



ANNALES DE L'INSTITUT FOURIER

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Article à paraître, mis en ligne le 5 février 2025, 35 p.

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Les *Annales de l'Institut Fourier* sont membres du
Centre Mersenne pour l'édition scientifique ouverte

www.centre-mersenne.org

e-ISSN : 1777-5310

RIGIDITY OF FIBONACCI REPRESENTATIONS OF MAPPING CLASS GROUPS

by Pierre GODFARD

ABSTRACT. — We prove that level 5 Witten–Reshetikhin–Turaev $\mathrm{SO}(3)$ quantum representations, also known as the Fibonacci representations, of mapping class groups are locally rigid. More generally, for any prime level ℓ , we prove that the level ℓ $\mathrm{SO}(3)$ quantum representations are locally rigid on all surfaces of genus $g \geq 3$ if and only if they are locally rigid on surfaces of genus 3 with at most 3 boundary components. This reduces local rigidity in prime level ℓ to a finite number of cases.

RÉSUMÉ. — Nous démontrons la rigidité locale des représentations quantiques $\mathrm{SO}(3)$ de Witten–Reshetikhin–Turaev de niveau 5 des groupes modulaires de surfaces, aussi connues sous le nom de représentations Fibonacci. Plus généralement, pour tout niveau premier ℓ , nous démontrons que les représentations quantiques $\mathrm{SO}(3)$ de Witten–Reshetikhin–Turaev de niveau ℓ sont localement rigides pour toutes les surfaces de genre $g \geq 3$ si et seulement si elles sont localement rigides pour les surfaces de genre 3 avec au plus 3 composantes de bord. Cela réduit la rigidité en niveau premier ℓ à un nombre fini de cas.

1. Introduction

It is expected that TQFT representations are locally rigid, either because of Kazhdan’s property (T) (yet to be proved for mapping class groups), or because of interpretations as complex variations of Hodge structures (yet to be constructed, see Section 1.2 below).

In this paper, we show that local rigidity at a prime level ℓ reduces to a finite number of cases. This enables us to prove local rigidity in the Fibonacci case (level 5).

1.1. The results

The quantum representations studied here are projective representations of the mapping class groups of surfaces, parametrized by a Lie group G , a positive integer ℓ called level, and a set of irreducible representations Λ of G , that depends on ℓ . We will restrict ourselves to the case where G is $\mathrm{SO}(3)$ and the level is prime, with a complete result only for $\ell = 5$. In quantum topology, these representations arise from the Witten–Reshetikhin–Turaev TQFT [13]. They are known to be related to spaces of conformal blocks.

The level has another meaning in Conformal Field Theory, but here the level will be the order of the root of unity considered. More precisely, these representations are defined over the cyclotomic field $\mathbb{Q}(\zeta_\ell)$ (and even over its ring of integers [9]).

In this context, for each compact surface S_g^n of genus g with n boundary components, odd integer ℓ and n -tuple of colors $\underline{\lambda} \in \Lambda^n$, there is a representation of the mapping class group $\mathrm{Mod}(S_g^n)$:

$$\rho_{g,n}(\underline{\lambda}) : \mathrm{Mod}(S_g^n) \longrightarrow \mathrm{PGL}_d(\mathbb{Q}(\zeta_\ell))$$

where d depends on g , n and $\underline{\lambda}$.

In this paper, we study the local rigidity of these representations. More precisely, with the notation $\mathcal{X}(\mathrm{Mod}(S_g^n), \mathrm{PGL}_d)$ for the character variety $\mathrm{Hom}(\mathrm{Mod}(S_g^n), \mathrm{PGL}_d) // \mathrm{PGL}_d$, the question is the following.

QUESTION. — *Is $\rho_{g,n}(\underline{\lambda})$ locally rigid? i.e. do we have*

$$H^1(\mathrm{Mod}(S_g^n), \mathrm{ad} \rho_{g,n}(\underline{\lambda})) = 0?$$

Equivalently, is $[\rho_{g,n}(\underline{\lambda})]$ an isolated smooth point of $\mathcal{X}(\mathrm{Mod}(S_g^n), \mathrm{PGL}_d)$?

This question was asked, for example, in the survey by L. Funar [8, Question 2.3]. Here, $\mathrm{ad} \rho_{g,n}(\underline{\lambda})$ is the adjoint representation of $\mathrm{Mod}(S_g^n)$ on the space $\mathfrak{sl}_d(\mathbb{C})$ of matrices with trace 0.

These representations factor through $\mathrm{Mod}^\ell(S_g^n)$, the quotient of $\mathrm{Mod}(S_g^n)$ by the ℓ -th powers of Dehn twists. Our main result is the following.

THEOREM 3.3. — *For $g \geq 3$ and $n \geq 0$, the representations of $\mathrm{Mod}(S_g^n)$ coming from the $\mathrm{SO}(3)$ TQFT in level 5 are locally rigid within $\mathrm{PGL}_d(\mathbb{C})$. Moreover, if we consider these representations as representations of the quotients $\mathrm{Mod}^5(S_g^n)$ of the mapping class groups, then they are all locally rigid within $\mathrm{PGL}_d(\mathbb{C})$, for any genus $g \geq 0$ and number of boundary components $n \geq 0$.*

Note that in this context, local rigidity never depends on the target group of the representation (see Proposition 4.2) and, when $g \geq 3$, does not depend on whether the source group is $\text{Mod}(S_g^n)$ or $\text{Mod}^\ell(S_g^n)$ (see Proposition 4.7).

The proof relies on a general induction on g and n and reduces the proof to a finite number of cases, where rigidity has then to be proved directly. This can be done either by computer, or by geometric arguments. For example, in the case S_0^5 of genus 0 with 5 boundary components, the representation is the monodromy of the Hirzebruch surface, and local rigidity follows from Weil's rigidity.

More generally, for any prime level $\ell \geq 5$, there is an induction process that reduces the study of local rigidity to the case of genus 3 with at most 3 boundary components:

THEOREM 3.1. — *Let $\ell \geq 5$ be a prime number.*

Assume that for $n \in \{0, 1, 2, 3\}$ and every coloring $\underline{\lambda}$ of ∂S_3^n , the corresponding $\text{SO}(3)$ representation of $\text{Mod}(S_3^n)$ is locally rigid within $\text{PGL}_d(\mathbb{C})$.

Then for all $g \geq 3$, $n \geq 0$ and coloring $\underline{\lambda}$ of ∂S_g^n , the associated $\text{SO}(3)$ representation of $\text{Mod}(S_g^n)$ is locally rigid within $\text{PGL}_d(\mathbb{C})$.

Remark 1.1. — In the rest of the paper, we need to use a linearized version of the representations to state the results. This amounts to take in account the “projective anomaly” of quantum representations. This approach is equivalent, see Section 4 for details.

Remark 1.2. — In prime level $\ell > 5$, the representation of $\text{Mod}^\ell(S_0^4)$ coming from the $\text{SO}(3)$ TQFT in level ℓ is not locally rigid for $\underline{\lambda} = (2, 2, 2, 2)$. This can be proved by a dimension count using the method of Lemma 7.3. Thus Theorem 3.3 fails at genus 0 for larger prime levels. This does not stop us from conjecturing that the representations are locally rigid in genus $g \geq 3$ for all prime levels.

1.2. Existence of complex variation of Hodge structures

For $g, n \geq 0$ with $3g - 3 + n \geq 0$ and $\ell \geq 5$ odd, the group $\text{Mod}^\ell(S_g^n)$ is the fundamental group of a compact Kähler uniformizable orbifold $\overline{\mathcal{M}}_{g,n}(\ell)$ ([6, Section 1.1]). It is constructed as a moduli space of ℓ -twisted stable curves.

As this orbifold is uniformizable and Kähler, it verifies the non-abelian Hodge correspondence. Thus any rigid reductive representation of its fundamental group supports a complex variation of Hodge structure [16, Lemma 4.5].

THEOREM 1.3. — *For $g \geq 0$ and $n \geq 0$ with $3g - 3 + n \geq 0$, the flat projective bundles over $\overline{\mathcal{M}}_{g,n}(5)$ induced by the level 5 $\mathrm{SO}(3)$ TQFT representations of $\mathrm{Mod}^5(S_g^n)$ support complex variations of Hodge structure.*

C. Simpson's motivicity conjecture states that the complex variations of Hodge structure of rigid representations have geometric origin. In our situation, this would mean that the flat projective bundle on $\overline{\mathcal{M}}_{g,n}(5)$ could be constructed from a sub flat bundle of the cohomology $R^m p_* \mathbb{C}$ of some fibration $p : E_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}(5)$.

The existence of a complex variation of Hodge structure thus makes a link between quantum representations and geometry. It would be interesting to study the Hodge decompositions and their compatibilities as a family in g and n .

1.3. Rigidity as unitary representations of the mapping class group

After the field embedding $i : \mathbb{Q}(\zeta_\ell) \rightarrow \mathbb{C}$ given by $\zeta_\ell = \exp(\pm i\pi \frac{\ell-1}{\ell})$, the representations are known to be unitary. Hence Theorem 3.3 has the following corollary.

THEOREM 1.4. — *For $g \geq 3$ and $n \geq 0$, the representations of $\mathrm{Mod}(S_g^n)$ coming from the $\mathrm{SO}(3)$ TQFT in level 5 are locally rigid as unitary representations.*

Local rigidity of unitary representations is implied by Kazhdan's property (T). It is not known if $\mathrm{Mod}(S_g)$ verifies property (T) for $g \geq 3$. However, $\mathrm{Mod}(S)$ does not verify property (T) if the genus of S is at most 2, as it has a finite index subgroup that surjects onto \mathbb{Z} (see, for example, [1, Section 2.3]).

1.4. Outline of the proofs

Here we outline the proofs of Theorem 3.1 and Theorem 3.3. Both proofs rely on Lemma 5.1. Given a surface S and 2 simple closed curves a_1, a_2 on S , we denote by S_{a_i} the surface obtained by cutting S along a_i . Under some assumptions, the lemma relates the deformation space of a representation of $\mathrm{M\ddot{o}d}^\ell(S)$ to the deformation spaces of its restrictions to $\mathrm{M\ddot{o}d}^\ell(S_{a_1})$ and $\mathrm{M\ddot{o}d}^\ell(S_{a_2})$. As a consequence, we will see that if the quantum representations associated to S_{a_1} and S_{a_2} are locally rigid, then so are those

associated to S (see Corollary 6.3). The proof of Lemma 5.1 is the content of Section 5.

Lemma 5.1 enables us to prove Theorem 3.1 by induction on the genus g and the number n of boundary components. The proof of Theorem 3.1 is the content of Section 6.

The proof of Theorem 3.3 is in 2 steps. The first step is to prove rigidity for some small surfaces. More precisely, local rigidity for S_0^2 , S_0^3 , S_1 and S_1^1 is deduced from the finiteness of the associated mapping class group quotients. Local rigidity for S_0^4 is proved with a direct dimension computation. For S_0^5 , the proof relies on Weil's rigidity. The second step is to perform an induction on g and n as in the proof of Theorem 3.1. However, because the assumptions of Lemma 5.1 are not always verified, we have to adapt the techniques of the lemma to the various cases. The proof of Theorem 3.3 is the content of Section 7.

Acknowledgment

This paper forms part of the PhD thesis of the author. The author thanks J. Marché for his help in writing this paper. The author also thanks B. Deroin, R. Detcherry, L. Funar, G. Masbaum and R. Santharoubane for helpful discussions. Finally, the author would like to thank the Reviewer for taking the necessary time and effort to review the manuscript.

2. Modular functors

2.1. Definition

Even though our proof applies essentially only to the $SO(3)$ representations at prime levels, we will write it in the context of modular functors to make clear which properties of the TQFT are used. In particular, with the exception of Appendix A, we will not refer explicitly to the construction of the TQFT. For the general theory of modular functor, see Turaev's book [17, Chapter 5]. We will use a stripped down version of modular functors, similar to the one used by B. Deroin and J. Marché in [5, Section 4].

DEFINITION 2.1. — *A set of colors is a finite set Λ with a preferred element $0 \in \Lambda$.*

Remark 2.2. — In the usual definition of a modular functor, there is the extra datum of an involution $\lambda \mapsto \lambda^*$ on Λ . Here, for simplicity, the involution is assumed to be the identity, as it is trivial in our examples.

DEFINITION 2.3. — Let $g, n \geq 0$. We define S_g^n to be the compact surface of genus g with n boundary components, and $S_{g,n}$ to be the surface of genus g with n punctures and no boundary.

DEFINITION 2.4. — Let S be a compact surface, which can have non-empty boundary. Let \widehat{S} be the closed surface obtained from S by capping off each boundary component S^1 with D^2 . A Lagrangian L on S is a subspace of $H_1(\widehat{S}; \mathbb{Q})$ of half dimension on which the intersection form vanishes. A split Lagrangian on S is a Lagrangian L that is a direct sum of Lagrangians on the connected components of S .

DEFINITION 2.5. — Let S be a compact oriented surface and L_0, L_1 and L_2 be 3 split Lagrangians on S . Their Maslov index, denoted $\mu(L_0, L_1, L_2)$, is the signature of the quadratic form q defined as follows. Let K be the kernel of the sum map:

$$L_0 \oplus L_1 \oplus L_2 \longrightarrow H_1(\widehat{S}; \mathbb{Q}).$$

Then we define q as:

$$q : K \longrightarrow \mathbb{Q}, (u_0, u_1, u_2) \longmapsto u_0 \cdot u_1$$

where \cdot is the intersection form.

We can now define the source category of modular functors.

DEFINITION 2.6. — Let Λ be a set of colors. The category of surfaces colored with Λ is such that:

- (1) its objects are compact oriented surfaces S together with a Lagrangian L on S , an identification $\varphi_B : B \simeq S^1$ and a color $\lambda_B \in \Lambda$ for every component B of ∂S ;
- (2) its morphisms from $\Sigma_1 = (S_1, L_1, \varphi^1, \lambda^1)$ to $\Sigma_2 = (S_2, L_2, \varphi^2, \lambda^2)$ are pairs (f, n) with $f : S_1 \rightarrow S_2$ an homeomorphism preserving orientation such for every component $B_1 \subset \partial S_1$ and its image $f(B_1) = B_2 \subset \partial S_2$, we have $\lambda_{B_1} = \lambda_{B_2}$ and $\varphi_{B_2}^2 \circ f = \varphi_{B_1}^1$. The second element n of the pair is an integer in \mathbb{Z} .
- (3) the composition of $(f_1, n_1) : \Sigma_0 \rightarrow \Sigma_1$ and $(f_2, n_2) : \Sigma_1 \rightarrow \Sigma_2$ is given by:

$$(f_2 \circ f_1, n_1 + n_2 - \mu(f_1(L_0), L_1, f_2^{-1}(L_2)))$$

where L_0, L_1 and L_2 are the respective Lagrangians of Σ_0, Σ_1 and Σ_2 .

This category has a natural monoidal structure induced by the disjoint union \sqcup .

In the rest of the paper, $\Sigma = (S, L, \varphi, \underline{\lambda})$ will be abbreviated $(S, \underline{\lambda})$, or even $(S, \lambda_1, \lambda_2, \dots)$ where $\lambda_1, \lambda_2, \dots$ are the colors relevant to the argument and the other colors are omitted.

DEFINITION 2.7. — Let S be a surface with boundary. We note by $\text{Mod}(S)$ its mapping class group, i.e. the group of connected components of the group of orientation preserving homeomorphisms of S fixing the boundary ∂S pointwise.

Let L be a Lagrangian for S , and $\text{Aut}(S, L)$ the group of pairs (f, n) with $f \in \text{Mod}(S)$ and $n \in \mathbb{Z}$ with composition as in Definition 2.6(3).

The isomorphism class of $\text{Aut}(S, L)$ does not depend on L . Hence we will use the notation $\text{Möd}(S)$ for it. The group $\text{Möd}(S)$ is a central extension of $\text{Mod}(S)$ by \mathbb{Z} :

$$1 \longrightarrow \mathbb{Z} \longrightarrow \text{Möd}(S) \longrightarrow \text{Mod}(S) \longrightarrow 1.$$

Let S be a surface and $\partial_+ S \sqcup \partial_- S \subset \partial S$ be two components of its boundary. Let $\varphi_{\partial_{\pm} S} : \partial_{\pm} S \simeq S^1$ be identifications of these components with S^1 .

Let S_{\pm} be the surface obtained from S by gluing $\partial_+ S$ to $\partial_- S$ along $\varphi_{\partial_- S}^{-1} \circ \varphi_{\partial_+ S}$. Then S_{\pm} is called the gluing of S along $\partial_{\pm} S$.

One can check that if S has a Lagrangian L , one defines a Lagrangian in S_{\pm} as follows. There exists a 3-manifold M with boundary \widehat{S} such that L is the kernel of $H_1(\widehat{S}; \mathbb{Q}) \rightarrow H_1(M; \mathbb{Q})$. Let M_{\pm} be the 3 manifold obtained by gluing together the discs bounding $\partial_+ S$ and $\partial_- S$ on the boundary of M . Then M_{\pm} bounds \widehat{S}_{\pm} and the desired Lagrangian is the kernel of $H_1(\widehat{S}_{\pm}; \mathbb{Q}) \rightarrow H_1(M_{\pm}; \mathbb{Q})$.

We now introduce the notion of modular functor.

DEFINITION 2.8 (Modular Functor). — Let Λ be a set of colors and \mathbb{C} be the associated category of colored surfaces as defined in Definition 2.6. Then a modular functor is the data of a monoidal functor:

$$\mathcal{V} : \mathbb{C} \longrightarrow \mathbb{C} - \text{vector spaces}$$

where the monoidal structure on \mathbb{C} -vector spaces is understood to be the tensor product. This data is augmented by the following isomorphisms.

- (G) For any surface with Lagrangian (S, L) and pair of boundary components $\partial_{\pm} S$, let (S_{\pm}, L_{\pm}) be the gluing of S along $\partial_{\pm} S$. For any coloring $\underline{\lambda}$ of the components of ∂S_{\pm} , an isomorphism as below is

given:

$$\mathcal{V}(S_{\pm}, L_{\pm}, \underline{\lambda}) \simeq \bigoplus_{\mu \in \Lambda} \mathcal{V}(S, L, \mu, \mu, \underline{\lambda}).$$

The isomorphisms of (G) are assumed to be functorial and compatible with disjoint unions. This rule, also sometimes called fusion or factorization rule, is the most important property of modular functors. The functor is also assumed to verify two more axioms:

- (1) $\dim \mathcal{V}(S_0^1, \lambda) = 1$ if $\lambda = 0$ and 0 otherwise;
- (2) $\dim \mathcal{V}(S_0^2, \lambda, \mu) = 1$ if $\lambda = \mu$ and 0 otherwise.

Remark 2.9. — Let S_{\pm} be a colored surface constructed as a gluing of S along $\partial_{\pm}S$. Let γ denote the simple closed curve that is the image of $\partial_{\pm}S$ in S_{\pm} . Then the Dehn twist T_{γ} acts block-diagonally on the decomposition (G) .

Moreover, one can easily see that it acts on the block $\mathcal{V}(S, L, \mu, \mu, \underline{\lambda})$ by a scalar r_{μ} , that depends only on μ , and not on the surface S . Indeed, r_{μ} is given by the action of the unique Dehn twist of S_0^2 on $\mathcal{V}(S_0^2, \mu, \mu)$, which is 1-dimensional.

From the fact that on any finite-dimensional representation of $\text{M\ddot{o}d}(S)$ with S of genus at least 3, the Dehn twists act with quasi-unipotent matrices [1, Section 2.5], one deduces that for all μ , r_{μ} is a root of unity.

Similarly, one can prove that $(\text{id}_{\Sigma}, 1)$ acts by a scalar κ on $\mathcal{V}(\Sigma)$ and that κ is independent of Σ .

DEFINITION 2.10. — *Let \mathcal{V} be a modular functor. A level for \mathcal{V} is an integer $\ell \geq 1$ such that $\forall \lambda \in \Lambda$, $r_{\lambda}^{\ell} = 1$ and $\kappa^{4\ell} = 1$. For $\ell > 1$ an integer and S a surface, the group $\text{M\ddot{o}d}^{\ell}(S)$ is defined as the quotient of $\text{M\ddot{o}d}(S)$ by the subgroup generated by $(\text{id}, 1)^{4\ell}$ and the ℓ -th powers of the Dehn twists $(T_{\gamma}, 0)^{\ell}$ for every γ simple closed curve in the Lagrangian L such that $\text{M\ddot{o}d}(S) = \text{Aut}(S, L)$.*

We can now define the representations studied in this article:

DEFINITION 2.11 (Quantum representations). — *Let \mathcal{V} be a modular functor and ℓ be a level for \mathcal{V} . Then, for any surface S and coloring $\underline{\lambda}$ of its boundary components, the functor yields a representation:*

$$\rho_{g,n}(\underline{\lambda}) : \text{M\ddot{o}d}(S) \longrightarrow \text{GL}(\mathcal{V}(S, \underline{\lambda}))$$

which factors as a representation:

$$\rho_{g,n}^{\ell}(\underline{\lambda}) : \text{M\ddot{o}d}^{\ell}(S) \longrightarrow \text{GL}(\mathcal{V}(S, \underline{\lambda})).$$

These latter representations will be called the representations associated to the modular functor \mathcal{V} .

We shall say that \mathcal{V} is rigid on $(S, \underline{\lambda})$ if $\rho_{g,n}^\ell(\underline{\lambda})$ is cohomologically rigid, i.e. if $H^1(\text{Möd}^\ell(S), \text{ad } \rho_{g,n}^\ell(\underline{\lambda})) = 0$.

We now introduce extra properties that we will need in our proof of rigidity:

- (I) For every $\lambda \in \Lambda$, $\mathcal{V}(S_1^1, \lambda) \neq 0$;
 - (II) For every $\lambda, \mu, \nu \in \Lambda$, $\mathcal{V}(S_0^3, \lambda, \mu, \nu)$ has dimension 0 or 1;
 - (III) For every $\lambda, \mu \in \Lambda$, if $\lambda \neq \mu$, then $r_\lambda \neq r_\mu$.
- (I) can be equivalently rephrased:

- (I) For every $\lambda \in \Lambda$, there exists $\mu \in \Lambda$ such that $\mathcal{V}(S_0^3, \lambda, \mu, \mu) \neq 0$.

To the author’s knowledge, these properties essentially restrict to the case of the $\text{SO}(3)$ modular functors at prime levels.

2.2. Bases

We will now explain how to construct bases of the $\mathcal{V}(S, \underline{\lambda})$ for a modular functor \mathcal{V} satisfying (II).

DEFINITION 2.12. — *Let \mathcal{V} be a modular functor satisfying (II). For $\lambda, \mu, \nu \in \Lambda$, we say that (λ, μ, ν) is admissible if $\mathcal{V}(S_0^3, \lambda, \mu, \nu) \neq 0$.*

Let $(S, \underline{\lambda})$ be a colored surface and $\{e_i\}$ be a set of disjoint simple closed curves on S that induces a pair of pants decomposition, i.e. such that cutting along the curves yields a disjoint union of surfaces, each homeomorphic to S_0^3 . Let us denote S_{cut} this disjoint union.

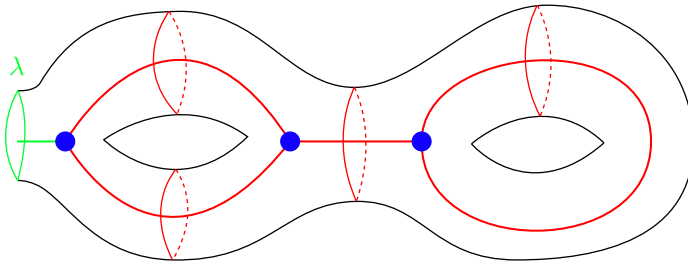


Figure 2.1. Pair of pants decomposition of a surface and associated graph, with leg in green.

Let $G = (V, H, v : H \rightarrow V, \iota : H \rightarrow H)$ be the trivalent graph defined by vertices and half-edges such that:

- the vertices V are the components of S_{cut} ;
- the half-edges H are the components of ∂S_{cut} ;
- the attaching map $v : H \rightarrow V$ sends a half edge h to the component of S_{cut} on which it sits;
- ι is an involution of H ;
- The set E of edges of G is the set of 2-cycles of ι . It identifies with the chosen set of curves $\{e_i\}$ on S ;
- The set L of legs of G is the set of fixed points of ι . It identifies with the set of components of ∂S .

An example is given on Figure 2.1.

By a coloring of G , we mean a map $c : E \rightarrow \Lambda$. We extend a coloring to L by sending a leg to its associated color in S . Alternatively, we can see c as a map $c : H \rightarrow \Lambda$ such that $c \circ \iota = c$ and $c|_L$ coincides with the coloring of the components of ∂S .

Such a coloring c is said to be admissible if for any vertex v and adjacent half-edges h_1, h_2 and h_3 , the triplet $(c(h_1), c(h_2), c(h_3))$ is admissible. We denote by $C(G)$ the set of admissible colorings of G .

By using the gluing axiom (G) repeatedly, one gets a decomposition:

$$\mathcal{V}(S, \underline{\lambda}) = \bigoplus_{c: E \rightarrow \Lambda} \bigotimes_{v \in V} \mathcal{V}(S_0^3, c(h_1), c(h_2), c(h_3))$$

where the h_i are the half-edges adjacent to v .

We can remove the non-admissible colorings to obtain a decomposition of $\mathcal{V}(S, \underline{\lambda})$ in vector spaces of dimension 1:

$$\mathcal{V}(S, \underline{\lambda}) = \bigoplus_{c \in C(G)} \bigotimes_{v \in V} \mathcal{V}(S_0^3, c(h_1), c(h_2), c(h_3)).$$

In particular, $\dim \mathcal{V}(S, \underline{\lambda}) = |C(G)|$.

2.3. The $\text{SO}(3)$ modular functors

Let $\ell \geq 5$ be an odd integer. Let $\Lambda = \{0, 2, \dots, \ell - 3\}$.

Then there exists a modular functor \mathcal{V}_ℓ on the surfaces colored with Λ , called the $\text{SO}(3)$ modular functor of level ℓ .

We refer to the work of C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel [3] for a construction of the $\text{SO}(3)$ modular functors. Only the following properties of these functors will be used in this paper.

PROPOSITION 2.13. — *Let $\ell \geq 5$ be an odd number. Then the modular functor \mathcal{V}_ℓ satisfies properties (I) and (II).*

Moreover, if ℓ is a prime number, \mathcal{V}_ℓ satisfies property (III).

Proof. — For properties (I) and (II), see [3]. Property (III) is easily deduced from the fact that $r_\lambda = \zeta_\ell^{\lambda(\lambda+2)}$. See [2, Section 3] or [12, Lemma 2.5] for the computation of r_λ . \square

PROPOSITION 2.14. — *Let $a, b, c \in \Lambda = \{0, 2, \dots, \ell - 3\}$. Then (a, b, c) is admissible for \mathcal{V}_ℓ if and only if:*

- a, b, c verify triangular inequalities, i.e. $|a - b| \leq c \leq a + b$;
- $a + b + c < 2\ell - 2$.

The following proposition follows from a result of [11], where they apply the method of [14] to a cleverly chosen special case of a proposition of [3]. See Appendix A for details.

PROPOSITION A.3. — *Let $\ell \geq 5$ be a prime number. Then for any colored surface $(S, \underline{\lambda})$, the representation of $\text{M\ddot{o}d}(S)$ on $\mathcal{V}_\ell(S, \underline{\lambda})$ is irreducible.*

3. Main results

3.1. General induction results

We say that a modular functor \mathcal{V} is irreducible on a surface S if for any coloring $\underline{\lambda}$ of the components of ∂S , the representation of $\text{M\ddot{o}d}(S)$ on $\mathcal{V}(S, \underline{\lambda})$ is irreducible. More generally, if $\partial S = \partial_c S \sqcup \partial_u S$ and $\underline{\mu}$ is a coloring of the components of $\partial_c S$, we say that \mathcal{V} is irreducible on $(S, \underline{\mu})$ if for any coloring $\underline{\lambda}$ of $\partial_u S$, the representation of $\text{M\ddot{o}d}(S)$ on $\mathcal{V}(S, \underline{\mu}, \underline{\lambda})$ is irreducible.

By extension, we say that \mathcal{V} is irreducible in genus $g \geq k$ if for any surface S of genus at least k , \mathcal{V} is irreducible on S .

The following theorem reduces local rigidity in genus $g \geq 3$ to local rigidity in genus 3 with at most 3 boundary components.

THEOREM 3.1. — *Let \mathcal{V} be a modular functor satisfying (I), (II) and (III) that is irreducible in genus $g \geq 1$. Let $\ell > 1$ be a level for \mathcal{V} .*

Assume that for $n \in \{0, 1, 2, 3\}$ and every coloring $\underline{\lambda}$ of ∂S_3^n , the representation of $\text{M\ddot{o}d}(S_3^n)$ on $\mathcal{V}(S_3^n, \underline{\lambda})$ is locally rigid within $\text{GL}_d(\mathbb{C})$.

Then for all $g \geq 3$, $n \geq 0$ and coloring $\underline{\lambda}$ of ∂S_g^n , the representation of $\text{M\ddot{o}d}(S_g^n)$ on $\mathcal{V}(S_g^n, \underline{\lambda})$ is locally rigid within $\text{GL}_d(\mathbb{C})$.

Note that in this context, local rigidity never depends on the target group of the representation (see Proposition 4.2) and, when $g \geq 3$, does not depend on whether the source group is $\text{Mod}(S_g^n)$ or $\text{Mod}^\ell(S_g^n)$ (see Proposition 4.7).

Remark 3.2. — For a fixed prime level ℓ , this theorem reduces the local rigidity of the level ℓ $\text{SO}(3)$ representations in genus $g \geq 3$ to a finite number of cases. Notice that as the mapping class groups have known finite presentations, the vanishing of $H^1(\text{Möd}(S), \text{ad } \rho_{g,n}(\underline{\lambda}))$ on a fixed colored surface $(S, \underline{\lambda})$ can be translated into the computation of the rank of a finite linear system. Such a computation can be fed into a computer.

Thus, given a prime level ℓ , the theorem reduces the question of local rigidity to a problem solvable by computation (that is, if the answer happens to be yes, otherwise the computation will just show that some of these representations are not rigid).

3.2. The case of the $\text{SO}(3)$ TQFT in level 5

THEOREM 3.3. — *For $g \geq 0$ and $n \geq 0$, the representations of $\text{Möd}^5(S_g^n)$ coming from the $\text{SO}(3)$ TQFT of level 5 are locally rigid within $\text{GL}_d(\mathbb{C})$. They are also locally rigid as representations of $\text{Möd}(S_g^n)$ when $g \geq 3$.*

Remark 3.4. — In level 5 the $\text{SO}(3)$ TQFT has only 2 colors: 0 and 2. Lemma 7.1 shows that if a boundary component is colored with 0, then it can be capped off without changing the deformation space. Thus, when proving Theorem 3.3, we can assume all the boundary components are colored with 2.

The method of the proof is in 2 steps:

- Prove local rigidity directly for $S_0^2, S_0^3, S_0^4, S_0^5, S_1$ and S_1^1 , using algebraic or geometric arguments;
- Use Theorem 3.1 or similar arguments to perform an induction on the genus and the number of boundary components.

4. Deformations

In this section, we relate the different versions of local rigidity for the quantum representations.

Let us first relate local rigidity as linear representation and projective representation. If $\rho : G \rightarrow \text{GL}_d(\mathbb{C})$ is a linear representation, we denote by $\text{ad } \rho$

the adjoint representation of G on the space $\mathfrak{gl}_d(\mathbb{C})$. If $\rho : G \rightarrow \mathrm{PGL}_d(\mathbb{C})$ is a projective representation, we denote by $\mathrm{ad} \rho$ the adjoint representation of G on the space $\mathfrak{sl}_d(\mathbb{C})$ of matrices with trace 0. This is because the tangent space to $\mathrm{PGL}_d(\mathbb{C})$ at I_d is isomorphic to $\mathfrak{sl}_d(\mathbb{C})$.

PROPOSITION 4.1. — *Let $g, n \geq 0$ and $\ell > 1$. Let $\tilde{\rho} : \mathrm{M\ddot{o}d}^\ell(S_g^n) \rightarrow \mathrm{GL}_d(\mathbb{C})$ be a representation such that $(\mathrm{id}, 1)$ acts by a scalar. Let $\rho : \mathrm{Mod}^\ell(S_g^n) \rightarrow \mathrm{PGL}_d(\mathbb{C})$ denote the associated projective representation.*

Then $\tilde{\rho}$ is locally rigid if and only if ρ is. More precisely, we have:

$$H^1(\mathrm{M\ddot{o}d}^\ell(S_g^n), \mathrm{ad} \tilde{\rho}) = 0 \text{ if and only if } H^1(\mathrm{Mod}^\ell(S_g^n), \mathrm{ad} \rho) = 0.$$

Proof. — First, we have an isomorphism of $\mathrm{M\ddot{o}d}^\ell(S_g^n)$ -modules:

$$\mathfrak{gl}_d(\mathbb{C}) \simeq \mathfrak{sl}_d(\mathbb{C}) \oplus \mathbb{C}.$$

As $\mathrm{M\ddot{o}d}^\ell(S_g^n)^{\mathrm{ab}}$ is finite, we have $H^1(\mathrm{M\ddot{o}d}^\ell(S_g^n), \mathbb{C}) = 0$. Hence:

$$H^1(\mathrm{M\ddot{o}d}^\ell(S_g^n), \mathfrak{gl}_d(\mathbb{C})) = H^1(\mathrm{M\ddot{o}d}^\ell(S_g^n), \mathfrak{sl}_d(\mathbb{C})).$$

The kernel of $\mathrm{M\ddot{o}d}^\ell(S_g^n) \rightarrow \mathrm{Mod}^\ell(S_g^n)$ is a finite group R . The inflation restriction exact sequence in Proposition 5.4 is:

$$0 \longrightarrow H^1(\mathrm{Mod}^\ell(S_g^n), \mathfrak{sl}_d(\mathbb{C})) \longrightarrow H^1(\mathrm{M\ddot{o}d}^\ell(S_g^n), \mathfrak{sl}_d(\mathbb{C})) \longrightarrow H^1(R, \mathfrak{sl}_d(\mathbb{C})).$$

But, as R is finite, $H^1(R, \mathfrak{sl}_d(\mathbb{C})) = 0$. Hence the result. \square

We now show that proving local rigidity in $\mathrm{PU}(p, q)$ is sufficient.

PROPOSITION 4.2. — *Let $\rho : G \rightarrow \mathrm{PU}(p, q)$ be a representation. Let $\bar{\rho} : G \rightarrow \mathrm{PGL}_{p+q}(\mathbb{C})$ be the extension given by the inclusion $\mathrm{PU}(p, q) \subset \mathrm{PGL}_{p+q}(\mathbb{C})$.*

Then ρ is locally rigid if and only if $\bar{\rho}$ is.

Proof. — Let $\mathfrak{su}(p, q) \subset \mathfrak{sl}_n(\mathbb{C})$ be the inclusion of real Lie algebras. One notices that there is a decomposition of G -modules (but not of Lie algebras):

$$\mathfrak{su}(p, q) \oplus i\mathfrak{su}(p, q) = \mathfrak{sl}_n(\mathbb{C}).$$

Now, $i\mathfrak{su}(p, q)$ is isomorphic to $\mathfrak{su}(p, q)$ as a G -module. Hence the result. \square

We now turn to the proof of Proposition 4.7 mentioned in Section 1.1. The proof is based on a generalization to deformations of the following result.

THEOREM 4.3 ([1, Section 2.5]). — *Let $g \geq 3$ and $n \geq 0$. Then for any finite dimensional representation ρ of $\mathrm{Mod}(S_g^n)$, and any Dehn twist T_γ , its image $\rho(T_\gamma)$ is quasi-unipotent.*

We will denote by $\mathrm{TU}(d)$ the tangent space to the unitary group $\mathrm{U}(d)$. More precisely, it is the group of matrices $V(\mathrm{I}_d + \epsilon A)$ with $V \in \mathrm{U}(d)$ and $A \in \mathfrak{u}_d$ (i.e. A is antisymmetric).

We will need the following two lemmas to generalize Theorem 4.3.

LEMMA 4.4. — *Any matrix in $\mathrm{TU}(d)$ is conjugate in $\mathrm{TU}(d)$ to a diagonal matrix.*

Proof. — Let $V(\mathrm{I}_d + \epsilon A)$ be an element of $\mathrm{TU}(d)$. Conjugating by elements of $\mathrm{U}(d) \subset \mathrm{TU}(d)$, we can assume that V is diagonal, say $V = \mathrm{diag}(\lambda_1 \mathrm{I}_{d_1}, \dots, \lambda_n \mathrm{I}_{d_n})$ with $\lambda_i \neq \lambda_j$ for $i \neq j$. For $B \in \mathfrak{u}_d$, $(\mathrm{I}_d + \epsilon B)V(\mathrm{I}_d + \epsilon A)(\mathrm{I}_d + \epsilon B)^{-1} = V(\mathrm{I}_d + \epsilon(A + V^{-1}BV - B))$. Hence we may assume that A is bloc diagonal of the form $A = \mathrm{diag}(A_1, \dots, A_n)$ with $A_i \in \mathfrak{u}_{d_i}$. Now for each i , $A_i = U_i D_i U_i^{-1}$ for $U_i \in \mathrm{U}(d_i)$ and D_i diagonal. Setting $D = \mathrm{diag}(D_1, \dots, D_n)$, as $U = \mathrm{diag}(U_1, \dots, U_n)$ commutes with V , we see that it conjugates $V(\mathrm{I}_d + \epsilon A)$ to the diagonal matrix $V(\mathrm{I}_d + \epsilon D)$. \square

LEMMA 4.5. — *For $g \geq 2$ and $n \geq 0$, $\mathrm{M\ddot{o}d}(S_g^n)$ has finite abelianization.*

Proof. — It is well known that in this range $\mathrm{Mod}(S_g^n)$ has finite abelianization (see for example [7, Section 5.1.2]). Because abelianization is right exact, we get an exact sequence:

$$\mathbb{Z} \longrightarrow \mathrm{M\ddot{o}d}(S_g^n)^{\mathrm{ab}} \longrightarrow \mathrm{Mod}(S_g^n)^{\mathrm{ab}} \longrightarrow 1.$$

We only need to show that the map $\mathbb{Z} \rightarrow \mathrm{M\ddot{o}d}(S_g^n)^{\mathrm{ab}}$ is not injective. Let us assume by contradiction that it is. Then the map $\mathbb{Q} \rightarrow \mathrm{M\ddot{o}d}(S_g^n)^{\mathrm{ab}} \otimes \mathbb{Q}$ would be an isomorphism, so that the cocycle $\tau \in \mathrm{H}^2(\mathrm{Mod}(S_g^n); \mathbb{Z})$ corresponding to the extension $\mathrm{M\ddot{o}d}(S_g^n)$ would be of torsion, as the corresponding extension by \mathbb{Q} would split. However, the restriction of τ to $\mathrm{H}^2(\mathrm{Mod}(S_g^1); \mathbb{Z}) \simeq \mathbb{Z}$ by an inclusion $S_g^1 \subset S_g^n$ is 4 times a generator (see [10]) and is thus not a torsion element. \square

THEOREM 4.6. — *Let $g \geq 3$, $n \geq 0$ and $d \geq 0$, then for any representation $\rho : \mathrm{M\ddot{o}d}(S_g^n) \rightarrow \mathrm{TU}(d)$ and any lift T_γ of a Dehn twist to $\mathrm{M\ddot{o}d}(S_g^n)$, its image $\rho(T_\gamma)$ has finite order equal to that of its projection to $\mathrm{U}(d)$.*

Proof. — Let γ be a simple closed curve and $T_\gamma \in \mathrm{M\ddot{o}d}(S_g^n)$ a lift of the Dehn twist around γ . Denote by S' the compact surface obtained by cutting S_g^n along γ . It has a component $S'' \subset S'$ of genus $g'' \geq 2$. Let $\rho'' : \mathrm{M\ddot{o}d}(S'') \rightarrow \mathrm{TU}(d)$ be the restriction of ρ . One may assume, by Lemma 4.4 that $\rho(T_\gamma) = \rho''(T_\gamma)$ is diagonal. Let us denote for $\lambda \in \mathbb{C}^*$ and $\mu \in \mathbb{C}$ by $E_{\lambda, \mu} \subset \mathbb{C}[\epsilon]^d$ the subspace generated by the coordinate vectors with

diagonal coefficient $\lambda + \epsilon\mu$ in $\rho(T_\gamma)$. Clearly $\mathbb{C}[\epsilon]^d = \bigoplus_{\lambda,\mu} E_{\lambda,\mu}$. We will use the notation $E_\lambda = \bigoplus_\mu E_{\lambda,\mu}$.

The twist T_γ commutes with the image of $\text{Möd}(S'')$ in $\text{Möd}(S_g^n)$, so that $\rho''(\text{Möd}(S''))$ commutes to $\rho(T_\gamma)$. Now the commutator of $\rho(T_\gamma)$ in $\text{TU}(d)$ preserves each $\ker(\rho(T_\gamma) - (\lambda + \epsilon\mu)\text{I}_d) = E_{\lambda,\mu} + \epsilon E_\lambda$ and hence also each $\epsilon E_{\lambda,\mu}$. Let us fix $\lambda \in \mathbb{C}^*$ and $\mu \in \mathbb{C}$. The representation ρ'' preserves $\bigoplus_{\mu' \neq \mu} \epsilon E_{\lambda,\mu'}$ and thus acts on the quotient $E_{\lambda,\mu} + \epsilon E_\lambda / \bigoplus_{\mu' \neq \mu} \epsilon E_{\lambda,\mu'}$, that we identify with $E_{\lambda,\mu}$. Let $\rho''_{\lambda,\mu} : \text{Möd}(S'') \rightarrow \text{U}(E_{\lambda,\mu})$ be this action. As by Lemma 4.5, $\text{Möd}(S'')^{\text{ab}}$ is finite, $\det(\rho''_{\lambda,\mu}) : \text{Möd}(S'') \rightarrow \mathbb{C}[\epsilon]^\times$ factors through roots of unity in \mathbb{C}^* . So, if $d_{\lambda,\mu} = \dim E_{\lambda,\mu} > 0$, $\det(\rho''_{\lambda,\mu})(T_\gamma) = (\lambda + \epsilon\mu)^{d_{\lambda,\mu}}$ must be a root of unity, i.e. λ is a root of unity and $\mu = 0$. So $\rho(T_\gamma) = \rho''(T_\gamma)$ has finite order and the order is the same after quotient by ϵ . \square

PROPOSITION 4.7. — *Let $\ell \geq 5$ be a prime number, $g \geq 3$ and $n \geq 0$. Let $\underline{\lambda}$ be any coloring of the boundary components of S_g^n . Then the $\text{SO}(3)$ quantum representation associated to S_g^n and $\underline{\lambda}$ is locally rigid as a representation of $\text{Mod}(S_g^n)$ if and only if it is as a representation of $\text{Mod}^\ell(S_g^n)$.*

Proof. — Because of Propositions 4.1 and 4.2, we may restrict to deformations as unitary representations of the central extension. We need to show that any deformation $\widehat{\rho} : \text{Möd}(S_g^n) \rightarrow \text{TU}(d)$ of the quantum representation ρ factors through $\text{Möd}^\ell(S_g^n)$. Let $T_\gamma \in \text{Möd}^\ell(S_g^n)$ be the lift of a Dehn twist. By Theorem 4.6, $\widehat{\rho}(T_\gamma)$ has finite order equal to that of $\rho(T_\gamma)$. Let c be the generator of the central extension. Then for any γ , $c = (cT_\gamma)T_\gamma^{-1}$ is the quotient of 2 commuting lifts of Dehn twists. Hence again $\widehat{\rho}(c)$ has finite order equal to that of $\rho(c)$. Thus $\widehat{\rho}$ factors through $\text{Möd}^\ell(S_g^n)$. \square

Finally, we mention that embedding the representation in a larger linear group does not change the space of deformations.

PROPOSITION 4.8. — *Let $\rho : \text{Möd}^\ell(S) \rightarrow \text{GL}_d(\mathbb{C})$ be a representation such that $\rho(\text{id}, 1)$ is non-trivial. Let $\rho' : \text{Möd}^\ell(S) \rightarrow \text{GL}_{d+N}(\mathbb{C})$ be the post-composition of the representation with the inclusion $\text{GL}_d(\mathbb{C}) \rightarrow \text{GL}_{d+N}(\mathbb{C})$. Then the induced morphism:*

$$H^1(\text{Möd}^\ell(S), \text{ad } \rho) \longrightarrow H^1(\text{Möd}^\ell(S), \text{ad } \rho')$$

is an isomorphism.

The hypothesis on $\rho(\text{id}, 1)$ is always verified for representations coming from the $\text{SO}(3)$ TQFT.

Proof. — As a representation of $\text{M\ddot{o}d}^\ell(S)$, $\text{ad } \rho'$ has a decomposition:

$$\mathfrak{gl}_{d+N} = \mathfrak{gl}_d \oplus \rho^{\oplus N} \oplus (\rho^*)^{\oplus N} \oplus 1^{\oplus N^2}$$

where 1 denotes the trivial representation. Now, as $\text{M\ddot{o}d}^\ell(S)^{\text{ab}}$ is finite:

$$H^1(\text{M\ddot{o}d}^\ell(S), 1) = 0.$$

As $(\text{id}, 1)$ is central and $\rho((\text{id}, 1))$ is non-trivial, by Lemma 6.1:

$$H^1(\text{M\ddot{o}d}^\ell(S), \rho) = 0 \text{ and } H^1(\text{M\ddot{o}d}^\ell(S), \rho^*) = 0. \quad \square$$

5. Proof of the main Lemma

From this point onwards, we will use the following notation for the adjoint representations. For $(S, \underline{\lambda})$ a colored surface, we will denote $\text{ad } \mathcal{V}(S, \underline{\lambda})$ or even just $\text{ad } \mathcal{V}(S)$ the adjoint of the representation of $\text{M\ddot{o}d}(S)$ on $\mathcal{V}(S, \underline{\lambda})$, i.e. the space $\text{End}(\mathcal{V}(S, \underline{\lambda}))$ with action of $\text{M\ddot{o}d}(S)$ by conjugation. This notation will be useful when dealing with mapping class groups of subsurfaces.

Let $(S, \underline{\lambda})$ be a colored surface, a_1, \dots, a_n some disjoint oriented simple closed curves on S and $\underline{\mu} \in \Lambda^n$ some colors. We will denote by $(S_{a_1, \dots, a_n}, \underline{\mu}, \underline{\lambda})$, or even just by $(S_{a_1, \dots, a_n}, \mu_1, \dots, \mu_n)$, the colored surface obtained by cutting S along a_1, \dots, a_n and coloring the new boundary components on either sides of a_i with μ_i . See Figure 5.1 for an example.

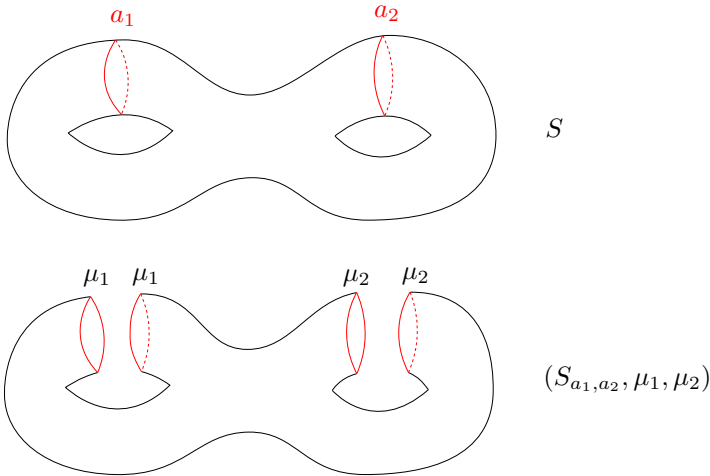


Figure 5.1. Surfaces S and $(S_{a_1, a_2}, \mu_1, \mu_2)$

One of the central technical results of this paper is the following lemma.

LEMMA 5.1. — Let \mathcal{V} be a modular functor. Let $\ell > 1$ be a level for \mathcal{V} .

Let $(S, \underline{\lambda})$ be a connected colored surface, and a_1, a_2 disjoint simple closed curves on S such that $\text{Mod}(S)$ is generated by the stabilizer of a_1 and the stabilizer of a_2 . Let c be a set of disjoint simple closed curves separating S into 2 components, containing a_1 and a_2 respectively.

We assume that \mathcal{V} is irreducible on the 2 components of S_c and those of S_{a_1, a_2} .

Assume there exist colors $\underline{\lambda}_c$ such that for all $\mu_1, \mu_2 \in \Lambda$:

$$\mathcal{V}(S_{a_1, a_2}, \mu_1, \mu_2) \neq 0 \implies \mathcal{V}(S_{a_1, a_2, c}, \mu_1, \mu_2, \underline{\lambda}_c) \neq 0.$$

Then the natural map:

$$\begin{aligned} H^1(\text{Mod}^\ell(S), \text{ad } \mathcal{V}(S)) \\ \longrightarrow H^1(\text{Mod}^\ell(S_{a_1}), \text{ad } \mathcal{V}(S)) \oplus H^1(\text{Mod}^\ell(S_{a_2}), \text{ad } \mathcal{V}(S)) \end{aligned}$$

is injective.

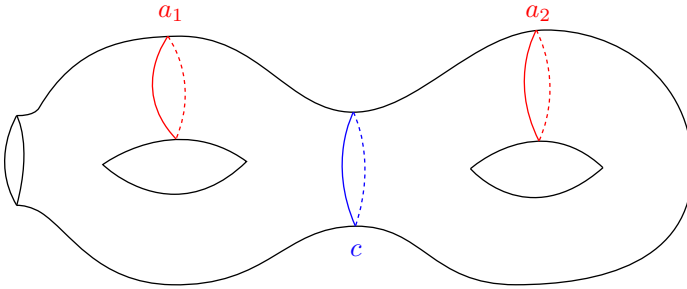


Figure 5.2. Surface S and curves c, a_1, a_2 as in Lemma 5.1.

Remark 5.2. — When applying Lemma 5.1, we will usually not give details on why $\text{Mod}(S)$ is generated by the stabilizer of a_1 and the stabilizer of a_2 , as it will easily follow from well known generating sets for $\text{Mod}(S)$. For details on these generators, we refer the reader to [7, Sections 4.4 and 9.3].

The proof of the lemma relies on the following group cohomology results.

PROPOSITION 5.3 (MV-sequence, [4, Chapter II (7.7), (7.8)]). — Let G_1, G_2, A be groups with inclusions $i_1 : A \rightarrow G_1, i_2 : A \rightarrow G_2$. Let $G = G_1 *_A G_2$ be their amalgamated sum. Let M be a G -module. Then one has an exact sequence of G -modules:

$$0 \longrightarrow \mathbb{Z}[G/A] \longrightarrow \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2] \xrightarrow{\begin{pmatrix} 1 & -1 \end{pmatrix}} \mathbb{Z} \longrightarrow 0$$

which induces a long exact sequence in cohomology:

$$\begin{aligned} \longrightarrow H^n(G, M) &\longrightarrow H^n(G_1, M) \oplus H^n(G_2, M) \\ &\longrightarrow H^n(A, M) \longrightarrow H^{n+1}(G, M) \longrightarrow . \end{aligned}$$

PROPOSITION 5.4 (IR-sequence, [15, Section VII, Proposition 4]). — Let G be a group, R a normal subgroup and M a G -module. Then we have the following exact sequence:

$$0 \longrightarrow H^1(G/R, M^R) \longrightarrow H^1(G, M) \longrightarrow H^1(R, M).$$

Where M^R denotes the set of elements of M fixed by R .

The idea behind the proof of Lemma 5.1 is the following. If ρ is a representation of $\text{Mod}^\ell(S)$ that is rigid when restricted to $\text{Mod}^\ell(S_{a_1})$ and $\text{Mod}^\ell(S_{a_2})$, one could hope that, as these groups generate $\text{Mod}^\ell(S)$, ρ is rigid. However this is not the case in general. The defect of rigidity of such a ρ can be controlled by the Mayer–Vietoris and Inflation–Restriction sequences (see step (1) below). In the situation of the Lemma, we use the technical assumption on c and the irreducibility of \mathcal{V} to control this defect.

Proof of Lemma 5.1.

(1) *Reformulation.* — For $i \in \{1, 2\}$, let $\Gamma_i = \text{Mod}^\ell(S, [a_i])$ denote the stabilizer of a_i , and let $\Gamma_{12} = \text{Mod}^\ell(S, [a_1], [a_2])$ their intersection. Let $G = \Gamma_1 *_{\Gamma_{12}} \Gamma_2$ be their amalgamated sum.

As by assumption the stabilizers generate $\Gamma = \text{Mod}^\ell(S)$, we have $\Gamma = G/R$ for a normal subgroup R . Let $S_c = S_1 \sqcup S_2$ such that $a_i \subset S_i$. For $\Gamma_1^c = \text{Mod}^\ell(S_2)$ and $\Gamma_2^c = \text{Mod}^\ell(S_1)$, one has $\Gamma_i^c \subset \Gamma_i$, which justifies the notation. As Γ_1^c and Γ_2^c commute in Γ , we have an inclusion $[\Gamma_1^c, \Gamma_2^c] \subset R$.

From Mayer–Vietoris sequence (Proposition 5.3) and the Inflation–Restriction sequence (Proposition 5.4), we get the following diagram with exact vertical and horizontal sequences.

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & H^1(\Gamma, M) & & & \\ & & & \downarrow & & & \\ M^{\Gamma_1} \oplus M^{\Gamma_2} & \longrightarrow & M^{\Gamma_{12}} & \xrightarrow{\delta} & H^1(G, M) & \longrightarrow & H^1(\Gamma_1, M) \oplus H^1(\Gamma_2, M) \\ & & & & \downarrow f & & \\ & & & & \text{Hom}(R, M) & & \end{array}$$

Here, $M = \text{ad } \mathcal{V}(S)$. From the diagram we can see that if f is injective on the image of δ , then $H^1(\Gamma, M)$ injects into $H^1(\Gamma_1, M) \oplus H^1(\Gamma_2, M)$.

Note that $\Gamma_i = \text{Mod}^\ell(S_{a_i})/\langle T_{a_i} \rangle$, where T_{a_i} is the Dehn twist along a_i . As $\langle T_{a_i} \rangle$ is finite, one has $\text{Hom}(\langle T_{a_i} \rangle, M) = 0$. Thus from the Inflation-Restriction sequence (Proposition 5.4):

$$0 \longrightarrow H^1(\Gamma_i, M) \longrightarrow H^1(\text{Mod}^\ell(S_{a_i}), M) \longrightarrow \text{Hom}(\langle T_{a_i} \rangle, M)$$

we can see that $H^1(\Gamma_i, M) = H^1(\text{Mod}^\ell(S_{a_i}), M)$.

Hence, if we can show that f is injective on the image of δ , we are done.

(2) *Computation of $f \circ \delta$.* — To compute δ , one has to choose acyclic resolutions of \mathbb{Z} , $\mathbb{Z}[G/G_1]$, \dots . We will choose the resolutions giving the canonical description of chains in group cohomology (see [15, Section VII.3]). With these resolutions, the chains for the G -module $\mathbb{Z}[G/H]$ are given by:

$$C^n(H, M) = \text{Hom}_{\text{Set}}(H^n, M).$$

The first 2 differentials are:

$$\begin{aligned} \partial_0 : M &\longrightarrow \text{Hom}_{\text{Set}}(H, M) \\ m &\longmapsto (g \longmapsto g \cdot m - m) \\ \partial_1 : \text{Hom}_{\text{Set}}(H, M) &\longrightarrow \text{Hom}_{\text{Set}}(H^2, M) \\ \varphi &\longmapsto ((g_1, g_2) \longmapsto g_1 \cdot \varphi(g_2) - \varphi(g_1 g_2) + \varphi(g_1)). \end{aligned}$$

Let $m \in M^{\Gamma_{12}} \subset M = C^0(\Gamma_{12}, M)$. We can lift m to $m + 0 \in M \oplus M = C^0(\Gamma_1, M) \oplus C^0(\Gamma_2, M)$. Now:

$$\partial_0(m+0) = \varphi + 0 \in \text{Hom}(\Gamma_1, M) \oplus \text{Hom}(\Gamma_1, M) = C^1(\Gamma_1, M) \oplus C^1(\Gamma_2, M)$$

where $\varphi(g_1) = g_1 \cdot m - m$. Hence $\delta(m) \in \text{Hom}(G, M)$ is the unique cocycle $\psi : G \rightarrow M$ such that $\psi(g_1) = \varphi(g_1)$ for all $g_1 \in \Gamma_1$ and $\psi(g_2) = 0$ for all $g_2 \in \Gamma_2$.

Let $g_1 \in \Gamma_1^c$ and $g_2 \in \Gamma_2^c$. We want to compute $f(\delta(m))(g_1 g_2 g_1^{-1} g_2^{-1}) = \psi(g_1 g_2 g_1^{-1} g_2^{-1})$. From the cocycle condition:

$$\begin{aligned} \psi(g_1 g_2 g_1^{-1} g_2^{-1}) &= \psi(g_1) + g_1 \cdot \psi(g_2) + g_1 g_2 \cdot \psi(g_1^{-1}) + g_1 g_2 g_1^{-1} \cdot \psi(g_2^{-1}) \\ &= \psi(g_1) + g_1 \cdot \psi(g_2) - g_2 \cdot \psi(g_1) - \psi(g_2) \\ &= g_1 \cdot m - m - g_2 g_1 \cdot m + g_2 \cdot m. \end{aligned}$$

We used $\psi(h^{-1}) = -h^{-1} \cdot \psi(h)$, the fact that g_1 and g_2 commute in Γ , $\psi(g_1) = g_1 \cdot m - m$ and $\psi(g_2) = 0$. Now M is by definition $\text{ad } \mathcal{V}(S)$. More precisely, $M = \text{End}(\mathcal{V}(S))$ with the action $g \cdot m = \rho(g)m\rho(g^{-1})$, where $\rho : \text{M\ddot{o}d}(S) \rightarrow \text{GL}(\mathcal{V}(S))$ is the quantum representation. In what follows we will simply denote the endomorphism $\rho(g) \in \text{GL}(\mathcal{V}(S))$ by g . We will

also use the notation $[\cdot, \cdot]$ for the Lie bracket $[u, v] = u \circ v - v \circ u$ in $M = \text{End}(\mathcal{V}(S))$. Let us continue our computation.

$$\begin{aligned} \psi(g_1 g_2 g_1^{-1} g_2^{-1}) &= g_1 m g_1^{-1} - m - g_2 g_1 m g_1^{-1} g_2^{-1} + g_2 m g_2^{-1} \\ &= (g_1 m g_2 - m g_1 g_2 - g_2 g_1 m + g_2 m g_1) g_1^{-1} g_2^{-1} \\ &= -[g_1, [g_2, m]] g_1^{-1} g_2^{-1}. \end{aligned}$$

Hence, if m is in the kernel of $f \circ \delta$, for all $g_1 \in \Gamma_1^c$ and $g_2 \in \Gamma_2^c$, $[g_1, [g_2, m]] = 0$.

(3) *Reduction to Lemma 5.5.* — From the gluing axiom applied along c , we have a decomposition:

$$\mathcal{V}(S) = \bigoplus_{\underline{\lambda}} \mathcal{V}(S_1, \underline{\lambda}) \otimes \mathcal{V}(S_2, \underline{\lambda}).$$

Let $M' = \text{End}(S_1, \underline{\lambda}_c) \otimes \text{End}(S_2, \underline{\lambda}_c)$. As $M = \text{End}(\mathcal{V}(S))$, we can decompose M as:

$$M = \bigoplus_{\underline{\lambda}} \bigoplus_{\underline{\mu}} \text{Hom}(\mathcal{V}(S_1, \underline{\lambda}), \mathcal{V}(S_1, \underline{\mu})) \otimes \text{Hom}(\mathcal{V}(S_2, \underline{\lambda}), \mathcal{V}(S_2, \underline{\mu})).$$

One of these summands is M' . Thus we have a natural projection $M \rightarrow M'$. Moreover, as elements of Γ_1^c and Γ_2^c preserve this decomposition of M , their actions commute with this projection $M \rightarrow M'$.

Hence if we denote by m' the image of m in M' , provided m is in the kernel of $f \circ \delta$, we have $[g_1, [g_2, m']] = 0$ in M' for any $g_i \in \Gamma_i^c$, $i \in \{1, 2\}$.

As \mathcal{V} is irreducible on S_1 , the image of $\mathbb{C}[\Gamma_1^c]$ in M' is $E_2 = \text{id} \otimes \text{End}(S_2, \underline{\lambda}_c)$. Similarly, the image of $\mathbb{C}[\Gamma_2^c]$ in M' is $E_1 = \text{End}(S_1, \underline{\lambda}_c) \otimes \text{id}$.

Let m be in the kernel of $f \circ \delta$. Then, from the gluing axiom applied to S along a_1 and a_2 we have the decomposition:

$$\mathcal{V}(S) = \bigoplus_{\mu_1, \mu_2} \mathcal{V}(S_{a_1, a_2}, \mu_1, \mu_2).$$

As \mathcal{V} is irreducible on the components of S_{a_1, a_2} and as m commutes to Γ_{12} , m decomposes as:

$$m = \sum_{\mu_1, \mu_2} c_{\mu_1 \mu_2} \text{id}_{\mathcal{V}(S_{a_1, a_2}, \mu_1, \mu_2)}$$

with $c_{\mu_1 \mu_2} \in \mathbb{C}$. Thus m' decomposes as:

$$m' = \sum_{\mu_1, \mu_2} c_{\mu_1 \mu_2} \text{id}_{\mathcal{V}(S_1, a_1, \underline{\lambda}_c, \mu_1)} \otimes \text{id}_{\mathcal{V}(S_2, a_2, \underline{\lambda}_c, \mu_2)}.$$

Now, applying Lemma 5.5 with $V_1 = \mathcal{V}(S_1, \underline{\lambda}_c)$ and $V_2 = \mathcal{V}(S_2, \underline{\lambda}_c)$, we get that there exists a and b such that whenever $\mathcal{V}(S_{a_1, a_2, c}, \mu_1, \mu_2, \underline{\lambda}_c) \neq 0$, one has $c_{\mu_1, \mu_2} = a_{\mu_1} + b_{\mu_2}$. As, by hypothesis, $\mathcal{V}(S_{a_1, a_2, c}, \mu_1, \mu_2, \underline{\lambda}_c) \neq 0$

whenever $\mathcal{V}(S_{a_1, a_2}, \mu_1, \mu_2) \neq 0$, this just means that m is in the image of $M^{\Gamma_1} \oplus M^{\Gamma_2} \rightarrow M^{\Gamma_{12}}$, i.e. in the kernel of δ .

Thus f is injective on the image of δ , and the lemma is proved. \square

LEMMA 5.5. — *Let $V_i = \bigoplus_{\mu_i} V_i(\mu_i)$, $i = 1, 2$, be vector spaces. Set $M_i = \{\mu_i \mid V_i(\mu_i) \neq 0\}$. Let:*

$$u = \sum_{\mu_1, \mu_2} c_{\mu_1 \mu_2} \text{id}_{V_1(\mu_1)} \otimes \text{id}_{V_2(\mu_2)} \in \text{End}(V_1 \otimes V_2)$$

with $c_{\mu_1 \mu_2} \in \mathbb{C}$. Let $E_1 = \text{End}(V_1) \otimes \text{id}$ and $E_2 = \text{id} \otimes \text{End}(V_2)$. Then if:

$$[E_2, [E_1, u]] = 0$$

there exist $(a_\mu)_{\mu \in M_1}$, $(b_\mu)_{\mu \in M_2}$ such that for all $\mu_i \in M_i$, $i = 1, 2$:

$$c_{\mu_1 \mu_2} = a_{\mu_1} + b_{\mu_2}.$$

Here $[\cdot, \cdot]$ denotes the Lie bracket.

Proof. — Let $\lambda_i, \nu_i \in M_i$ for $i = 1, 2$. Let $m_{\lambda_i \nu_i} \in \text{Hom}(V_i(\lambda_i), V_i(\nu_i))$ for $i = 1, 2$. Define $x_1 = m_{\lambda_1 \nu_1} \otimes \text{id}_{V_2}$ and $x_2 = \text{id}_{V_1} \otimes m_{\lambda_2 \nu_2}$. Now:

$$\begin{aligned} [x_1, u] &= x_1 \circ u - u \circ x_1 \\ &= \sum_{\mu_2} c_{\nu_1 \mu_2} m_{\lambda_1 \nu_1} \otimes \text{id}_{V_2(\mu_2)} - c_{\lambda_1 \mu_2} m_{\lambda_1 \nu_1} \otimes \text{id}_{V_2(\mu_2)} \\ &= m_{\lambda_1 \nu_1} \otimes \left(\sum_{\mu_2} (c_{\nu_1 \mu_2} - c_{\lambda_1 \mu_2}) \text{id}_{V_2(\mu_2)} \right). \end{aligned}$$

And:

$$\begin{aligned} [x_2, [x_1, u]] &= x_2 \circ [x_1, u] - [x_1, u] \circ x_2 \\ &= m_{\lambda_1 \nu_1} \otimes (c_{\nu_1 \nu_2} - c_{\lambda_1 \nu_2}) m_{\lambda_2 \nu_2} - m_{\lambda_1 \nu_1} \otimes (c_{\nu_1 \lambda_2} - c_{\lambda_1 \lambda_2}) m_{\lambda_2 \nu_2} \\ &= (c_{\nu_1 \nu_2} - c_{\lambda_1 \nu_2} - c_{\nu_1 \lambda_2} + c_{\lambda_1 \lambda_2}) m_{\lambda_1 \nu_1} \otimes m_{\lambda_2 \nu_2}. \end{aligned}$$

As the last line must be 0, one has for all $\lambda_i, \nu_i \in M_i$, $i = 1, 2$:

$$c_{\nu_1 \nu_2} - c_{\lambda_1 \nu_2} - c_{\nu_1 \lambda_2} + c_{\lambda_1 \lambda_2} = 0.$$

One can check that for fixed $\nu_i \in M_i$ for $i = 1, 2$, $a_{\mu_1} = c_{\mu_1 \nu_2}$ and $b_{\mu_2} = c_{\nu_1 \mu_2} - c_{\nu_1 \nu_2}$ verify the claim of the lemma. \square

6. Proof of Theorem 3.1

The idea behind the proof of Theorem 3.1 is to proceed by induction on the genus and number of marked points by repeated use of Lemma 5.1.

LEMMA 6.1 (Center Kills). — *Let G be a group and M be a $\mathbb{C}[G]$ -module. Let $Z \subset G$ be a finite central subgroup of G such that its action on M is given by a non-trivial character $\chi : Z \rightarrow \mathbb{C}^*$. Then $H^1(G, M) = 0$.*

Proof. — As Z is central, it is a normal subgroup of G . Moreover, as χ is non-trivial, we have $M^Z = 0$. Hence, applying Proposition 5.4, we get an exact sequence:

$$0 \longrightarrow H^1(G/Z, 0) \longrightarrow H^1(G, M) \longrightarrow H^1(Z, M).$$

Now, as Z is finite, any $\mathbb{C}[Z]$ -module is projective. Hence $H^1(Z, M) = 0$.

Thus $H^1(G, M) = 0$. \square

COROLLARY 6.2. — *Let S be a compact surface with $\partial S \neq \emptyset$. Let $\underline{\lambda}$ and $\underline{\mu}$ be two colorings of ∂S . Assume \mathcal{V} is a modular functor of level ℓ satisfying assumption (III). If $\underline{\lambda} \neq \underline{\mu}$, then:*

$$H^1(\text{Mod}^\ell(S), \mathcal{V}(S, \underline{\lambda})^* \otimes \mathcal{V}(S, \underline{\mu})) = 0.$$

Proof. — Let T_γ be a Dehn twist along a boundary component of S on which $\underline{\lambda}$ and $\underline{\mu}$ differ. Then, by (III), $Z = \langle T_\gamma \rangle$ acts non-trivially by scalars on $\mathcal{V}(S, \underline{\lambda})^* \otimes \mathcal{V}(S, \underline{\mu})$. As Z is central and finite in $\text{Mod}^\ell(S)$, the result is a consequence of Lemma 6.1. \square

COROLLARY 6.3. — *We assume \mathcal{V} , ℓ , S , a_i , c , $\underline{\lambda}_c$ satisfy the hypotheses of Lemma 5.1.*

We also assume \mathcal{V} verifies (III).

Then for $i = 1, 2$:

$$H^1(\text{Mod}^\ell(S_{a_i}), \text{ad } \mathcal{V}(S)) = \bigoplus_{\mu_i} H^1(\text{Mod}^\ell(S_{a_i}), \text{ad } \mathcal{V}(S_{a_i}, \mu_i)).$$

In particular, we have an injective map:

$$H^1(\text{Mod}^\ell(S), \text{ad } \mathcal{V}(S)) \longrightarrow \bigoplus_{i, \mu_i} H^1(\text{Mod}^\ell(S_{a_i}), \text{ad } \mathcal{V}(S_{a_i}, \mu_i)).$$

Proof. — Fix $i \in \{1, 2\}$. From axiom **(G)**, we have:

$$\mathcal{V}(S) = \bigoplus_{\mu_i} \mathcal{V}(S_{a_i}, \mu_i).$$

And thus:

$$\text{ad } \mathcal{V}(S) = \bigoplus_{\mu_i, \mu'_i} \mathcal{V}(S_{a_i}, \mu'_i)^* \otimes \mathcal{V}(S_{a_i}, \mu_i).$$

Now apply Corollary 6.2 to conclude. \square

LEMMA 6.4 (Künneth). — *Let G_1 and G_2 be 2 groups. For $i = 1, 2$, let M_i be a $\mathbb{C}[G_i]$ -module. Then we have an isomorphism of graded vector spaces:*

$$H^*(G_1, M_1) \otimes H^*(G_2, M_2) \simeq H^*(G_1 \times G_2, M_1 \otimes M_2).$$

Proof. — For $i = 1, 2$, let X_i be a CW-complex modeling the classifying space of G_i , and let \mathcal{L}_i be the local system of coefficients on X_i corresponding to M_i . Then $X_1 \times X_2$ is a CW-complex modeling $G_1 \times G_2$, and $M_1 \otimes M_2$ corresponds to the local system of coefficients $\mathcal{L}_1 \otimes \mathcal{L}_2$. Now, as the cells of $X_1 \times X_2$ are products of cells, we have an isomorphism of cellular complexes:

$$C^*(X_1 \times X_2; \mathcal{L}_1 \otimes \mathcal{L}_2) \simeq C^*(X_1; \mathcal{L}_1) \otimes C^*(X_2; \mathcal{L}_2).$$

Hence, as these are complexes of vector spaces, the homological Künneth formula yields an isomorphism:

$$H^*(X_1 \times X_2; \mathcal{L}_1 \otimes \mathcal{L}_2) \simeq H^*(X_1; \mathcal{L}_1) \otimes H^*(X_2; \mathcal{L}_2). \quad \square$$

From now on, we will use the following corollary of the Künneth formula without mentioning it. It enables us to work component by component when computing local rigidity.

COROLLARY 6.5. — *Let $S = S_1 \sqcup S_2$ be a surface with 2 connected components. For $i = 1, 2$, let ρ_i be an irreducible complex representation of $\text{Möd}^\ell(S_i)$. Then we have an isomorphism:*

$$H^1(\text{Möd}^\ell(S), \text{ad}(\rho_1 \otimes \rho_2)) \simeq H^1(\text{Möd}^\ell(S_1), \text{ad} \rho_1) \oplus H^1(\text{Möd}^\ell(S_2), \text{ad} \rho_2).$$

Proof. — For $i = 1, 2$, let $G_i = \text{Möd}^\ell(S_i)$ and $M_i = \text{ad} \rho_i$. Now $\text{Möd}^\ell(S) = G_1 \times G_2$ and $\text{ad}(\rho_1 \otimes \rho_2) = \text{ad} \rho_1 \otimes \text{ad} \rho_2$. Hence, applying Lemma 6.4, we have:

$$\begin{aligned} H^1(G_1 \times G_2, M_1 \otimes M_2) \\ \simeq H^0(G_1, M_1) \otimes H^1(G_2, M_2) \oplus H^1(G_1, M_1) \otimes H^0(G_2, M_2). \end{aligned}$$

But as ρ_i is irreducible, $H^0(G_i, \text{ad} \rho_i) = \mathbb{C}$. Hence the result. \square

We can now proceed with the proof of the theorem.

Proof of Theorem 3.1. Since $g \geq 3$, by Proposition 4.7, we only need to prove local rigidity as representations of $\text{Mod}^\ell(S_g^n)$.

(1). — Let us first reduce to the case $g = 3$.

Let $g \geq 4$. Assume that for all $g' \in \{3, \dots, g-1\}$, $n' \geq 0$ and $\underline{\lambda}'$ coloring of $\partial S_{g'}^{n'}$:

$$H^1(\text{Mod}^\ell(S_{g'}^{n'}), \text{ad} \mathcal{V}(S_{g'}^{n'}, \underline{\lambda}')) = 0.$$

Let $n \geq 0$ and $\underline{\lambda}$ be any coloring of ∂S_g^n . We assume that $\mathcal{V}(S_g^n, \underline{\lambda}) \neq 0$. Otherwise, the result is trivial.

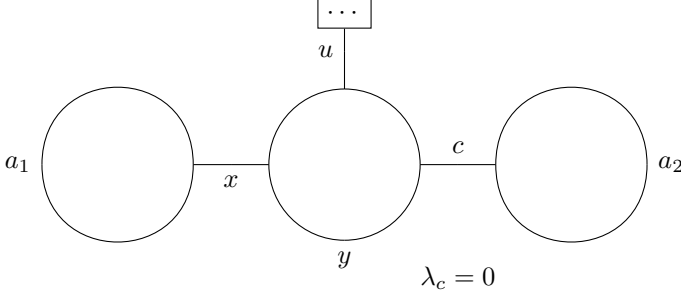


Figure 6.1. Graph of a pair of pants decomposition. Edges correspond to simple closed curves, vertices to pairs of pants. See the end of Section 2.2 for more details.

Then S_g^n has a pair of pants decomposition with associated graph of the form described on Figure 6.1. Let us check that the cuts a_1 , a_2 and c verify the hypotheses of Lemma 5.1 for $\lambda_c = 0$. Let $\mu_1, \mu_2 \in \Lambda$. We show that:

$$\mathcal{V}(S_{a_1, a_2, c}, \mu_1, \mu_2, \lambda_c) \neq 0.$$

Let ν_u be a color that appears on the edge u in an admissible coloring of the graph. Then, by (I), there exists $\nu_y \in \Lambda$ such that $\mathcal{V}(S_0^3, \nu_u, \nu_y, \nu_y) \neq 0$. Hence, there exists an admissible coloring of the graph with u colored with ν_u , y colored with ν_y , x and c colored with 0, a_1 colored with μ_1 and a_2 colored with μ_2 . Thus $\mathcal{V}(S_{a_1, a_2, c}, \mu_1, \mu_2, \lambda_c) \neq 0$.

Now, as any component of S_c or S_{a_1, a_2} has genus at least 1, \mathcal{V} is irreducible on these for any colorings.

Hence the hypotheses of Lemma 5.1 are verified. Moreover, as \mathcal{V} is assumed to verify (III), the conclusion of Corollary 6.3 holds, i.e. the map:

$$\mathrm{H}^1(\mathrm{Mod}^\ell(S), \mathrm{ad} \mathcal{V}(S)) \longrightarrow \bigoplus_{i, \mu_i} \mathrm{H}^1(\mathrm{Mod}^\ell(S_{a_i}), \mathrm{ad} \mathcal{V}(S_{a_i}, \mu_i))$$

is injective. But, by the induction hypothesis we made, the right hand side is 0. Hence $\mathrm{H}^1(\mathrm{Mod}^\ell(S), \mathrm{ad} \mathcal{V}(S)) = 0$.

(2). — The case $g = 3$ remains. Let $n > 4$. Assume that for all $n' \in \{0, \dots, n-1\}$ and any coloring $\underline{\lambda}'$ of $\partial S_3^{n'}$, $\mathrm{H}^1(\mathrm{Mod}^\ell(S_3^{n'}), \mathcal{V}(S_3^{n'}, \underline{\lambda}')) = 0$.

Let $\underline{\lambda}$ be any coloring of ∂S_3^n . Again, we assume $\mathcal{V}(S_3^n, \underline{\lambda}) \neq 0$. Otherwise, the result is trivial.

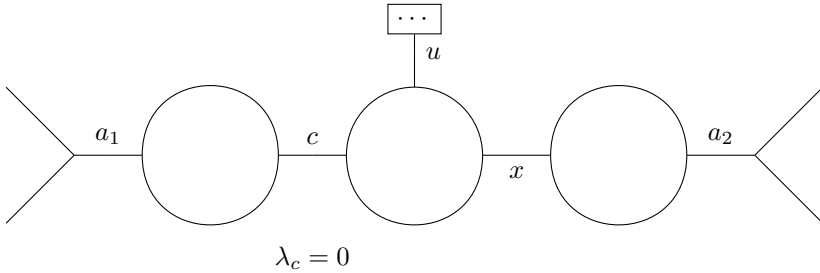


Figure 6.2. Graph of a pair of pants decomposition. Edges correspond to simple closed curves, legs to boundary components and vertices to pairs of pants. See the end of Section 2.2 for more details.

Then S_3^n has a pair of pants decomposition with associated graph of the form described on Figure 6.2. Let us verify that the cuts a_1 , a_2 and c verify the hypotheses of Lemma 5.1 for $\lambda_c = 0$. Let $\mu_1, \mu_2 \in \Lambda$. We show that:

$$\mathcal{V}(S_{a_1, a_2, c}, \mu_1, \mu_2, \lambda_c) \neq 0.$$

Let ν_u be a color that appears on the edge u in an admissible coloring of the graph. Then, as above, by (I), there exists an admissible coloring of the graph with u colored with ν_u , x and c colored with 0, a_1 colored with μ_1 and a_2 colored with μ_2 . Thus $\mathcal{V}(S_{a_1, a_2, c}, \mu_1, \mu_2, \lambda_c) \neq 0$.

Now, every component of S_c or S_{a_1, a_2} has genus at least 1 or is homeomorphic to S_0^3 . Since \mathcal{V} has dimension at most 1 on any coloring of S_0^3 , \mathcal{V} is irreducible on the components of S_c and S_{a_1, a_2} for any colorings.

As above, by Corollary 6.3 and the induction hypothesis, we have:

$$H^1(\text{Mod}^\ell(S), \text{ad } \mathcal{V}(S)) = 0.$$

We are thus reduced to the cases where $g = 3$ and $n \in \{0, 1, 2, 3\}$. \square

7. Proof of Theorem 3.3

In this section, we prove Theorem 3.3. The proof is in 2 steps. The first step is to prove rigidity for some small surfaces, and is the content of Lemma 7.2, Lemma 7.3 and Lemma 7.5. The second step is to perform an induction on g and n as in the proof of Theorem 3.1. However, because the assumptions of Lemma 5.1 are not always verified, we have to adapt the techniques of the lemma to the various cases, see Lemma 7.6, Lemma 7.7 and proof of the theorem.

In the context of modular functors, we know that a boundary component colored with 0 can be removed without changing the module. However, when considering rigidity, the source group changes. The following lemma shows that this does not change the space of deformations.

LEMMA 7.1. — *Let \mathcal{V} be a modular functor of level $\ell > 1$. Let S be a colored surface and $B \subset \partial S$ a boundary component colored with 0. Let \widehat{S} be the surface obtained by capping S with a disk along B , i.e. $\widehat{S} = S \cup_B D^2$.*

There is a natural isomorphism of $\text{M\ddot{o}d}^\ell(S)$ -modules $\mathcal{V}(S) \simeq \mathcal{V}(\widehat{S})$ and a group morphism $\text{Mod}^\ell(S) \rightarrow \text{Mod}^\ell(\widehat{S})$, which induce an isomorphism:

$$H^1(\text{Mod}^\ell(S), \text{ad } \mathcal{V}(S)) \simeq H^1(\text{Mod}^\ell(\widehat{S}), \text{ad } \mathcal{V}(\widehat{S})).$$

Proof. — Let us remind the Birman exact sequence ([7, Section 4.2.5]):

$$1 \longrightarrow \pi_1(U\widehat{S}) \xrightarrow{\text{Push}} \text{Mod}(S) \longrightarrow \text{Mod}(\widehat{S}) \longrightarrow 1.$$

Here $U\widehat{S}$ denotes the unitary tangent bundle of \widehat{S} . Taking the quotient by the ℓ -th powers of Dehn twists, we get an exact sequence:

$$\pi_1(U\widehat{S}) \xrightarrow{\text{Push}} \text{Mod}^\ell(S) \longrightarrow \text{Mod}^\ell(\widehat{S}) \longrightarrow 1.$$

Let us denote by K the image of *Push*.

The Inflation-Restriction sequence (Proposition 5.4) is in this setting:

$$\begin{aligned} 0 \longrightarrow H^1(\text{Mod}^\ell(\widehat{S}), \text{ad } \mathcal{V}(\widehat{S})) &\longrightarrow H^1(\text{Mod}^\ell(S), \text{ad } \mathcal{V}(S)) \\ &\longrightarrow \text{Hom}(K, \text{ad } \mathcal{V}(S)). \end{aligned}$$

Now $\pi_1(U\widehat{S})$ is generated by z and $\gamma_1, \dots, \gamma_d$ where z is a simple loop on any fiber of $U\widehat{S} \rightarrow \widehat{S}$ and the γ_i are lifts of simple closed curves. As for their images in $\text{Mod}^\ell(S)$, $\text{Push}(z)$ is the Dehn twist along B and $\text{Push}(\gamma_i)$ is a product of powers of two commuting Dehn twists. In particular, $\text{Push}(z)^\ell = 1$ and $\text{Push}(\gamma_i)^\ell = 1$ in $\text{Mod}^\ell(S)$. Thus K^{ab} is finite and $\text{Hom}(K, \text{ad } \mathcal{V}(S)) = 0$. Hence:

$$H^1(\text{Mod}^\ell(\widehat{S}), \text{ad } \mathcal{V}(\widehat{S})) \simeq H^1(\text{Mod}^\ell(S), \text{ad } \mathcal{V}(S)). \quad \square$$

From now on in this section, \mathcal{V} will denote the modular functor associated to the $\text{SO}(3)$ TQFT in level 5. All boundary components will be colored with 2.

LEMMA 7.2. — *The $\text{SO}(3)$ TQFT in level 5 is locally rigid on S_0^2 , S_0^3 , S_1 and S_1^1 .*

Proof. — The groups $\text{Mod}^5(S_0^2)$ and $\text{Mod}^5(S_0^3)$ are finite. The groups $\text{Mod}(S_1)$ and $\text{Mod}(S_{1,1})$ are isomorphic to $\text{SL}_2(\mathbb{Z})$, via the linear action

on $\mathbb{R}^2/\mathbb{Z}^2$. Moreover, one has $SL_2(\mathbb{Z}) \simeq \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$. This isomorphism is explicitly given by S^{-1} and ST , where:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then $T = S^{-1}ST$ is a Dehn twist in $Mod(S_{1,1})$ and $Mod(S_1)$. Thus T^5 is trivial in $Mod^5(S_{1,1})$ and $Mod^5(S_1)$. Hence the central quotients $Mod^5(S_{1,1})/(\mathbb{Z}/2)$ and $Mod^5(S_1)/(\mathbb{Z}/2)$ are quotients of the triangular group $TG(2, 3, 5)$. As $1/2 + 1/3 + 1/5 > 1$, $TG(2, 3, 5)$ is spherical and thus finite. Hence $Mod^5(S_{1,1})$ and $Mod^5(S_1)$ are finite. As $Mod^5(S_1^1)$ is an extension of $Mod^5(S_{1,1})$ by $\mathbb{Z}/5$ or $\{1\}$, it is also finite.

Finite groups do not have cohomology on \mathbb{C} -vector spaces, so the $SO(3)$ TQFT in level 5 is necessarily locally rigid on S_0^2, S_0^3, S_1 and S_1^1 . \square

Here we used some explicit computations of Mapping Class Groups. They can be found in [7, Section 2.2].

LEMMA 7.3. — *The $SO(3)$ TQFT in level 5 is locally rigid on S_0^4 .*

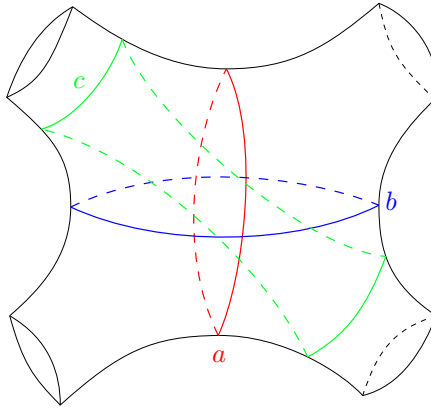


Figure 7.1. Surface S_0^4 and curves.

Proof. — Here $S = S_0^4$. Let a and b be the curves of Figure 7.1. Let Γ_a be the stabilizer of a and Γ_b the stabilizer of b . Let Γ_{ab} be their intersection. Define $G = \Gamma_a *_{\Gamma_{ab}} \Gamma_b$, and R such that $Mod^5(S_0^4) = G/R$. Notice that Γ_a and Γ_b are finite groups. Hence, the Mayer–Vietoris sequence for G is:

$$0 \longrightarrow M^G \longrightarrow M^{\Gamma_a} \oplus M^{\Gamma_b} \longrightarrow M^{\Gamma_{ab}} \longrightarrow H^1(G, M) \longrightarrow 0$$

where $M = \text{ad } \mathcal{V}(S)$. Now, as $\mathcal{V}(S)$ has dimension 2, a simple count on basis elements shows that M^G has dimension 1, M^{Γ_a} and M^{Γ_b} dimension

2 and $M = M^{\Gamma_{ab}}$ dimension 4. Hence $H^1(G, M)$ has dimension 1. Now the inflation restriction exact sequence for $R \subset G$ is:

$$0 \longrightarrow H^1(\text{Mod}^5(S_0^4), M) \longrightarrow H^1(G, M) \longrightarrow \text{Hom}(R, M).$$

Let $\varphi \in H^1(G, M)$ be non-zero. As $M^{\Gamma_{ab}} \rightarrow H^1(G, M)$ is surjective, a computation as in the proof of Lemma 5.1 gives that there exists $m \in M$ such that:

$$\varphi(T_a) = T_a \cdot m - m \text{ and } \varphi(T_b) = 0.$$

Now, notice that $(T_a T_b)^{-1}$ is the Dehn twist T_c on Figure 7.1. Hence $(T_a T_b)^5$ is in R . Let $\Phi(X) = 1 + X + X^2 + X^3 + X^4$. We have:

$$\varphi((T_a T_b)^5) = \Phi(\text{ad}(T_a T_b))(\varphi(T_a T_b)) = \Phi(\text{ad}(T_a T_b))(T_a \cdot m - m).$$

As $(u - 1)\Phi(u) = u^5 - 1 = 0$ for $u = \text{ad}(T_a T_b)$, we have $\varphi((T_a T_b)^5) = 0$ if and only if $T_a \cdot m - m$ is in the image of $\text{ad}(T_a T_b) - \text{id}$. Let us now prove that:

$$\dim(\text{SL}(\text{ad } T_a - \text{id}) \cap \text{SL}(\text{ad}(T_a T_b) - \text{id})) = 1.$$

Notice that for the usual trace quadratic form $(A, B) \mapsto \text{Tr}(AB)$, we have:

$$\text{SL}(\text{ad } T_a - \text{id})^\perp = \text{Ker}(\text{ad } T_a^{-1} - \text{id}) = \text{Ker}(\text{ad } T_a - \text{id}).$$

Similarly for $\text{SL}(\text{ad}(T_a T_b) - \text{id})$. As scalar matrices are fixed by $\text{ad } T_a$ and $\text{ad}(T_a T_b)$, we have:

$$\dim(\text{Ker}(\text{ad } T_a - \text{id}) + \text{Ker}(\text{ad}(T_a T_b) - \text{id})) \leq 3.$$

As T_a and $T_a T_b$ generate $\text{Mod}^5(S_0^4)$ and the representation is irreducible, the dimension is actually equal to 3. Hence the dimension result. This shows that there exists $m \in M$ such that $T_a \cdot m - m \neq 0$ and $\varphi((T_a T_b)^5) \neq 0$ for φ associated to m as above.

From this we get that the map $M = M^{\Gamma_{ab}} \rightarrow H^1(G, M) \rightarrow \text{Hom}(R, M)$ has rank at least 1. Hence, as $H^1(G, M)$ has dimension 1, $H^1(G, M) \rightarrow \text{Hom}(R, M)$ is injective. From the inflation-restriction exact sequence above, we see that the space of deformations $H^1(\text{Mod}^5(S_0^4), M)$ is 0, as desired. \square

For the case of $S_{0,5}$, we use that the representation $\rho_{0,5}^5$ is the monodromy of the Hirzebruch surface. This is proved in the paper of B. Deroin and J. Marché:

PROPOSITION 7.4 ([5, Proposition 9]). — *The $\text{SO}(3)$ -quantum representation of level 5 associated to $S_{0,5}$ and $\zeta_5 = e^{\frac{2i\pi}{5}}$ is the holonomy of a $\mathbb{H}_{1,2}$ -structure on the compact orbifold $\overline{\mathcal{M}}_{0,5}(5)$.*

We now retrieve local rigidity from Weil's rigidity.

LEMMA 7.5. — *The $\text{SO}(3)$ TQFT in level 5 is locally rigid on S_0^5 .*

Proof. — A consequence of Proposition 7.4 is that the representation is given by the inclusion of a cocompact lattice $\Gamma \subset \text{PU}(1, 2)$, and an isomorphism $\Gamma \simeq \text{Mod}^5(S_{0,5})$. From the proof of local rigidity given by A. Weil [18], we have that $H^1(\Gamma, \mathfrak{pu}(1, 2)) = 0$, where $\mathfrak{pu}(1, 2)$ is the Lie algebra of $\text{PU}(1, 2)$. Hence the representation is locally rigid in $\text{PU}(1, 2)$. We conclude with Proposition 4.2. \square

LEMMA 7.6. — *The $\text{SO}(3)$ TQFT in level 5 is locally rigid on S_0^6 .*

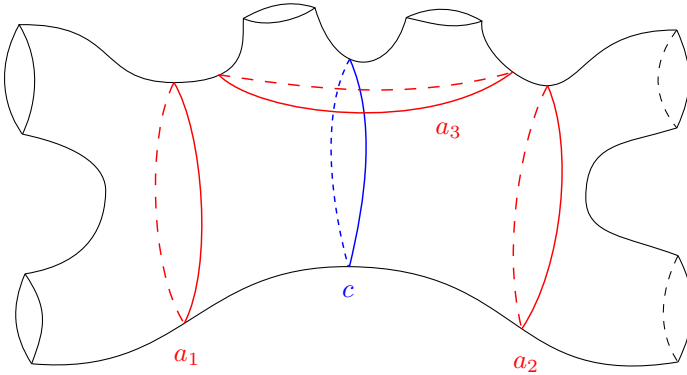


Figure 7.2. Surface S_0^6 and curves.

Proof. — Here $S = S_0^6$. We will need a slight variation of Lemma 5.1 where the mapping class group is not generated by the stabilizer of a_1 and the stabilizer of a_2 . The proof is exactly the same if we replace the full mapping class group by the subgroup generated by the stabilizers.

Let a_1, a_2, a_3 , and c be the simple closed curves on Figure 7.2.

Let Γ_i be the stabilizer of a_i for $i = 1, 2, 3$. Let Γ_{ij} be the stabilizer of a_i and a_j for $i, j \in \{1, 2, 3\}$. Let $G_{ij} = \Gamma_i *_{\Gamma_{ij}} \Gamma_j$ for $i, j \in \{1, 2, 3\}$. Let L_{ij} be the image of $G_{ij} \rightarrow \text{Mod}^\ell(S_0^6)$.

The curves a_1, a_2 and c verify all the hypotheses of Lemma 5.1 for $\lambda_c = 2$ if we replace $\text{Mod}^\ell(S_0^6)$ by L_{12} . Hence, applying the proof of the lemma, we get:

$$H^1(L_{12}, \text{ad } \mathcal{V}(S)) = 0.$$

By symmetry, the same result holds for L_{13} .

Let $G = L_{12} *_{\langle \Gamma_1, \Gamma_{23} \rangle} L_{13}$. Now $\text{Mod}^\ell(S_0^6)$ is a quotient of G . Moreover, the Mayer–Vietoris sequence is:

$$0 \longrightarrow M^G \longrightarrow M^{L_{12}} \oplus M^{L_{13}} \longrightarrow M^{\langle \Gamma_1, \Gamma_{23} \rangle} \longrightarrow H^1(G, M) \longrightarrow 0$$

where $M = \text{ad } \mathcal{V}(S)$. Clearly, by Corollary A.3, M^G has dimension 1.

Now, as all 4 colorings $(0, 0)$, $(0, 2)$, $(2, 0)$ and $(2, 2)$ of (a_1, a_2) are admissible, $\mathcal{V}(S)$ is irreducible as a representation of $G_{12} = \Gamma_1 *_{\Gamma_{12}} \Gamma_2$. The same holds for G_{13} . Thus, $M^{L_{12}}$ and $M^{L_{13}}$ have dimension 1.

To conclude, we must prove that $M^{(\Gamma_1, \Gamma_{23})}$ also has dimension 1. Let φ be an element of M stabilized by Γ_1 and Γ_{23} . Then, φ is diagonal in the basis associated to the pair of pants decomposition formed by the a_i . Let $\lambda(c_1, c_2, c_3)$ be its diagonal coefficient on the basis element where a_i is colored with c_i .

As Γ_{23} fixes φ , by Corollary A.3, $\lambda(c_1, c_2, c_3)$ only depends on (c_2, c_3) . But the colorings $(2, 2, 2)$ and $(0, 2, 2)$ are admissible, so $\lambda(0, 2, 2) = \lambda(2, 2, 2)$. As Γ_1 fixes φ , by Corollary A.3, $\lambda(c_1, c_2, c_3)$ only depends on c_1 . So φ is scalar. Hence the result. \square

LEMMA 7.7. — For $n \geq 7$, the $SO(3)$ TQFT in level 5 is locally rigid on S_0^n .

Proof. — Here $S = S_0^n$. The proof is very similar to the one given for S_0^6 . We consider the pair of pants decomposition associated to the graph of Figure 7.3. Let a_1, a_2, a_3 , and c be the simple closed curves associated to the edges marked on the figure.

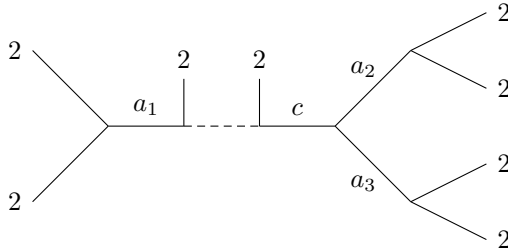


Figure 7.3. Graph of a pair of pants decomposition of S_0^n for $n \geq 7$.

Let $\Gamma_i, \Gamma_{ij}, L_{ij}$ and G be as in the proof of Lemma 7.6.

As for S_0^6 , the curves a_1, a_2 and c verify all the hypotheses of Lemma 5.1 for $\lambda_c = 2$ if we replace $\text{Mod}^\ell(S_0^n)$ by L_{12} . Hence, we get:

$$H^1(L_{12}, \text{ad } \mathcal{V}(S)) = H^1(L_{13}, \text{ad } \mathcal{V}(S)) = 0.$$

The Mayer–Vietoris sequence is:

$$0 \longrightarrow M^G \longrightarrow M^{L_{12}} \oplus M^{L_{13}} \longrightarrow M^{(\Gamma_1, \Gamma_{23})} \longrightarrow H^1(G, M) \longrightarrow 0$$

where $M = \text{ad } \mathcal{V}(S)$. As for S_0^6 , $M^G, M^{L_{12}}$ and $M^{L_{13}}$ have dimension 1.

Now as all 8 possible colorings of (a_1, a_2, a_3) appear in the basis, $\mathcal{V}(S)$ is irreducible as a representation of $\langle \Gamma_1, \Gamma_{23} \rangle$. Thus, $M^{\langle \Gamma_1, \Gamma_{23} \rangle}$ has dimension 1.

The result follows by dimension count on the exact sequence. □

We now have all the base cases necessary to prove the theorem.

Proof of Theorem 3.3. The second statement of the theorem follows from the first and Proposition 4.7.

In this proof, for any of the figures we will use the following notations. For $i \in \{1, 2\}$, $\Gamma_i = \text{Mod}^\ell(S, [a_i])$ will denote the stabilizer of a_i . $\Gamma_{12} = \text{Mod}^\ell(S, [a_1], [a_2])$ is their intersection. $G = \Gamma_1 *_{\Gamma_{12}} \Gamma_2$ will be their amalgamated sum.

(1) S_1^2 . — Here $S = S_1^2$. We use Figure 7.4 as a reference. By Lemma 7.2 and Lemma 7.3, \mathcal{V} is locally rigid on the components of $S_{a_1} \simeq S_0^3 \sqcup S_1^1$ and on $S_{a_2} \simeq S_0^4$ for any coloring of the boundary components. Hence, by Corollary 6.2:

$$H^1(\Gamma_1, \text{ad } \mathcal{V}(S)) = 0 \text{ and } H^1(\Gamma_2, \text{ad } \mathcal{V}(S)) = 0.$$

Now, according to Proposition 5.3, we have an exact sequence:

$$0 \longrightarrow M^G \longrightarrow M^{\Gamma_1} \oplus M^{\Gamma_2} \longrightarrow M^{\Gamma_{12}} \longrightarrow H^1(G, M) \longrightarrow 0.$$

Here $M = \text{ad } \mathcal{V}(S)$. Now, a quick count on the basis using Corollary A.3 shows that M^{Γ_1} and M^{Γ_2} have dimension 2, whereas $M^{\Gamma_{12}}$ has dimension 3 and M^G has dimension 1. Hence $H^1(G, M) = 0$, and by Proposition 5.4, $H^1(\Gamma, M) = 0$, as required.

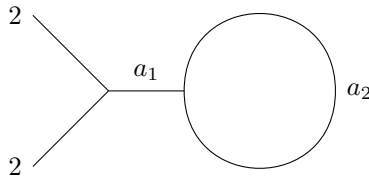


Figure 7.4. Graph of a pair of pants decomposition of S_1^2 .

(2) S_1^3 . — Here $S = S_1^3$. We use Figure 7.5 as a reference. By (1), Lemma 7.2 and Lemma 7.5, \mathcal{V} is locally rigid on the components of $S_{a_1} \simeq S_0^3 \sqcup S_1^2$ and on $S_{a_2} \simeq S_0^5$ for any coloring of the boundary components.

As above, we get the exact sequence:

$$0 \longrightarrow M^G \longrightarrow M^{\Gamma_1} \oplus M^{\Gamma_2} \longrightarrow M^{\Gamma_{12}} \longrightarrow H^1(G, M) \longrightarrow 0.$$

And again M^{Γ_1} and M^{Γ_2} have dimension 2, $M^{\Gamma_{12}}$ has dimension 3 and M^G has dimension 1. So $H^1(\Gamma, M) = 0$.

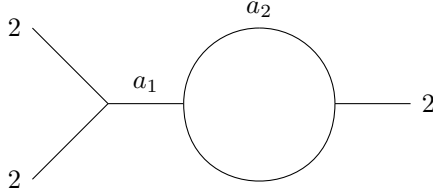


Figure 7.5. Graph of a pair of pants decomposition of S_1^3 .

(3) S_1^n with $n \geq 4$. — Fix $n \geq 4$. Here $S = S_1^n$. We use Figure 7.6 as a reference. We apply Lemma 5.1 with a_1 , a_2 and c as on the figure. One checks that the hypotheses of the lemma are verified when the components of c are colored with 2. The only non-trivial part is the irreducibility. But as we are in level 5, all the representations are irreducible, see Corollary A.3.

As $S_{a_1} \simeq S_{a_2} \simeq S_0^3 \sqcup S_1^{n-1}$, we are reduced to the case of S_1^{n-1} . We conclude by (2) and induction on n .

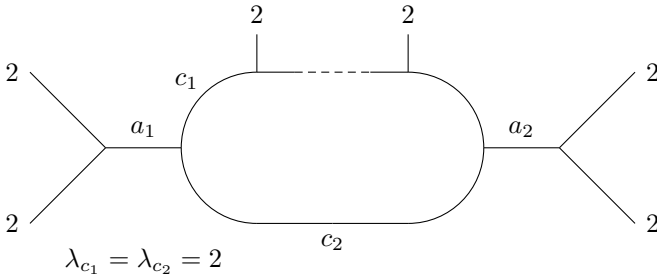


Figure 7.6. Graph of a pair of pants decomposition of S_1^n .

(4) S_2^1 . — Here $S = S_2^1$. We use Figure 7.7 as a reference. We proceed as in (1) and (2). Using Lemma 7.2 and (2), we see that \mathcal{V} is locally rigid on $S_{a_1} \simeq S_{a_2} \simeq S_1^3$ for any coloring of the a_i . Hence, we get the exact sequence:

$$0 \longrightarrow M^G \longrightarrow M^{\Gamma_1} \oplus M^{\Gamma_2} \longrightarrow M^{\Gamma_{12}} \longrightarrow H^1(G, M) \longrightarrow 0.$$

And again M^{Γ_1} and M^{Γ_2} have dimension 2, $M^{\Gamma_{12}}$ has dimension 3 and M^G dimension 1. So $H^1(\Gamma, M) = 0$.

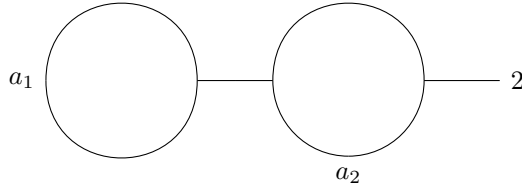


Figure 7.7. Graph of a pair of pants decomposition of S_2^1 .

(5) S_2^n with $n = 0$ or $n \geq 2$. — Fix $n \geq 2$. The cas $n = 0$ is treated similarly. Here $S = S_2^n$. We use Figure 7.8 as a reference in (5). We easily check that a_1, a_2 and c as on the figure verify the hypotheses of Lemma 5.1 for $\lambda_c = 0$. As in (3), we use Corollary A.3.

As $S_{a_1} \simeq S_{a_1} \simeq S_1^{n+2}$, (3) and Corollary 6.3 imply that \mathcal{V} is locally rigid on S .

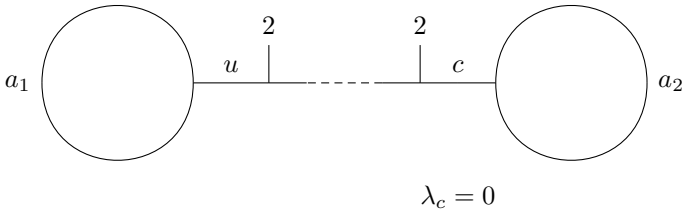


Figure 7.8. Graph of a pair of pants decomposition of $S_2^n, n \geq 2$.

(6) S_g^n with $g \geq 3$. — Choose $g \geq 3$ and $n \geq 0$. We proceed by induction on g . Here $S = S_g^n$ and we use Figure 6.1 as a reference. The hypotheses of Lemma 5.1 can be checked exactly as in the proof of Theorem 3.1. As $S_{a_1} \simeq S_{a_1} \simeq S_{g-1}^{n+2}$, the induction hypothesis, (4), (5) and Corollary 6.3 imply that \mathcal{V} is locally rigid on S . \square

Appendix A. Irreducibility of quantum representations

In this appendix, we prove irreducibility of quantum $SO(3)$ representations at prime levels. We deduce it as a corollary of a result from the paper [11] of T. Koberda and R. Santharoubane. In this section, we denote \mathcal{V}_ℓ the TQFT of level ℓ constructed in [3].

Here is the result we need from the paper of T. Koberda and R. Santharoubane, stated in the case of a $\mathrm{SO}(3)$ TQFT. Note that this is not the main result of their paper.

PROPOSITION A.1 ([11, Section 3.2]). — *Let $\ell \geq 5$ be an odd integer. Let S_g^n denote the compact surface of genus g with n boundary components. Denote by $\mathcal{S}(S_g^n)$ the Kauffman bracket skein module of $S_g^n \times [0, 1]$.*

Let $\lambda_1, \dots, \lambda_n \in \{0, 2, \dots, \ell - 3\}$. Denote by $(S_{g,n}, \lambda_1, \dots, \lambda_n)$ the closed surface of genus g with n marked points labelled $\lambda_1, \dots, \lambda_n$.

Then the map:

$$\mathcal{S}(S_g^n) \longrightarrow \mathrm{End}(\mathcal{V}_\ell(S_{g,n}, \lambda_1, \dots, \lambda_n))$$

is surjective.

Remark A.2. — Let $\beta \subset S_g^n$ be a simple closed curve. Let C_β be the curve operator that acts on $\mathcal{V}_\ell(S_{g,n}, \lambda_1, \dots, \lambda_n)$ via the element e_β of $\mathcal{S}(S_g^n)$ corresponding to β . If ℓ is prime, the operator C_β is a polynomial in the Dehn twist T_β (see [12, Section 4.2]).

We can now apply the proof of irreducibility of Roberts [14] to the case of surfaces with boundary.

COROLLARY A.3. — *Let $\ell \geq 5$ be a prime number. Let $g, n \geq 0$ and $\lambda \in \Lambda^n$. Then the associated quantum $\mathrm{SO}(3)$ representation of $\mathrm{Mod}(S_g^n)$ is irreducible.*

Proof. — The set $\{e_\beta \mid \beta \text{ simple closed curve}\}$ generates the skein module $\mathcal{S}(S_g^n)$ as an algebra. Hence, by Proposition A.1, the endomorphism algebra $\mathrm{End}(\mathcal{V}_\ell(S_{g,n}, \lambda_1, \dots, \lambda_n))$ is generated by the curve operators. Moreover, every such operator C_β is a polynomial in the associated Dehn twist T_β , so that $\mathrm{End}(\mathcal{V}_\ell(S_{g,n}, \lambda_1, \dots, \lambda_n))$ is generated as an algebra by the action of the Dehn twists. Thus, the map:

$$\mathbb{C}[\mathrm{Mod}(S_g^n)] \longrightarrow \mathrm{End}(\mathcal{V}_\ell(S_{g,n}, \lambda_1, \dots, \lambda_n))$$

is surjective. This shows that $\mathcal{V}_\ell(S_{g,n}, \lambda_1, \dots, \lambda_n)$ is irreducible. \square

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Manuscrit reçu le 18 juin 2023,
révisé le 2 décembre 2023,
accepté le 21 décembre 2023.

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