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Gastón Andrés GARCÍA & Fabio GAVARINI

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and specializations**

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FORMAL MULTIPARAMETER QUANTUM GROUPS, DEFORMATIONS AND SPECIALIZATIONS

by Gastón Andrés GARCÍA & Fabio GAVARINI (*)

ABSTRACT. — We introduce the notion of *formal multiparameter QUEA* – in short *FoMpQUEA* – as a straightforward generalization of Drinfeld’s quantum group $U_h(\mathfrak{g})$. Then we show that the class of FoMpQUEAs is closed under deformations by (“toral”) twists and deformations by (“toral”) 2-cocycles: as a consequence, all “multiparameter formal QUEAs” considered so far are recovered, as falling within this class. In particular, we prove that any FoMpQUEA is isomorphic to a suitable deformation, by twist or by 2-cocycle, of Drinfeld’s standard QUEA.

We introduce also multiparameter Lie bialgebras (in short, MpLbA’s), and we consider their deformations, by twist and by 2-cocycle. The semiclassical limit of every FoMpQUEA is a suitable MpLbA, and conversely each MpLbA can be quantized to a suitable FoMpQUEA. In the end, we prove that, roughly speaking, the two processes of “specialization” – of a FoMpQUEA to a MpLbA – and of “deformation (by toral twist or toral 2-cocycle)” – at the level of FoMpQUEAs or of MpLbA’s – do commute with each other.

RÉSUMÉ. — Nous introduisons la de *algèbre enveloppante universelle quantifiée* (=AEUQ) *multiparamétrique formelle* – en bref AEUQMpFo – comme généralisation directe du groupe quantique de Drinfeld $U_h(\mathfrak{g})$. Ensuite, nous prouvons que la classe des AEUQMpFo est fermée par rapport aux déformations par torseurs (“toraux”) et aux déformations 2-cocycles (“toraux”): par conséquent, toutes les “AEUQ multiparamétrique formelle” considérées jusqu’à ce jour sont retrouvées, comme incluses en cette classe. En particulier, nous prouvons que toute AEUQMpFo est isomorphe à une déformation convenable, par torseur ou par 2-cocycle, de la AEUQ standard de Drinfeld.

Nous introduisons aussi des bigèbres de Lie multiparamétriques (en bref, bGLMp), et nous considérons leur déformations, par torseur et par 2-cocycle. La limite semiclassique de chaque AEUQMpFo est une bGLMp convenable, et à l’envers chaque bGLMp peut être quantifiée à une AEUQMpFo convenable. Finalement, nous montrons que, en gros, les deux procédures de “specialisation” – d’une AEUQMpFo à une bGLMp – et de “déformation (par torseur toral ou 2-cocycle toral)” – au niveau des AEUQMpFo ou des bGLMp – commutent l’une avec l’autre.

Keywords: Quantum Groups, Quantum Enveloping Algebras.

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1. Introduction

Quantum groups can be thought of, roughly speaking, as Hopf algebras depending on one “parameter” such that, for a “special value” of this parameter, they turn isomorphic either to the universal enveloping algebra of some Lie algebra \mathfrak{g} or to the function algebra of some algebraic group G . In the first case the quantum group is called “quantized universal enveloping algebra” (or QUEA in short) and in the second “quantized function algebra” (or QFA in short).

Quite soon, people also began to introduce new quantum groups depending on two or more parameters, whence the terminology “multiparameter quantum groups” came in use: see, e.g., [7, 8, 9, 12, 14, 15, 19, 25, 31, 34, 37, 38, 42, 45, 46, 49, 51, 53], and the list might be longer. Nevertheless, one can typically describe a multiparameter quantum group so that one single parameter stands “distinguished”, as the *continuous* one that can be specialized. The other parameters instead (seen as *discrete*) parametrize different structures on a common “socle” underlying the semiclassical limit of the quantum group, that is achieved when the continuous parameter is specialized. Indeed, this already occurs with one-parameter quantum groups: for example, the celebrated Drinfeld’s QUEA $U_{\hbar}(\mathfrak{g})$ associated with a complex, finite-dimensional, semisimple Lie algebra \mathfrak{g} has a description where the *continuous* parameter \hbar bears the quantization nature of $U_{\hbar}(\mathfrak{g})$, while other *discrete* parameters, namely the entries of the Cartan matrix of \mathfrak{g} , describe the Lie algebra structure on \mathfrak{g} itself.

In this paper we focus onto the study of multiparameter QUEAs; then it will be possible to realize a parallel study and to achieve the corresponding results for multiparameter QFA’s by suitably applying duality. Recall that QUEAs (and QFA’s alike) are usually considered in two versions: the so-called “formal” one, dealing with topological Hopf algebras over $\mathbb{k}[[\hbar]]$, and the “polynomial” one, dealing with Hopf algebras over a field \mathbb{K} with some $q \in \mathbb{K}$ entering the game as parameter. One of the first general examples of multiparameter QUEA, hereafter mentioned as MpQUEA, was provided by Reshetikhin in [49]. This extends Drinfeld’s definition of $U_{\hbar}(\mathfrak{g})$ to a new object $U_{\hbar}^{\Psi}(\mathfrak{g})$ that shares the same algebra structure of $U_{\hbar}(\mathfrak{g})$ but bears a new *coalgebra* structure, depending on a matrix Ψ that collects the new, *discrete* parameters of $U_{\hbar}^{\Psi}(\mathfrak{g})$. At the semiclassical limit, these new parameters (hence Ψ) describe the new Lie coalgebra structure inherited by \mathfrak{g} from $U_{\hbar}^{\Psi}(\mathfrak{g})$ itself. Note that $U_{\hbar}^{\Psi}(\mathfrak{g})$ is defined from scratch as being the outcome of a *deformation by twist* of Drinfeld’s $U_{\hbar}(\mathfrak{g})$, using a twist of a specific type (that we shall call “toral”) defined via Ψ . It follows that

the class of all Reshetikhin’s MpQUEAs is stable under deformation by toral twists, i.e. any such deformation of an object of this kind is again an object of the same kind. Even more, this class is “homogeneous”, in that each $U_{\hbar}^{\Psi}(\mathfrak{g})$ is nothing but a twist deformation of Drinfeld’s $U_{\hbar}(\mathfrak{g})$.

With a parallel approach, a *polynomial* version of Reshetikhin’s MpQUEAs was introduced and studied by Costantini–Varagnolo: see [14, 15], and also [27]; on the other hand, these works do not consider deformations. Alternatively, using the duality with quantum coordinate algebras, two-parameters quantum enveloping algebras of polynomial type are considered in Dobrev–Parashar [16] and in Dobrev–Tahri [17]. The effect of the twist can be seen in the description of the coproduct after a change the presentation à la Drinfeld–Jimbo type.

In another direction, a different version of *polynomial* MpQUEA (still working over \mathfrak{g} as above), call it $U_{\mathbf{q}}(\mathfrak{g})$, has been developed in the works of Andruskiewitsch–Schneider, Rosso, and many others (see for instance [1, 2, 46, 50]). In this case, the “multiparameter” is cast into a matrix $\mathbf{q} = (q_{ij})_{i,j \in I}$ whose entries take part in the description of the *algebra* structure of $U_{\mathbf{q}}(\mathfrak{g})$. Under mild, additional conditions, this yields a very general family of MpQUEAs which is very well-behaved: in particular, it is stable under deformations by 2-cocycles of “toral” type. Even better, this family is “homogeneous”, in that each $U_{\mathbf{q}}(\mathfrak{g})$ is a 2-cocycle deformation of Jimbo–Lusztig’s polynomial version $U_q(\mathfrak{g})$ of Drinfeld’s $U_{\hbar}(\mathfrak{g})$.

Note that, in Hopf theory, *twist* and *2-cocycle* are notions dual to each other. Thus the constructions of MpQUEAs by Reshetikhin and by Andruskiewitsch–Schneider (besides the difference in being “formal” or “polynomial”) are somehow *dual* to each other, and, as such, seem definitely different from each other.

The purpose of this paper is to introduce a new notion of MpQUEA that encompass both Reshtikhin’s one and Andruskiewitsch–Schneider’s one. Indeed, we achieve this goal introducing a new family of MpQUEAs which incorporates Andruskiewitsch–Schneider’s one, hence in particular it includes Drinfeld’s standard example (see Definition 4.4, Theorem 4.13 and Section 4.5). We show that this new family is stable by toral 2-cocycle deformations (Theorem 5.13), just as Andruskiewitsch–Schneider’s, *and* it is also stable by toral twist deformations (Theorem 5.2), hence it incorporates Reshetikhin’s family as well. In particular, we show that every MpQUEA of the Reshetikhin’s family is actually isomorphic to one of the Andruskiewitsch–Schneider’s family, and viceversa: the isomorphism is especially meaningful in itself, in that it amounts to a suitable change of

presentation via a well-focused change of generators (see Theorem 5.2). In this sense, we really end up with a single, *homogeneous* family – not just a collage of two distinct families; this can be seen as a byproduct of the intrinsic “self-duality” of Drinfeld’s standard $U_{\hbar}(\mathfrak{g})$.

For each one of these MpQUEAs, then, one can decide to focus the dependence on the *discrete* multiparameters either on the coalgebra structure (which amounts to adopt Reshetikhin’s point of view) or on the algebra structure (thus following Andruskiewitsch–Schneider’s approach). In our definition we choose to adopt the latter point of view, as it is definitely closer to the classical Serre’s presentation of $U(\mathfrak{g})$ – or even to the presentation of Drinfeld’s standard $U_{\hbar}(\mathfrak{g})$ – where the *discrete* multiparameters given by the Cartan matrix entries rule the algebra structure.

Technically speaking, we adopt the setting and language of *formal* quantum groups, thus our newly minted objects are “formal MpQUEAs”, in short “FoMpQUEAs”. This is indeed a necessary option: in fact, the setup of polynomial MpQUEAs is well-suited when one deals with (toral) 2-cocycle deformations, but behaves quite poorly under deformations by (toral) twists. Roughly speaking, the toral part in a polynomial MpQUEA (in the sense of Andruskiewitsch–Schneider, say) happens to be too rigid, in general, under twist deformations; this is shown in our previous paper [24], where we pursued the same goal by means of “polynomial MpQUEAs”, which eventually prove to be a somewhat less suitable tool.

Thus, one needs to allow a more flexible notion of “toral part” in our would-be MpQUEA in order to get a notion that is stable under deformation by (toral) twists. We obtain this by choosing to define our *formal* MpQUEA as having a toral part with two distinguished sets of “coroots” and “roots”, whose mutual interaction is encrypted in a “multiparameter matrix” P whose role generalizes that of the Cartan matrix. We formalize all this via the notion of *realization* of the matrix P , which is a natural extension of Kac’ notion of realization of a generalized Cartan matrix (cf. Definition 2.1); our FoMpQUEA then is defined much like Drinfeld’s standard one, with the entries of P playing the role of *discrete* multiparameters.

By looking at semiclassical limits, we find that our new class of FoMpQUEAs gives rise to a new family of multiparameter Lie bialgebras (in short MpLbA’s) that come equipped with a presentation “à la Serre” in which the parameters (i.e., the entries of P , again) rule the Lie algebra structure (cf. Section 3.2.3). Again, we prove that this family is stable by deformations, in Lie bialgebra theoretical sense, both via “toral” 2-cocycles

and via “toral” twists (see Theorem 3.17 and Theorem 3.12). In particular, every such MpLbA admits an alternative presentation in which the Lie algebra structure stands fixed (always being ruled by a fixed generalized Cartan matrix) while the Lie *coalgebra* structure does vary according to the multiparameter matrix P . Like in the quantum setup, the isomorphism between the two presentations is quite meaningful, as it boils down to a well-chosen change of generators (cf. Theorem 3.12). The very definition of these MpLbA’s, as well as the just mentioned results about them, can be deduced as byproducts of those for FoMpQUEAs (via the process of specialization); otherwise, they can be introduced and proved directly; in short, we do both (cf. Section 3 and Theorem 6.3). These MpLbA’s were possibly known in literature, at least in part: yet our construction yields a new, systematic presentation of their whole family in its full extent, also proving its stability under deformations by (toral) 2-cocycles and (toral) twists.

As a final, overall comment, we recall that a close relation between multiparameters and deformations is ubiquitous in several applications, e.g. in the classification of complex finite-dimensional pointed Hopf algebras over abelian groups [2, 3], where deformations by 2-cocycle play a central role. Moreover, MpQUEAs may also serve as interpolating objects in the study of the representation theory of quantum groups associated with Langlands dual semi-simple Hopf algebras [22], where deformations by twist instead are a key tool.

A last word about the organization of the paper.

In Section 2, we introduce the “combinatorial data” underlying our constructions of MpLbA’s and FoMpQUEAs alike: the notion of *realization* of a multiparameter matrix, and the process of deforming realizations either by twists or by 2-cocycles.

Section 3 is dedicated to introduce our MpLbA’s and study their deformations by (toral) twists and by (toral) 2-cocycles.

In Section 4 we introduce our newly minted FoMpQUEAs, in particular using different, independent approaches, and to prove their basic properties.

With Section 5 we discuss deformations of FoMpQUEAs by (toral) twists and by (toral) 2-cocycles: we prove that these deformations turn FoMpQUEAs into new FoMpQUEAs again, the case by twist being possibly the more surprising.

Finally, in Section 6 we perform specializations of FoMpQUEAs and look at their resulting semiclassical limit: we find that this limit is always a MpLbA (in short, by the very definition of MpLbA’s), with the same multiparameter matrix P as the FoMpQUEA it comes from. Conversely, any

possible MpLbA does arise as such a limit – in other words, any MpLbA has a FoMpQUEA which is quantization of it. Then – more important – we compare deformations (by toral twists or 2-cocycle) before and after specialization: the outcome is, in a nutshell, that “*specialization and deformation (of either type) commute with each other*” (cf. Theorem 6.4 and Theorem 6.4). In fact, this last result can be deduced also as a special instance of a more general one, which in turn is an outcome of a larger study about deformations (of either type) of formal quantum groups, i.e., Drinfeld’s-like QUEAs and their dual, the so-called QFSHA’s, and of their semiclassical limits. This is a more general chapter in quantum group theory, with its own reasons of interest, thus we shall treat it in a separate publication (cf. [26]).

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2. Multiparameters and their realizations

In this section we fix the basic combinatorial data that we need later on. The definition of our multiparameter Lie bialgebras and formal multiparameter quantum groups requires a full lot of related material that we now present. In particular, $\mathbb{N} = \{0, 1, \dots\}$ and $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$, while \mathbb{k} will be a field of characteristic zero.

2.1. Multiparameter matrices, Cartan data, and realizations

We introduce hereafter the “multiparameters”, which we will use to construct (semi)classical and quantum objects as well. The theory can be developed more in general, but we stick to the case of “Cartan type” as more relevant to us; accordingly, this will keep us close to the common setup of Lie algebras of Kac–Moody type, in particular those whose Cartan matrix is symmetrisable.

2.1.1. Cartan data and associated Lie algebras

Hereafter we fix $n \in \mathbb{N}_+$ and $I := \{1, \dots, n\}$. Let $A := (a_{ij})_{i,j \in I}$ be a generalized, symmetrisable Cartan matrix; then there exists a unique diagonal matrix $D := (d_i \delta_{ij})_{i,j \in I}$ with positive integral, pairwise coprime entries such that DA is symmetric. Let $\mathfrak{g} = \mathfrak{g}_A$ be the Kac–Moody algebra over \mathbb{C} associated with A (cf. [35]); we consider a split integral \mathbb{Z} -form of \mathfrak{g} , and for the latter the scalar extension from \mathbb{Z} to any field \mathbb{k} : by abuse of notation, *the resulting Lie algebra over \mathbb{k} will be denoted by \mathfrak{g} again.*

Let Φ be the root system of \mathfrak{g} , with $\Pi = \{\alpha_i | i \in I\}$ as a set of simple roots, $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ the associated root lattice, Φ^+ the set of positive roots with respect to Π , $Q^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i$ the positive root (semi)lattice.

Fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} , whose associated set of roots identifies with Φ (so $\mathbb{k}Q \subseteq \mathfrak{h}^*$); then for all $\alpha \in \Phi$ we call \mathfrak{g}_α the corresponding root space. Now set $\mathfrak{h}' := \mathfrak{g}' \cap \mathfrak{h}$ where $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ is the derived Lie subalgebra of \mathfrak{g} : then $(\mathfrak{h}')^* = \mathbb{k}Q \subseteq \mathfrak{h}^*$. We fix a \mathbb{k} -basis $\Pi^\vee := \{h_i := \alpha_i^\vee\}_{i \in I}$ of \mathfrak{h}' so that $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a *realization* of A , as in [35, Chapter 1]; in particular, $\alpha_i(h_j) = a_{ji}$ for all $i, j \in I$.

Let \mathfrak{h}'' be any vector space complement of \mathfrak{h}' inside \mathfrak{h} . Then there exists a unique symmetric \mathbb{k} -bilinear pairing on \mathfrak{h} , denoted $(\ , \)$, such that $(h_i, h_j) = a_{ij}d_j^{-1}$, $(h_i, h_2'') = \alpha_i(h_2'')$ and $(h_1'', h_2'') = 0$, for all $i, j \in I$, $h_1'', h_2'' \in \mathfrak{h}''$; in addition, this pairing is invariant and non-degenerate (cf. [35, Chapter 2]). By non-degeneracy, this pairing induces a \mathbb{k} -linear isomorphism $t : \mathfrak{h}^* \xrightarrow{\cong} \mathfrak{h}$, and this in turn defines a similar pairing on \mathfrak{h}^* , again denoted $(\ , \)$, via pull-back, namely $(t^{-1}(h_1), t^{-1}(h_2)) := (h_1, h_2)$; in particular, on simple roots this gives $(\alpha_i, \alpha_j) := d_i a_{ij}$ for all $i, j \in I$. In fact, this pairing on \mathfrak{h}^* restricts to a (symmetric, \mathbb{Z} -valued, \mathbb{Z} -bilinear) pairing on Q ; note that, in terms of the latter pairing on Q , one has $d_i = (\alpha_i, \alpha_i)/2$ and $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$ ($i, j \in I$). Moreover $t : \mathfrak{h}^* \xrightarrow{\cong} \mathfrak{h}$ restricts to another isomorphism $t' : (\mathfrak{h}')^* \xrightarrow{\cong} \mathfrak{h}'$ for which we will use notation $t_\alpha := t'(\alpha) = t(\alpha)$.

Let \mathfrak{n}_+ , resp. \mathfrak{n}_- , be the nilpotent subalgebra in \mathfrak{g} containing all positive, resp. negative, root spaces, and set $\mathfrak{b}_\pm := \mathfrak{h} \oplus \mathfrak{n}_\pm$ be the corresponding Borel subalgebras. There is a canonical, non-degenerate pairing between \mathfrak{b}_+ and \mathfrak{b}_- , using which one can construct a *Manin double* $\mathfrak{g}^D = \mathfrak{b}_+ \oplus \mathfrak{b}_-$, automatically endowed with a structure of Lie bialgebra – roughly, \mathfrak{g}^D is like \mathfrak{g} but with *two copies* of \mathfrak{h} inside it (cf. [11], Section 1.4), namely $\mathfrak{h}_+ := \mathfrak{h} \oplus 0$ inside \mathfrak{b}_+ and $\mathfrak{h}_- := 0 \oplus \mathfrak{h}$ inside \mathfrak{b}_- ; accordingly, we set $\mathfrak{h}'_+ := \mathfrak{h}' \oplus 0$ and $\mathfrak{h}'_- := 0 \oplus \mathfrak{h}'$. By construction both \mathfrak{b}_+ and \mathfrak{b}_- lie in \mathfrak{g}^D as Lie sub-bialgebras.

Moreover, there exists a Lie bialgebra epimorphism $\pi_{\mathfrak{g}^D} : \mathfrak{g}^D \rightarrow \mathfrak{g}$ which maps the copy of \mathfrak{b}_{\pm} inside \mathfrak{g}^D identically onto its copy in \mathfrak{g} .

For later use we fix generators $e_i, h_i, f_i (i \in I)$ in \mathfrak{g} as in the usual Serre's presentation of \mathfrak{g} . Moreover, for the corresponding elements inside $\mathfrak{g}^D = \mathfrak{b}_+ \oplus \mathfrak{b}_-$ we adopt notation $e_i := (e_i, 0)$, $h_i^+ := (h_i, 0)$, $h_i^- := (0, h_i)$ and $f_i := (0, f_i)$, for all $i \in I$. Notice that we have by construction

$$e_i \in \mathfrak{g}_{+\alpha_i}, \quad h_i = d_i^{-1} t_{\alpha_i} \in \mathfrak{h}, \quad f_i \in \mathfrak{g}_{-\alpha_i} \quad \forall i \in I.$$

In sight of applications to Lie theory, we introduce, mimicking [35, Chapter 1], the notion of *realization* of a multiparameter matrix:

DEFINITION 2.1. — *Let \hbar be a formal variable, and $\mathbb{k}[[\hbar]]$ the ring of formal power series in \hbar with coefficients in \mathbb{k} . Let \mathfrak{h} be a free $\mathbb{k}[[\hbar]]$ -module of finite rank, and pick subsets $\Pi^{\vee} := \{T_i^+, T_i^-\}_{i \in I} \subseteq \mathfrak{h}$, and $\Pi := \{\alpha_i\}_{i \in I} \subseteq \mathfrak{h}^* := \text{Hom}_{\mathbb{k}[[\hbar]]}(\mathfrak{h}, \mathbb{k}[[\hbar]])$. For later use, we also introduce the elements $S_i := 2^{-1}(T_i^+ + T_i^-)$ and $\Lambda_i := 2^{-1}(T_i^+ - T_i^-)$ (for $i \in I$) and the sets $\Sigma := \{S_i\}_{i \in I} \subseteq \mathfrak{h}$ and $\Lambda := \{\Lambda_i\}_{i \in I} \subseteq \mathfrak{h}$.*

Let $P \in M_n(\mathbb{k}[[\hbar]])$ be any $(n \times n)$ -matrix with entries in $\mathbb{k}[[\hbar]]$.

- (a) We call the triple $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^{\vee})$ a realization of P over $\mathbb{k}[[\hbar]]$, with rank defined as $\text{rk}(\mathcal{R}) := \text{rk}_{\mathbb{k}[[\hbar]]}(\mathfrak{h})$, if:
- (a.1) $\alpha_j(T_i^+) = p_{ij}$, $\alpha_j(T_i^-) = p_{ji}$, for all $i, j \in I$;
 - (a.2) the set $\bar{\Sigma} := \{\bar{S}_i := S_i \pmod{\hbar\mathfrak{h}}\}_{i \in I}$ is \mathbb{k} -linearly independent in $\bar{\mathfrak{h}} := \mathfrak{h}/\hbar\mathfrak{h}$.

N.B. — This is equivalent to saying that Σ itself can be completed to a $\mathbb{k}[[\hbar]]$ -basis of \mathfrak{h} , hence in particular Σ is $\mathbb{k}[[\hbar]]$ -linearly independent in \mathfrak{h} .

- (b) We call a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^{\vee})$ of the matrix P
- (b.1) straight if the set $\bar{\Pi} := \{\bar{\alpha}_i := \alpha_i \pmod{\hbar\mathfrak{h}^*}\}_{i \in I}$ is \mathbb{k} -linearly independent in $\bar{\mathfrak{h}}^* := \mathfrak{h}^*/\hbar\mathfrak{h}^*$.

N.B. — This is equivalent to saying that Π can be completed to a $\mathbb{k}[[\hbar]]$ -basis of \mathfrak{h}^* , thus in particular Π is $\mathbb{k}[[\hbar]]$ -linearly independent in \mathfrak{h}^* ;

- (b.2) small if $\text{Span}_{\mathbb{k}[[\hbar]]}(\{S_i\}_{i \in I}) = \text{Span}_{\mathbb{k}[[\hbar]]}(\{T_i^+, T_i^-\}_{i \in I})$;
- (b.3) split if the set $\bar{\Pi}^{\vee} := \{\bar{T}^{\pm}_i := T_i^{\pm} \pmod{\hbar\mathfrak{h}}\}_{i \in I}$ is \mathbb{k} -linearly independent in $\bar{\mathfrak{h}} := \mathfrak{h}/\hbar\mathfrak{h}$.

N.B. — This is equivalent to saying that Π^{\vee} can be completed to a $\mathbb{k}[[\hbar]]$ -basis of \mathfrak{h} , hence in particular it is $\mathbb{k}[[\hbar]]$ -linearly independent in \mathfrak{h} ;

(b.4) minimal if $\text{Span}_{\mathbb{k}[[\hbar]]}(\{T_i^+, T_i^-\}_{i \in I}) = \mathfrak{h}$.

N.B. — In particular, the realization \mathcal{R} is *split* and *minimal* if and only if $\{T_i^+, T_i^-\}_{i \in I}$ is a $\mathbb{k}[[\hbar]]$ -basis of \mathfrak{h} .

(c) For any pair of realizations $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ and $\dot{\mathcal{R}} := (\dot{\mathfrak{h}}, \dot{\Pi}, \dot{\Pi}^\vee)$ of the same matrix P , a (homo)morphism $\underline{\phi} : \mathcal{R} \rightarrow \dot{\mathcal{R}}$ is the datum of any $\mathbb{k}[[\hbar]]$ -module morphism $\phi : \mathfrak{h} \rightarrow \dot{\mathfrak{h}}$ such that $\phi(T_i^\pm) = \dot{T}_{\sigma(i)}^\pm$ (for all $i \in I$) for some permutation $\sigma \in \mathbb{S}_I$ – the symmetric group over I – hence, in particular, $\phi(\Pi^\vee) = \dot{\Pi}^\vee$, and also that $\phi^*(\dot{\Pi}) = \Pi$.

N.B. — Realizations along with their morphisms form a category, in which the iso-/epi-/mono-morphisms are those morphisms ϕ as above that actually are $\mathbb{k}[[\hbar]]$ -module iso-/epi-/mono-morphisms.

(d) Let $A := (a_{ij})_{i,j \in I} \in M_n(\mathbb{k})$ be any symmetrisable generalized Cartan matrix, and $D := (d_i \delta_{ij})_{i,j \in I}$ the associated diagonal matrix, as in Section 2.1.1. We say that a matrix $P \in M_n(\mathbb{k}[[\hbar]])$ is of Cartan type with corresponding Cartan matrix A if $P_s := 2^{-1}(P + P^T) = DA$.

N.B. — Condition (b.3) is equivalent to requiring that $\bar{\Sigma} \cup \bar{\Lambda}$ be \mathbb{k} -linearly independent in $\bar{\mathfrak{h}} := \mathfrak{h}/\hbar\mathfrak{h}$; in turn, this is equivalent to saying that $\Sigma \cup \Lambda$ itself can be completed to a $\mathbb{k}[[\hbar]]$ -basis of \mathfrak{h} , hence in particular it is $\mathbb{k}[[\hbar]]$ -linearly independent. Similarly, the condition $\phi(T_i^\pm) = \dot{T}_{\sigma(i)}^\pm - i \in I$, for some permutation $\sigma \in \mathbb{S}(I)$ – in (c) can be replaced by $\phi(S_i) = \dot{S}_{\sigma(i)}$ and $\phi(A_i) = \dot{A}_{\sigma(i)}$.

(e) In an entirely similar way, one may define realizations of a matrix $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k})$ over a ground field \mathbb{k} . Such a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ consists of a \mathbb{k} -vector space \mathfrak{h} and distinguished subsets $\Pi^\vee := \{T_i^+, T_i^-\}_{i \in I} \subseteq \mathfrak{h}$ and $\Pi := \{\alpha_i\}_{i \in I} \subseteq \mathfrak{h}^* := \text{Hom}_{\mathbb{k}}(\mathfrak{h}, \mathbb{k})$: then condition (a.1) reads the same, while (a.2) instead says that the set $\{S_i = 2^{-1}(T_i^+ + T_i^-)\}_{i \in I}$ is linearly independent, and the rank of the realization is $\text{rk}(\mathcal{R}) := \dim_{\mathbb{k}}(\mathfrak{h})$. Also, \mathcal{R} is straight, resp. split, if Π , resp. Π^\vee , is linearly independent.

Basing on the context, we shall possibly stress the ring we are working over, namely $\mathbb{k}[[\hbar]]$ for $P \in M_n(\mathbb{k}[[\hbar]])$ and the field \mathbb{k} for $P \in M_n(\mathbb{k})$.

Remark 2.2. — In the present language, if $P = P^T$ is symmetric a realization of it in the sense of [35, Chapter 1, Section 1.1], is also a realization, in the sense of Definition 2.1, of P which has rank $2n - r$, is straight and small with $A_i = 0$ for all $i \in I$.

The following consequence of the definitions yields another link with Kac' notion of "realization":

LEMMA 2.3. — *Let $P \in M_n(\mathbb{k}[[\hbar]])$ be a matrix as above. If $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ is a straight realization of P , then the triple $(\mathfrak{h}, \Pi, \Pi_S^\vee)$ – with $\Pi_S^\vee := \{S_i\}_{i \in I}$ – is a realization of $P_s := 2^{-1}(P + P^T)$ – over the ring $\mathbb{k}[[\hbar]]$ – in the sense of [35, Chapter 1, Section 1.1], but for condition (1.1.3).*

Note that condition (1.1.3) in [35, Chapter 1, Section 1.1], is fulfilled whenever $\text{rk}(\mathfrak{h}) = 2n - \text{rk}(P_s)$; in particular, we can always achieve that condition up to suitably enlarging or restricting \mathfrak{h} . In any case, from now on with any straight realization of a matrix P of Cartan type, for some Cartan matrix A , we shall always associate the realization of $P_s = DA$ given by Lemma 2.3, hence also the corresponding realization of A and then all the related data and machinery mentioned in Section 2.1.

We need now a few technical results:

PROPOSITION 2.4.

- (a) *For every $P \in M_n(\mathbb{k}[[\hbar]])$ and every $\ell \geq 3n - \text{rk}(P + P^T)$, there exists a straight split realization of P with $\text{rk}(\mathfrak{h}) = \ell$, which is unique up to isomorphisms.*
- (b) *Claim (a) still holds true if we drop the condition "straight" and pick $\ell \geq 2n$.*

Proof.

- (a). — Let $r := \text{rk}(P + P^T)$ and $\ell \geq 3n - r$ be fixed. We set

$$S_i := 2^{-1}(T_i^+ + T_i^-), \quad A_i := 2^{-1}(T_i^+ - T_i^-) \quad \forall i \in I$$

for any choice of elements $T_i^\pm (i \in I)$ in any $\mathbb{k}[[\hbar]]$ -module \mathfrak{h} ; then $T_i^\pm = S_i \pm A_i$ for all $i \in I$, so we have

$$\text{Span}_{\mathbb{k}[[\hbar]]} \left(\{T_i^+, T_i^-\}_{i \in I} \right) = \text{Span}_{\mathbb{k}[[\hbar]]} \left(\{S_i, A_i\}_{i \in I} \right).$$

Therefore, the existence of a split realization of P amounts to the same as the existence of the slightly modified notion where:

- instead of the T_i^+ 's and the T_i^- 's one considers the S_i 's and the A_i 's,
 - condition (a.1) in Definition 2.1 is replaced by condition (a.1+)
- $$(a.1+) \quad \alpha_j(S_i) = 2^{-1}(p_{ij} + p_{ji}), \quad \alpha_j(A_i) = 2^{-1}(p_{ij} - p_{ji}) \quad \forall i, j \in I;$$

Therefore, we look now for such a "realization" in this alternative sense. We consider the matrices (respectively symmetric and antisymmetric)

$$P_s := 2^{-1}(P + P^T), \quad P_a := 2^{-1}(P - P^T)$$

and then, reordering the indices in I if necessary, we re-write the matrix P_s in the block form $P_s = \begin{pmatrix} P_s^\top & P_s^\top \\ P_s^\perp & P_s^\perp \end{pmatrix}$ where P_s^\top has size $r \times r$, $\text{rk}(P_s^\top) = r$, and the other blocks have the corresponding sizes; according to the same reordering of the indices (if any), we also re-write P_a as $P_a = \begin{pmatrix} P_a^\top & P_a^\top \\ P_a^\perp & P_a^\perp \end{pmatrix}$ with P_a^\top of size $r \times r$, and so on.

Now we consider the $\ell \times \ell$ matrix

$$(2.1) \quad G_P = \begin{pmatrix} P_s^\top & P_s^\top & 0 & 0 & 0 & 0 \\ P_s^\perp & P_s^\perp & I_{n-r} & 0 & 0 & 0 \\ 0 & I_{n-r} & 0 & 0 & 0 & 0 \\ P_a^\top & P_a^\top & 0 & I_r & 0 & 0 \\ P_a^\perp & P_a^\perp & 0 & 0 & I_{n-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\ell-(3n-r)} \end{pmatrix}$$

that is non-degenerate, as $\det(G_P) = \pm \det(P_s^\top) \neq 0$. Now, set $\mathfrak{h}_\bullet := \mathbb{k}[[\hbar]]^{3n-r}$, fix as S_i 's, respectively Λ_i 's ($i \in I$), the rows of G_P (as vectors in \mathfrak{h}_\bullet) from 1 to n , respectively from $2n - r + 1$ to $3n - r$, and fix as α_j 's ($j \in I$) the first n linear coordinate functions on \mathfrak{h}_\bullet (as vectors in \mathfrak{h}_\bullet^*). Now set $\Pi_{S,\Lambda}^\vee := \{S_i, \Lambda_i\}_{i \in I}$ and let \mathfrak{h} be the $\mathbb{k}[[\hbar]]$ -span (inside \mathfrak{h}_\bullet) of the rows of G_P ; then the α_j 's restrict to elements of \mathfrak{h}^* (that we still denote by α_j) hence we consider $\Pi := \{\alpha_j\}_{j \in I}$ as a subset in \mathfrak{h}^* . Now the triple $\mathcal{R}_{S,\Lambda} := (\mathfrak{h}, \Pi, \Pi_{S,\Lambda}^\vee)$ is a "realization" (in the present, modified sense) of P which is straight split, thus proving the existence part of claim (a).

As to uniqueness, we reverse the previous line of arguing. Indeed, given a split "realization", in modified sense, $\mathcal{R}_{S,\Lambda} := (\mathfrak{h}, \Pi, \Pi_{S,\Lambda}^\vee)$ of P , we complete $\Pi_{S,\Lambda}^\vee$ to a $\mathbb{k}[[\hbar]]$ -basis of \mathfrak{h} adding extra elements $Y_1, \dots, Y_{\ell-2n}$; moreover, we define additional $\alpha_{n+1}, \dots, \alpha_\ell \in \mathfrak{h}^*$ such that the matrix of all values of the α_j 's on the elements of the ordered basis

$$\{S_1, \dots, S_n, Y_1, \dots, Y_{n-r}, \Lambda_1, \dots, \Lambda_n, Y_{n-r+1}, \dots, Y_{\ell-2n}\}$$

is given by

$$(2.2) \quad N_P = \begin{pmatrix} P_s^\top & P_s^\top & 0 & 0 & 0 & 0 \\ P_s^\perp & P_s^\perp & I_{n-r} & 0 & 0 & 0 \\ B_{<} & B_{>} & 0 & 0 & 0 & 0 \\ P_a^\top & P_a^\top & 0 & I_r & 0 & 0 \\ P_a^\perp & P_a^\perp & 0 & 0 & I_{n-r} & 0 \\ D_{<} & D_{>} & 0 & 0 & 0 & I_{\ell-(3n-r)} \end{pmatrix}$$

which by construction is non-degenerate. Now, let us extend scalars for a while from $\mathbb{k}[[\hbar]]$ to $\mathbb{k}((\hbar))$: then by Gauss' elimination algorithm on the rows (involving in particular the first r rows) we can modify the matrix

N_P in (2.2) till it gets a new form where $B_{<} = 0$ and $D_{<} = 0$; moreover, the “new” submatrix $B_{>}$ fulfills

$$\det(P_s^\top) \det(B_{>}) = \pm \det(N_P) \neq 0, \quad \text{hence } B_{>} \in GL_{n-r}(\mathbb{k}(\!(\hbar)\!))$$

hence we can choose another basis in $\text{Span}_{\mathbb{k}(\!(\hbar)\!)}(Y_1, \dots, Y_{n-r})$ so to get $B_{>} = I_{n-r}$. Then another Gauss’ elimination process involving the rows from $n+1$ to $2n-r$ allows us to modify the last $\ell - (3n-r)$ rows so as to get $D_{>} = 0$.

All this gives us a new split realization (in modified sense) of P over $\mathbb{k}(\!(\hbar)\!)$ and a specific basis, including the S_i ’s and the Λ_i ’s, of the $\mathbb{k}(\!(\hbar)\!)$ -vector space $\mathfrak{h}_{\mathbb{k}(\!(\hbar)\!)}$ considered in it: eventually, taking as \mathfrak{h} the $\mathbb{k}[[\hbar]]$ -span of that basis we can easily read off that “realization over $\mathbb{k}(\!(\hbar)\!)$ ” is a genuine realization over $\mathbb{k}[[\hbar]]$, which is isomorphic to the original one, by construction (indeed, we only modified a direct sum complement of $\text{Span}_{\mathbb{k}(\!(\hbar)\!)}(\{S_i, \Lambda_i\}_{i \in I})$ by a sheer rescaling, at most). But now, for this final realization the matrix N_P in (2.2) takes the same form as G_P in (2.1): so taking as $\phi : \mathfrak{h} \rightarrow \mathfrak{h}$ the isomorphism given by change of bases, we are eventually done.

(b). — As claim (a) already guarantees the existence of straight realizations, the relevant part of claim (b) concerns the uniqueness, that is proved again like for (a), up to minimal changes. Namely, instead of the matrix in (2.2) we deal with

$$N'_P = \begin{pmatrix} P_s^\top & P_s^\top & 0 & 0 & 0 & 0 \\ P_s^\top & P_s^\top & I_{n-r} & 0 & 0 & 0 \\ P_a^\top & P_a^\top & 0 & I_r & 0 & 0 \\ P_a^\top & P_a^\top & 0 & 0 & I_{n-r} & 0 \\ D_{<} & D_{>} & 0 & 0 & 0 & I_{\ell-2n} \end{pmatrix}$$

and then we observe that we can again reduce it to a similar matrix where $D_{<} = 0$ – acting by Gauss’ elimination on the rows, exploiting the nonsingular square submatrix P_s^\top – and $D_{>} = 0$ – where we perform another Gauss’ elimination *on the columns* (which in the end amounts to modifying the α_j ’s), exploiting the nonsingular square submatrix $I_{\ell-2n}$. \square

Remark 2.5. — It follows from definitions that a *necessary* condition for a small realization of any $P \in M_n(\mathbb{k}[[\hbar]])$ to exist is $\text{rk}(P_s \mid P_a) = \text{rk}(P_s)$. Conversely, with much the same arguments used in the proof of Proposition 2.4, we can prove that such a condition is also *sufficient*, as the following holds true, indeed:

PROPOSITION 2.6. — *If $P \in M_n(\mathbb{k}[[\hbar]])$ is such that $\text{rk}(P_s \mid P_a) = \text{rk}(P_s)$, then, for all $\ell \geq 2n - \text{rk}(P_s)$, there exists a straight small realization of P with $\text{rk}(\mathfrak{h}) = \ell$, and such a realization is unique up to isomorphisms.*

After this existence results concerning realizations of special type, we can achieve a more general result with two additional steps. The first one tells in short that every realization can be “lifted” to a *split* one:

LEMMA 2.7. — *Let $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of $P \in M_n(\mathbb{k}[[\hbar]])$. Then there exists a split realization $\dot{\mathcal{R}} := (\dot{\mathfrak{h}}, \dot{\Pi}, \dot{\Pi}^\vee)$ of the same matrix P and an epimorphism of realizations $\underline{\pi} : \dot{\mathcal{R}} \rightarrow \mathcal{R}$ such that, if $\mathfrak{h}_T := \text{Span}(\{T_i^\pm\}_{i \in I})$ and $\dot{\mathfrak{h}}_T := \text{Span}(\{\dot{T}_i^\pm\}_{i \in I})$, then $\underline{\pi}$ induces an isomorphism $\pi_* : \dot{\mathfrak{h}}/\dot{\mathfrak{h}}_T \cong \mathfrak{h}/\mathfrak{h}_T$.*

If in addition \mathcal{R} is straight, resp. minimal, then a split realization $\dot{\mathcal{R}}$ as above can be found that is straight, resp. minimal, as well.

Proof. — We proceed in two steps, first working over scalar extensions from $\mathbb{k}[[\hbar]]$ to $\mathbb{k}((\hbar))$ and then “pulling back” our result to the original setup. To this end, hereafter, for any $\mathbb{k}[[\hbar]]$ -module \mathfrak{m} we write $\mathfrak{m}^{(\hbar)} := \mathbb{k}((\hbar)) \otimes_{\mathbb{k}[[\hbar]]} \mathfrak{m}$.

Let $\mathfrak{h}_T := \text{Span}(\{T_i^+, T_i^-\}_{i \in I})$. Then $\mathfrak{h}_T^{(\hbar)}$ embeds into $\mathfrak{h}^{(\hbar)}$ and the latter splits into $\mathfrak{h}^{(\hbar)} = \mathfrak{h}_T^{(\hbar)} \oplus \mathfrak{h}'_o$ for some $\mathbb{k}((\hbar))$ -submodule \mathfrak{h}'_o in $\mathfrak{h}^{(\hbar)}$. Now fix formal vectors $\dot{T}_i^\pm (i \in I)$, the free $\mathbb{k}((\hbar))$ -module \mathfrak{h}'_T with $\mathbb{k}((\hbar))$ -basis $\Pi'^\vee := \{\dot{T}_i^+, \dot{T}_i^-\}_{i \in I}$, and the $\mathbb{k}((\hbar))$ -module epimorphism $\pi' : \mathfrak{h}'_\oplus := \mathfrak{h}'_T \oplus \mathfrak{h}'_o \rightarrow \mathfrak{h}^{(\hbar)}$ given by $\pi'(\dot{T}_i^\pm) := T_i^\pm (i \in I)$ and $\pi'|_{\mathfrak{h}'_o} := id_{\mathfrak{h}'_o}$. If we let $\alpha_j^{(\hbar)}$ be the natural scalar extension of $\alpha_j (j \in I)$, then every such $\alpha_j^{(\hbar)}$ is a $\mathbb{k}((\hbar))$ -linear function from $\mathfrak{h}^{(\hbar)}$ to $\mathbb{k}((\hbar))$, and altogether the $\alpha_j^{(\hbar)}$'s are linearly independent over $\mathbb{k}((\hbar))$ if the α_j 's are; therefore, the set $\Pi' := \{\alpha'_j := \alpha_j^{(\hbar)} \circ \pi'\}_{j \in I}$ lies inside the $\mathbb{k}((\hbar))$ -dual module of \mathfrak{h}'_\oplus , and it is also $\mathbb{k}((\hbar))$ -linearly independent if Π is – in other words, if we are in the *straight* case.

Now look at \mathfrak{h} embedded into $\mathfrak{h}^{(\hbar)}$ and set

$$\begin{aligned} \dot{\mathfrak{h}} &:= (\pi')^{-1}(\mathfrak{h}), & \dot{\Pi}^\vee &:= (\pi'|_{\dot{\mathfrak{h}}})^{-1}(\Pi^\vee) = \Pi'^\vee \\ \dot{\Pi} &:= (\pi'|_{\dot{\mathfrak{h}}})^*(\Pi) = \left\{ (\pi'|_{\dot{\mathfrak{h}}})^*(\alpha_j^{(\hbar)}) := \alpha_j^{(\hbar)} \circ \pi'|_{\dot{\mathfrak{h}}} = \alpha'_j|_{\dot{\mathfrak{h}}} \right\}_{j \in I}. \end{aligned}$$

Then $\dot{\mathfrak{h}}$ is a free $\mathbb{k}[[\hbar]]$ -module containing $\dot{\Pi}^\vee$ and such that $\dot{\mathfrak{h}}^{(\hbar)} = \mathfrak{h}'_\oplus$, while $\dot{\Pi}$ is a subset in the $\mathbb{k}[[\hbar]]$ -dual of $\dot{\mathfrak{h}}$, that is even $\mathbb{k}[[\hbar]]$ -linearly independent if Π is – i.e., if we are in the *straight* case. Even more, looking in depth we find that $\dot{\mathcal{R}} := (\dot{\mathfrak{h}}, \dot{\Pi}, \dot{\Pi}^\vee)$ is indeed a *split* realization of the matrix P – which is also *straight*, resp. *minimal* if such is the original – that together with the epimorphism $\pi := \pi'|_{\dot{\mathfrak{h}}} : \dot{\mathfrak{h}} \rightarrow \mathfrak{h}$ yields all that is prescribed in

the claim. Indeed, we only have to point out the last step, noting that π induces an isomorphism $\pi_* : \dot{\mathfrak{h}}/\pi^{-1}(\mathfrak{h}_T) \cong \mathfrak{h}/\mathfrak{h}_T$ and then observing that, by construction, we have $\pi^{-1}(\mathfrak{h}_T) = \dot{\mathfrak{h}}_T$. \square

A last result concerns morphisms between realizations.

LEMMA 2.8. — *Let $\widehat{\mathcal{R}} := (\widehat{\mathfrak{h}}, \widehat{\Pi}, \widehat{\Pi}^\vee)$ and $\check{\mathcal{R}} := (\check{\mathfrak{h}}, \check{\Pi}, \check{\Pi}^\vee)$ be two realizations of a same $P \in M_n(\mathbb{k}[[\hbar]])$, and let $\underline{\phi} : \widehat{\mathcal{R}} \rightarrow \check{\mathcal{R}}$ be a morphism between them. Then $\text{Ker}(\phi : \widehat{\mathfrak{h}} \rightarrow \check{\mathfrak{h}}) \subseteq \bigcap_{j \in I} \text{Ker}(\widehat{\alpha}_j)$.*

Proof. — Since $\widehat{\alpha}_j = \phi^*(\check{\alpha}_j) = \check{\alpha}_j \circ \phi$ ($j \in I$) by assumption, for all $k \in \text{Ker}(\phi)$ we have $\widehat{\alpha}_j(k) = (\check{\alpha}_j \circ \phi)(k) = \check{\alpha}_j(\phi(k)) = \check{\alpha}_j(0) = 0$ ($j \in I$), whence the claim. \square

Remark 2.9. — Working with matrices in $M_n(\mathbb{k})$ and realizations of them over \mathbb{k} (for some field \mathbb{k}), all the previous constructions still make sense; some results (e.g., Proposition 2.4 and Lemma 2.7) even get stronger and/or have simpler proofs.

2.2. Twist deformations of multiparameters and realizations

In this subsection we introduce the notion of *deformation by twist* of realizations, which will be needed later when dealing with deformations of multiparameter Lie bialgebras and formal multiparameter quantum universal enveloping algebras.

2.2.1. Deforming realizations (and matrices) by twist

Fix $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ and a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$, possibly (up to changing minimal details in what follows) over \mathbb{k} if $P \in M_n(\mathbb{k})$; in particular $d_i := p_{ii}/2$ for all $i \in I$.

Recall that \mathfrak{h} is, by assumption, a free $\mathbb{k}[[\hbar]]$ -module of finite rank $t := \text{rk}(\mathfrak{h})$. We fix in \mathfrak{h} any $\mathbb{k}[[\hbar]]$ -basis $\{H_g\}_{g \in \mathcal{G}}$, where \mathcal{G} is an index set with $|\mathcal{G}| = \text{rk}(\mathfrak{h}) = t$.

Fix an antisymmetric square matrix $\Phi = (\phi_{gk})_{g,k \in \mathcal{G}} \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ — indeed, we might work with any $\Phi \in M_t(\mathbb{k}[[\hbar]])$, but at some point we should single out its antisymmetric part $\Phi_a := 2^{-1}(\Phi - \Phi^T)$ which would be all that matters. We define the *twisted “distinguished toral elements”* (also shortly called “coroots”)

$$(2.3) \quad T_{\Phi, \ell}^\pm := T_\ell^\pm \pm \sum_{g,k=1}^t \alpha_\ell(H_g) \phi_{kg} H_k.$$

As a matter of notation, let $\mathbf{T} := \begin{pmatrix} T^+ \\ T^- \end{pmatrix}$ be the $(2n \times 1)$ -matrix given by the column vectors $\underline{T}^\pm = (T_i^\pm)_{i \in I}$. Similarly, let $\mathbf{T}_\Phi := \begin{pmatrix} T_\Phi^+ \\ T_\Phi^- \end{pmatrix}$ be the $(2n \times 1)$ -matrix given by the (superposed) two column vectors $\underline{T}_\Phi^\pm = (T_{\Phi,i}^\pm)_{i \in I}$, and \mathbf{H} the column vector $\mathbf{H} := (H_g)_{g \in \mathcal{G}}$. Moreover, denote by \mathfrak{A} the $(n \times t)$ -matrix with entries in $\mathbb{k}[[\hbar]]$ given by $\mathfrak{A} := (\alpha_\ell(H_g))_{\ell \in I}^{g \in \mathcal{G}}$, and set $\mathfrak{A}_\bullet := \begin{pmatrix} +\mathfrak{A} \\ -\mathfrak{A} \end{pmatrix}$, a matrix of size $(2n \times t)$.

Now, using matrix notation we have $\mathbf{T}_\Phi = \mathbf{T} - \mathfrak{A}_\bullet \Phi \mathbf{H}$. Eventually, define also

$$(2.4) \quad P_\Phi = (p_{i,j}^\Phi)_{i,j \in I} := P - \mathfrak{A} \Phi \mathfrak{A}^T.$$

Now, using the above notation, a direct computation yields

$$\begin{aligned} S_{\Phi,i} &:= 2^{-1}(T_{\Phi,i}^+ + T_{\Phi,i}^-) = 2^{-1}(T_i^+ + T_i^-) = S_i & \forall i \in I \\ \alpha_j(T_{\Phi,i}^+) &= p_{i,j}^\Phi, & \alpha_j(T_{\Phi,i}^-) &= p_{j,i}^\Phi & \forall i, j \in I \end{aligned}$$

so that the triple $\mathcal{R}_\Phi := (\mathfrak{h}_\Phi, \Pi_\Phi, \Pi_\Phi^\vee)$ with $\mathfrak{h}_\Phi := \mathfrak{h}$, $\Pi_\Phi := \Pi$ and $\Pi_\Phi^\vee := \{T_{\Phi,i}^+, T_{\Phi,i}^- \mid i \in I\}$, is a realization of the matrix $P_\Phi = (p_{i,j}^\Phi)_{i,j \in I}$, as in Definition 2.1; also, by construction \mathcal{R}_Φ is also *straight*, resp. *small*, if and only if such is \mathcal{R} . Moreover, P_Φ is the sum of P plus an antisymmetric matrix, so *the symmetric part of P_Φ is the same as P* , i.e. $(P_\Phi)_s = P_s$. In particular, if P is of Cartan type, then so is P_Φ , and they are associated with the same Cartan matrix. In short, we get:

PROPOSITION 2.10. — *With notation as above, the following holds true:*

- (a) *the matrix $P_\Phi := P - \mathfrak{A} \Phi \mathfrak{A}^T$ obeys $(P_\Phi)_s = P_s$; in particular, if P is of Cartan type, then so is P_Φ , and they are associated with the same Cartan matrix.*
- (b) *the triple*

$$\mathcal{R}_\Phi := (\mathfrak{h}_\Phi := \mathfrak{h}, \Pi_\Phi := \Pi, \Pi_\Phi^\vee := \{T_{\Phi,i}^+, T_{\Phi,i}^-\}_{i \in I})$$

is a realization of P_Φ ; moreover, \mathcal{R}_Φ is straight, resp. small, if and only if \mathcal{R} itself is straight, resp. small.

DEFINITION 2.11. — *The realization $\mathcal{R}_\Phi := (\mathfrak{h}, \Pi, \Pi_\Phi^\vee)$ of the matrix $P_\Phi = (p_{i,j}^\Phi)_{i,j \in I}$ is called a twist deformation (via Φ) of the realization $\mathcal{R} = (\mathfrak{h}, \Pi, \Pi^\vee)$ of P .*

Similarly, the matrix P_Φ is called a twist deformation of the matrix P .

Remarks 2.12.

- (a) Observe that, by the very definition of twisting one has that $(P_\Phi)_{\Phi'} = P_{\Phi+\Phi'}$, and $(\mathcal{R}_\Phi)_{\Phi'} = \mathcal{R}_{\Phi+\Phi'}$ for all $\Phi, \Phi' \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$. Therefore, the additive group $\mathfrak{so}_t(\mathbb{k}[[\hbar]])$ acts on the set of (multi-parameter) matrices of size $n := |I|$ with fixed symmetric part, as well as on the set of their realizations of (any) fixed rank. When two matrices $P, P' \in M_n(\mathbb{k}[[\hbar]])$ belong to the same orbit of this $\mathfrak{so}_t(\mathbb{k}[[\hbar]])$ -action we say that P and P' are twist equivalent.
- (b) It follows from Proposition 2.10(a) that if two multiparameter matrices P and P' are twist equivalent, then their symmetric part is the same, that is $P_s = P'_s$. Next result shows the converse holds true as well, up to taking the group $\mathfrak{so}_t(\mathbb{k}[[\hbar]])$ big enough, namely with $t \geq 3n - \text{rk}(P_s)$.

LEMMA 2.13. — *With notation as above, let $P, P' \in M_n(\mathbb{k}[[\hbar]])$, and consider the aforementioned action by twist on $M_n(\mathbb{k}[[\hbar]])$ by any additive group $\mathfrak{so}_t(\mathbb{k}[[\hbar]])$ with $t \geq 3n - \text{rk}(P_s)$. Then P and P' are twist equivalent if and only if $P_s = P'_s$.*

Proof. — We have to prove the “if” part of the statement, so we assume that $P_s = P'_s$ and we look for $\Phi \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ such that $P' = P_\Phi$, that is $P' = P - \mathfrak{A}\Phi\mathfrak{A}^T$. By assumption $P' = P + \Lambda$ with $\Lambda := P' - P$ antisymmetric, and we want

$$(2.5) \quad \Lambda = -\mathfrak{A}\Phi\mathfrak{A}^T$$

for some $\Phi \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ – in other words, we have to show that the equation (2.5) in the indeterminate Φ has a solution.

By Proposition 2.4(a), there exists a straight split realization of P , say $\mathcal{R} = (\mathfrak{h}, \Pi, \Pi^\vee)$, of rank t . By the straightness assumption, the α_ℓ 's are linearly independent in \mathfrak{h}^* , while the H_g 's form a basis of \mathfrak{h} , so the matrix $\mathfrak{A} := (\alpha_\ell(H_g))_{\ell \in I}^{g \in \mathcal{G}}$ has rank $|I| = n$; therefore, we can write it as a block matrix $\mathfrak{A} = (G \mid Q)M_\sigma$ where G, Q and M_σ are matrices of size $n \times n$, $n \times (t - n)$ and $t \times t$, respectively, and moreover G is invertible and M_σ is a permutation matrix. Then (2.5) reads

$$(2.6) \quad \Lambda = -(G \mid Q)M_\sigma\Phi M_\sigma^T \begin{pmatrix} G^T \\ Q^T \end{pmatrix}.$$

Now let us replace the indeterminate matrix Φ with $\Psi := M_\sigma\Phi M_\sigma^T$ and accordingly let us read (2.6) as an equation in Ψ , namely

$$(2.7) \quad \Lambda = -(G \mid Q)\Psi \begin{pmatrix} G^T \\ Q^T \end{pmatrix}$$

then writing the latter in block form as $\Psi = \begin{pmatrix} A|B \\ \hline C|D \end{pmatrix}$ where A, B, C and D has size $n \times n, n \times (t-n), (t-n) \times n$ and $(t-n) \times (t-n)$, respectively, we see at once that a possible solution for (2.7) is $\Psi = \begin{pmatrix} A|0 \\ \hline 0|0 \end{pmatrix}$ with $A := G^{-1} \Lambda G^{-t}$. Thus (2.7) has a solution, hence (2.5) has one too. \square

Remark 2.14. — A similar notion of twist-equivalence of matrices can be found in [1] and [50] for matrices corresponding to diagonal braidings.

Next result “upgrades” the previous one to the level of realizations.

PROPOSITION 2.15. — *Let P and P' be two matrices in $M_n(\mathbb{k}[[\hbar]])$ with the same symmetric part, i.e. such that $P_s = P'_s$.*

- (a) *Let \mathcal{R} be a straight realization of P of rank t . Then there exists a matrix $\Phi \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ such that $P' = P_\Phi$ and the corresponding realization \mathcal{R}_Φ is straight. In short, if $P'_s = P_s$ then from any straight realization of P we can obtain by twist deformation a straight realization (of the same rank) for P' , and viceversa.*
- (b) *Let \mathcal{R} and \mathcal{R}' be straight small realizations of P and P' , such that $\text{rk}(\mathcal{R}) = \text{rk}(\mathcal{R}') =: t$. Then there exists a matrix $\Phi \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ such that $\mathcal{R}' \cong \mathcal{R}_\Phi$. In short, if $P'_s = P_s$ then any straight small realization of P is isomorphic to a twist deformation of a straight small realization of P' of same rank, and viceversa.*
- (c) *Every straight small realization \mathcal{R} of P is isomorphic to some twist deformation of the standard realization of P_s of the same rank as \mathcal{R} , as in Lemma 2.3.*

Proof.

(a). — We can resume the same argument used in the proof of Lemma 2.13 to show that there exists a suitable $\Phi \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ such that $P' = P_\Phi$, the difference being only that now the starting point is the given realization \mathcal{R} of P . But then Proposition 2.10 ensures also that $\mathcal{R}_\Phi := (\mathfrak{h}, \Pi, \Pi_\Phi^\vee)$ is a realization of $P' = P_\Phi$, which is straight because \mathcal{R} itself is.

(b). — This follows from claim (a), along with the uniqueness (up to isomorphisms) of straight small realizations (cf. Proposition 2.4(a)).

(c). — This follows as an application of claim (b), taking $P' := P_s$ and $\mathcal{R}' := (\mathfrak{h}, \Pi, \Pi_S^\vee)$ – assuming $\mathcal{R} = (\mathfrak{h}, \Pi, \Pi^\vee)$ – as the standard Kac’ realization over $\mathbb{k}[[\hbar]]$ which is straight and small, see Remark 2.2 and Lemma 2.3. \square

2.2.2. Stability issues for twisted realizations

Keep notation as above; in particular, from Section 2.2 we consider $\underline{T}^\pm = (T_i^\pm)_{i \in I}$ and $\mathsf{T} := \begin{pmatrix} T^+ \\ T^- \end{pmatrix}$, we fix a vector of $\mathbb{k}[[\hbar]]$ -basis elements $\mathsf{H} := (H_g)_{g \in \mathcal{G}}$ for \mathfrak{h} and accordingly we set $\mathfrak{A} := (\alpha_\ell(H_g))_{\ell \in I}^{g \in \mathcal{G}}$ and $\mathfrak{A}_\bullet := \begin{pmatrix} +\mathfrak{A} \\ -\mathfrak{A} \end{pmatrix}$. Finally, given $\Phi \in \mathfrak{so}_\ell(\mathbb{k}[[\hbar]])$ we consider the new strings of “coroot vectors” $\underline{T}_\Phi^\pm = (T_{\Phi,i}^\pm)_{i \in I}$ and $\mathsf{T}_\Phi := \begin{pmatrix} T_\Phi^+ \\ T_\Phi^- \end{pmatrix}$ that are linked to the old coroot vectors – as in Section 2.2.1 – by the formulas

$$(2.8) \quad \underline{T}_\Phi^\pm := \underline{T}^\pm \mp \mathfrak{A}\Phi\mathsf{H}, \quad \mathsf{T}_\Phi = \mathsf{T} - \mathfrak{A}_\bullet\Phi\mathsf{H}.$$

Eventually, recall also the notation $P_\Phi = (p_{i,j}^\Phi)_{i,j \in I} := P - \mathfrak{A}\Phi\mathfrak{A}^T$.

From Proposition 2.10 we have that the class of (small) minimal realizations is stable under twist deformations. We look now instead at the *split* case.

Assume that the realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of P is *split*, i.e. the T_i^\pm 's are part of a $\mathbb{k}[[\hbar]]$ -basis of \mathfrak{h} . From (2.8) we see that we cannot give for granted the same property for the $T_{\Phi,i}^\pm$'s, hence we cannot say either that the realization $\mathcal{R}_\Phi := (\mathfrak{h}, \Pi, \Pi_\Phi^\vee)$ of P_Φ be split as well – in fact, all that depends on the matrix $\mathfrak{A}_\bullet\Phi$. We shall now discuss this issue in detail in a more restricted setting.

We assume now that \mathcal{R} is *split minimal*, so $\{T_i^+, T_i^-\}_{i \in I}$ is a $\mathbb{k}[[\hbar]]$ -basis of \mathfrak{h} (cf. Definition 2.1(d)). Again from Definition 2.1, let us consider the elements $S_i := 2^{-1}(T_i^+ + T_i^-)$ and $\Lambda_i := 2^{-1}(T_i^+ - T_i^-)$ – for all $i \in I$ – and similarly $S_{\Phi,i} := 2^{-1}(T_{\Phi,i}^+ + T_{\Phi,i}^-)$ and $\Lambda_{\Phi,i} := 2^{-1}(T_{\Phi,i}^+ - T_{\Phi,i}^-)$ – for all $i \in I$; set also $\underline{S} := 2^{-1}(\underline{T}^+ + \underline{T}^-)$ and $\underline{\Lambda} := 2^{-1}(\underline{T}^+ - \underline{T}^-)$, and similarly $\underline{S}_\Phi := 2^{-1}(\underline{T}_\Phi^+ + \underline{T}_\Phi^-)$ and $\underline{\Lambda}_\Phi := 2^{-1}(\underline{T}_\Phi^+ - \underline{T}_\Phi^-)$. In matrix terms, we have

$$\begin{pmatrix} \underline{S} \\ \underline{\Lambda} \end{pmatrix} = \begin{pmatrix} +2^{-1}I_n & +2^{-1}I_n \\ +2^{-1}I_n & -2^{-1}I_n \end{pmatrix} \begin{pmatrix} \underline{T}^+ \\ \underline{T}^- \end{pmatrix}$$

and

$$\begin{pmatrix} \underline{S}_\Phi \\ \underline{\Lambda}_\Phi \end{pmatrix} = \begin{pmatrix} +2^{-1}I_n & +2^{-1}I_n \\ +2^{-1}I_n & -2^{-1}I_n \end{pmatrix} \begin{pmatrix} \underline{T}_\Phi^+ \\ \underline{T}_\Phi^- \end{pmatrix}$$

and conversely also

$$\begin{pmatrix} \underline{T}^+ \\ \underline{T}^- \end{pmatrix} = \begin{pmatrix} +I_n & +I_n \\ +I_n & -I_n \end{pmatrix} \underline{S}\underline{\Lambda}, \quad \underline{T}_\Phi^+\underline{T}_\Phi^- = \begin{pmatrix} +I_n & +I_n \\ +I_n & -I_n \end{pmatrix} \underline{S}_\Phi\underline{\Lambda}_\Phi.$$

In particular we have $\text{Span}_{\mathbb{k}[[\hbar]]}(\{S_i, \Lambda_i\}_{i \in I}) = \text{Span}_{\mathbb{k}[[\hbar]]}(\{T_i^+, T_i^-\}_{i \in I})$ and similarly $\text{Span}_{\mathbb{k}[[\hbar]]}(\{S_{\Phi,i}, \Lambda_{\Phi,i}\}_{i \in I}) = \text{Span}_{\mathbb{k}[[\hbar]]}(\{T_{\Phi,i}^+, T_{\Phi,i}^-\}_{i \in I})$.

Now, with respect to the previous analysis we pick our fixed $\mathbb{k}[[\hbar]]$ -basis of \mathfrak{h} to be $\{H_g\}_{g \in \mathcal{G}} := \{T_i^+, T_i^-\}_{i \in I}$: then \mathfrak{A} reads as a block $(n \times 2n)$ -matrix $\mathfrak{A} = (P^T P)$, hence the first identity in (2.8) yields, via straightforward computations,

$$\underline{S}_\Phi = \underline{S}, \quad \underline{A}_\Phi = \underline{A} - (P^T P) \Phi \begin{pmatrix} +I_n & +I_n \\ +I_n & -I_n \end{pmatrix} \begin{pmatrix} \underline{S} \\ \underline{A} \end{pmatrix}$$

which in matrix terms reads

$$(2.9) \quad \begin{pmatrix} \underline{S}_\Phi \\ \underline{A}_\Phi \end{pmatrix} = \begin{pmatrix} I_n & 0_n \\ -B' & (I_n - B'') \end{pmatrix} \begin{pmatrix} \underline{S} \\ \underline{A} \end{pmatrix}$$

where B' and B'' are blocks in the matrix $(P^T P) \Phi \begin{pmatrix} +I_n & +I_n \\ +I_n & -I_n \end{pmatrix} = (B' B'')$, i.e. they are the $(n \times n)$ -matrices $B' := P^T (\Phi^{+,+} + \Phi^{+,-}) + P (\Phi^{-,+} + \Phi^{-,-})$ and $B'' := P^T (\Phi^{+,+} - \Phi^{+,-}) + P (\Phi^{-,+} - \Phi^{-,-})$, with notation as follows: we write Φ in block form $\Phi := \begin{pmatrix} \Phi^{++} & \Phi^{+-} \\ \Phi^{-+} & \Phi^{--} \end{pmatrix}$ with $\Phi^{\varepsilon_1, \varepsilon_2} = (\phi_{ij}^{\varepsilon_1, \varepsilon_2})_{i, j \in I}$ for all $\varepsilon_i, \varepsilon_j \in \{+, -\}$.

Now, it is clear that the set $\{T_{\Phi, i}^+, T_{\Phi, i}^-\}_{i \in I}$ is $\mathbb{k}[[\hbar]]$ -linearly independent if and only if

$$\text{Span}_{\mathbb{k}[[\hbar]]} \left(\{T_{\Phi, i}^+, T_{\Phi, i}^-\}_{i \in I} \right) = \text{Span}_{\mathbb{k}[[\hbar]]} \left(\{T_i^+, T_i^-\}_{i \in I} \right)$$

and the latter is true if and only if

$$\text{Span}_{\mathbb{k}[[\hbar]]} \left(\{S_{\Phi, i}, A_{\Phi, i}\}_{i \in I} \right) = \text{Span}_{\mathbb{k}[[\hbar]]} \left(\{S_i, A_i\}_{i \in I} \right).$$

But the latter holds true, by (2.9), if and only if the matrix $(I_n - B'')$ is invertible in $M_n(\mathbb{k}[[\hbar]])$. Finally, from the explicit form of B'' , we find the following criterion:

Assume that the realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of P be split minimal. If the matrix $M_P^\Phi := I_n - P^T (\Phi^{+,+} - \Phi^{+,-}) - P (\Phi^{-,+} - \Phi^{-,-})$ is invertible in $M_n(\mathbb{k}[[\hbar]])$, then the realization $\mathcal{R}_\Phi := (\mathfrak{h}, \Pi, \Pi_\Phi^\vee)$ of P_Φ is split minimal too (and viceversa).

As an outcome, this proves that the twist deformation of a split realization may be not split (counterexamples do exist, see below), hence the subclass of all split realizations is *not* stable under twist deformations.

Examples 2.16. — Note that M_P^Φ has the following matrix form

$$M_P^\Phi = I_n - (P^T P) \Phi \begin{pmatrix} +I_n \\ -I_n \end{pmatrix}.$$

Using this, we may find examples where M_P^Φ is invertible or not. For instance, we have:

- (a) if $\Phi = 0_{2n}$, then $M_P^\Phi = I_n$ is invertible;
- (b) if $(P^T P)\Phi \begin{pmatrix} +I_n \\ -I_n \end{pmatrix}$ is nilpotent, then M_P^Φ is clearly invertible;
- (c) Let $\Phi = \begin{pmatrix} 0 & \Phi \\ -\Phi^T & 0 \end{pmatrix}$ with $\Phi \in M_n(\mathbb{k}[[\hbar]])$. Then $M_P^\Phi = I_n + P^T \Phi + P\Phi^T$. If we take Φ antisymmetric, we find $M_P^\Phi = I_n - 2P_a\Phi$ where $P_a := 2^{-1}(P - P^T)$ is the antisymmetric part of P . Hence, by taking the canonical multiparameter $P = DA$, we get $M_P^\Phi = I_n$. On the other hand, there are plenty of examples such that $(I_n - 2P_a\Phi)$ is non-invertible; for example, take n even, Φ invertible and antisymmetric, and $P_a = 2^{-1}\Phi^{-1}$. In addition, in this last case is $M_P^\Phi = 0$, hence

$$\begin{aligned} \text{Span}_{\mathbb{k}[[\hbar]]} \left(\{S_{\Phi,i}, A_{\Phi,i}\}_{i \in I} \right) \\ = \text{Span}_{\mathbb{k}[[\hbar]]} \left(\{S_i\}_{i \in I} \right) \subsetneq \text{Span}_{\mathbb{k}[[\hbar]]} \left(\{S_i, A_i\}_{i \in I} \right). \end{aligned}$$

By our analysis, this proves that *in this case the realization \mathcal{R}_Φ is definitely not split* – on the contrary it is (small) minimal.

2.3. 2-cocycle deformations of multiparameters and realizations

In this subsection we introduce the notion of *deformation by 2-cocycles* of realizations (as well as of multiparameters), which is dual to that of deformation by twist.

2.3.1. Deforming realizations (and matrices) by 2-cocycles

Fix again $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ and a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of it, setting $d_i := p_{ii}/2$ for all $i \in I$ and $D_P := \text{diag}(d_1, \dots, d_n)$. We consider special deformations of realizations, called “2-cocycle deformations”. To this end, like in Section 2.2, we fix in \mathfrak{h} a $\mathbb{k}[[\hbar]]$ -basis $\{H_g\}_{g \in \mathcal{G}}$, where \mathcal{G} is an index set with $|\mathcal{G}| = \text{rk}(\mathfrak{h}) = t$.

Let $\chi : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{k}[[\hbar]]$ be any $\mathbb{k}[[\hbar]]$ -bilinear map: note that it bijectively corresponds to some $X = (\chi_{g\gamma})_{g,\gamma \in \mathcal{G}} \in M_t(\mathbb{k}[[\hbar]])$ via $\chi_{g\gamma} = \chi(H_g, H_\gamma)$. We assume that χ is antisymmetric, which means $\chi^T(x, y) = -\chi(x, y)$ where $\chi^T(x, y) := \chi(y, x)$, for all $x, y \in \mathfrak{h}$; this is equivalent to saying that X is antisymmetric, i.e. $X \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$. We denote by $\text{Alt}_{\mathbb{k}[[\hbar]]}(\mathfrak{h} \times \mathfrak{h}, \mathbb{k}[[\hbar]])$ the

set of all antisymmetric, $\mathbb{k}[[\hbar]]$ -bilinear maps from $\mathfrak{h} \times \mathfrak{h}$ to $\mathbb{k}[[\hbar]]$. We assume also that χ obeys

$$(2.10) \quad \chi(S_i, -) = 0 = \chi(-, S_i) \quad \forall i \in I$$

where $S_i := 2^{-1}(T_i^+ + T_i^-)$ for all $i \in I$. In particular, this implies (for $i \in I$, $T \in \mathfrak{h}$) that $\chi(+T_i^+, T) = \chi(-T_i^-, T)$, $\chi(T, +T_i^+) = \chi(T, -T_i^-)$, hence

$$(2.11) \quad +\chi(-, T_i^+) = -\chi(-, T_i^-) \quad \forall i \in I.$$

For later use, we introduce also the notation

$$(2.12) \quad \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h}) := \left\{ \chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}(\mathfrak{h} \times \mathfrak{h}, \mathbb{k}[[\hbar]]) \mid \chi \text{ obeys (2.10)} \right\}$$

and to each $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ we associate $\mathring{X} := \left(\mathring{\chi}_{ij} = \chi(T_i^+, T_j^+) \right)_{i,j \in I}$, which is a matrix in $\mathfrak{so}_n(\mathbb{k}[[\hbar]])$.

Basing on the above, we define

$$P_{(\chi)} := P + \mathring{X} = \left(p_{ij}^{(\chi)} := p_{ij} + \mathring{\chi}_{ij} \right)_{i,j \in I}$$

$$\Pi_{(\chi)} := \left\{ \alpha_i^{(\chi)} := \alpha_i \pm \chi(-, T_i^\pm) \right\}_{i \in I}.$$

We are now ready for our key result on 2-cocycle deformations.

PROPOSITION 2.17. — *Keep notation as above. Then:*

- (a) $P_{(\chi)} := P + \mathring{X}$ obeys $(P_{(\chi)})_s = P_s$; in particular, if P is of Cartan type, then so is $P_{(\chi)}$, and they are associated with the same Cartan matrix.
- (b) the triple $\mathcal{R}_{(\chi)} = (\mathfrak{h}, \Pi_{(\chi)}, \Pi^\vee)$ is a realization of the matrix $P_{(\chi)}$, which is minimal, resp. split, if so is \mathcal{R} .

Proof.

(a). — This is obvious, as \mathring{X} is antisymmetric.

(b). — Since the set Π^\vee does not change, condition (a.2) is trivially satisfied. In particular, $\mathcal{R}_{(\chi)}$ is minimal, resp. split, if so is \mathcal{R} . The conditions on (a.1) follows easily by definition and (2.11): namely,

$$\begin{aligned} \alpha_j^{(\chi)}(T_i^+) &= \alpha_j(T_i^+) + \chi(T_i^+, T_j^+) = p_{ij} + \mathring{\chi}_{ij} = p_{ij}^{(\chi)} \\ \alpha_j^{(\chi)}(T_i^-) &= \alpha_j(T_i^-) + \chi(T_i^-, T_j^+) = p_{ji} - \chi(T_j^+, T_i^-) \\ &= p_{ji} + \chi(T_j^+, T_i^+) = p_{ji} + \mathring{\chi}_{ji} = p_{ji}^{(\chi)} \end{aligned}$$

for all $i, j \in I$. This shows that $\mathcal{R}_{(\chi)}$ is a realization of $P_{(\chi)}$. \square

DEFINITION 2.18. — *The realization $\mathcal{R}_{(\chi)} = (\mathfrak{h}, \Pi_{(\chi)}, \Pi^\vee)$ of $P_{(\chi)} = (p_{ij}^{(\chi)})_{i,j \in I}$ is called a 2-cocycle deformation of the (initial) realization $\mathcal{R} = (\mathfrak{h}, \Pi, \Pi^\vee)$ of P . Similarly, the matrix $P_{(\chi)}$ is called a 2-cocycle deformation of the matrix P .*

Remarks 2.19.

(a) The very definitions give

$$(P_{(\chi)})_{(\chi')} = P_{(\chi+\chi')}, \quad (\mathcal{R}_{(\chi)})_{(\chi')} = \mathcal{R}_{(\chi+\chi')} \quad \forall \chi, \chi' \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h}).$$

Thus, the additive group $\text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ acts on the set of (multiparameter) matrices of size $n := |I|$ with fixed symmetric part, as well as on the set of their realizations of (any) fixed rank. When two matrices P and P' in $M_n(\mathbb{k}[[\hbar]])$ belong to the same orbit of this $\text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ -action, we say that P and P' are 2-cocycle equivalent.

(b) It follows from Proposition 2.17(a) that if two multiparameter matrices P and P' are 2-cocycle equivalent, then their symmetric part is the same, i.e. $P_s = P'_s$. As a consequence of the next result, the converse holds true as well (cf. Lemma 2.21 below), *under mild, additional assumptions*.

Next result concerns the aforementioned $\text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ -action on realizations; indeed, up to minor details it can be seen as the “2-cocycle analogue” of Proposition 2.15:

PROPOSITION 2.20. — *Let $P, P' \in M_n(\mathbb{k}[[\hbar]])$ be two matrices with the same symmetric part, i.e. such that $P_s = P'_s$. Moreover, let \mathcal{R} be a split realization of P .*

- (a) *There exists a map $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ such that $P' = P_{(\chi)}$ and the realization $\mathcal{R}_{(\chi)} = (\mathfrak{h}, \Pi_{(\chi)}, \Pi^\vee)$ of $P' = P_{(\chi)}$ is split. In a nutshell, if $P'_s = P_s$ then from any split realization of P we can obtain a split realization (of the same rank) of P' by 2-cocycle deformation, and viceversa.*
- (b) *Assume in addition that \mathcal{R} be minimal. Then \mathcal{R} is isomorphic to a 2-cocycle deformation of the split minimal realization of P_s .*

Proof.

(a). — Since P and P' share the same symmetric part, we have $\Lambda := P' - P \in \mathfrak{so}_n(\mathbb{k}[[\hbar]])$, so $P' = P + \mathring{X}$ with $\mathring{X} = (\mathring{\chi}_{ij})_{i,j \in I} := \Lambda$. Let $\mathfrak{h}'' := \text{Span}_{\mathbb{k}[[\hbar]]}(\{T_i^+\}_{i \in I})$. Then \mathring{X} defines a unique antisymmetric, $\mathbb{k}[[\hbar]]$ -bilinear map

$$\chi'' \in \text{Alt}_{\mathbb{k}[[\hbar]]}(\mathfrak{h}'' \times \mathfrak{h}'', \mathbb{k}[[\hbar]]) \quad \text{with} \quad \chi''(T_i^+, T_j^+) := \mathring{\chi}_{ij} \quad \forall i, j \in I.$$

Imposing (2.11), this χ'' extends to a map, non-unique, in general, $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h} \times \mathfrak{h}, \mathbb{k}[[\hbar]])$, obeying (2.11) and such that $\chi|_{\mathfrak{h}'' \times \mathfrak{h}''} = \chi''$. Now choosing $\Pi_{(\chi)} := \{\alpha_i^{(\chi)} := \alpha_i \pm \chi(-, T_i^\pm)\}_{i \in I} \subseteq \mathfrak{h}^*$, we get, thanks to Proposition 2.17, that $\mathcal{R}_{(\chi)} := (\mathfrak{h}, \Pi_{(\chi)}, \Pi^\vee)$ is a split realization of $P' = P_{(\chi)}$.

(b). — Let’s write the split minimal realization of P_s as

$$\mathcal{R}_{st} = (\mathfrak{h}, \Pi_{st}, \Pi_{st}^\vee)$$

with $\Pi_{st}^\vee = \{T_i^\pm\}_{i \in I}$ and $\Pi_{st} = \{\alpha_i^{(st)}\}_{i \in I}$. Since $P = P_s + P_a$ – with $P_a := 2^{-1}(P - P^T)$ – applying the arguments in (a) above we fix the matrix $\mathring{X} := P_a = (\mathring{X}_{ij})_{i,j \in I} \in \mathfrak{so}_n(\mathbb{k}[[\hbar]])$ and $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h} \times \mathfrak{h}, \mathbb{k}[[\hbar]])$ obeying (2.11); moreover, for all $i \in I$ we set $\alpha_i^{(\chi)} := \alpha_i^{(st)} \pm \chi(-, T_i^\pm)$. As \mathcal{R} is split minimal and

$$\alpha_j^{(\chi)}(T_i^\pm) = \alpha_j^{(st)}(T_i^\pm) + \chi(T_i^\pm, T_j^\pm) = (P_s)_{ij} + \mathring{X}_{ij} = (P_s)_{ij} + (P_a)_{ij} = p_{ij}$$

for all $i, j \in I$, we get $\alpha_j^{(\chi)} = \alpha_j$ for all $j \in I$. Thus the realization $(\mathcal{R}_{st})_{(\chi)}$ obtained from the 2-cocycle deformation of \mathcal{R} afforded by χ coincides with \mathcal{R} . Finally, the assumption “split minimal” implies $\text{rk}(\mathfrak{h}) = 2n$, so the uniqueness property in Proposition 2.4(b) gives $\mathcal{R} \cong \mathcal{R}_{st}$ as desired. \square

As a byproduct, we find this “2-cocycle counterpart” of Lemma 2.13:

LEMMA 2.21. — *With notation as above, let $P, P' \in M_n(\mathbb{k}[[\hbar]])$. Then P and P' are 2-cocycle equivalent, for the aforementioned 2-cocycle action on $M_n(\mathbb{k}[[\hbar]])$ of some additive group $\mathfrak{so}_t(\mathbb{k}[[\hbar]])$, if and only if $P_s = P'_s$.*

Proof. — The “if” part is Proposition 2.17, so we are left to prove the “only if”. By the existence result for realizations (cf. Proposition 2.4), we can pick a realization \mathcal{R} of P of rank $\text{rk}(\mathcal{R}) = t$: then Proposition 2.20(a) applies, and we are done. \square

Remark 2.22. — To sum up, we wish to stress the following, remarkable fact. Consider two matrices $P, P' \in M_n(\mathbb{k}[[\hbar]])$ with the same symmetric part $P_s = P'_s$, and a realization $\mathcal{R} = (\mathfrak{h}, \Pi, \Pi^\vee)$ of P that is *split* and *straight*. Then, by Proposition 2.15 and Proposition 2.20, one can construct two realizations \mathcal{R}_Φ and $\mathcal{R}_{(\chi)}$ of P' by a twist deformation, respectively a 2-cocycle deformation, of \mathcal{R} that affects only the coroot set Π^\vee or the root set Π , respectively; in particular, \mathcal{R}_Φ is still straight (yet possibly not split) and $\mathcal{R}_{(\chi)}$ is still split (yet possibly not straight), while both have the same rank as \mathcal{R} .

3. Multiparameter Lie bialgebras and their deformations

In this section we introduce multiparameter Lie bialgebras, i.e. Lie bialgebra structures on a given vector space that depend on a multiparameter, and their deformations. Indeed, these will be the semiclassical objects corresponding to the specialization of our formal multiparameter quantum enveloping algebras at $\hbar = 0$.

3.1. Lie bialgebras and their deformations

We recall hereafter a few notions concerning Lie bialgebras and their deformations; all this is classic, so we rely on references for more details.

3.1.1. Generalities on Lie bialgebras

A *Lie bialgebra* is any triple $(\mathfrak{g}; [\ , \], \delta)$ such that \mathfrak{g} is a \mathbb{k} -module – for some ground field \mathbb{k} – $[\ , \]$ is a Lie bracket on \mathfrak{g} (making the latter into a Lie algebra), the map $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ is a *Lie cobracket* on \mathfrak{g} (making it into a *Lie coalgebra*, i.e. $\delta^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is a Lie algebra bracket on \mathfrak{g}^*), and the two structures are linked by the constraint that δ is a 1-cocycle – for the Chevalley–Eilenberg cohomology of the Lie algebra $(\mathfrak{g}; [\ , \])$ with coefficients in $\mathfrak{g} \wedge \mathfrak{g}$. As a matter of notation, we set $x \wedge y := 2^{-1}(x \otimes y - y \otimes x)$ and thus we identify $\mathfrak{g} \wedge \mathfrak{g}$ with the subspace of antisymmetric tensors in $\mathfrak{g} \otimes \mathfrak{g}$. Moreover, we use a Sweedler’s-like notation $\delta(x) = x_{[1]} \otimes x_{[2]}$ for any $x \in \mathfrak{g}$.

For example, the compatibility condition between both structures reads

$$(3.1) \quad \begin{aligned} \delta([x, y]) &= \text{ad}_x(\delta(y)) - \text{ad}_y(\delta(x)) \\ &= [x, y_{[1]}] \otimes y_{[2]} + y_{[1]} \otimes [x, y_{[2]}] - [y, x_{[1]}] \otimes x_{[2]} - x_{[1]} \otimes [y, x_{[2]}]. \end{aligned}$$

When $(\mathfrak{g}; [\ , \], \delta)$ is a Lie bialgebra, the same holds for $(\mathfrak{g}^*; \delta^*, [\ , \]^*)$, up to topological technicalities, if \mathfrak{g} is infinite-dimensional, which is called the *dual* Lie bialgebra to $(\mathfrak{g}; [\ , \], \delta)$. We shall usually denote a Lie bialgebra simply by \mathfrak{g} , hence its dual by \mathfrak{g}^* .

We need some more notation. Given $r = r_1 \otimes r_2$ and $s = s_1 \otimes s_2$ in $\mathfrak{g} \otimes \mathfrak{g}$ (and similarly in $\mathfrak{g} \wedge \mathfrak{g}$) we write $r_{2,1} := r_2 \otimes r_1$ and

$$\llbracket r, s \rrbracket := [r_1, s_1] \otimes r_2 \otimes s_2 + r_1 \otimes [r_2, s_1] \otimes s_2 + r_1 \otimes s_1 \otimes [r_2, s_2]$$

which in compact form reads

$$\llbracket r, s \rrbracket := [r_{1,2}, s_{1,3}] + [r_{1,2}, s_{2,3}] + [r_{1,3}, s_{2,3}].$$

Further details can be found in [11, 41], and references therein.

3.1.2. Deformations of Lie bialgebras

A general theory of *deformations* for Lie bialgebras exists, which clearly springs up as a sub-theory of that of Lie algebras: see, for instance, [13, 43], and references therein. In the present work, we are mainly interested in two special kinds of deformations, where either the Lie cobracket or the Lie bracket alone is deformed, leaving the “other side” of the overall structure untouched. We begin by deforming the Lie cobracket. Let $(\mathfrak{g}; [\ , \], \delta)$ be a Lie bialgebra. Let then $c \in \mathfrak{g} \otimes \mathfrak{g}$, identified with a 0-cochain, be such that

$$(3.2) \quad \text{ad}_x ((\text{id} \otimes \delta)(c) + \text{c.p.} + \llbracket c, c \rrbracket) = 0, \text{ad}_x (c + c_{2,1}) = 0 \quad \forall x \in \mathfrak{g}$$

where ad_x denotes the standard adjoint action of x and “c.p.” means “cyclic permutations (on the tensor factors of the previous summand)”.

Then *the map* $\delta^c : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ *defined by*

$$(3.3) \quad \delta^c := \delta - \partial(c), \quad \text{i.e.} \quad \delta^c(x) := \delta(x) - \text{ad}_x(c) \quad \forall x \in \mathfrak{g}$$

is a new Lie cobracket on the Lie algebra $(\mathfrak{g}; [\ , \])$ *making* $(\mathfrak{g}; [\ , \], \delta^c)$ *into a new Lie bialgebra* (cf. [41, Theorem 8.1.7]).

DEFINITION 3.1. — *Every* $c \in \mathfrak{g} \otimes \mathfrak{g}$ *that obeys* (3.2) *is called a twist of the Lie bialgebra* \mathfrak{g} , *and the associated Lie bialgebra* $\mathfrak{g}^c := (\mathfrak{g}; [\ , \], \delta^c)$ *is called a deformation by twist (or “twist deformation”) of the original Lie bialgebra* \mathfrak{g} .

Now we go and deform the Lie bracket. Let again $(\mathfrak{g}; [\ , \], \delta)$ be a Lie bialgebra. Let now $\chi \in \text{Hom}_{\mathbb{k}}(\mathfrak{g} \otimes \mathfrak{g}, \mathbb{k})$ and identify $\text{Hom}_{\mathbb{k}}(\mathfrak{g} \otimes \mathfrak{g}, \mathbb{k}) = (\mathfrak{g} \otimes \mathfrak{g})^* = \mathfrak{g}^* \otimes \mathfrak{g}^*$ – up to technicalities in the infinite-dimensional case (yet the outcome is always the same). Then condition (3.2) with \mathfrak{g}^* replacing \mathfrak{g} and χ in the role of c reads

$$(3.4) \quad \text{ad}_\psi (\partial_*(\chi) + \llbracket \chi, \chi \rrbracket_*) = 0, \text{ad}_\psi (\chi + \chi_{2,1}) = 0 \quad \forall \psi \in \mathfrak{g}^*$$

where $\chi_{2,1} := \chi^T$, ∂_* is the coboundary map for the Lie algebra \mathfrak{g}^* and similarly the symbol $\llbracket \ , \ \rrbracket_*$ has the same meaning as above but with respect to \mathfrak{g}^* .

For example, the condition $\text{ad}_\psi (\chi + \chi_{2,1}) = 0$ for all $\psi \in \mathfrak{g}^*$ reads

$$\psi(x_{[1]})(\chi(x_{[2]}, y) + \chi(y, x_{[2]})) + \psi(y_{[1]})(\chi(x, y_{[2]}) + \chi(y_{[2]}, x)) = 0$$

for all $x, y \in \mathfrak{g}$. This is clearly satisfied, for instance, whenever χ is anti-symmetric, i.e. it is a 2-cochain for the usual Lie algebra cohomology.

Then *the map* $[\ , \]_\chi : \mathfrak{g} \wedge \mathfrak{g} \rightarrow \mathfrak{g}$ *defined by*

$$(3.5) \quad [x, y]_\chi := [x, y] + \chi(x_{[1]}, y)x_{[2]} - \chi(y_{[1]}, x)y_{[2]} \quad \forall x, y \in \mathfrak{g}$$

is a new Lie bracket on the Lie coalgebra $(\mathfrak{g}; \delta)$ making $(\mathfrak{g}; [\cdot, \cdot]_\chi, \delta)$ into a new Lie bialgebra (cf. [41, Exercise 8.1.8]).

DEFINITION 3.2. — Every $\chi \in \text{Hom}_{\mathbb{k}}(\mathfrak{g} \wedge \mathfrak{g}, \mathbb{k})$ obeying (3.4) is called a 2-cocycle of the Lie bialgebra \mathfrak{g} , and the Lie bialgebra $\mathfrak{g}_\chi := (\mathfrak{g}; [\cdot, \cdot]_\chi, \delta)$ is called a deformation by 2-cocycle (or “2-cocycle deformation”) of the Lie bialgebra \mathfrak{g} .

At last, we point out that the two notions of “twist” and of “2-cocycle” for Lie bialgebras, as well as the associated deformations, are so devised as to be dual to each other. The following result then holds, whose proof is left to the reader:

PROPOSITION 3.3. — Let \mathfrak{g} be a Lie bialgebra, and \mathfrak{g}^* the dual Lie bialgebra.

- (a) Let c be a twist for \mathfrak{g} , and χ_c the image of c in $(\mathfrak{g}^* \otimes \mathfrak{g}^*)^*$ for the natural composed embedding $\mathfrak{g} \otimes \mathfrak{g} \hookrightarrow \mathfrak{g}^{**} \otimes \mathfrak{g}^{**} \hookrightarrow (\mathfrak{g}^* \otimes \mathfrak{g}^*)^*$. Then χ_c is a 2-cocycle for \mathfrak{g}^* , and there exists a canonical isomorphism of Lie bialgebras $(\mathfrak{g}^*)_{\chi_c} \cong (\mathfrak{g}^c)^*$.
- (b) Let χ be a 2-cocycle for \mathfrak{g} ; assume that \mathfrak{g} is finite-dimensional, and let c_χ be the image of χ in the natural identification $(\mathfrak{g} \otimes \mathfrak{g})^* = \mathfrak{g}^* \otimes \mathfrak{g}^*$. Then c_χ is a twist for \mathfrak{g}^* , and there exists a canonical isomorphism of Lie bialgebras $(\mathfrak{g}^*)^{c_\chi} \cong (\mathfrak{g}_\chi)^*$.

3.2. Multiparameter Lie bialgebras (=MpLbA’s)

Let $A := (a_{i,j})_{i,j \in I}$ be some fixed generalized symmetrizable Cartan matrix, and let $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k})$ be a matrix of Cartan type with associated Cartan matrix A : about the latter, hereafter we refer to the notions in Definition 2.1 and all what follows in Section 2, *but* working now with \mathbb{k} as ground ring instead of $\mathbb{k}[[\hbar]]$. Thus $P + P^T = 2DA$, i.e. $p_{ij} + p_{ji} = 2d_i a_{ij}$ for all $i, j \in I$, which implies $p_{ii} = 2d_i \neq 0$ for all $i \in I$. Let $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ be a split minimal realization of P , as in Definition 2.1(b.4) – so \mathfrak{h} is free over \mathbb{k} with $\{T_i^+, T_i^-\}_{i \in I}$ as \mathbb{k} -basis.

Out of these data, we introduce the “multiparameter Borel Lie bialgebras” \mathfrak{b}_\pm^P and a suitable Lie bialgebra pairing among them; then out of this pairing we construct the associated *Manin double*, that is a suitable, canonical structure of Lie bialgebra onto $\mathfrak{b}_+^P \oplus \mathfrak{b}_-^P$ depending on that of \mathfrak{b}_\pm^P and on the pairing itself. Our recipe follows in the footsteps of Halbout’s construction in [30], that we are just slightly generalizing: indeed,

all proofs in [30] easily adapt to the present situation, the only assumptions which are relevant in the calculations being that $(\alpha_i, \alpha_i) = 2d_i = p_{ii}$ and $(\alpha_i, \alpha_j) + (\alpha_j, \alpha_i) = d_i a_{ij} + d_j a_{ji} = p_{ij} + p_{ji}$ for all $i, j \in I$.

N.B. — As a matter of notation, as we are dealing with \mathbb{k} rather than $\mathbb{k}[[\hbar]]$, comparing with Section 2.1 we identify the space \mathfrak{h} with $\bar{\mathfrak{h}}$, the roots α_j with $\bar{\alpha}_j$, etc.

3.2.1. Pre-Borel multiparameter Lie bialgebras

We define the *positive, resp. negative, pre-Borel multiparameter Lie bialgebra* with multiparameter P as being the free Lie algebra over \mathbb{k} , denoted by $\widehat{\mathfrak{b}}_+^P$, resp. by $\widehat{\mathfrak{b}}_-^P$, with generators T_i^+, E_i , resp. T_i^-, F_i ($i \in I$). Moreover, we give $\widehat{\mathfrak{b}}_+^P$, resp. $\widehat{\mathfrak{b}}_-^P$, the unique structure of Lie bialgebra over \mathbb{k} whose Lie cobracket is uniquely defined – still using shorthand notation $x \wedge y := 2^{-1}(x \otimes y - y \otimes x)$ – for all $i \in I$, by

$$\begin{aligned} \delta(T) &= 0, \delta(E_i) = -2T_i^+ \wedge E_i = -(T_i^+ \otimes E_i - E_i \otimes T_i^+) \\ \text{resp. by } \delta(T) &= 0, \delta(F_i) = +2T_i^- \wedge F_i = +(T_i^- \otimes F_i - F_i \otimes T_i^-). \end{aligned}$$

One can prove – like in [30] – that there exists a Lie bialgebra pairing

$$\langle \cdot, \cdot \rangle : \widehat{\mathfrak{b}}_+^P \times \widehat{\mathfrak{b}}_-^P \longrightarrow \mathbb{k}$$

uniquely given – for all and $i, j \in I$ – by

$$\begin{aligned} \langle T_i^+, T_j^- \rangle &= p_{ij} = \alpha_i(T_j^-) = \alpha_j(T_i^+), \\ \langle T_i^+, F_j \rangle &= 0 = \langle E_i, T_j^- \rangle \\ \langle E_i, F_j \rangle &= \delta_{ij} p_{ii}^{-1} = \delta_{ij} 2d_i^{-1}. \end{aligned}$$

3.2.2. Borel multiparameter Lie bialgebras

We introduce a Lie ideal \mathfrak{l}_\pm of $\widehat{\mathfrak{b}}_\pm^P$ as follows. On the one hand, \mathfrak{l}_+ is the Lie ideal generated by the elements

$$\begin{aligned} T_{i,j}^+ &:= [T_i^+, T_j^+], & E_{i,j}^{(T)} &:= [T_i^+, E_j] - \alpha_j(T_i^+)E_j & \forall i, j \in I \\ E_{i,j} &:= (\text{ad}(E_i))^{1-a_{ij}}(E_j) & & & \forall i \neq j \end{aligned}$$

on the other hand, \mathfrak{l}_- is the Lie ideal generated by the elements

$$\begin{aligned} T_{i,j}^- &:= [T_i^-, T_j^-], & F_{i,j}^{(T)} &:= [T_i^-, F_j] + \alpha_j(T_i^-)F_j & \forall i, j \in I \\ F_{i,j} &:= (\text{ad}(F_i))^{1-a_{ij}}(F_j) & & & \forall i \neq j. \end{aligned}$$

Now, acting once again like in [30], one sees that \mathfrak{l}_+ is contained in the left radical and \mathfrak{l}_- is contained in the right radical of the pairing $\langle \cdot, \cdot \rangle : \widehat{\mathfrak{b}}_+^P \times \widehat{\mathfrak{b}}_-^P \rightarrow \mathbb{k}$ introduced above. This has two consequences:

- (a) first, \mathfrak{l}_\pm is in fact a Lie bi-ideal in the Lie bialgebra $\widehat{\mathfrak{b}}_\pm^P$, hence either quotient $\mathfrak{b}_\pm^P := \widehat{\mathfrak{b}}_\pm^P / \mathfrak{l}_\pm$ is a quotient Lie bialgebra indeed: we call \mathfrak{b}_+^P , resp. \mathfrak{b}_-^P the positive, resp. the negative, Borel multiparameter Lie bialgebra with multiparameter P ;
- (b) second, the (non-degenerate) Lie bialgebra pairing $\langle \cdot, \cdot \rangle : \widehat{\mathfrak{b}}_+^P \times \widehat{\mathfrak{b}}_-^P \rightarrow \mathbb{k}$ boils down to a (possibly degenerate) Lie bialgebra pairing $\langle \cdot, \cdot \rangle : \mathfrak{b}_+^P \times \mathfrak{b}_-^P \rightarrow \mathbb{k}$ of (multiparameter) Borel Lie bialgebras.

3.2.3. Multiparameter Lie bialgebras as doubles

Still following [30], we proceed now to introduce our multiparameter Lie bialgebras, in two consecutive steps.

The split minimal case. — By general theory we can use the two Lie bialgebras \mathfrak{b}_+^P and \mathfrak{b}_-^P and the pairing among them to define their *generalized double* (as in [41, Section 8.3]). Namely, we endow $\mathfrak{g}_P^D := \mathfrak{b}_+^P \oplus \mathfrak{b}_-^P$ with a Lie algebra structure described in the same way as in [30, Théorème 1.5], and the unique Lie coalgebra structure such that $(\mathfrak{b}_+^P)^{\text{cop}} \hookrightarrow \mathfrak{g}_P^D \hookleftarrow \mathfrak{b}_-^P$ are Lie coalgebra embeddings, where a superscript “cop” means that we are taking the opposite Lie coalgebra structure – cf. [11, Chapter 1, Section 1.4], for further details, or even [5, Section 2.3] (and references therein), for a quick recap of this topic and its generalizations. This makes \mathfrak{g}_P^D into a Lie bialgebra; in addition, when the pairing $\langle \cdot, \cdot \rangle : \mathfrak{b}_+ \times \mathfrak{b}_- \rightarrow \mathbb{k}$ is non-degenerate, the Lie bialgebra \mathfrak{g}_P^D is even *quasitriangular*.

Finally, from the previous description of \mathfrak{b}_+^P and \mathfrak{b}_-^P one also deduces an explicit presentation for \mathfrak{g}_P^D . Namely, \mathfrak{g}_P^D is the Lie bialgebra generated (as Lie algebra) by the \mathbb{k} -subspace \mathfrak{h} together with elements E_i and F_i ($i \in I$), with relations

$$\begin{aligned} [T', T''] &= 0, & [T, E_j] - \alpha_j(T)E_j &= 0, & [T, F_j] + \alpha_j(T)F_j &= 0 \\ (\text{ad}(E_i))^{1-a_{ij}}(E_j) &= 0, & (\text{ad}(F_i))^{1-a_{ij}}(F_j) &= 0 \\ [E_i, F_j] &= \delta_{ij} \frac{T_i^+ + T_i^-}{2d_i} \end{aligned}$$

for all $T', T'', T \in \mathfrak{h}$, $i, j \in I$, $i \neq j$, with Lie bialgebra structure given on generators – for all $T \in \mathfrak{h}$ and all $i \in I$ – by

$$\delta(T) = 0, \quad \delta(E_i) = 2T_i^+ \wedge E_i, \quad \delta(F_i) = 2T_i^- \wedge F_i.$$

As a last remark, we stress that in \mathfrak{g}_P^D the Lie algebra structure does depend on the multiparameter P , while the Lie coalgebra structure is actually independent of it.

The general case. — Let now P be again a multiparameter (of Cartan type) and $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ be any realization of it – not necessarily split nor minimal. Then we define a Lie bialgebra $\mathfrak{g}_P^{\mathcal{R}}$ generalizing the notion of \mathfrak{g}_P^D introduced above.

Namely, as a Lie algebra we define $\mathfrak{g}_P^{\mathcal{R}}$ by generators and relations with (formally) the very same presentation as for \mathfrak{g}_P^D right above. The Lie coalgebra also has the same form, but we introduce it indirectly.

First of all, using Lemma 2.7 we fix a realization $\hat{\mathcal{R}} := (\hat{\mathfrak{h}}, \hat{\Pi}, \hat{\Pi}^\vee)$ of P that is *split*, and we also fix $\hat{\mathfrak{h}}_T := \text{Span}(\{T_i^\pm\}_{i \in I})$ inside $\hat{\mathfrak{h}}$. Then we consider also $\hat{\mathcal{R}} := (\hat{\mathfrak{h}}_T, \hat{\Pi}, \hat{\Pi}^\vee)$ – where $\hat{\Pi} := \{\hat{\alpha}_i := \alpha_i|_{\hat{\mathfrak{h}}_T}\}_{i \in I}$ – that is again a realization of P , which is now *split and minimal*. Out of $\hat{\mathcal{R}}$ and \mathcal{R} we define two Lie algebras – denoted $\mathfrak{g}_P^{\hat{\mathcal{R}}}$, resp. $\mathfrak{g}_P^{\mathcal{R}}$ – via an explicit presentation, which is again (formally) like the one given above for \mathfrak{g}_P^D , up to the obvious changes. Clearly, the inclusion $\hat{\mathfrak{h}}_T \subseteq \hat{\mathfrak{h}}$ induces an embedding of Lie algebras $\mathfrak{g}_P^{\hat{\mathcal{R}}} \hookrightarrow \mathfrak{g}_P^{\mathcal{R}}$.

Now, as $\hat{\mathcal{R}}$ is *split minimal*, the Lie algebra $\mathfrak{g}_P^{\hat{\mathcal{R}}}$ actually is one of the form \mathfrak{g}_P^D , and as such it bears a structure of Lie bialgebra as given above. But then, it follows by construction that there is a unique way to extend the Lie cobracket of $\mathfrak{g}_P^{\hat{\mathcal{R}}}$ to $\mathfrak{g}_P^{\mathcal{R}}$ in such a way that the embedding $\mathfrak{g}_P^{\hat{\mathcal{R}}} \hookrightarrow \mathfrak{g}_P^{\mathcal{R}}$ mentioned above is actually one of Lie bialgebras. In short, $\mathfrak{g}_P^{\mathcal{R}}$ bears a Lie bialgebra structure that is again described by the very same formulas as for \mathfrak{g}_P^D , up to replacing everywhere \mathfrak{h} with $\hat{\mathfrak{h}}$.

Finally, again by Lemma 2.7 there exists also an epimorphism of realizations $\pi : \hat{\mathcal{R}} \twoheadrightarrow \mathcal{R}$. Then, from the presentation of both $\mathfrak{g}_P^{\hat{\mathcal{R}}}$ and $\mathfrak{g}_P^{\mathcal{R}}$, this π induces an epimorphism of Lie algebras $\mathcal{L}_\pi : \mathfrak{g}_P^{\hat{\mathcal{R}}} \twoheadrightarrow \mathfrak{g}_P^{\mathcal{R}}$, such that $\text{Ker}(\mathcal{L}_\pi)$ is generated by $\text{Ker}(\pi : \hat{\mathfrak{h}} \twoheadrightarrow \mathfrak{h})$, and the latter lies in the center of $\mathfrak{g}_P^{\hat{\mathcal{R}}}$, by definitions and by Lemma 2.8; moreover, the Lie cobracket of $\mathfrak{g}_P^{\hat{\mathcal{R}}}$ kills $\text{Ker}(\pi)$, so the latter is a Lie bi-ideal in the Lie bialgebra $\mathfrak{g}_P^{\hat{\mathcal{R}}}$. Thus $\mathfrak{g}_P^{\mathcal{R}}$ inherits via \mathcal{L}_π a quotient Lie bialgebra structure from $\mathfrak{g}_P^{\hat{\mathcal{R}}}$, again described by the same formulas given above for \mathfrak{g}_P^D .

Every such Lie bialgebra $\mathfrak{g}_P^{\mathcal{R}}$ will be called *multiparameter Lie bialgebra*, in short *MpLbA*. In addition, we say that the MpLbA $\mathfrak{g}_P^{\mathcal{R}}$ is *straight*, or

small, or minimal, or split, if such is \mathcal{R} , and we define rank of $\mathfrak{g}_P^{\mathcal{R}}$ as

$$\text{rk}(\mathfrak{g}_P^{\mathcal{R}}) := \text{rk}(\mathcal{R}) = \text{rk}_{\mathbb{k}}(\mathfrak{h}).$$

For later use, we remark that every $\mathfrak{g}_P^{\mathcal{R}}$ has two obvious triangular decompositions

$$(3.6) \quad \mathfrak{g}_P^{\mathcal{R}} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-, \quad \mathfrak{g}_P^{\mathcal{R}} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

as a direct sum of Lie algebras, where \mathfrak{n}_+ , resp. \mathfrak{n}_- , is the Lie subalgebra of $\mathfrak{g}_P^{\mathcal{R}}$ generated by all the E_i 's, resp. all the F_i 's.

The following result points out the fact that the dependence of MpLbA's on realizations (for a common, fixed multiparameter matrix) is functorial:

PROPOSITION 3.4. — *Let $P \in M_n(\mathbb{k})$. If both \mathcal{R}' and \mathcal{R}'' are realizations of P and $\underline{\phi} : \mathcal{R}' \rightarrow \mathcal{R}''$ is a morphism between them, then there exists a unique morphism of Lie bialgebras $\mathcal{L}_{\underline{\phi}} : \mathfrak{g}_P^{\mathcal{R}'} \rightarrow \mathfrak{g}_P^{\mathcal{R}''}$ that extends the morphism $\phi : \mathfrak{h}' \rightarrow \mathfrak{h}''$ given by $\underline{\phi}$; moreover, $\mathcal{L}_{\text{id}_{\mathcal{R}}} = \text{id}_{\mathfrak{g}_P^{\mathcal{R}}}$ and $\mathcal{L}_{\phi' \circ \phi} = \mathcal{L}_{\phi'} \circ \mathcal{L}_{\phi}$ (whenever $\phi' \circ \phi$ is defined). Thus, the construction $\mathcal{R} \mapsto \mathfrak{g}_P^{\mathcal{R}}$ – for any fixed P – is functorial in \mathcal{R} .*

Moreover, if $\underline{\phi}$ is an epimorphism, resp. a monomorphism, then $\mathcal{L}_{\underline{\phi}}$ is an epimorphism, resp. a monomorphism, as well. Finally, for any morphism $\underline{\phi} : \mathcal{R}' \rightarrow \mathcal{R}''$, the kernel $\text{Ker}(\mathcal{L}_{\underline{\phi}})$ of $\mathcal{L}_{\underline{\phi}}$ coincides with $\text{Ker}(\phi)$, and the latter is central in $\mathfrak{g}_P^{\mathcal{R}'}$.

In particular, when $\underline{\phi}$, and hence also $\mathcal{L}_{\underline{\phi}}$, is an epimorphism, we have – setting $\mathfrak{k} := \text{Ker}(\phi)$ – a central exact sequence of Lie bialgebras

$$0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{g}_P^{\mathcal{R}'} \xrightarrow{\mathcal{L}_{\underline{\phi}}} \mathfrak{g}_P^{\mathcal{R}''} \longrightarrow 0.$$

Proof. — The existence of $\mathcal{L}_{\underline{\phi}}$ is obvious, as well as all the other claims; we only spend a moment on the centrality of $\text{Ker}(\phi)$. Lemma 2.8 imply $\text{Ker}(\phi) \subseteq \bigcap_{j \in I} \text{Ker}(\alpha'_j)$; then from the relations among the generators of $\mathfrak{g}_P^{\mathcal{R}'}$ (cf. Section 3.2.3) we get that each element in $\text{Ker}(\phi)$ commutes with all generators of $\mathfrak{g}_P^{\mathcal{R}'}$, thus $\text{Ker}(\phi)$ is central. \square

COROLLARY 3.5. — *With notation as above, if $\mathcal{R}' \cong \mathcal{R}''$ then there exists a Lie bialgebra isomorphism $\mathfrak{g}_P^{\mathcal{R}'} \cong \mathfrak{g}_P^{\mathcal{R}''}$.*

In particular, all MpLbA's built upon split realizations, respectively small realizations, of the same matrix P and sharing the same rank of \mathfrak{h} , are isomorphic to each other, hence they are independent (up to isomorphisms) of the specific realization, but only depend on P and on the rank of \mathfrak{h} .

Proof. — This follows at once from Proposition 3.4 together with the uniqueness result in Proposition 2.4 and Proposition 2.6. \square

Remark 3.6. — We expect that our definition (and construction) of MpLbA’s, as well as the related results presented below, can be extended to the case where the symmetrizable generalized Cartan matrix A is replaced by a more general symmetrizable Borchers–Cartan matrix, as in [10]. However, due to additional technical difficulties, we do not pursue such a goal in the present paper.

3.2.4. Construction via double cross sums

In this subsection we give another construction of our MpLbA’s, as suitable *double cross sums*; the latter can be seen as a semiclassical version of the double cross products of FoMpQUEAs given in Section 4.5.4. We follow [41, Section 8.3] for the exposition.

DEFINITION 3.7 ([41, Definition 8.3.1]). — *Two Lie algebras $(\mathfrak{g}, \mathfrak{m})$ form a right-left matched pair if there is a right action of \mathfrak{g} on \mathfrak{m} and a left action of \mathfrak{m} on \mathfrak{g} , denoted*

$$\triangleleft : \mathfrak{m} \otimes \mathfrak{g} \longrightarrow \mathfrak{m} \quad \text{and} \quad \triangleright : \mathfrak{m} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$$

obeying the following conditions (for all $m, n \in \mathfrak{m}$ and $x, y \in \mathfrak{g}$):

$$\begin{aligned} [m, n] \triangleleft x &= [m \triangleleft x, n] + [m, n \triangleleft x] + n \triangleleft (m \triangleright x) - m \triangleleft (n \triangleright x) \\ m \triangleright [x, y] &= [m \triangleright x, y] + [x, m \triangleright y] + (m \triangleleft y) \triangleright x - (m \triangleleft x) \triangleright y. \end{aligned}$$

After the previous definition, the key fact is the following result:

PROPOSITION 3.8 ([41, Proposition 8.3.2]).

- (a) *Given a matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{m})$, there exists a Lie algebra, called double cross sum Lie algebra and denoted by $\mathfrak{g} \bowtie \mathfrak{m}$, whose socle is the vector space $\mathfrak{g} \oplus \mathfrak{m}$ and whose Lie bracket is (for all $x, y \in \mathfrak{g}$ and $m, n \in \mathfrak{m}$)*

$$[x \oplus m, y \oplus n] = ([x, y] + m \triangleright y - n \triangleright x) \oplus ([m, n] + m \triangleleft y - n \triangleleft x).$$

- (b) *Conversely, if the direct sum space $\mathfrak{h} := \mathfrak{g} \oplus \mathfrak{m}$ bears a structure of Lie bialgebra such that the copies of \mathfrak{g} and \mathfrak{m} inside it are Lie subalgebras, then $(\mathfrak{g}, \mathfrak{m})$ is a matched pair of Lie algebras whose associated double cross sum obeys $\mathfrak{g} \bowtie \mathfrak{m} \cong \mathfrak{h}$, i.e. it is isomorphic to \mathfrak{h} . The required actions are recovered from the identities*

$$[j(m), i(x)] = i(m \triangleright x) + j(m \triangleleft x) \quad \forall x \in \mathfrak{g}, m \in \mathfrak{m}$$

where $i : \mathfrak{g} \rightarrow \mathfrak{g} \oplus \mathfrak{m} =: \mathfrak{h} (y \mapsto i(y) := (y, 0_{\mathfrak{m}}))$ and likewise $j : \mathfrak{m} \rightarrow \mathfrak{g} \oplus \mathfrak{m} =: \mathfrak{h} (n \mapsto j(n) := (0_{\mathfrak{g}}, n))$ are the natural Lie algebra monomorphisms.

In order to extend the notion of matched pair to Lie bialgebras, it is necessary to have a compatibility of the left-right actions with the Lie coalgebra structures. Thus assume now that \mathfrak{g} and \mathfrak{m} are Lie bialgebras: we say that \mathfrak{m} is a *right \mathfrak{g} -module Lie coalgebra* if it is a right \mathfrak{g} -module and in addition (for $m \in \mathfrak{m}$ and $x \in \mathfrak{g}$) one has

$$\delta_{\mathfrak{m}}(m \triangleleft x) = (m_{[1]} \triangleleft x) \otimes m_{[2]} + m_{[1]} \otimes (m_{[2]} \triangleleft x) =: \delta_{\mathfrak{m}}(m) \triangleleft x$$

the notion of left \mathfrak{m} -module Lie coalgebra is defined analogously.

PROPOSITION 3.9 ([41, Proposition 8.3.4]). — *Let $(\mathfrak{g}, \mathfrak{m})$ be a matched pair of Lie algebras, with both \mathfrak{g} and \mathfrak{m} being Lie bialgebras and with \triangleleft , resp. \triangleright , making \mathfrak{g} into a left \mathfrak{m} -module Lie coalgebra, resp. \mathfrak{m} into a right \mathfrak{g} -module Lie coalgebra, such that, for all $m \in \mathfrak{m}$ and $x \in \mathfrak{g}$, we have*

$$0 = m \triangleleft \delta_{\mathfrak{g}}(x) + \delta_{\mathfrak{m}}(m) \triangleright x$$

$$= (m \triangleleft x_{[1]}) \otimes x_{[2]} + x_{[1]} \otimes (m \triangleleft x_{[2]}) + (m_{[1]} \triangleright x) \otimes m_{[2]} + m_{[1]} \otimes (m_{[2]} \triangleright x).$$

Then the direct sum Lie coalgebra structure makes $\mathfrak{g} \bowtie \mathfrak{m}$ into a Lie bialgebra, which is called the double cross sum Lie bialgebra. \square

3.2.5. Multiparameter Lie bialgebras as double cross sums

Let $A := (a_{i,j})_{i,j \in I}$ be some fixed generalized symmetrizable Cartan matrix, and let $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k})$ be a matrix of Cartan type with associated Cartan matrix A . Then one defines, as in Subsection 3.2.2, the Borel multiparameter Lie bialgebras \mathfrak{b}_+^P and \mathfrak{b}_-^P , dually paired by a Lie bialgebra pairing, that we denote hereafter by $\langle \cdot, \cdot \rangle : \mathfrak{b}_+^P \times \mathfrak{b}_-^P \rightarrow \mathbb{k}$. Using this pairing one may define left-right actions

$$\begin{aligned} \triangleleft : \mathfrak{b}_+^P \otimes (\mathfrak{b}_-^P)^{\text{op}} &\longrightarrow \mathfrak{b}_+^P & \triangleright : \mathfrak{b}_+^P \otimes (\mathfrak{b}_-^P)^{\text{op}} &\longrightarrow (\mathfrak{b}_-^P)^{\text{op}} \\ m \triangleleft x &:= m_{[1]} \langle m_{[2]}, x \rangle & m \triangleright x &:= x_{[1]} \langle m, x_{[2]} \rangle \end{aligned}$$

for all $m \in \mathfrak{b}_+^P$ and $x \in (\mathfrak{b}_-^P)^{\text{op}}$, cf. [41, Example 8.3.7]. Then these Borel multiparameter Lie bialgebras form a matched pair $\left((\mathfrak{b}_-^P)^{\text{op}}, \mathfrak{b}_+^P \right)$, whence the double cross sum Lie bialgebra $(\mathfrak{b}_-^P)^{\text{op}} \bowtie \mathfrak{b}_+^P$ is defined. By the very construction, there exist also Lie bialgebra monomorphisms

$$(\mathfrak{b}_-^P)^{\text{op}} \hookrightarrow (\mathfrak{b}_-^P)^{\text{op}} \bowtie \mathfrak{b}_+^P \hookleftarrow \mathfrak{b}_+^P.$$

An entirely similar construction can be made transposing the opposite Lie algebra structure on \mathfrak{b}_-^P to the co-opposite Lie coalgebra structure on \mathfrak{b}_+^P : in other words, the pairing $\langle \cdot, \cdot \rangle$ induces skew-parings both on $(\mathfrak{b}_-^P)^{\text{op}} \otimes \mathfrak{b}_+^P$ and on $\mathfrak{b}_-^P \otimes (\mathfrak{b}_+^P)^{\text{cop}}$. In this case, we get the matched pair of Lie

bialgebras $(\mathfrak{b}_-^P, (\mathfrak{b}_+^P)^{\text{cop}})$ and the double cross sum $\mathfrak{b}_-^P \bowtie (\mathfrak{b}_+^P)^{\text{cop}}$; the latter also admits the Lie bialgebra monomorphisms

$$\mathfrak{b}_-^P \hookrightarrow \mathfrak{b}_-^P \bowtie (\mathfrak{b}_+^P)^{\text{cop}} \hookleftarrow (\mathfrak{b}_+^P)^{\text{cop}}.$$

Moreover, by the very definition, this double cross sum is isomorphic to the Drinfeld double \mathfrak{g}_P^D as defined in Section 3.2.3, that is $\mathfrak{b}_-^P \bowtie (\mathfrak{b}_+^P)^{\text{cop}} \cong \mathfrak{g}_P^D$.

3.3. Deformations of MpLbA's by toral twists

Let $\mathfrak{g}_P^{\mathcal{R}}$ be a MpLbA as in Section 3.2.3 above; then \mathfrak{h} is a free \mathbb{k} -module of finite rank $t := \text{rk}(\mathfrak{h})$: we fix in it a \mathbb{k} -basis $\{H_g\}_{g \in \mathcal{G}}$, where \mathcal{G} is an index set with $|\mathcal{G}| = \text{rk}(\mathfrak{h}) =: t$.

We begin introducing the so-called “toral” twists for $\mathfrak{g}_P^{\mathcal{R}}$.

DEFINITION 3.10. — *For any antisymmetric matrix $\Theta = (\theta_{gk})_{g,k \in \mathcal{G}} \in \mathfrak{so}_t(\mathbb{k})$, we set*

$$(3.7) \quad j_\Theta := \sum_{g,k=1}^t \theta_{gk} H_g \otimes H_k \in \mathfrak{h} \otimes \mathfrak{h} \subseteq \mathfrak{g} \otimes \mathfrak{g}.$$

We call j_Θ the toral twist (or “twist of toral type”) associated with Θ .

N.B. — When \mathfrak{g} is a simple Lie algebra, in the classification of [20] via Belavin–Drinfeld triples the above twist is associated with the empty datum; moreover, it turns out to be a semiclassical limit of a twist for $U_{\hbar}(\mathfrak{g})$, see Section 6.

Next result – which explains our use of terminology – follows by construction; in particular, it makes use of the antisymmetry condition on Θ .

LEMMA 3.11. — *For any matrix $\Theta = (\theta_{gk})_{g,k \in \mathcal{G}} \in \mathfrak{so}_t(\mathbb{k})$, the element j_Θ given in Definition 3.10 is a twist element for the Lie bialgebra $\mathfrak{g}_P^{\mathcal{R}}$, in the sense of (3.2).*

Concerning deformations of MpLbA's by toral twists, our main result is the next one. To settle its content, let $P \in M_t(\mathbb{k})$ be a multiparameter matrix of Cartan type with associated Cartan matrix A , let \mathcal{R} be a realization of it, and let $\mathfrak{g}_P^{\mathcal{R}}$ be the associated multiparameter Lie bialgebra; then, for any given antisymmetric matrix $\Theta = (\theta_{gk})_{g,k \in \mathcal{G}} \in \mathfrak{so}_t(\mathbb{k})$, let $j_\Theta := \sum_{g,k=1}^t \theta_{gk} H_g \otimes H_k$ be the associated twist as in (3.7). Moreover, we consider the “deformed” multiparameter matrix $P_\Theta = (p_{i,j}^\Theta)_{i,j \in I} := P - \mathfrak{A} \Theta \mathfrak{A}^T$ as in (2.4) – again of Cartan type, the same as P – and its corresponding “deformed” realization $\mathcal{R}_\Theta := (\mathfrak{h}, \Pi, \Pi_\Theta^\vee := \{T_{\Theta,i}^+, T_{\Theta,i}^-\}_{i \in I})$.

THEOREM 3.12. — *There exists a Lie bialgebra isomorphism $f_P^\Theta : \mathfrak{g}_{P^\Theta}^{\mathcal{R}} \xrightarrow{\cong} (\mathfrak{g}_P^{\mathcal{R}})^{j_\Theta}$ (notation as above, with in right-hand side the twist deformation $(\mathfrak{g}_P^{\mathcal{R}})^{j_\Theta}$ of $\mathfrak{g}_P^{\mathcal{R}}$ by j_Θ occurs) given by $E_i \mapsto E_i, T \mapsto T$ and $F_i \mapsto F_i$ for all $i \in I, T \in \mathfrak{h}$.*

In particular, the class of all MpLbA's of any fixed Cartan type and of fixed rank is stable by toral twist deformations. Moreover, inside it the subclass of all such MpLbA's associated with straight, resp. small, realizations is stable as well.

Proof. — By Section 3.2.3, the Lie algebra structure in $(\mathfrak{g}_P^{\mathcal{R}})^{j_\Theta}$ is the same as in $\mathfrak{g}_P^{\mathcal{R}}$, and the latter only depends on the α_j 's and the sums $S_j := 2^{-1}(T_i^+ + T_i^-)$ ($j \in I$). Now, both the α_j 's and the S_j 's do not change (see above) when we pass from $\mathfrak{g}_P^{\mathcal{R}}$ to $\mathfrak{g}_{P^\Theta}^{\mathcal{R}}$ or viceversa; therefore, the formulas in the claim (mapping each generator of $\mathfrak{g}_{P^\Theta}^{\mathcal{R}}$ onto the same name generator of $\mathfrak{g}_P^{\mathcal{R}} = (\mathfrak{g}_P^{\mathcal{R}})^{j_\Theta}$) provide an isomorphism of Lie algebras.

Now consider the toral twist $j_\Theta := \sum_{g,k=1}^t \theta_{gk} H_g \otimes H_k$ given in (3.7). By (3.3)

$$\delta^{j_\Theta}(x) := \delta(x) - \text{ad}_x(j_\Theta) = \delta(x) - \sum_{g,k=1}^t \theta_{gk} ([x, H_g] \otimes H_k + H_g \otimes [x, H_k])$$

for all $x \in \mathfrak{g}$. Now take $x := E_\ell$ ($\ell \in I$): then the previous formula reads

$$\begin{aligned} \delta^{j_\Theta}(E_\ell) &:= \delta(E_\ell) - \sum_{g,k=1}^t \theta_{gk} ([E_\ell, H_g] \otimes H_k + H_g \otimes [E_\ell, H_k]) \\ &= \delta(E_\ell) - \sum_{g,k=1}^t \theta_{gk} (-\alpha_\ell(H_g)E_\ell \otimes H_k - H_g \otimes \alpha_\ell(H_k)E_\ell) \\ &= T_\ell^+ \otimes E_\ell - E_\ell \otimes T_\ell^+ \\ &\quad + \sum_{g,k=1}^t \theta_{gk} (\alpha_\ell(H_g)E_\ell \otimes H_k + \alpha_\ell(H_k)H_g \otimes E_\ell) \\ &= \left(T_\ell^+ + \sum_{g,k=1}^t \theta_{kg} \alpha_\ell(H_g) H_k \right) \otimes E_\ell \\ &\quad - E_\ell \otimes \left(T_\ell^+ + \sum_{g,k=1}^t \theta_{kg} \alpha_\ell(H_g) H_k \right) \\ &= T_{\Theta,\ell}^+ \otimes E_\ell - E_\ell \otimes T_{\Theta,\ell}^+ = 2T_{\Theta,\ell}^+ \wedge E_\ell \end{aligned}$$

hence in short we get $\delta^{j_\Theta}(E_\ell) = 2T_{\Theta,\ell}^+ \wedge E_\ell$. Similar computations give

$$\delta^{j_\Theta}(E_i) = 2T_{\Theta,i}^+ \wedge E_i, \quad \delta^{j_\Theta}(T) = 0, \quad \delta^{j_\Theta}(F_i) = 2T_{\Theta,i}^- \wedge F_i, \quad \forall i \in I, T \in \mathfrak{h}.$$

This means that, through the Lie algebra isomorphism f_P^Θ , the Lie coalgebra structure of $(\mathfrak{g}_P^{\mathcal{R}})^{j_\Theta}$ is described on generators exactly like that of $\mathfrak{g}_{P_\Theta}^{\mathcal{R}_\Theta}$, with the new ‘‘coroots’’ $T_{\Theta,i}^\pm$ ($i \in I$) in $(\mathfrak{g}_P^{\mathcal{R}})^{j_\Theta}$ playing the role of the coroots T_i^\pm ($i \in I$) in $\mathfrak{g}_{P_\Theta}^{\mathcal{R}_\Theta}$. Therefore $f_P^\Theta : \mathfrak{g}_{P_\Theta}^{\mathcal{R}_\Theta} \rightarrow (\mathfrak{g}_P^{\mathcal{R}})^{j_\Theta}$ is an isomorphism of Lie bialgebras. \square

In fact, *the previous result can be reversed*, somehow. Namely, our next result shows, in particular, that every *straight small MpLbA* can be realized as a *toral twist deformation* of the ‘‘standard’’ MpLbA \mathfrak{g}_{DA}^D (as in Section 3.2.3) (cf. claim (c) below).

THEOREM 3.13. — *With assumptions as above, let P and P' be two matrices of Cartan type with the same associated Cartan matrix A , i.e. such that $P_s = P'_s$.*

- (a) *Let \mathcal{R} be a straight realization of P and let $\mathfrak{g}_P^{\mathcal{R}}$ be the associated MpLbA. Then there exists a matrix $\Theta \in \mathfrak{so}_t(\mathbb{k})$ such that $P' = P_\Theta$, the corresponding \mathcal{R}_Θ is a straight realization of $P' = P_\Theta$, and for the twist element j_Θ as in (3.7) we have*

$$\mathfrak{g}_{P'}^{\mathcal{R}_\Theta} \cong (\mathfrak{g}_P^{\mathcal{R}})^{j_\Theta}.$$

In a nutshell, if $P'_s = P_s$ then from any straight MpLbA over P we can obtain by toral twist deformation a straight MpLbA (of the same rank) over P' again.

- (b) *Let \mathcal{R} and \mathcal{R}' be straight small realizations of P and P' respectively, with $\text{rk}(\mathcal{R}) = \text{rk}(\mathcal{R}') =: t$, and let $\mathfrak{g}_P^{\mathcal{R}}$ and $\mathfrak{g}_{P'}^{\mathcal{R}'}$ be the associated MpLbA’s. Then there exists a matrix $\Theta \in \mathfrak{so}_t(\mathbb{k})$ such that for the twist element j_Θ as in (3.7) we have*

$$\mathfrak{g}_{P'}^{\mathcal{R}'} \cong (\mathfrak{g}_P^{\mathcal{R}})^{j_\Theta}.$$

In a nutshell, if $P'_s = P_s$ then any straight small MpLbA over P' is isomorphic to a toral twist deformation of any straight small MpLbA over P of the same rank.

- (c) *Every straight small MpLbA is isomorphic to some toral twist deformation of the standard MpLbA \mathfrak{g}_{DA}^D (over $DA = P_s$, cf. Section 3.2.3) of the same rank.*

Proof.

(a). — By Theorem 3.12 it is enough to find $\Theta \in \mathfrak{so}_t(\mathbb{k})$ such that $P' = P_\Theta$, that is $P' = P - \mathfrak{A}\Theta\mathfrak{A}^T$; but this is guaranteed by Lemma 2.13, so we are done.

(b). — This follows from claim (a), along with the uniqueness of straight small realizations – cf. Proposition 2.4(b) – and Proposition 3.4.

(c). — This follows from (b), taking as $\mathfrak{g}_{P'}^{\mathcal{R}'}$ the given straight small MpLbA and as $\mathfrak{g}_P^{\mathcal{R}}$ the “standard” MpLbA $\mathfrak{g}_P^{\mathcal{D}}$ over $P := DA = P'_s$. \square

Remarks 3.14.

- (a) Theorems 3.12 and 3.13 have the following interpretation. Our MpLbA’s $\mathfrak{g}_P^{\mathcal{R}}$ depend on the multiparameter P ; at a further level, once we perform onto $\mathfrak{g}_P^{\mathcal{R}}$ a deformation by toral twist the outcome $\mathfrak{g}_{P,\Theta}^{\mathcal{R}} := (\mathfrak{g}_P^{\mathcal{R}})^{j_\Theta}$ depends on *two* multiparameters, namely P and Θ . Thus all these $\mathfrak{g}_{P,\Theta}^{\mathcal{R}}$ ’s form a seemingly richer family of “twice-multiparametric” MpLbA’s. Nonetheless, Theorem 3.12 above proves that this *coincides* with the family of all MpLbA’s, though the latter seems smaller.

In short, Theorems 3.12 and 3.13 show the following. The dependence of the Lie bialgebra structure of $\mathfrak{g}_{P,\Theta}^{\mathcal{R}}$ on the “double parameter” (P, Θ) is “split” in the algebraic structure – ruled by P – and in the coalgebraic structure – ruled by Θ . Now, Theorems 3.12 and 3.13 enable us to “polarize” this dependence so to codify it either entirely within the algebraic structure (while the coalgebraic one is reduced to a “canonical form”) – so the single multiparameter P_Θ is enough to describe it – or entirely within the coalgebraic structure (with the algebraic one being reduced to the “standard” one) – so the one multiparameter Θ_P is enough indeed.

- (b) As we saw at the end of Section 2.2.2, the (sub)class of *split* realizations is *not closed* under twist deformation; as a consequence, the subclass of all MpLbA’s that are “split” is not closed either under twist deformation.

3.4. Deformations of MpLbA’s by toral 2-cocycles

Let $\mathfrak{g}_P^{\mathcal{R}}$ be a MpLbA as in Section 3.2.3, and keep notation as above. Dually to what we did before, we consider now the so-called “toral” 2-cocycles for $\mathfrak{g}_P^{\mathcal{R}}$.

DEFINITION 3.15. — Fix an antisymmetric \mathbb{k} -linear map $\chi : \mathfrak{h} \otimes \mathfrak{h} \rightarrow \mathbb{k}$ such that

$$(3.8) \quad \chi(S_i, -) = 0 = \chi(-, S_i) \quad \forall i \in I$$

where $S_i := 2^{-1}(T_i^+ + T_i^-)$ for all $i \in I$ (cf. Definition 2.1); in other words, we have $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$, cf. (2.12). Moreover, let $\pi_{\mathfrak{h}}^{\mathfrak{g}_P^{\mathcal{R}}} : \mathfrak{g}_P^{\mathcal{R}} \twoheadrightarrow \mathfrak{h}$ be the canonical projection induced by any one of the triangular decompositions in (3.6). We define

$$\chi_{\mathfrak{g}} := \chi \circ \left(\pi_{\mathfrak{h}}^{\mathfrak{g}_P^{\mathcal{R}}} \otimes \pi_{\mathfrak{h}}^{\mathfrak{g}_P^{\mathcal{R}}} \right) : \mathfrak{g}_P^{\mathcal{R}} \otimes \mathfrak{g}_P^{\mathcal{R}} \longrightarrow \mathfrak{h} \otimes \mathfrak{h} \longrightarrow \mathbb{k}$$

and we call it the toral 2-cocycle (or “the 2-cocycle of toral type”) associated with χ .

Next result follows at once by construction, and explains our use of terminology:

LEMMA 3.16. — For any antisymmetric \mathbb{k} -linear map $\chi : \mathfrak{h} \otimes \mathfrak{h} \rightarrow \mathbb{k}$ obeying (3.8), the map

$$\chi_{\mathfrak{g}} := \chi \circ \left(\pi_{\mathfrak{h}}^{\mathfrak{g}_P^{\mathcal{R}}} \otimes \pi_{\mathfrak{h}}^{\mathfrak{g}_P^{\mathcal{R}}} \right) : \mathfrak{g}_P^{\mathcal{R}} \otimes \mathfrak{g}_P^{\mathcal{R}} \longrightarrow \mathfrak{h} \otimes \mathfrak{h} \longrightarrow \mathbb{k}$$

is a 2-cocycle for the Lie bialgebra $\mathfrak{g}_P^{\mathcal{R}}$, in the sense of (3.4).

Proof. — We have to check that $\chi_{\mathfrak{g}} \in \text{Hom}_{\mathbb{k}}(\mathfrak{g}_P^{\mathcal{R}} \otimes \mathfrak{g}_P^{\mathcal{R}}, \mathbb{k})$ satisfies (3.4), that is

$$\text{ad}_{\psi}(\partial_*(\chi_{\mathfrak{g}}) - \llbracket \chi_{\mathfrak{g}}, \chi_{\mathfrak{g}} \rrbracket_*) = 0, \quad \text{ad}_{\psi}(\chi_{\mathfrak{g}} + (\chi_{\mathfrak{g}})_{2,1}) = 0 \quad \forall \psi \in (\mathfrak{g}_P^{\mathcal{R}})^*.$$

Since χ is antisymmetric, we have that $\chi_{\mathfrak{g}} + (\chi_{\mathfrak{g}})_{2,1} = 0$, hence the second condition is trivially satisfied. On the other hand, the first equality follows from the fact that actually one has $\partial_*(\chi_{\mathfrak{g}}) = 0$ and $\llbracket \chi_{\mathfrak{g}}, \chi_{\mathfrak{g}} \rrbracket_* = 0$. To see it, let us describe $\partial_*(\chi_{\mathfrak{g}})$, $\llbracket \chi_{\mathfrak{g}}, \chi_{\mathfrak{g}} \rrbracket_* \in (\mathfrak{g}_P^{\mathcal{R}} \otimes \mathfrak{g}_P^{\mathcal{R}} \otimes \mathfrak{g}_P^{\mathcal{R}})^*$ explicitly. For $x, y, z \in \mathfrak{g}_P^{\mathcal{R}}$ we have

$$\begin{aligned} \partial_*(\chi_{\mathfrak{g}})(x, y, z) &= ((\text{id} \otimes \delta_*)(\chi_{\mathfrak{g}}) + \text{c.p.})(x, y, z) \\ &= \chi_{\mathfrak{g}}(x \otimes [y, z]) + \text{c.p.} \\ &= \chi_{\mathfrak{g}}(x, [y, z]) + \chi_{\mathfrak{g}}(z, [x, y]) + \chi_{\mathfrak{g}}(y, [z, x]) \\ \llbracket \chi_{\mathfrak{g}}, \chi_{\mathfrak{g}} \rrbracket_*(x, y, z) &= \chi_{\mathfrak{g}}(x_{[1]}, y)\chi_{\mathfrak{g}}(x_{[2]}, z) + \chi_{\mathfrak{g}}(y_{[1]}, z)\chi_{\mathfrak{g}}(y_{[2]}, x) \\ &\quad + \chi_{\mathfrak{g}}(z_{[1]}, x)\chi_{\mathfrak{g}}(z_{[2]}, y). \end{aligned}$$

Using (3.6), for $g \in \mathfrak{g}_P^{\mathcal{R}}$ write $g = g_+ + g_0 + g_-$ with $g_{\pm} \in \mathfrak{n}_{\pm}$ and $g_0 \in \mathfrak{h}$. As $\chi_{\mathfrak{g}}$ is defined through the map $\pi_{\mathfrak{h}}^{\mathfrak{g}_P^{\mathcal{R}}}$, the map $\partial_*(\chi_{\mathfrak{g}})$ vanishes

when evaluated at elements in \mathfrak{n}_+ or \mathfrak{n}_- . Moreover, since the bracket on \mathfrak{h} is trivial and $[\mathfrak{h}, \mathfrak{n}_\pm] \subseteq \mathfrak{n}_\pm$, we get

$$\partial_*(\chi_{\mathfrak{g}})(x, y, z) = \chi(x_0, [y_+, z_-] + [y_-, z_+]) + \text{c.p.}$$

But $[\mathfrak{n}_+, \mathfrak{n}_-]$ is contained in the Lie subalgebra spanned by the S_i 's ($i \in I$), so we eventually get $\partial_*(\chi_{\mathfrak{g}})(x, y, z) = 0$ – for all $x, y, z \in \mathfrak{g}_P^{\mathcal{R}}$ – by condition (3.8).

To prove that $\llbracket \chi_{\mathfrak{g}}, \chi_{\mathfrak{g}} \rrbracket_*(x, y, z) = 0$, we use the Lie coalgebra structure. From (3.1) it follows that $\delta(\mathfrak{n}_\pm) \subseteq \mathfrak{b}_\pm^P \otimes \mathfrak{n}_\pm$. Since $\delta(\mathfrak{h}) = 0$ by definition and $\partial_*(\chi_{\mathfrak{g}})$ vanishes on \mathfrak{n}_\pm , we get that each summand of $\llbracket \chi_{\mathfrak{g}}, \chi_{\mathfrak{g}} \rrbracket_*(x, y, z)$ is zero. \square

The second result is the dual analog of Theorem 3.12 and of Theorem 3.13.

We start with some preliminaries. Let $P \in M_t(\mathbb{k})$ be a multiparameter matrix of Cartan type with associated Cartan matrix A , let \mathcal{R} be a realization of it, and let $\mathfrak{g}_P^{\mathcal{R}}$ be the associated multiparameter Lie bialgebra; then, given any $\chi \in \text{Alt}_{\mathbb{k}[\llbracket \mathfrak{h} \rrbracket]}^S(\mathfrak{h})$ as in (2.12), let $\chi_{\mathfrak{g}} : \mathfrak{g}_P^{\mathcal{R}} \otimes \mathfrak{g}_P^{\mathcal{R}} \rightarrow \mathbb{k}$ be the 2-cocycle map for $\mathfrak{g}_P^{\mathcal{R}}$ as in Lemma 3.16.

Consider the antisymmetric matrix $\mathring{X} := (\mathring{\chi}_{ij} = \chi(T_i^+, T_j^+))_{i,j \in I} \in \mathfrak{so}_n(\mathbb{k})$. By Proposition 2.17 we have a matrix $P_{(\chi)}$ and a realization $\mathcal{R}_{(\chi)}$ of it, given by

$$P_{(\chi)} := P + \mathring{X} = \left(p_{ij}^{(\chi)} := p_{ij} + \mathring{\chi}_{ij} \right)_{i,j \in I}$$

and

$$\Pi_{(\chi)} := \left\{ \alpha_i^{(\chi)} := \alpha_i \pm \chi(-, T_i^\pm) \right\}_{i \in I}.$$

In particular, if P is of Cartan type, then so is $P_{(\chi)}$ as well, and they are associated with the same Cartan matrix.

THEOREM 3.17. — *Keep notation as above.*

- (a) *There exists a Lie bialgebra isomorphism $f_P^{(\chi)} : \mathfrak{g}_{P_{(\chi)}}^{\mathcal{R}_{(\chi)}} \xrightarrow{\cong} (\mathfrak{g}_P^{\mathcal{R}})_{\chi_{\mathfrak{g}}}$, where $(\mathfrak{g}_P^{\mathcal{R}})_{\chi_{\mathfrak{g}}} = (\mathfrak{g}_P^{\mathcal{R}}; [\cdot, \cdot]_{\chi_{\mathfrak{g}}}, \delta)$ is the 2-cocycle deformation of $\mathfrak{g}_P^{\mathcal{R}}$ as in Definition 3.2.*

In particular, the class of all MpLbA's of any fixed Cartan type and of fixed rank is stable by toral 2-cocycle deformations. Moreover, inside it the subclass of all such MpLbA's associated with split, resp. minimal, realizations is stable as well.

(b) Let P and P' be two matrices of Cartan type with the same associated Cartan matrix A , i.e. such that $P_s = P'_s$. Then the following holds:

(b.1) Let \mathcal{R} be a split realization of P and let $\mathfrak{g}_P^{\mathcal{R}}$ be the associated MpLbA. Then there is $\chi \in \text{Alt}_{\mathbb{k}}^S(\mathfrak{h})$ such that $P' = P_{(\chi)}$, the corresponding $\mathcal{R}_{(\chi)}$ is a split realization of $P' = P_{(\chi)}$, and for the 2-cocycle $\chi_{\mathfrak{g}}$ as in Definition 3.15 we have

$$\mathfrak{g}_{P'}^{\mathcal{R}_{(\chi)}} \cong (\mathfrak{g}_P^{\mathcal{R}})_{\chi_{\mathfrak{g}}}$$

In a nutshell, if $P'_s = P_s$ then from any split MpLbA over P we can obtain by toral 2-cocycle deformation a split MpLbA (of the same rank) over P' .

(b.2) Let \mathcal{R} and \mathcal{R}' be split minimal realizations of P and P' respectively, and let $\mathfrak{g}_P^{\mathcal{R}}$ and $\mathfrak{g}_{P'}^{\mathcal{R}'}$ be the associated MpLbA's. Then there exists $\chi \in \text{Alt}_{\mathbb{k}}^S(\mathfrak{h})$ such that for the 2-cocycle $\chi_{\mathfrak{g}}$ as in Definition 3.15 we have

$$\mathfrak{g}_{P'}^{\mathcal{R}'} \cong (\mathfrak{g}_P^{\mathcal{R}})_{\chi_{\mathfrak{g}}}$$

In a nutshell, if $P'_s = P_s$ then any split minimal MpLbA over P' is isomorphic to a toral 2-cocycle deformation of any split minimal MpLbA over P .

(b.3) Every split minimal MpLbA is isomorphic to some toral 2-cocycle deformation of the Manin double $\mathfrak{g}_{DA}^D := \mathfrak{b}_+ \oplus \mathfrak{b}_-$ associated with DA endowed with the canonical Lie bialgebra structure given in Section 3.2.3.

Proof.

(a). — By Section 3.1.2, the Lie coalgebra structure in $(\mathfrak{g}_P^{\mathcal{R}})_{\chi_{\mathfrak{g}}}$ is the same as in $\mathfrak{g}_P^{\mathcal{R}}$, and the latter coincides with the one in $\mathfrak{g}_{P_{(\chi)}}^{\mathcal{R}_{(\chi)}}$. In particular, we have a Lie coalgebra isomorphism among them. With respect to the Lie algebra structure, we know that the Lie bracket in $(\mathfrak{g}_P^{\mathcal{R}})_{\chi_{\mathfrak{g}}}$ is a deformation of that of $\mathfrak{g}_P^{\mathcal{R}}$ according to (3.5). Let us see that the modified defining relations in $(\mathfrak{g}_P^{\mathcal{R}})_{\chi_{\mathfrak{g}}}$ coincide with the ones in $\mathfrak{g}_{P_{(\chi)}}^{\mathcal{R}_{(\chi)}}$, cf. Section 3.2.3; this will imply that both objects are isomorphic as Lie bialgebras.

As the Lie cobracket on the subalgebra \mathfrak{h} of $\mathfrak{g}_{P_{(\chi)}}^{\mathcal{R}_{(\chi)}}$ is trivial, we get

$$[T', T'']_{\chi_{\mathfrak{g}}} = [T', T''] - T'_{[1]} \chi_{\mathfrak{g}}(T'_{[2]}, T'') + T''_{[1]} \chi_{\mathfrak{g}}(T'_{[2]}, T') = [T', T''] = 0$$

for all $T', T'' \in \mathfrak{h}$. Take now $i \in I$: since $\delta(T) = 0$, $\delta(E_i) = 2T_i^+ \wedge E_i$ and $\delta(F_i) = 2T_i^- \wedge F_i$, we get that

$$\begin{aligned} [T, E_i]_{\chi_{\mathfrak{g}}} &= [T, E_i] - E_i \chi_{\mathfrak{g}}(T_i^+, T) \\ &= \alpha_i(T)E_i + \chi(T, T_i^+)E_i = \alpha_i^{(\chi)}(T)E_i \\ [T, F_i]_{\chi_{\mathfrak{g}}} &= [T, F_i] - F_i \chi_{\mathfrak{g}}(T_i^-, T) = -\alpha_i(T)F_i + \chi_{\mathfrak{g}}(T, T_i^-)F_i \\ &= -\left(\alpha_i(T)F_i - \chi(T, T_i^-)\right)F_i = -\alpha_i^{(\chi)}(T)F_i \\ [E_i, F_j]_{\chi_{\mathfrak{g}}} &= [E_i, F_j] + E_i \chi_{\mathfrak{g}}(T_i^+, F_j) - F_j \chi_{\mathfrak{g}}(T_j^-, E_i) \\ &= [E_i, F_j] = \delta_{ij} \frac{T_i^+ + T_i^-}{2d_i}. \end{aligned}$$

Note that only the relations involving the roots are changed by the cocycle. Now, with respect to the Serre relations, as the Lie subalgebras \mathfrak{n}_{\pm} are contained in the right and left radical of $\chi_{\mathfrak{g}}$, analogous calculations as above yield that

$$\begin{aligned} (\mathrm{ad}_{\chi_{\mathfrak{g}}}(E_i))^{1-a_{ij}}(E_j) &= (\mathrm{ad}(E_i))^{1-a_{ij}}(E_j) = 0 \\ (\mathrm{ad}_{\chi_{\mathfrak{g}}}(F_i))^{1-a_{ij}}(F_j) &= (\mathrm{ad}(F_i))^{1-a_{ij}}(F_j) = 0 \end{aligned}$$

where hereafter the symbol $\mathrm{ad}_{\chi_{\mathfrak{g}}}$ denotes the adjoint action with respect to the deformed bracket $[-, -]_{\chi_{\mathfrak{g}}}$.

(b.1). — By claim (a), it is enough to find an antisymmetric \mathbb{k} -linear $\chi : \mathfrak{h} \otimes \mathfrak{h} \rightarrow \mathbb{k}$ obeying (3.8) such that $P' = P_{(\chi)}$; this is guaranteed by Proposition 2.20.

(b.2). — This follows from claim (b.1), along with the uniqueness of split realizations – cf. Proposition 2.4(b).

(b.3). — This follows as an application of claim (b.2), taking as $\mathfrak{g}_{P'}^{\mathcal{R}}$, the given split minimal MpLbA and for $\mathfrak{g}_P^{\mathcal{R}}$ the “standard” MpLbA \mathfrak{g}_P^D over $P := DA = P'_s$ (cf. Section 3.2.3), which by definition is straight and split minimal. \square

4. Formal multiparameter QUEAs (=FoMpQUEAs)

This section is devoted to introduce formal multiparameter quantized universal enveloping algebras (in short, FoMpQUEAs) and to study their deformations.

4.1. The Hopf algebra setup

Our main references for the theory of Hopf algebras are [44] and [48]. Concerning topological Hopf algebras, in particular \hbar -adically complete ones, see for instance [11, 36, 39].

4.1.1. Hopf notation

Let us fix our notation for Hopf algebra theory (mainly standard, indeed). The comultiplication is denoted Δ , the counit ϵ and the antipode \mathcal{S} ; for the first, we use the Heyneman–Sweedler notation, namely $\Delta(x) = x_{(1)} \otimes x_{(2)}$.

Hereafter by \mathbb{k} we denote the ground ring of our algebras, coalgebras, etc. In any coalgebra C , the set of group-like elements is denoted by $G(C)$; also, we denote by $C^+ := \text{Ker}(\epsilon)$ the augmentation ideal. If $g, h \in G(C)$, the set of (g, h) -primitive elements is defined to be $P_{g,h}(C) := \{x \in C \mid \Delta(x) = x \otimes g + h \otimes x\}$. In case C is a bialgebra, we write $\text{Prim}(C) = P_{1,1}(C)$ for the space of primitive elements.

For a Hopf algebra H (or just bialgebra), we write H^{op} , resp. H^{cop} , for the Hopf algebra (or bialgebra) given by taking in H the opposite product, resp. coproduct.

Given a Hopf algebra map $\pi : H \rightarrow K$, then H is a left and right K -comodule, with structure maps $\lambda := (\pi \otimes \text{id})\Delta : H \rightarrow K \otimes H$, $\rho := (\text{id} \otimes \pi)\Delta : H \rightarrow H \otimes K$. The space of left and right *coinvariants* then is defined, respectively, by

$$\begin{aligned} {}^{\text{co}}K H &:= {}^{\text{co}}\pi H = \{h \in H \mid (\pi \otimes \text{id})(\Delta(h)) = 1 \otimes h\} \\ H {}^{\text{co}}K &:= H {}^{\text{co}}\pi = \{h \in H \mid (\text{id} \otimes \pi)(\Delta(h)) = h \otimes 1\}. \end{aligned}$$

We recall the (essentially standard) notion of *skew-Hopf pairing* between two Hopf algebras and the construction of the Drinfeld’s double.

DEFINITION 4.1 ([4, Section 2.1]). — *Given two Hopf algebras H and K with bijective antipode over the ring \mathbb{k} , a \mathbb{k} -linear map $\eta : H \otimes_{\mathbb{k}} K \rightarrow \mathbb{k}$ is called a skew-Hopf pairing (between H and K) if, for all $h \in H$, $k \in K$, one has*

$$(4.1) \quad \eta(h, k'k'') = \eta(h_{(1)}, k')\eta(h_{(2)}, k'')$$

$$(4.2) \quad \eta(h'h'', k) = \eta(h', k_{(2)})\eta(h'', k_{(1)})$$

$$(4.3) \quad \eta(h, 1) = \epsilon(h), \quad \eta(1, k) = \epsilon(k)$$

$$(4.4) \quad \eta(\mathcal{S}^{\pm 1}(h), k) = \eta(h, \mathcal{S}^{\mp 1}(k)).$$

Note that the map η turns out to be convolution invertible: its inverse is given by $\eta^{-1}(h, k) = \eta(h, \mathcal{S}(k)) = \eta(\mathcal{S}^{-1}(h), k)$ for all $h \in H$ and $k \in K$.

In this setup, the Drinfeld double, or “quantum double”, $D(H, K, \eta)$ is the quotient algebra $T(H \oplus K)/\mathcal{I}$, where \mathcal{I} is the (two-sided) ideal generated by the relations

$$\begin{aligned} 1_H = 1 = 1_K, \quad a \otimes b = ab \\ x_{(1)} \otimes y_{(1)} \eta(y_{(2)}, x_{(2)}) = \eta(y_{(1)}, x_{(1)}) y_{(2)} \otimes x_{(2)} \end{aligned}$$

for all $a, b \in H$ or $a, b \in K$ and $x \in K, y \in H$. This is also endowed with a standard Hopf algebra structure, for which H and K are Hopf \mathbb{k} -subalgebras.

4.1.2. Topological issues

We will often deal with *topological* Hopf algebras, namely Hopf algebras over the ring $\mathbb{k}[[\hbar]]$ of formal power series over a field \mathbb{k} in a formal variable \hbar . This ring carries a natural topology, called the *\hbar -adic topology*, coming from the so-called *\hbar -adic norm* with respect to which it is complete, namely

$$\|a_n \hbar^n + a_{n+1} \hbar^{n+1} + \dots\| := C^{-n} \quad (a_n \neq 0)$$

where $C > 1$ is any fixed constant in \mathbb{R} . In this sense, we shall consider *topological* $\mathbb{k}[[\hbar]]$ -modules and the *completed* tensor products among them, which we denote by $\widehat{\otimes}_{\mathbb{k}[[\hbar]]}$ or simply by $\widehat{\otimes}$. For any \mathbb{k} -vector space V , set

$$V[[\hbar]] := \left\{ \sum_{n \geq 0} v_n \hbar^n \mid v_n \in V, \forall n \geq 0 \right\}$$

then $V[[\hbar]]$ is a complete $\mathbb{k}[[\hbar]]$ -module. We call a topological $\mathbb{k}[[\hbar]]$ -module *topologically free* if it is isomorphic to $V[[\hbar]]$ for some \mathbb{k} -vector space V .

For two topologically free modules $V[[\hbar]]$ and $W[[\hbar]]$ one has that

$$V[[\hbar]] \widehat{\otimes} W[[\hbar]] \cong (V \otimes W)[[\hbar]]$$

see [36, Proposition XVI.3.2]. Then all completed tensor products between $\mathbb{k}[[\hbar]]$ -modules will be denoted simply by $\widehat{\otimes}$, unless we intend to stress the topological aspect. In particular, we make no distinction on the notation between Hopf algebras and topological Hopf algebras; we assume it is well-understood from the context.

4.1.3. Hopf algebra deformations

There exist two standard methods to deform Hopf algebras, leading to so-called “2-cocycle deformations” and to “twist deformations”: hereafter we recall both procedures, adapting them to the setup of *topological* Hopf algebras, then later on we apply them to formal quantum groups.

Twist deformations. — Let H be a Hopf algebra (over a commutative ring), and let $\mathcal{F} \in H \otimes H$ be an invertible element in $H^{\otimes 2}$ (later called a “twist”) such that

$$\mathcal{F}_{12}(\Delta \otimes \text{id})(\mathcal{F}) = \mathcal{F}_{23}(\text{id} \otimes \Delta)(\mathcal{F}), \quad (\epsilon \otimes \text{id})(\mathcal{F}) = 1 = (\text{id} \otimes \epsilon)(\mathcal{F}).$$

Then H bears a second Hopf algebra structure, denoted $H^{\mathcal{F}}$ and called *twist deformation* of the old one, with the old product, unit and counit, but with new “twisted” coproduct $\Delta^{\mathcal{F}}$ and antipode $\mathcal{S}^{\mathcal{F}}$ given by

$$\Delta^{\mathcal{F}}(x) := \mathcal{F}\Delta(x)\mathcal{F}^{-1}, \quad \mathcal{S}^{\mathcal{F}}(x) := v\mathcal{S}(x)v^{-1} \quad \forall x \in H$$

where $v := \sum_{\mathcal{F}} \mathcal{S}(f'_1)f'_2 -$ with $\sum_{\mathcal{F}} f'_1 \otimes f'_2 = \mathcal{F}^{-1} -$ is invertible in H (see, [11, Section 4.2.E], for further details). *When H is in fact a topological Hopf algebra* – meaning that, in particular, its coproduct Δ takes values into $H \otimes H$ where now “ \otimes ” stands for a suitable topological tensor product – then the same notions still make sense, and the related results apply again, up to properly reading them.

Cocycle deformations. — Let $(H, m, 1, \Delta, \epsilon)$ be a bialgebra over a ring \mathbb{k} . A convolution invertible linear map σ in $\text{Hom}_{\mathbb{k}}(H \otimes H, \mathbb{k})$ is called a (normalized) *Hopf 2-cocycle* (or just a “2-cocycle” if no confusion arises) if

$$\sigma(b_{(1)}, c_{(1)})\sigma(a, b_{(2)}c_{(2)}) = \sigma(a_{(1)}, b_{(1)})\sigma(a_{(2)}b_{(2)}, c)$$

and $\sigma(a, 1) = \epsilon(a) = \sigma(1, a)$ for all $a, b, c \in H$, see [44, Section 7.1].

Using a 2-cocycle σ it is possible to define a new algebra structure on H by deforming the multiplication. Indeed, define $m_{\sigma} = \sigma * m * \sigma^{-1} : H \otimes H \rightarrow H$ by

$$m_{\sigma}(a, b) = a \cdot_{\sigma} b = \sigma(a_{(1)}, b_{(1)})a_{(2)}b_{(2)}\sigma^{-1}(a_{(3)}, b_{(3)}) \quad \forall a, b \in H.$$

If in addition H is a Hopf algebra with antipode \mathcal{S} , then define also $\mathcal{S}_{\sigma} : H \rightarrow H$ as $\mathcal{S}_{\sigma} : H \rightarrow H$ where

$$\mathcal{S}_{\sigma}(a) = \sigma(a_{(1)}, \mathcal{S}(a_{(2)}))\mathcal{S}(a_{(3)})\sigma^{-1}(\mathcal{S}(a_{(4)}), a_{(5)}) \quad \forall a \in H.$$

It is known that $(H, m_\sigma, 1, \Delta, \epsilon)$ is in turn a bialgebra, and similarly also that $(H, m_\sigma, 1, \Delta, \epsilon, \mathcal{S}_\sigma)$ is a Hopf algebra: we shall call such a new structure on H a *cocycle deformation* of the old one, and we shall graphically denote it by H_σ ; see [18] for more details.

4.1.4. Deformations and duality

The two notions of “2-cocycle” and of “twist” are so devised as to be dual to each other with respect to Hopf duality. The proof of the following result (an exercise in Hopf theory) is left to the reader:

PROPOSITION 4.2. — *Let H be a Hopf algebra over a field, and H^* its linear dual.*

- (a) *Let \mathcal{F} be a twist for H , and $\sigma_{\mathcal{F}}$ the image of \mathcal{F} in $(H \otimes H)^*$ for the natural composed embedding $H \otimes H \hookrightarrow H^{**} \otimes H^{**} \hookrightarrow (H^* \otimes H^*)^*$. Then $\sigma_{\mathcal{F}}$ is a 2-cocycle for H^* , and there exists a canonical isomorphism $(H^*)_{\sigma_{\mathcal{F}}} \cong (H^{\mathcal{F}})^*$.*
- (b) *Let σ be a 2-cocycle for H ; assume that H is finite-dimensional, and let \mathcal{F}_σ be the image of σ in the natural identification $(H \otimes H)^* = H^* \otimes H^*$. Then \mathcal{F}_σ is a twist for H^* , and there exists a canonical isomorphism $(H^*)^{\mathcal{F}_\sigma} \cong (H_\sigma)^*$.*

4.1.5. Some q -numbers

Let $\mathbb{Z}[q, q^{-1}]$ be the ring of Laurent polynomials with integral coefficients in the indeterminate q . For every $n \in \mathbb{N}_+$ we define

$$\begin{aligned} (0)_q &:= 1, & (n)_q &:= \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1} = \sum_{s=0}^{n-1} q^s \\ (n)_q! &:= (0)_q(1)_q \cdots (n)_q := \prod_{s=0}^n (s)_q, & \binom{n}{k}_q &:= \frac{(n)_q!}{(k)_q!(n-k)_q!} \\ [0]_q &:= 1, & [n]_q &:= \frac{q^n - q^{-n}}{q - q^{-1}} = q^{-(n-1)} + \cdots + q^{n-1} = \sum_{s=0}^{n-1} q^{2s-n+1} \\ [n]_q! &:= [0]_q[1]_q \cdots [n]_q = \prod_{s=0}^n [s]_q, & \left[\begin{matrix} n \\ k \end{matrix} \right]_q &:= \frac{[n]_q!}{[k]_q![n-k]_q!} \end{aligned}$$

where the objects in the first two lines are elements of $\mathbb{Z}[q]$, and those in the last two are elements of $\mathbb{Z}[q, q^{-1}]$. In particular, we have the identities

$$(n)_{q^2} = q^{n-1}[n]_q, \quad (n)_{q^2}! = q^{\frac{n(n-1)}{2}}[n]_q, \quad \binom{n}{k}_{q^2} = q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

Moreover, for any field \mathbb{F} we can think of Laurent polynomials as functions on \mathbb{F}^\times , hence for any $q \in \mathbb{F}^\times$ we shall read every symbol above as a suitable element in \mathbb{F} .

4.2. Formal multiparameter QUEAs

We introduce now the notion of *formal multiparameter quantum universal enveloping algebra*, or just “FoMpQUEA”, in short.

Hereafter, \mathbb{k} is a field of characteristic zero, $\mathbb{k}[[\hbar]]$ the ring of formal power series in \hbar with coefficients in \mathbb{k} . In any topological, $\mathbb{k}[[\hbar]]$ -adically complete $\mathbb{k}[[\hbar]]$ -algebra \mathcal{A} , if $X \in \mathcal{A}$ we use the standard notation

$$e^{\hbar X} := \exp(\hbar X) = \sum_{n=0}^{+\infty} \hbar^n X^n / n! \in \mathcal{A}.$$

DEFINITION 4.3. — Let $A := (a_{i,j})_{i,j \in I}$ be some fixed generalized symmetrizable Cartan matrix, and let $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ be a matrix of Cartan type associated with A as in the sense of Definition 2.1(d), that is $P + P^T = 2DA$, i.e. $p_{ij} + p_{ji} = 2d_i a_{ij}$ for all $i, j \in I$, which implies $p_{ii} = 2d_i \neq 0$ for all $i \in I$.

We define in $\mathbb{k}[[\hbar]]$ the following elements: $q := e^{\hbar} = \exp(\hbar) \in \mathbb{k}[[\hbar]]$, $q_i := e^{\hbar d_i}$ ($= q^{d_i}$), $q_{ij} := e^{\hbar p_{ij}}$ ($= q^{p_{ij}}$) for all $i, j \in I$, and also $q_{ij}^{1/2} := e^{\hbar p_{ij}/2}$ for all $i, j \in I$. In particular we have $q_{ii}^{1/2} = e^{\hbar d_i} = q_i$ and $q_{ij} q_{ji} = q_{ii}^{a_{ij}}$ for all $i, j \in I$.

We can now define our FoMpQUEAs, using notation as in Definition 4.3 above.

DEFINITION 4.4. — Let $A := (a_{i,j})_{i,j \in I}$ be a generalized symmetrizable Cartan matrix, and $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ a matrix of Cartan type associated with A . We fix a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of P as in Definition 2.1.

- (a) We define the formal multiparameter quantum universal enveloping algebra – in short formal MpQUEA, or simply FoMpQUEA – with multiparameter P and realization \mathcal{R} as follows. It is the unital, associative, topological, \hbar -adically complete $\mathbb{k}[[\hbar]]$ -algebra $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ generated by the $\mathbb{k}[[\hbar]]$ -submodule \mathfrak{h} together with elements E_i, F_i (for all $i \in I$), with relations (for all $T, T', T'' \in \mathfrak{h}, i, j \in I$)

$$\begin{aligned}
 TE_j - E_jT &= +\alpha_j(T)E_j, & TF_j - F_jT &= -\alpha_j(T)F_j \\
 T'T'' &= T''T', & E_iF_j - F_jE_i &= \delta_{i,j} \frac{e^{+\hbar T_i^+} - e^{-\hbar T_i^-}}{q_i^{+1} - q_i^{-1}}
 \end{aligned}$$

(4.5)
$$\begin{aligned}
 \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} E_i^{1-a_{ij}-k} E_j E_i^k &= 0 \quad (i \neq j) \\
 \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} F_i^{1-a_{ij}-k} F_j F_i^k &= 0 \quad (i \neq j).
 \end{aligned}$$

We say that the FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is straight, or small, or minimal, or split, if such is \mathcal{R} ; also, we define the rank of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ as $\text{rk}(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})) := \text{rk}(\mathcal{R}) = \text{rk}_{\mathbb{k}[[\hbar]]}(\mathfrak{h})$.

- (b) We define the Cartan subalgebra $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$, or just $U_{\hbar}(\mathfrak{h})$, of a FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ as being the unital, \hbar -adically complete topological $\mathbb{k}[[\hbar]]$ -subalgebra of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ generated by the $\mathbb{k}[[\hbar]]$ -submodule \mathfrak{h} .
- (c) We define the positive, resp. the negative, Borel subalgebra $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$, resp. $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$, of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ to be the unital, \hbar -adically complete topological $\mathbb{k}[[\hbar]]$ -subalgebra of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ generated by \mathfrak{h} and all the E_i 's, resp. by \mathfrak{h} and all the F_i 's ($i \in I$).
- (d) We define the positive, resp. negative, nilpotent subalgebra $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_+)$, resp. $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_-)$, of a FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ to be the unital, \hbar -adically complete topological $\mathbb{k}[[\hbar]]$ -subalgebra of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ generated by the E_i 's, resp. the F_i 's, with $i \in I$.

The following two results underscore that the dependence of FoMpQUEAs on realizations (which includes that on the multiparameter matrix) is functorial:

PROPOSITION 4.5. — Let $P \in M_n(\mathbb{k}[[\hbar]])$. If both \mathcal{R}' and \mathcal{R}'' are realizations of P and $\underline{\phi} : \mathcal{R}' \rightarrow \mathcal{R}''$ is a morphism between them, then there exists a unique morphism of unital topological $\mathbb{k}[[\hbar]]$ -algebras $U_{\underline{\phi}} : U_{P,\hbar}^{\mathcal{R}'}(\mathfrak{g}) \rightarrow U_{P,\hbar}^{\mathcal{R}''}(\mathfrak{g})$ that extends the morphism $\phi : \mathfrak{h}' \rightarrow \mathfrak{h}''$ of $\mathbb{k}[[\hbar]]$ -modules given by $\underline{\phi}$;

moreover, $U_{\text{id}_{\mathcal{R}}} = \text{id}_{U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})}$ and $U_{\underline{\phi}' \circ \underline{\phi}} = U_{\underline{\phi}'} \circ U_{\underline{\phi}}$ (whenever $\underline{\phi}' \circ \underline{\phi}$ is defined). Thus, the construction $\mathcal{R} \mapsto U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$ – for any fixed P – is functorial in \mathcal{R} .

Moreover, if ϕ is an epimorphism, resp. a monomorphism, then $U_{\underline{\phi}}$ is an epimorphism, resp. a monomorphism, as well.

Finally, for any morphism $\underline{\phi} : \mathcal{R}' \rightarrow \mathcal{R}''$, the kernel $\text{Ker}(U_{\underline{\phi}})$ of $U_{\underline{\phi}}$ is the two-sided ideal in $U_{P, \hbar}^{\mathcal{R}'}(\mathfrak{g})$ generated by $\text{Ker}(\phi)$, and the latter is central in $U_{P, \hbar}^{\mathcal{R}'}(\mathfrak{g})$.

Proof. — Everything is obvious, we only spend some words on the centrality of $\text{Ker}(\phi)$. Lemma 2.8 gives $\text{Ker}(\phi) \subseteq \bigcap_{j \in I} \text{Ker}(\alpha'_j)$; then (4.5) implies that each element in $\text{Ker}(\phi)$ commutes with all generators of the algebra $U_{P, \hbar}^{\mathcal{R}'}(\mathfrak{g})$, so $\text{Ker}(\phi)$ is central in $U_{P, \hbar}^{\mathcal{R}'}(\mathfrak{g})$. \square

COROLLARY 4.6. — *With notation as above, if $\mathcal{R}' \cong \mathcal{R}''$ then there exists an isomorphism $U_{P, \hbar}^{\mathcal{R}'}(\mathfrak{g}) \cong U_{P, \hbar}^{\mathcal{R}''}(\mathfrak{g})$ of (topological) Hopf $\mathbb{k}[[\hbar]]$ -algebras.*

In particular, all FoMpQUEAs built upon split realizations, respectively small realizations, of the same matrix P and sharing the same rank of \mathfrak{h} are isomorphic to each other, hence they are independent (up to isomorphisms) of the specific realization, but only depend on P and on the rank of \mathfrak{h} .

Proof. — This follows at once from Proposition 4.5 together with the uniqueness result in Proposition 2.4 and Proposition 2.6. \square

We conclude this subsection with an important structure result, namely the “triangular decomposition” for FoMpQUEAs. We begin with some preliminaries.

DEFINITION 4.7. — *Let $A := (a_{i,j})_{i,j \in I}$ be a generalized symmetrizable Cartan matrix, and $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ a matrix of Cartan type associated with A . We fix a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of P as in Definition 2.1.*

- (a) *We define \widehat{U}^+ , resp. \widehat{U}^- , to be the unital, associative, topological, \hbar -adically complete $\mathbb{k}[[\hbar]]$ -algebra with generators $E_i (i \in I)$, resp. $F_i (i \in I)$, and relations*

$$u_{ij}^X := \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} X_i^{1-a_{ij}-k} X_j X_i^k = 0 \quad (\forall i \neq j)$$

with $X := E$, resp. $X := F$.

- (b) *We define \widehat{U}^0 to be the unital, associative, commutative, topological, \hbar -adically complete $\mathbb{k}[[\hbar]]$ -algebra generated by \mathfrak{h} . In other words, it is $\widehat{U}^0 := \widehat{S}_{\hbar}(\mathfrak{h})$, the \hbar -adic completion of the symmetric $\mathbb{k}[[\hbar]]$ -algebra over the $\mathbb{k}[[\hbar]]$ -module \mathfrak{h} .*

- (c) We define $\overrightarrow{U}_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g}) := \widehat{U}^- \widehat{\otimes}_{\mathbb{k}[\hbar]} \widehat{U}^0 \widehat{\otimes}_{\mathbb{k}[\hbar]} \widehat{U}^+$, and we introduce notation $\widehat{U}_{\otimes}^- := \widehat{U}^- \otimes \mathbb{k}[\hbar] \otimes \mathbb{k}[\hbar]$, $\widehat{U}_{\otimes}^0(\mathfrak{h}) := \mathbb{k}[\hbar] \otimes \widehat{U}^0 \otimes \mathbb{k}[\hbar]$, $\widehat{U}_{\otimes}^+ := \mathbb{k}[\hbar] \otimes \mathbb{k}[\hbar] \otimes \widehat{U}^+$, as well as $F^{\otimes} := F \otimes 1 \otimes 1$, $H^{\otimes} := 1 \otimes H \otimes 1$, $E^{\otimes} := 1 \otimes 1 \otimes E$ for all $F \in \widehat{U}^-$, $H \in \widehat{U}^0$, $E \in \widehat{U}^+$.
 Similarly, we set $\overleftarrow{U}_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g}) := \widehat{U}^+ \widehat{\otimes}_{\mathbb{k}[\hbar]} \widehat{U}^0 \widehat{\otimes}_{\mathbb{k}[\hbar]} \widehat{U}^-$, and we consider parallel, suitable notation for elements inside it.

The following, key technical result is also interesting in itself:

LEMMA 4.8. — *There exists on $\overrightarrow{U}_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g})$ a unique structure of unital, associative, topological, \hbar -adically complete $\mathbb{k}[\hbar]$ -algebra such that \widehat{U}_{\otimes}^- , \widehat{U}_{\otimes}^0 and \widehat{U}_{\otimes}^+ are all $\mathbb{k}[\hbar]$ -subalgebras in $U_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g})$, and moreover*

$$F_i^{\otimes} \cdot T^{\otimes} = F_i \otimes T \otimes 1, \quad T^{\otimes} \cdot E_j^{\otimes} = 1 \otimes T \otimes E_j, \quad F_i^{\otimes} \cdot E_j^{\otimes} = F_i \otimes 1 \otimes E_j$$

$$T^{\otimes} \cdot F_i^{\otimes} = F_i \otimes T \otimes 1 - \alpha_i(T)F_i^{\otimes}, \quad E_j^{\otimes} \cdot T^{\otimes} = 1 \otimes T \otimes E_j - \alpha_j(T)E_j^{\otimes}$$

$$E_j^{\otimes} \cdot F_i^{\otimes} = F_i \otimes 1 \otimes E_j + \delta_{ij}1 \otimes \frac{e^{+\hbar T_i^+} - e^{-\hbar T_i^-}}{e^{+\hbar d_i} - e^{-\hbar d_i}} \otimes 1.$$

An parallel claim holds true for

$$\overleftarrow{U}_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g}) := \widehat{U}^+ \widehat{\otimes}_{\mathbb{k}[\hbar]} \widehat{U}^0 \widehat{\otimes}_{\mathbb{k}[\hbar]} \widehat{U}^-.$$

Proof. — It is enough to prove the statement about $\overrightarrow{U}_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g})$.

We introduce a structure of an \hbar -adically complete, topological $\mathbb{k}[\hbar]$ -algebra $\overrightarrow{U}_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g})$ as required by hands, somehow. First, we assume that in this algebra the submodules \widehat{U}_{\otimes}^- , \widehat{U}_{\otimes}^0 and \widehat{U}_{\otimes}^+ sit as $\mathbb{k}[\hbar]$ -subalgebras – there is no obstruction to such a requirement. After this, the structure will be uniquely determined once we fix the products among elements in any two (different) of these subalgebras. Moreover, as the subalgebra \widehat{U}_{\otimes}^- , resp. $\widehat{U}_{\otimes}^0(\mathfrak{h})$, resp. \widehat{U}_{\otimes}^+ , is (topologically) generated by the F_i^{\otimes} 's ($i \in I$), resp. the T^{\otimes} 's ($T \in \mathfrak{h}$), resp. the E_j^{\otimes} 's ($j \in I$), it is enough to fix the products among any two such generators (from different sets). Eventually, recall that the F_i 's, resp. the E_j 's, are indeed generators for \widehat{U}^- , resp. for \widehat{U}^+ , which are only subject to the “quantum Serre relations” in Definition 4.7(b), while the T 's are “commutative free” – but for the fact that they are related by obvious, built-in relations such as $T = c'T' + c''T''$ inside \mathfrak{h} implies $T = c'T' + c''T''$ in \widehat{U}_{\otimes}^0 , as \mathfrak{h} is naturally mapped (linearly) into \widehat{U}_{\otimes}^0 . Thus, one can define the values of the product among F_i^{\otimes} , E_j^{\otimes} and T^{\otimes} in any possible way as soon as all “quantum Serre relations” among the F_i^{\otimes} 's and among the E_j^{\otimes} 's, as well as the “obvious relations” among the T 's

from \mathfrak{h} – namely the “linear relations” (such as $T = c'T' + c''T''$) and the commutation relations (of the form $T'T'' = T''T'$) are respected.

By the above discussion, the following choices

$$\begin{aligned} F_i^\otimes \cdot T^\otimes &:= F_i \otimes T \otimes 1, & T^\otimes \cdot E_j^\otimes &:= 1 \otimes T \otimes E_j, & F_i^\otimes \cdot E_j^\otimes &:= F_i \otimes 1 \otimes E_j \\ T^\otimes \cdot F_i^\otimes &:= F_i \otimes T \otimes 1 - \alpha_i(T)F_i \otimes 1 \otimes 1 \\ E_j^\otimes \cdot T^\otimes &:= 1 \otimes T \otimes E_j - \alpha_j(T)1 \otimes 1 \otimes E_j \\ E_j^\otimes \cdot F_i^\otimes &:= F_i \otimes 1 \otimes E_j + \delta_{ij} \cdot 1 \otimes \frac{e^{+\hbar T_i^+} - e^{-\hbar T_i^-}}{e^{+\hbar d_i} - e^{-\hbar d_i}} \otimes 1 \end{aligned}$$

for the values of the product among two generators – from different subalgebras $\widehat{U}_\otimes^-, \widehat{U}_\otimes^0$ or \widehat{U}_\otimes^+ – are enough to determine a unique algebra structure as required: we only have still to check that, using these defining formulas for the product, the relations mentioned above among generators are respected.

First of all, we consider all linear relations and commutation relations among the T 's: in this case, the check is entirely trivial.

Second, we consider the case of quantum Serre relations. Concerning them, *in order to have more readable formulas, we simplify notation* (with a slight abuse) by writing, instead of “ F^\otimes ” ($\forall F \in \widehat{U}^-$) just “ F ” again, and similarly “ H ” instead of “ H^\otimes ” ($\forall H \in \widehat{U}^0$) and “ E ” instead of “ E^\otimes ” ($\forall E \in \widehat{U}^+$).

Our goal now is to check that the multiplication defined by the previous formulas “respects” the quantum Serre relations: In turn, this eventually boils down to verify the following: *all products between a factor chosen in $\{F_i, T, E_j \mid i, j \in I, T \in \mathfrak{h}\}$ and another (in either order) chosen in $\{u_{ij}^E, u_{ij}^F \mid i, j \in I, i \neq j\}$ is zero.*

Clearly all products of type $u_{ij}^F \cdot F_\ell$ and $F_\ell \cdot u_{ij}^F$, resp. $u_{ij}^E \cdot E_t$ and $E_t \cdot u_{ij}^E$, are zero because so they are in the subalgebra U^- , resp. U^+ . The non-trivial cases are

$$T \cdot u_{ij}^E, \quad u_{ij}^E \cdot T, \quad T \cdot u_{ij}^F, \quad u_{ij}^F \cdot T, \quad u_{ij}^E \cdot F_\ell, \quad F_\ell \cdot u_{ij}^E, \quad u_{ij}^F \cdot E_t, \quad E_t \cdot u_{ij}^F$$

but among these, four cases are indeed almost trivial, as definitions give

$$\begin{aligned} T \cdot u_{ij}^E &= 1 \otimes T \otimes u_{ij}^E = 1 \otimes T \otimes 0 = 0 \\ u_{ij}^F \cdot T &= u_{ij}^F \otimes T \otimes 1 = 0 \otimes T \otimes 1 = 0 \\ F_\ell \cdot u_{ij}^E &= F_\ell \otimes 1 \otimes u_{ij}^E = F_\ell \otimes 1 \otimes 0 = 0 \\ u_{ij}^F \cdot E_t &= u_{ij}^F \otimes 1 \otimes E_t = 0 \otimes 1 \otimes E_t = 0. \end{aligned}$$

Eventually, the remaining, really non-trivial cases are the following four

$$u_{ij}^E \cdot T, \quad T \cdot u_{ij}^F, \quad u_{ij}^E \cdot F_\ell, \quad E_t \cdot u_{ij}^F$$

that we now go and analyze in detail.

Let us consider the product $T \cdot u_{ij}^F$: straightforward calculations give

$$\begin{aligned} T \cdot u_{ij}^F &= T \left(\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} F_i^{1-a_{ij}-k} F_j F_i^k \right) \\ &= \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} T F_i^{1-a_{ij}-k} F_j F_i^k \\ &= \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} \\ &\quad \times F_i^{1-a_{ij}-k} F_j F_i^k \otimes \left(T - ((1-a_{ij})\alpha_i + \alpha_j)(T) \right) \otimes 1 \\ &= u_{ij}^F \otimes \left(T - ((1-a_{ij})\alpha_i + \alpha_j)(T) \right) \otimes 1 \\ &= 0 \otimes \left(T - ((1-a_{ij})\alpha_i + \alpha_j)(T) \right) \otimes 1 = 0 \end{aligned}$$

which is good. The product $u_{ij}^E \cdot T$ is dealt with in a similar way.

Let us now consider the product $u_{ij}^E \cdot F_\ell$: again, direct calculations yield different results, depending on whether $\ell \in \{i, j\}$ or not.

Case 1: $\ell \notin \{i, j\}$. —

$$\begin{aligned} \implies u_{ij}^E \cdot F_\ell &= \left(\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} E_i^{1-a_{ij}-k} E_j E_i^k \right) F_\ell \\ &= F_\ell \otimes 1 \otimes \left(\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} E_i^{1-a_{ij}-k} E_j E_i^k \right) \\ &= F_\ell \otimes 1 \otimes u_{ij}^E = F_\ell \otimes 1 \otimes 0 = 0 \end{aligned}$$

which stands good!

Case 2: $\ell = j$. —

$$\begin{aligned}
 \implies u_{ij}^E \cdot F_\ell &= \left(\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} E_i^{1-a_{ij}-k} E_j E_i^k \right) F_j \\
 &= \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} E_i^{1-a_{ij}-k} \cdot E_j \cdot F_j \cdot E_i^k \\
 &= \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} \\
 &\quad \times E_i^{1-a_{ij}-k} \cdot \left(F_j \cdot E_j + \frac{e^{+\hbar T_j^+} - e^{-\hbar T_j^-}}{e^{+\hbar d_j} - e^{-\hbar d_j}} \right) \cdot E_i^k \\
 &= F_j \cdot u_{ij}^E + \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} E_i^{1-a_{ij}-k} \\
 &\quad \times \frac{e^{+\hbar T_j^+} - e^{-\hbar T_j^-}}{e^{+\hbar d_j} - e^{-\hbar d_j}} \cdot E_i^k \\
 &= F_j \otimes 1 \otimes u_{ij}^E + \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} q_{ji}^{k-1+a_{ij}} \\
 &\quad \times \frac{e^{+\hbar T_j^+}}{e^{+\hbar d_j} - e^{-\hbar d_j}} \cdot E_i^{1-a_{ij}} \\
 &\quad - \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} q_{ij}^{1-a_{ij}-k} \\
 &\quad \times \frac{e^{-\hbar T_j^-}}{e^{+\hbar d_j} - e^{-\hbar d_j}} \cdot E_i^{1-a_{ij}} \\
 &= F_j \otimes 1 \otimes u_{ij}^E + 1 \otimes \frac{C_{ij}^+(q) e^{+\hbar T_j^+} - C_{ij}^-(q) e^{-\hbar T_j^-}}{e^{+\hbar d_j} - e^{-\hbar d_j}} \otimes E_i^{1-a_{ij}}
 \end{aligned}$$

where in the last line we have $F_j \otimes 1 \otimes u_{ij}^E = F_j \otimes 1 \otimes 0 = 0$, and

$$\begin{aligned}
 C_{ij}^+(q) &:= \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} q_{ji}^{k-1+a_{ij}} \\
 &= \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} (q_{ij} q_{ji})^{+k/2} q_{ji}^{-1+a_{ij}} \\
 &= q_{ji}^{a_{ij}-1} \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_i^{ka_{ij}} = 0
 \end{aligned}$$

where the very last identity follows from the general, combinatorial q -identity (see for example [33, Chapter 0], or [40, Section 1.3.4])

$$\sum_{k=0}^N (-1)^k \begin{bmatrix} N \\ k \end{bmatrix}_{q_i} q_i^{k(1-N)} = 0 \quad \forall N \in \mathbb{N}_+.$$

In a parallel way we get $C_{i\bar{j}}^-(q) = 0$, hence from the above analysis we conclude that $u_{i\bar{j}}^E \cdot F_\ell = u_{i\bar{j}}^E \cdot F_j = 0$ whenever $\ell = j$. The third and last case is when $\ell = i$. To deal with that, let us notice that standard computations give us, for all $n \in \mathbb{N}$,

$$\begin{aligned} E_i^n \cdot F_i &= F_i E_i^n + [E_i^n, F_i] \\ &= F_i E_i^n + \sum_{\ell=0}^n E_i^\ell [E_i, F_i] E_i^{n-1-\ell} \\ &= F_i E_i^n + \sum_{\ell=0}^{n-1} E_i^\ell \frac{e^{+\hbar T_i^+} - e^{-\hbar T_i^-}}{e^{+\hbar d_i} - e^{-\hbar d_i}} E_i^{n-1-\ell} \\ &= F_i \cdot E_i^n + \frac{\sum_{\ell=0}^{n-1} q_{ii}^{-\ell} e^{+\hbar T_i^+} - \sum_{\ell=0}^{n-1} q_{ii}^{+\ell} e^{-\hbar T_i^-}}{q_i^{+1} - q_i^{-1}} \cdot E_i^{n-1} \\ &= F_i \cdot E_i^n + [n]_{q_i} \frac{q_i^{-n+1} e^{+\hbar T_i^+} - q_i^{+n-1} e^{-\hbar T_i^-}}{q_i^{+1} - q_i^{-1}} \cdot E_i^{n-1}. \end{aligned}$$

Using this, we compute still more, and get, for all $r, s \in \mathbb{N}$,

$$\begin{aligned} E_i^r E_j E_i^s \cdot F_i &= E_i^r E_j \left(F_i \cdot E_i^s + [s]_{q_i} \frac{q_i^{-s+1} e^{+\hbar T_i^+} - q_i^{+s-1} e^{-\hbar T_i^-}}{q_i^{+1} - q_i^{-1}} \cdot E_i^{s-1} \right) \\ &= E_i^r \cdot F_i \cdot E_j E_i^s + [s]_{q_i} E_i^r E_j \cdot \frac{q_i^{-s+1} e^{+\hbar T_i^+} - q_i^{+s-1} e^{-\hbar T_i^-}}{q_i^{+1} - q_i^{-1}} \cdot E_i^{s-1} \\ &= F_i \cdot E_i^r E_j E_i^s + [r]_{q_i} \frac{q_i^{-r+1} e^{+\hbar T_i^+} - q_i^{+r-1} e^{-\hbar T_i^-}}{q_i^{+1} - q_i^{-1}} \cdot E_i^{r-1} E_j E_i^s \\ &\quad + [s]_{q_i} \frac{q_i^{-s+1-2r} q_{ij}^{-1} e^{+\hbar T_i^+} - q_i^{+s-1+2r} q_{ji}^{+1} e^{-\hbar T_i^-}}{q_i^{+1} - q_i^{-1}} \cdot E_i^r E_j E_i^{s-1}. \end{aligned}$$

Applying this result, we get the following:

$$\begin{aligned}
 u_{ij}^E \cdot F_i &= \sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} q_{ij}^{+s/2} q_{ji}^{-s/2} E_i^r E_j E_i^s \cdot F_i \\
 &= \sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} q_{ij}^{+s/2} q_{ji}^{-s/2} \cdot \left(F_i \cdot E_i^r E_j E_i^s \right. \\
 &\quad \left. + [r]_{q_i} \frac{q_i^{-r+1} e^{+\hbar T_i^+} - q_i^{+r-1} e^{-\hbar T_i^-}}{q_i^{+1} - q_i^{-1}} \cdot E_i^{r-1} E_j E_i^s \right. \\
 &\quad \left. + [s]_{q_i} \frac{q_i^{-s+1-2r} q_{ij}^{-1} e^{+\hbar T_i^+} - q_i^{+s-1+2r} q_{ji}^{+1} e^{-\hbar T_i^-}}{q_i^{+1} - q_i^{-1}} \cdot E_i^r E_j E_i^{s-1} \right) \\
 &= F_i \cdot \sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} q_{ij}^{+s/2} q_{ji}^{-s/2} E_i^r E_j E_i^s \\
 &\quad + \sum_{h+k=-a_{ij}} \frac{(-1)^k \Gamma_{h,k}^+}{q_i^{+1} - q_i^{-1}} e^{+\hbar T_i^+} \cdot E_i^h E_j E_i^k \\
 &\quad + \sum_{h+k=-a_{ij}} \frac{(-1)^k \Gamma_{h,k}^-}{q_i^{+1} - q_i^{-1}} e^{-\hbar T_i^-} \cdot E_i^h E_j E_i^k
 \end{aligned}$$

where the coefficients $\Gamma_{h,k}^+$ and $\Gamma_{h,k}^-$ are given by where the coefficients $\Gamma_{h,k}^+$ and $\Gamma_{h,k}^-$ are given by

$$\begin{aligned}
 \Gamma_{h,k}^\pm &= q_{ij}^{+k/2} q_{ji}^{-k/2} q_i^{\mp h} \left(\begin{bmatrix} h+k+1 \\ k \end{bmatrix}_{q_i} [h+1]_{q_i} - \begin{bmatrix} h+k+1 \\ k+1 \end{bmatrix}_{q_i} [k+1]_{q_i} \right) \\
 &= 0.
 \end{aligned}$$

Plugging this result in the previous formulas, we get

$$\ell = i. \text{ ---}$$

$$\begin{aligned}
 \implies u_{ij}^E \cdot F_\ell &= F_i \cdot \sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} q_{ij}^{+s/2} q_{ji}^{-s/2} E_i^r E_j E_i^s + 0 + 0 \\
 &= F_i u_{ij}^E + 0 + 0 = F_i \otimes 1 \otimes u_{ij}^E = F_i \otimes 1 \otimes 0 = 0
 \end{aligned}$$

which makes the job. The case of $E_t \cdot u_{ij}^F$ is entirely similar, hence it is left to the reader. \square

The next result shows that $\overrightarrow{U}_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g})$ and $\overleftarrow{U}_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g})$ are nothing but different, explicit realizations of our FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$; moreover, from this we deduce an explicit description of the nilpotent, Borel and Cartan quantum subalgebras in $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$.

THEOREM 4.9.

(a) *There exist natural isomorphisms of topological $\mathbb{k}[[\hbar]]$ -algebras*

$$\begin{aligned}\overrightarrow{U}_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g}) &:= \widehat{U}^- \widehat{\otimes}_{\mathbb{k}[[\hbar]]} \widehat{U}^0 \widehat{\otimes}_{\mathbb{k}[[\hbar]]} \widehat{U}^+ \cong U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \\ \overleftarrow{U}_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g}) &:= \widehat{U}^+ \widehat{\otimes}_{\mathbb{k}[[\hbar]]} \widehat{U}^0 \widehat{\otimes}_{\mathbb{k}[[\hbar]]} \widehat{U}^- \cong U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})\end{aligned}$$

induced by multiplication in $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$.

(b) *With notation as in Definition 4.4, the isomorphisms in claims (a) above induce by restriction similar isomorphisms*

$$\begin{aligned}U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_-) &\cong \widehat{U}^-, & U_{\hbar}(\mathfrak{h}) &\cong \widehat{U}^0, & U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_+) &\cong \widehat{U}^+ \\ \widehat{U}^\pm \widehat{\otimes}_{\mathbb{k}[[\hbar]]} \widehat{U}^0 &\cong U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\pm), & \widehat{U}^0 \widehat{\otimes}_{\mathbb{k}[[\hbar]]} \widehat{U}^\pm &\cong U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\pm).\end{aligned}$$

It follows then that $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_\pm)$, $U_{\hbar}(\mathfrak{h})$, and $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\pm)$ admit the obvious descriptions (in particular, by generators and relations) inherited from Definition 4.7.

Proof.

(a). — The two cases are entirely similar, so we prove the claim only for $\overrightarrow{U}_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g})$.

Consider the $\mathbb{k}[[\hbar]]$ -algebra $\overrightarrow{U}_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g}) := \widehat{U}^- \widehat{\otimes} \widehat{U}^0 \widehat{\otimes} \widehat{U}^+$ of Lemma 4.8. By construction, it is (topologically) generated by the elements F_i^\otimes , T^\otimes , E_j^\otimes ($i, j \in I, T \in \mathfrak{h}$), and these generators obey the same relations (up to “inserting/removing” any super/sub-script “ \otimes ”) as the analogous generators of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$. Therefore, there exists an epimorphism of topological $\mathbb{k}[[\hbar]]$ -algebras

$$\pi : U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \longrightarrow U_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g}), \quad F_i \longmapsto F_i^\otimes, T \longmapsto T^\otimes, E_j \longmapsto E_j^\otimes \quad (\forall i, j, T).$$

On the other hand, for each $\bullet \in \{-, 0, +\}$ there is a morphism of algebras $\widehat{U}^\bullet \xrightarrow{\eta_\bullet} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ mapping every generator of \widehat{U}^\bullet onto the corresponding generator in $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$. Composing these with “threefold-multiplication” m_3 in $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ we obtain a morphism

$$\mu_3 := m_3 \circ (\eta_- \otimes \eta_0 \otimes \eta_+) : U_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g}) := \widehat{U}^- \widehat{\otimes} \widehat{U}^0 \widehat{\otimes} \widehat{U}^+ \longrightarrow U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$$

of topological $\mathbb{k}[[\hbar]]$ -modules. Now, by construction, the map μ_3 is inverse to π , so the latter is bijective, hence it is a $\mathbb{k}[[\hbar]]$ -algebra isomorphism. As on both sides we consider \hbar -adic topology, this π is then an homeomorphism of topological spaces too, so it is an isomorphism of topological $\mathbb{k}[[\hbar]]$ -algebras. Therefore μ_3 , being the inverse of π , is an isomorphism of topological $\mathbb{k}[[\hbar]]$ -algebras too.

(b). — This follows quite easily from claim (a). □

Remark 4.10. — The proof above relies on an *ad-hoc* argument which has roots on well-known Hopf theoretic constructions: *bosonization* and *lifting*. Indeed, the algebras \widehat{U}^\pm admit a *braided* Hopf algebra structure whose comultiplication is defined by setting the generators E_i , resp. F_i , to be primitive elements, for all $i \in I$. Hereafter, by braided Hopf algebra we mean a Hopf algebra in a braided tensor category; in the present case, the category is the category ${}_{\widehat{U}^0}^{\widehat{U}^0}\mathcal{YD}$ of Yetter–Drinfeld modules over the topological Hopf algebra \widehat{U}^0 . Given a Hopf algebra B in a category of Yetter–Drinfeld modules ${}^H_H\mathcal{YD}$ over a Hopf algebra H , there is a process to construct a usual Hopf algebra, called the “Radford biproduct” or *bosonization*: it is kind of a semidirect product and coproduct, discovered by Radford and interpreted categorically by Majid. As a vector space, the bosonization $B\#H$ is just the vector space $B \otimes H$. In our case, the (completed) tensor product $\widehat{U}^- \otimes \widehat{U}^+$ of the braided Hopf algebras is again a braided Hopf algebra, and the bosonization $(\widehat{U}^- \otimes \widehat{U}^+)\#\widehat{U}^0$ is a topological, complete Hopf $\mathbb{k}[[\hbar]]$ -algebra. By construction, it can be presented by the generators $T \in \mathfrak{h}$ and E_i, F_i with $i \in I$, satisfying all the relations in (4.5) except the commuting relation between E_i and F_j ; in fact, the latter now is replaced simply by $[E_i, F_j] = 0$. Here enters into the picture the process of *lifting* or *deformation*: through this process, one deforms the relations in a specific way, in our case, the element $[E_i, F_j] \in \widehat{U}^- \otimes \widehat{U}^+$ becomes skew-primitive in the bosonization $(\widehat{U}^- \otimes \widehat{U}^+)\#\widehat{U}^0$ and one change the relation by setting it equal to the difference between the group-like elements appearing in the comultiplication, that is

$$E_i F_j - F_j E_i = \delta_{i,j} \frac{e^{+\hbar T_i^+} - e^{-\hbar T_i^-}}{q_i^{+1} - q_i^{-1}}.$$

In general, the lifting process can also be described by means cocycle deformations.

All these constructions are described explicitly in [23] in the case of polynomial multiparameter quantum groups. It is worth noting that, by means of a process of bosonization and/or taking quantum doubles, one can generalize triangular decompositions to more general families of Hopf algebras or quantum groups; this implies special features in representation theory, see for example [47].

Eventually, we can now prove the “triangular decomposition” theorem for our FoMpQUEAs:

THEOREM 4.11 (“Triangular Decompositions” in $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$). — *There exist natural isomorphisms of topological $\mathbb{k}[[\hbar]]$ -algebras*

$$\begin{aligned} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_{\mp}) \widehat{\otimes}_{\mathbb{k}[[\hbar]]} U_{\hbar}(\mathfrak{h}) &\cong U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\mp}) \\ U_{\hbar}(\mathfrak{h}) \widehat{\otimes}_{\mathbb{k}[[\hbar]]} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_{\mp}) &\cong U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\mp}) \\ U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_{-}) \widehat{\otimes}_{\mathbb{k}[[\hbar]]} U_{\hbar}(\mathfrak{h}) \widehat{\otimes}_{\mathbb{k}[[\hbar]]} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_{+}) &\cong U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \\ U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_{+}) \widehat{\otimes}_{\mathbb{k}[[\hbar]]} U_{\hbar}(\mathfrak{h}) \widehat{\otimes}_{\mathbb{k}[[\hbar]]} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_{-}) &\cong U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \end{aligned}$$

(notation as in Definition 4.4) induced by multiplication in $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$.

Proof. — This is a direct consequence of Theorem 4.9 above. □

4.3. Hopf structure on FoMpQUEAs

We introduce now on our FoMpQUEAs a structure of topological Hopf algebra. Another proof of its existence will follow from an alternative construction (cf. Section 4.5). To begin with, we explain the link between our “formal” and the “polynomial” one.

Remark 4.12. — The usual, formal QUEA $U_{\hbar}(\mathfrak{g})$ by Drinfeld has a “polynomial” sibling $U_q(\mathfrak{g})$ introduced by Jimbo and Lusztig: the latter is a (Hopf) subalgebra of the former, over the subring $\mathbb{k}[q, q^{-1}]$ of $\mathbb{k}[[\hbar]]$, with $q^{\pm 1} := e^{\pm \hbar}$.

Our notion of *formal* multiparameter QUEA is explicitly tailored so that this parallelism extend to the multiparameter setting, linking our formal multiparameter $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ with a “polynomial” multiparameter QUEA $U_{\mathbf{q}}(\mathfrak{g})$ as in [46] or [25].

To see this, we consider a matrix P of Cartan type, whose associated Cartan matrix is A , and a *split* realization $\mathcal{R} = (\mathfrak{h}, \Pi, \Pi^{\vee})$ of it: for this the algebra $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is defined and we begin by modifying the presentation of the latter. First, replace each F_i by $\dot{F}_i := q_i F_i$ for all $i \in I$. Then the fourth relation in (4.5) reads

$$E_i \dot{F}_j - \dot{F}_j E_i = \delta_{i,j} q_{ii} \frac{e^{+\hbar T_i^+} - e^{-\hbar T_i^-}}{q_{ii} - 1} \quad \forall i, j \in I.$$

Second, using the relation $\binom{n}{k}_{q_{ii}} = q_i^{k(n-k)} \binom{n}{k}_{q_i}$ (cf. Section 4.1.5) along with the identity $q_{ij} q_{ji} = q_{ii}^{a_{ij}}$ — which follows from the assumption $P + P^t = 2DA$ and definitions, see Definition 4.3, we can re-write the two last

relations in (4.5) as

$$\begin{aligned} \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} q_{ii}^{\binom{k}{2}} q_{ij}^k E_i^{1-a_{ij}-k} E_j E_i^k &= 0 \quad (i \neq j) \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} q_{ii}^{\binom{k}{2}} q_{ij}^k \dot{F}_i^{1-a_{ij}-k} \dot{F}_j \dot{F}_i^k &= 0 \quad (i \neq j). \end{aligned}$$

With this reformulation, $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ has the following, alternative presentation: it is the unital, associative, topological, \hbar -adically complete $\mathbb{k}[[\hbar]]$ -algebra generated by the $\mathbb{k}[[\hbar]]$ -submodule \mathfrak{h} and the elements E_i, \dot{F}_i (for all $i \in I$), with relations

$$\begin{aligned} TE_j - E_j T &= +\alpha_j(T)E_j, & T\dot{F}_j - \dot{F}_j T &= -\alpha_j(T)\dot{F}_j \\ T'T'' &= T''T', & E_i\dot{F}_k - \dot{F}_k E_i &= \delta_{i,k} q_{ii} \frac{e^{+\hbar T_i^+} - e^{-\hbar T_i^-}}{q_{ii} - 1} \\ (4.6) \quad \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} q_{ij}^{+k/2} q_{ji}^{-k/2} E_i^{1-a_{ij}-k} E_j E_i^k &= 0 \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} q_{ij}^{+k/2} q_{ji}^{-k/2} \dot{F}_i^{1-a_{ij}-k} \dot{F}_j \dot{F}_i^k &= 0 \end{aligned}$$

for all $T, T', T'' \in \mathfrak{h}$, $i, j, k \in I$, with $i \neq j$.

Now set

$$q^{\pm 1} := e^{\pm \hbar} \in \mathbb{k}[[\hbar]]$$

and consider the \mathbb{k} -subalgebra $\mathbb{k}[q, q^{-1}]$ of $\mathbb{k}[[\hbar]]$, and the elements $K_i^{\pm 1} := e^{\pm \hbar T_i^+}, L_i^{\pm 1} := e^{\mp \hbar T_i^-}$ ($i \in I$) in $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$. Then in the scalar extension

$$U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) := \mathbb{k}(q) \otimes_{\mathbb{k}[q, q^{\pm 1}]} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}),$$

we slightly abuse notation by writing $X := 1 \otimes X$ for any $X \in U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, and we define $U_{\mathbf{q}}(\mathfrak{g})$ to be the unital $\mathbb{k}(q)$ -subalgebra of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ generated by $\{K_i^{\pm 1}, L_i^{\pm 1}, E_i, \dot{F}_i\}_{i \in I}$. By construction and by (4.6), we can describe $U_{\mathbf{q}}(\mathfrak{g})$ as being the unital, associative algebra over $\mathbb{k}(q)$ with generators

$K_i^{\pm 1}, L_i^{\pm 1}, E_i, \dot{F}_i$ (for all $i \in I$) and relations

$$\begin{aligned}
 K_i^{\pm 1} K_j^{\pm 1} &= K_j^{\pm 1} K_i^{\pm 1}, & K_i^{\pm 1} L_j^{\pm 1} &= L_j^{\pm 1} K_i^{\pm 1}, & L_i^{\pm 1} L_j^{\pm 1} &= L_j^{\pm 1} L_i^{\pm 1} \\
 K_i^{+1} K_i^{-1} &= 1 = K_i^{-1} K_i^{+1}, & L_i^{+1} L_i^{-1} &= 1 = L_i^{-1} L_i^{+1} \\
 K_i^{\pm 1} E_j K_i^{\mp 1} &= q_{i,j}^{\pm 1} E_j, & L_i^{\pm 1} E_j L_i^{\mp 1} &= q_{j,i}^{\mp 1} E_j \\
 K_i^{\pm 1} \dot{F}_j K_i^{\mp 1} &= q_{i,j}^{\mp 1} \dot{F}_j, & L_i^{\pm 1} \dot{F}_j L_i^{\mp 1} &= q_{j,i}^{\pm 1} \dot{F}_j \\
 (4.7) \quad E_i \dot{F}_j - \dot{F}_j E_i &= \delta_{i,j} q_{ii} \frac{K_i - L_i}{q_{ii} - 1} \\
 \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\binom{k}{2}} q_{ij}^k E_i^{1-a_{ij}-k} E_j E_i^k &= 0 \quad (i \neq j) \\
 \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\binom{k}{2}} q_{ij}^k \dot{F}_i^{1-a_{ij}-k} \dot{F}_j \dot{F}_i^k &= 0 \quad (i \neq j).
 \end{aligned}$$

Next we consider the scalar extension $\mathbb{U}_{\hbar}(\mathfrak{h}) := \mathbb{k}(q) \otimes_{\mathbb{k}[q, q^{\pm 1}]} U_{\hbar}(\mathfrak{h})$ of $U_{\hbar}(\mathfrak{h})$ (cf. Definition 4.4(b)) which clearly embeds as a ‘‘Cartan subalgebra’’ into $\mathbb{U}_{P, \hbar}^{\mathcal{R}}(\mathfrak{g}) := \mathbb{k}(q) \otimes_{\mathbb{k}[q, q^{\pm 1}]} U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$; let also $U_{\mathbf{q}}(\mathfrak{h})$ be the $\mathbb{k}(q)$ -subalgebra – inside $U_{\mathbf{q}}(\mathfrak{g})$, $U_{\hbar}(\mathfrak{h})$ and $\mathbb{U}_{\hbar}(\mathfrak{h})$ – generated by all the $K_i^{\pm 1}$ ’s and all the $L_i^{\pm 1}$ ’s. Note that inside $\mathbb{U}_{\hbar}(\mathfrak{h})$ we have $\pm T_i^+ = \hbar^{-1} \log(K_i^{\pm 1})$ and $\pm T_i^- = \hbar^{-1} \log(L_i^{\pm 1})$. By construction, both $\mathbb{U}_{\hbar}(\mathfrak{h})$ and $U_{\mathbf{q}}(\mathfrak{g})$ are modules (on the right and on the left, respectively) for the commutative algebra $U_{\mathbf{q}}(\mathfrak{h})$; therefore, the $U_{\mathbf{q}}(\mathfrak{h})$ -module $\mathbb{U}_{\hbar}(\mathfrak{h}) \otimes_{U_{\mathbf{q}}(\mathfrak{h})} U_{\mathbf{q}}(\mathfrak{g})$ is well-defined. Finally, the \hbar -adic completion of the latter actually identifies with its closure inside $\mathbb{U}_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$, which is exactly all of $\mathbb{U}_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$: in a nutshell, we have a (complete) tensor product factorization

$$\mathbb{U}_{P, \hbar}^{\mathcal{R}}(\mathfrak{g}) = \mathbb{U}_{\hbar}(\mathfrak{h}) \widehat{\otimes}_{U_{\mathbf{q}}(\mathfrak{h})} U_{\mathbf{q}}(\mathfrak{g}).$$

The previous observation is our bridge to achieve the key point about the notion of formal multiparameter QUEA, that is the following:

THEOREM 4.13. — *Every FoMpQUEA $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$ as in Definition 4.4 bears a well-defined structure of topological Hopf algebra over $\mathbb{k}[[\hbar]]$ – with coproduct taking values into the \hbar -adically completed tensor product $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g}) \widehat{\otimes}_{\mathbb{k}[[\hbar]]} U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$ – given by ($\forall T \in \mathfrak{h}$, $\ell \in I$)*

$$\begin{aligned}
 (4.8) \quad \Delta(E_{\ell}) &= E_{\ell} \otimes 1 + e^{\hbar T_{\ell}^+} \otimes E_{\ell} \\
 \Delta(T) &= T \otimes 1 + 1 \otimes T \\
 \Delta(F_{\ell}) &= F_{\ell} \otimes e^{-\hbar T_{\ell}^-} + 1 \otimes F_{\ell}
 \end{aligned}$$

$$(4.9) \quad \epsilon(E_\ell) = 0, \quad \epsilon(T) = 0, \quad \epsilon(F_\ell) = 0$$

$$(4.10) \quad \mathcal{S}(E_\ell) = -e^{-\hbar T_\ell^+} E_\ell, \quad \mathcal{S}(T) = -T, \quad \mathcal{S}(F_\ell) = -F_\ell e^{+\hbar T_\ell^-}.$$

Proof. — We provide hereafter two proofs; a third, independent one will follow from another approach, that is detailed in Section 4.5 later on.

First Proof. — Via a direct approach, the proof is a matter of computation. First, in the free, topological, \hbar -adically complete, unital, associative $\mathbb{k}[[\hbar]]$ -algebra $\mathcal{F}_\mathcal{R}$ generated by the $\mathbb{k}[[\hbar]]$ -submodule \mathfrak{h} together with the E_i 's and the F_i 's, the formulas (4.8) and (4.9) define a structure of (topological) bialgebra. Then one has to check that the two-sided ideal $\mathcal{I}_\mathcal{R}$ in $\mathcal{F}_\mathcal{R}$ generated by relations (4.5) is a bi-ideal for that bialgebra. Second, one has to check that the map $\mathcal{S} : U_{P,\hbar}^\mathcal{R}(\mathfrak{g}) \rightarrow U_{P,\hbar}^\mathcal{R}(\mathfrak{g})^{\text{op}}$ defined on $\mathcal{F}_\mathcal{R}/\mathcal{I}_\mathcal{R} =: U_{P,\hbar}^\mathcal{R}(\mathfrak{g})$ by the second line in (4.10) is an algebra anti-morphism with the “right” properties for the antipode map on the generators. In all cases, computations are the same as for [11, Definition-Proposition 6.5.1] which treats Drinfeld’s special case with \mathfrak{h} of minimal rank. It is worth stressing, though, a key feature of this generalized result: namely, the assumption that P be of Cartan type is what one uses to prove that the quantum Serre’s relations (i.e., the last two relations in (4.5)) are mapped into $\mathcal{F}_\mathcal{R} \widehat{\otimes} \mathcal{I}_\mathcal{R} + \mathcal{I}_\mathcal{R} \widehat{\otimes} \mathcal{F}_\mathcal{R}$ – where $\widehat{\otimes}$ denotes the \hbar -adic completion of the algebraic tensor product – by the given coproduct on $\mathcal{F}_\mathcal{R}$. Actually, one shows that the elements $E_{i,j}$ and $F_{i,j}$ represented by the left hand side of these equalities are skew-primitives, namely

$$\Delta(E_{i,j}) = E_{i,j} \otimes 1 + e^{+(1-a_{ij})\hbar T_i^+ + \hbar T_j^+} \otimes E_{i,j}$$

and similarly

$$\Delta(F_{i,j}) = F_{i,j} \otimes e^{-(1-a_{ij})\hbar T_i^- - \hbar T_j^-} + 1 \otimes F_{i,j}.$$

Second Proof. — This alternative method goes through an indirect approach, based on multiparameter QUEA of “polynomial type”.

First of all, we assume the realization \mathcal{R} to be split. In this case, Observation 4.12 provides a factorization $U_{P,\hbar}^\mathcal{R}(\mathfrak{g}) := \mathbb{k}(q) \otimes_{\mathbb{k}[q,q^{\pm 1}]} U_{P,\hbar}^\mathcal{R}(\mathfrak{g}) = U_\hbar(\mathfrak{h}) \widehat{\otimes}_{U_{\mathfrak{q}}(\mathfrak{h})} U_{\mathfrak{q}}(\mathfrak{g})$.

Due to its presentation in Observation 4.12, our $U_{\mathfrak{q}}(\mathfrak{g})$ is a “multiparameter quantum group” in the sense of [46] – cf. also [25], where such an example of multiparameter quantum group is referred to as being “rational”, in that each q_{ij} is a power of a single, common parameter q . The key point then is that any such (“polynomial”) multiparameter quantum group $U_{\mathfrak{q}}(\mathfrak{g})$ has a specific Hopf algebra structure (cf. [46] and [23]) given

(for all $\ell \in I$) by

$$\begin{aligned}
 \Delta(E_\ell) &= E_\ell \otimes 1 + K_\ell \otimes E_\ell, & \epsilon(E_\ell) &= 0, & \mathcal{S}(E_\ell) &= -K_\ell^{-1} E_\ell \\
 \Delta(K_\ell^{\pm 1}) &= K_\ell^{\pm 1} \otimes K_\ell^{\pm 1}, & \epsilon(K_\ell^{\pm 1}) &= 1, & \mathcal{S}(K_\ell^{\pm 1}) &= K_\ell^{\mp 1} \\
 \Delta(L_\ell^{\pm 1}) &= L_\ell^{\pm 1} \otimes L_\ell^{\pm 1}, & \epsilon(L_\ell^{\pm 1}) &= 1, & \mathcal{S}(L_\ell^{\pm 1}) &= L_\ell^{\mp 1} \\
 \Delta(\dot{F}_\ell) &= \dot{F}_\ell \otimes L_\ell + 1 \otimes \dot{F}_\ell, & \epsilon(\dot{F}_\ell) &= 0, & \mathcal{S}(\dot{F}_\ell) &= -\dot{F}_\ell L_\ell^{-1}.
 \end{aligned}
 \tag{4.11}$$

Let now $\tilde{U}_{\mathfrak{q}}(\mathfrak{g})$ be the $\mathbb{k}[q, q^{-1}]$ -subalgebra of $U_{\mathfrak{q}}(\mathfrak{g})$, hence of $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$, generated by $\{E_i, K_i^{\pm 1}, L_i^{\pm 1}, \dot{F}_i\}_{i \in I}$: it is \hbar -adic dense in $U_{\mathfrak{q}}(\mathfrak{g})$, and restricting the coproduct Δ to $\tilde{U}_{\mathfrak{q}}(\mathfrak{g})$ yields $\Delta(\tilde{U}_{\mathfrak{q}}(\mathfrak{g})) \subseteq \tilde{U}_{\mathfrak{q}}(\mathfrak{g}) \otimes_{\mathbb{k}[q, q^{-1}]} \tilde{U}_{\mathfrak{q}}(\mathfrak{g})$. Therefore, since $K_i = \exp(+\hbar T_i^+)$ and $L_i = \exp(-\hbar T_i^-)$, there exists one and only one way to extend (continuously) $\Delta|_{\tilde{U}_{\mathfrak{q}}(\mathfrak{g})}$ to $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$, which gives a map

$$\Delta : U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g}) \longrightarrow U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g}) \widehat{\otimes} U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})
 \tag{4.12}$$

described by (4.11) together with the additional constraint that all the T_i^{\pm} 's be primitive.

Since the original map Δ on $U_{\mathfrak{q}}(\mathfrak{g})$ obey the axioms of a coproduct, the same holds true for the map in (4.12) as well – though in a topological framework. With similar arguments, we deal with counit and antipode map, so that we end up with a (topological) Hopf structure for $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$, uniquely induced from the one on $U_{\mathfrak{q}}(\mathfrak{g})$. Tracking the whole construction, we eventually see that such a structure is described on generators by (4.8), (4.9) and (4.10).

Now we consider the case of a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of any kind. By Lemma 2.7, we can then pick a *split* realization (of the same matrix P as \mathcal{R}), say $\dot{\mathcal{R}} := (\dot{\mathfrak{h}}, \dot{\Pi}, \dot{\Pi}^\vee)$ and an epimorphism of realizations $\pi : \dot{\mathcal{R}} \rightarrow \mathcal{R}$. By functoriality (cf. Proposition 4.5), such a π induces an epimorphism of FoMpQUEAs $U_\pi : U_{P, \hbar}^{\dot{\mathcal{R}}}(\mathfrak{g}) \rightarrow U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$, whose kernel $\text{Ker}(U_\pi)$ is generated by $\text{Ker}(\pi)$, and the latter is central in $U_{P, \hbar}^{\dot{\mathcal{R}}}(\mathfrak{g})$; moreover, every element in $\text{Ker}(\pi)$ is primitive. Therefore $\text{Ker}(U_\pi)$ is indeed a *Hopf ideal* in the Hopf algebra $U_{P, \hbar}^{\dot{\mathcal{R}}}(\mathfrak{g})$, hence $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$ automatically inherits via U_π a quotient Hopf algebra structure from $U_{P, \hbar}^{\dot{\mathcal{R}}}(\mathfrak{g})$, which is again described by the formulas in the statement. \square

The following is now immediate:

COROLLARY 4.14. — *The Cartan subalgebra $U_\hbar(\mathfrak{h})$ and the Borel subalgebras $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ and $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ are actually (topological) Hopf subalgebras of $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$, their Hopf structure being described again via formulas (4.8)*

and (4.10). In addition, when \mathcal{R} is split we have that

$$U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm}) \cong U_{\hbar}(\mathfrak{h}_{\pm}) \underset{U_{\mathfrak{q}}(\mathfrak{b}_{\pm})}{\widehat{\times}} U_{\mathfrak{q}}(\mathfrak{b}_{\pm})$$

where $U_{\mathfrak{q}}(\mathfrak{b}_{\pm})$ is the multiparameter quantum group corresponding to the Borel subalgebras.

The next result follows at once from the second proof of Theorem 4.13.

PROPOSITION 4.15. — *Let $\underline{\phi} : \mathcal{R}' \rightarrow \mathcal{R}''$ be a morphism between realizations of a same matrix $P \in M_n(\mathbb{k}[[\hbar]])$. Then the morphism of unital topological $\mathbb{k}[[\hbar]]$ -algebras $U_{\underline{\phi}} : U_{P,\hbar}^{\mathcal{R}'}(\mathfrak{g}) \rightarrow U_{P,\hbar}^{\mathcal{R}''}(\mathfrak{g})$ granted by Proposition 4.5 is indeed a morphism of (unital topological) Hopf $\mathbb{k}[[\hbar]]$ -algebras. If we set $\mathfrak{k} := \text{Ker}(\phi)$, then $U_{\hbar}(\mathfrak{k})$ is a unital, \hbar -adically complete $\mathbb{k}[[\hbar]]$ -subalgebra of $U_{P,\hbar}^{\mathcal{R}'}(\mathfrak{g})$ which is a central Hopf subalgebra, isomorphic to a quantum Cartan (in the sense of Definition 4.4(b)), and $\text{Ker}(U_{\underline{\phi}}) = U_{P,\hbar}^{\mathcal{R}'}(\mathfrak{g})U_{\hbar}(\mathfrak{k})^+$, with $U_{\hbar}(\mathfrak{k})^+$ the augmentation ideal of $U_{\hbar}(\mathfrak{k})$.*

In particular, if $U_{\underline{\phi}}$ is an epimorphism, then we have that

$$U_{P,\hbar}^{\mathcal{R}''}(\mathfrak{g}) \cong U_{P,\hbar}^{\mathcal{R}'}(\mathfrak{g}) / U_{P,\hbar}^{\mathcal{R}'}(\mathfrak{g})U_{\hbar}(\mathfrak{k})^+.$$

Example 4.16. — Fix $P := DA$, $r := \text{rk}(DA)$ and let $\widehat{\mathcal{R}} := (\widehat{\mathfrak{h}}, \widehat{\Pi}, \widehat{\Pi}^{\vee})$ and $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^{\vee})$ be realizations of DA , where $\widehat{\mathcal{R}}$ is straight and split with $\text{rk}(\widehat{\mathfrak{h}}) = 2(2n - r)$ while \mathcal{R} is straight and small with $\text{rk}(\mathfrak{h}) = 2n - r$; more precisely, we assume $T_i^+ = T_i^-$ in \mathcal{R} , for all $i \in I$. With this setup, $U_{DA,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is the usual Drinfeld’s QUEA $U_{\hbar}(\mathfrak{g}_A)$ for the Kac–Moody algebra \mathfrak{g}_A associated with the Cartan matrix A as in Section 2.1.1; in particular, its semiclassical limit is $U(\mathfrak{g})$. Instead, $U_{DA,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g})$ has semiclassical limit $U(\mathfrak{g}^D)$, with \mathfrak{g}^D the Manin double of $\mathfrak{g} = \mathfrak{g}_A$ (cf. Section 2.1.1).

Now, there exists a (non-unique, if $r \not\leq n$) epimorphism $\underline{\phi} : \widehat{\mathcal{R}} \twoheadrightarrow \mathcal{R}$ such that $\phi(\widehat{T}_i^{\pm}) = T_i^{\pm}$ ($i \in I$); then $\mathfrak{z} := \text{Ker}(\phi) \subseteq \bigcap_{j \in I} \text{Ker}(\widehat{\alpha}_j)$. Since every element of $\widehat{\mathfrak{h}}$ is primitive inside $U_{DA,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g})$, the subalgebra $U_{\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{z})$ of $U_{DA,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g})$ generated by \mathfrak{z} is indeed a Hopf subalgebra; moreover, it is central in $U_{DA,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g})$ because $\mathfrak{z} \subseteq \bigcap_{j \in I} \text{Ker}(\widehat{\alpha}_j)$. Also, $U_{\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{z})$ is the \hbar -adic completion of the polynomial $\mathbb{k}[[\hbar]]$ -algebra over \mathfrak{z}^* , so we can loosely think of it as being a “quantum Cartan algebra” of “rank” $2n - r$.

By Proposition 4.15, $\underline{\phi}$ yields a (Hopf) epimorphism of (topological) Hopf algebras $U_{\underline{\phi}} : U_{DA,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g}) \twoheadrightarrow U_{DA,\hbar}^{\mathcal{R}}(\mathfrak{g})$; by construction, the kernel of latter is

the two-sided ideal generated by \mathfrak{z} , that is

$$\begin{aligned} \text{Ker}(U_\phi) &= U_{DA,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g})U_{\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{z})^+U_{DA,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g}) = U_{DA,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g})U_{\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{z})^+ \\ &= U_{\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{z})^+U_{DA,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g}) \end{aligned}$$

and we have that $U_{DA,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g}) \cong U_{DA,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g})/U_{DA,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g})U_{\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{z})^+$.

Finally, if we deal instead with \mathcal{R} small and $\widehat{\mathcal{R}}$ split such that $\text{rk}(\widehat{\mathfrak{h}}) = 2n$ and $\text{rk}(\mathfrak{h}) = n$, then $\mathfrak{g} = \mathfrak{g}_A$ has to be replaced by the *derived* algebra \mathfrak{g}' associated with A , and \mathfrak{g}^D by the Manin double of \mathfrak{g}' . The previous analysis then works again.

Remark 4.17. — Let us now take any matrix P (of Cartan type), a realization \mathcal{R} of it that is minimal and small with $\text{rk}(\mathfrak{h}) = 2n - r$ — with $r := \text{rk}(P + P^T)$ — and the associated FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$; then we can still find another realization $\widehat{\mathcal{R}}$ of P that is split with $\text{rk}(\mathfrak{h}) = 2(2n - r)$ and an epimorphism of realizations $\pi : \widehat{\mathcal{R}} \rightarrow \mathcal{R}$ so that $\mathfrak{z}_\pi := \text{Ker}(\pi)$ is again free of rank $2n - r$ (see Lemma 2.7 and its proof). Then the previous analysis (that was based upon $\phi : \widehat{\mathcal{R}} \rightarrow \mathcal{R}$ and $U_\phi : U_{DA,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g}) \rightarrow U_{DA,\hbar}^{\mathcal{R}}(\mathfrak{g})$) can be repeated now, step by step, basing instead upon $\pi : \widehat{\mathcal{R}} \rightarrow \mathcal{R}$ and the associated epimorphism $U_\pi : U_{P,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g}) \rightarrow U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ of FoMpQUEAs: this leads to the sequence of Hopf algebra maps

$$(4.13) \quad U_{\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{z}_\pi) \longrightarrow U_{P,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g}) \xrightarrow{U_\pi} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$$

where $U_{\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{z}_\pi)$ is the (central) subalgebra of $U_{P,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g})$ generated by \mathfrak{z}_π , which is again a “quantum Cartan algebra” of “rank” $2n - r$. Again, we obtain that

$$U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \cong U_{P,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g})/U_{P,\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{g})U_{\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{z}_\pi)^+.$$

Therefore, the situation in general is much similar to what happens in the special, “standard” case of $P = DA$; what does actually change, indeed, is the explicit description of \mathfrak{z}_π — with respect to that of \mathfrak{z} , that was quite clear — hence of the “quantum Cartan algebra of rank $2n - r$ ” $U_{\hbar}^{\widehat{\mathcal{R}}}(\mathfrak{z}_\pi)$.

4.3.1. The case of split (and) minimal FoMpQUEAs

We consider now the special case of a FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, as defined in Definition 4.4, for which the realization \mathcal{R} is split and minimal — in short, *split minimal*. In this case, it follows by definition that $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ can be described as follows: it is the unital, associative, topological, \hbar -adically

complete algebra over $\mathbb{k}[[\hbar]]$ with generators E_i, T_i^+, T_i^-, F_i (for all $i \in I$) and relations

$$\begin{aligned} T_i^+ E_j - E_j T_i^+ &= +p_{i,j} E_j, & T_i^- E_j - E_j T_i^- &= +p_{j,i} E_j \\ T_i^+ F_j - F_j T_i^+ &= -p_{i,j} F_j, & T_i^- F_j - F_j T_i^- &= -p_{j,i} F_j \\ T_i^\pm T_j^\pm &= T_j^\pm T_i^\pm, & T_i^\pm T_j^\mp &= T_j^\mp T_i^\pm \\ E_i F_j - F_j E_i &= \delta_{i,j} \frac{e^{+\hbar T_i^+} - e^{-\hbar T_i^-}}{q_i^{+1} - q_i^{-1}} \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} X_i^{1-a_{ij}-k} X_j X_i^k &= 0 & X &\in \{E, F\} \\ & & & i \neq j \end{aligned}$$

and bearing the (topological) Hopf $\mathbb{k}[[\hbar]]$ -algebra structure given (for all $\ell \in I$) by

$$\begin{aligned} \Delta(E_\ell) &= E_\ell \otimes 1 + e^{+\hbar T_\ell^+} \otimes E_\ell, & \epsilon(E_\ell) &= 0, & \mathcal{S}(E_\ell) &= -e^{-\hbar T_\ell^+} E_\ell \\ \Delta(T_\ell^\pm) &= T_\ell^\pm \otimes 1 + 1 \otimes T_\ell^\pm, & \epsilon(T_\ell^\pm) &= 0, & \mathcal{S}(T_\ell^\pm) &= -T_\ell^\pm \\ \Delta(F_\ell) &= F_\ell \otimes e^{-\hbar T_\ell^-} + 1 \otimes F_\ell, & \epsilon(F_\ell) &= 0, & \mathcal{S}(F_\ell) &= -F_\ell e^{+\hbar T_\ell^-}. \end{aligned}$$

Note then that in this case $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ depends only on the matrix P . Indeed, in the spirit of Observation 4.12, in this special case the formal MpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is just a “logarithmic version” of the polynomial MpQUEA $U_{\mathbf{q}}(\mathfrak{g})$ in Observation 4.12.

In addition, in this case the FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ admits an alternative, somewhat significant presentation, as follows. Consider inside $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ the vectors $S_i := 2^{-1}(T_i^+ + T_i^-)$ and $A_i := 2^{-1}(T_i^+ - T_i^-)$ – for all $i \in I$; these clearly form yet another $\mathbb{k}[[\hbar]]$ -basis of \mathfrak{h} . Then $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ admits the obvious presentation given by construction and taking into account that $\{T_i^+, T_i^-\}_{i \in I}$ is a $\mathbb{k}[[\hbar]]$ -basis of \mathfrak{h} , but also the following, alternative one: it is the \hbar -adically complete, unital $\mathbb{k}[[\hbar]]$ -algebra with generators E_i, F_i, S_i, A_i ($i \in I$) and relations

$$\begin{aligned} [S_i, E_j] &= \frac{+(p_{ij} + p_{ji})}{2} E_j = +d_i a_{ij} E_j \\ [S_i, F_j] &= \frac{-(p_{ij} + p_{ji})}{2} F_j = -d_i a_{ij} F_j \\ [A_i, E_j] &= +(p_{ij} - p_{ji}) E_j, & [A_i, F_j] &= -(p_{ij} - p_{ji}) F_j \\ [A_i, S_j] &= 0, & E_i F_j - F_j E_i &= \delta_{i,j} e^{+\hbar A_i^+} \frac{e^{+\hbar S_i} - e^{-\hbar S_i}}{q_i^{+1} - q_i^{-1}} \end{aligned}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} X_i^{1-a_{ij}-k} X_j X_i^k = 0 \quad \begin{array}{l} X \in \{E, F\} \\ i \neq j. \end{array}$$

Moreover, the Hopf structure of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is then described (for $i \in I$) by

$$\begin{aligned} \Delta(E_\ell) &= E_\ell \otimes 1 + e^{+\hbar\Lambda_\ell} e^{+\hbar S_\ell} \otimes E_\ell \\ \Delta(S_\ell) &= S_\ell \otimes 1 + 1 \otimes S_\ell, \quad \Delta(\Lambda_\ell) = \Lambda_\ell \otimes 1 + 1 \otimes \Lambda_\ell \\ \Delta(F_\ell) &= F_\ell \otimes e^{-\hbar S_\ell} e^{+\hbar\Lambda_\ell} + 1 \otimes F_\ell. \end{aligned}$$

In particular, this implies that the \hbar -adically complete, unital subalgebra $U_\hbar(\mathfrak{b}_+)$, rep. $U_\hbar(\mathfrak{b}_-)$, of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ generated by all the E_i 's, resp. all the F_i 's, and all the S_i 's is isomorphic to the “standard” positive, resp. negative, Borel subalgebra in the derived version of Drinfeld’s QUEA $U_\hbar(\mathfrak{g})$. On the other hand, both subalgebras $U_\hbar(\mathfrak{b}_\pm)$ are *not Hopf* subalgebras inside $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, contrary to what happens in Drinfeld’s setup.

4.4. Further results on FoMpQUEAs

We present now a few more techniques, which provide alternative proofs for our results about the structure of quantum nilpotent, Cartan and Borel subalgebras, as well as the triangular decomposition results. This mainly follows in the footpath of a standard strategy, already used for one-parameter QUEA’s, yet we also provide a totally new idea for a proof of the triangular decomposition theorem.

4.4.1. Preformal multiparameter QUEAs and special representations

We introduce now some “preliminary versions of FoMpQUEAs”, essentially defined like the FoMpQUEAs but for dropping from their definition the quantum Serre relations. These “pre-FoMpQUEAs” will be a key tool in our analysis, as well as some special representations of them that we also introduce presently.

Let U be the unital, associative $\mathbb{k}[[\hbar]]$ -algebra generated by the $\mathbb{k}[[\hbar]]$ -submodule \mathfrak{h} together with elements E_i, F_i ($i \in I$), subject to the same relations as in (4.5), *except the last two* (the *quantum Serre relations*). Let $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ be the \hbar -adic completion of U . From the proof of Theorem 4.13, it follows at once that $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is a topological Hopf algebra over $\mathbb{k}[[\hbar]]$.

Let $V := \bigoplus_{i \in I} \mathbb{k}[[\hbar]] \cdot v_i$ be the free $\mathbb{k}[[\hbar]]$ -module with basis $\{v_i\}_{i \in I}$, let $T_\hbar(V)$ be the tensor algebra of V over $\mathbb{k}[[\hbar]]$, and let $\widehat{T}_\hbar(V)$ be the \hbar -adic completion of $T_\hbar(V)$. Then, $T_\hbar(V)$ is a free $\mathbb{k}[[\hbar]]$ -module with basis

$\{v_J\}_{J \in \mathcal{J}}$, where \mathcal{J} is the set of all finite sequences of elements in I and $v_J := v_{j_1} \otimes \cdots \otimes v_{j_r}$ – or simply $v_J := v_{j_1} \cdots v_{j_r}$ – is standard monomial notation for all $J := (j_1, \dots, j_r) \in \mathcal{J}$. For $J = (j_1, \dots, j_r) \in \mathcal{J}$ and $1 \leq k \leq r$, write $\widehat{J}_k := (j_1, \dots, j_{k-1}, j_{k+1}, \dots, j_r)$, $J_k := (j_{k+1}, \dots, j_r)$ and $\alpha_J := \sum_{\ell=1}^r \alpha_{j_\ell}$.

LEMMA 4.18. — *For every $\lambda \in \mathfrak{h}^*$, there exists a unique representation of U onto $T_{\hbar}(V)$ – which is then denoted $T_{\hbar}^{\lambda}(V)$ – such that, for all $J = (j_1, \dots, j_r) \in \mathcal{J}$*

$$\begin{aligned} F_i \cdot v_J &= v_{(i,J)}, & T \cdot v_J &= (\lambda(T) - \alpha_J(T))v_J \\ E_i \cdot v_J &= \sum_{\substack{1 \leq \ell \leq r \\ j_\ell = i}} \frac{q^{+\lambda(T_i^+) - \alpha_{J_\ell}(T_i^+) - q^{-\lambda(T_i^-) + \alpha_{J_\ell}(T_i^-)}}}{q_i^{+1} - q_i^{-1}} v_{j_\ell}. \end{aligned}$$

In addition, this representation – of U onto $T_{\hbar}(V)$ – induces by continuity a unique representation $\widehat{U}_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$ onto $\widehat{T}_{\hbar}(V)$, which is hereafter denoted by $\widehat{T}_{\hbar}^{\lambda}(V)$.

Proof. — We must prove that the equalities above do endow $T_{\hbar}(V)$ with a structure of U -module: to this end, let us performe a quick check to show that such an action is well-defined. For every $T, T' \in \mathfrak{h}$ we have

$$\begin{aligned} T' \cdot (T \cdot v_J) &= (\lambda(T) - \alpha_J(T))T' \cdot v_J \\ &= (\lambda(T) - \alpha_J(T))(\lambda(T') - \alpha_J(T'))v_J = T \cdot (T' \cdot v_J). \end{aligned}$$

Take now $T \in \mathfrak{h}$ and F_i with $i \in I$. Then

$$\begin{aligned} T \cdot (F_i \cdot v_J) - F_i \cdot (T \cdot v_J) &= T \cdot v_{(i,J)} - F_i \cdot ((\lambda(T) - \alpha_J(T))v_J) \\ &= (\lambda(T) - \alpha_{(i,J)}(T) - \lambda(T) + \alpha_J(T))v_{(i,J)} \\ &= (-\alpha_{(i,J)}(T) + \alpha_J(T))v_{(i,J)} \\ &= -\alpha_i(T)v_{(i,J)} = -\alpha_i(T)F_i \cdot v_J. \end{aligned}$$

Similarly, for $T \in \mathfrak{h}$ and E_i with $i \in I$ we find

$$\begin{aligned} T \cdot (E_i \cdot 0v_J) &= \sum_{\substack{1 \leq \ell \leq r \\ i = j_\ell}} (\lambda(T) - \alpha_{j_\ell}(T)) \frac{q^{+\lambda(T_i^+) - \alpha_{J_\ell}(T_i^+) - q^{-\lambda(T_i^-) + \alpha_{J_\ell}(T_i^-)}}}{q_i^{+1} - q_i^{-1}} v_{j_\ell} \\ E_i \cdot (T \cdot v_J) &= \sum_{\substack{1 \leq \ell \leq r \\ i = j_\ell}} (\lambda(T) - \alpha_J(T)) \frac{q^{+\lambda(T_i^+) - \alpha_{J_\ell}(T_i^+) - q^{-\lambda(T_i^-) + \alpha_{J_\ell}(T_i^-)}}}{q_i^{+1} - q_i^{-1}} v_{j_\ell}. \end{aligned}$$

Since $\alpha_J(T) - \alpha_{j_\ell}(T) = \alpha_{j_\ell}(T) = \alpha_i(T)$ for all $j_\ell = i$, we obtain

$$\begin{aligned} [T, E_i] \cdot v_J &= \sum_{\substack{1 \leq \ell \leq r \\ i = j_\ell}} \alpha_i(T) \frac{q^{+\lambda(T_i^+) - \alpha_{J_\ell}(T_i^+)} - q^{-\lambda(T_i^-) + \alpha_{J_\ell}(T_i^-)}}{q_i^{+1} - q_i^{-1}} v_{j_\ell} \\ &= \alpha_i(T) E_i \cdot v_J. \end{aligned}$$

Finally, to check the commuting relation between E_i and F_j we note first that $\sum_{n=0}^m \frac{\hbar^n}{n!} T^n \cdot v_J = \sum_{n=0}^m \frac{\hbar^n}{n!} (\lambda(T) - \alpha_J(T))^n \cdot v_J$ for all $T \in \mathfrak{h}$ and $m \geq 1$. Then, by the continuity of the linear action, we get that

$$e^{+\hbar T_i^+} \cdot v_J = q^{+(\lambda(T_i^+) - \alpha_J(T_i^+))} v_J$$

and

$$e^{-\hbar T_i^-} \cdot v_J = q^{-(\lambda(T_i^-) - \alpha_J(T_i^-))} v_J.$$

Then, for $i, j \in I$ with $i \neq j$, we have

$$\begin{aligned} E_i \cdot (F_j \cdot v_J) - F_j \cdot (E_i \cdot v_J) &= E_i \cdot v_{(j, J)} - F_j \cdot \left(\sum_{\substack{1 \leq \ell \leq r \\ i = j_\ell}} \frac{q^{+\lambda(T_i^+) - \alpha_{J_\ell}(T_i^+)} - q^{-\lambda(T_i^-) + \alpha_{J_\ell}(T_i^-)}}{q_i^{+1} - q_i^{-1}} v_{j_\ell} \right) \\ &= E_i \cdot v_{(j, J)} - \sum_{\substack{1 \leq \ell \leq r \\ i = j_\ell}} \frac{q^{+\lambda(T_i^+) - \alpha_{J_\ell}(T_i^+)} - q^{-\lambda(T_i^-) + \alpha_{J_\ell}(T_i^-)}}{q_i^{+1} - q_i^{-1}} v_{(j, j_\ell)}. \end{aligned}$$

First, if $i \neq j$, then $j_\ell = i \neq j$ and $(\widehat{j, J})_\ell = (j, \widehat{J}_\ell)$, $\alpha_{(j, J)_\ell} = \alpha_{J_\ell}$ for all $1 \leq \ell \leq r$. Hence

$$E_i \cdot v_{(j, J)} = \sum_{\substack{1 \leq \ell \leq r \\ i = j_\ell}} (q_i^{+1} - q_i^{-1})^{-1} (q^{+\lambda(T_i^+) - \alpha_{J_\ell}(T_i^+)} - q^{-\lambda(T_i^-) + \alpha_{J_\ell}(T_i^-)}) v_{(j, j_\ell)}$$

which implies that $E_i \cdot (F_j \cdot v_J) - F_j \cdot (E_i \cdot v_J) = 0$.

Assume now that $i = j$. Then for $i = j_\ell$ we have $(\widehat{i, J})_\ell = J$ and $\alpha_{(i, J)_\ell} = \alpha_J$ for $\ell = 1$, while $(\widehat{i, J})_\ell = (i, \widehat{J}_\ell)$ and $\alpha_{(i, J)_\ell} = \alpha_{J_\ell}$ for $\ell > 1$. This implies that

$$\begin{aligned} E_i \cdot v_{(i, J)} &= (q_i^{+1} - q_i^{-1})^{-1} (q^{+\lambda(T_i^+) - \alpha_J(T_i^+)} - q^{-\lambda(T_i^-) + \alpha_J(T_i^-)}) v_J \\ &\quad + \sum_{\substack{1 \leq \ell \leq r \\ i = j_\ell}} (q_i^{+1} - q_i^{-1})^{-1} (q^{+\lambda(T_i^+) - \alpha_{J_\ell}(T_i^+)} - q^{-\lambda(T_i^-) + \alpha_{J_\ell}(T_i^-)}) v_{(i, j_\ell)} \end{aligned}$$

and consequently

$$\begin{aligned} E_i \cdot (F_i \cdot v_J) - F_i \cdot (E_i \cdot v_J) &= \frac{q^{\lambda(T_i^+) - \alpha_J(T_i^+)} - q^{-\lambda(T_i^-) + \alpha_J(T_i^-)}}{q_i^{+1} - q_i^{-1}} \cdot v_J \\ &= \frac{e^{\hbar T_i^+} - e^{\hbar T_i^-}}{q_i^{+1} - q_i^{-1}} \cdot v_J. \end{aligned}$$

Therefore, the formulas above define indeed an action of U onto $T_{\hbar}(V)$.

Finally, since this action is $\mathbb{k}[[\hbar]]$ -linear, it induces a unique (topological) action of $\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ on $\widehat{T}_{\hbar}(V)$ by completion. This completes the proof. \square

With an entirely analogous proof, we obtain the following lemma:

LEMMA 4.19. — *For every $\lambda \in \mathfrak{h}^*$, there exists a unique representation of U onto $T_{\hbar}(V)$ – which is then denoted ${}^{\lambda}T_{\hbar}(V)$ – such that, for all $J = (j_1, \dots, j_r) \in \mathcal{J}$*

$$\begin{aligned} E_i \cdot v_J &= v_{(i,J)}, & T \cdot v_J &= (\lambda(T) + \alpha_J(T))v_J \\ F_i \cdot v_J &= \sum_{\substack{1 \leq \ell \leq r \\ j_{\ell} = i}} \frac{q^{-\lambda(T_i^-) - \alpha_{J_{\ell}}(T_i^-)} - q^{+\lambda(T_i^+) + \alpha_{J_{\ell}}(T_i^+)}}{q_i^{+1} - q_i^{-1}} v_{j_{\ell}}. \end{aligned}$$

Moreover, this representation, of U onto $T_{\hbar}(V)$, induces by continuity a representation $\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ onto $\widehat{T}_{\hbar}(V)$, hereafter denoted by ${}^{\lambda}\widehat{T}_{\hbar}(V)$. \square

Denote by U^0 , resp. U^+ , resp. U^- , the unital associative $\mathbb{k}[[\hbar]]$ -subalgebras of U generated by \mathfrak{h} , resp. by all the E_i 's, resp. by all the F_i 's ($i \in I$). Write $\widetilde{U}^0 (= \widetilde{U}_{\hbar}(\mathfrak{h}) := \widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}))$, resp. \widetilde{U}^+ , resp. \widetilde{U}^- , for the \hbar -adic completion of them: all these are topological $\mathbb{k}[[\hbar]]$ -subalgebras of $\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$.

As a first consequence of Lemma 4.18, we get the following:

PROPOSITION 4.20. — *The Cartan subalgebra $\widetilde{U}^0 = \widetilde{U}_{\hbar}(\mathfrak{h}) = \widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$ is the \hbar -adic completion of $S_{\hbar}(\mathfrak{h})$, the symmetric algebra of \mathfrak{h} over $\mathbb{k}[[\hbar]]$. In particular, $\widetilde{U}_{\hbar}(\mathfrak{h})$ is independent of P and \mathcal{R} (though not of \mathfrak{h}) – whence the simplified notation.*

Proof. — We provide two different, independent proofs.

First Proof. — Let $T_{\hbar}(\mathfrak{h})$, resp. $S_{\hbar}(\mathfrak{h})$, be the tensor algebra, resp. the symmetric algebra, of \mathfrak{h} over $\mathbb{k}[[\hbar]]$, and let $\widehat{T}_{\hbar}(\mathfrak{h})$, resp. $\widehat{S}_{\hbar}(\mathfrak{h})$ be the \hbar -adic completion of them. By the commutation relations among elements of \mathfrak{h} in

$\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, we have a diagram of morphisms

$$\begin{array}{ccccccc}
 \widehat{T}_{\hbar}(\mathfrak{h}) & \xrightarrow{p_{\hbar,S}} & \widehat{S}_{\hbar}(\mathfrak{h}) & \xrightarrow{p_{\hbar,U}} & \widetilde{U}_{\hbar}(\mathfrak{h}) & \hookrightarrow & \widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \\
 \uparrow & & \uparrow & \nearrow & \uparrow & & \\
 T_{\hbar}(\mathfrak{h}) & \xrightarrow{p_S} & S_{\hbar}(\mathfrak{h}) & \xrightarrow{p_{U^0}} & U^0 & & \\
 \uparrow & \nearrow & & & & & \\
 \mathfrak{h} & & & & & &
 \end{array}$$

where the maps $p_{\hbar,S}$, $p_{\hbar,U}$, p_S and p_{U^0} are the canonical epimorphisms, ι_{\hbar} is the canonical inclusion, all vertical arrows are canonical embeddings, and the diagonal arrows $\mathfrak{h} \rightarrow S_{\hbar}(\mathfrak{h})$ too. We want to show that $p_{\hbar,U}$ is in fact an isomomorphism.

Let $V := \bigoplus_{i \in I} \mathbb{k}[[\hbar]].v_i$ be the free $\mathbb{k}[[\hbar]]$ -module with basis $\{v_i\}_{i \in I}$. Choosing $\lambda \in \mathfrak{h}^*$ and restricting the action defined in Lemma 4.18 to the image of i_{\hbar} , we have that $\widetilde{U}_{\hbar}(\mathfrak{h})$ acts on $\widehat{T}_{\hbar}^{\lambda}(V)$ by the character λ via $T.v_J = (\lambda(T) - \alpha_J(T))v_J$, for all $T \in \mathfrak{h}$ and for all $J = (j_1, \dots, j_r) \in \mathcal{J}$. In particular, $T.v_{\emptyset} = T \cdot 1 = \lambda(T)$ for all $T \in \mathfrak{h}$. This action induces an action of $\widehat{S}_{\hbar}(\mathfrak{h})$, and of its $\mathbb{k}[[\hbar]]$ -subalgebra $S_{\hbar}(\mathfrak{h})$, on $\widehat{T}_{\hbar}^{\lambda}(V)$ via the epimorphism $p_{\hbar,U}$. By the very definition, this action coincides with the unique action of $S_{\hbar}(\mathfrak{h})$ on $\widehat{T}_{\hbar}^{\lambda}(V)$ defined by the character $\lambda \in \mathfrak{h}^*$ by the universal properties of the symmetric and the tensor algebras.

Let $t \in S_{\hbar}(\mathfrak{h})$ be such that $p_{\hbar,U}(t) = 0$: then $0 = t.v_{\emptyset} = \lambda_S(t)$ – extending $\lambda \in \mathfrak{h}^*$ to a $\mathbb{k}[[\hbar]]$ -algebra character λ_S of $S_{\hbar}(\mathfrak{h})$. Since λ is arbitrary, we get $\lambda_S(t) = 0$ for all $\lambda \in \mathfrak{h}^*$, so that $t = 0$. Similarly, λ_S further extends, canonically and uniquely, to a \hbar -adically continuous character of $\widehat{S}_{\hbar}(\mathfrak{h})$, denoted $\lambda_{\widehat{S}}$. Then for any $\widehat{t} \in \widehat{S}_{\hbar}(\mathfrak{h})$ such that $p_{\hbar,U}(\widehat{t}) = 0$ we have $0 = t.v_{\emptyset} = \lambda_{\widehat{S}}(\widehat{t})$, thus $\lambda_{\widehat{S}}(\widehat{t}) = 0$ for all $\lambda \in \mathfrak{h}^*$, which implies $\widehat{t} = 0$. Hence $p_{\hbar,U}$ is injective and $\widehat{S}_{\hbar}(\mathfrak{h}) \cong \widetilde{U}_{\hbar}(\mathfrak{h})$.

Second Proof. — By definition there is a $\mathbb{k}[[\hbar]]$ -linear morphism from \mathfrak{h} to $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, whose image we denote by \mathfrak{h}' ; in other words, \mathfrak{h}' is the $\mathbb{k}[[\hbar]]$ -submodule of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ spanned by the generators $T \in \mathfrak{h}$. By definition, $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$ is (topologically) generated by \mathfrak{h}' , which in turn is a Lie subalgebra inside the Lie algebra of primitive elements of the Hopf algebra $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$. Since we are in characteristic zero, by Milnor–Moore’s Theorem we deduce that $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$ is indeed nothing but the \hbar -adic completion of the universal

enveloping algebra $U(\mathfrak{h}')$ of \mathfrak{h}' ; in turn, the latter coincides with the symmetric $\mathbb{k}[[\hbar]]$ -algebra $S_{\hbar}(\mathfrak{h}')$ – hence its completion coincides with $\widehat{S}_{\hbar}(\mathfrak{h}')$ – because the multiplication therein is commutative, by construction!

Finally, we observe that the built-in epimorphism $\mathfrak{h} \twoheadrightarrow \mathfrak{h}'$ is indeed an isomorphism, so that $\mathfrak{h}' \cong \mathfrak{h}$. This is proved again making use of Lemma 4.18 above, in particular looking at how \mathfrak{h} acts on each representation $\widehat{T}_{\hbar}^{\lambda}(V)$, for all $\lambda \in \mathfrak{h}^*$, along the same lines as in the last part of the *First Proof* above. \square

Let U' be the $\mathbb{k}[[\hbar]]$ -subalgebra of $\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ generated by U^- , U^+ and $\widetilde{U}^0 = \widetilde{U}_{\hbar}(\mathfrak{h})$ – the \hbar -adic completion of U^0 . By Theorem 4.13, it follows that U' is a (topological) Hopf $\mathbb{k}[[\hbar]]$ -subalgebra of $\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$.

We fix some more notation. For any finite sequence $J = (j_1, j_2, \dots, j_r)$ – with $r \geq 1$ – of elements in I , we set $T_{\emptyset}^{\pm} := 0$ and $E_{\emptyset} := 1 =: F_{\emptyset}$, and in general

$$T_J^{\pm} := T_{j_1}^{\pm} + T_{j_2}^{\pm} + \dots + T_{j_r}^{\pm}, \quad E_J := E_{j_1} E_{j_2} \dots E_{j_r}, \quad F_J := F_{j_1} F_{j_2} \dots F_{j_r}.$$

As the coproduct of the elements E_i 's and F_i 's in U' and in $\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ coincides with the one defined for the one-parameter polynomial QUEA, the following lemma follows at once, from [33, Lemma 4.12]:

LEMMA 4.21. — *Let J be a finite sequence as above. Then there exist Laurent polynomials $c_{A,B}^J \in \mathbb{Z}[x, x^{-1}]$, indexed by finite sequences of elements of I , with $\alpha_J = \alpha_A + \alpha_B$ and such that, both in U' and in $\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, one has*

$$(4.14) \quad \begin{aligned} \Delta(E_J) &= \sum_{A,B} c_{A,B}^J(q) E_A e^{\hbar T_B^+} \otimes E_B \\ \Delta(F_J) &= \sum_{A,B} c_{A,B}^J(q^{-1}) F_A \otimes e^{\hbar T_A^-} F_B. \end{aligned}$$

Moreover, one has $c_{A,\emptyset}^J = \delta_{A,J}$ and $c_{\emptyset,B}^J = \delta_{B,J}$. \square

As an intermediate result, we get a triangular decomposition for $\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$:

PROPOSITION 4.22. — *The multiplication maps*

$$\overrightarrow{\mu}_3 : \widetilde{U}^- \otimes \widetilde{U}_{\hbar}(\mathfrak{h}) \otimes \widetilde{U}^+ \longrightarrow \widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$$

and

$$\overleftarrow{\mu}_3 : \widetilde{U}^+ \otimes \widetilde{U}_{\hbar}(\mathfrak{h}) \otimes \widetilde{U}^- \longrightarrow \widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$$

induced by restriction of multiplication in $\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ – the tensor products here being the \hbar -adically completed ones – are both isomorphisms of topological $\mathbb{k}[[\hbar]]$ -modules.

Proof. — It is enough to prove one case, say that of $\vec{\mu}_3$, the other one being entirely similar.

Consider the map $\mu_3' : U^- \otimes \tilde{U}_{\hbar}(\mathfrak{h}) \otimes U^+ \rightarrow U'$ induced by restriction of multiplication in U' . We show that this map is bijective.

We prove first that μ_3' is *surjective*. Let $\{H_g\}_{g \in \mathcal{G}}$ be a $\mathbb{k}[[\hbar]]$ -basis of \mathfrak{h} . Thanks to the defining relations in the first two lines of (4.5), we see at once that U' is $\mathbb{k}[[\hbar]]$ -spanned (in \hbar -adic sense) by the set of “monomials”

$$\left\{ F_{i_1} \cdots F_{i_n} \cdot H_{\underline{i}, \underline{j}} \cdot E_{j_1} \cdots E_{j_m} \mid n, s, m \in \mathbb{N}, i_a, j_b \in I \forall a, b, H_{\underline{i}, \underline{j}} \in \tilde{U}_{\hbar}(\mathfrak{h}) \right\}$$

and then this guarantees that μ_3' is onto, since

$$F_{i_1} \cdots F_{i_n} \cdot H_{\underline{i}, \underline{j}} \cdot E_{j_1} \cdots E_{j_m} = \mu_3'((F_{i_1} \cdots F_{i_n}) \otimes H_{\underline{i}, \underline{j}} \otimes (E_{j_1} \cdots E_{j_m})).$$

Our second, last task is to prove that $\vec{\mu}_3$ is *injective*. Let $\lambda, \theta \in \mathfrak{h}^*$ and consider the $\tilde{U}_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$ -modules $\hat{T}_{\hbar}^{\lambda}(V)$ and ${}^{\theta}\hat{T}_{\hbar}(V)$ given by Lemmas 4.18 and 4.19, respectively. Then $\tilde{U}_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$ acts on the tensor product $\hat{T}_{\hbar}^{\lambda}(V) \otimes {}^{\theta}\hat{T}_{\hbar}(V)$ too, and this yields by restriction a U' -action as well. Assume we have a linear dependence relation

$$\sum_{J, L} a_{J, L} F_J H_{J, L} E_L = 0$$

for finitely many elements $a_{J, L} \in \mathbb{k}[[\hbar]]$, where J , and L are finite sequences of elements in I and $H_{J, L} \in \tilde{U}_{\hbar}(\mathfrak{h})$.

In the set of finite sequences of elements in I , we consider the partial order given by $J = (j_1, j_2, \dots, j_r) > (\ell_1, \ell_2, \dots, \ell_s) =: L$ if and only if $\alpha_J - \alpha_L = \sum_k \alpha_{j_k} - \sum_k \alpha_{\ell_k} = \sum_t \alpha_{i_t}$ for some simple roots α_{i_t} . Choose J_0 such that $a_{J_0, L} \neq 0$ for some L , and such that J_0 is maximal with respect to the order given above.

Now, for $v_{\emptyset} \in \hat{T}_{\hbar}^{\lambda}(V) \setminus \{0\}$ and $w_{\emptyset} \in {}^{\theta}\hat{T}_{\hbar}(V) \setminus \{0\}$ being “highest weight vector” and “lowest weight vector”, respectively, definitions give

$$\begin{aligned} 0 &= \left(\sum_{J, K, L} a_{J, L} F_J H_{J, L} E_L \right) \cdot (v_{\emptyset} \otimes w_{\emptyset}) \\ &= \sum_{J, L} a_{J, L} (F_J H_{J, L}) \cdot (E_L \cdot (v_{\emptyset} \otimes w_{\emptyset})) \\ &= \sum_{J, L} a_{J, L} (F_J H_{J, L}) \cdot \left(\sum_{A, B} c_{A, B}^L(q) \left(E_A e^{\hbar T_B^+} \right) \cdot v_{\emptyset} \otimes (E_B \cdot w_{\emptyset}) \right) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{J,L} a_{J,L} (F_J H_{J,L}) \cdot \left(\sum_{A,B} c_{A,B}^L(q) q^{\lambda(T_B^+)} E_A \cdot v_\emptyset \otimes w_B \right) \\
 &= \sum_{J,L} a_{J,L} c_{\emptyset,L}^L(q) q^{\lambda(T_L^+)} (F_J H_{J,L}) \cdot (v_\emptyset \otimes w_L) \\
 &= \sum_{J,L} a_{J,L} q^{\lambda(T_L^+)} \lambda((H_{J,L})_{(1)}) (\theta + \alpha_L) ((H_{J,L})_{(2)}) F_J \cdot (v_\emptyset \otimes w_L) \\
 &= \sum_{J,L} a_{J,L} q^{\lambda(T_L^+)} \lambda((H_{J,L})_{(1)}) (\theta + \alpha_L) ((H_{J,L})_{(2)}) \\
 &\quad \cdot \left(\sum_{A,B} c_{A,B}^J(q^{-1}) F_A \cdot v_\emptyset \otimes (e^{\hbar T_A^-} F_B) \cdot w_L \right) \\
 &= \sum_{J,L} a_{J,L} q^{\lambda(T_L^+)} \lambda((H_{J,L})_{(1)}) (\theta + \alpha_L) ((H_{J,L})_{(2)}) \\
 &\quad \times \sum_{A,B} c_{A,B}^J(q^{-1}) v_A \otimes (e^{\hbar T_A^-} F_B) \cdot w_L
 \end{aligned}$$

where we get third and fifth equality from Lemma 4.21, and we recall $c_{\emptyset,L}^L(q) = 1$.

Consider now those coefficients with $J = J_0$. Since $\alpha_{J_0} = \alpha_A + \alpha_B$, we have that $A, B \leq J_0$ and $A = J_0$ if and only if $B = \emptyset$; therefore

$$\begin{aligned}
 0 &= \sum_L a_{J_0,L} q^{\lambda(T_L^+)} \lambda((H_{J_0,L})_{(1)}) (\theta + \alpha_L) ((H_{J_0,L})_{(2)}) \\
 &\quad \times c_{J_0,\emptyset}^{J_0}(q^{-1}) v_{J_0} \otimes e^{\hbar T_{J_0}^-} \cdot w_L \\
 &= \sum_L a_{J_0,L} q^{\lambda(T_L^+)} \lambda((H_{J_0,L})_{(1)}) (\theta + \alpha_L) ((H_{J_0,L})_{(2)}) q^{(\theta + \alpha_L)(T_{J_0}^-)} v_{J_0} \otimes w_L.
 \end{aligned}$$

Since $\{v_J \otimes w_L\}_{J,L}$ is a basis of the free $\mathbb{k}[[\hbar]]$ -module $\widehat{T}_\hbar^\lambda(V) \otimes^\theta \widehat{T}_\hbar^-(V)$, for all L such that $a_{J_0,L} \neq 0$ we must have

$$\begin{aligned}
 0 &= a_{J_0,L} q^{\lambda(T_L^+)} \lambda((H_{J_0,L})_{(1)}) (\theta + \alpha_L) ((H_{J_0,L})_{(2)}) q^{(\theta + \alpha_L)(T_{J_0}^-)} \\
 &= a_{J_0,L} q^{\lambda(T_L^+) + (\theta + \alpha_L)(T_{J_0}^-)} (\lambda * (\theta + \alpha_L)) (H_{J_0,L})
 \end{aligned}$$

where “ $*$ ” denotes the convolution product between characters; this implies that $0 = a_{J_0,L} (\lambda * (\theta + \alpha_L)) (H_{J_0,L})$ for all $\lambda, \theta \in \mathfrak{h}^*$. Since $\widetilde{U}_\hbar(\mathfrak{h}) \cong \widehat{S}_\hbar(\mathfrak{h})$, this holds true if and only if $a_{J_0,L} = 0$ for all L , a contradiction. Thus, μ'_3 is injective.

Finally, as μ'_3 is a $\mathbb{k}[[\hbar]]$ -linear map, it is an isomorphism between topological algebras which extends uniquely to an isomorphism $\overrightarrow{\mu}_3$ on their

completions. Since the completion of U' is exactly our $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, we eventually obtain the isomorphism $\vec{\mu}_3 : \tilde{U}^- \otimes \tilde{U}_{\hbar}(\mathfrak{h}) \otimes \tilde{U}^+ \rightarrow \tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ which is described just like in the claim. \square

For the last steps, we need more notation: if $X \in \{F, E\}$, $i, j \in I$ ($i \neq j$), set

$$u_{ij}^X := \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_j^{-k/2} X_i^{1-a_{ij}-k} X_j X_i^k \in \tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}).$$

Let \mathcal{E}^+ , resp. \mathcal{F}^- , be the closed, two-sided ideal of \tilde{U}^+ , resp. of \tilde{U}^- , generated by all the u_{ij}^E 's, resp. u_{ij}^F 's ($i \neq j$). Denote by $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_{\pm})$ the unital, \hbar -adically complete topological $\mathbb{k}[[\hbar]]$ -subalgebra of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ generated by all the E_i 's, resp. F_i 's ($i \in I$).

Next result describes explicitly the structure of the Cartan and of the (positive/negative) nilpotent subalgebras in our FoMpQUEAs:

PROPOSITION 4.23.

- (a) The closed two-sided ideal of $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ generated by all the u_{ij}^E 's, resp. all the u_{ij}^F 's, is equal to the image of $\tilde{U}^- \otimes \tilde{U}_{\hbar}(\mathfrak{h}) \otimes \mathcal{E}^+$, resp. $\mathcal{F}^- \otimes \tilde{U}_{\hbar}(\mathfrak{h}) \otimes \tilde{U}^+$, under the multiplication map $\vec{\mu}_3 : \tilde{U}^- \otimes \tilde{U}_{\hbar}(\mathfrak{h}) \otimes \tilde{U}^+ \rightarrow \tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$.
 An entirely similar claim holds true as well with $\overleftarrow{\mu}_3$ replacing $\vec{\mu}_3$.
- (b) $\tilde{U}_{\hbar}(\mathfrak{h}) \cong U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$ through the obvious canonical epimorphism. In particular, $U_{\hbar}(\mathfrak{h}) := U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$ is the \hbar -adic completion of $S_{\hbar}(\mathfrak{h})$, the symmetric algebra of \mathfrak{h} over $\mathbb{k}[[\hbar]]$, hence it is independent of P and \mathcal{R} (it only depends on \mathfrak{h}).
- (c) The algebra $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_+)$, resp. $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_-)$, is isomorphic to the unital, \hbar -adically complete topological $\mathbb{k}[[\hbar]]$ -algebra generated by the E_i 's ($i \in I$), resp. the F_i 's ($i \in I$), with relations $u_{ij}^E = 0$, resp. $u_{ij}^F = 0$, for $i \neq j$.

Proof.

(a). — This follows *mutatis mutandis* from [33, Lemma 4.20].

(b). — The first part of the claim follows from the proof of [33, Theorem 4.21 d)]. Indeed, let \mathcal{J} be the kernel of the canonical, \hbar -adically continuous epimorphism $p : \tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \twoheadrightarrow U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$; by construction, it is the closed two-sided ideal generated by the u_{ij}^E and u_{ij}^F for all $i \neq j$ in I . By (a), we have

$$(4.15) \quad \mathcal{J} = \vec{\mu}_3 \left(\tilde{U}^- \otimes \tilde{U}_{\hbar}(\mathfrak{h}) \otimes \mathcal{E}^+ + \mathcal{F}^- \otimes \tilde{U}_{\hbar}(\mathfrak{h}) \otimes \tilde{U}^+ \right).$$

Now, the kernel of the canonical epimorphism $p|_{\tilde{U}_\hbar(\mathfrak{h})} : \tilde{U}_\hbar(\mathfrak{h}) \rightarrow U_\hbar(\mathfrak{h}) := U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$ is nothing but $\mathcal{J} \cap \tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$; then by (4.15) it is equal to the image under $\vec{\mu}_3^\rightarrow$ of

$$\left(\tilde{U}^- \otimes \tilde{U}_\hbar(\mathfrak{h}) \otimes \mathcal{E}^+ + \mathcal{F}^- \otimes \tilde{U}_\hbar(\mathfrak{h}) \otimes \tilde{U}^+ \right) \cap (\mathbb{k}[[\hbar]] \otimes \tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}) \otimes \mathbb{k}[[\hbar]])$$

and since the latter is obviously trivial, we get that $\mathcal{J} \cap \tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}) = 0$. Therefore, $p|_{\tilde{U}_\hbar(\mathfrak{h})} : \tilde{U}_\hbar(\mathfrak{h}) \rightarrow U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$ is injective, hence it is an isomorphism.

The second part of the claim follows from the first, coupled with Proposition 4.20.

(c). — This is proved much like item (b), but using that

$$\begin{aligned} & \mathcal{J} \cap \tilde{U}^+ \\ &= \vec{\mu}_3^\rightarrow \left((\tilde{U}^- \otimes \tilde{U}_\hbar(\mathfrak{h}) \otimes \mathcal{E}^+ + \mathcal{F}^- \otimes \tilde{U}_\hbar(\mathfrak{h}) \otimes \tilde{U}^+) \cap (\mathbb{k}[[\hbar]] \otimes \mathbb{k}[[\hbar]] \otimes \tilde{U}^+) \right) \\ &= \vec{\mu}_3^\rightarrow (\mathbb{k}[[\hbar]] \otimes \mathbb{k}[[\hbar]] \otimes \mathcal{E}^+) = \mathcal{E}^+ \end{aligned}$$

$$\begin{aligned} & \mathcal{J} \cap \tilde{U}^- \\ &= \vec{\mu}_3^\rightarrow \left((\tilde{U}^- \otimes \tilde{U}_\hbar(\mathfrak{h}) \otimes \mathcal{E}^+ + \mathcal{F}^- \otimes \tilde{U}_\hbar(\mathfrak{h}) \otimes \tilde{U}^+) \cap (\tilde{U}^- \otimes \mathbb{k}[[\hbar]] \otimes \mathbb{k}[[\hbar]]) \right) \\ &= \vec{\mu}_3^\rightarrow (\mathcal{F}^- \otimes \mathbb{k}[[\hbar]] \otimes \mathbb{k}[[\hbar]]) = \mathcal{F}^-. \quad \square \end{aligned}$$

Remark 4.24. — An alternative, independent argument which also leads to prove Proposition 4.22 and Proposition 4.23 goes as follows.

First, we can state a strict analogue of Lemma 4.8 where we replace the $\mathbb{k}[[\hbar]]$ -module $U_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g}) := U^- \otimes U_\hbar(\mathfrak{h}) \otimes U^+$ – taking *complete* tensor product – with its “parent” $\tilde{U}_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g}) := \tilde{U}^- \otimes \tilde{U}_\hbar(\mathfrak{h}) \otimes \tilde{U}^+$: the claim will be that $\tilde{U}_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g})$ bears a structure of \hbar -adically complete, topological $\mathbb{k}[[\hbar]]$ -algebra which is uniquely determined by the same recipe (and formulas) as in Lemma 4.8 – the proof will be quite the same, only a bit simpler because there will be less relations to deal with.

Second, we provide a strict analogue of Theorem 4.9, now concerning $\tilde{U}_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g})$ and $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ instead of $U_{P,\hbar}^{\mathcal{R},\otimes}(\mathfrak{g})$ and $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$; here again, the proof will follow in the footsteps of the one for the previous case. This last result eventually will be just a reformulation of Proposition 4.22 and Proposition 4.23 altogether.

4.4.2. Triangular decomposition – revisited

Using the previous constructions, we present now alternative proofs of “triangular decomposition” for FoMpQUEAs. Again, it is enough to prove

one of the various isomorphisms in the statement of Theorem 4.11, so now we shall deal with

$$U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_-) \widehat{\otimes}_{\mathbb{k}[[\hbar]]} U_{\hbar}(\mathfrak{h}) \widehat{\otimes}_{\mathbb{k}[[\hbar]]} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_+) \cong U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$$

where the (would-be) isomorphism is induced by multiplication.

Second Proof. — By construction and by the results reported in Proposition 4.23, we have isomorphisms of (topological) $\mathbb{k}[[\hbar]]$ -modules

$$\begin{aligned} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) &\cong \left(\widetilde{U}^- \otimes \widetilde{U}_{\hbar}(\mathfrak{h}) \otimes \widetilde{U}^+ \right) / \left(\mathcal{F}^- \otimes \widetilde{U}_{\hbar}(\mathfrak{h}) \otimes \widetilde{U}^+ + \widetilde{U}^- \otimes \widetilde{U}_{\hbar}(\mathfrak{h}) \otimes \mathcal{E}^+ \right) \\ &\cong \left(\widetilde{U}^- / \mathcal{F}^- \right) \otimes \widetilde{U}_{\hbar}(\mathfrak{h}) \otimes \left(\widetilde{U}^+ / \mathcal{E}^+ \right) \\ &\cong U^- \otimes \widetilde{U}_{\hbar}(\mathfrak{h}) \otimes U^+ = U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_-) \otimes U_{\hbar}(\mathfrak{h}) \otimes U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_+) \end{aligned}$$

(using simplified notation for the tensor product), where the isomorphism (from right to left) is induced by multiplication, hence we are done. \square

Third Proof. — The claim amount to saying that the $\mathbb{k}[[\hbar]]$ -linear map

$$(4.16) \quad \mu_{\mathfrak{z}} : U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_-) \widehat{\otimes}_{\mathbb{k}[[\hbar]]} U_{\hbar}(\mathfrak{h}) \widehat{\otimes}_{\mathbb{k}[[\hbar]]} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_+) \longrightarrow U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$$

induced by multiplication (on three factors) is in fact *bijective*.

To begin with, let U^- , resp. U^0 , resp. U^+ , be the $\mathbb{k}[[\hbar]]$ -subalgebra of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ generated by all the F_i ' ($i \in I$), resp. all of \mathfrak{h} , resp. E_i 's ($i \in I$); then let $U_{\hbar}^- := U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_-)$, resp. $U_{\hbar}^0 := U_{\hbar}(\mathfrak{h})$, resp. $U_{\hbar}^+ := U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_+)$. Furthermore, let us define $U^{\downarrow} := \text{Span}_{\mathbb{k}[[\hbar]]} \left(\{ \underline{F} \cdot \underline{H} \cdot \underline{E} \}_{\underline{F}, \underline{H}, \underline{E}} \right)$ where the \underline{F} , resp. \underline{H} , resp. \underline{E} , are all possible monomials in the F_i 's ($i \in I$), resp. in the H 's ($H \in \mathfrak{h}$), resp. in the E_j 's ($j \in I$); note that U^{\downarrow} is a $\mathbb{k}[[\hbar]]$ -submodule of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, but *not* a $\mathbb{k}[[\hbar]]$ -subalgebra. Finally, we let U be the $\mathbb{k}[[\hbar]]$ -subalgebra of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ generated by U^{\downarrow} .

Clearly, the map $\mu_{\mathfrak{z}}$ in (4.16) restricts to a similar map

$$(4.17) \quad \mu : U^- \otimes_{\mathbb{k}[[\hbar]]} U^0 \otimes_{\mathbb{k}[[\hbar]]} U^+ \longrightarrow U^{\downarrow}$$

which again is induced by multiplication. We shall presently prove the following

CLAIM 4.25. — *The map μ in (4.17) is bijective.*

Once this is settled, we have that both μ and its inverse μ^{-1} will be (mutually inverse) isomorphisms of $\mathbb{k}[[\hbar]]$ -modules, hence in particular continuous for the \hbar -adic topology. Then, taking \hbar -adic completion on both sides, μ and μ^{-1} will canonically induce (bicontinuous) mutually inverse isomorphisms of topological $\mathbb{k}[[\hbar]]$ -modules, denoted μ_{\hbar} and μ_{\hbar}^{-1} , between the \hbar -adic completion of $U^- \otimes_{\mathbb{k}[[\hbar]]} U^0 \otimes_{\mathbb{k}[[\hbar]]} U^+$ and the \hbar -adic completion

of U^\downarrow . Now, by construction the \hbar -adic completion of $U^- \otimes_{\mathbb{k}[[\hbar]]} U^0 \otimes_{\mathbb{k}[[\hbar]]} U^+$ is just $U_{\hbar}^- \widehat{\otimes}_{\mathbb{k}[[\hbar]]} U_{\hbar}^0 \widehat{\otimes}_{\mathbb{k}[[\hbar]]} U_{\hbar}^+$, while the \hbar -adic completion of U^\downarrow (which coincides with the \hbar -adic completion of U) is nothing but $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$. In particular, again by construction μ_{\hbar} coincides with $\underline{\mu}_3$ in (4.16), hence the latter in turn is an isomorphism of topological $\mathbb{k}[[\hbar]]$ -modules.

Thus we are left with the task to prove the *Claim* 4.25 above. As it is clear that μ is *surjective*, hence we only have to prove that it is *injective* too. For this, we use the Hopf structure of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, in particular its coproduct, adapting an argument that does work in the uniparameter case, see e.g. [32, Section 3.1], or [39, Section 6.1.5].

We saw above that U^\downarrow is $\mathbb{k}[[\hbar]]$ -spanned by the set of monomials of the form $\underline{F} \cdot \underline{H} \cdot \underline{E}$ where each single factor in turn is of type $\underline{F} = F_{i_1} \cdots F_{i_n}$, resp. $\underline{H} = H_{\ell_1} \cdots H_{\ell_s}$, resp. $\underline{E} = E_{j_1} \cdots E_{j_m}$, with $i_1, \dots, i_n, j_1, \dots, j_m \in I$ ($n, m \in \mathbb{N}$) and $H_{\ell_1}, \dots, H_{\ell_s}$ ($s \in \mathbb{N}$) ranging in some fixed, ordered $\mathbb{k}[[\hbar]]$ -basis $\{H_\ell\}_{\ell \in \mathcal{L}}$ of \mathfrak{h} . Similarly (with same notation), $U^- \otimes_{\mathbb{k}[[\hbar]]} U^0 \otimes_{\mathbb{k}[[\hbar]]} U^+$ is $\mathbb{k}[[\hbar]]$ -spanned by the set of (tensor) monomial of the form $\underline{F} \otimes \underline{H} \otimes \underline{E}$, with $\underline{F}, \underline{H}$ and \underline{E} as before.

To begin with, let \mathcal{G} be the free Abelian group with basis $\{\varepsilon_i\}_{i \in I}$: we endow it with the product order, hereafter denoted by \preceq , induced by the standard order in \mathbb{Z} . We define on U a $\mathbb{k}[[\hbar]]$ -algebra \mathcal{G} -grading $U = \bigoplus_{\gamma \in \mathcal{G}} U_\gamma$ given on generators by

$$\partial(E_i) := +\varepsilon_i, \quad \partial(\underline{H}) := 0, \quad \partial(F_j) := -\varepsilon_j$$

for all $i, j \in I$, $\underline{H} \in U_{\hbar}(\mathfrak{h}) \setminus \{0\}$ – the reader can easily check that these formulas on the generators are indeed compatible with the relations in (4.5) among them. This restricts to similar \mathcal{G} -gradings on U^\downarrow as well as on U^-, U^0 and U^+ – which then are *graded* subalgebras of U with respect to this \mathcal{G} -grading – hence on $U^- \otimes_{\mathbb{k}[[\hbar]]} U^0 \otimes_{\mathbb{k}[[\hbar]]} U^+$ too. Note then that each one of the monomials $\underline{F} \cdot \underline{H} \cdot \underline{E} = F_{i_1} \cdots F_{i_n} \cdot H_{\ell_1} \cdots H_{\ell_s} \cdot E_{j_1} \cdots E_{j_m}$ and similarly $\underline{F} \otimes \underline{H} \otimes \underline{E}$ considered above is \mathcal{G} -homogeneous of degree

$$\partial(F_{i_1} \cdots F_{i_n} \cdot H_{\ell_1} \cdots H_{\ell_s} \cdot E_{j_1} \cdots E_{j_m}) = \varepsilon_{j_1} + \cdots + \varepsilon_{j_m} - \varepsilon_{i_1} - \cdots - \varepsilon_{i_n} \in \mathcal{G}.$$

Now consider the twofold iteration $\Delta^{(2)} := (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$ of the coproduct map Δ of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$. By the very definition of Δ we easily see that

$$(4.18) \quad \begin{aligned} \Delta^{(2)}(F_{i_1} \cdots F_{i_n}) &= 1 \otimes 1 \otimes (F_{i_1} \cdots F_{i_n}) + \sum_t u_t \otimes v_t \otimes w_t \\ \Delta^{(2)}(E_{j_1} \cdots E_{j_m}) &= (E_{j_1} \cdots E_{j_m}) \otimes 1 \otimes 1 + \sum_k a_k \otimes b_k \otimes c_k \end{aligned}$$

where the w_t 's in the first line of (4.18) are elements in $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ which are \mathcal{G} -homogeneous of degree strictly *greater* – for the order \preceq in \mathcal{G} – than $\partial(F_{i_1} \cdots F_{i_n})$, while similarly the a_k 's in second line are elements in $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ which are \mathcal{G} -homogeneous of degree strictly *smaller* than $\partial(E_{j_1} \cdots E_{j_m})$; in short,

$$(4.19) \quad \partial(w_t) \succneq \partial(F_{j_1} \cdots F_{j_m}) \quad \forall t, \quad \partial(a_k) \precneq \partial(E_{i_1} \cdots E_{i_n}) \quad \forall k.$$

Eventually, from (4.18) and (4.19) together we get

$$\begin{aligned} & \Delta^{(2)}(F_{i_1} \cdots F_{i_n} H_{\ell_1} \cdots H_{\ell_s} E_{j_1} \cdots E_{j_m}) \\ &= \Delta^{(2)}(F_{i_1} \cdots F_{i_n} \cdot \underline{H} \cdot E_{j_1} \cdots E_{j_m}) \\ &= \Delta^{(2)}(F_{i_1} \cdots F_{i_n}) \Delta^{(2)}(\underline{H}) \Delta^{(2)}(E_{j_1} \cdots E_{j_m}) \\ &= (F_{i_1} \cdots F_{i_n} \cdot \underline{H}_{(1)}) \otimes \underline{H}_{(2)} \otimes (\underline{H}_{(3)} \cdot E_{j_1} \cdots E_{j_m}) + \sum_r \Phi_r \otimes \Xi_r \otimes \Omega_r \end{aligned}$$

where each tensor $\Phi_r \otimes \Xi_r \otimes \Omega_r$ lies in $U_\phi \otimes U_\xi \otimes U_\omega$ – with ϕ , ξ and ω being degrees for the \mathcal{G} -grading – and obeys the following condition:

$$(4.20) \quad \partial(F_{i_1} \cdots F_{i_n}) \precneq \phi \quad \text{or} \quad \omega \precneq \partial(E_{j_1} \cdots E_{j_m}).$$

We have to prove that the map μ in (4.17) is injective. As $U^- \otimes U^0 \otimes U^+$ is $\mathbb{k}[[\hbar]]$ -spanned by all the (tensor) monomials of the form $\underline{F} \otimes \underline{H} \otimes \underline{E}$ (notation as above, with the monomial \underline{H} being ordered): so we assume

$$(4.21) \quad \mu \left(\sum_{\sigma \in S} \kappa_\sigma \underline{F}_\sigma \otimes \underline{H}_\sigma \otimes \underline{E}_\sigma \right) = 0$$

for finitely many $\kappa_\sigma \in \mathbb{k}[[\hbar]]$, and we prove that

$$\sum_{\sigma \in S} \kappa_\sigma \underline{F}_\sigma \otimes \underline{H}_\sigma \otimes \underline{E}_\sigma = 0.$$

First of all, (4.21) yields

$$\sum_{\sigma \in S} \kappa_\sigma \underline{F}_\sigma \underline{H}_\sigma \underline{E}_\sigma = \mu \left(\sum_{\sigma \in S} \kappa_\sigma \underline{F}_\sigma \otimes \underline{H}_\sigma \otimes \underline{E}_\sigma \right) = 0.$$

Second, the previous analysis for $\Delta^{(2)}$ gives, for all indices σ ,

$$\begin{aligned} & \Delta^{(2)}(\underline{F}_\sigma \underline{H}_\sigma \underline{E}_\sigma) \\ &= \left(\underline{F}_\sigma \cdot (\underline{H}_\sigma)_{(1)} \right) \otimes (\underline{H}_\sigma)_{(2)} \otimes \left((\underline{H}_\sigma)_{(3)} \cdot \underline{E}_\sigma \right) + \sum_r \Phi_r^\sigma \otimes \Xi_r^\sigma \otimes \Omega_r^\sigma \end{aligned}$$

with the $\Phi_r^\sigma \otimes \Xi_r^\sigma \otimes \Omega_r^\sigma$'s obeying (4.20) above. Then

$$\begin{aligned} 0 &= \Delta^{(2)}(0) = \Delta^{(2)}\left(\sum_{\sigma \in S} \kappa_\sigma \underline{F}_\sigma \underline{H}_\sigma \underline{E}_\sigma\right) \\ &= \sum_{\sigma \in S} \kappa_\sigma \Delta^{(2)}\left(\underline{F}_\sigma \underline{H}_\sigma \underline{E}_\sigma\right) \\ &= \sum_{\sigma \in S} \kappa_\sigma \left(\left(\underline{F}_\sigma \cdot (\underline{H}_\sigma)_{(1)}\right) \otimes (\underline{H}_\sigma)_{(2)} \otimes \left((\underline{H}_\sigma)_{(3)} \cdot \underline{E}_\sigma\right) + \sum_r \Phi_r^\sigma \otimes \Xi_r^\sigma \otimes \Omega_r^\sigma \right). \end{aligned}$$

Now, we select those σ in S for which $\partial(\underline{F}_\sigma)$ has some minimal value – in $(\mathcal{G}; \preceq)$ – say $\check{\mu}$, and simultaneously $\partial(\underline{E}_\sigma)$ has a maximal value, say $\hat{\mu}$; we denote by $S_\mu^\check{\mu}$ the subset of such indices. Then by degree comparison, we see (cf. (4.20)) that

$$\sum_{\sigma \in S_\mu^\check{\mu}} \kappa_\sigma \left(\underline{F}_\sigma \cdot (\underline{H}_\sigma)_{(1)} \right) \otimes (\underline{H}_\sigma)_{(2)} \otimes \left((\underline{H}_\sigma)_{(3)} \cdot \underline{E}_\sigma \right)$$

is the whole homogeneous summand in

$$\sum_{\sigma \in S} \kappa_\sigma \left(\left(\underline{F}_\sigma \cdot (\underline{H}_\sigma)_{(1)}\right) \otimes (\underline{H}_\sigma)_{(2)} \otimes \left((\underline{H}_\sigma)_{(3)} \cdot \underline{E}_\sigma\right) + \sum_r \Phi_r^\sigma \otimes \Xi_r^\sigma \otimes \Omega_r^\sigma \right)$$

of degree $(\check{\mu}, 0, \hat{\mu})$ with respect to the grading by $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$ in $U \otimes U \otimes U$ canonically induced by the \mathcal{G} -grading of U . For this reason, the identity

$$\begin{aligned} \sum_{\sigma \in S} \kappa_\sigma \left(\left(\underline{F}_\sigma \cdot (\underline{H}_\sigma)_{(1)}\right) \otimes (\underline{H}_\sigma)_{(2)} \otimes \left((\underline{H}_\sigma)_{(3)} \cdot \underline{E}_\sigma\right) \right. \\ \left. + \sum_r \Omega_r^\sigma \otimes \Xi_r^\sigma \otimes \Phi_r^\sigma \right) = 0 \end{aligned}$$

found above implies at once

$$(4.22) \quad \sum_{\sigma \in S_\mu^\check{\mu}} \kappa_\sigma \left(\underline{F}_\sigma \cdot (\underline{H}_\sigma)_{(1)} \right) \otimes (\underline{H}_\sigma)_{(2)} \otimes \left((\underline{H}_\sigma)_{(3)} \cdot \underline{E}_\sigma \right) = 0.$$

Now observe that U^0 admits as $\mathbb{k}[[\hbar]]$ -basis the set of all *ordered* monomials \underline{H} in the H_ℓ 's (i.e., we assume that $H_{\ell_1} \preceq \cdots \preceq H_{\ell_s}$), directly by construction and by Proposition 4.20. Let \mathcal{D} be the Abelian group of rank $t := \text{rk}(\mathfrak{h})$ with basis $\{\delta_\ell\}_{\ell \in \mathcal{L}}$, that we endow with the product order, again denoted by \preceq , induced by the standard order in \mathbb{Z} . There is a natural \mathcal{D} -grading on U^0 such that $\partial(H_\ell) = \delta_\ell$ for all elements of the

fixed basis $\{H_\ell\}_{\ell \in \mathcal{L}}$, whence $\underline{H} = H_{\ell_1} \cdots H_{\ell_s}$ is \mathcal{D} -homogeneous of degree $\delta_{\ell_1} + \cdots + \delta_{\ell_s}$. Also, as the elements of \mathfrak{h} are primitive in $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$, we have

$$(4.23) \quad \Delta^{(2)}(H_{\ell_1} \cdots H_{\ell_s}) = 1 \otimes (H_{\ell_1} \cdots H_{\ell_s}) \otimes 1 + \sum_r x_r \otimes y_r \otimes z_r$$

where the y_r 's are elements in U^0 which are homogeneous (for the \mathcal{D} -grading mentioned above) of degree strictly lower than that of $H_{\ell_1} \cdots H_{\ell_s}$, that is

$$(4.24) \quad \partial(y_r) \prec \partial(H_{\ell_1} \cdots H_{\ell_s}) = s \quad \forall r.$$

The left-hand side of (4.22) belongs to $U_{\check{\mu}}^- \otimes U^0 \otimes U_{\hat{\mu}}^+$; taking into account the \mathcal{D} -grading in U^0 mentioned above, the identity (4.22) implies that each homogeneous component – with respect to the obvious grading of $U^- \otimes U^0 \otimes U^+$ by $\mathcal{G} \times \mathcal{D} \times \mathcal{G}$ – in the left-hand side of (4.22) must be zero as well. In particular, let us focus on a single monomial $\underline{H}_{\bar{\sigma}} = H_{\ell_1^{\bar{\sigma}}} \cdots H_{\ell_s^{\bar{\sigma}}}$ which actually occurs in (4.22), having *maximal* degree in $(\mathcal{D}; \preceq)$: then for the $(\mathcal{G} \times \mathcal{D} \times \mathcal{G})$ -homogeneous component of degree $(\check{\mu}, \partial(\underline{H}_{\bar{\sigma}}), \hat{\mu}) = (\check{\mu}, \delta_{\ell_1^{\bar{\sigma}}} + \cdots + \delta_{\ell_s^{\bar{\sigma}}}, \hat{\mu})$ in (4.22) we find, by (4.23) and (4.24)

$$(4.25) \quad \sum_{\sigma \in S_{\check{\mu}}^{\hat{\mu}}(\ell)} \kappa_{\sigma} \underline{E}_{\sigma} \otimes \underline{H}_{\sigma} \otimes \underline{E}_{\sigma} = 0$$

where $S_{\check{\mu}}^{\hat{\mu}}(\ell) := \{\sigma \in S_{\check{\mu}}^{\hat{\mu}} \mid \underline{H}_{\sigma} = \underline{H}_{\bar{\sigma}}\}$ is a non-empty subset of $S_{\check{\mu}}^{\hat{\mu}}$. But now (4.21) and (4.25) jointly provide the new, *shorter* linear combination

$$\sum_{\sigma \in S \setminus S_{\check{\mu}}^{\hat{\mu}}} \kappa_{\sigma} \underline{E}_{\sigma} \otimes \underline{H}_{\sigma} \otimes \underline{F}_{\sigma} = \sum_{\sigma} \kappa_{\sigma} \underline{E}_{\sigma} \otimes \underline{H}_{\sigma} \otimes \underline{F}_{\sigma} - \sum_{\sigma \in S_{\check{\mu}}^{\hat{\mu}}} \kappa_{\sigma} \underline{E}_{\sigma} \otimes \underline{H}_{\sigma} \otimes \underline{F}_{\sigma}$$

that still belongs to $\text{Ker}(\mu)$: applying again the same argument, and iterating, we end up with $\sum_{\sigma \in S} \kappa_{\sigma} \underline{E}_{\sigma} \otimes \underline{H}_{\sigma} \otimes \underline{F}_{\sigma} = 0$. \square

4.5. Construction of FoMpQUEAs

In this section we provide two new, independent constructions (with respect to what is done in Section 4) of FoMpQUEAs. Namely, when \mathcal{R} is *split minimal* we construct a FoMpQUEA $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$ – with its whole structure of Hopf algebra – first as (a slight variation of) a *quantum double* (or “*Drinfeld’s double*”), and then as a *double cross product*. Then from this special case (via Proposition 2.4, Lemma 2.7 and Proposition 4.15) we deduce the existence – and explicit presentation – of $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$ for *any* realization \mathcal{R} as a quotient of $U_{P, \hbar}^{\mathcal{R}'}(\mathfrak{g})$ for a suitable, split realization \mathcal{R}' .

It is worth explaining a bit what is the general scheme beneath our presentation. The construction of the quantum double applies to any pair of Hopf algebras (possibly topological) over a ring R , with an R -valued skew-Hopf pairing between them. Typically, this applies to any pair of Hopf algebras which are dual to each other, and their canonical (evaluation) pairing. Now assume we do that for some QUEA, say $U_{\hbar}(\mathfrak{g})$, together with its dual (in topological sense) $U_{\hbar}(\mathfrak{g})^* =: F_{\hbar}[[G]]$ – the latter being a “quantum formal series Hopf algebra” (= QFSHA), in Drinfeld’s terminology. Then the corresponding quantum double $D(U_{\hbar}(\mathfrak{g}), F_{\hbar}[[G]])$ will be a “quantum object” – isomorphic to $U_{\hbar}(\mathfrak{g}) \otimes F_{\hbar}[[G]]$ as a coalgebra – whose specialization at $\hbar = 0$ will be the Drinfeld’s double $D(U(\mathfrak{g}), F[[G]])$; this means that, roughly speaking, $D(U_{\hbar}(\mathfrak{g}), F_{\hbar}[[G]])$ is indeed “half a QUEA” and “half a QFSHA”. Therefore, if one aims instead to get a new, full QUEA out of the initial QUEA $U_{\hbar}(\mathfrak{g})$, then one has to modify the previous construction; indeed, there exists a general recipe to perform such a modification (see [21], Section 12.2) which in turn relies on Drinfeld’s “Quantum Duality Principle” which allows one to “extract” a suitable QUEA out of a QFSHA (cf. [28] and references therein). In our presentation we will *not* formally apply this general recipe: instead, we will present an ad hoc construction, tailored to the specific situation we have at hand. However, whatever we do is directly dictated, step by step, by the general recipe, only we display our construction in layman’s terms just to spare the reader some extra theoretical tools that are not needed in full generality. Nevertheless, it is worth stressing that we are actually applying the general recipe, even we do not show it in full light: yet it is there, standing in the backstage.

As a first goal, we aim to construct a suitable quantum double of Borel-like FoMpQUEAs, starting from a pairing among Borel FoMpQUEAs: to this end, we need to step back and introduce “pre-Borel” FoMpQUEAs instead, a pairing *with values in $\mathbb{k}((\hbar))$* among them, and Borel FoMpQUEAs as quotients of pre-Borel ones. As second step, we show that this pairing “pushes-forward” to Borel FoMpQUEAs, hence can be used to perform a quantum double construction; actually, *a priori* this would not be feasible, because the pairing is valued in $\mathbb{k}((\hbar))$ rather than $\mathbb{k}[[\hbar]]$: nevertheless, we prove that in the present case the quantum double construction indeed does work (in a suitable sense) over $\mathbb{k}[[\hbar]]$ as well, hence we are done.

Along the way, another obstruction we encounter is that the construction of the pairing we would need actually clashes with \hbar -completeness of (pre-)Borel FoMpQUEAs; therefore, we scale down to constructing a pairing defined on some dense, non-complete subalgebras – of (pre-)Borel

FoMpQUEAs – and then we manage to carry out the quantum double construction we are looking for.

Finally, we will present yet another construction – parallel to that via quantum doubles – of FoMpQUEAs in terms of *double cross products* of Borel FoMpQUEAs.

4.5.1. Pre-Borel FoMpQUEAs and their pairings

Our first purpose is to construct quantum doubles of Borel FoMpQUEAs of split, minimal type; for this, we need a suitable pairing among such Borel FoMpQUEAs. To this end, we need to step back and introduce “pre-Borel” FoMpQUEAs instead, a pairing with values in $\mathbb{k}(\!(\hbar)\!)$ among them, and Borel FoMpQUEAs as quotients of pre-Borel ones.

DEFINITION 4.26. — *Let $A := (a_{i,j})_{i,j \in I}$ be a generalized symmetrizable Cartan matrix, $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ a matrix of Cartan type with associated Cartan matrix A and $\mathcal{R} = (\mathfrak{h}, \Pi, \Pi^\vee)$ a split minimal realization of P , so that $\mathfrak{h} = \text{Span}_{\mathbb{k}[[\hbar]]}(\{T_i^+, T_i^-\}_{i \in I})$ and it has rank $2n$ (cf. Definition 2.1).*

We define the positive, resp. negative, pre-Borel formal multiparameter quantum universal enveloping algebra – in short positive, resp. negative, pre-Borel FoMpQUEA – with multiparameter P as being the free unital, associative, topological, \hbar -adically complete algebra over $\mathbb{k}[[\hbar]]$, denoted by $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$, resp. by $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$, with generators T_i^+, E_i ($i \in I$), resp. T_i^-, F_i ($i \in I$).

Moreover, we give to $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$, resp. to $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$, the unique, topological Hopf $\mathbb{k}[[\hbar]]$ -algebra structure uniquely defined by (for all $i \in I$)

$$\begin{aligned} \Delta(T_i^+) &= T_i^+ \otimes 1 + 1 \otimes T_i^+, & \mathcal{S}(T_i^+) &= -T_i^+, & \epsilon(T_i^+) &= 0 \\ \Delta(E_i) &= E_i \otimes 1 + e^{+\hbar T_i^+} \otimes E_i, & \mathcal{S}(E_i) &= -e^{-\hbar T_i^+} E_i, & \epsilon(E_i) &= 0 \end{aligned}$$

for $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$, and for $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ in turn by (for all $i \in I$)

$$\begin{aligned} \Delta(T_i^-) &= T_i^- \otimes 1 + 1 \otimes T_i^-, & \mathcal{S}(T_i^-) &= -T_i^-, & \epsilon(T_i^-) &= 0 \\ \Delta(F_i) &= F_i \otimes e^{-\hbar T_i^-} + 1 \otimes F_i, & \mathcal{S}(F_i) &= -F_i e^{+\hbar T_i^-}, & \epsilon(F_i) &= 0. \end{aligned}$$

From now on, we work with fixed positive and negative pre-Borel FoMpQUEAs $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\pm)$ as above; in the following construction the two will play asymmetric roles, but one can also reverse those roles – switching $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ and $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ among them – and eventually get exactly the same outcome.

DEFINITION 4.27. — Let us consider $\bar{T}_t^- := \hbar T_t^-, \bar{F}_t := \hbar F_t \in \tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$, for $t \in I$. We define $\tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ to be the \hbar -adic closure in $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ of the unital $\mathbb{k}[[\hbar]]$ -subalgebra generated by $\{\bar{T}_\ell^-, \bar{F}_\ell\}_{\ell \in I}$.

Similarly, we define $\tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ to be the \hbar -adic closure in $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ of the unital $\mathbb{k}[[\hbar]]$ -subalgebra generated by $\{\bar{T}_k^+ := \hbar T_k^+, \bar{E}_k := \hbar E_k\}_{k \in I}$.

The following, technical result is obvious from definitions

LEMMA 4.28. — Let $\tilde{\mathcal{E}}_{\pm}^{(\hbar)} := \text{Ker}(\epsilon_{\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})}) \cap \tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm}) + \hbar \tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})$. Then:

- (a) $\tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})$ is complete with respect to the $\tilde{\mathcal{E}}_{\pm}^{(\hbar)}$ -adic topology, and $\{\bar{T}_i^+, \bar{E}_i\}_{i \in I}$, resp. $\{\bar{T}_j^-, \bar{F}_j\}_{j \in I}$, is a set of topological generators of it;
- (b) $\tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})$ is a (topological) Hopf $\mathbb{k}[[\hbar]]$ -subalgebra of $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})$.

The key point with pre-Borel FoMpQUEAs is our next result, whose proof is more or less standard in Hopf theory.

PROPOSITION 4.29. — There exists a $\mathbb{k}[[\hbar]]$ -linear skew-Hopf pairing

$$\tilde{\pi} : \tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+) \hat{\otimes}_{\mathbb{k}[[\hbar]]} \tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-) \longrightarrow \mathbb{k}[[\hbar]]$$

uniquely given – for all $i, j \in I$ – by

$$\begin{aligned} \tilde{\pi}(T_i^+, \bar{T}_j^-) &= \alpha_i(T_j^-) = \alpha_j(T_i^+) = p_{ij}, & \tilde{\pi}(T_i^+, 1) &= 0 = \tilde{\pi}(1, \bar{T}_j^-) \\ \tilde{\pi}(T_i^+, \bar{F}_j) &= 0 = \tilde{\pi}(E_i, \bar{T}_j^-), & \tilde{\pi}(1, \bar{F}_j) &= 0 = \tilde{\pi}(E_i, 1) \\ \tilde{\pi}(1, 1) &= 1, & \tilde{\pi}(E_i, \bar{F}_j) &= \frac{\delta_{ij} \hbar}{q_i^{+1} - q_i^{-1}}. \end{aligned}$$

Proof. — Assume first that such a skew-Hopf pairing exists: then it is uniquely determined by its values on the (topological) algebra generators of $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ and $\tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ chosen in the sets $\{T_i^+, E_i\}_{i \in I}$ and $\{\bar{T}_j^-, \bar{F}_j\}_{j \in I}$, respectively; indeed, this follows from repeated applications of formulas (4.1) and (4.2) in Definition 4.1 along with the fact that $\tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ is a Hopf (sub)algebra. Therefore, once the above mentioned values are specified as in the statement, this proves uniqueness.

To show that such a pairing exists, it is equivalent to prove, as usual, that there exists a anti-homomorphism $\gamma : \tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+) \rightarrow \tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)^*$ of algebras, where $\tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)^* := \text{Hom}_{\mathbb{k}[[\hbar]]}(\tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-), \mathbb{k}[[\hbar]])$. This is known once it is assigned on the free (topological) generators of $\tilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ picked from $\{T_i^+, E_i\}_{i \in I}$, and to define γ on those elements we use the coproduct on them, because $\tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ is freely generated by $\{\bar{T}_j^-, \bar{F}_j\}_{j \in I}$.

Given an augmented R -algebra (A, ϵ) over a ring R , a map $d : A \rightarrow R$ is called a *derivation* if $d(xy) = d(x)\epsilon(y) + \epsilon(x)d(y)$ for all $x, y \in A$. Also, for two algebra maps $\alpha, \beta \in \text{Alg}_R(A, R)$, an (α, β) -*derivation* is a map $d : A \rightarrow R$ such that $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in A$.

Taking into account that the T_i^+ 's are primitive, the $K_i := e^{+\hbar T_i^+}$ are group-like and the E_i 's are $(1, K_i)$ -primitive, for all $i \in I$, we define in $\tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_-)^*$ the derivation τ_i , the algebra morphism κ_i and the (ϵ, κ_i) -derivation η_i by (for all $i, j \in I$)

$$\begin{aligned} \tau_i(\bar{T}_j^-) &:= p_{ij}, & \tau_i(\bar{F}_j) &:= 0, & \kappa_i(\bar{T}_j^-) &:= p_{ij}, & \kappa_i(\bar{F}_j) &:= 0 \\ \eta_i(\bar{T}_j^-) &:= 0, & \eta_i(\bar{F}_j) &:= \delta_{ij}\hbar(q_i^{+1} - q_i^{-1})^{-1}. \end{aligned}$$

Now consider $\tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_-)^*$ as an algebra with the convolution product, that is, $(fg)(x) := f(x_{(1)})g(x_{(2)})$ for all $f, g \in \tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_-)^*$ and $x \in \tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_-)$. As $\tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ is the free (topological) algebra generated by $\{T_i^+, E_i\}_{i \in I}$, one has an algebra anti-homomorphism $\gamma : \tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_+) \rightarrow \tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_-)^*$ given by $\gamma(T_i^+) := \tau_i$ and $\gamma(E_i) := \eta_i$ for all $i \in I$. Let $\tilde{\pi} : \tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_+) \otimes_{\mathbb{k}[[\hbar]]} \tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_-) \rightarrow \mathbb{k}[[\hbar]]$ be the linear map defined by $\tilde{\pi}(x, y) := (\gamma(x))(y)$ for all $x \in \tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ and $y \in \tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_-)$; then by the very construction of γ , condition (4.2) is satisfied. On the other hand, condition (4.1) is satisfied because it is satisfied on the generators T_i^+ and E_i ($i \in I$) and the comultiplication is an algebra map; the same holds for the conditions (4.3). Finally, one may prove conditions (4.4) concerning the antipode using again the values on the generators, as both \mathcal{S} and \mathcal{S}^{-1} are algebra and coalgebra anti-homomorphisms. \square

Remark 4.30. — It is clear by construction that one can also introduce a topological Hopf subalgebra $\tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ of $\tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_+)$, for which the analog of Lemma 4.28 holds true, and a suitable skew-Hopf pairing $\tilde{\pi} : \tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_+) \hat{\otimes}_{\mathbb{k}[[\hbar]]} \tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_-) \rightarrow \mathbb{k}[[\hbar]]$ similar to the one in Proposition 4.29, and denoted again by $\tilde{\pi}$. Moreover, both $\tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_+) \hat{\otimes}_{\mathbb{k}[[\hbar]]} \tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ and $\tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_+) \hat{\otimes}_{\mathbb{k}[[\hbar]]} \tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ will embed in $\tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_+) \hat{\otimes}_{\mathbb{k}[[\hbar]]} \tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_-)$, their intersection will coincide with $\tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_+) \hat{\otimes}_{\mathbb{k}[[\hbar]]} \tilde{\mathcal{U}}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_-)$, and the restrictions to this last submodule of the two pairings considered so far will coincide.

4.5.2. From pre-Borel FoMpQUEAs to Borel FoMpQUEAs

We introduce now *Borel FoMpQUEAs*, as quotients of pre-Borel FoMpQUEAs: indeed, the former are carefully devised so to inherit from the latter all possible “good” properties.

DEFINITION 4.31. — Let pre-Borel FoMpQUEAs $\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})$ be given as in Definition 4.26. We define \widetilde{I}_+ to be the closure – in the \hbar -adic topology – of the two-sided ideal in $\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ generated by the elements $(i, j \in I)$

$$\begin{aligned} T_{i,j}^+ &:= T_i^+ T_j^+ - T_j^+ T_i^+, & E_{i,j}^{(T)} &:= T_i^+ E_j - E_j T_i^+ - \alpha_j(T_i^+) E_j \\ E_{i,j} &:= \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} E_i^{1-a_{ij}-k} E_j E_i^k & (i \neq j) \end{aligned}$$

and we define \widetilde{I}_- to be the closure – in the \hbar -adic topology – of the two-sided ideal in $\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ generated by all the elements $(i, j \in I)$

$$\begin{aligned} T_{i,j}^- &:= T_i^- T_j^- - T_j^- T_i^-, & F_{i,j}^{(T)} &:= T_i^- F_j - F_j T_i^- + \alpha_j(T_i^-) E_j \\ F_{i,j} &:= \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} F_i^{1-a_{ij}-k} F_j F_i^k & (i \neq j). \end{aligned}$$

Moreover, we define the positive, resp. negative, Borel formal multiparameter quantum universal enveloping algebra – in short positive, resp. negative, Borel FoMpQUEA – with multiparameter P as being the quotient $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+) := \widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)/\widetilde{I}_+$, resp. $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-) := \widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)/\widetilde{I}_-$. With a standard abuse of notation, hereafter we shall denote with the same symbol any element in $\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})$ as well as its image (via the quotient map) in the quotient algebra $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm}) := \widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})/\widetilde{I}_{\pm}$.

We need also similar definitions for the Hopf subalgebras $\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})$:

DEFINITION 4.32. — Let $\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})$ be defined as in Definition 4.27, and consider in it $\widetilde{\mathcal{E}}_{\pm}^{(\hbar)} := \text{Ker}(\epsilon_{\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})}) \cap \widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm}) + \hbar \widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})$. We define \widetilde{I}_+ to be the closure, in the $\widetilde{\mathcal{E}}_+^{(\hbar)}$ -adic topology, of the two-sided ideal in $\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ generated by the elements (for all $i, j \in I$)

$$\begin{aligned} \bar{T}_{i,j}^+ &:= \bar{T}_i^+ \bar{T}_j^+ - \bar{T}_j^+ \bar{T}_i^+, & \bar{E}_{i,j}^{(T)} &:= \bar{T}_i^+ \bar{E}_j - \bar{E}_j \bar{T}_i^+ - \hbar \alpha_j(\bar{T}_i^+) \bar{E}_j \\ \bar{E}_{i,j} &:= \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} \bar{E}_i^{1-a_{ij}-k} \bar{E}_j \bar{E}_i^k & (i \neq j) \end{aligned}$$

and we define $\widetilde{\mathcal{I}}_-$ to be the closure – in the $\widetilde{\mathcal{E}}_-^{(\hbar)}$ -adic topology – of the two-sided ideal in $\widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ generated by the elements (for all $i, j \in I$)

$$\begin{aligned} \overline{T}_{i,j}^- &:= \overline{T}_i^- \overline{T}_j^- - \overline{T}_j^- \overline{T}_i^-, & \overline{F}_{i,j}^{(T)} &:= \overline{T}_i^- \overline{F}_j - \overline{F}_j \overline{T}_i^- + \hbar \alpha_j (\overline{T}_i^-) \overline{F}_j \\ \overline{F}_{i,j} &:= \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} \overline{F}_i^{1-a_{ij}-k} \overline{F}_j \overline{F}_i^k & (i \neq j). \end{aligned}$$

Accordingly, we consider the quotients $\mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm}) := \widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})/\widetilde{\mathcal{I}}_{\pm}$, and, with standard abuse of notation, we shall denote with the same symbol any element in $\widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})$ as well as its coset in the quotient algebra $\mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm}) := \widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})/\widetilde{\mathcal{I}}_{\pm}$.

The key point concerning Borel FoMpQUEAs is the following:

PROPOSITION 4.33.

- (a) $\widetilde{\mathcal{I}}_{\pm}$ is a Hopf ideal of $\widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})$, so that $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm}) := \widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})/\widetilde{\mathcal{I}}_{\pm}$ is a quotient Hopf algebra. Similarly, $\widetilde{\mathcal{I}}_{\pm}$ is a Hopf ideal of $\widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})$, therefore the quotient $\mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm}) := \widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})/\widetilde{\mathcal{I}}_{\pm}$ is in fact a Hopf algebra.

Moreover, $\mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})$ is a (topological) Hopf subalgebra in $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})$.

- (b) The ideal $\widetilde{\mathcal{I}}_+$, resp. $\widetilde{\mathcal{I}}_-$, is contained in the left, resp. right, radical of the pairing $\widetilde{\pi} : \widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+) \widehat{\otimes} \widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-) \rightarrow \mathbb{k}[[\hbar]]$ in Proposition 4.29.

Similarly, the ideal $\widetilde{\mathcal{I}}_+$, resp. $\widetilde{\mathcal{I}}_-$, is contained in the left, resp. right, radical of the pairing $\widetilde{\pi} : \widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+) \widehat{\otimes} \widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-) \rightarrow \mathbb{k}[[\hbar]]$ in Remark 4.30.

- (c) The two skew-Hopf pairings $\widetilde{\pi} : \widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+) \widehat{\otimes} \widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-) \rightarrow \mathbb{k}[[\hbar]]$ and $\widetilde{\pi} : \widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+) \widehat{\otimes} \widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-) \rightarrow \mathbb{k}[[\hbar]]$ mentioned in (b) uniquely induce skew-Hopf pairings $\pi : U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+) \widehat{\otimes} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-) \rightarrow \mathbb{k}[[\hbar]]$ and $\pi : \mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+) \widehat{\otimes} \mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-) \rightarrow \mathbb{k}[[\hbar]]$ described by obvious formulas as in Proposition 4.29.

Proof. — Claim (c) follows at once from (a) and (b), so now we prove the latter ones.

As to (a), computations show that the $T_{i,j}^+$'s are primitive, while the $E_{i,j}^{(T)}$'s are skew-primitive, namely $\Delta(E_{i,j}^{(T)}) = E_{i,j}^{(T)} \otimes 1 + e^{+\overline{T}_j^+} \otimes E_{i,j}^{(T)}$; similarly, again direct computations prove also that $\Delta(E_{i,j}) = E_{i,j} \otimes 1 + e^{+(1-a_{ij})\hbar T_i^+ + \hbar T_j^+} \otimes E_{i,j}$. This implies that $\widetilde{\mathcal{I}}_+$ is a Hopf ideal of $\widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$.

With similar arguments, one proves the claim for $\widetilde{\mathcal{I}}_-$ and for $\widetilde{\mathcal{I}}_{\pm}$ as well.

Finally, the statement about being a Hopf subalgebra follows by construction.

As to (b), again direct computation shows that \tilde{I}_+ lies in the left radical of the pairing $\tilde{\pi}$. For instance, the functional $\tilde{\pi}(T_i^+ T_j^+, -)$ is non-zero only when it is evaluated in elements of the form $\bar{T}_k^- \bar{T}_\ell^-$ for some $1 \leq k, \ell \leq n$; in this case,

$$\begin{aligned} \tilde{\pi}(T_i^+ T_j^+, \bar{T}_k^- \bar{T}_\ell^-) &= \tilde{\pi}(T_i^+, \bar{T}_\ell^-) \tilde{\pi}(T_j^+, \bar{T}_k^-) + \tilde{\pi}(T_i^+, \bar{T}_k^-) \tilde{\pi}(T_j^+, \bar{T}_\ell^-) \\ &= p_{i\ell} p_{jk} + p_{ik} p_{j\ell} = \tilde{\pi}(T_j^+ T_i^+, \bar{T}_k^- \bar{T}_\ell^-) \end{aligned}$$

so the generators $T_{i,j}^+ := T_i^- T_j^- - T_j^- T_i^-$ of \tilde{I}_+ lie in the left radical for all $i, j \in I$.

Similarly, we saw that the generators $E_{i,j}^{(T)} := T_i^+ E_j - E_j T_i^+ - \alpha_j(T_i^+) E_j$ are skew-primitive again, namely $\Delta(E_{i,j}^{(T)}) = E_{i,j}^{(T)} \otimes 1 + e^{+\bar{T}_j^+} \otimes E_{i,j}^{(T)}$. Thanks to this, in order to prove that the $E_{i,j}^{(T)}$'s are contained in the left radical it is enough to show that they kill the generators of $\tilde{\mathcal{U}}_{P,h}^{\mathcal{R}}(\mathfrak{b}_-)$, because for all $x, y \in \tilde{\mathcal{U}}_{P,h}^{\mathcal{R}}(\mathfrak{b}_-)$ we have

$$\tilde{\pi}(E_{i,j}^{(T)}, xy) = \tilde{\pi}(E_{i,j}^{(T)}, x) \tilde{\pi}(1, y) + \tilde{\pi}(e^{+\bar{T}_j^+}, x) \tilde{\pi}(E_{i,j}^{(T)}, y).$$

Now, from $\tilde{\pi}(E_\ell, 1) = 0$ for all $\ell \in I$, the properties of the skew-Hopf pairing imply that $\tilde{\pi}(E_{i,j}^{(T)}, 1) = 0$ too, for all $i, j \in I$. Similarly, direct computation gives, using notation $\tilde{\pi}_\otimes(a \otimes b, u \otimes v) := \tilde{\pi}(a, u) \tilde{\pi}(b, v)$,

$$\begin{aligned} \tilde{\pi}(E_{i,j}^{(T)}, \bar{T}_k^-) &= \tilde{\pi}(T_i^+ E_j - E_j T_i^+ - \alpha_j(T_i^+) E_j, \bar{T}_k^-) \\ &= \tilde{\pi}_\otimes(T_i^+ \otimes E_j - E_j \otimes T_i^+ - \alpha_j(T_i^+) E_j \otimes 1, \Delta(\bar{T}_k^-)) \\ &= \tilde{\pi}_\otimes(T_i^+ \otimes E_j - E_j \otimes T_i^+ - \alpha_j(T_i^+) E_j \otimes 1, \bar{T}_k^- \otimes 1 + 1 \otimes \bar{T}_k^-) \\ &= 0 \end{aligned}$$

exactly because $\tilde{\pi}(E_j, 1) = 0 = \tilde{\pi}(E_j, \bar{T}_k^-)$ for all $j, k \in I$. Likewise, we also have

$$\begin{aligned} \tilde{\pi}(E_{i,j}^{(T)}, \bar{F}_k) &= \tilde{\pi}(T_i^+ E_j - E_j T_i^+ - \alpha_j(T_i^+) E_j, \bar{F}_k) \\ &= \tilde{\pi}_\otimes(T_i^+ \otimes E_j - E_j \otimes T_i^+ - \alpha_j(T_i^+) E_j \otimes 1, \Delta(\bar{F}_k)) \\ &= \tilde{\pi}_\otimes(T_i^+ \otimes E_j - E_j \otimes T_i^+ - \alpha_j(T_i^+) E_j \otimes 1, \bar{F}_k \otimes e^{-\bar{T}_k^-} + 1 \otimes \bar{F}_k) \\ &= 0 \end{aligned}$$

because when we expand the last line the only non-trivial summands are

$$-\tilde{\pi}(E_j, \bar{F}_k) \cdot \tilde{\pi}\left(T_i^+, e^{-\bar{T}_k^-}\right) = +\delta_{j,k} \hbar (q_j^{+1} - q_j^{-1})^{-1} \cdot \alpha_k(T_i^+)$$

and

$$-\alpha_j(T_i^+) \tilde{\pi}(E_j, \bar{F}_k) \cdot \tilde{\pi}\left(1, e^{-\bar{T}_k^-}\right) = -\alpha_j(T_i^+) \delta_{j,k} \hbar (q_j^{+1} - q_j^{-1})^{-1} \cdot 1$$

which add up to zero.

Finally, the $E_{i,j}$'s are skew-primitives too, so again it is enough to show that they kill the generators of $\tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$. This follows again by direct calculation, for instance

$$\begin{aligned} & \tilde{\pi}(E_{i,j}, \bar{F}_k) \\ &= \tilde{\pi}\left(\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} q_{ij}^{+s/2} q_{ji}^{-s/2} E_i^{1-a_{ij}-s} E_j E_i^s, \bar{F}_k\right) \\ &= \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} q_{ij}^{+s/2} q_{ji}^{-s/2} \tilde{\pi}_{\otimes}^{(3)}\left(E_i^{1-a_{ij}-s} \otimes E_j \otimes E_i^s, \Delta^{(3)}(\bar{F}_k)\right) \\ &= \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} q_{ij}^{+s/2} q_{ji}^{-s/2} \\ & \quad \times \tilde{\pi}_{\otimes}^{(3)}\left(E_i^{1-a_{ij}-s} \otimes E_j \otimes E_i^s, \bar{F}_k \otimes e^{-\bar{T}_k^-} \otimes e^{-\bar{T}_k^-} + 1 \otimes \bar{F}_k \otimes e^{-\bar{T}_k^-} + 1 \otimes 1 \otimes \bar{F}_k\right) \end{aligned}$$

so that for all s we get

$$\tilde{\pi}_{\otimes}^{(3)}\left(E_i^{1-a_{ij}-s} \otimes E_j \otimes E_i^s, \bar{F}_k \otimes e^{-\bar{T}_k^-} \otimes e^{-\bar{T}_k^-} + 1 \otimes \bar{F}_k \otimes e^{-\bar{T}_k^-} + 1 \otimes 1 \otimes \bar{F}_k\right) = 0$$

because $\tilde{\pi}(E_j, e^{-\bar{T}_k^-}) = 0 = \tilde{\pi}(E_j, 1)$ and $\tilde{\pi}(E_j^{1-a_{ij}-s}, 1) = 0$. □

Remarks 4.34.

- (a) Constructions imply that $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm})$ in Proposition 4.33(a) coincide with the Borel FoMpQUEAs of Definition 4.4(c), with their whole Hopf structure (cf. Proposition 4.14), so we use again same notation and terminology.
- (b) Denote by \mathfrak{r}_+ and \mathcal{R}_- , respectively \mathcal{R}_+ and \mathfrak{r}_- , the left and the right radical of the skew-Hopf pairing $\tilde{\pi} : \tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+) \widehat{\otimes} \tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-) \rightarrow \mathbb{k}[[\hbar]]$, respectively $\tilde{\pi} : \tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+) \widehat{\otimes} \tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-) \rightarrow \mathbb{k}[[\hbar]]$; all these are Hopf ideals, and then – in both cases – the pairings $\tilde{\pi}$ induce similar skew Hopf pairings between the quotient Hopf algebras $\tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)/\mathfrak{r}_+$ and $\tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)/\mathcal{R}_-$, resp. $\tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)/\mathcal{R}_+$ and $\tilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)/\mathfrak{r}_-$, which are non-degenerate. When the matrix P is symmetric, hence equal to

DA , basing on [35, Theorem 9.11], one can prove that the relations in Definition 4.31, resp. in Definition 4.32, generate the Hopf ideals \mathfrak{r}_\pm , resp. \mathcal{R}_\mp , hence one has that $\widetilde{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\pm)/\mathfrak{r}_\pm \cong U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\pm)$ and $\widetilde{\mathcal{U}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\pm)/\mathcal{R}_\pm \cong \mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\pm)$.

Next result points out a technical property of the Borel FoMpQUEAs.

LEMMA 4.35. — *The algebras $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\pm)$ are topologically free, i.e. they are torsion-free as $\mathbb{k}[[\hbar]]$ -modules and they are separated and complete for the \hbar -adic topology.*

Proof. — It is proved in Theorem 5.15(b) later on, in a way independent of whatever follows from here to there, that in the split minimal case the FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is just a deformation (in a proper sense) of Drinfeld’s $U_\hbar(\mathfrak{g})$ – in “double version”, i.e. with Cartan subalgebra of rank $2|I|$; in particular, $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ and $U_\hbar(\mathfrak{g})$ have the same $\mathbb{k}[[\hbar]]$ -module structure. Now $U_\hbar(\mathfrak{g})$ is known to be topologically free, so the same holds for $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, and then this property is inherited by the subalgebras $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\pm)$ too. Alternatively, the proof of Theorem 5.15(b) also applies directly to $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\pm)$, proving that the former are suitable deformations of Drinfeld’s $U_\hbar(\mathfrak{b}_\pm)$: the latter are known to be topologically free, so the same holds for $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\pm)$ too. \square

4.5.3. FoMpQUEAs as quasi-doubles of Borel FoMpQUEAs

The analysis carried on from Section 4.5.2 on provides skew-Hopf pairings between Borel FoMpQUEAs $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\pm)$ and $\mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\mp)$. Following the recipe in Definition 4.1, we can then consider the associated (Drinfeld’s) quantum doubles, that we denote by

$$(4.26) \quad \begin{aligned} \overrightarrow{\mathcal{D}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) &:= D(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+), \mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-), \pi) \\ \overleftarrow{\mathcal{D}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) &:= D(\mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+), U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-), \pi). \end{aligned}$$

By definition of Drinfeld’s quantum double, there exists an isomorphism of (topological) $\mathbb{k}[[\hbar]]$ -coalgebras $\overrightarrow{\mathcal{D}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \cong U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+) \widehat{\otimes}_{\mathbb{k}[[\hbar]]} \mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$. Even more, both $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ and $\mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ embed into $\overrightarrow{\mathcal{D}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ – via $u \mapsto u \otimes 1$ and $\bar{v} \mapsto 1 \otimes \bar{v}$, respectively – as Hopf subalgebras, and these (Hopf) subalgebras actually generate all of $\overrightarrow{\mathcal{D}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, as a topological algebra. A similar analysis applies to $\overleftarrow{\mathcal{D}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$. Note that here we apply Lemma 4.35: by it, $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\pm)$ is topologically free, thus also $\mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\pm)$ is, hence the products $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\pm) \widehat{\otimes}_{\mathbb{k}[[\hbar]]} \mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\mp)$ are topologically free too.

Our next result is an explicit description of these quantum double Hopf algebras.

PROPOSITION 4.36. — *With assumptions as above, the quantum double Hopf algebra $\overrightarrow{\mathcal{D}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ in (4.26) admits the following presentation: it is the unital, associative, topological, \hbar -adically complete algebra over $\mathbb{k}[[\hbar]]$ with generators $E_i, T_i^+, \overline{T}_j^-, \overline{F}_j, (i, j \in I)$ and relations (for $i, j \in I$)*

$$\begin{aligned}
 & T_i^+ T_j^+ = T_j^+ T_i^+, \quad T_i^+ E_j - E_j T_i^+ = +p_{ij} E_j \\
 & \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} E_i^{1-a_{ij}-k} E_j E_i^k = 0 \quad (i \neq j) \\
 & \overline{T}_i^- \overline{T}_j^- = \overline{T}_j^- \overline{T}_i^-, \quad \overline{T}_i^- \overline{F}_j - \overline{F}_j \overline{T}_i^- = -\hbar p_{ji} \overline{F}_j \\
 (4.27) \quad & \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} \overline{F}_i^{1-a_{ij}-k} \overline{F}_j \overline{F}_i^k = 0 \quad (i \neq j) \\
 & \overline{T}_i^- E_j - E_j \overline{T}_i^- = +\hbar p_{ji} E_j, \quad T_i^+ \overline{F}_j - \overline{F}_j T_i^+ = -p_{ij} \overline{F}_j \\
 & T_i^+ \overline{T}_j^- = \overline{T}_j^- T_i^+, \quad E_i \overline{F}_j - \overline{F}_j E_i = \delta_{i,j} \hbar \frac{e^{+\hbar T_i^+} - e^{-\overline{T}_i^-}}{q_i^{+1} - q_i^{-1}}
 \end{aligned}$$

with Hopf structure given on the above generators (for all $i \in I$) by

$$\begin{aligned}
 \Delta(E_i) &= E_i \otimes 1 + e^{+\hbar T_i^+} \otimes E_i, & \Delta(T_i^+) &= T_i^+ \otimes 1 + 1 \otimes T_i^+ \\
 \Delta(\overline{T}_i^-) &= \overline{T}_i^- \otimes 1 + 1 \otimes \overline{T}_i^-, & \Delta(\overline{F}_i) &= \overline{F}_i \otimes e^{-\overline{T}_i^-} + 1 \otimes \overline{F}_i \\
 (4.28) \quad \epsilon(E_i) &= 0, \quad \epsilon(T_i^+) = 0, & \epsilon(\overline{T}_i^-) &= 0, \quad \epsilon(\overline{F}_i) = 0 \\
 \mathcal{S}(E_i) &= -e^{-\hbar T_i^+} E_i, & \mathcal{S}(T_i^+) &= -T_i^+ \\
 \mathcal{S}(\overline{T}_i^-) &= -\overline{T}_i^-, & \mathcal{S}(\overline{F}_i) &= -e^{+\overline{T}_i^-} \overline{F}_i.
 \end{aligned}$$

A similar result provides a likewise presentation of $\overleftarrow{\mathcal{D}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$.

Proof. — Recall that we have an isomorphism of (topological) $\mathbb{k}[[\hbar]]$ -coalgebras $\overrightarrow{\mathcal{D}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \cong U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+) \widehat{\otimes}_{\mathbb{k}[[\hbar]]} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$. Moreover, through it both $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ and $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ embed into $\overrightarrow{\mathcal{D}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ – via $u \mapsto u \otimes 1$ and via $\overline{v} \mapsto 1 \otimes \overline{v}$ – as Hopf subalgebras, which generate $\overrightarrow{\mathcal{D}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, as a topological algebra. In particular, as a matter of notation we shall write u for $u \otimes 1$ and \overline{v} for $1 \otimes \overline{v}$. From all this it follows that $\overrightarrow{\mathcal{D}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ admits a presentation with generators $E_i, T_i^+, \overline{T}_j^-, \overline{F}_j, (i, j \in I)$ – as these generate $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ and $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ – and relations given by the first two lines in (4.27) – because these are the relations among the E_i ’s and T_i^+ ’s inside $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ –

and the mid two lines in (4.27) – since these are those among the \bar{T}_j^- 's and \bar{F}_j 's inside $\mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ – plus the additional relations, given at the end of Definition 4.1, that link the generators E_i and T_i^+ ($i \in I$) inside $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ with the generators \bar{T}_j^- and \bar{F}_j ($j \in I$) inside $\mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$. Concerning these last set of relations, direct computation proves that they are given by the last two lines in (4.27).

For example, taking $x = \bar{T}_j^-$ and $y = T_i^+$ we have that

$$\begin{aligned} x_{(1)}y_{(1)}\pi(y_{(2)},x_{(2)}) &= \bar{T}^-T_i^+\pi(1,1) + \bar{T}_j^-\pi(T_i^+,1) + T_i^+\pi(1,\bar{T}_j^-) + \pi(T_i^+,\bar{T}_j^-) \\ &= \bar{T}_j^-T_i^+ + p_{ij} \\ \pi(y_{(1)},x_{(1)})y_{(2)}x_{(2)} &= \pi(T_i^+,\bar{T}_j^-) + \pi(T_i^+,1)\bar{T}_j^- + \pi(1,\bar{T}_j^-)T_i^+ + \pi(1,1)T_i^+\bar{T}_j^- \\ &= p_{ij} + T_i^+\bar{T}_j^- \end{aligned}$$

which yields $T_i^+\bar{T}_j^- = \bar{T}_j^-T_i^+$ for all $i, j \in I$; similarly, for $x = \bar{F}_j$ and $y = E_i$, we get

$$\begin{aligned} x_{(1)}y_{(1)}\pi(y_{(2)},x_{(2)}) &= \bar{F}_jE_i\pi(1,e^{-\bar{T}_j^-}) + \bar{F}_je^{\hbar T_i^+}\pi(E_i,e^{-\bar{T}_j^-}) + E_i\pi(1,\bar{F}_j) + e^{\hbar T_i^+}\pi(E_i,\bar{F}_j) \\ &= \bar{F}_jE_i + \delta_{i,j}\hbar e^{\hbar T_i^+}(q_i^{+1} - q_i^{-1})^{-1} \\ \pi(y_{(1)},x_{(1)})y_{(2)}x_{(2)} &= \pi(E_i,\bar{F}_j)e^{-\bar{T}_j^-} + \pi(E_i,1)\bar{F}_j + \pi(e^{+\hbar T_i^+},\bar{F}_j)E_ie^{-\bar{T}_j^-} + \pi(e^{+\hbar T_i^+},1)E_i\bar{F}_j \\ &= \delta_{i,j}\hbar e^{-\bar{T}_j^-}(q_i^{+1} - q_i^{-1})^{-1} + E_i\bar{F}_j \end{aligned}$$

which yields the relation $E_i\bar{F}_j - \bar{F}_jE_i = \delta_{i,j}\hbar \frac{e^{\hbar T_i^+} - e^{\hbar T_i^-}}{q_i^{+1} - q_i^{-1}}$ ($i, j \in I$).

Finally, the Hopf structure is given once we know how it looks on generators, hence it is given by (4.28) because $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ and $\mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ are both Hopf subalgebras.

A parallel argument yields a similar presentation for $\overleftarrow{\mathcal{D}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$. □

We still need some auxiliary ingredients:

DEFINITION 4.37.

- (a) We denote by $\overrightarrow{\mathcal{D}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ the \hbar -adic completion of the $\mathbb{k}[[\hbar]]$ -subalgebra generated in $\overrightarrow{\mathcal{D}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ by $\{E_i, T_i^+, T_i^- = \hbar^{-1}\bar{T}_i^-, F_i = \hbar^{-1}\bar{F}_i\}_{i \in I}$.
- (b) We denote by $\overleftarrow{\mathcal{D}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ the \hbar -adic completion of the $\mathbb{k}[[\hbar]]$ -subalgebra generated in $\overleftarrow{\mathcal{D}}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ by $\{E_i = \hbar^{-1}\bar{E}_i, T_i^+ = \hbar^{-1}\bar{T}_i^+, T_i^-, F_i\}_{i \in I}$.

We are finally ready for the main result we are looking for:

THEOREM 4.38. — *Let $A := (a_{i,j})_{i,j \in I}$ be a generalized symmetrizable Cartan matrix, and let $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ be a matrix of Cartan type with associated Cartan matrix A . With assumptions as above, both $\overrightarrow{D}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ and $\overleftarrow{D}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ are topological, \hbar -adically complete Hopf $\mathbb{k}[[\hbar]]$ -algebras, which are isomorphic to the FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ given in Definition 4.4.*

Proof. — The claim follows directly from the construction of $\overrightarrow{D}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ and $\overleftarrow{D}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, and from Proposition 4.36 above: in fact, all this yields a presentation for $\overrightarrow{D}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ and one for $\overleftarrow{D}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ – with generating set $\{E_i, T_i^+, T_i^-, F_i\}_{i \in I}$, in both cases – that just coincide, hence these two algebras are isomorphic. At the same time, the formulas for the Hopf structure in $\overrightarrow{D}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ and $\overleftarrow{D}_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ show that these algebras inherit the Hopf structure as well. Comparing this presentation with the one defining $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ one sees that they coincide again, whence the last part of the claim. \square

4.5.4. Construction as double cross products

In this subsection we implement an alternative construction of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ as a subalgebra of a *double cross product*, which is also an alternative way of constructing a quantum double. We follow Majid [41, Section 7.2] for the description of the double cross product. We begin by introducing the construction in the general context of matched pairs of Hopf algebras.

DEFINITION 4.39 ([41, Definition 7.2.1]). — *Two bialgebras or Hopf algebras A and H form a right-left matched pair if H is a right A -module coalgebra and A is a left H -module coalgebra with mutual actions $\triangleleft : H \otimes A \rightarrow H, \triangleright : H \otimes A \rightarrow A$ that obey the compatibility conditions*

$$\begin{aligned} (hg) \triangleleft a &= (h \triangleleft (g_{(1)} \triangleright a_{(1)}))(g_{(2)} \triangleleft a_{(2)}), \quad 1 \triangleleft a = \epsilon(a) \\ h \triangleright (ab) &= (h_{(1)} \triangleright a_{(1)})((h_{(2)} \triangleleft a_{(2)}) \triangleright b), \quad h \triangleright 1 = \epsilon(h) \\ (h_{(1)} \triangleleft a_{(1)}) \otimes (h_{(2)} \triangleright a_{(2)}) &= (h_{(2)} \triangleleft a_{(2)}) \otimes (h_{(1)} \triangleright a_{(1)}). \end{aligned}$$

THEOREM 4.40 ([41, Theorem 7.2.2]). — *Given a matched pair of bialgebras (A, H) , there exists a double cross product bialgebra $A \bowtie H$ built on the vector space $A \otimes H$ with product*

$$(a \otimes h) \cdot (b \otimes g) := a(h_{(1)} \triangleleft b_{(1)}) \otimes (h_{(2)} \triangleright b_{(2)})g \quad \forall a, b \in A, h, g \in H$$

and tensor product unit, counit and coproduct maps. Moreover, A and H are subbialgebras via the canonical inclusions, and $A \bowtie H$ is generated by them with relations

$$h \cdot a = (h_{(1)} \triangleleft b_{(1)}) \otimes (h_{(2)} \triangleright b_{(2)}) \quad \forall h \in H, a \in A.$$

If in addition A and H are Hopf algebras, then so is their double cross product, with antipode

$$\mathcal{S}(a \otimes h) = (1 \otimes \mathcal{S}(h))(\mathcal{S}(a) \otimes 1) = (\mathcal{S}(h_{(2)}) \triangleleft \mathcal{S}(b_{(2)})) \otimes (\mathcal{S}(h_{(1)}) \triangleright \mathcal{S}(b_{(1)})).$$

4.5.5. From skew-Hopf pairings to double cross products

Let R be a ring, let A, H be two R -bialgebras and let $\eta : H \otimes A \rightarrow R$ be a skew-Hopf pairing which is convolution invertible. Then H is a right A -module coalgebra and A is a left H -module coalgebra via the actions

$$\begin{aligned} h \triangleright a &:= h_{(2)} \eta^{-1}(h_{(1)}, a_{(1)}) \eta(h_{(3)}, a_{(2)}) \\ h \triangleleft a &:= a_{(2)} \eta^{-1}(h_{(1)}, a_{(1)}) \eta(h_{(2)}, a_{(3)}) \end{aligned}$$

for all $h \in H$ and $a \in A$. In particular, then, there exists a double cross product bialgebra $A \bowtie H$ built upon $A \otimes H$; as we know, it has the tensor product unit, counit and coproduct, while its product now explicitly reads, in terms of the pairing, as follows (see [41, Example 7.2.7]):

$$(a \otimes h) \cdot (b \otimes g) := \eta^{-1}(h_{(1)}, b_{(1)}) a b_{(2)} \otimes h_{(2)} g \eta(h_{(3)}, b_{(3)}).$$

In addition, when both A and H are Hopf algebras then such is $A \bowtie H$ as well.

4.5.6. FoMpQUEAs as double cross products

Using the skew-Hopf pairing between our Borel FoMpQUEAs given by Proposition 4.33, namely

$$\pi : U_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_+) \widehat{\otimes} \mathcal{U}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_-) \longrightarrow \mathbb{k}[[\hbar]]$$

and

$$\pi : \mathcal{U}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_+) \widehat{\otimes} U_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_-) \longrightarrow \mathbb{k}[[\hbar]]$$

we may apply the general construction in Section 4.5.5 above and define two new, double crossed-product Hopf algebras

$$\mathcal{U}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_-) \bowtie U_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_+) \quad \text{and} \quad U_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_-) \bowtie \mathcal{U}_{P, \hbar}^{\mathcal{R}}(\mathfrak{b}_+).$$

Following the recipe in Section 4.5.5, the actions of $\mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ on $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ and of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ on $\mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ via the skew-Hopf paring π are given by

$$\begin{aligned} T_i^+ \triangleright \bar{T}_j^- &= 0, & T_i^+ \triangleleft \bar{T}_j^- &= 0, & T_i^+ \triangleleft \bar{F}_j &= 0, & E_i \triangleright \bar{T}_j^- &= 0 \\ T_i^+ \triangleright \bar{F}_j &= -p_{ij}\bar{F}_j, & E_i \triangleleft \bar{T}_j^- &= -\hbar p_{ij}E_i \\ E_i \triangleright \bar{F}_j &= \delta_{ij} \frac{\hbar(1 - e^{-\bar{T}_j^-})}{q_i^{+1} - q_i^{-1}}, & E_i \triangleleft \bar{F}_j &= \delta_{ij} \frac{\hbar(e^{+\hbar T_i^+} - 1)}{q_i^{+1} - q_i^{-1}} \\ X \triangleright 1 &= 0, & 1 \triangleleft X &= 0 & \forall X \in \{E_i, T_i^+, \bar{F}_i, \bar{T}_i^-\}_{i \in I} \end{aligned}$$

(for all $i, j \in I$). It is clear that these formulae completely define the cross product structure on $\mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-) \bowtie U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$. For example, let us compute $E_i \triangleleft \bar{F}_j$ explicitly. Set $L_j = e^{-\bar{T}_j^-}$ and $K_i = e^{\bar{T}_i^+}$: then computations give

$$\begin{aligned} E_i \triangleright \bar{F}_j &= (\bar{F}_j)_{(2)} \eta^{-1}((E_i)_{(1)}, (\bar{F}_j)_{(1)}) \eta((E_i)_{(2)}, (\bar{F}_j)_{(3)}) \\ &= L_j \eta^{-1}(E_i, \bar{F}_j) \eta(1, L_j) + L_j \eta^{-1}(K_i, \bar{F}_j) \eta(E_i, L_j) \\ &\quad + \bar{F}_j \eta^{-1}(E_i, 1) \eta(1, L_j) + \bar{F}_j \eta^{-1}(K_i, 1) \eta(E_i, L_j) \\ &\quad + 1 \eta^{-1}(E_i, 1) \eta(1, \bar{F}_j) + 1 \eta^{-1}(K_i, 1) \eta(E_i, \bar{F}_j) \\ &= L_j \eta^{-1}(E_i, \bar{F}_j) + \eta(E_i, \bar{F}_j) = L_j \eta(E_i, \mathcal{S}(\bar{F}_j)) + \eta(E_i, \bar{F}_j) \\ &= L_j \eta(E_i, -\bar{F}_j L_j^{-1}) + \eta(E_i, \bar{F}_j) = (1 - L_j) \eta(E_i, \bar{F}_j) \\ &= \delta_{ij} \frac{\hbar(1 - L_j)}{q_i^{+1} - q_i^{-1}}. \end{aligned}$$

Now, the formulae above show that actually even $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-), U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+))$ is indeed a matched pair of Hopf algebras, with actions uniquely induced in the obvious way from the actions for the pair $(\mathcal{U}_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-), U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+))$ which are explicitly given by

$$\begin{aligned} T_i^+ \triangleright T_j^- &= 0, & T_i^+ \triangleleft T_j^- &= 0, & T_i^+ \triangleleft F_j &= 0, & E_i \triangleright T_j^- &= 0 \\ T_i^+ \triangleright F_j &= -p_{ij}F_j, & E_i \triangleleft T_j^- &= -p_{ij}E_i \\ E_i \triangleright F_j &= \delta_{ij} \frac{(1 - e^{-\hbar T_j^-})}{q_i^{+1} - q_i^{-1}}, & E_i \triangleleft F_j &= \delta_{ij} \frac{(e^{+\hbar T_i^+} - 1)}{q_i^{+1} - q_i^{-1}} \\ Y \triangleright 1 &= 0, & \triangleleft Y &= 0 & \forall Y \in \{E_i, T_i^+, F_i, T_i^-\}_{i \in I}. \end{aligned}$$

Therefore, a well-defined double cross product $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-) \bowtie U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ exists, which is a Hopf algebra containing both $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ and $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-)$ as Hopf subalgebras.

With a similar situation as for Theorem 4.38, we may then obtain our FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ as a double cross product Hopf algebra, namely the following holds:

THEOREM 4.41. — *Let $A := (a_{i,j})_{i,j \in I}$ be a generalized symmetrizable Cartan matrix, and let $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ be a matrix of Cartan type with associated Cartan matrix A . With assumptions as above, $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-) \bowtie U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ is a topological, \hbar -adically complete Hopf $\mathbb{k}[[\hbar]]$ -algebra, which is isomorphic to the FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ given in Definition 4.4. \square*

Remark 4.42. — Here again, it is worth pointing out that the procedure we followed above to construct $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_-) \bowtie U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_+)$ follows a general recipe. Namely, starting with a QUEA U_{\hbar} and its dual QFSHA $U_{\hbar}^* := F_{\hbar}$, one has the right-left matched pair (U_{\hbar}, F_{\hbar}) , with U_{\hbar} acting on F_{\hbar} by coadjoint action, and viceversa; thus one can construct $U_{\hbar} \bowtie F_{\hbar}$, which is isomorphic to the quantum double $D(U_{\hbar}, F_{\hbar})$ and, as such, is not yet the kind of object we are looking for. Then one observes (see [6, Section A.5]) that the right-left matched pair (U_{\hbar}, F_{\hbar}) induces another similar right-left matched pair $(U_{\hbar}, U_{\hbar}^{\vee})$, where U_{\hbar}^{\vee} denotes (in notation of [6]) the QUEA that is associated by Drinfeld’s *Quantum Duality Principle* with the QFSHA F_{\hbar} . Finally, we can consider the double cross product $U_{\hbar} \bowtie U_{\hbar}^{\vee}$ – isomorphic to $(U_{\hbar}, U_{\hbar}^{\vee})$ – which is now exactly the kind of QUEA we are looking for.

Instead of applying *verbatim* the recipe sketched above, in the previous construction we followed an explicit, concrete approach that seems totally independent; however, it is important to understand that what we did is in fact nothing but a concrete “realization” of the general recipe, even though it is not formally apparent.

4.5.7. The general case: third proof of Theorem 4.13

The previous analysis provides an explicit construction of any FoMpQUEA defined on a realization \mathcal{R} which is *split minimal*. Out of this, we now deduce a construction of a FoMpQUEA on \mathcal{R} of *any type*, by a process of “extension and quotient”. In the end, we find another proof for Theorem 4.13.

Let P be a multiparameter matrix (of Cartan type), let $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^{\vee})$ be any realization of it, and let $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ be the associated (topological, unital, associative) $\mathbb{k}[[\hbar]]$ -algebra, as in Definition 4.4.

By Lemma 2.7, we can also pick a *split* realization of P , say $\mathcal{R} := (\dot{\mathfrak{h}}, \mathring{\Pi}, \mathring{\Pi}^\vee)$, and $\dot{\mathfrak{h}}_T := \text{Span}(\{T_i^\pm\}_{i \in I})$ inside $\dot{\mathfrak{h}}$. Then we take the FoMpQUEA $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$ associated with \mathcal{R} : inside it, we consider the Cartan subalgebras $U_{\dot{\mathfrak{h}}}^{\mathcal{R}}(\dot{\mathfrak{h}}) := U_{P, \hbar}^{\mathcal{R}}(\dot{\mathfrak{h}})$ and $U_{\dot{\mathfrak{h}}_T}^{\mathcal{R}}(\dot{\mathfrak{h}}_T) := U_{P, \hbar}^{\mathcal{R}}(\dot{\mathfrak{h}}_T)$ – both independent of \mathcal{R} and P , as for every Cartan subalgebra – and also the complete, unital $\mathbb{k}[[\hbar]]$ -subalgebra generated by $\dot{\mathfrak{h}}_T$, the E_i 's and the F_i 's: the latter is clearly yet another FoMpQUEA, namely $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$, where $\mathcal{R} := (\dot{\mathfrak{h}}_T, \mathring{\Pi}, \mathring{\Pi}^\vee)$ – with $\mathring{\Pi} := \{\hat{\alpha}_i := \alpha_i|_{\dot{\mathfrak{h}}_T}\}_{i \in I}$ – is again a realization of P , now *split and minimal*. Thanks to Theorem 4.38 then, there exists a Hopf algebra structure on $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$, which is described by formulas as in (4.8), (4.9), (4.10).

On the other hand, definitions imply that $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g}) \cong U_{\dot{\mathfrak{h}}}^{\mathcal{R}}(\dot{\mathfrak{h}}) \widehat{\otimes}_{U_{\dot{\mathfrak{h}}_T}^{\mathcal{R}}(\dot{\mathfrak{h}}_T)} U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$. Then there is only one way to extend the Hopf structure in $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$ mentioned above to a Hopf structure on $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g}) \cong U_{\dot{\mathfrak{h}}}^{\mathcal{R}}(\dot{\mathfrak{h}}) \widehat{\otimes}_{U_{\dot{\mathfrak{h}}_T}^{\mathcal{R}}(\dot{\mathfrak{h}}_T)} U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$ so that all elements in $\dot{\mathfrak{h}}$ are primitive; in other words, there exists a unique (topological) Hopf structure in $U_{\dot{\mathfrak{h}}}^{\mathcal{R}}(\dot{\mathfrak{h}}) \widehat{\otimes}_{U_{\dot{\mathfrak{h}}_T}^{\mathcal{R}}(\dot{\mathfrak{h}}_T)} U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g}) \cong U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$ which coincides with the given one on the right-hand factor and makes all elements of $\dot{\mathfrak{h}}$ primitive in the left-hand factor.

Finally, again by Lemma 2.7, there exists an epimorphism of realizations $\pi : \mathcal{R} \twoheadrightarrow \mathcal{R}$. By functoriality, we get an epimorphism $U_\pi : U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g}) \twoheadrightarrow U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$ with $\text{Ker}(U_\pi)$ generated by $\text{Ker}(\pi)$ (cf. Proposition 4.5); moreover, every element in $\text{Ker}(\pi)$ is primitive and is central in $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$. Thus $\text{Ker}(U_\pi)$ is a *Hopf ideal* in the Hopf algebra $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$, hence $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$ inherits via U_π a quotient Hopf algebra structure from $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$, again described by the formulas in (4.8)–(4.10).

Remark 4.43. — We expect that our definition (and construction) of FoMpQUEAs, and all related results presented hereafter, can be extended to the case when the symmetrizable generalized Cartan matrix A is replaced by a more general symmetrizable Borcherds–Cartan matrix, see [5] and references therein. However, due to additional technical difficulties, we do not pursue such a goal in this paper.

5. Deformations of formal multiparameter QUEAs

After introducing formal MpQUEAs, in the previous section, now in the present section we go and study their deformations, either by twist or by 2-cocycle – both of “toral type”, say.

5.1. Deformations of FoMpQUEAs by toral twists

We discuss now suitable twist deformations (of “toral type”) of FoMpQUEAs, proving that they are again FoMpQUEAs. By the results in [20] one can show that all possible twist elements \mathcal{F} for Drinfeld’s $U_{\hbar}(\mathfrak{g})$ can be constructed from data associated with Belavin–Drinfeld triples which classify classical r -matrices for \mathfrak{g} itself: in this respect, our “toral” twists correspond to the trivial Belavin–Drinfeld triples.

5.1.1. Toral twist deformations of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$

We fix a matrix $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ of Cartan type with associated Cartan matrix A , a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of it and the FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, as in Section 2 and Section 4; in particular, $d_i := p_{ii}/2$ ($i \in I$), and \mathfrak{h} is a free $\mathbb{k}[[\hbar]]$ -module of finite rank $t := \text{rk}(\mathfrak{h})$. We fix in \mathfrak{h} any $\mathbb{k}[[\hbar]]$ -basis $\{H_g\}_{g \in \mathcal{G}}$, where $|\mathcal{G}| = \text{rk}(\mathfrak{h}) = t$. Pick $\Phi = (\phi_{gk})_{g,k \in \mathcal{G}} \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$, and set

$$\mathfrak{J}_\Phi := \sum_{g,k=1}^t \phi_{gk} H_g \otimes H_k \in \mathfrak{h} \otimes \mathfrak{h} \subseteq U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}) \otimes U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}).$$

By direct check, we see that the element

$$(5.1) \quad \mathcal{F}_\Phi := e^{\hbar 2^{-1} \mathfrak{J}_\Phi} = \exp \left(\hbar 2^{-1} \sum_{g,k=1}^t \phi_{gk} H_g \otimes H_k \right)$$

in $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}) \widehat{\otimes} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$ is actually a *twist* for $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ in the sense of Section 4.1.3. Using it, we construct a new (topological) Hopf algebra $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))^{\mathcal{F}_\Phi}$, isomorphic to $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ as an algebra but with a new, twisted coalgebra structure, as in Section 4.1.3. A direct calculation yields explicit formulas for the new coproduct on generators, namely

$$\begin{aligned} \Delta^\Phi(E_\ell) &= E_\ell \otimes \mathcal{L}_{\Phi,\ell}^{+1} + e^{+\hbar T_\ell^+} \mathcal{K}_{\Phi,\ell}^{+1} \otimes E_\ell \quad (\forall \ell \in I) \\ \Delta^\Phi(T) &= T \otimes 1 + 1 \otimes T \quad (\forall T \in \mathfrak{h}) \\ \Delta^\Phi(F_\ell) &= F_\ell \otimes \mathcal{L}_{\Phi,\ell}^{-1} e^{-\hbar T_\ell^-} + \mathcal{K}_{\Phi,\ell}^{-1} \otimes F_\ell \quad (\forall \ell \in I) \end{aligned}$$

with

$$\mathcal{L}_{\Phi,\ell} := e^{+\hbar 2^{-1} \sum_{g,k=1}^t \alpha_\ell(H_g) \phi_{gk} H_k}, \quad \mathcal{K}_{\Phi,\ell} := e^{+\hbar 2^{-1} \sum_{g,k=1}^t \alpha_\ell(H_g) \phi_{kg} H_k}$$

for all $\ell \in I$. Similarly, the “twisted” antipode $\mathcal{S}^\Phi := \mathcal{S}^{\mathcal{F}^\Phi}$ and the counit $\epsilon^\Phi := \epsilon$ are given by

$$\begin{aligned} \mathcal{S}^\Phi(E_\ell) &= -e^{-\hbar T_\ell^+} \mathcal{K}_{\Phi,\ell}^{-1} E_\ell \mathcal{L}_{\Phi,\ell}^{-1}, & \epsilon^\Phi(E_\ell) &= 0 \quad (\forall \ell \in I) \\ \mathcal{S}^\Phi(T) &= -T, & \epsilon^\Phi(T) &= 0 \quad (\forall T \in \mathfrak{h}) \\ \mathcal{S}^\Phi(F_\ell) &= -\mathcal{K}_{\Phi,\ell}^{+1} F_\ell \mathcal{L}_{\Phi,\ell}^{+1} e^{+\hbar T_\ell^-}, & \epsilon^\Phi(F_\ell) &= 0 \quad (\forall \ell \in I). \end{aligned}$$

Remark 5.1. — The twist \mathcal{F}_Φ is an example of Reshetikhin’s twist as in [49], only “adapted” to the present case of our more general FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$. When \mathfrak{g} is a simple Lie algebra, this twist corresponds to empty datum of the Belavin–Drinfeld triple with respect to the classification in [20] of twists for $U_\hbar(\mathfrak{g})$.

5.1.2. Twisted generators

From the explicit description of the coproduct Δ^Φ , it follows that $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))^{\mathcal{F}^\Phi}$ is generated by group-likes and skew-primitive elements; in particular, it is a pointed Hopf algebra. Moreover, both Hopf algebras $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ and $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))^{\mathcal{F}^\Phi}$ have the same coradical and the same space of skew-primitive elements. As the coproduct is changed by the twist, one sees that the skew-primitive generators of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, which are $(1, g)$ - or $(g, 1)$ -primitive for some $g \in G(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))$, with respect to Δ , become (h, k) -primitive for Δ^Φ . Looking at the coradical filtration, and the associated graded Hopf algebra, one may find from that set of generators some new $(1, \ell)$ - or $(\ell, 1)$ -primitives for $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))^{\mathcal{F}^\Phi}$. This leads to devise (new) *twisted generators* and a corresponding presentation for $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))^{\mathcal{F}^\Phi}$, which yields a Hopf algebra isomorphism between $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))^{\mathcal{F}^\Phi}$ and a new FoMpQUEA with suitable multiparameter matrix and realization.

Motivated by the above analysis, we introduce now in $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))^{\mathcal{F}^\Phi}$ the “twisted” generators (for all $\ell \in I$) $E_\ell^\Phi := \mathcal{L}_{\Phi,\ell}^{-1} E_\ell$, $F_\ell^\Phi := F_\ell \mathcal{K}_{\Phi,\ell}^{+1}$ and the twisted “distinguished toral elements” (or “coroots”) that were already defined in (2.3), i.e. $T_{\Phi,\ell}^\pm := T_\ell^\pm \pm \sum_{g,k=1}^t \alpha_\ell(H_g) \phi_{kg} H_k$. Still from Section 2.2.1, we recall also $P_\Phi := (p_{i,j}^\Phi)_{i,j \in I}$ and $\mathcal{R}_\Phi := (\mathfrak{h}, \Pi, \Pi_\Phi^\vee)$, the latter being a realization of the former.

Second, the commutation relations in the algebra $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))^{\mathcal{F}^\Phi}$ give new commutation relations between twisted generators. Namely, by straightforward computations, for instance using that

$$\mathcal{K}_{\Phi,j} E_i = e^{\hbar 2^{-1} \sum_{g,k=1}^t \alpha_j(H_g) \phi_{kg} \alpha_i(H_k)} E_i \mathcal{K}_{\Phi,j}$$

and that

$$e^{+\hbar T_{\Phi,i}^{\pm}} = e^{+\hbar T_i^{\pm}} (\mathcal{K}_{\Phi,i} \mathcal{L}_{\Phi,i}^{-1})^{\pm 1},$$

one proves that inside $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))^{\mathcal{F}^{\Phi}}$ the following identities hold true (for all $T, T', T'' \in \mathfrak{h}$, $i, j, \ell \in I$, $i \neq j$, $X \in \{E, F\}$):

$$TE_j^{\Phi} - E_j^{\Phi}T = +\alpha_j(T)E_j^{\Phi}, \quad TF_j^{\Phi} - F_j^{\Phi}T = -\alpha_j(T)F_j^{\Phi}$$

$$T'T'' = T''T', \quad E_i^{\Phi}F_{\ell}^{\Phi} - F_{\ell}^{\Phi}E_i^{\Phi} = \delta_{i,\ell} \frac{e^{+\hbar T_{\Phi,i}^+} - e^{-\hbar T_{\Phi,i}^-}}{q_i^{+1} - q_i^{-1}}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} (q_{ij}^{\Phi})^{+k/2} (q_{ji}^{\Phi})^{-k/2} (X_i^{\Phi})^{1-a_{ij}-k} X_j^{\Phi} (X_i^{\Phi})_i^k = 0$$

with $q_{i,j}^{\Phi} := e^{\hbar p_{i,j}^{\Phi}}$ ($i, j \in I$) – so that $q_{i,i}^{\Phi} = e^{\hbar p_{i,i}^{\Phi}} = e^{\hbar p_{i,i}} = e^{\hbar 2d_i} = q_i^2$.

Third, the Hopf operations on the twisted generators read ($\ell \in I$, $T \in \mathfrak{h}$)

$$\Delta^{\Phi}(E_{\ell}^{\Phi}) = E_{\ell}^{\Phi} \otimes 1 + e^{+\hbar T_{\Phi,\ell}^+} \otimes E_{\ell}^{\Phi}, \quad \epsilon^{\Phi}(E_{\ell}^{\Phi}) = 0$$

$$\Delta^{\Phi}(T) = T \otimes 1 + 1 \otimes T, \quad \epsilon^{\Phi}(T) = 0$$

$$\Delta^{\Phi}(F_{\ell}^{\Phi}) = F_{\ell}^{\Phi} \otimes e^{-\hbar T_{\Phi,\ell}^-} + 1 \otimes F_{\ell}^{\Phi}, \quad \epsilon^{\Phi}(F_{\ell}^{\Phi}) = 0$$

$$\mathcal{S}^{\Phi}(E_{\ell}^{\Phi}) = -e^{-\hbar T_{\Phi,\ell}^+} E_{\ell}^{\Phi}, \quad \mathcal{S}^{\Phi}(T) = -T, \quad \mathcal{S}^{\Phi}(F_{\ell}^{\Phi}) = -F_{\ell}^{\Phi} e^{+\hbar T_{\Phi,\ell}^-}.$$

In a nutshell, the above analysis proves the following result:

THEOREM 5.2. — *There exists an isomorphism of topological Hopf algebras*

$$f_P^{\Phi} : U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \xrightarrow{\cong} (U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))^{\mathcal{F}^{\Phi}}$$

given by $E_i \mapsto E_i^{\Phi}$, $T \mapsto T$ and $F_i \mapsto F_i^{\Phi}$ for all $i \in I$, $T \in \mathfrak{h}$.

In particular, the class of all FoMpQUEAs of any fixed Cartan type and of fixed rank is stable by toral twist deformations. Moreover, inside it the subclass of all such FoMpQUEAs associated with straight, resp. small, realizations is stable as well.

Similar, parallel statements hold true for the Borel FoMpQUEAs, namely there exist isomorphisms $f_{P,\pm}^{\Phi} : U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm}) \xrightarrow{\cong} (U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_{\pm}))^{\mathcal{F}^{\Phi}}$ given by formulas as above. □

In fact, the previous result can be somehow reversed, as the following shows: in particular, loosely speaking, we end up finding that every straight small FoMpQUEA can be realized as a toral twist deformation of the “standard” FoMpQUEA by Drinfeld (cf. claim (c) in Theorem 5.3 here below).

THEOREM 5.3. — *With assumptions as above, let P and P' be two matrices of Cartan type with the same associated Cartan matrix A .*

- (a) Let \mathcal{R} be a straight realization of P and let $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ be the associated FoMpQUEA. Then there exists a straight realization $\tilde{\mathcal{R}}'$ of P' and a matrix $\Phi \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ such that for the associated twist element \mathcal{F}_Φ as in (5.1) we have

$$U_{P',\hbar}^{\tilde{\mathcal{R}}'}(\mathfrak{g}) \cong (U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))^{\mathcal{F}_\Phi}.$$

In a nutshell, if $P'_s = P_s$ then from any straight FoMpQUEA over P we can obtain by toral twist deformation a straight FoMpQUEA (of the same rank) over P' .

Conversely, if \mathcal{R}' is any straight realization of P' and $U_{P',\hbar}^{\mathcal{R}'}(\mathfrak{g})$ is the associated FoMpQUEA, then there exists a straight realization $\hat{\mathcal{R}}$ of P and a matrix $\Phi \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ such that for the associated twist element \mathcal{F}_Φ as in (5.1) we have

$$U_{P',\hbar}^{\mathcal{R}'}(\mathfrak{g}) \cong (U_{P,\hbar}^{\hat{\mathcal{R}}}(\mathfrak{g}))^{\mathcal{F}_\Phi}.$$

- (b) Let \mathcal{R} and \mathcal{R}' be straight small realizations of P and P' respectively, with $\text{rk}(\mathcal{R}) = \text{rk}(\mathcal{R}') = t$, and let $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ and $U_{P',\hbar}^{\mathcal{R}'}(\mathfrak{g})$ be the associated FoMpQUEAs. Then there exists a matrix $\Phi \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ such that for \mathcal{F}_Φ as in (5.1) we have

$$U_{P',\hbar}^{\mathcal{R}'}(\mathfrak{g}) \cong (U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))^{\mathcal{F}_\Phi}.$$

In other words, if $P'_s = P_s$ any straight small FoMpQUEA over P' is isomorphic to a toral twist deformation of any straight small FoMpQUEA over P of same rank.

- (c) Every straight small FoMpQUEA is isomorphic to some toral twist deformation of Drinfeld’s standard FoMpQUEA (over $DA = P_s$) of the same rank.
 (d) Similar, parallel statements hold true for the Borel FoMpQUEAs.

Proof.

(a). — By Theorem 5.2 it is enough to find $\Phi \in \mathfrak{so}_t(\mathbb{k})$ such that $P' = P_\Phi$, that is $P' = P - \mathfrak{A}\Phi\mathfrak{A}^T$; but this is guaranteed by Lemma 2.13, so we are done.

(b). — Like for Theorem 3.13(b), this follows from claim (a), along with the uniqueness of straight small realizations, by Proposition 2.4(b), and Proposition 3.4.

(c). — This follows applying (b), with $U_{P',\hbar}^{\mathcal{R}'}(\mathfrak{g})$ the given straight small FoMpQUEA and $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ the “standard” FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}_P^D)$ over $P := DA = P'_s$ as in Drinfeld’s definition (up to “taking the double”), which is straight and split minimal. \square

Remark 5.4. — Theorems 5.2 and 5.3 have the following interpretation. Our FoMpQUEAs $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ are quantum objects depending on the multiparameter P ; but when we perform onto $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ a deformation by twist as in Section 5.1.1, the output $U_{P,\hbar}^{\mathcal{R},\Phi}(\mathfrak{g}) := (U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))^{\mathcal{F}_\Phi}$ depends on two multiparameters, namely P and Φ . Thus all these $U_{P,\hbar}^{\mathcal{R},\Phi}(\mathfrak{g})$'s form a seemingly *richer* family of “twice-multiparametric” formal QUEAs. Nonetheless, Theorem 5.2 above proves that this family actually *coincides* with the family of all FoMpQUEAs, although the latter seems a priori smaller.

In short, Theorems 5.2 and 5.3 show the following. The dependence of the Hopf structure of $U_{P,\hbar}^{\mathcal{R},\Phi}(\mathfrak{g})$ on the “double parameter” (P, Φ) is “split” in the algebraic structure (ruled by P) and in the coalgebraic structure (ruled by Φ); now Theorems 5.2 and 5.3 enable us to “polarize” this dependence so to codify it either entirely within the algebraic structure (while the coalgebraic one is reduced to a “canonical form”), so that the single multiparameter P_Φ is enough to describe it, or entirely within the coalgebraic structure (with the algebraic one being reduced to the “standard” Drinfeld’s one), so that the multiparameter Φ_P alone is enough.

Remark 5.5. — As the subclass of *split* realizations is *not closed* under twist (cf. the end of Section 2.2.2), the subclass of all “split” FoMpQUEAs is not closed too under twist deformation; this is a quantum analogue of Remark 3.14(b).

5.2. Deformations of FoMpQUEAs by toral 2-cocycles

We consider now some 2-cocycle deformations (called “of toral type” again) of the formal MpQUEAs $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, and we prove that these are again formal MpQUEAs.

5.2.1. Special 2-cocycles of $U_{P,\hbar}(\mathfrak{h})$

Fix again $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ of Cartan type with associated Cartan matrix A , a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of it and the (topological) Hopf algebra $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, as in Section 2 and Section 4, setting $d_i := p_{ii}/2$ for all $i \in I$ and $D_P := \text{diag}(d_1, \dots, d_n)$. We consider special 2-cocycles of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, called “toral” as they are induced from the quantum torus. To this end, like in Section 5.1.1, we fix in \mathfrak{h} a $\mathbb{k}[[\hbar]]$ -basis $\{H_g\}_{g \in \mathcal{G}}$, where \mathcal{G} is an index set with $|\mathcal{G}| = \text{rk}(\mathfrak{h}) = t$.

Like in Section 2.3.1, we fix an antisymmetric, $\mathbb{k}[[\hbar]]$ -bilinear map $\chi : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{k}[[\hbar]]$, that corresponds to some $X = (\chi_{g\gamma})_{g,\gamma \in \mathcal{G}} \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ via $\chi_{g\gamma} = \chi(H_g, H_\gamma)$. We also consider the antisymmetric matrix $\mathring{X} := (\mathring{\chi}_{ij} = \chi(T_i^+, T_j^+))_{i,j \in I} \in \mathfrak{so}_n(\mathbb{k}[[\hbar]])$. Any such map χ induces uniquely an antisymmetric, $\mathbb{k}[[\hbar]]$ -bilinear map

$$\tilde{\chi}_U : U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}) \times U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}) \longrightarrow \mathbb{k}[[\hbar]]$$

as follows. By definition, $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$ is an \hbar -adically complete topologically free Hopf algebra isomorphic to $\widehat{S}_{\mathbb{k}[[\hbar]]}(\mathfrak{h}) := \widehat{\bigoplus_{n \in \mathbb{N}} S_{\mathbb{k}[[\hbar]]}^n(\mathfrak{h})}$ – the \hbar -adic completion of the symmetric algebra $S_{\mathbb{k}[[\hbar]]}(\mathfrak{h}) = \bigoplus_{n \in \mathbb{N}} S_{\mathbb{k}[[\hbar]]}^n(\mathfrak{h})$ – hence the following makes sense:

DEFINITION 5.6. — We define $\tilde{\chi}_U$ as the unique $\mathbb{k}[[\hbar]]$ -linear (hence \hbar -adically continuous) map $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}) \otimes U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}) \xrightarrow{\tilde{\chi}_U} \mathbb{k}[[\hbar]]$ such that (with identifications as above)

$$\begin{aligned} \tilde{\chi}_U(z, 1) &:= \epsilon(z) =: \tilde{\chi}_U(1, z) & \forall z \in \widehat{S}_{\mathbb{k}[[\hbar]]}(\mathfrak{h}) \\ \tilde{\chi}_U(x, y) &:= \chi(x, y) & \forall x, y \in S_{\mathbb{k}[[\hbar]]}^1(\mathfrak{h}) \\ \tilde{\chi}_U(x, y) &:= 0 & \forall x \in S_{\mathbb{k}[[\hbar]]}^r(\mathfrak{h}), y \in S_{\mathbb{k}[[\hbar]]}^s(\mathfrak{h}) : r, s \geq 1, r + s > 2. \end{aligned}$$

By construction, $\tilde{\chi}_U$ is a normalized Hochschild 2-cocycle on $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$, that is for all $x, y, z \in U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$ we have

$$\epsilon(x)\tilde{\chi}_U(y, z) - \tilde{\chi}_U(xy, z) + \tilde{\chi}_U(x, yz) - \tilde{\chi}_U(x, y)\epsilon(z) = 0.$$

Now recall that, given two linear maps $\eta, \vartheta : U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}) \otimes U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}) \rightarrow \mathbb{k}[[\hbar]]$, one may define the convolution product map $\eta * \vartheta : U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})^{\otimes 2} \rightarrow \mathbb{k}[[\hbar]]$ – using on $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})^{\otimes 2} \cong \widehat{S}_{\mathbb{k}[[\hbar]]}(\mathfrak{h})^{\otimes 2} \cong \widehat{S}_{\mathbb{k}[[\hbar]]}(\mathfrak{h} \oplus \mathfrak{h})$ the standard coalgebra structure – by the formula

$$(\eta * \vartheta)(x \otimes y) := \eta(x_{(1)}, y_{(1)})\vartheta(x_{(2)}, y_{(2)}) \quad \forall x, y \in U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}).$$

Then by η^{*m} we denote the m -th power with respect to the convolution product of any map η as above; in particular, we set $\eta^{*0} := \epsilon \otimes \epsilon$.

The following result describes the powers of our map $\tilde{\chi}_U$:

LEMMA 5.7. — For all $H_+, H_- \in \mathfrak{h}$ and $k, \ell, m \in \mathbb{N}_+$, we have

$$\tilde{\chi}_U^{*m}(H_+^k, H_-^\ell) = \begin{cases} \delta_{k,m}\delta_{\ell,m}(m!)^2 \chi(H_+, H_-)^m & \text{form} \geq 1, \\ \delta_{k,0}\delta_{\ell,0} & \text{form} = 0. \end{cases}$$

Proof. — The proof follows by a direct computation. □

DEFINITION 5.8. — *Keep notation as above. We define χ_U as the unique $\mathbb{k}[[\hbar]]$ -linear map from $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}) \otimes_{\mathbb{k}[[\hbar]]} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$ to $\mathbb{k}((\hbar))$ given by the exponentiation of $\hbar^{-1}2^{-1}\tilde{\chi}_U$, i.e.*

$$\chi_U := e^{\hbar^{-1}2^{-1}\tilde{\chi}_U} = \sum_{m \geq 0} \hbar^{-m} \tilde{\chi}_U^{*m} / 2^m m!.$$

LEMMA 5.9. — *The map χ_U is a well defined, normalized, $\mathbb{k}((\hbar))$ -valued Hopf 2-cocycle for $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$, such that, for all $H_+, H_- \in \mathfrak{h}$, and setting $K_{\pm} := e^{\hbar H_{\pm}}$,*

$$\chi_U^{\pm 1}(H_+, H_-) = \pm \hbar^{-1}2^{-1}\chi(H_+, H_-), \quad \chi_U(K_+, K_-) = e^{\hbar 2^{-1}\chi(H_+, H_-)}.$$

Proof. — The identities follow from Lemma 5.7, hence χ_U is well defined. The other claims follow from the proof of [52, Theorem 4.1], see also [29, Lemma 4.1]. □

5.2.2. Toral 2-cocycles of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$

The previous construction provides, starting from χ , a normalized Hopf 2-cocycle that will be denoted $\chi_U : U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}) \times U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}) \rightarrow \mathbb{k}((\hbar))$.

We assume now that the map χ satisfies the additional requirement (2.10), in other words we require that $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ – notation of Section 2.3. The latter map canonically induces a $\mathbb{k}[[\hbar]]$ -bilinear map $\bar{\chi} : \bar{\mathfrak{h}} \times \bar{\mathfrak{h}} \rightarrow \mathbb{k}[[\hbar]]$, where $\bar{\mathfrak{h}} := \mathfrak{h}/\mathfrak{s}$ with $\mathfrak{s} := \text{Span}_{\mathbb{k}[[\hbar]]}(\{S_i\}_{i \in I})$, given by

$$\bar{\chi}(T' + \mathfrak{s}, T'' + \mathfrak{s}) := \chi(T', T'') \quad \forall T', T'' \in \mathfrak{h}.$$

Now, replaying the construction above but with $\bar{\mathfrak{h}}$ and $\bar{\chi}$ replacing \mathfrak{h} and χ , we can construct a normalized Hopf 2-cocycle $\bar{\chi}_U : U_{P,\hbar}^{\mathcal{R}}(\bar{\mathfrak{h}}) \times U_{P,\hbar}^{\mathcal{R}}(\bar{\mathfrak{h}}) \rightarrow \mathbb{k}((\hbar))$; for the latter, the analogue of Lemma 5.9 holds true again. Moreover, note that $U_{P,\hbar}^{\mathcal{R}}(\bar{\mathfrak{h}}) \cong \widehat{S}_{\mathbb{k}[[\hbar]]}(\bar{\mathfrak{h}})$, and, thanks to (2.10), there exists a unique Hopf algebra epimorphism $\pi : U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \rightarrow U_{P,\hbar}^{\mathcal{R}}(\bar{\mathfrak{h}})$ given by $\pi(E_i) := 0$, $\pi(F_i) := 0$ – for $i \in I$ – and $\pi(T) := (T + \mathfrak{s}) \in \bar{\mathfrak{h}} \subseteq U_{P,\hbar}^{\mathcal{R}}(\bar{\mathfrak{h}})$ – for $T \in \mathfrak{h}$. Then we consider

$$\sigma_{\chi} := \bar{\chi}_U \circ (\pi \times \pi) : U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \times U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \rightarrow \mathbb{k}((\hbar))$$

which, with a slight abuse of terminology, is *automatically* a normalized, $\mathbb{k}((\hbar))$ -valued Hopf 2-cocycle on $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$.

DEFINITION 5.10. — *We shall call all normalized Hopf 2-cocycles σ_{χ} of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ obtained, from all $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$, via the above construction as “of toral type”, or “toral 2-cocycles”; we denote by $\mathcal{Z}_2(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))$ the set of all of them, which is actually independent of P , indeed.*

5.2.3. Formulas for the σ_χ -deformed product

Let a toral 2-cocycle $\sigma_\chi \in \mathcal{Z}_2(U_{P,\hbar}^R(\mathfrak{g}))$ as above be given. Following Section 4.1.3, using σ_χ we introduce in $U_{P,\hbar}^R(\mathfrak{g})$ a “deformed product”, hereafter denoted by $\dot{\sigma}_\chi$; then $X^{(n)\sigma_\chi} = X \dot{\sigma}_\chi \cdots \dot{\sigma}_\chi X$ will denote the n -th power of any $X \in U_{P,\hbar}^R(\mathfrak{g})$ with respect to this deformed product.

Directly from definitions, sheer computation yields the following formulas, relating the deformed product with the old one ($T', T'', T \in \mathfrak{h}$, $i, j \in I$):

$$\begin{aligned}
T' \dot{\sigma}_\chi T'' &= T' T'', & E_i \dot{\sigma}_\chi F_j &= E_i F_j, & F_j \dot{\sigma}_\chi E_i &= F_j E_i \\
T \dot{\sigma}_\chi E_j &= T E_j + 2^{-1} \chi(T, T_j^+) E_j, & E_j \dot{\sigma}_\chi T &= E_j T + 2^{-1} \chi(T_j^+, T) E_j \\
T \dot{\sigma}_\chi F_j &= T F_j + 2^{-1} \chi(T, T_j^-) F_j, & F_j \dot{\sigma}_\chi T &= F_j T + 2^{-1} \chi(T_j^-, T) F_j \\
E_i^{(m)\sigma_\chi} &= \prod_{\ell=1}^{m-1} \sigma_\chi \left(e^{+\hbar \ell T_i^+}, e^{+\hbar T_i^+} \right) E_i^m = E_i^m \\
E_i^m \dot{\sigma}_\chi E_j^n &= \sigma_\chi \left(e^{+\hbar m T_i^+}, e^{+\hbar n T_j^+} \right) E_i^m E_j^n = e^{+\hbar m n 2^{-1} \chi_{ij}} E_i^m E_j^n \\
E_i^{(m)\sigma_\chi} \dot{\sigma}_\chi E_j \dot{\sigma}_\chi E_k^{(n)\sigma_\chi} &= \left(\prod_{\ell=1}^{m-1} \sigma_\chi \left(e^{+\hbar \ell T_i^+}, e^{+\hbar T_i^+} \right) \right) \left(\prod_{t=1}^{n-1} \sigma_\chi \left(e^{+\hbar t T_k^+}, e^{+\hbar T_k^+} \right) \right) \\
&\quad \cdot \sigma_\chi \left(e^{+\hbar m T_i^+}, e^{+\hbar T_j^+} \right) \sigma_\chi \left(e^{+\hbar(m T_i^+ + T_j^+)}, e^{+\hbar n T_k^+} \right) E_i^m E_j E_k^n \\
F_i^{(m)\sigma_\chi} &= \prod_{\ell=1}^{m-1} \sigma_\chi^{-1} \left(e^{-\hbar \ell T_i^-}, e^{-\hbar T_i^-} \right) F_i^m = F_i^m \\
F_i^m \dot{\sigma}_\chi F_j^n &= \sigma_\chi^{-1} \left(e^{-\hbar m T_i^-}, e^{-\hbar n T_j^-} \right) F_i^m F_j^n = e^{-\hbar m n 2^{-1} \chi_{ij}} F_i^m F_j^n \\
F_i^{(m)\sigma_\chi} \dot{\sigma}_\chi F_j \dot{\sigma}_\chi F_k^{(n)\sigma_\chi} &= \left(\prod_{\ell=1}^{m-1} \sigma_\chi^{-1} \left(e^{-\hbar \ell T_i^-}, e^{-\hbar T_i^-} \right) \right) \left(\prod_{t=1}^{n-1} \sigma_\chi^{-1} \left(e^{-\hbar t T_k^-}, e^{-\hbar T_k^-} \right) \right) \\
&\quad \cdot \sigma_\chi^{-1} \left(e^{-\hbar m T_i^-}, e^{-\hbar T_j^-} \right) \sigma_\chi^{-1} \left(e^{-\hbar(m T_i^- + T_j^-)}, e^{-\hbar n T_k^-} \right) F_i^m F_j F_k^n \\
F_i^{(m)\sigma_\chi} \dot{\sigma}_\chi F_j \dot{\sigma}_\chi F_k^{(n)\sigma_\chi} &= \left(\prod_{\ell=1}^{m-1} \sigma_\chi^{-1} \left(e^{-\hbar \ell T_i^-}, e^{-\hbar T_i^-} \right) \right) \left(\prod_{t=1}^{n-1} \sigma_\chi^{-1} \left(e^{-\hbar t T_k^-}, e^{-\hbar T_k^-} \right) \right) \\
&\quad \cdot \sigma_\chi^{-1} \left(e^{-\hbar m T_i^-}, e^{-\hbar T_j^-} \right) \sigma_\chi^{-1} \left(e^{-\hbar(m T_i^- + T_j^-)}, e^{-\hbar n T_k^-} \right) F_i^m F_j F_k^n.
\end{aligned}$$

It is also worth remarking that the identity $T'_{\sigma_\chi} T'' = T' T''$ (for $T', T'' \in \mathfrak{h}$) also implies that $T^{(n)\sigma_\chi} = T^n$ (for $T \in \mathfrak{h}$ and $n \in \mathbb{N}$) – i.e., each “deformed” power of any toral element coincides with the corresponding “undeformed” power. It follows then that the exponential of any toral element with respect to the deformed product σ_χ is the same as with respect to the old one.

Remark 5.11. — Observe that the whole procedure of 2-cocycle deformation by σ_χ should apply to the scalar extension $\mathbb{k}(\!(\hbar)\!) \otimes_{\mathbb{k}[\![\hbar]\!]} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, since *a priori* the 2-cocycle σ_χ takes values in $\mathbb{k}(\!(\hbar)\!)$ rather than in $\mathbb{k}[\![\hbar]\!]$. Nevertheless, the formulas above show that $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is actually *closed* for the deformed product “ σ_χ ” provided by this procedure, hence the deformation eventually “restricts” to $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ itself, so that we eventually end up with a well-defined 2-cocycle deformation $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))_{\sigma_\chi}$.

A first, direct consequence of these formulas is the following:

PROPOSITION 5.12. — *With notations as above, the deformed algebra $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))_{\sigma_\chi}$ is still generated by the elements E_i, F_i and T (with $i \in I$ and $T \in \mathfrak{h}$) of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$.*

5.2.4. Toral 2-cocycle deformations of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$

Our key result concerns 2-cocycle deformations by means of toral 2-cocycles. In order to state it, we need some more notation, which we now settle.

Let $P := (p_{ij})_{i,j \in I} \in M_n(\mathbb{k}[\![\hbar]\!])$ be a multiparameter matrix of Cartan type with associated Cartan matrix A (cf. Definition 2.1) and fix a realization $\mathcal{R} = (\mathfrak{h}, \Pi, \Pi^\vee)$ of P . Fix an antisymmetric $\mathbb{k}[\![\hbar]\!]$ -bilinear map $\chi : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{k}[\![\hbar]\!]$ enjoying (2.10) – that is, $\chi \in \text{Alt}_{\mathbb{k}[\![\hbar]\!]}^S(\mathfrak{h})$ – we associate with it the matrix $\mathring{X} := (\mathring{\chi}_{ij} = \chi(T_i^+, T_j^+))_{i,j \in I}$ as above. Note that $+\chi(-, T_j^+) = -\chi(-, T_j^-)$ for all $j \in I$, as direct consequence of (2.10). Basing on the above, like in Section 2.3 we define

$$P_{(\chi)} := P + \mathring{X} = \left(p_{ij}^{(\chi)} := p_{ij} + \mathring{\chi}_{ij} \right)_{i,j \in I}$$

and

$$\Pi_{(\chi)} := \left\{ \alpha_i^{(\chi)} := \alpha_i \pm \chi(-, T_i^\pm) \right\}_{i \in I}.$$

Then, still from Section 2.3 we know that $P_{(\chi)}$ is a matrix of Cartan type, the same of P indeed, and $\mathcal{R}_{(\chi)} = (\mathfrak{h}, \Pi_{(\chi)}, \Pi^\vee)$ is a realization of it.

We are now ready for our result on toral 2-cocycle deformations. Note in particular that, though toral 2-cocycles have values in $\mathbb{k}(\!(\hbar)\!)$, the deformed multiplication is still well defined within our initial FoMpQUEA, which is defined over $\mathbb{k}[[\hbar]]$.

THEOREM 5.13. — *There exists an isomorphism of topological Hopf algebras*

$$(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))_{\sigma_\chi} \cong U_{P^{(\chi)},\hbar}^{\mathcal{R}^{(\chi)}}(\mathfrak{g})$$

which is the identity on generators. In short, every toral 2-cocycle deformation of a FoMpQUEA is another FoMpQUEA, whose multiparameter $P^{(\chi)}$ and realization $\mathcal{R}^{(\chi)}$ depend on the original P and \mathcal{R} , as well as on χ , as explained in Section 2.3.

Similar statements hold true for the Borel FoMpQUEAs and their deformations by σ_χ , namely there exist isomorphisms

$$(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{b}_\pm))_{\sigma_\chi} \cong U_{P^{(\chi)},\hbar}^{\mathcal{R}^{(\chi)}}(\mathfrak{b}_\pm).$$

Proof. — We begin by noting the following key

FACT. — *The generators of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, when thought of as elements of the deformed algebra $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))_{\sigma_\chi}$, obey the defining relations of the (same name) generators of $U_{P^{(\chi)},\hbar}^{\mathcal{R}^{(\chi)}}(\mathfrak{g})$, with respect to the deformed product $\dot{\sigma}_\chi$.*

Indeed, most relations follow at once from the formulas in Section 5.2.3. Namely, the identities $T'_{\dot{\sigma}_\chi} T'' = T' T''$ imply $T'_{\dot{\sigma}_\chi} T'' = T''_{\dot{\sigma}_\chi} T' -$ for all $T', T'' \in \mathfrak{h}$. Also, from $T_{\dot{\sigma}_\chi} E_j = T E_j + 2^{-1} \chi(T, T_j^+) E_j$ and $E_j \dot{\sigma}_\chi T = E_j T + 2^{-1} \chi(T_j^+, T) E_j -$ for all $T \in \mathfrak{h}$ and $j \in I -$ we get (since χ is antisymmetric)

$$T_{\dot{\sigma}_\chi} E_j - E_j \dot{\sigma}_\chi T = \left(\alpha_j(T) + 2^{-1} (\chi - \chi^T)(T, T_j^+) \right) E_j = +\alpha_j^{(\chi)}(T) E_j.$$

A similar, direct analysis also yields $T_{\dot{\sigma}_\chi} F_j - F_j \dot{\sigma}_\chi T = -\alpha_j^{(\chi)}(T) F_j.$

The identities $E_i \dot{\sigma}_\chi F_j = E_i F_j$ and $F_j \dot{\sigma}_\chi E_i = F_j E_i$ in turn imply

$$E_i \dot{\sigma}_\chi F_j - F_j \dot{\sigma}_\chi E_i = E_i F_j - F_j E_i = \delta_{i,j} \frac{e^{+\hbar T_i^+} - e^{-\hbar T_i^-}}{e^{+\hbar p_{ii}/2} - e^{-\hbar p_{ii}/2}}.$$

Eventually, since $p_{ii} = p_{ii}^{(\chi)}$ by definition, and the exponential of toral elements with respect to $\dot{\sigma}_\chi$ is the same as with respect to the old product, we conclude that

$$E_i \dot{\sigma}_\chi F_j - F_j \dot{\sigma}_\chi E_i = \delta_{i,j} \frac{e_{\sigma_\chi}^{+\hbar T_i^+} - e_{\sigma_\chi}^{-\hbar T_i^-}}{e^{+\hbar p_{ii}^{(\chi)}/2} - e^{-\hbar p_{ii}^{(\chi)}/2}}$$

where $e_{\sigma_\chi}^X$ denotes the exponential of any X with respect to $\dot{\sigma}_\chi$.

What is less trivial is proving the quantum Serre relations for the deformed product; we do this for the relation involving the E_i 's, leaving the relation involving the F_i 's as an exercise for the reader. To this end, set

$$q_{ij}^{(\chi)} := e^{\hbar p_{ij}^{(\chi)}} = e^{\hbar(p_{ij} + \check{\chi}_{ij})} \quad \text{for all } i, j \in I.$$

As $(P_{(\chi)})_s = P_s = DA$, we have $q_{ii}^{(\chi)} = q_{ii}$ and $q_i^{(\chi)} = e^{+\hbar p_{ii}^{(\chi)}/2} = q_i$ for all $i \in I$. Then, for all $i \neq j \in I$ we have to prove that

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} (q_{ij}^{(\chi)})^{+k/2} (q_{ji}^{(\chi)})^{-k/2} E_i^{(1-a_{ij}-k)\sigma_\chi} E_{j\check{\sigma}_\chi} E_i^{(k)\sigma_\chi} = 0.$$

To prove that, we analyze the factors in the summands separately.

Claim 1. — For all $i \neq j \in I$, we have

$$(q_{ij}^{(\chi)})^{+k/2} (q_{ji}^{(\chi)})^{-k/2} = q_{ij}^{+k/2} q_{ji}^{-k/2} e^{\hbar k \check{\chi}_{ij}}.$$

This follows by direct computation. Next claim instead is a consequence of Section 5.2.3:

Claim 2. — Fix $i \neq j \in I$ and write $m := 1 - a_{ij}$. Then

$$E_i^{(m-k)\sigma_\chi} E_{j\check{\sigma}_\chi} E_i^{(k)\sigma_\chi} = \sigma_\chi(K_i^{m-k}, K_j) \sigma_\chi(K_i^{m-k} K_j, K_i^k) E_i^{m-k} E_j E_i^k.$$

Now we evaluate the value of the toral 2-cocycle using the exponentials.

Claim 3. — For all $i, j \in I$ and $m, k, \ell \in \mathbb{N}$, we have

- (a) $\sigma_\chi(K_i^\ell, K_j) = e^{\hbar \ell 2^{-1} \check{\chi}_{ij}}$,
- (b) $\sigma_\chi(K_i^{m-k} K_j, K_i^k) = e^{\hbar k 2^{-1} \check{\chi}_{ji}}$,
- (c) $\sigma_\chi(K_i^{m-k}, K_j) \sigma_\chi(K_i^{m-k} K_j, K_i^k) = e^{\hbar(m-2k)2^{-1} \check{\chi}_{ij}}$.

All assertions follow by computation using Lemma 5.9. Indeed, for (a) we have

$$\sigma_\chi(K_i^\ell, K_j) = \bar{\chi}_U(K_i^\ell, K_j) = e^{\hbar 2^{-1} \chi(\ell T_i^+, T_j^+)} = e^{\hbar \ell 2^{-1} \check{\chi}_{ij}}.$$

For item (b), Lemma 5.9 yields

$$\begin{aligned} \sigma_\chi(K_i^{m-k} K_j, K_i^k) &= \bar{\chi}_U(K_i^{m-k} K_j, K_i^k) = \bar{\chi}_U(e^{\hbar((m-k)T_i^+ + T_j^+)}, e^{\hbar k T_i^+}) \\ &= e^{\hbar 2^{-1} \chi((m-k)T_i^+ + T_j^+, k T_i^+)} = e^{\hbar k 2^{-1} \check{\chi}_{ji}}. \end{aligned}$$

Now, putting altogether (a) and (b) we eventually get (c), because

$$\sigma_\chi(K_i^{m-k}, K_j) \sigma_\chi(K_i^{m-k} K_j, K_i^k) = e^{\hbar(m-k)2^{-1} \check{\chi}_{ij}} e^{\hbar k 2^{-1} \check{\chi}_{ji}} = e^{\hbar(m-2k)2^{-1} \check{\chi}_{ij}}.$$

Finally, Claim 1, 2 and 3 altogether yield, for $m := 1 - a_{ij}$,

$$\begin{aligned} & \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_{q_i} (q_{ij}^{(\chi)})^{+k/2} (q_{ji}^{(\chi)})^{-k/2} E_i^{(m-k)\sigma_\chi} E_j \dot{\sigma}_\chi E_i^{(k)\sigma_\chi} \\ &= \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} e^{\hbar k \hat{\chi}_{ij}} e^{\hbar(m-2k)2^{-1}\hat{\chi}_{ij}} E_i^{m-k} E_j E_i^k \\ &= e^{\hbar m 2^{-1}\hat{\chi}_{ij}} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_{q_i} q_{ij}^{+k/2} q_{ji}^{-k/2} E_i^{m-k} E_j E_i^k = 0 \end{aligned}$$

where the last equality follows from the quantum Serre relation.

Now, the Fact above implies that there exists a well-defined homomorphism of topological Hopf algebras $\ell : U_{P(\chi), \hbar}^{\mathcal{R}(\chi)}(\mathfrak{g}) \rightarrow (U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g}))_{\sigma_\chi}$ given on generators by $\ell(E_i) := E_i, \ell(F_i) := F_i, \ell(T) := T (i \in I, T \in \mathfrak{h})$ – in short, it is the identity on generators. Moreover, thanks to Proposition 5.12 this is in fact an epimorphism. As an application of this result, there exists also an epimorphism of topological Hopf algebras $\ell' : U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g}) \rightarrow (U_{P(\chi), \hbar}^{\mathcal{R}(\chi)}(\mathfrak{g}))_{\sigma_{-\chi}}$ which again is the identity on generators – just replace χ with $-\chi$ and \check{P} with $P(\chi)$.

Mimicking what we did for $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$, we can construct, out of χ , a normalized Hopf 2-cocycle $\dot{\sigma}_\chi$ for $(U_{P(\chi), \hbar}^{\mathcal{R}(\chi)}(\mathfrak{g}))_{\sigma_{-\chi}}$; then we also have a similar 2-cocycle σ'_χ on $U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g})$ defined as the pull-back of $\dot{\sigma}_\chi$ via ℓ' , and a unique, induced Hopf algebra homomorphism

$$\ell'_{\dot{\sigma}_\chi} : (U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g}))_{\sigma'_\chi} \longrightarrow \left((U_{P(\chi), \hbar}^{\mathcal{R}(\chi)}(\mathfrak{g}))_{\sigma_{-\chi}} \right)_{\dot{\sigma}_\chi}$$

between deformed Hopf algebras, which is again the identity on generators. Now, tracking the whole construction one sees at once that $\sigma'_\chi = \sigma_\chi$, so that

$$(U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g}))_{\sigma'_\chi} = (U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g}))_{\sigma_\chi}, \quad \text{and} \quad \left((U_{P(\chi), \hbar}^{\mathcal{R}(\chi)}(\mathfrak{g}))_{\sigma_{-\chi}} \right)_{\dot{\sigma}_\chi} = U_{P(\chi), \hbar}^{\mathcal{R}(\chi)}(\mathfrak{g}).$$

But then composition gives two homomorphisms

$$\begin{aligned} \ell'_{\dot{\sigma}_\chi} \circ \ell &: U_{P(\chi), \hbar}^{\mathcal{R}(\chi)}(\mathfrak{g}) \longrightarrow (U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g}))_{\sigma_\chi} \longrightarrow U_{P(\chi), \hbar}^{\mathcal{R}(\chi)}(\mathfrak{g}) \\ \ell \circ \ell'_{\dot{\sigma}_\chi} &: (U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g}))_{\sigma_\chi} \longrightarrow U_{P(\chi), \hbar}^{\mathcal{R}(\chi)}(\mathfrak{g}) \longrightarrow (U_{P, \hbar}^{\mathcal{R}}(\mathfrak{g}))_{\sigma_\chi} \end{aligned}$$

which (both) are the identity on generators: hence we get $\ell'_{\dot{\sigma}_\chi} \circ \ell = \text{id}$ and $\ell \circ \ell'_{\dot{\sigma}_\chi} = \text{id}$, thus in particular ℓ is an isomorphism. \square

Remark 5.14. — With notation of Theorem 5.13, we have $P_{(\chi)} - P = \Lambda$ for some *antisymmetric* matrix $\Lambda \in \mathfrak{so}_n(\mathbb{k}[[\hbar]])$. Conversely, under mild assumptions on P , this result can be “reversed” as it is shown below.

THEOREM 5.15. — *Let $P, P' \in M_n(\mathbb{k}[[\hbar]])$ be two matrices of Cartan type with the same associated Cartan matrix A .*

- (a) *Let \mathcal{R} be a split realization of P and $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ be the associated FoMpQUEA. Then there exists a split realization $\check{\mathcal{R}}'$ of P' , a matrix $\check{X} = (\check{\chi}_{ij})_{i,j \in I} \in \mathfrak{so}_n(\mathbb{k}[[\hbar]])$ and a toral 2-cocycle $\sigma_{\check{X}}$ such that*

$$U_{P',\hbar}^{\check{\mathcal{R}}'}(\mathfrak{g}) \cong (U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))_{\sigma_{\check{X}}}.$$

In a nutshell, if $P'_s = P_s$ then from any split FoMpQUEA over P we can obtain a split FoMpQUEA (of the same rank) over P' by a toral 2-cocycle deformation.

- (b) *Let \mathcal{R} be a split minimal realization of P . Then the FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is isomorphic to a toral 2-cocycle deformation of the Drinfeld’s standard double QUEA, that is there exists some bilinear map $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ such that*

$$U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \cong (U_{DA,\hbar}(\mathfrak{g}))_{\sigma_{\chi}}.$$

- (c) *Similar, parallel statements hold true for the Borel FoMpQUEAs.*

Proof.

(a). — By Proposition 2.20(a), there exists $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ such that $P' = P_{(\chi)}$ and the triple – constructed as in Section 2.3 – $\mathcal{R}' := \mathcal{R}_{(\chi)} = (\mathfrak{h}, \Pi_{(\chi)}, \Pi^{\vee})$ is a split realization of $P' = P_{(\chi)}$. Then

$$U_{P',\hbar}^{\mathcal{R}'}(\mathfrak{g}) \cong (U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))_{\sigma_{\chi}}$$

by Theorem 5.13.

(b). — Drinfeld’s $U_{DA,\hbar}(\mathfrak{g})$ is – in our language – nothing but the FoMpQUEA built upon a split minimal realization $\mathcal{R}_{st} = (\mathfrak{h}, \Pi_{st}, \Pi_{st}^{\vee})$ of DA , for which we write $\Pi_{st}^{\vee} = \{T_i^{\pm}\}_{i \in I}$ and $\Pi_{st} = \{\alpha_i^{(st)}\}_{i \in I}$. From Proposition 2.20(b) we have a suitable form $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ such that the realization $(\mathcal{R}_{st})_{(\chi)}$ obtained as toral 2-cocycle deformation of \mathcal{R}_{st} through χ coincides with \mathcal{R} . Then, by Theorem 5.13 we get eventually $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \cong (U_{DA,\hbar}(\mathfrak{g}^D))_{\sigma_{\chi}}$ as desired. \square

6. Specialization and quantization: FoMpQUEAs vs. MpLbA's

This section dwells upon the interplay of specialization – applied to quantum objects as our FoMpQUEAs – and, conversely, of quantization – performed onto such semiclassical objects as our MpLbA's.

First of all, we shall see that the specialization of a FoMpQUEA yields a suitable MpLbA; conversely, any MpLbA has at least one quantization, in the form of a well defined FoMpQUEA. Then, we shall investigate the interaction between the process of specialization – at $\hbar = 0$ – of any FoMpQUEA and the process of deformation – either by (toral) twist or by (toral) 2-cocycle – of the same FoMpQUEA or of the MpLbA which is its semiclassical limit. In particular we will find out that, in a suitable, natural sense, *the two processes commute with each other*.

6.1. Deformation vs. specialization for FoMpQUEAs and MpLbA's

Recall that a *deformation algebra* is a topological, unital, associative $\mathbb{k}[[\hbar]]$ -algebra A which is topologically free as a $\mathbb{k}[[\hbar]]$ -module. Conversely, a deformation of a (unital, associative) \mathbb{k} -algebra A_0 is by definition a deformation algebra A such that $A_0 \cong A/\hbar A$. The same criteria apply to the notion of “deformation Hopf algebra”, just replacing “topological, unital, associative $\mathbb{k}[[\hbar]]$ -algebra” with “topological Hopf $\mathbb{k}[[\hbar]]$ -algebra”. Following Drinfeld, we say that a deformation Hopf algebra H is a *quantized universal enveloping algebra* (or *QUEA* in short) if $H/\hbar H \cong U(\mathfrak{g})$ for some Lie algebra \mathfrak{g} . In particular, the Lie bracket in \mathfrak{g} comes from the multiplication in $U(\mathfrak{g}) \cong H/\hbar H$. Moreover, this \mathfrak{g} inherits a Lie coalgebra structure from the QUEA, making it into a Lie *bialgebra*, thanks to the following result:

THEOREM 6.1 (cf. [11, Proposition 6.2.7], [21, Theorem 9.1]). — *Let H be a quantized universal enveloping algebra with $H/\hbar H \cong U(\mathfrak{g})$. Then the Lie algebra \mathfrak{g} is naturally equipped with a Lie bialgebra structure, whose Lie cobracket is defined by*

$$(6.1) \quad \delta(x) := \frac{\Delta(x) - \Delta^{\text{op}}(x)}{\hbar} \pmod{\hbar}$$

where $x \in H$ is any lifting of $x \in \mathfrak{g} \subseteq U(\mathfrak{g}) \cong H/\hbar H$. □

DEFINITION 6.2 ([11, 21]). — *The semiclassical limit of a quantized universal enveloping algebra H is the Lie bialgebra $(\mathfrak{g}, [-, -], \delta)$ where \mathfrak{g} is the*

Lie algebra *s. t.* $H/\hbar H \cong U(\mathfrak{g})$ and δ is defined as above. Conversely, we say that H is a quantization of the Lie bialgebra $(\mathfrak{g}, [-, -], \delta)$.

6.1.1. Formal MpQUEAs vs. MpLbA's

In this section we finally compare our FoMpQUEAs with our MpLbA's. Mainly, we show that the FoMpQUEAs are indeed quantized universal enveloping algebras; in particular, we prove that their specialization at $\hbar = 0$ is a universal enveloping algebra of a MpLbA as those in Section 3. Thus the specialization of each FoMpQUEA is a MpLbA; conversely, any FoMpQUEA is the quantization of some MpLbA. The other way round is true as well: every MpLbA admits a FoMpQUEA as its quantization.

Second, we describe the interplay between the process of specializing/quantizing (switching between FoMpQUEAs and MpLbA's) and the process of deforming within either family of FoMpQUEAs or MpLbA's, separately – by twist or by 2-cocycle.

We fix a matrix $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ of Cartan type with associated Cartan matrix A , and a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of P . Then we have the associated (topological) Hopf algebra $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, as in Definition 2.1 and Section 4. Similarly, we also have the MpLbA $\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}}}$ introduced in Section 3.2.3, where we use (loose) obvious notation such as $\overline{\mathcal{R}} := \mathcal{R}(\text{mod } \hbar)$ and $\overline{P} := P(\text{mod } \hbar)$.

Our first result points out that FoMpQUEAs and MpLbA's are in bijection through the specialization/quantization process, as one might expect:

THEOREM 6.3. — $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is a quantized universal enveloping algebra in the sense of Section 6.1, whose semiclassical limit is $U(\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}}})$.

In short, for each pair (P, \mathcal{R}) as above – \mathcal{R} being a realization of P – and for the FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ and the MpLbA $\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}}}$ associated with (P, \mathcal{R}) , we have: $\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}}}$ is the specialization of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, or – equivalently – $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is a quantization of $\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}}}$.

Proof. — First of all, we note that $H := U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is topologically free. We prove that by reducing the problem to the case of Drinfeld's standard double QUEA. Namely, we begin assuming that $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is *split minimal*, i.e. such is the realization \mathcal{R} . Then by Theorem 5.15(c) we have an isomorphism $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \cong (U_{DA,\hbar}(\mathfrak{g}))_{\sigma_x}$ as topological Hopf $\mathbb{k}[[\hbar]]$ -algebras, where $(U_{DA,\hbar}(\mathfrak{g}))_{\sigma_x}$ is a suitable 2-cocycle deformation of Drinfeld's standard double QUEA $U_{DA,\hbar}(\mathfrak{g})$. As the latter is known to be a topologically free

$\mathbb{k}[[\hbar]]$ -module, and 2-cocycle deformation does not affect the $\mathbb{k}[[\hbar]]$ -module structure, we conclude that $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is topologically free as well.

Now assume that $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is just split (possibly not minimal): then, by definition, $\Pi^{\vee} := \{T_i^+, T_i^-\}_{i \in I}$ can be completed to a $\mathbb{k}[[\hbar]]$ -basis of \mathfrak{h} , hence $\mathfrak{h}' := \text{Span}_{\mathbb{k}[[\hbar]]}(\Pi^{\vee})$ has a direct sum complement \mathfrak{h}'' so that $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$; therefore $U_{\hbar}(\mathfrak{h}) = U_{\hbar}(\mathfrak{h}') \widehat{\otimes} U_{\hbar}(\mathfrak{h}'')$ as algebras. Furthermore, the realization \mathcal{R} clearly “restricts” to another realization \mathcal{R}' of P whose Cartan (sub)algebra is \mathfrak{h}' , which in addition is split *minimal*: then as $U_{\hbar}(\mathfrak{h}) = U_{\hbar}(\mathfrak{h}') \widehat{\otimes} U_{\hbar}(\mathfrak{h}'')$ one also gets easily

$$(6.2) \quad U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \cong U_{P,\hbar}^{\mathcal{R}'}(\mathfrak{g}) \widehat{\otimes} U_{\hbar}(\mathfrak{h}'') \quad \text{as } \mathbb{k}[[\hbar]]\text{-modules}$$

(by construction). Indeed, by definition we have a natural monomorphism of realizations $\mathcal{R}' \hookrightarrow \mathcal{R}$ induced by the monomorphism $\mathfrak{h}' \hookrightarrow \mathfrak{h}$ of Cartan (sub)algebras; by Propositions 4.5 and 4.15, this induces a monomorphism $U_{P,\hbar}^{\mathcal{R}'}(\mathfrak{g}) \hookrightarrow U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ between FoMpQUEAs – the image of the latter is the (complete) $\mathbb{k}[[\hbar]]$ -subalgebra generated by the E_i ’s, the T_i^{\pm} ’s and the F_i ’s ($i \in I$). Now, applying twice Theorem 4.11 – yielding triangular decompositions for $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ and $U_{P,\hbar}^{\mathcal{R}'}(\mathfrak{g})$ – we get

$$\begin{aligned} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) &\cong U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_-) \widehat{\otimes} U_{\hbar}(\mathfrak{h}) \widehat{\otimes} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_+) \\ &\cong U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_-) \widehat{\otimes} U_{\hbar}(\mathfrak{h}') \widehat{\otimes} U_{\hbar}(\mathfrak{h}'') \widehat{\otimes} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_+) \\ &\cong U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_-) \widehat{\otimes} U_{\hbar}(\mathfrak{h}') \widehat{\otimes} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_+) \widehat{\otimes} U_{\hbar}(\mathfrak{h}'') \cong U_{P,\hbar}^{\mathcal{R}'}(\mathfrak{g}) \widehat{\otimes} U_{\hbar}(\mathfrak{h}'') \end{aligned}$$

where we applied also $U_{\hbar}(\mathfrak{h}'') \widehat{\otimes} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_+) \cong U_{P,\hbar}^{\mathcal{R}}(\mathfrak{n}_+) \widehat{\otimes} U_{\hbar}(\mathfrak{h}'')$, which is clear, and $U_{\hbar}(\mathfrak{h}) = U_{\hbar}(\mathfrak{h}') \widehat{\otimes} U_{\hbar}(\mathfrak{h}'')$, which proves our claim.

Therefore, as $U_{P,\hbar}^{\mathcal{R}'}(\mathfrak{g})$ is topologically free by the previous analysis (as \mathcal{R}' is split minimal) and $U_{\hbar}(\mathfrak{h}'')$ is also topologically free by construction, from (6.2) we deduce the same for $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ as well.

Finally, let’s cope with the general case. By Lemma 2.7 there exists an epimorphism $\underline{\pi} : \dot{\mathcal{R}} \rightarrow \mathcal{R}$, where $\dot{\mathcal{R}} := (\dot{\mathfrak{h}}, \dot{\Pi}, \dot{\Pi}^{\vee})$ is a *split* realization of P : by Proposition 4.15, this induces an epimorphism of Hopf algebras (though, for us, it is enough to be one of algebras, indeed) $U_{P,\hbar}^{\dot{\mathcal{R}}}(\mathfrak{g}) \xrightarrow{U_{\underline{\pi}}} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ whose kernel is generated by $U_{\hbar}(\mathfrak{k})^+$ with $\mathfrak{k} := \text{Ker}(\pi)$, where $\pi : \dot{\mathfrak{h}} \rightarrow \mathfrak{h}$ is the epimorphism of Cartan (sub)algebras associated with $\underline{\pi}$. As \mathfrak{h} is free of finite rank, we have $\dot{\mathfrak{h}} = \mathfrak{k} \oplus \mathfrak{h}' \cong \mathfrak{k} \oplus \mathfrak{h}$ for some free submodule $\mathfrak{h}' \cong \mathfrak{h}$ inside $\dot{\mathfrak{h}}$; therefore $U_{\hbar}(\dot{\mathfrak{h}}) = U_{\hbar}(\mathfrak{k}) \widehat{\otimes} U_{\hbar}(\mathfrak{h}) \cong U_{\hbar}(\mathfrak{k}) \otimes U_{\hbar}(\mathfrak{h})$ as algebras, whence, as in (6.2), one gets $U_{P,\hbar}^{\dot{\mathcal{R}}}(\mathfrak{g}) \cong U_{\hbar}(\mathfrak{k}) \widehat{\otimes} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ as $\mathbb{k}[[\hbar]]$ -modules. As $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is topologically free as seen before and $U_{\hbar}(\mathfrak{k})$ is too, we deduce the same for $U_{P,\hbar}^{\dot{\mathcal{R}}}(\mathfrak{g})$ too.

Second, we must prove that $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})/\hbar U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$, as a co-Poisson Hopf algebra, is isomorphic to $U(\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}}})$. Indeed, from the presentation of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ we get that $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})/\hbar U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is generated by the cosets (modulo $\hbar U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$) of the F_i 's, T 's and E_i 's ($i \in I, T \in \mathfrak{h}$); moreover, these cosets $\overline{X} := X \bmod \hbar U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ obey all relations induced modulo \hbar by the defining relations among the original generators X of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$. On the other hand, by construction the Lie bialgebra $\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}}}$ is endowed with a built-in presentation (as a Lie algebra) by generators – the F_i 's, T 's and E_i 's – and relations, and explicit formulas for the value of the Lie cobracket δ on the given generators. From this, a presentation of $U(\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}}})$ is obtained in the obvious way, where the generators are again the F_i 's, T 's and E_i 's as before, as well as explicit formulas for the value of the Poisson cobracket δ on each one of those generators.

Comparing, the presentation of $U(\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}}})$ with that of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})/\hbar U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ we find that all the given relations among generators of the latter algebra do correspond to identical relations among the corresponding generators in the former: namely, mapping X to \overline{X} – for all $X \in \{E_i, F_i \mid i \in I\} \cup \mathfrak{h}$ – turns every relation among the X 's into a same-look relation among the \overline{X} 's. This implies that a well-defined epimorphism of Hopf algebras

$$(6.3) \quad \begin{aligned} \phi : U(\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}}}) &\longrightarrow U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})/\hbar U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \\ E_i &\longmapsto \overline{E}_i, \quad T \longmapsto \overline{T}, \quad F_i \longmapsto \overline{F}_i \end{aligned}$$

($i \in I, T \in \mathfrak{h}$) exists; moreover, comparing the formulas on both sides for the co-Poisson bracket on generators we see that this is also a *co-Poisson* Hopf epimorphism.

On the other hand, we can make $U(\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}}})$ into a $\mathbb{k}[[\hbar]]$ -algebra by scalar restriction – via $\mathbb{k}[[\hbar]] \twoheadrightarrow \mathbb{k}[[\hbar]]/\hbar\mathbb{k}[[\hbar]] \cong \mathbb{k}$. Then the same remark about relations implies that there exists a well-defined $\mathbb{k}[[\hbar]]$ -algebra epimorphism

$$\psi' : U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \longrightarrow U(\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}}}), \quad E_i \longmapsto E_i, T \longmapsto T, F_i \longmapsto F_i$$

($i \in I, T \in \mathfrak{h}$) with $\text{Ker}(\psi') \supseteq \hbar U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$; so a \mathbb{k} -algebra epimorphism

$$(6.4) \quad \begin{aligned} \psi : U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})/\hbar U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) &\longrightarrow U(\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}}}) \\ \overline{E}_i &\longmapsto E_i, \quad \overline{T} \longmapsto T, \quad \overline{F}_i \longmapsto F_i \end{aligned}$$

($i \in I, T \in \mathfrak{h}$) is induced too. Direct comparison between (6.3) and (6.4) shows that ϕ and ψ are inverse to each other, hence ψ is a *Hopf* morphism too and we are done. □

6.2. Blending specialization and deformation

In this section we compare the process of deformation at the quantum level or at the semiclassical level. The outcome, in a nutshell, is as follows: *deformation (by twist or by 2-cocycle) commutes with specialization.*

6.2.1. Blending specialization and twist deformation

Once more, we fix again $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ of Cartan type, a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of it, and the associated FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ and MpLbA $\mathfrak{g}_{\overline{P}}^{\mathcal{R}}$. As \mathfrak{h} is a free $\mathbb{k}[[\hbar]]$ -module of finite rank t , we fix a $\mathbb{k}[[\hbar]]$ -basis $\{H_g\}_{g \in \mathcal{G}}$, with $|\mathcal{G}| = \text{rk}(\mathfrak{h}) = t$.

Pick a matrix $\Psi = (\psi_{gk})_{g,k \in \mathcal{G}} \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$. Out of it, we define

$$j_{\overline{\Psi}} := \sum_{g,k=1}^t \overline{\psi_{gk} H_g} \otimes \overline{H_k} \in \overline{\mathfrak{h}} \otimes \overline{\mathfrak{h}} \quad \text{as in (3.7)}$$

$$\mathcal{F}_{\overline{\Psi}} := \exp \left(\hbar 2^{-1} \sum_{g,k=1}^t \psi_{gk} H_g \otimes H_k \right) \in U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \widehat{\otimes} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \quad \text{as in (5.1)}$$

with $j_{\overline{\Psi}}$ being a (toral) twist for the Lie bialgebra $\mathfrak{g}_{\overline{P}}^{\mathcal{R}}$ and $\mathcal{F}_{\overline{\Psi}}$ is a (toral) twist for the Hopf algebra $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$. Then we consider the deformation $(\mathfrak{g}_{\overline{P}}^{\mathcal{R}})^{j_{\overline{\Psi}}}$ of $\mathfrak{g}_{\overline{P}}^{\mathcal{R}}$ by the (Lie) twist $j_{\overline{\Psi}}$ and the deformation $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))^{\mathcal{F}_{\overline{\Psi}}}$ of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ by the (Hopf) twist $\mathcal{F}_{\overline{\Psi}}$.

Again out of Ψ , we define also the matrix P_Ψ and its realization $\mathcal{R}_\Psi := (\mathfrak{h}, \Pi, \Pi_\Psi^\vee)$, as in Proposition 2.10. Then the FoMpQUEA $U_{P_\Psi,\hbar}^{\mathcal{R}_\Psi}(\mathfrak{g})$ and the MpLbA $\mathfrak{g}_{\overline{P}_\Psi}^{\mathcal{R}_\Psi}$ both exist, again linked with each other by a quantization/specialization relationship.

We can now state our result, which in particular claims (roughly speaking) that “*deformation by twist commutes with specialization*”.

THEOREM 6.4. — *With assumptions as above, $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))^{\mathcal{F}_{\overline{\Psi}}}$ is a quantized universal enveloping algebra, whose semiclassical limit is $U((\mathfrak{g}_{\overline{P}}^{\mathcal{R}})^{j_{\overline{\Psi}}})$. More precisely, we have $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))^{\mathcal{F}_{\overline{\Psi}}} \cong U_{P_\Psi,\hbar}^{\mathcal{R}_\Psi}(\mathfrak{g})$ and $(\mathfrak{g}_{\overline{P}}^{\mathcal{R}})^{j_{\overline{\Psi}}} \cong \mathfrak{g}_{\overline{P}_\Psi}^{\mathcal{R}_\Psi}$.*

Proof. — The claim follows, as direct application, from the three isomorphisms $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))^{\mathcal{F}_{\overline{\Psi}}} \stackrel{5.2}{\cong} U_{P_\Psi,\hbar}^{\mathcal{R}_\Psi}(\mathfrak{g})$, $U_{P_\Psi,\hbar}^{\mathcal{R}_\Psi}(\mathfrak{g})/\hbar U_{P_\Psi,\hbar}^{\mathcal{R}_\Psi}(\mathfrak{g}) \stackrel{6.3}{\cong} U_\hbar(\mathfrak{g}_{\overline{P}_\Psi}^{\mathcal{R}_\Psi})$ and $\mathfrak{g}_{\overline{P}_\Psi}^{\mathcal{R}_\Psi} \stackrel{3.12}{\cong} (\mathfrak{g}_{\overline{P}}^{\mathcal{R}})^{j_{\overline{\Psi}}}$ which respectively come from Theorem 5.2, Theorem 6.3 and Theorem 3.12. □

6.2.2. Blending specialization and 2-cocycle deformation

Now we analyze what happens when combining *deformations by 2-cocycle*, for both FoMpQUEAs and MpLbA's, with the specialization process (from the former to the latter ones).

We start with $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ of Cartan type, a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of P , and a fixed $\mathbb{k}[[\hbar]]$ -basis $\{H_g\}_{g \in \mathcal{G}}$ of \mathfrak{h} , with $|\mathcal{G}| = \text{rk}(\mathfrak{h}) = t$. Then we have also $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ and $\mathfrak{g}_{\overline{P}}$, interlocked via quantization/specialization.

Like in Section 5.2.1, let $\chi : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{k}[[\hbar]]$ be an antisymmetric $\mathbb{k}[[\hbar]]$ -bilinear map which obeys (3.8). Taking everything modulo \hbar , this χ defines a similar antisymmetric, \mathbb{k} -bilinear map $\gamma := (\chi \bmod \hbar) : \mathfrak{h}_0 \times \mathfrak{h}_0 \rightarrow \mathbb{k}$, where $\mathfrak{h}_0 := \mathfrak{h}/\hbar\mathfrak{h} = \overline{\mathfrak{h}}$, which obeys (3.8) again, up to replacing “ χ ” with “ γ ”. Following Section 5.2.2, we construct out of χ a $\mathbb{k}((\hbar))$ -valued toral 2-cocycle $\sigma_\chi : U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \otimes U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \rightarrow \mathbb{k}((\hbar))$, and then out of this the 2-cocycle deformed Hopf algebra $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))_{\sigma_\chi}$. Similarly, out of γ we construct, as in Section 3.4 (but replacing “ χ ” with “ γ ”) a toral 2-cocycle $\gamma_{\mathfrak{g}}$ for the Lie bialgebra $\mathfrak{g}_{\overline{P}}$, and out of it the 2-cocycle deformed Lie bialgebra $(\mathfrak{g}_{\overline{P}})_{\gamma_{\mathfrak{g}}}$.

Still out of χ , we define the matrix $P_{(\chi)}$ and its realization $\mathcal{R}_{(\chi)} := (\mathfrak{h}, \Pi_{(\chi)}, \Pi^\vee)$, as in Proposition 2.17; similarly, out of γ we get the matrix $P_{(\gamma)}$ and its realization $\mathcal{R}_{(\gamma)}$: then by construction $P_{(\gamma)} = \overline{P}_{(\chi)}$ and $\mathcal{R}_{(\gamma)} = \overline{\mathcal{R}}_{(\chi)}$. Attached to these we have $U_{P_{(\chi)},\hbar}^{\mathcal{R}_{(\chi)}}(\mathfrak{g})$ and $\mathfrak{g}_{P_{(\gamma)}}^{\mathcal{R}_{(\gamma)}} = \mathfrak{g}_{\overline{P}_{(\chi)}}$, again connected via quantization/specialization.

Next result claims that, roughly speaking, “*deformation by 2-cocycle commutes with specialization*”.

THEOREM 6.5. — *With assumptions as above, $(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))_{\sigma_\chi}$ is a quantized universal enveloping algebra, with semiclassical limit $U((\mathfrak{g}_{\overline{P}})_{\gamma_{\mathfrak{g}}})$.*

Proof. — The claim follows, as direct application, from the isomorphisms

$$(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}))_{\sigma_\chi} \stackrel{5.13}{\cong} U_{P_{(\chi)},\hbar}^{\mathcal{R}_{(\chi)}}(\mathfrak{g}),$$

$$U_{P_{(\chi)},\hbar}^{\mathcal{R}_{(\chi)}}(\mathfrak{g})/\hbar U_{P_{(\chi)},\hbar}^{\mathcal{R}_{(\chi)}}(\mathfrak{g}) \stackrel{6.3}{\cong} U_{\hbar}(\mathfrak{g}_{P_{(\gamma)}}^{\mathcal{R}_{(\gamma)}})$$

and

$$\mathfrak{g}_{P_{(\gamma)}}^{\mathcal{R}_{(\gamma)}} \stackrel{3.17}{\cong} (\mathfrak{g}_{\overline{P}})_{\gamma_{\mathfrak{g}}}$$

which respectively come from Theorem 5.13, Theorem 6.3 and Theorem 3.17. □

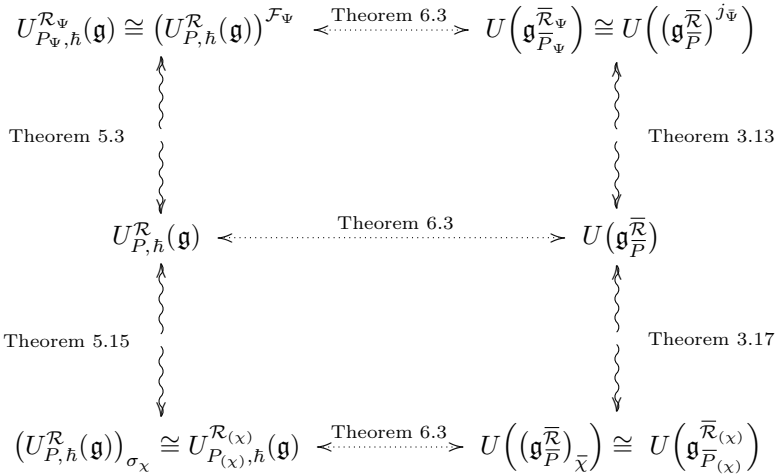
6.3. Final overview

In this paper we studied multiparametric versions of formal quantum universal enveloping algebras and their semiclassical limits. As these are presented by generators and relations, their very definition highlights the relation between the multiparameters and the action of a fixed commutative subalgebra: like for Kac–Moody algebras, this is encoded in the notion of *realization* of a multiparameter matrix P related to a symmetrizable Cartan matrix. This tool allows us to relate the quantum objects with the semiclassical limit, and the multiparameter objects with the standard ones: the latter is done via deformation(s) and an explicit change of generators.

In conclusion, loosely speaking, one may say that:

- (a) multiparameters are encoded in realizations;
- (b) FoMpQUEAs are quantizations of MpLbAs;
- (c) multiparameter objects are given by deformation of either the algebra or the coalgebra structure (both options being available) of *standard* objects.

Finally, we provide a “pictorial sketch” of the global picture. What we have in one single glance is summed up in the following diagram:



Note that in this diagram each horizontal arrow (with dotted shaft) denotes a “quantization/specialization (rightwards/leftwards) relationship” – which involves the “continuous parameter” \hbar – whereas each vertical arrow (having a waving shaft) denotes a relationship “via deformation” – which involves “discrete parameters”.

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Gastón Andrés GARCÍA

Departamento de Matemática, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, CMaLP-CIC-CONICET, 1900 La Plata (Argentina)
ggarcia@mate.unlp.edu.ar

Fabio GAVARINI

Dipartimento di Matematica, Università degli Studi di Roma “Tor Vergata”, Via della ricerca scientifica 1, I-00133 Roma (Italy)
gavarini@mat.uniroma2.it