Clearly, if (M, g) admits an essential conformal transformation, then its conformal group is essential, however the converse, that an essential conformal group contains an essential conformal transformation, is not obvious. For Riemannian conformal structures this implication follows from the confirmed Lichnerowicz conjecture [13, 29, 33].

For homotheties, there is a sufficient condition for being essential.

PROPOSITION 2.5. — *Let (M, g) be a semi-Riemannian manifold and let ϕ be a strict homothety with a finite orbit point p, i.e. for some $k > 0$, $\phi^k(p) = p$. Then ϕ is essential. In particular, a strict homothety with a fixed point is essential.*

Proof. — Let $\phi^* g = e^{2s} g$ for $s \in \mathbb{R}$. Assume $\phi^k(p) = p$ for $p \in M$ and that ϕ is not essential: let f be a smooth function on M such that ϕ is an isometry of $e^{2f} g$. Then we evaluate at p,

$$(e^{2f} g)|_p = (\phi^k)^*(e^{2f} g)|_p = e^{2f \circ \phi^k}(p)((\phi^k)^* g)|_p$$
$$= e^{2ks} e^{2f(p)} g|_p = e^{2ks}(e^{2f} g)|_p.$$

But this implies $s = 0$, and so ϕ is an isometry. $\qquad\square$

We will see in Theorem 3.14 that for Cahen–Wallach spaces also the converse holds, so that the essential homotheties are *exactly* those with a fixed point.

Remark 2.6. — In [2, Theorem 2.1] the converse of Proposition 2.5 is claimed for causal Lorentzian manifolds, i.e. that every fixed point free homothety of a causal Lorentzian manifold is inessential. This relied on a claim in [1], the proof of which has a gap. An example of an essential homothety without fixed point on a causal Lorentzian manifold is easily constructed.

2.3. Properly discontinuous cocompact and conformal group actions

Let Γ be a group of diffeomorphisms acting on a smooth manifold M. The group action is *properly discontinuous* if it satisfies the following two conditions

(PD1) For each point $x \in M$, there is a neighbourhood U of x such that if γU meets U, i.e. $\gamma U \cap U \neq \emptyset$ for $\gamma \in \Gamma$, then $\gamma = e$, where e is the identity element.

(PD2) For all pairs of points $x, y \in M$ in different orbits, there are neighbourhoods U of x, and V of y, such that for all $\gamma \in \Gamma$, γU and V are disjoint, i.e. $\gamma U \cap V = \emptyset$.

Clearly, (PD1) implies that Γ acts freely, that is, its elements act without fixed points. If a group Γ acts properly discontinuously on a manifold M, then there is a unique smooth manifold structure on the orbit space M/Γ

and $\pi : M \to M/\Gamma$ is a covering map, see for example [34, Proposition 7 in Chapter 7].

If a group Γ acts by diffeomorphisms on M such that the orbit space M/Γ is compact we say that Γ acts *cocompactly*. If a group Γ acts by diffeomorphisms on M and admits a fundamental region with compact closure, then M/Γ is compact. The converse is not true in general in the sense that a Γ acting on M can have a compact orbit space but admits fundamental domains with non-compact closures. For metric spaces the converse holds if the fundamental region is assumed to be locally finite [39, Chapter 6]. In [28, 41] we prove a converse that holds also for non-isometric (in the metric space sense) group actions, but requires a stronger assumption on the fundamental region. Although this notion may shorten some of our proofs here, we will not use it in the following for the sake of being self-contained.

Now let $(\widetilde{M}, \widetilde{g})$ be a semi Riemannian manifold and Γ a group that acts properly discontinuously on $\widetilde{M}$. If Γ is contained in the isometry group of $(\widetilde{M}, \widetilde{g})$, then the orbit space $M = \widetilde{M}/\Gamma$ is equipped with a unique semi-Riemannian metric g such that $\pi^*g = \widetilde{g}$, where $\pi : \widetilde{M} \to M$ is the covering map. Similarly, when Γ is a group of conformal transformations, the orbit space $M = \widetilde{M}/\Gamma$ is equipped with a conformal structure $\mathbf{c}$ such that $\pi : \widetilde{M} \to M$ is a conformal covering map, that is, π is a covering map and for each $g \in \mathbf{c}$ there is a function $f \in C^\infty(\widetilde{M})$ such that $\pi^*g = e^{2f}\widetilde{g}$. Note that the original metric $\widetilde{g}$ on $\widetilde{M}$ in general is not a lift of a metric in $\mathbf{c}$. This is only the case if Γ is consists of isometries. For more details and results, see [41].

3. Conformal transformations of Cahen–Wallach spaces

3.1. Conformal flatness of Cahen–Wallach space

In [4, 10] M. Cahen and N. Wallach have shown that an indecomposable simply connected Lorentzian symmetric space either has constant curvature or is isometric to a *Cahen–Wallach space*, which is defined as a Lorentzian manifold $(\mathbb{R}^{n+2}, g_S)$ with $n \geqslant 1$ and g_S is the metric in (1.1) defined by a symmetric $(n \times n)$-matrix $S = (S_{ij})$ with non-zero determinant. The condition that S_{ij} is invertible is to ensure that (M, g) is indecomposable. If S_{ij} is not invertible, then the metric g_S is a product of Euclidean space $\mathbb{R}^k$ and a Cahen–Wallach space of dimension $n + 2 - k$. Some of our results remain valid when S is not invertible, and we will point out when this is

the case. Clearly, if S is the zero matrix, g_S is just the Minkowski metric. We call the metric g_S a *Cahen–Wallach metric*, even when it is only defined on an open subset of $\mathbb{R}^{n+2}$.

We denote by Σ the spectrum of the matrix S and by $\Sigma_\pm$ the positive and negative eigenvalues. A Cahen–Wallach space is of *real type* if $\Sigma = \Sigma_+$ and of *imaginary type* if $\Sigma = \Sigma_-$. Otherwise it is of *mixed type*, see [19]. Two Cahen–Wallach spaces are isometric if and only if the corresponding matrices S and $\widehat{S}$ have the same spectrum with the same multiplicities up to multiplication by a positive number, so that $\widehat{\Sigma} = a\Sigma$, with $a > 0$.

For the metric g_S, even when S is degenerate, the vector field $\partial_v = \frac{\partial}{\partial v}$, and consequently the one-form $\mathrm{d}t = g(\partial_v, .)$, are parallel and null. Moreover,

$$(3.1) \qquad \nabla \partial_i = x^j S_{ij} \, \mathrm{d}t \otimes \partial_v, \qquad \nabla \partial_t = x^i S_{ij} \left(\mathrm{d}x^j \otimes \partial_v - \mathrm{d}t \otimes \partial_j \right),$$

where we use Einstein's summation convention. By slightly abusing notation, we define the symmetric bilinear form $S = S_{ij}\mathrm{d}x^i\mathrm{d}x^j$ on M, so that the curvature of g_S is given as

$$(3.2) \qquad R = -S \owedge (\mathrm{d}t)^2,$$

and consequently $\nabla R = 0$, and the Ricci curvature is

$$(3.3) \qquad \mathrm{Ric} = -\mathrm{tr}(S)(\mathrm{d}t)^2,$$

where $\mathrm{tr}(S)$ is the trace of the matrix S. Hence, g_S has vanishing scalar curvature and Weyl tensor

$$(3.4) \qquad W = -S \owedge (\mathrm{d}t)^2 + \frac{\mathrm{tr}(S)}{n} g_S \owedge (\mathrm{d}t)^2 = \left(\frac{\mathrm{tr}(S)}{n} \mathrm{I} - S \right) \owedge \mathrm{d}t^2,$$

where we define $\mathrm{I} = \delta_{ij} \, \mathrm{d}x^i \, \mathrm{d}x^j$ and use that

$$g_S \owedge (\mathrm{d}t)^2 = \mathrm{I} \owedge (\mathrm{d}t)^2.$$

This yields the following result:

PROPOSITION 3.1. — *The metric g_S is conformally flat if and only if S is a scalar matrix.*

Note that this result includes the case of dimension 3, i.e. when $n = 1$: since the Ricci tensor is parallel, the Cotton tensor of a Cahen–Wallach metric always vanishes. It also implies that in each dimension there are exactly two non-isometric Weyl-flat Cahen–Wallach spaces, namely those with $S = \pm\mathbf{1}$, where $\mathbf{1}$ is the identity matrix of n dimensions. We denote their metrics by $g_\pm$. Since $g_\pm$ are conformally flat, every local conformal

transformation is given, via conjugation with the local conformal diffeomorphism to $\mathbb{R}^{1,n+1}$, by a local conformal transformation of Minkowski space, that is, by the composition of a similarity and a local inversion of $\mathbb{R}^{1,n+1}$.

Due to Kuiper's result about the conformal development map [24], see also [23] for a survey, a conformally flat Cahen–Wallach space $(\mathbb{R}^{n+2}, g_\pm)$ embeds into the conformally flat model space, the Einstein universe $\mathbb{S}^1 \times \mathbb{S}^{n+1}$ with conformal class defined by the product metric $-\mathrm{d}\theta^2 + g_{\mathbb{S}^{n+1}}$, which has the conformal group $\mathbf{PO}(2, n+2)$, see [15] for a survey. Hence, the Lie algebra of conformal vector fields of a conformally flat Cahen–Wallach space $(\mathbb{R}^{n+2}, g_\pm)$ has maximal dimension, that is $\frac{1}{2}(n+4)(n+3)$. An explicit basis for the Lie algebra of conformal vector fields was given in [6, Proposition 4.4].

Since our focus is on the conformally curved case, we will not study the conformal group of $(\mathbb{R}^{n+2}, g_\pm)$ further, we will only make a few more comments on the difference between the real and the imaginary case. In terms of global rescaling to a flat metric we have:

PROPOSITION 3.2. — *Let $(\mathbb{R}^{n+2}, g_\pm)$ be the conformally flat Cahen–Wallach spaces of dimension $n + 2$. Then any conformal rescaling to a Ricci-flat metric is a rescaling to a flat metric and*

(1) *the metric $g_0 = \mathrm{e}^{2t} g_+$ on $\mathbb{R}^{n+2}$ is flat;*
(2) *there is no global rescaling $f \in C^\infty(\mathbb{R}^{n+2})$ such that $\mathrm{e}^{2f} g_-$ is flat.*

Proof. — With $S = \epsilon \mathrm{I}$, with $\epsilon = \pm 1$, the curvature of g_ϵ is

$$R = -\epsilon \, \mathrm{I} \oslash (\mathrm{d}t)^2 = -\epsilon \, g_S \oslash (\mathrm{d}t)^2.$$

Now assume that f is a rescaling to a Ricci-flat metric $\widehat{g} = \mathrm{e}^{2f} g_\epsilon$. Both metrics have vanishing scalar curvature, so Lemma 2.1 and equations (3.2) and (3.3) yield

$$\mathrm{e}^{-2f} \widehat{R} = R - \frac{1}{n} g_S \oslash \mathrm{Ric} = \left(-\epsilon + \frac{\mathrm{tr}(S)}{n} \right) g_S \oslash (\mathrm{d}t)^2 = 0,$$

as $\mathrm{tr}(S) = n\epsilon$. Hence, $\widehat{g}$ is not only Ricci-flat, but also flat.

By equations (2.1), a conformal rescaling f to a Ricci-flat metric satisfies $2\Delta f = ng(\nabla f, \nabla f)$ and hence is a solution to

(3.5) $$0 = \epsilon(\mathrm{d}t)^2 + \left(\nabla \mathrm{d}f - (\mathrm{d}f)^2 \right) + \frac{1}{2} g(\nabla f, \nabla f) g.$$

If $\epsilon = 1$, then a solution to this equation is $f = t$. Indeed, $\mathrm{d}f = \mathrm{d}t$, and hence $\nabla \mathrm{d}f = 0$ and $g(\nabla f, \nabla f) = 0$.

Now assume that $\epsilon = -1$ and that $f \in C^\infty(M)$ is a global solution to equation (3.5). We consider the function $h(t) = f(t, 0, \ldots, 0)$ which is

defined on $\mathbb{R}$. Evaluating the equation in the ∂_t direction and only along $(t, 0, \ldots, 0)$ yields

$$0 = -1 + \partial_t^2 f - x^i \partial_i f - (\partial_t f)^2 - \frac{\|\boldsymbol{x}\|^2}{2} g(\nabla f, \nabla f) = -1 + \partial_t^2 f - (\partial_t f)^2,$$

which shows that h satisfies the ODE

$$\ddot{h} = \dot{h}^2 + 1.$$

Its derivative $y = \dot{h} \in C^\infty(\mathbb{R})$ satisfies the first order separable equation

$$\dot{y} = y^2 + 1.$$

Since $\int \frac{1}{y^2+1} dy = \arctan(x)$ is bounded, the maximal domain of the solutions y is also bounded (see for example [42, p. 9]), which contradicts the assumption. $\qquad\square$

In the real type case this proposition shows that g_+ is *globally* conformally equivalent to a flat Lorentzian metric $\widehat{g}$ on $\mathbb{R}^{n+2}$. We will see that this flat metric is in fact a *geodesically incomplete* Lorentzian metric. For this, use (2.1) and (3.1) to show that the vector fields

$$\partial_v, \quad Y_i := e^{-t} \left(\partial_i + x^i \partial_v \right),$$

and

$$Z := e^{-2t} \partial_t - e^{-t} x^k Y_k = e^{-2t} \left(\partial_t - x^k \partial_k - \|\boldsymbol{x}\|^2 \partial_v \right)$$

on $\mathbb{R}^{n+2}$ are parallel for $\widehat{\nabla}$ and satisfy

$$\widehat{g}(\partial_v, \partial_v) = \widehat{g}(\partial_v, Y_i) = \widehat{g}(Y_i, Z) = \widehat{g}(Z, Z) = 0,$$

as well as

$$\widehat{g}(\partial_v, Z) = 1, \quad \widehat{g}(Y_i, Y_j) = \delta_{ij}.$$

Observe now that the vector field Z is not complete. For example, its maximal integral curve through the origin is given as

$$\gamma(s) = \begin{pmatrix} t(s) = \frac{1}{2} \ln(2s + 1) \\ 0 \\ 0 \end{pmatrix}.$$

Since Z is parallel, this is also a maximal geodesic for $\widehat{g}$, which consequently is a geodesically *incomplete* flat Lorentzian metric on $\mathbb{R}^{n+2}$.

We will find an explicit conformal transformation between $(\mathbb{R}^{n+2}, g_+)$ and an open set in Minkowski space $\mathbb{R}^{1,n+1}$. Since ∂_v, Y_i and Z are parallel, their metric duals are closed 1-forms and we can find a diffeomorphism

$$\phi = \begin{pmatrix} u \\ y^i \\ z \end{pmatrix}$$

of $\mathbb{R}^{n+2}$ by integrating the equations

$$du = \widehat{g}(\partial_v, .) = e^{2t}dt,$$
$$dy^i = \widehat{g}(Y_i, .) = e^t\, dx^i + e^t x^i\, dt,$$
$$dz = \widehat{g}(Z, .) = dv - x^k\, dx^k.$$

A solution that yields a diffeomeorphism $\phi : \mathbb{R}^{n+2} \to \{u > 0\} \subset \mathbb{R}^{1,n+1}$ is given by

$$u = \frac{e^{2t}}{2}, \quad y^i = e^t x^i, \quad z = v - \frac{\|\boldsymbol{x}\|^2}{2}.$$

Hence we arrive at:

PROPOSITION 3.3. — *Let* $(\mathbb{R}^{n+2}, g_+)$ *be the Weyl-flat Cahen–Wallach space of real type and let* (M, g_0) *be the Minkowski half space,*

$$M = \{(u, y^1, \ldots, y^n, z) \in \mathbb{R}^{n+2} \mid u > 0\}, \quad g_0 = 2dudz + \delta_{ij}dy^i dy^j.$$

Then ϕ *defined by*

$$\mathbb{R}^{n+2} \ni \begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \overset{\phi}{\longmapsto} \begin{pmatrix} u = \frac{e^{2t}}{2} \\ \boldsymbol{y} = e^t\boldsymbol{x} \\ z = v - \frac{\|\boldsymbol{x}\|^2}{2} \end{pmatrix} \in M$$

is a global conformal diffeomorphism between $(\mathbb{R}^{n+2}, g_+)$ *and* (M, g_0) *with* $\phi^* g_0 = e^{2t} g_+$.

Remark 3.4. — Note that the inverse of ϕ is

$$\begin{pmatrix} u \\ \boldsymbol{y} \\ z \end{pmatrix} \overset{\phi^{-1}}{\longmapsto} \begin{pmatrix} t = \frac{1}{2}\ln(2u) \\ \boldsymbol{x} = \frac{\boldsymbol{y}}{\sqrt{2u}} \\ v = z + \frac{\|\boldsymbol{y}\|^2}{4u} \end{pmatrix},$$

so that $(\phi^{-1})^* g_+ = \frac{1}{2u} g_0$. Under conjugation by ϕ, the isometries of $(\mathbb{R}^{n+2}, g_+)$ that are given by a translation in the t-component by c (see next section),

$$\begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \longmapsto \begin{pmatrix} t + c \\ \boldsymbol{x} \\ v \end{pmatrix},$$

are mapped to strict homotheties of (M, g_0) of the form

$$\begin{pmatrix} u \\ \boldsymbol{y} \\ z \end{pmatrix} \longmapsto \begin{pmatrix} e^{2c}u \\ e^c\boldsymbol{y} \\ z \end{pmatrix},$$

whereas the isometry of g_+,

$$\begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \longmapsto \begin{pmatrix} -t \\ \boldsymbol{x} \\ -v \end{pmatrix},$$

is mapped to the non-homothetic conformal transformation of g_0 given by

$$\begin{pmatrix} u \\ \boldsymbol{y} \\ z \end{pmatrix} \xrightarrow{\eta} \begin{pmatrix} \frac{1}{4u} \\ \frac{\boldsymbol{y}}{2u} \\ -z - \frac{\|\boldsymbol{y}\|^2}{2u} \end{pmatrix},$$

that satisfies $\eta^* g_0 = \frac{1}{4u^2} g_0$.

In the imaginary case, only *local* rescalings to a flat metric $\widehat{g}$ exist, for example,

$$\widehat{g} = \frac{1}{\cos^2(t)} \, g_-.$$

Similarly to the real case, one can show that the parallel vector fields of $\widehat{g}$ on $\{t \neq \frac{(2k+1)\pi}{2}\}$ are

$$\partial_v, \quad Y_i = \cos(t)\, \partial_i + x^i \sin(t)\, \partial_v,$$

and

$$Z = \cos^2(t)\partial_t - x^i \sin(t) Y_i + \frac{\|x\|^2}{2}\partial_v$$

$$= \cos^2(t)\partial_t - \frac{x^i}{2} \sin(2t)\partial_i + \frac{\|x\|^2}{2} \cos(2t)\partial_v.$$

Note that now the integral curves of Z through the origin are given by

$$\gamma(s) = \begin{pmatrix} \arctan(s) \\ 0 \\ 0 \end{pmatrix}$$

and hence are defined for all s. As before, for finding a diffeomeorphism $\phi = (u, y^i, z)$ we can integrate the equations

$$\mathrm{d}u = \widehat{g}(\partial_v, .) = \frac{1}{\cos^2(t)}\mathrm{d}t,$$

$$\mathrm{d}y^i = \widehat{g}(Y_i, .) = \frac{1}{\cos(t)}\mathrm{d}x^i + \frac{\tan(t)}{\cos(t)}x^i\mathrm{d}t,$$

$$\mathrm{d}z = \widehat{g}(Z, .) = \mathrm{d}v - x^k \tan(t)\mathrm{d}x^k - \frac{\|x\|^2}{2\cos^2(t)}\mathrm{d}t$$

and get a diffeomeorphism $\phi : \{-\frac{\pi}{2} < t < \frac{\pi}{2}\} \to \mathbb{R}^{1,n+1}$, given by

$$u = \tan(t), \quad y^i = \frac{x^i}{\cos(t)}, \quad z = v - \frac{\|x\|^2}{2} \tan(t).$$

Indeed, for the Minkowski metric $g_0 = 2\mathrm{d}u\mathrm{d}z + \sum_{i=1}^{n}(\mathrm{d}y^i)^2$ on $\mathbb{R}^{1,n+1}$, we have that

$$\phi^* g_0 = \frac{1}{\cos^2(t)} g_-.$$

3.2. Isometries, homotheties and conformal transformations

In this section we are going to determine the homothety group of a Cahen–Wallach space, and consequently by virtue of Corollary 2.3, in the conformally curved case also its conformal group. First we describe its isometry group, which is well-known since [10], see also [19].

Let $(\mathbb{R}^{n+2}, g_S)$ be a Cahen–Wallach space defined by the matrix S, which, for the moment, is not assumed to be invertible. We denote by $C_{\mathbf{O}(n)}(S)$ the orthogonal matrices commuting with S and by V_S the $2n$-dimensional solution space of the ODE system $\ddot{\beta} = S\beta$,

$$V_S := \{\beta : \mathbb{R} \longrightarrow \mathbb{R}^n \mid \ddot{\beta} = S\beta\}.$$

It is straightforward to check that the following diffeomorphisms are isometries of g_S,

$$(3.6) \qquad \psi = \psi_{c,\epsilon,b,\beta,A} : \begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \longmapsto \begin{pmatrix} \epsilon\,t + c \\ A\boldsymbol{x} + \beta(t) \\ \epsilon\left(v + b - \langle\dot{\beta}(t), A\boldsymbol{x} + \tfrac{1}{2}\beta(t)\rangle\right) \end{pmatrix},$$

where $c \in \mathbb{R}$, $\epsilon \in \{\pm 1\}$, $A \in C_{\mathbf{O}(n)}(S)$, $b \in \mathbb{R}$, $\beta \in V_S$ and $\langle .,. \rangle$ is the standard Euclidean inner product on $\mathbb{R}^n$. Moreover, when S is invertible, using the fact that an isometry preserves the parallel null vector field ∂_v, one can show that every isometry of g_S is of this form (see [41, Section 4.2] for an explicit calculation).

In order to describe the group structure of the isometry group we will identify several subgroups of the isometry group and their relation to each other. All of these groups come with their natural action on $\mathbb{R}^{n+2}$ via formula (3.6).

First note that $\mathbb{R} \times \{\pm 1\}$ is a subgroup of the isometries with the group structure of the Euclidean group of $\mathbb{R}$, $\mathbf{Euc}(1) = \mathbb{R} \rtimes \mathbb{Z}_2$. We denote its elements by either (c, ϵ) or by $E_{c,\epsilon}$ when we refer to the Euclidean motion $E_{c,\epsilon}(t) = \epsilon\,t + c$ of $\mathbb{R}$. Next, note that $\mathbf{Euc}(1)$ and $C_{\mathbf{O}(n)}(S)$ commute with each other and that $\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S)$ forms a subgroup of $\mathrm{Isom}(\mathbb{R}^{n+2}, g_S)$, and we denote its elements by pairs $(E_{c,\epsilon}, A)$.

Furthermore, also $\mathbb{R} \times V_S$, with its elements denoted by (b, β), forms a subgroup with group operation

$$(b, \beta) +_\omega (\widehat{b}, \widehat{\beta}) := (b + \widehat{b} + \omega(\beta, \widehat{\beta}), \beta + \widehat{\beta}),$$

where ω is the symplectic form on V_S, defined by

$$\omega(\beta, \widehat{\beta}) := \frac{1}{2} \left(\langle \beta(0), \dot{\widehat{\beta}}(0) \rangle - \langle \dot{\beta}(0), \widehat{\beta}(0) \rangle \right).$$

Note that the function $t \mapsto \langle \beta(t), \dot{\widehat{\beta}}(t) \rangle - \langle \dot{\beta}(t), \widehat{\beta}(t) \rangle$ is actually constant, so in order to define ω we could have evaluated it at any other $t \neq 0$. In particular, ω satisfies

(3.7) $\qquad \omega(\beta, \widehat{\beta} \circ E_{c,\epsilon}) = \epsilon \omega(\beta \circ E_{c,\epsilon}^{-1}, \widehat{\beta}), \qquad$ for all $E_{c,\epsilon} \in \mathbf{Euc}(1)$,

which will turn out to be useful, as well as

(3.8) $\qquad\qquad \omega(A\beta, A\widehat{\beta}) = \omega(\beta, \widehat{\beta}), \qquad$ for all $A \in \mathbf{O}(n)$.

This shows that $\mathbb{R} \times V_S$ has the group structure of the $(2n+1)$-dimensional Heisenberg group

$$\mathbf{Hei}_n := \mathbb{R} \times_\omega V_S,$$

which is the central extension of V_S by $\mathbb{R}$.

The subgroup $\mathbf{Hei}_n$ is normal in $\mathrm{Isom}(\mathbb{R}^{n+2}, g_S)$. In fact, if $E_{c,\epsilon} \in \mathbf{Euc}(1)$ and $A \in C_{\mathbf{O}(n)}(S)$, we have for $(b, \beta) \in \mathbf{Hei}_n$ that

$$\begin{aligned}(E_{c,\epsilon}, A)(b, \beta)(E_{c,\epsilon}, A)^{-1} &= (E_{c,\epsilon}, A)(b, \beta)(E_{\epsilon,-\epsilon c}, A^\top) \\ &= (\epsilon b, A\beta \circ E_{-\epsilon c, \epsilon}) \ \in \ \mathbf{Hei}_n.\end{aligned}$$

Note that the translations $\mathbb{R} \subset \mathbf{Euc}(1) = \mathbb{R} \ltimes \mathbb{Z}_2$ act on solutions $\beta \in V_S \subset \mathbf{Hei}(n)$ by a shift $\beta(t) \mapsto \beta(t - c)$. In the case of a Cahen–Wallach space of imaginary type, (i.e. when S has only negative eigenvalues), the shifted solutions are given by rotations of the initial conditions (see the proof of Lemma 4.5 for details). Hence, in this case $\mathbb{R} \subset \mathbf{Euc}(1)$ acts by rotations of the elements in $V_S \subset \mathbf{Hei}(n)$. This will be important in the proof of Theorem 1.1.

Finally, any isometry ψ as in (3.6) is a product of elements from $\mathbf{Hei}_n$ and $\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S)$. Indeed, it is

$$\psi = \psi_{c,\epsilon,b,\beta,A} = \underbrace{E_{\epsilon,c}}_{\in \mathbf{Euc}(1)} \underbrace{(b, \beta)}_{\in \mathbf{Hei}_n} \underbrace{A}_{\in C_{\mathbf{O}(n)}(S)} = \underbrace{(\epsilon b, \beta \circ E_{-\epsilon c, \epsilon})}_{\in \mathbf{Hei}_n} \underbrace{(E_{c,\epsilon}, A)}_{\in \mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S)}.$$

The reader may have noticed that, in order to keep the notation brief, we use it quite flexibly: for example by A we refer to $\psi_{0,1,0,0,A}$, by $(E_{c,\epsilon}, A)$ to $\psi_{c,\epsilon,0,0,A}$, by (b, β) to $\Psi_{0,1,b,\beta,\mathbf{1}}$, etc., and the group product the is the

composition when acting on $\mathbb{R}^{n+2}$. Hence, we have arrived at the well known fact [10, 19]:

PROPOSITION 3.5. — *The isometry group of a Cahen–Wallach space* $(\mathbb{R}^{n+2}, g_S)$ *is isomorphic to the semidirect product*

$$\mathbf{Hei}_n \rtimes_\alpha \left(\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \right),$$

where $\mathbf{Hei}_n$ *is the* $(2n+1)$-*dimensional Heisenberg group,* $\mathbf{Euc}(1) = \mathbb{R} \rtimes \mathbb{Z}_2$ *is the Euclidean group in one dimension,* $C_{\mathbf{O}(n)}(S)$ *is the centraliser of the matrix* S *in* $\mathbf{O}(n)$, *and the homomorphism* $\alpha : \mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \to \mathrm{Aut}(\mathbf{Hei}_n)$ *is defined as*

$$\alpha_{(c,\epsilon,A)}(b,\beta) := \left(\epsilon b, A\beta \circ E_{c,\epsilon}^{-1} \right).$$

The isomorphism maps $\psi_{c,\epsilon,b,\beta,A}$ *in* (3.6) *to*

$$((\epsilon b, \beta \circ E_{-\epsilon c, \epsilon}), E_{c,\epsilon}, A) \in \mathbf{Hei}_n \rtimes(\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S)).$$

To be very explicit, let us emphasise again that the action of an element $((b,\beta),(E_{c,\epsilon},A))$ of $\mathbf{Hei}_n \rtimes_\alpha \left(\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \right)$ on $\mathbb{R}^{n+2}$ is given via (3.6) as

$$((b,\beta),(E_{c,\epsilon},A))(t,\boldsymbol{x},v) = \psi_{0,1,b,\beta,\mathbf{1}}\left(\psi_{c,\epsilon,0,0,A}(t,\boldsymbol{x},v)\right).$$

Moreover, the explicit formula for the group product in

$$\mathbf{Hei}_n \rtimes_\alpha \left(\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \right)$$

is

$$(3.9) \quad \left((b,\beta),(E_{c,\epsilon},A)\right)\left((\widehat{b},\widehat{\beta}),(E_{\hat{c},\hat{\epsilon}},\widehat{A})\right)$$
$$= \left(\left((b,\beta) +_\omega \alpha_{c,\epsilon,A}(\widehat{b},\widehat{\beta})\right),\left(E_{c,\epsilon} \circ E_{\hat{c},\hat{\epsilon}}, A\widehat{A}\right)\right)$$
$$= \left(\left((b + \epsilon\widehat{b} + \omega(\beta, A\widehat{\beta} \circ E_{-\epsilon c,\epsilon}), \beta + A\widehat{\beta} \circ E_{-\epsilon c,\epsilon}\right),(E_{c+\epsilon\hat{c},\epsilon\hat{\epsilon}}, A\widehat{A})\right),$$

where (b,β) and $(\widehat{b},\widehat{\beta})$ are elements from $\mathbf{Hei}_n$ and $(E_{c,\epsilon},A)$ and $(E_{\hat{c},\hat{\epsilon}},\widehat{A})$ from $\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S)$. The formula for the inverse is

$$(3.10) \quad \left((b,\beta),(E_{c,\epsilon},A)\right)^{-1} = \left((-\epsilon b, -A^\top\beta \circ E_{c,\epsilon}),(E_{-\epsilon c,\epsilon}, A^\top)\right).$$

Remark 3.6. — We should also point out that if S is not invertible, then the isometry group of g_S contains the group $\mathbf{Hei}_n \rtimes_\alpha \left(\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \right)$, but in general is larger, for example when $S = 0$, in which case g_S is the Minkowski metric.

When S is not zero but has a kernel of dimension $k \geqslant 1$, the metric g_S is isometric to a product of an indecomposable Cahen–Wallach space of dimension $n-k+2$ and Euclidean space of dimension k. Interestingly, since it

contains $\mathbf{Hei}_n \rtimes_\alpha \left(\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S)\right)$, the isometry group is larger than the product of the isometry groups of both manifolds, whose dimension is

$$\dim(C_{\mathbf{O}(n-k)}(S)) + 2(n+1) + \frac{1}{2}k(k-3).$$

On the other hand, since the centraliser of S in $\mathbf{O}(n)$ is $C_{\mathbf{O}(n-k)}(S) \times \mathbf{O}(k)$, the dimension of $\mathbf{Hei}_n \rtimes_\alpha \left(\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S)\right)$ is

$$\dim(C_{\mathbf{O}(n-k)}(S)) + 2(n+1) + \frac{1}{2}k(k-1).$$

Moreover, observe that if $(\mathbb{R}^{n+2}, g_S)$ is conformally flat, then

$$\mathrm{Isom}(\mathbb{R}^{n+2}, g_S) = \mathbf{Hei}_n \rtimes_\alpha \left(\mathbf{Euc}(1) \times \mathbf{O}(n)\right),$$

and hence the dimension of the isometry group is reduced by $n+1$ from the dimension of the isometry group of Minkowski space $\mathbb{R}^{1,n+1}$, which is $\frac{1}{2}(n+2)(n+3)$.

Remark 3.7. — The transvection group within the isometry group is the solvable group $\mathbf{Hei} \ltimes_\alpha \mathbb{R}$, where $\mathbb{R}$ are the translations in $\mathbf{Euc}(1)$. The stabiliser in $\mathrm{Isom}(\mathbb{R}^{n+2}, g_S)$ of the origin is given as

$$L_S \rtimes_\alpha (\mathbb{Z}_2 \times C_{\mathbf{O}(n)}(S)),$$

where

$$L_S := \{\beta \in V_S \mid \beta(0) = 0\} \subset \mathbf{Hei}_n$$

is a Lagrangian subspace in V_S and hence a subgroup of $\mathbf{Hei}_n$. Similarly the stabiliser in the transvections is the abelian group L_S and we have

$$(\mathbb{R}^{n+2}, g_S) = (\mathbf{Hei} \ltimes_\alpha \mathbb{R})/L_S.$$

Now we turn to the homotheties of $(\mathbb{R}^{n+2}, g_S)$. Clearly, for each $s \in \mathbb{R}$ the linear map given by the matrix $h_s := \mathrm{diag}(1, \mathrm{e}^s, \ldots, \mathrm{e}^s, \mathrm{e}^{2s})$,

$$(3.11) \qquad \begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \xmapsto{h_s} \begin{pmatrix} t \\ \mathrm{e}^s \boldsymbol{x} \\ \mathrm{e}^{2s} v \end{pmatrix}$$

is a homothety of g_S. We call h_s a *pure homothety*. The pure homotheties are a subgroup in the homotheties which we denote by $\mathbb{R}$. The isometries are normal in the homotheties and we have that

$$(3.12) \qquad h_s\left((b, \beta) \cdot (c, \epsilon, A)\right) h_s^{-1} = (\mathrm{e}^{2s} b, \mathrm{e}^s \beta) \cdot (c, \epsilon, A).$$

In particular, the pure homotheties commute with $\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S)$. This yields

PROPOSITION 3.8. — *The homothety group of a Cahen–Wallach space is isomorphic to*

$$\mathbf{Hei}_n \rtimes_\varphi \left(\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \times \mathbb{R}\right),$$

where $\varphi : \mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \times \mathbb{R} \to \mathrm{Aut}(\mathbf{Hei}_n)$ *is defined as*

$$\varphi(c, \epsilon, A, s)(b, \beta) := \left(\epsilon e^{2s} b, e^s A\beta \circ E_{c,\epsilon}^{-1}\right).$$

In the following we will denote

$$(3.13) \qquad H_S := \mathbf{Hei}_n \rtimes_\varphi \left(\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \times \mathbb{R}\right),$$

and identify it with the the homothety group of the Cahen–Wallach space $(\mathbb{R}^{n+2}, g_S)$. From the proposition it follows that there is a surjective group homomorphism

$$H_S \longrightarrow H_S/\mathbf{Hei}_n \simeq \mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \times \mathbb{R},$$

and for a homothety ϕ we denote the image under this projection by

$$(E_\phi, A_\phi, s_\phi) = (c_\phi, \epsilon_\phi, A_\phi, s_\phi)$$

which is an element in

$$\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \times \mathbb{R} = (\mathbb{R} \ltimes \mathbb{Z}_2) \times C_{\mathbf{O}(n)}(S) \times \mathbb{R}.$$

Remark 3.9. — It was already noted in [2], see also [37, 38], that the diffeomorphism h_s in (3.11) is a homothety for many Lorentzian metrics, namely those of the form

$$2\mathrm{d}t \left(\mathrm{d}v + P_{ij}(t)x^i\mathrm{d}x^j + \left(Q_{ij}(t)x^ix^j + R(t)v\right)\mathrm{d}t\right) + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j,$$

including the so-called pp-waves, of which the Cahen–Wallach metrics are a special case.

In the non Weyl-flat case with $n \geqslant 2$, the conformal group of $(\mathbb{R}^{n+2}, g_S)$, reduces to the homotheties by Corollary 2.3.

COROLLARY 3.10. — *Let* $(\mathbb{R}^{n+2}, g_S)$ *be a Cahen–Wallach space of dimension* $n+2 \geqslant 4$ *such that* S *has at least two different eigenvalues, i.e.* g_S *is not Weyl-flat. If* $\phi : U \to \phi(U)$ *is a conformal transformation on an open set* U, *then* ϕ *is a homothety. In particular,*

$$\mathrm{Conf}(\mathbb{R}^{n+2}, g_S) = H_S = \mathbf{Hei}_n \rtimes_\varphi \left(\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \times \mathbb{R}\right).$$

3.3. Fixpoints and essential homotheties

In this section, we will prove Theorem 1.2. For this, we detail various sufficient conditions for a homothety of a Cahen–Wallach space to have a fixed point and hence be essential. Using this, we will show the converse of Proposition 2.5. Throughout this section, we denote by $\phi = \phi_{c,\epsilon,b,\beta,A,s}$ the homothety

(3.14)

$$\phi = \phi_{c,\epsilon,b,\beta,A,s} \; : \; \begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \longmapsto \begin{pmatrix} \epsilon\,t + c \\ \mathrm{e}^s A\boldsymbol{x} + \beta(t) \\ \epsilon\left(\mathrm{e}^{2s}v + b - \langle \dot\beta(t), \mathrm{e}^s A\boldsymbol{x} + \tfrac{1}{2}\beta(t)\rangle\right) \end{pmatrix},$$

and by E_ϕ the Euclidean motion $E_{c,\epsilon}$ that maps t to $\epsilon\,t + c$.

PROPOSITION 3.11. — *A strict homothety ϕ of $(\mathbb{R}^{n+2}, g_S)$ has a fixed point if and only if the Euclidean motion E_ϕ of $\mathbb{R}$ has a fixed point, i.e. if and only if $\epsilon = -1$ or $c = 0$.*

Proof. — Recall that E_ϕ is defined by $E_\phi(t) = \epsilon\,t + c$. If E_ϕ has no fixed point, then ϕ cannot fix any point.

Conversely, let t be a fixed point of E_ϕ. Then one can verify that

$$\begin{pmatrix} t \\ -(\mathrm{e}^s A - \mathbf{1})^{-1}\beta(t) \\ -(\mathrm{e}^{2s}a - 1)^{-1}a\left(b - \langle\dot\beta(t), -\mathrm{e}^s A(\mathrm{e}^s A - I_n)^{-1}\beta(t) + \tfrac{1}{2}\beta(t)\rangle\right) \end{pmatrix}$$

is a fixed point of ϕ. The assumption that ϕ is a strict homothety is crucial for the inverses of $(\mathrm{e}^{2s}\epsilon - 1)$ and $(\mathrm{e}^s A - \mathbf{1})$ to exist.   □

LEMMA 3.12. — *Let ϕ be an isometry of $(\mathbb{R}^{n+2}, g_S)$ with $\epsilon = -1$. Then ϕ fixes a point if and only if $\boldsymbol{x} \mapsto A\boldsymbol{x} + \beta(\tfrac{c}{2})$ fixes a point.*

Proof. — Note that $\tfrac{c}{2}$ is the unique fixed point of $E_\phi = E_{c,-1} : t \mapsto -t + c$, so if $A\boldsymbol{x} + \beta(\tfrac{c}{2})$ does not have a fixed point, then ϕ cannot fix any point.

Conversely, let $\boldsymbol{y} = A\boldsymbol{y} + \beta(\tfrac{c}{2})$. Then one can check that

$$\begin{pmatrix} \tfrac{c}{2} \\ \boldsymbol{y} \\ -\tfrac{1}{2}\left(b - \langle\dot\beta(\tfrac{c}{2}), A\boldsymbol{y} + \tfrac{1}{2}\beta(\tfrac{c}{2})\rangle\right) \end{pmatrix}$$

is a fixed point of ϕ.   □

PROPOSITION 3.13. — *Let ϕ be a homothety of $(\mathbb{R}^{n+2}, g_S)$ with $\phi^k = \mathrm{id}$ for some $k > 0$. Then ϕ fixes a point.*

Proof. — We construct a point y fixed by ϕ. We start with the t component of y: If $\epsilon = 1$, then $t \circ \phi^k = t + kc$. This can only fix a point for $k > 0$ if $c = 0$, so we conclude either $c = 0$ or $\epsilon = -1$. In either case

$$t(y) := \frac{c}{2}$$

is a fixed point of $t \mapsto \epsilon t + c$. From this we also get by Proposition 3.11, that ϕ either has a fixed point or is an isometry. So we assume that ϕ is an isometry. Define $\beta_0 := \beta(\frac{c}{2})$ and $\dot\beta_0 := \dot\beta(\frac{c}{2})$.

Next we consider the $\boldsymbol{x}$ component: since $\phi^k(\frac{c}{2}, 0, 0) = (\frac{c}{2}, 0, 0)$ we get that the Euclidean motion $E_{\beta_0, A}(\boldsymbol{x}) = A\boldsymbol{x} + \beta_0$ satisfies $E_{\beta_0, A}^k \equiv \mathrm{Id}$. In particular,

$$E_{\beta_0, A}^k(0) = \sum_{i=0}^{k-1} A^i \beta_0 = 0.$$

In general any euclidean motion E satisfying $E^k(\boldsymbol{x}) = \boldsymbol{x}$ fixes a point. This fixed point is given by the centre of mass (in our case, $\boldsymbol{x} = 0$),

$$\boldsymbol{y} := \frac{1}{k} \sum_{i=1}^{k} E_{\beta_0, A}^i(0).$$

Now when $\epsilon = -1$, Lemma 3.12 gives us a fixed point of ϕ. When $\epsilon = 1$, to have a fixed point we require that $b - \langle \dot\beta_0, A\boldsymbol{y} + \frac{1}{2}\beta_0 \rangle = 0$. But since $\phi(\frac{c}{2}, \boldsymbol{y}, .) = (\frac{c}{2}, \boldsymbol{y}, \ldots)$, we get

$$\phi^k \begin{pmatrix} \frac{c}{2} \\ \boldsymbol{y} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{c}{2} \\ \boldsymbol{y} \\ k(b - \langle \dot\beta_0, A\boldsymbol{y} + \frac{1}{2}\beta_0 \rangle) \end{pmatrix} = \begin{pmatrix} \frac{c}{2} \\ \boldsymbol{y} \\ 0 \end{pmatrix}.$$

Hence $b - \langle \dot\beta_0, A\boldsymbol{y} + \frac{1}{2}\beta_0 \rangle = 0$, and so any choice of $v(\boldsymbol{y})$, for example $v(\boldsymbol{y}) := 0$ makes $(\frac{c}{2}, \boldsymbol{y}, 0)$ a fixed point of ϕ. □

Now we give a characterisation of essential homotheties of Cahen–Wallach spaces. The non-trivial part of the proof is a special case of the results in [28, 41], however we will present it here for the sake of completeness.

THEOREM 3.14. — *A strict homothety ϕ of a Cahen–Wallach space is essential if and only if it fixes a point.*

Proof. — First, if ϕ has a fixed point, then by Proposition 2.5, ϕ is essential. For the converse, assume that ϕ is essential but without fixed point. Then by Proposition 3.11, $t \circ \phi(x) = t(x) + c$ for some $c > 0$. We will now construct a function f such that ϕ is an ismometry for the metric $\mathrm{e}^{2f} g$.

The group $\langle \phi \rangle$ admits a fundamental domain $D = \mathbb{R}^{n+1} \times (0, c)$. Let $h : \mathbb{R} \to \mathbb{R}_{>0}$ be a smooth function of t such that $h|_{[0,c]} \equiv 1$ and with support in $(-\frac{c}{4}, \frac{5c}{4})$. Then define functions $\{h_k\}_{k \in \mathbb{Z}}$ on $\mathbb{R}^{n+2}$ by

$$h_0(t, \boldsymbol{x}, v) = h(t), \qquad h_k := h_0 \circ \phi^{-k},$$

so that $h_k = h_{k+1} \circ \phi$. Since $\{\mathrm{supp}(h_k)\}_{k \in \mathbb{Z}}$ is locally finite, the function $\sum_{k \in \mathbb{Z}} h_k$ is well defined. Since D is a fundamental domain and $h_k \geqslant 0$, the function $\sum_{k \in \mathbb{Z}} h_k$ has no zeros. Hence,

$$f_k := \frac{h_k}{\sum_{k \in \mathbb{Z}} h_k}$$

is a partition of unity on $\mathbb{R}^{n+2}$ that satisfies $f_k = f_{k+1} \circ \phi$. If $\phi^* g = \mathrm{e}^{2s} g$, we set

$$f := -s \sum_{k \in \mathbb{Z}} k f_k,$$

which yields that ϕ is an isometry for $\mathrm{e}^{2f} g$ as

$$f \circ \phi = -s \sum_{k \in \mathbb{Z}} k f_{k-1} = f - s \sum_{k \in \mathbb{Z}} f_k = f - s.$$

Therefore, ϕ is not essential and we arrive at a contradiction, so that ϕ must have a fixed point. $\square$

4. Non-existence results for compact conformal quotients of Cahen–Wallach spaces

4.1. Conformal compact quotients of imaginary type

In this section we show that conformal compact quotients of Cahen–Wallach spaces of imaginary type must be isometric quotients.

We start with some technical results about cocompact conformal group actions of Cahen–Wallach spaces. Recall the definition of H_S in (3.13), which is the group of homotheties of the Cahen–Wallach space $(\mathbb{R}^{n+2}, g_S)$, i.e. when S is invertible. The technical statements that will lead up to Theorem 4.7 however will not require that S is invertible. First we show that cyclic groups of homotheties cannot act cocompactly.

LEMMA 4.1. — *If $\gamma \in H_S$, then $\langle \gamma \rangle$ does not act cocompactly.*

Proof. — Assume that $\langle \gamma \rangle$ acts cocompactly. Then $\Gamma := \langle \gamma^2 \rangle$ acts cocompactly and we have that $t \circ \gamma^2 = t + c$, where t is the coordinate

$$t : \mathbb{R}^{n+2} \longrightarrow \mathbb{R}, \qquad (t, \boldsymbol{x}, v) \longmapsto t.$$

If $c = 0$, then the smooth map t is invariant under Γ and descends to a smooth, surjective map $f : \mathbb{R}^{n+2}/\Gamma \to \mathbb{R}$. This contradicts the compactness of $\mathbb{R}^{n+2}/\Gamma$.

If $c \neq 0$, consider $D = \mathbb{R}^{n+1} \times (0, c)$ and choose an unbounded continuous function f on the closure of D with support in D. This defines an unbounded function $\widetilde{f}$ on $\mathbb{R}^{n+2}$ by

$$\widetilde{f}(t, \boldsymbol{x}, v) = f(\gamma^{-2n}(t, \boldsymbol{x}, v)), \quad \text{if } t \in [nc, (n+1)c].$$

This function descends to an unbounded continuous function on the compact quotient $\mathbb{R}^{n+2}/\Gamma$, which is a contradiction. $\qquad \square$

In regards to a group of homotheties acting properly discontinuously, we obtain from Proposition 3.11 the following corollary.

COROLLARY 4.2. — *If Γ is a group of homotheties acting properly discontinuously, then every strict homothety $\gamma \in \Gamma \setminus \mathrm{Isom}$ must have $\epsilon = 1$ and $c \neq 0$.*

Next, we find an obstruction for a homothety group acting properly discontinuously and cocomapactly. Recall that Σ_+ denotes the set of positive eigenvalues of S.

PROPOSITION 4.3. — *Let $\Gamma \subset H_S$ be a subgroup that acts cocompactly. If Γ contains a strict homothety*

$$\gamma = (c_\gamma, A_\gamma, s_\gamma) \in \mathbb{R} \times C_{\mathbf{O}(n)}(S) \times \mathbb{R}$$

with the property

(4.1) $$s_\gamma^2 - \lambda_i^2 c_\gamma^2 > 0 \quad \text{for all } \lambda_i^2 \in \Sigma_+,$$

then Γ cannot act properly discontinuously.

Proof. — Since Γ acts cocompactly, by Lemma 4.1 it cannot be cyclic, so there is a homothety $\phi \in \Gamma \setminus \langle \gamma \rangle$, which we fix. Without loss of generality, we can assume that ϕ is a strict homothety, otherwise we multiply ϕ by γ. Let $\phi = \phi_{c,\epsilon,b,\beta,A,s}$ be a homothety as in (3.14). For a proof by contradiction, assume that Γ acts properly discontinuously. This implies that $\epsilon = 1$ and $c \neq 0$, as otherwise, by Proposition 3.11, ϕ would have a fixed point and Γ could not act properly discontinuously.

Furthermore, if β, b are both zero, then for any sequence of rational numbers $p_i/q_i \to c/c_\gamma$, we have that $\gamma^{p_i}\phi^{-q_i}(0) \to 0$, contradicting PD1. Hence at least one of b or β is non-zero.

The assumption that

$$0 < s_\gamma^2 - \lambda_i^2 c_\gamma^2 = (s_\gamma + \lambda_i c_\gamma)(s_\gamma - \lambda_i c_\gamma)$$

means that $(s_\gamma + \lambda c_\gamma)$ and $(s_\gamma - \lambda c_\gamma)$ have the same sign. Without loss of generality, we assume that both are positive (if both are negative, we use γ^{-1} in what follows). Now consider the sequence

$$y_k = \gamma^{-k}\phi\gamma^k(0) = \begin{pmatrix} c \\ \mathrm{e}^{-ks_\gamma}(A_\gamma^\top)^k\beta(kc_\gamma) \\ \mathrm{e}^{-2ks_\gamma}\left(b - \langle \dot\beta(kc_\gamma), \tfrac{1}{2}\beta(kc_\gamma)\rangle\right) \end{pmatrix}.$$

Let $\beta^i : \mathbb{R} \to \mathbb{R}$, $i = 1,\dots,n$ be the components of $\beta \in V_S$. For those i for which the eigenvalues of S are negative, the sequence $\beta^j(kc_\gamma)$ is bounded and hence $\mathrm{e}^{-ks_\gamma}\beta^j(kc_\gamma)$ converges to zero when k goes to infinity.

For those j that correspond to the kernel of S, β^i is linear and again $\mathrm{e}^{-ks_\gamma}\beta^i(kc_\gamma)$ converges to zero when k goes to infinity.

For those i for which the eigenvalues λ_i^2 of S is positive we have

(4.2)
$$\beta^i(kc_\gamma) = b^i \cosh(\lambda_i kc_\gamma) + c^i \sinh(\lambda_i kc_\gamma)$$
$$= \frac{b^i + c^i}{2}\mathrm{e}^{\lambda_i kc_\gamma} + \frac{b^i - c^i}{2}\mathrm{e}^{-\lambda_i kc_\gamma},$$

for some constants b^i and c^i, not both zero. By the assumption that both $(s_\gamma + \lambda_i c_\gamma)$ and $(s_\gamma - \lambda_i c_\gamma)$ are positive, we have that

$$\mathrm{e}^{-ks_\gamma}\beta^i(kc_\gamma) = \frac{b^i + c^i}{2}\mathrm{e}^{-k(s_\gamma - \lambda_i c_\gamma)} + \frac{b^i - c^i}{2}\mathrm{e}^{-k(s_\gamma + \lambda_i c_\gamma)}$$

(no summation over i), is a non-constant sequence that converges to zero when $k \to \infty$. Since A_γ is an orthogonal matrix this implies that y_k^i is a non-constant sequence converging to zero for $k \to \infty$.

Finally for $v_k = v(y_k) = \mathrm{e}^{-2ks_\gamma}(b - \langle \dot\beta(kc_\gamma), \tfrac{1}{2}\beta(kc_\gamma)\rangle$, using the formula (4.2) for the β^is, we have that

$$\lim_{k\to\infty} v_k = \frac{1}{4}\sum_{i=1}^{p} \lim_{k\to\infty}\left(\mathrm{e}^{-2k(s_\gamma - \lambda_i c_\gamma)} - \mathrm{e}^{-2k(s_\gamma + \lambda_i c_\gamma)}\right) = 0,$$

since both $(s_\gamma \pm \lambda c_\gamma)$ are positive.

Hence, $y_k = \gamma^{-k}\phi\gamma^k(0)$ is a non-constant sequence in the Γ-orbit of 0 that converges to $(c,0,0)$. This contradicts the assumption that Γ acts properly discontinuously. $\qquad\square$

Remark 4.4. — The assumption of Proposition 4.3 can be formulated as

$$\frac{s^2}{c^2} > \lambda_{\max}^2,$$

where $\lambda_{\max}^2$ is the largest positive eigenvalue of S. Note that we have not assumed that S is non-degenerate. Also, this condition is invariant in the isometry class of g_S. Indeed, for $a \in R_{>0}$ and $\alpha(t, x, v) = (at, x, a^{-1}v)$, we get that $\alpha^* g_S = g_{a^2 S}$ and the homotheties of $g_{a^2 S}$ are given by conjugation with α, i.e. by $\widehat{\gamma} = \alpha^{-1} \circ \gamma \circ \alpha$. Then $\widehat{c}$ of $\widehat{\gamma}$ is given by $\frac{c_\gamma}{a}$, s is not changed, and the largest eigenvalue of $a^2 S$ is $a^2 \lambda_{\max}$, so that

$$\left(\frac{\widehat{s}}{\widehat{c}} \right)^2 = a^2 \left(\frac{s}{c} \right)^2 > a^2 \lambda_{\max}.$$

The proof also shows that if Γ contains a $\gamma = (c_\gamma, A_\gamma, s_\gamma) \in \mathbb{R} \times C_{\mathbf{O}(n)}(S) \times \mathbb{R}$ and acts cocompactly and properly discontinuously, then for all $\phi \in \Gamma \setminus \langle \gamma \rangle$ the corresponding β must have at least one exponentially growing component, that is, β must have a non-vanishing component in an eigenspace for a positive eigenvalue.

As the next step we show that under certain conditions, every strict homothety is conjugated to a homothety $\gamma \in \mathbb{R} \times C_{\mathbf{O}(n)}(S) \times \mathbb{R}$. This requires the following technical result.

LEMMA 4.5. — *Let* $\widehat{\beta} \in V_S$, $A \in C_{\mathbf{O}(n)}(S)$, $s \in \mathbb{R}$, $s \neq 0$, *and* $c \in \mathbb{R}$. *Unless* $(\frac{s}{c})^2$ *is an eigenvalue of* S, *there is a* $\beta \in V_S$ *such that*

$$(4.3) \qquad\qquad e^s A \beta \circ \sigma_c - \beta = \widehat{\beta},$$

where $\sigma_c = E_{c,1} \in \mathbf{Euc}(1)$ *is the shift by* c *in* $\mathbb{R}$, $\sigma_c(t) = t + c$.

Proof. — Since A commutes with S, it preserves the eigenspaces of S and it suffices to determine β on each eigenspace separately. We abuse notation by denoting by β and $\widehat{\beta}$ their component on each eigenspace and by n the dimension of an eigenspace. Since $\beta \in V_S$ and $\widehat{\beta} \in V_S$, both are determined by their initial values $\beta_0 := \beta(0)$, $\widehat{\beta}_0 := \widehat{\beta}(0)$ and derivatives $\beta_1 := \beta'(0)$ and $\widehat{\beta}_1 := \widehat{\beta}'(0)$ we have to show that we can choose β_0 and β_1 such that the corresponding solution β satisfies (4.3). First we consider the case of a negative eigenvalue $-\mu^2$. A solution β is given as

$$\beta(t) = \beta_0 \cos(\mu t) + \frac{\beta_1}{\mu} \sin(\mu t).$$

Hence we are looking for initial conditions β_0 and β_1 satisfying the linear system of $2n$ equations

(4.4)
$$\underbrace{\begin{pmatrix} \mu(\mathrm{e}^s \cos(\mu c)A - \mathbf{1}) & \mathrm{e}^s \sin(\mu c)A \\ -\mu \mathrm{e}^s \sin(\mu c)A & \mathrm{e}^s \cos(\mu c)A - \mathbf{1} \end{pmatrix}}_{=:M} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \mu \widehat{\beta}_0 \\ \widehat{\beta}_1 \end{pmatrix}$$

This system has a solution for any given right-hand-side if the matrix M is invertible. Since the bottom two blocks commute, we get the determinant

$$\det(M) = \mu^n \det\left(\mathrm{e}^{2s}A^2 - 2\mathrm{e}^s \cos(\mu c)A + \mathbf{1}\right),$$

see [40] for details. We know that A is orthogonal and hence is diagonalisable over $\mathbb{C}$ with complex eigenvalues $z_1, \ldots, z_n$ which lie on the unit circle, $|z_i| = 1$. So we compute this determinant by diagonalisation to obtain

$$\det(M) = \prod_{i=1}^{n} \left(\mathrm{e}^{2s}z_i^2 - 2\mathrm{e}^s \cos(\mu c)z_i + 1\right).$$

Hence, the determinant of M can only be zero if one of the z_i's is a root of

$$\mathrm{e}^{2s}z^2 - 2\mathrm{e}^s \cos(\mu c)z + 1.$$

However, the roots of this quadratic polynomial are given by the complex numbers $z = \mathrm{e}^{-s \pm \mathrm{i}\mu c}$, which do not lie on the unit circle as $s \neq 0$. Hence M is invertible and by inverting it we find suitable initial conditions β_0 and β_1 so that β solves (4.3).

Secondly, we consider the case of a positive eigenvalue λ^2. Now a solution β is given as

$$\beta(t) = \beta_0 \cosh(\lambda t) + \frac{\beta_1}{\lambda} \sinh(\lambda t).$$

The corresponding linear system for β_0 and β_1 that replaces (4.4) now is given by the matrix

$$M = \begin{pmatrix} \lambda(\mathrm{e}^s \cosh(\lambda c)A - \mathbf{1}) & \mathrm{e}^s \sinh(\lambda c)A \\ \lambda \mathrm{e}^s \sinh(\lambda c)A & \mathrm{e}^s \cosh(\lambda c)A - \mathbf{1} \end{pmatrix}$$

with determinant

$$\det(M) = \lambda^n \det\left(\mathrm{e}^{2s}A^2 - 2\mathrm{e}^s \cosh(\lambda c)A + \mathbf{1}\right)$$

$$= \prod_{i=1}^{n} \left(\mathrm{e}^{2s}z_i^2 - 2\mathrm{e}^s \cosh(\lambda c)z_i + 1\right).$$

Again, the determinant of M can only be zero if one of the z_i's is a root of

$$\mathrm{e}^{2s}z^2 - 2\mathrm{e}^s \cosh(\lambda c)z + 1.$$

Now the roots are real numbers $z = e^{-s \pm \lambda c}$ which do not lie on the unit circle *unless* $s = \pm \lambda c$. This however was excluded in the assumption and so M is invertible and yields a solution β to (4.3).

Finally, in the case of eigenvalue zero, the solutions are affine, $\beta(t) = \beta_1 t + \beta_0$, so that (4.3) is equivalent to

$$(e^s A - \mathbf{1})\beta_1 = \widehat{\beta}_1, \quad (e^s A - \mathbf{1})\beta_0 = \widehat{\beta}_0 - ce^s A\beta_1.$$

Inverting $(e^s A - \mathbf{1})$ gives the result also in this case. $\qquad\qquad\square$

PROPOSITION 4.6. — *Let $\widehat{\phi} \in H_S$ be a strict homothety such that $\epsilon_{\widehat{\phi}} = 1$. Unless $\left(\frac{s_{\widehat{\phi}}}{c_{\widehat{\phi}}}\right)^2$ is an eigenvalue of S, the homothety $\widehat{\phi}$ is conjugate by an isometry from $\mathbf{Hei}_n \rtimes \mathbb{Z}_2$ to a strict homothety $\phi \in \mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \times \mathbb{R}$ with $\epsilon_\phi = 1$ and $c_\phi \geqslant 0$.*

Proof. — Let $\widehat{\phi}$ be a strict homothety with $\epsilon_{\widehat{\phi}} = 1$. Then $\widehat{\phi} = \widehat{\psi} \circ h_{\widehat{s}}$ with an isometry $\widehat{\psi} = (\widehat{b}, \widehat{\beta}) \circ (\widehat{c}, 1, \widehat{A})$ with $(\widehat{b}, \widehat{\beta}) \in \mathbf{Hei}_n$ and $(\widehat{c}, 1, \widehat{A}) \in \mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S)$, and $h_{\widehat{s}}$ a pure homothety with $\widehat{s} = s_{\widehat{\phi}} \neq 0$. First, we are searching for an isometry $\psi = (b, \beta) \in \mathbf{Hei}_n$ such that

$$(4.5) \qquad \psi\widehat{\phi}\psi^{-1} = \psi\widehat{\psi}h_{\widehat{s}}\psi^{-1} \in \mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \times \mathbb{R}.$$

Note that, since $\epsilon_{\widehat{\phi}} = 1$ we automatically have that $\epsilon_{\psi\widehat{\phi}\psi^{-1}} = 1$. Denoting by

$$(4.6) \qquad\qquad\qquad \varphi_s := \varphi(0, 1, \mathbf{1}, s)$$

with $\varphi : \mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \times \mathbb{R} \to \mathrm{Aut}(\mathbf{Hei}_n)$ from Proposition 3.8 and using using 3.12), we have

$$\psi\widehat{\phi}\psi^{-1} = \psi\widehat{\psi}h_{\widehat{s}}\psi^{-1} = \psi\widehat{\psi}\varphi_{\widehat{s}}(\psi^{-1})h_{\widehat{s}}.$$

Hence condition (4.5) is equivalent to

$$\psi\widehat{\psi}\varphi_{\widehat{s}}(\psi^{-1}) \in \mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S).$$

Now we compute using the formulas (3.9), (3.10) and (3.12),

$$\psi\widehat{\psi}\varphi_{\widehat{s}}(\psi^{-1}) = \left(\left((b + \widehat{b} + \omega(\beta, \widehat{\beta}), \beta + \widehat{\beta}\right), (\widehat{c}, 1, \widehat{A})\right)(-be^{2\widehat{s}}, -e^{\widehat{s}}\beta),$$

which is equal to

$$\left((1 + \omega(\beta, \widehat{\beta}) - e^{2\widehat{s}})b + \widehat{b} - e^{\widehat{s}}\omega(\beta + \widehat{\beta}, \widehat{A}\beta \circ \sigma_{-\widehat{c}}), \beta + \widehat{\beta} - e^{\widehat{s}}\widehat{A}\beta \circ \sigma_{-\widehat{c}}\right)\left(\widehat{c}, 1, \widehat{A}\right),$$

where $\sigma_{-\hat{c}}(t) = t - \hat{c}$ denotes the shift by $-\hat{c}$. By Lemma 4.5, there is a solution $\beta \in V_S$ to

$$e^{\hat{s}} \widehat{A} \beta \circ \sigma_{-\hat{c}} - \beta = \widehat{\beta}.$$

Given this β, we can solve for b such that the first entry in the above display is zero, i.e. such that $\psi \widehat{\psi} \varphi_{\hat{s}}(\psi^{-1}) \in \mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S)$. Finally, we can conjugate $\psi \widehat{\psi} \varphi_{\hat{s}}(\psi^{-1})$ to the required ϕ by $\epsilon \in \mathbb{Z}_2$ to achieve that $c_\phi = \epsilon \widehat{c} \geqslant 0$. $\qquad \square$

This leads to the following result:

THEOREM 4.7. — *Let (M, g_S) be a Cahen–Wallach space and Γ be a group of homotheties acting properly discontinuously and cocompactly. Then Γ is contained in the isometries, unless S has at least one positive eigenvalue and all elements in $\Gamma \setminus \mathrm{Isom}$ satisfy*

$$(4.7) \qquad \left(\frac{s_\gamma}{c_\gamma}\right)^2 \leqslant \lambda_{\max}^2,$$

where $\lambda_{\max}^2$ is the largest positive eigenvalue.

In particular, if (M, g_S) is of imaginary type, then Γ is contained in the isometries.

Proof. — Assume for contradiction that Γ acts properly discontinuously and contains a strict homothety γ, i.e. with $s_\gamma \neq 0$. By Corollary 4.2, γ must have $\epsilon_\gamma = 1$ and $c_\gamma \neq 0$. If all eigenvalues of S are nonpositive (including zero), or if $\left(\frac{s_\gamma}{c_\gamma}\right)^2 > \lambda_{\max}^2$, we can use Proposition 4.6 to conjugate by an isometry Γ to $\widehat{\Gamma}$, which still acts properly discontinuously and cocompactly but now contains a strict homothety with the same s_γ and c_γ but in $\mathbb{R} \times C_{\mathbf{O}(n)}(S) \times \mathbb{R}$. Then we can apply Proposition 4.3 to get a contradiction. Hence, unless all $\gamma \in \Gamma \setminus \mathrm{Isom}$ satisfy inequality (4.7), we get that Γ is contained in the isometries.

For (M, g_S) of imaginary type, S has no positive eigenvalue and hence Γ is contained in the isometries. $\qquad \square$

For the remainder of the proof of Theorem 1.1, recall from by Proposition 2.2 that if $(\mathbb{R}^{n+2}, g_S)$ is conformally curved, Γ is a group of homotheties and by Theorem 2.2 a group of isometries. Then the metric endowed to the quotient is locally isometric to $(\mathbb{R}^{n+2}, g_S)$ and hence locally symmetric, so by Corollary 2.4, the conformal group of the quotient is equal to its isometry group.

4.2. Cocompact groups in the centraliser of an essential homothety

In this section we are going to provide another non-existence result. We will show that given an essential conformal transformation η on a conformally curved Cahen–Wallach space, there is no subgroup in the centraliser of η that acts cocompactly and properly discontinuously. The motivation for this was explained in the introduction: if there was such a subgroup, then the essential conformal transformation would descend to the compact quotient (see [41] for details) and hence would provide a counterexample to the Lorentzian Lichnerowicz conjecture. The counterexamples to the conjecture in signatures beyond Lorentzian in [16] are constructed in this way. Our result excludes this possibility.

Recall the definition of H_S in (3.13) and consider a pure homothety

$$h_s = \operatorname{diag}(1, \mathrm{e}^s \mathbf{1}, \mathrm{e}^{2s}) \in H_S.$$

It is straightforward to compute its centraliser in H_S as

$$C_{H_S}(h_s) = \mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \times \mathbb{R}.$$

Furthermore, denote by p the projection

$$p : H_S \longrightarrow \mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \times \mathbb{R} \simeq H_S / \mathbf{Hei}_n,$$

which is a group homomorphism with kernel $\mathbf{Hei}_n$.

PROPOSITION 4.8. — *Let $\eta \in H_S$ a strict homothety fixing the origin in $\mathbb{R}^{n+2}$ and with $\epsilon_\eta = 1$ and let $C_{H_S}(\eta)$ be its centraliser in the homotheties. Then the group homomorphism*

$$q := p|_{C_{H_S}(\eta)} : C_{H_S}(\eta) \longmapsto \mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \times \mathbb{R},$$

is injective.

Moreover, let $\gamma_n = (b_n, \beta_n) \cdot (c_n, \epsilon_n, A_n, s_n) \in C_{H_S}(\eta)$ such that $c_n \to_{n\to\infty} 0$. Then $b_n \to 0$, and $\beta_n(0) \to 0$. In particular, if $c_n \to 0$, then $\gamma_n(0) \to 0$.

Proof. — First, observe that since $\eta(0) = 0$ and $\epsilon_\eta = 1$, we have that $c_\eta = 0$ and $\eta = \psi h_s$ with

$$\psi = (0, \beta_\eta) \cdot A_\eta \in V_S \rtimes_\varphi C_{\mathbf{O}(n)}(S) \subset \mathbf{Hei}_n \rtimes_\varphi C_{\mathbf{O}(n)}(S)$$

an isometry fixing the origin, i.e. with $\beta_\eta(0) = 0$.

Let $\gamma = (b, \beta) \in \ker(q) = \mathbf{Hei}_n \cap C_{H_S}(\eta)$. By (3.12) we have that $\gamma \in C_{H_S}(\eta)$ if and only if

$$(4.8) \qquad \psi = \gamma \cdot \psi \cdot \varphi_s(\gamma^{-1}) = (b, \beta) \cdot \psi \cdot \left(-\mathrm{e}^{2s} b, -\mathrm{e}^s \beta\right),$$

where φ_s was defined in (4.6), and equivalently by (3.9) that

$$0 = (1 - e^{2s})b + \langle \beta(0), \dot{\beta}_\eta(0) \rangle \quad \text{and} \quad e^s A_\eta \beta(t) = \beta(t) \quad \text{for all } t.$$

Since $s \neq 0$, the second equation implies that $\beta \equiv 0$, and with that the first gives $b = 0$. Therefore, $\gamma = \mathrm{id}$ and q is injective.

To show the second part of the proposition, we first determine the inverse of q on its image. For $(c, \epsilon, A, r) = q(\gamma)$ in the image of q we need to find $(b, \beta) \in \mathbf{Hei}_n$ such that $\gamma = \phi h_r \in C_{H_S}(h_s)$ with

$$\phi = (b, \beta) \cdot (c, \epsilon, A) \in \mathbf{Hei}_n \rtimes_\varphi (\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S)).$$

Since homotheties commute and using (3.12) again, $\eta = \gamma \eta \gamma^{-1}$ yields the equation

$$\psi = \phi h_r \psi h_s h_r^{-1} \phi^{-1} h_s^{-1} = \phi \varphi_r(\psi) \varphi_s(\phi^{-1}).$$

Using (3.9) this can be seen to be equivalent to $A_\eta = A A_\eta A^\top$,

$$(\mathbf{1} - e^s A_\eta)\beta(t) = \beta_\eta(t) - e^r A \beta_\eta(\epsilon(t - c))$$

and

$$b(1 - \epsilon e^{2s}) = -e^r \omega \left((1 - \epsilon e^s A_\eta)\beta, A\beta_\eta \circ E_{-\epsilon c, \epsilon} \right) + e^s \omega \left(\beta, A_\eta \beta \right),$$

where we use (3.7) and (3.8). Evaluating these equations at $t = 0$ and taking into account that $\beta_\eta(0) = 0$ we get

$$(\mathbf{1} - e^s A_\eta)\beta(0) = -e^r A\beta_\eta(-\epsilon c))$$

and

$$2b(1 - \epsilon e^{2s})$$
$$= -e^r \left(\langle (1 - \epsilon e^s A_\eta)\beta(0), A\dot{\beta}_\eta(-\epsilon c) \rangle - \langle (1 - \epsilon e^s A_\eta)\dot{\beta}(0), A\beta_\eta(-\epsilon c) \rangle \right)$$
$$+ e^s \langle (\beta(0), A_\eta \dot{\beta}(0) \rangle.$$

If γ_n is a sequence as in the proposition, i.e. with $c_n \to 0$, using that $s \neq 0$, the first equation implies that $\beta_n(0) \to 0$ and consequently the second implies that $b_n \to 0$. Hence, $\gamma_n(0)$ converges to 0. $\qquad \square$

THEOREM 4.9. — *Let η be a strict homothety in H_S that fixes zero. Let Γ be a subgroup of the centraliser of η in H_S. Then Γ does not act properly discontinuously and cocompactly.*

Proof. — Assume for contradiction that Γ is a subgroup of $C_{H_S}(\eta)$ acting properly discontinuously and cocompactly. Without loss of generality we can assume that $\epsilon_\eta = 1$. If not, η^2 has this property and we still have that Γ is contained in the centraliser of η. We will derive a contradiction to (PD1).

We can apply Proposition 4.8 to get an isomorphism between Γ and a subgroup $\widehat{\Gamma} = q(\Gamma)$ of $\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \times \mathbb{R}$. We briefly justify that $\widehat{\Gamma} = q(\Gamma)$ is discrete when the homotheties are given the product topology:

Let Γ be equipped with a topology such that the action of Γ is continuous, i.e. the map $(\gamma, x) \mapsto \gamma(x)$ is continuous. Note that the map $\Theta(\gamma, x) = (x, \gamma(x))$ is then also continuous. Because of (PD1), there is an open set U such that $\Theta^{-1}(U \times U) = \{\mathrm{id}\} \times U$. Then, with with Θ being continuous, $\{\mathrm{id}\}$ is open in Γ and hence Γ is discrete. Hence, since the homotheties act continuously when given the product topology, Γ is discrete with respect to this topology. Then we note that p is a projection map and hence is an open map. Therefore $\widehat{\Gamma} = q(\Gamma) = p(\Gamma)$ is discrete. We remark that this argument does not require us to claim that the compact-open topology on the homotheties coincides with the product topology.

By the second part of Proposition 4.8, if we have $\gamma \in \Gamma$ such that $c_\gamma = 0$, then $\gamma(0) = 0$. Then by freeness of Γ, $\gamma = \mathrm{id}$. So for all non-identity elements $\gamma \in \Gamma$ we have $c_\gamma \neq 0$. Note that by Proposition 3.13, Γ has no torsion elements. Then we also conclude that $\epsilon_\gamma = 1$ for all non-identity elements $\gamma \in \Gamma$, since otherwise $c_{\gamma^2} = 0$.

Now we observe that the projection

$$\rho : \widehat{\Gamma} = q(\Gamma) \longrightarrow \mathbb{R}^2, \qquad \gamma = (c_\gamma, \epsilon_\gamma, A_\gamma, s_\gamma) \longmapsto (c_\gamma, s_\gamma)$$

is an injective homomorphism. Indeed, its kernel is contained in the compact group $K = \mathbb{Z}_2 \times C_{\mathbf{O}(n)}(S)$. Since $\widehat{\Gamma}$ is discrete, any non-identity element in the kernel must be torsion, which would imply that Γ does not act freely by Proposition 3.13. Therefore ρ is injective.

Since ρ is a projection map, it is also an open map, i.e. $\rho(q(\Gamma))$ is a discrete subgroup of $\mathbb{R}^2$. Hence, using that Γ cannot be cyclic and also act cocompactly by Lemma 4.1, Γ is a discrete non-cyclic subgroup of $\mathbb{R}^2$. Therefore Γ must contain a subgroup that is isomorphic to $\mathbb{Z}^2$. Let $\gamma, \phi \in \Gamma$ be two generators of $\mathbb{Z}^2$, i.e. such that $\langle \gamma \rangle \cap \langle \phi \rangle = \{\mathrm{id}\}$. By earlier in this proof, $c_\phi \neq 0$, and so we take a sequence of rational numbers p_n/q_n approaching c_γ/c_ϕ. Then we consider the sequence $\gamma^{p_n} \phi^{-q_n}$. This is a sequence of elements for which the component c_n approaches 0. Hence by the second part in Proposition 4.8, $\gamma^{p_n} \phi^{-q_n}(0) \to 0$, contradicting PD1 in the definition of proper discontinuity. $\qquad \square$

THEOREM 4.10. — *A group of homotheties of a Cahen–Wallach space centralising an essential homothety cannot act properly discontinuously and cocompactly.*

Proof. — Let η be the essential homothety. By Theorem 3.14, η has a fixed point. Since Cahen–Wallach spaces are homogeneous, the isometry group acts transitively. We conjugate η and Γ by the isometry that sends 0 to the fixed point of η. Note that Γ acts properly discontinuously and cocompactly if and only if its conjugate acts properly discontinuously and cocompactly. Hence, without loss of generality, we assume that η fixes 0. Since Γ centralises η, by Theorem 4.9, Γ does not act properly discontinuously and cocompactly. □

Combining this result with Proposition 2.2, we obtain Theorem 1.3.

4.3. Examples

In this last section we are going to illustrate some of the difficulties that arise when attempting to construct compact quotients of Cahen–Wallach spaces by groups of conformal transformations. We start with some examples of isometric quotients.

Example 4.11 (Compact isometric quotient of imaginary type). — For Cahen–Wallach spaces of imaginary type, the function β in an isometry is given by trigonometric functions. This makes it relatively straightforward to find groups of isometries that act properly discontinuously and cocompactly. For simplicity, let $(\mathbb{R}^4, g_-)$ be a conformally flat Cahen–Wallach space of imaginary type of dimension 4.

Solutions to $\ddot{\beta} = S\beta$ are of the form $\boldsymbol{u}\cos(t) + \boldsymbol{w}\sin(t)$, where $\boldsymbol{u}, \boldsymbol{w} \in \mathbb{R}^2$. Let Γ be generated by the following isometries,

$$\gamma \begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} := \begin{pmatrix} t + \frac{\pi}{2} \\ \boldsymbol{x} \\ v \end{pmatrix}, \qquad \eta \begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} := \begin{pmatrix} t \\ \boldsymbol{x} \\ v + 1 \end{pmatrix},$$

where $\boldsymbol{x} = (x^1, x^2)$ and

$$\zeta \begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} := \begin{pmatrix} t \\ \boldsymbol{x} + \beta(t) \\ v - \langle \beta(t), \boldsymbol{x} \rangle \end{pmatrix}, \qquad \text{with} \quad \beta(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

Consider the diffeomorphism $f : \mathbb{R}^4 \to (\mathbb{R}^4, g_-)$ given by

$$f \begin{pmatrix} u \\ x \\ y \\ v \end{pmatrix} := \begin{pmatrix} u \\ x \begin{pmatrix} \cos(u) \\ \sin(u) \end{pmatrix} + y \begin{pmatrix} -\sin(u) \\ \cos(u) \end{pmatrix} \\ -v - xy \end{pmatrix},$$

with inverse

$$f^{-1}\begin{pmatrix} t \\ x^1 \\ x^2 \\ v \end{pmatrix} = \begin{pmatrix} t \\ x^1 \begin{pmatrix} \cos(-t) \\ \sin(-t) \end{pmatrix} + x^2 \begin{pmatrix} -\sin(-t) \\ \cos(-t) \end{pmatrix} \\ -v - \left(x^1 \cos(-t) - x^2 sin(-t)\right)\left(x^1 \sin(-t) + x^2 \cos(-t)\right) \end{pmatrix}.$$

Then the action on $\mathbb{R}^4$ by Γ is given by

$$f^{-1}\gamma f \begin{pmatrix} u \\ x \\ y \\ v \end{pmatrix} = \begin{pmatrix} u + \frac{\pi}{2} \\ y \\ -x \\ v \end{pmatrix}, \qquad f^{-1}\eta f \begin{pmatrix} u \\ x \\ y \\ v \end{pmatrix} = \begin{pmatrix} u \\ x \\ y \\ v - 1 \end{pmatrix},$$

and

$$f^{-1}\zeta f \begin{pmatrix} u \\ x \\ y \\ v \end{pmatrix} = \begin{pmatrix} u \\ x + 1 \\ y \\ v \end{pmatrix}.$$

The conjugation has removed the dependence of the group action on t. Γ acts properly discontinuously and cocompactly on $\mathbb{R}^4$: observe that

$$\Lambda := f \langle \gamma^4, \ \zeta, \ \gamma^{-1}\zeta\gamma, \ \eta \rangle f^{-1} = \langle 2\pi e_1, e_2, e_3, -e_4 \rangle$$

is a subgroup of $f\Gamma f^{-1}$ of index 4 and a lattice isomorphic to $\mathbb{Z}^4$, so Λ acts properly and cocompactly. Hence $f\Gamma f^{-1}$ acts properly and cocompactly. Then we observe that $f\Gamma f^{-1}$ and Λ both act freely. Hence $f\Gamma f^{-1}$ acts properly discontinuously and cocompactly on $\mathbb{R}^4$. Therefore, Γ acts properly discontinuously and cocompactly on $(\mathbb{R}^4, g_-)$ and $\mathbb{R}^4/\Gamma$ is fourfold covered by the 4-torus $\mathbb{R}^4/\Lambda$.

Example 4.12 (Compact isometric quotient of real type). — Here we give an example of a group Γ acting properly discontinuously and cocompactly by isometries on a conformally flat Cahen–Wallach space of real type and of dimension 4. We will also show why an attempt to generalise this to a group of homotheties fails. We follow the construction in [19, Chapter 5]. For $r \in \mathbb{N}_{\geqslant 3}$, consider the polynomial

$$f(x) = x^2 - rx + 1,$$

and let $\rho \neq \frac{1}{\rho}$ be its roots. Let $(\mathbb{R}^4, g_\rho)$ be the conformally flat Cahen–Wallach space of real type defined by $g_\rho := g_S$ with $S = (\ln |\rho|)^2 \mathbf{1}$, i.e.

$$g_\rho = 2\,\mathrm{d}t\,\mathrm{d}v + (\ln |\rho|)^2(x^2 + y^2)\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2.$$

According to [19, Proposition 8.8], $(\mathbb{R}^4, g_\rho)$ admits a subgroup of the transvections, $\Gamma \subset \mathbf{Hei}_2 \rtimes_\varphi \mathbb{R}$, such that $\mathbb{R}^4/\Gamma$ is a compact manifold. Such Γ can be given as follows:

Let

$$\beta(t) = \begin{pmatrix} \rho^t \\ \rho^{-t} \end{pmatrix} \quad \text{and} \quad \widehat{\beta}(t) = \beta(t+1)$$

be solutions to $\ddot{\beta} = S\beta$ and denote the corresponding isometries also by η and $\widehat{\eta}$. Let α_b be the translation in the v-component by b and define γ_c as the translation by c in the t-component,

$$\begin{pmatrix} t \\ x \\ y \\ v \end{pmatrix} \xrightarrow{\alpha_b} \begin{pmatrix} t \\ x \\ y \\ v+b \end{pmatrix}, \qquad \begin{pmatrix} t \\ x \\ y \\ v \end{pmatrix} \xrightarrow{\gamma_c} \begin{pmatrix} t+c \\ x \\ y \\ v \end{pmatrix}$$

and let Γ be the group of isometries generated by $\alpha := \alpha_1$, η, $\widehat{\eta}$ and $\gamma := \gamma_1$. An arbitrary group element in Γ is given as

$$\begin{pmatrix} t \\ x \\ y \\ v \end{pmatrix} \xrightarrow{\phi} \begin{pmatrix} t+k \\ x + \rho^t(n+m\rho) \\ y + \rho^{-t}(n+\frac{m}{\rho}) \\ \phi_v \end{pmatrix}$$

with $k, l, m, n \in \mathbb{Z}$ and where ϕ_v is equal to

$$v + l - \ln|\rho| \left((n+m\rho)x\rho^t + \frac{(n+m\rho)^2}{2}\rho^{2t} + \left(n+\frac{m}{\rho}\right)y\rho^{-t} + \frac{(n+\frac{m}{\rho})^2}{2}\rho^{-2t} \right).$$

In order to show that Γ acts cocompactly and properly discontinuously, we use [19, Proposition 4.8]). First we note that

$$\beta(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \widehat{\beta}(0) = \begin{pmatrix} \rho \\ \frac{1}{\rho} \end{pmatrix}$$

are linearly independent. This is condition (a) in [19, Proposition 4.8]). Note that $L = \operatorname{span}_{\mathbb{R}}(\beta, \widehat{\beta})$ is a Lagrangian subspace for ω, i.e. $\omega(\beta, \widehat{\beta}) = 0$. For condition (b) [19, Proposition 4.8]) we need to find a lattice Λ in $\mathbb{R} \times L \subset \mathbf{Hei}_2$ that is invariant under the shift $\tau : t \mapsto t+1$. Note that

$$\widehat{\beta}(t+1) - r\widehat{\beta}(t) + \beta(t) = \begin{pmatrix} \rho^t f(\rho) \\ \rho^{-t} f(\rho^{-1}) \end{pmatrix} = 0,$$

so that

$$\tau(\eta) = \widehat{\eta}, \quad \tau(\widehat{\eta}) = -\eta + r\widehat{\eta}, \quad \tau^{-1}(\widehat{\eta}) = \eta, \quad \tau^{-1}(\eta) = r\eta - \widehat{\eta}.$$

Hence, the lattice $\Lambda_0 = \mathrm{span}_{\mathbb{Z}}\{\eta, \widehat{\eta}\}$ is stable under the action of the group $\langle \tau \rangle$ and so is the lattice $\Lambda = \mathrm{span}_{\mathbb{Z}}\{\alpha, \eta, \widehat{\eta}\}$ in $\mathbb{R} \times L \subset \mathbf{Hei}_2$. Hence, by [19, Proposition 4.8]), Γ is a group of isometries that acts properly discontinously and cocompactly on $\mathbb{R}^4$.

In order to generalise this to a group $\widehat{\Gamma}$ of homotheties acting cocompactly and properly discontinuously one could try to replace α by the translation α_ρ by ρ in the v-component and γ by the homothety $\widehat{\gamma}$,

$$\begin{pmatrix} t \\ x \\ y \\ v \end{pmatrix} \xmapsto{\gamma_c} \begin{pmatrix} t+1 \\ \rho x \\ \rho y \\ \rho^2 v \end{pmatrix}$$

and $\widehat{\Gamma}$ be the subgroup of $H = (\mathbf{Hei} \rtimes_\varphi (\mathbb{R} \times \mathbb{R}))$ that is generated by α_ρ, η, $\widehat{\eta}$, $\widehat{\gamma}$,

$$\Gamma = \langle \alpha, \eta, \widehat{\eta}, \widehat{\gamma} \rangle.$$

Then one may try use ideas in [19, Chapters 3 & 4] to show that Γ acts properly discontinuously and cocompactly on $\mathbb{R}^4$. For this we need a group G that is a semidirect product of a nilpotent group with $\mathbb{R}$. Note that also $\mathbf{Hei}_2$ is invariant under conjugation with $\mathbb{R}\widehat{\gamma}$ in $H = (\mathbf{Hei} \rtimes_\varphi (\mathbb{R} \times \mathbb{R}))$ and we set

$$G = \mathbf{Hei}_2 \rtimes_\varphi \mathbb{R}\widehat{\gamma}.$$

Note that even though $\mathbf{Hei}_2$ is normal in G, Λ_3 is not normal in Γ, in fact

$$\widehat{\gamma}^k (m\alpha + n\eta + \widehat{n}\widehat{\eta})\widehat{\gamma}^l = \widehat{\gamma}^k \left(m\alpha + n\sigma_l\eta + \widehat{n}\sigma_l\widehat{\eta} + \widehat{\gamma}_l \right)$$
$$= m\alpha_{\rho^{1+2k}} + n\rho^k\sigma_l\eta + \widehat{n}\rho^k\sigma_l\widehat{\eta} + \widehat{\gamma}_{k+l}.$$

This shows that the group Γ is not discrete. Indeed, the sequence

$$\widehat{\gamma}^{-k}\alpha\widehat{\gamma}^k = \alpha_{\rho^{1-2k}}$$

for $k \in \mathbb{N}$ converges to the identity.

Example 4.13. — The previous example demonstrates the issues that arise when attempting to modify a properly discontinuous and cocompact group of isometries of a Weyl-flat Cahen–Wallach space of real type to a group of homotheties by maintaining the translations in the v-direction. In this example we will try a different approach that avoids these translations.

For simplicity, let $(\mathbb{R}^3, g_1)$ be a three dimensional Cahen–Wallach space of real type. Solutions to $\ddot{\beta} = S\beta$ are of the form $ae^t + be^{-t}$, where $a, b \in \mathbb{R}$.

As before, we consider the homothety γ and the isometry η

$$\gamma : \begin{pmatrix} t \\ x \\ v \end{pmatrix} \longmapsto \begin{pmatrix} t+1 \\ ex \\ e^2 v \end{pmatrix}, \quad \eta : \begin{pmatrix} t \\ x \\ v \end{pmatrix} \longmapsto \begin{pmatrix} x \\ x + k e^t \\ v - \langle k e^t, x + \frac{1}{2} k e^t \rangle \end{pmatrix}.$$

We define a diffeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ and its inverse by

$$f : \begin{pmatrix} t \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} t \\ e^t y \\ e^{2t}(z - y^2/2) \end{pmatrix}, \quad f^{-1} : \begin{pmatrix} t \\ x \\ v \end{pmatrix} \longmapsto \begin{pmatrix} t \\ e^{-t} x \\ e^{-2t}(v + x^2/2) \end{pmatrix}.$$

The conjugates are

$$f^{-1} \gamma f : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x+1 \\ y \\ z \end{pmatrix}, \quad f^{-1} \eta f : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y+k \\ z \end{pmatrix}.$$

At this stage it looks promising, but we still have a remaining direction to compactify. The issue that occurs in general at this stage is that when we have a strict homothety γ of the simplest form possible without admitting fixed points as in Proposition 3.11, introducing an element α that translates in the v-direction will not help us, for the same reason as in the previous example: $\gamma^{-i} \alpha \gamma^i(0)$ will approach 0. What this means is that it seems we will require β-terms to compactify in $n+1$ directions. However this turns out to be difficult: if

$$\zeta : \begin{pmatrix} t \\ x \\ v \end{pmatrix} \longmapsto \begin{pmatrix} t \\ x + k e^{-t} \\ v - k e^{-t} \left(x + \frac{1}{2} k e^{-t} \right) \end{pmatrix},$$

then

$$f^{-1} \zeta f : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y + l e^{-2x} \\ z \end{pmatrix}.$$

This demonstrates two issues: first that the conjugated element fails to act at all on the z direction — which is the direction still remaining to be compactified. And secondly, that when we have a homothety in the form of γ, we see immediately that only n of the β dimensions are able to grow fast enough to avoid $\gamma^{-i} \zeta \gamma^i(0) \to 0$. This is because it is not sufficient that β be exponential, it must grow exponentially in the same direction as γ. So this example too cannot lead to a properly discontinuous and cocompact action.

Example 4.14 (Compact homothetic quotient of an open subset). — In this example, we produce a compact quotient of an open submanifold of a Cahen–Wallach space by homotheties.

Consider the metric g_S on $U := \mathbb{R}^{n+2} \setminus \{(t, 0, 0) \mid t \in \mathbb{R}\}$. We have removed all fixed points of a pure homothety, allowing us to use a pure homothety to compactify: set

$$\gamma \begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} := \begin{pmatrix} t+1 \\ \boldsymbol{x} \\ v \end{pmatrix}, \qquad \eta \begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} := \begin{pmatrix} t \\ 2\boldsymbol{x} \\ 4v \end{pmatrix},$$

and $\Gamma := \langle \gamma, \eta \rangle$. We now show that Γ acts properly discontinuously and cocompactly on U.

A fundamental region for this action is a product of the unit interval and an annulus in the last $n+1$ dimensions. Define

$$R := (0, 1) \times ((-2, 2)^n \times (-4, 4)) \setminus [-1, 1]^{n+1}.$$

It is not hard to see that $\phi(R)$, $\phi \in \Gamma$ does not meet R, so R is a fundamental region. We take a neighbourhood V of $\bar{R}$:

$$V := (-1, 2) \times ((-4, 4)^n \times (-16, 16)) \setminus \left(\left[-\frac{1}{2}, \frac{1}{2} \right]^n \times \left[-\frac{1}{4}, \frac{1}{4} \right] \right).$$

Note that $\phi(V)$ meets V only for

$$\{\gamma^i \eta^j \mid i, j \in \{-2, -1, 0, 1, 2\}\}.$$

In particular, by results in [28, 41], R is locally finite, so U/Γ is homeomorphic to $\bar{R}/\Gamma$. Then, since $\bar{R}/\Gamma$ is a manifold, we get that Γ acts properly discontinuously and since $\bar{R}$ is compact, Γ also acts cocompactly. Hence the action on the open submanifold U is properly discontinuous and cocompact. However, the homotheties centralised by Γ are not essential. We have

$$C_{H_S}(\Gamma) = \mathbb{R} \times C_{\mathbf{O}(n)}(S) \times \mathbb{R},$$

and define

$$f(t, \boldsymbol{x}, v) = (\|\boldsymbol{x}\|^4 + (v)^2)^{-1/2}.$$

Then for $\phi \in C_{H_S}(\Gamma)$,

$$\phi^*(fg)|_x = (\|e^s A\boldsymbol{x}\|^4 + (e^{2s}v)^2)^{-1/2} e^{2s} g|_x = (\|\boldsymbol{x}\|^2 + (v)^2)^{-1/2} g|_x = (fg)|_x,$$

so ϕ is inessential on U. Note that this same choice of f works for all such ϕ, and thus $C_{H_S}(\Gamma)$ is inessential. In this example we can go further and conclude that the normaliser of Γ is inessential as well, since $\phi \gamma^r \eta^t = \gamma^{r'} \eta^{t'} \phi$ implies already that $t = t'$, and that $r' = ar$. Hence, we see that

the normaliser is simply $\mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \times \mathbb{R}$, and the same f as before makes the normaliser inessential.

We stress that this is not a proof that U/Γ has an inessential conformal structure: it is possible to have an essential transformation on the quotient which does not lift or whose lift is not essential, or a transformation may be preserved without normalising Γ. For details see [41, Section 5.4].

Remark 4.15. — In this final remark we address the fact that our results are about compact quotients of a *complete* Cahen–Wallach space $(\mathbb{R}^{n+2}, g_S)$, whereas the construction in [16] starts with an *incomplete* locally symmetric space that has the origin removed. That construction however, relies on the existence of a pair of strict homotheties with different sets of fixed points (one fixes a line, the other fixes only a point). Such a pair of homotheties is not available to us in Cahen–Wallach spaces. For this, note that every non-isometric conformal transformation of a conformally curved Cahen–Wallach space is a homothety and hence either has no fixed points or it has a line of finite-orbit points parameterised by t. Now assume that we have two strict homotheties γ and ϕ, each with finite-orbit points, such that ϕ descends to the quotient by a group Γ containing γ. Then γ and ϕ must have the same line of finite-orbit points, since ϕ must map finite-orbit points of γ to finite-orbit points of γ. Consequently, if Γ is acting on an open subset of a Cahen–Wallach space that has the finite-orbit points removed, so that it acts properly discontinuously, then also ϕ has had its finite-orbit points removed and therefore can no longer be expected to be essential.

BIBLIOGRAPHY

[1] D. V. ALEKSEEVSKIĬ, "Groups of conformal transformations of Riemannian spaces", *Math. USSR, Sb.* **89(131)** (1972), p. 280-296.

[2] ———, "Self-similar Lorentzian manifolds", *Ann. Global Anal. Geom.* **3** (1985), no. 1, p. 59-84.

[3] A. L. BESSE, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 10, Springer, 1987, xii+510 pages.

[4] M. CAHEN, "Lorentzian symmetric spaces", *Bull. Cl. Sci., VI. Sér., Acad. R. Belg.* **9** (1998), no. 7-12, p. 325-330.

[5] M. CAHEN & Y. KERBRAT, "Transformations conformes des espaces symétriques pseudo-riemanniens", *C. R. Math. Acad. Sci. Paris* **285** (1977), no. 5, p. A383-A385.

[6] ———, "Champs de vecteurs conformes et transformations conformes des espaces lorentziens symétriques", *J. Math. Pures Appl.* **57** (1978), no. 2, p. 99-132.

[7] ———, "Transformations conformes des espaces symétriques pseudo-riemanniens", *Ann. Mat. Pura Appl.* **132** (1982), p. 275-289.

[8] M. CAHEN, Y. KERBRAT & G. PRAET, "Conformational completion of Lorentz symmetric spaces", *Lett. Math. Phys.* **1** (1975/77), no. 5, p. 417-422.

[9] ———, "Complétions équivariantes conformes d'espaces lorentziens symétriques", *Bull. Cl. Sci., V. Sér., Acad. R. Belg.* **62** (1976), no. 10, p. 767-783.

[10] M. CAHEN & N. WALLACH, "Lorentzian symmetric spaces", *Bull. Am. Math. Soc.* **79** (1970), p. 585-591.

[11] E. CALABI & L. MARKUS, "Relativistic space forms", *Ann. Math.* **75** (1962), p. 63-76.

[12] Y. CARRIÈRE, "Autour de la conjecture de L. Markus sur les variétés affines", *Invent. Math.* **95** (1989), no. 3, p. 615-628.

[13] J. FERRAND, "The action of conformal transformations on a Riemannian manifold", *Math. Ann.* **304** (1996), no. 2, p. 277-291.

[14] C. FRANCES, "Sur les variétés lorentziennes dont le groupe conforme est essentiel", *Math. Ann.* **332** (2005), no. 1, p. 103-119.

[15] ———, "Essential conformal structures in Riemannian and Lorentzian geometry", in *Recent developments in pseudo-Riemannian geometry*, ESI Lectures in Mathematics and Physics, European Mathematical Society, 2008, p. 231-260.

[16] ———, "About pseudo-Riemannian Lichnerowicz conjecture", *Transform. Groups* **20** (2015), no. 4, p. 1015-1022.

[17] C. FRANCES & K. MELNICK, "Conformal actions of nilpotent groups on pseudo-Riemannian manifolds", *Duke Math. J.* **153** (2010), no. 3, p. 511-550.

[18] ———, "The Lorentzian Lichnerowicz conjecture for real-analytic, three-dimensional manifolds", *J. Reine Angew. Math.* **803** (2023), p. 183-218.

[19] I. KATH & M. OLBRICH, "Compact quotients of Cahen–Wallach spaces", *Mem. Am. Math. Soc.* **262** (2019), no. 1264, p. v+84.

[20] B. KLINGLER, "Complétude des variétés lorentziennes à courbure constante", *Math. Ann.* **306** (1996), no. 2, p. 353-370.

[21] W. KÜHNEL & H.-B. RADEMACHER, "Essential conformal fields in pseudo-Riemannian geometry", *J. Math. Pures Appl.* **74** (1995), no. 5, p. 453-481.

[22] ———, "Essential conformal fields in pseudo-Riemannian geometry. II", *J. Math. Sci., Tokyo* **4** (1997), no. 3, p. 649-662.

[23] ———, "Conformal transformations of pseudo-Riemannian manifolds", in *Recent developments in pseudo-Riemannian geometry*, ESI Lectures in Mathematics and Physics, European Mathematical Society, 2008, p. 261-298.

[24] N. H. KUIPER, "On conformally-flat spaces in the large", *Ann. Math.* **50** (1949), p. 916-924.

[25] R. S. KULKARNI, "Proper actions and pseudo-Riemannian space forms", *Adv. Math.* **40** (1981), no. 1, p. 10-51.

[26] T. LEISTNER & T. MUNN, "Completeness of certain compact Lorentzian locally symmetric spaces", *C. R. Math. Acad. Sci. Paris* **361** (2023), p. 819-824.

[27] T. LEISTNER & D. SCHLIEBNER, "Completeness of compact Lorentzian manifolds with abelian holonomy", *Math. Ann.* **364** (2016), no. 3-4, p. 1469-1503.

[28] T. LEISTNER & S. TEISSEIRE, "Fundamental regions for non-isometric group actions", *Geom. Dedicata* **218** (2024), no. 5, article no. 95 (13 pages).

[29] J. LELONG-FERRAND, "Transformations conformes et quasi-conformes des variétés riemanniennes compactes (démonstration de la conjecture de A. Lichnerowicz)", *Acad. Roy. Belg. Cl. Sci. Mém. Collect. Octavo (2)* **39** (1971), no. 5, p. 3-44.

[30] A. LICHNEROWICZ, "Sur les transformations conformes d'une variété riemannienne compacte", *C. R. Acad. Sci. Paris* **259** (1964), p. 697-700.

[31] K. MELNICK, "Rigidity of transformation groups in differential geometry", *Notices Am. Math. Soc.* **68** (2021), no. 5, p. 721-732.

[32] V. Melnick, Karin and Pecastaing, "The conformal group of a compact simply connected Lorentzian manifold", *J. Am. Math. Soc.* **35** (2021), no. 1, p. 81-122.

[33] M. Obata, "The conjectures on conformal transformations of Riemannian manifolds", *J. Differ. Geom.* **6** (1971), p. 247-258.

[34] B. O'Neill, *Semi-Riemannian geometry. With applications to relativity*, Pure and Applied Mathematics, vol. 103, Academic Press Inc., 1983, xiii+468 pages.

[35] V. Pecastaing, "Essential conformal actions of $\mathrm{PSL}(2, \mathbf{R})$ on real-analytic compact Lorentz manifolds", *Geom. Dedicata* **188** (2017), p. 171-194.

[36] ———, "Lorentzian manifolds with a conformal action of $\mathrm{SL}(2, \mathbb{R})$", *Comment. Math. Helv.* **93** (2018), no. 2, p. 401-439.

[37] M. N. Podoksënov, "A Lorentzian manifold with a one-parameter group of homotheties that has a closed isotropic orbit", *Sib. Mat. Zh.* **30** (1989), no. 5, p. 135-137.

[38] ———, "Conformally homogeneous Lorentzian manifolds. II", *Sib. Mat. Zh.* **33** (1992), no. 6, p. 154-161.

[39] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*, second ed., Graduate Texts in Mathematics, vol. 149, Springer, 2006, xii+779 pages.

[40] J. R. Silvester, "Determinants of Block Matrices", *Math. Gaz.* **84** (2000), no. 501, p. 460-467.

[41] S. Teisseire, "Conformal Group Actions on Cahen–Wallach Spaces", Master thesis, University of Adelaide, School of Mathematical Sciences, 2021, `http://hdl.handle.net/2440/131752`.

[42] G. Teschl, *Ordinary differential equations and dynamical systems*, Graduate Studies in Mathematics, vol. 140, American Mathematical Society, 2012, xii+356 pages.

Thomas LEISTNER
School of Computer
and Mathematical Sciences,
University of Adelaide,
SA 5005, (Australia)
thomas.leistner@adelaide.edu.au

Stuart TEISSEIRE
Department of Mathematics,
University of Auckland,
Private Bag 92019,
Auckland 1142 (New Zealand)
stuart.teisseire@gmail.com