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Florin Catrina,
Sofiya Ostrovska & Mikhail I. Ostrovskii

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DVORETZKY-TYPE THEOREM FOR LOCALLY FINITE SUBSETS OF A HILBERT SPACE

**by Florin CATRINA,
Sofiya OSTROVSKA & Mikhail I. OSTROVSKII (*)**

Abstract. — The main result of the paper: Given any $\varepsilon > 0$, every locally finite subset of ℓ_2 admits a $(1+\varepsilon)$-bilipschitz embedding into an arbitrary infinite-dimensional Banach space. The result is based on two results which are of independent interest:

(1) A direct sum of two finite-dimensional Euclidean spaces contains a sub-sum of a controlled dimension which is ε-close to a direct sum with respect to a 1-unconditional basis in a two-dimensional space.

(2) For any finite-dimensional Banach space Y and its direct sum X with itself with respect to a 1-unconditional basis in a two-dimensional space, there exists a $(1+\varepsilon)$-bilipschitz embedding of Y into X which on a small ball coincides with the identity map onto the first summand and on the complement of a large ball coincides with the identity map onto the second summand.

Résumé. — Le résultat principal de l'article : étant donné $\varepsilon > 0$, chaque sous-ensemble localement fini de ℓ_2 admet un plongement $(1+\varepsilon)$-bilipschitz dans n'importe quel espace de Banach de dimension infinie. Le résultat est basé sur deux résultats qui présentent un intérêt indépendant :

(1) Une somme directe de deux espaces euclidiens de dimension finie contient une sous-somme de dimension contrôlée qui est ε-proche d'une somme directe par rapport à une base 1-inconditionnelle dans un espace à deux dimensions.

(2) Pour tout espace de Banach de dimension finie Y et sa somme directe X avec lui-même par rapport à une base 1-inconditionnelle dans un espace à deux dimensions, il existe un plongement $(1+\varepsilon)$-bilipschitz de Y dans X qui coïncide, sur une petite boule, avec l'identité sur la première composante, et qui coïncide, sur le complément d'une grosse boule, avec l'identité sur la deuxième composante.

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1. Introduction

All normed vector spaces considered in this paper are over the reals.

Recall the classical Dvoretzky Theorem [12, 13] which proved Grothendieck's conjecture [15, Section 7].

THEOREM 1.1 ([13, Section 7]). — *Let $k \in \mathbb{N}$, $k \geqslant 2$, and $0 < \varepsilon < 1$. There exists $N = N(k, \varepsilon) \in \mathbb{N}$ so that every normed space having more than N dimensions, in particular every infinite-dimensional normed space, has a k-dimensional subspace whose Banach–Mazur distance from the k-dimensional Hilbert space is less than $(1 + \varepsilon)$.*

In this connection, it is natural to call a result establishing the significant presence of Hilbert space structures in an arbitrary infinite-dimensional Banach space a *Dvoretzky-type theorem*.

The following classes of spaces and embeddings are very important in applications, see [7, 30, 37].

Recall that a metric space is called *locally finite* if each ball of finite radius in it contains finitely many elements. A map $F : \mathcal{M} \to \mathcal{L}$ between two metric spaces $(\mathcal{M}, d_{\mathcal{M}})$ and $(\mathcal{L}, d_{\mathcal{L}})$ is called a *bilipschitz embedding* if there exist constants $C_1, C_2 > 0$ so that for all $u, v \in \mathcal{M}$

$$C_1 d_{\mathcal{M}}(u, v) \leqslant d_{\mathcal{L}}(F(u), F(v)) \leqslant C_2 d_{\mathcal{M}}(u, v).$$

The *distortion* of F is defined as $\mathrm{Lip}(F) \cdot \mathrm{Lip}(F^{-1}|_{F(\mathcal{M})})$, where $\mathrm{Lip}(\cdot)$ denotes the Lipschitz constant. A bilipschitz embedding whose distortion does not exceed $C \in [1, \infty)$ is called *C-bilipschitz*. An embedding satisfying $d_{\mathcal{L}}(F(u), F(v)) = d_{\mathcal{M}}(u, v)$ is called an *isometric embedding*.

A map $F : (\mathcal{M}, d_{\mathcal{M}}) \to (\mathcal{L}, d_{\mathcal{L}})$ between two metric spaces is called a *coarse embedding* if there exist non-decreasing functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ such that $\lim_{t\to\infty} \rho_1(t) = \infty$ and

$$\forall u, v \in \mathcal{M} \ \rho_1(d_{\mathcal{M}}(u, v)) \leqslant d_{\mathcal{L}}(F(u), F(v)) \leqslant \rho_2(d_{\mathcal{M}}(u, v)).$$

The main goal of this paper is to prove the following Dvoretzky-type theorem:

THEOREM 1.2. — *Given any $\varepsilon > 0$, every locally finite subset of ℓ_2 admits a $(1+\varepsilon)$-bilipschitz embedding into an arbitrary infinite-dimensional Banach space.*

Note that there exist locally finite subsets of ℓ_2 which do not admit isometric embeddings into some infinite-dimensional Banach spaces, see [33, Theorem 1.8].

At this point, it is appropriate to present a short overview of the available Dvoretzky-type results and related open problems.

First, we recall the open problem on the validity of a *finite isometric Dvoretzky Theorem* for all infinite-dimensional Banach spaces.

Problem 1.3 ([38], published in [18]). — *Do there exist a finite subset F of ℓ_2 and an infinite-dimensional Banach space X such that F does not admit an isometric embedding into X?*

A related negative result for spaces ℓ_p, $1 < p < \infty$, $p \neq 2$ was proved in [18].

The following weaker version of Theorem 1.2 was proved in [35, Theorem 1].

Theorem 1.4. — *Each locally finite subset of ℓ_2 admits a coarse embedding into an arbitrary infinite-dimensional Banach space.*

Using the technique of [5], different from that employed in [35], Theorem 1.4 was strengthened to

Theorem 1.5 ([36, Theorem 4.3]). — *Each locally finite subset of ℓ_2 admits a bilipschitz embedding into arbitrary infinite-dimensional Banach space.*

The upper estimate for the distortion of embeddings of a locally finite subspace of ℓ_2 into an arbitrary infinite-dimensional Banach space obtained in [36] is 100. The present paper aims to prove the best possible result in this direction.

As another development, Nowak [29] showed that the embedding techniques of [10] can be used to find coarse embeddings of Hilbert space into Banach spaces for which such an embeddability appeared to be somewhat unexpected. Later, Ostrovskii [36] combined the technique of Nowak [29] with the results of [31] and strengthened Nowak's result as follows:

Theorem 1.6 ([36, Theorem 5.1]). — *Let X be a Banach space containing a subspace with an unconditional basis which does not contain ℓ_∞^n uniformly. Then ℓ_2 embeds coarsely into X.*

Theorem 1.6 together with Theorem 1.4 led to the problem: *Is it true that ℓ_2 embeds coarsely into an arbitrary infinite-dimensional Banach space?* This problem was posed in [35, pp. 1–2] and published in [36, Problem 4.1].

A positive answer to this problem would be a significant strengthening of Theorem 1.4, yet, as the matter stands, it was answered in the negative in [6, Corollary B], a typical counterexample is the Tsirelson space constructed in [41].

One of the most important directions related to the Dvoretzky Theorem is finding optimal estimates for the function $N(k,\varepsilon)$ in the statement of Theorem 1.1 (see [1, 2, 25, 26, 39, 40]).

Starting with the paper of Bourgain–Figiel–Milman [8], a parallel theory for metric spaces was developed. In this theory the main goal is estimating from below the size (defined either as cardinality or in some measure-theoretic ways) of subsets of a metric space which admit low-distortion embeddings into a Hilbert space. We list a representative selection of papers devoted to the results of this type and their applications: [3, 4, 23, 24, 28]. See also a short survey in [27, Section 8].

Our proof of the main Theorem 1.2 will be presented according to the scheme below:

- First, an almost-unconditionality result for sums of two Euclidean spaces will be established in Theorem 2.2.
- Next, Theorem 3.4 provides a bending result for two-dimensional unconditional sums.
- Finally, combining these results in the spirit of [32], Theorem 1.2 will be proved in Section 4.

In addition, we prove a non-bending result, see Theorem 5.2. It is related to the following open problem:

Problem 1.7 ([32, Problem 5.1]). — *Do there exist $\alpha > 1$, a locally finite metric space $\mathcal{M}$, and a Banach space X such that all finite subsets of $\mathcal{M}$ admit isometric embeddings into X, but any bilipschitz embedding of $\mathcal{M}$ into X has distortion at least α?*

We use the standard terminology and notation of Banach space theory [7, 17, 20, 21], local theory [1, 2, 26], and theory of metric embeddings [22, 37].

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2. Almost-unconditionality result

Definition 2.1. — *Let $Y_1 \oplus Y_2$ be a direct sum in which the subspaces Y_1 and Y_2 are Euclidean, and let $\varepsilon \in [0,1)$. The sum $Y_1 \oplus Y_2$ is endowed with a norm whose restrictions to Y_1 and Y_2 are the Euclidean norms. We say $Y_1 \oplus Y_2$ is* ε-invariant *if for any orthogonal operator O_1 on Y_1 and any orthogonal operator O_2 on Y_2, the inequality*

$$(1-\varepsilon)\|y_1+y_2\| \leqslant \|O_1y_1+O_2y_2\| \leqslant (1+\varepsilon)\|y_1+y_2\|$$

holds.

As it will be shown below, this invariance is related to unconditionality, see Lemmas 2.3 and 2.4.

For a direct sum $X = X_1 \oplus X_2$ by *direct sum projections* we mean projections $P_1 : X \to X_1$ and $P_2 : X \to X_2$ given by $P(x_1,x_2) = x_1$ and $P(x_1,x_2) = x_2$, respectively.

Theorem 2.2. — *Given $n \in \mathbb{N}$, $\varepsilon \in (0,1)$, and $A \in [1,\infty)$ there exists $N \in \mathbb{N}$, such that, for every direct sum $X = X_1 \oplus X_2$ with both X_1 and X_2 isometric to ℓ_2^N, and the direct sum projections having norms $\leqslant A$, there are n-dimensional subspaces $Y_1 \subset X_1$ and $Y_2 \subset X_2$, such that the norm on $Y_1 \oplus Y_2$ induced from X is ε-invariant.*

To see that Theorem 2.2 can be understood as an almost-unconditionality result we need the following two lemmas.

Lemma 2.3. — *Let $Y = Y_1 \oplus Y_2$ be a direct sum of Euclidean subspaces with an ε-invariant norm $\|\cdot\|$. Let*

$$|||y_1+y_2||| = \sup_{O_1,O_2 \text{ orthogonal on } Y_1,Y_2} \|O_1y_1+O_2y_2\|, \quad y_1 \in Y_1, y_2 \in Y_2.$$

Then $|||\cdot|||$ is a norm on $Y_1 \oplus Y_2$ satisfying

$$\|y_1+y_2\| \leqslant |||y_1+y_2||| \leqslant (1+\varepsilon)\|y_1+y_2\|$$

and

$$|||V_1y_1+V_2y_2||| = |||y_1+y_2|||$$

for every orthogonal operators V_1 on Y_1 and V_2 on Y_2. Also, the norms $\|\cdot\|$ and $|||\cdot|||$ coincide on Y_1 and Y_2. Thus, the norm $|||\cdot|||$ is 0*-invariant on $Y = Y_1 \oplus Y_2$.*

Proof. — Proof is straightforward. □

A norm $\|(a,b)\|$ on $\mathbb{R}^2$ is called 1-*unconditional* if $\|(\pm a, \pm b)\| = \|(a,b)\|$ for every $(a,b) \in \mathbb{R}^2$.

Lemma 2.4. — *If a norm on a direct sum $Y = Y_1 \oplus Y_2$ of two Euclidean spaces satisfies*

$$\|O_1 y_1 + O_2 y_2\| = \|y_1 + y_2\| \tag{2.1}$$

for all $y_1 \in Y_1$, $y_2 \in Y_2$ and all orthogonal operators O_1 on Y_1 and O_2 on Y_2, then there exists a 1-unconditional norm $\|\cdot\|_Z$ on $\mathbb{R}^2$ such that

$$\|y_1 + y_2\| = \|(\|y_1\|, \|y_2\|)\|_Z.$$

Proof. — If the norm of $Y_1 \oplus Y_2 = \ell_2^{n_1} \oplus \ell_2^{n_2}$ satisfies (2.1), we can define a nonnegative function f on the nonnegative quadrant of $\mathbb{R}^2$ by

$$f(a_1, a_2) = \|y_1 + y_2\|,$$

where $y_1 \in Y_1$ is such that $\|y_1\| = a_1$ and $y_2 \in Y_2$ is such that $\|y_2\| = a_2$. Equality (2.1) in combination with the transitivity of the group of orthogonal operators on any 0-centered sphere implies that the resulting function $f(a_1, a_2)$ is well-defined.

We extend f to $\mathbb{R}^2$ by

$$f(a_1, a_2) = f(|a_1|, |a_2|).$$

It remains to verify that the resulting function $f(a_1, a_2)$ is a 1-unconditional norm on $\mathbb{R}^2$.

The only norm property that needs checking is the triangle inequality since the others are immediate from the definition of f.

Let us verify the triangle inequality. Clearly,

$$f(a_1+b_1, a_2+b_2) = f(|a_1+b_1|, |a_2+b_2|) = f(\rho_1|a_1|+\sigma_1|b_1|, \rho_2|a_2|+\sigma_2|b_2|),$$

for some ρ's and σ's belonging to the set $\{-1, +1\}$. Hence, taking $u_1 \in Y_1$ and $u_2 \in Y_2$ to be unit vectors, one has:

$$\begin{aligned} &f(a_1 + b_1, a_2 + b_2) \\ &\qquad = \|(\rho_1|a_1| + \sigma_1|b_1|)u_1 + (\rho_2|a_2| + \sigma_2|b_2|)u_2\| \\ &\qquad \leqslant \|\rho_1|a_1|u_1 + \rho_2|a_2|u_2\| + \|\sigma_1|b_1|u_1 + \sigma_2|b_2|u_2\| \\ &\qquad = \||a_1|(\rho_1 u_1) + |a_2|(\rho_2 u_2)\| + \||b_1|(\sigma_1 u_1) + |b_2|(\sigma_2 u_2)\| \\ &\qquad = f(|a_1|, |a_2|) + f(|b_1|, |b_2|) = f(a_1, a_2) + f(b_1, b_2). \qquad \square \end{aligned}$$

Proof of Theorem 2.2. — We start by picking $N \in \mathbb{N}$, $\varepsilon > 0$, $A \in [1, \infty)$, and a direct sum $X_1 \oplus X_2$ satisfying the conditions of Theorem 2.2. Our goal is to find n such that the conditions of Theorem 2.2 are satisfied, and to establish that $n \to \infty$ as $N \to \infty$.

We will consider two metric structures on $X_1 \oplus X_2$. One of them is induced by the norm of X, the other is a Euclidean structure on $X_1 \oplus X_2$

for which X_1 and X_2 are orthogonal and have the same Euclidean norms as in X.

To find the subspaces Y_1 and Y_2 for a given ε and A, we start with an asymmetric problem. More precisely, for some $x_1 \in S(X_1)$ (the unit sphere of X_1, it is the same in both norms), consider the space $\mathrm{lin}(X_2 \cup \{x_1\})$ where lin denotes the linear span of $X_2 \cup \{x_1\}$. The "asymmetric problem" to which we refer above is to find a subspace $E(X_2, x_1)$ of X_2 such that the closed unit ball B (in the norm of the space X) of the space $\mathrm{lin}(E(X_2, x_1) \cup \{x_1\})$ is *ω-invariant with respect to orthogonal operators on the space* $E(X_2, x_1)$, in the sense that

$$(1-\omega)\|\alpha x_1 + y_1\| \leqslant \|\alpha x_1 + O y_1\| \leqslant (1+\omega)\|\alpha x_1 + y_1\|$$

for every $\alpha \in \mathbb{R}$, every $y_1 \in E(X_2, x_1)$, and every orthogonal operator O on $E(X_2, x_1)$. A selection of $\omega > 0$ needed to get an ε-invariant norm will be specified later. As the first step in the desired direction, we observe that an application of [14, Theorem 7 and Remark 8], which is a quantitative version for the result of [19, Corollary of Theorem 2], yields Lemma 2.5 below.

By a *pointed convex body* in a k-dimensional affine space L we mean a pair consisting of a full-dimensional bounded convex body and a point in its interior. We say that a pointed convex body (K, z) in an affine space L with a Euclidean structure is *δ-equivalent* ($\delta > 0$) *to a Euclidean ball* if there exists $r > 0$ such that the following inclusion holds for Euclidean balls in L centered at z:

$$B(z, r) \subset K \subset B(z, (1+\delta)r).$$

Lemma 2.5. — *For any $x_1 \in S(X_1)$, for any $0 < \delta < 1$, there exists a subspace $E(X_2, x_1)$ of X_2 satisfying the conditions:*

(1) *Its dimension can be estimated from below in terms of N (recall that $X_2 = \ell_2^N$) and δ; and this dimension tends to ∞ if δ is fixed and $N \to \infty$. For convenience, δ will be chosen in such a way that $k := \frac{1}{\delta} \in \mathbb{N}$.*

(2) *Pointed convex bodies whose components are sections of B by affine subspaces $E(X_2, x_1) + s\delta x_1$ and points $s\delta x_1$, where $s = 0, 1, \ldots, \frac{1}{\delta} - 1$, are δ-equivalent to Euclidean balls in the Euclidean structure described above. If $((E(X_2, x_1) + x_1) \cap B, x_1)$ is a pointed convex body in $E(X_2, x_1) + x_1$, it is also required to satisfy the same condition.*

Proof. — We use [14, Theorem 7] to construct the subspace $E(X_2, x_1)$ by reducing the space X_2 to $E(X_2, x_1)$ in $k = \frac{1}{\delta}$ steps. Let step m be such that after this step the condition of δ-equivalence to Euclidean balls is satisfied for levels $0, 1, \ldots, m$ (that is, for subspaces $E(X_2, x_1) + s\delta x_1$ with $s = 0, 1, \ldots, m$).

Observe that the intersection of the ball of X with X_2 is a Euclidean ball, therefore the condition of the item (2) for $m = 0$ is satisfied.

After that we start reducing the subspace X_2 as follows.

Step 1 corresponding to $m = 1$. — We start with the subspace $E_0 = X_2$ and denote the unit ball of $\mathrm{lin}(E_0 \cup \{x_1\})$ in the X-norm by B_0. Consider the intersection of B_0 and the affine subspace $\delta x_1 + E_0$. Since $\delta < 1$, it is clear that δx_1 is an interior point of this section (recall that x_1 is a unit vector in X_1). By [14, Theorem 7 and Remark 8], there is a linear subspace $E_1 \subset E_0$ such that the intersection of B_0 with $\delta x_1 + E_1$ is δ-equivalent to a Euclidean ball (centered at δx_1) and $\dim E_1 \geqslant g(\dim E_0, \delta)$, where g is given by

$$g(N, \delta) = \delta^2 \ln(\sigma N)/\beta \tag{2.2}$$

for some universal constants $\sigma > 0$ and $0 < \beta < \infty$. Step 1 is complete.

Denote by $g^{\{s\}}$ the function obtained as the s^{th} iteration of g, that is, $g^{\{s\}}(N, \delta) = g(g \ldots g(g(N, \delta), \delta) \ldots, \delta), \delta)$, s times.

Step m. — We start with a subspace $E_{m-1} \subset X_2$ whose dimension is at least $g^{\{m-1\}}(N, \delta)$. Denote the unit ball of $\mathrm{lin}(E_{m-1} \cup \{x_1\})$ in the X-norm by B_{m-1}.

Note that the intersections of B_{m-1} with the affine subspaces $i\delta x_1 + E_1$ for $i = 1, \ldots, m-1$ are δ-equivalent to Euclidean balls centered at $i\delta x_1$.

Now, consider the intersection of B_{m-1} and the affine subspace $m\delta x_1 + E_{m-1}$. It is clear that $m\delta x_1$ is an interior point of this section (if $m < \frac{1}{\delta}$). By [14, Theorem 7], there exists a linear subspace $E_m \subset E_{m-1}$ such that the intersection of B_{m-1} with $m\delta x_1 + E_m$ is δ-equivalent to the Euclidean ball centered at $m\delta x_1$ and $\dim E_m \geqslant g(\dim E_{m-1}, \delta)$.

If x_1 is an interior point of $B_{k-1} \cap (x_1 + E_{k-1})$, we stop after doing Step $k = \frac{1}{\delta}$. Otherwise, we stop one step earlier.

We denote the subspace obtained at the end of this procedure by $E(X_2, x_1)$. It is clear that $\dim E(X_2, x_1) \geqslant g^{\{k\}}(N, \delta)$. Since k depends only on δ, the condition (1) of Lemma 2.5 is satisfied.

It is clear that after this procedure Condition (2) is satisfied for all levels, except, possibly, level k. $\square$

We are going to prove that the established in Lemma 2.5 properties of the ball B imply its $\omega(\delta)$-invariance with respect to orthogonal operators in $E(X_2, x_1)$, where $\omega(\delta)$ is a function defined for positive δ and satisfying $\lim_{\delta\downarrow 0}\omega(\delta) = 0$.

To do this, we define the function r for $t \in [0,1]$ in the following manner. Let $B_h(tx_1)$ be the largest Euclidean ball in the affine subspace $E(X_2, x_1)+tx_1$ centered at tx_1 which is contained in B. The value $r(t)$ is defined to be the radius of this ball.

Consider the union $C^+ := \bigcup_{t\in[0,1]} B_h(tx_1)$, and let C^- be its image under the central symmetry about 0. Since we consider the Euclidean structure in which x_1 is orthogonal to X_2 and $B_h(tx_1)$ is a Euclidean ball centered at tx_1, the sets C^+ and C^- are reflections of each other in the subspace $E(X_2, x_1)$. Their union will be denoted by D, that is, $D = C^+ \cup C^-$. The function r is extended as an even function on $[-1, 1]$.

The following statement holds:

Lemma 2.6. — *The function r is concave and continuous on $[-1,1]$, and it is non-increasing on $[0,1]$.*

Proof. — Consider $-1 \leqslant t_1 < t_2 \leqslant 1$. By the convexity of B, the ball $B_h(t_3x_1)$ with $t_3 = \alpha t_1 + (1-\alpha)t_2$ (for some $0 < \alpha < 1$) contains the ball at level t_3x_1 of radius $\alpha r(t_1) + (1-\alpha)r(t_2)$. Therefore, $r(t_3) \geqslant \alpha r(t_1) + (1-\alpha)r(t_2)$, and r is concave.

The continuity of r on the interval $(-1, 1)$ follows from concavity: A function concave on an interval is continuous everywhere except, possibly, at the endpoints of the interval (see, e.g., the introductory chapter in [16]).

Monotonicity on $[0,1]$ follows from the concavity of r and the fact that it is even on $[-1,1]$.

The continuity of r at $t = 1$ (and therefore at $t = -1$ also) follows since r is a decreasing function bounded below by 0 and therefore has a limit L from the left at $t = 1$. This limit coincides with $r(1)$ because B is closed and its intersection with $E(X_2, x_1) + tx_1$ contains a ball of radius L centered at tx_1 for all $t \in [0, 1)$. □

Next we prove:

Lemma 2.7. — *For some $\omega(\delta) > 0$ satisfying $\lim_{\delta\downarrow 0}\omega(\delta) = 0$, the inclusion $B \subset (1 + \omega(\delta))D$ holds and thus, B is $\omega(\delta)$-invariant with respect to orthogonal operators on $E(X_2, x_1)$.*

The proof of Lemma 2.7 will be given below following two preparatory propositions. These propositions will be applied to two-dimensional sections of B and D. Results about the set A below will be applied to two-dimensional sections of D.

In $\mathbb{R}^2$ endowed with a Cartesian system of coordinates (x, y) consider a closed convex domain A symmetric about the coordinate axes and containing the points $(\pm 1, 0)$ and $(0, \pm 1)$ on its boundary. The boundary of A is represented by the thick black curves in Figure 2.1. For some natural number $k \geqslant 5$ let $\delta = 1/k$ and denote by $\{(r_i, 1 - i\delta) \mid 1 \leqslant i \leqslant k\}$ the coordinates of the points in the first quadrant at the intersection of the boundary of A with the horizontal lines $y = 1 - i\delta$, $1 \leqslant i \leqslant k$.

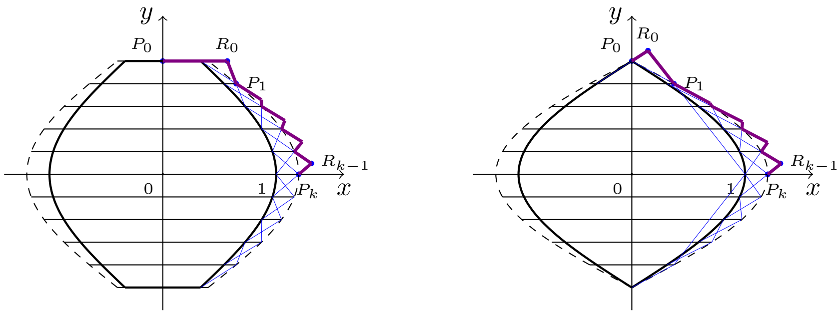

Figure 2.1. Flat top $r_0 > 0$ (Left) and Sharp top $r_0 = 0$ (Right)

For $i = 0$, let r_0 be the largest x-coordinate among the points at the intersection of A and the line $y = 1$. Note that r_0 may be positive if A has a flat top, or it may be zero. We will distinguish between the two cases.

Define a polygonal line with vertices alternating from the set of points P_i with coordinates $((1+\delta)r_i, 1-i\delta)$, $1 \leqslant i \leqslant k$ and the set of points R_i at the intersection of pairs of lines through $(r_{i-1}, 1-(i-1)\delta)$, $((1+\delta)r_i, 1-i\delta)$ and $((1+\delta)r_{i+1}, 1-(i+1)\delta)$, $(r_{i+2}, 1-(i+2)\delta)$, $1 \leqslant i \leqslant k-2$ (see Figures 2.1 and 2.2). Let P_0 be the point of coordinates $(0, 1)$.

We define R_{k-1} as the intersection of the line through $(r_{k-1}, -\delta)$ and $(1+\delta, 0)$ with the line through $(r_{k-2}, 2\delta)$ and $((1+\delta)r_{k-1}, \delta)$.

According to the cases $r_0 > 0$ and $r_0 = 0$, we define:

(i) In the case $r_0 > 0$, let R_0 be the intersection of the line through $((1+\delta)r_1, 1-\delta)$ and $(r_2, 1-2\delta)$ with the line $y = 1$.

(ii) In the case $r_0 = 0$, let R_0 be the intersection of the line through $(-r_1, 1-\delta)$ and $(0, 1)$ with the line through $(1, 0)$ and $((1+\delta)r_1, 1-\delta)$.

Comment. — This choice of R_0 could look artificial, but it works for our goals and makes the computation easier.

Consider the closed domain C bounded by the polygon obtained from the reflections of the polygonal line $P_0, R_0, P_1, R_1, P_2, R_2, \dots, P_{k-1}, R_{k-1}, P_k$ about the coordinate axes and about the origin.

Proposition 2.8. — *For $0 < \delta < 1/4$, the domain C is contained in $(1+\omega(\delta))A$ (the homothetic dilation of A) where $\omega(\delta) = 4\delta + \delta^{\frac{1}{2}}$.*

Proof. — Because both A and C are symmetric about the coordinate axes, we will focus on the parts of A and C contained in the first quadrant.

The proof will proceed in two steps:

(1) We show that the region of C with points (x, y) with $y \leqslant 1 - \delta^{\frac{1}{2}}$ is contained in the $(1+\omega(\delta))$ horizontal stretch of A, i.e. in the set

$$\{((1+\omega(\delta))x, y) \mid (x, y) \in A\}.$$

(2) We show that the points of C above the line $y = 1 - \delta^{\frac{1}{2}}$ are covered by the full dilation $(1+\omega(\delta))A$.

The proof of the proposition follows from the convexity of A and from items (1) and (2).

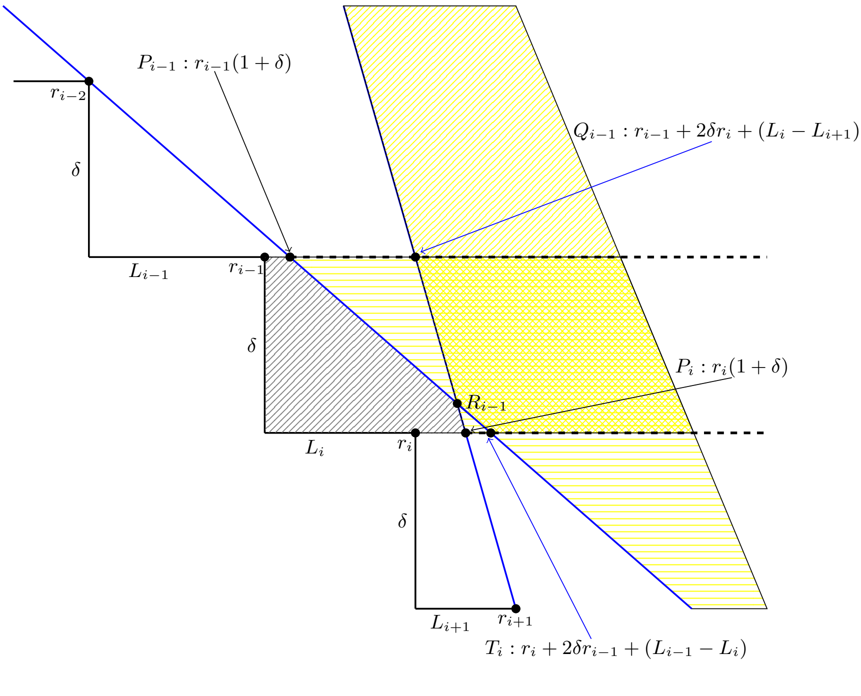

Figure 2.2. Analysis at the boundary of C.

Figure 2.2 displays:

- Labels r_i which indicate the points of coordinates $(r_i, 1-i\delta)$ on the boundary of A.
- Horizontal intervals of length $L_i := r_i - r_{i-1}$ ending at the points r_i.
- Labels $(1+\delta)r_i$ which indicate the points of coordinates $((1+\delta)r_i, 1-i\delta)$ denoted above by P_i.
- For $1 \leqslant i \leqslant k-1$, the number $r_{i-1} + 2\delta r_i + (L_i - L_{i+1})$ which represents the x-coordinate of the point Q_{i-1} at the intersection of the line through $((1+\delta)r_i, 1-i\delta)$, $(r_{i+1}, 1-(i+1)\delta)$ and the line $y = 1-(i-1)\delta$. We define the point Q_{k-1} as the intersection of the line through the points $(1+\delta, 0)$, $(r_{k-1}, -\delta)$ and the line $y=\delta$; it has coordinates $(2+2\delta - r_{k-1}, \delta)$.
- The number $r_i + 2\delta r_{i-1} + (L_{i-1} - L_i)$ which represents the x-coordinate of the point T_i at the intersection of the line through $(r_{i-2}, 1-(i-2)\delta)$, $((1+\delta)r_{i-1}, 1-(i-1)\delta)$ and the line $y = 1-i\delta$.
- The point R_{i-1} at the intersection of the line through P_{i-1}, T_i and the line through Q_{i-1}, P_i. The intersection looks as is shown in Figures 2.1 and 2.2 because of the monotonicity of $\{L_i\}_{i=1}^k$ observed below.

The points P_{i-1}, R_{i-1}, P_i are consecutive vertices of the polygon that bounds the region C described above.

Define $L_0 := r_0$. Observe that the convexity of the domain A implies that $L_1 \geqslant L_2 \geqslant \cdots \geqslant L_k$, while nothing of this type can be claimed about L_0.

To prove step (1) we show that for each $\delta^{-1/2} + 1 \leqslant i \leqslant k-1$ the following inequalities hold:

$$(1+\omega(\delta))r_{i-1} \geqslant r_{i-1} + 2\delta r_i + (L_i - L_{i+1}), \tag{2.3}$$

$$(1+\omega(\delta))r_i \geqslant r_i + 2\delta r_{i-1} + (L_{i-1} - L_i). \tag{2.4}$$

Our proof shows (2.4) for $i = k$ also. We prove a suitable version of (2.3) below, see (2.5).

Indeed, if (2.3) fails, we get

$$(4\delta + \delta^{1/2})r_{i-1} < 2\delta r_i + (L_i - L_{i+1}),$$

or

$$\delta^{1/2} r_{i-1} < 4\delta\,(r_i/2 - r_{i-1}) + (L_i - L_{i+1}).$$

For $i \geqslant 2$, the inequality $L_i \leqslant L_{i-1}$ implies $r_i/2 \leqslant r_{i-1}$, and thus the last inequality implies

$$\delta^{1/2} r_{i-1} < (L_i - L_{i+1}) \leqslant L_i \leqslant L_{i-1} \leqslant \cdots \leqslant L_1.$$

Therefore $(i-1)\delta^{1/2}r_{i-1} < \sum_{j=1}^{i-1} L_j = r_{i-1} - r_0 \leqslant r_{i-1}$, implying $(i-1) < \delta^{-1/2}$, contrary to the assumption.

Meanwhile, if (2.4) fails, we get

$$(4\delta + \delta^{1/2})r_i < 2\delta r_{i-1} + (L_{i-1} - L_i).$$

Since $r_{i-1} \leqslant r_i$, we get

$$(2\delta + \delta^{1/2})r_{i-1} \leqslant (4\delta + \delta^{1/2})r_i < 2\delta r_{i-1} + (L_{i-1} - L_i).$$

This inequality implies

$$\delta^{1/2}r_{i-1} < (L_{i-1} - L_i) \leqslant L_{i-1} \leqslant \cdots \leqslant L_1.$$

As in the previous case, this implies $(i-1) < \delta^{-1/2}$, contrary to the assumption.

Thus, for $\delta^{-1/2} + 1 \leqslant i \leqslant k-1$ the inequalities (2.3) and (2.4) hold, and this implies that the points Q_{i-1}, T_i, and consequently R_{i-1}, are covered by the horizontal $(1+\omega(\delta))$ stretch of A.

Now we show that the point Q_{k-1} is also covered by the horizontal $(1+\omega(\delta))$ stretch of A.

For any $\delta < 1/4$ it holds that

$$\delta^{1/2}(1 - 4\delta^{3/2} - \delta) \geqslant 0.$$

This is equivalent to

$$(2 + 4\delta + \delta^{1/2})(1-\delta) \geqslant 2 + 2\delta.$$

Since $r_k = 1$ and $r_0 \geqslant 0$, the convexity of A requires that $r_{k-1} \geqslant 1 - \delta$. Therefore,

$$(2 + 4\delta + \delta^{1/2})r_{k-1} \geqslant 2 + 2\delta$$

and thus

$$(1 + \omega(\delta))r_{k-1} \geqslant 2 + 2\delta - r_{k-1}, \tag{2.5}$$

which is what we needed to show.

Consequently, for each $\delta^{-1/2} + 1 \leqslant i \leqslant k$ the corners R_{i-1}, P_i of the polygon bounding C are covered by the horizontal $(1+\omega(\delta))$ stretch of A.

We now prove step (2). We use the fact that multiplication of A by $(1+\omega(\delta))$ stretches A in all directions, not only horizontally.

After the horizontal stretch of A, on level i (i.e., on the line $y = 1 - i\delta$), we obtain an interval of length $(1+\omega(\delta))r_i$. It is easy to see that it is at least as long as the interval of level $(i-1)$ of length $r_{i-1} + 2\delta r_i + (L_i - L_{i+1})$. Indeed,

$$(1+\omega(\delta))r_i > (1+4\delta)r_i = r_{i-1} + 4\delta r_i + r_i - r_{i-1} \geqslant r_{i-1} + 2\delta r_i + (L_i - L_{i+1}).$$

By the convexity of the image of A, it is enough to cover the interval of this length, provided that the vertical stretch moves level i on or above the level $(i-1)$, that is, we need the inequality

$$(1+\omega(\delta))(1-i\delta) \geqslant (1-(i-1)\delta) \tag{2.6}$$

for each i satisfying $(i-1) < \delta^{-1/2}$.

We assumed that $\delta < \frac{1}{4}$. With this in mind, $1-(i-1)\delta > 1-\delta^{-1/2}\delta = 1-\delta^{1/2} \geqslant 1/2$. Since the fraction $\frac{a-\delta}{a}$ is increasing in a for $a>0$, we have

$$\frac{1-i\delta}{1-(i-1)\delta} = \frac{(1-(i-1)\delta)-\delta}{1-(i-1)\delta} \geqslant \frac{\frac{1}{2}-\delta}{\frac{1}{2}}.$$

To prove (2.6) it suffices to show that

$$(1+\omega(\delta))\frac{\frac{1}{2}-\delta}{\frac{1}{2}} \geqslant 1.$$

Since $\omega(\delta) > 4\delta$, the left-hand side in the last inequality is at least

$$(1+4\delta)(1-2\delta) = 1+2\delta-8\delta^2 > 1$$

if $\delta < \frac{1}{4}$.

Now it remains to consider points of C which are above level $y=1$. Note that such points can exist only if $r_0 = 0$. In this case we have $L_1 = r_1 > 0$. We show that the point R_0 (sketched in Figure 2.1 (Right)) at the intersection of the line through $(-r_1, 1-\delta)$, $(0,1)$ and the line through $(1,0)$, $((1+\delta)r_1, 1-\delta)$, is covered by $(1+\omega(\delta))A$. For this it suffices to show that the coordinates of R_0 satisfy

$$(1+\omega(\delta))r_1 \geqslant x_{R_0}, \tag{2.7}$$

and

$$(1+\omega(\delta))(1-\delta) \geqslant y_{R_0}. \tag{2.8}$$

Direct calculation gives that $x_{R_0} = r_1\alpha$ and $y_{R_0} = \delta\alpha + 1$ where $\alpha = \frac{(1+\delta)r_1-\delta}{(1-2\delta-\delta^2)r_1+\delta}$. Thus, the inequality (2.7) is equivalent to $(1+\omega(\delta)) \geqslant \alpha$ and the inequality (2.8) is equivalent to $(1-\delta)\omega(\delta) \geqslant \delta(1+\alpha)$.

If (2.7) holds, then necessarily (2.8) is also true. Indeed, since by direct verification $(1-\delta)\omega(\delta) \geqslant \delta(2+\omega(\delta))$, it follows from $(1+\omega(\delta)) \geqslant \alpha$ that $(1-\delta)\omega(\delta) \geqslant \delta(1+\alpha)$ which is equivalent to (2.8). Therefore, it will suffice to show that (2.7) holds.

Now we check that (2.7) holds. Note that $(1+\omega(\delta)) \geqslant \alpha$ is equivalent to

$$\omega(\delta) \geqslant \delta\frac{(3+\delta)r_1-2}{(1-2\delta-\delta^2)r_1+\delta}, \tag{2.9}$$

which leads to the following two cases:

(1) the case $(3+\delta)r_1 \leqslant 2$ when the inequality (2.9) holds trivially, and
(2) the case $(3+\delta)r_1 > 2$, which means $r_1 > \frac{2}{3+\delta}$.

Since in case (2), if we increase r_1 to 1 in the numerator of the fraction on the right hand side and decrease r_1 to $\frac{2}{3+\delta}$ in the denominator we obtain

$$\delta \frac{3+4\delta+\delta^2}{2-\delta-\delta^2} \geqslant \delta \frac{(3+\delta)r_1 - 2}{(1-2\delta-\delta^2)r_1+\delta}.$$

Since $0 < \delta < 1/4$, a direct check shows that 4δ is larger than the left-hand side of the inequality above. Thus, since $\omega(\delta) > 4\delta$, the inequality (2.9) holds in both cases.

Therefore, we obtain that the portion of C above level 1 is contained inside the $(1+\omega(\delta))$ homothetic image of A. □

Define a family $\mathcal{I}$ of $4(k-1)$ closed horizontal intervals, which consists of the intervals with endpoints $(r_i, 1-i\delta)$ and $((1+\delta)r_i, 1-i\delta)$, $1 \leqslant i \leqslant k-1$, together with their reflection about the coordinate axes and about the origin.

Proposition 2.9. — *Any closed convex domain H which contains A, and whose boundary intersects all the intervals in $\mathcal{I}$ and passes through the points $(\pm 1, 0)$, $(0, \pm 1)$, is contained in $(1+\omega(\delta))A$ (the homothetic dilation of A) where $\omega(\delta) = 4\delta + \delta^{\frac{1}{2}}$.*

Proof. — From the requirement that the domain H is convex it follows that H is contained in the polygonal domain C described in Proposition 2.8. Since C is contained in $(1+\omega(\delta))A$, it follows that so is H. □

Proof of Lemma 2.7. — Consider any unit vector x_2 in $E(X_2, x_1)$. In the plane spanned by x_1 and x_2 we consider the Cartesian system of coordinates with the x-axis in the direction of x_2 and the y-axis in the direction of x_1. Define the domains A and H as the respective intersections of D and the unit ball B with this plane. The hypotheses of Proposition 2.9 are satisfied and therefore H is contained in $(1+\omega(\delta))A$. Since this holds for any choice of x_2, the inclusion $B \subset (1+\omega(\delta))D$ follows. Because $D \subset B$ and D is invariant under any orthogonal map on $E(X_2, x_1)$ we conclude that B is $\omega(\delta)$-invariant. □

Let G be a function, $G : \mathbb{N} \times (0,\infty) \to \mathbb{N} \cup \{0\}$. We say that G is *indefinitely growing* (IG) if $\lim_{N\to\infty} G(N,\delta) = \infty$ for every $\delta > 0$.

Observation 2.10. — *Finite iterations of* IG *functions with fixed* $\delta > 0$*, that is, iterations of the form*

$$G(G(\dots G(G(N,\delta),\delta)\dots,\delta),\delta),$$

are also IG *functions.*

Proposition 2.11. — *There exists an* IG *function* $G(= G(N,\alpha))$ *such that, for each subspace* U *of* X_2*, each* $\alpha > 0$*, and each* $v \in S(X_1)$*, there exists a subspace* $E(U,v) \subset U$ *with* $\dim E(U,v) \geqslant G(\dim U,\alpha)$ *and such that the* X*-norm on* $\operatorname{lin}(E(U,v) \cup \{v\})$ *is* α*-invariant with respect to the orthogonal operators on* $E(U,v)$*.*

Proof. — For the given $\alpha > 0$ select $\delta > 0$ so that $k := \frac{1}{\delta} \in \mathbb{N}$ and $\omega(\delta) < \alpha$. Applying Lemmas 2.5 and 2.7 for this value of δ, $X_2 = U$ and $x_1 = v$, we obtain the subspace $E(U,v) \subset U$ with dimension bounded below by $g^{\{k\}}(\dim U,\delta)$, the k^{th} iteration of the function (2.2), such that the X-norm on $\operatorname{lin}(E(U,v) \cup \{v\})$ is α-invariant with respect to the orthogonal operators on $E(U,v)$.

Since we may assume that $\delta = \frac{1}{k}$ satisfies both $\omega(\delta) < \alpha$ and $\omega\left(\frac{1}{k-1}\right) \geqslant \alpha$ we can regard $g^{\{k\}}(\dim U,\delta)$ as $G(\dim U,\alpha)$ for some IG function G. □

Combining Observation 2.10 and Proposition 2.11 with the known fact that the cardinality of an α-net in the unit sphere of a t-dimensional normed space can be estimated from above in terms of t and $\alpha > 0$ only, we arrive at the following statements.

(1) For every $\alpha > 0$ and $n, M \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that if we apply Proposition 2.11 to X_1 with $\dim X_1 = N$ and all points x_2 in an α-net N_2 of the unit sphere of an M-dimensional subspace $U \subset X_2$, we obtain a subspace Y_1 in X_1 of dimension at least n, such that

$$(1-\alpha)\|y_1 + y_2\| \leqslant \|O_1 y_1 + y_2\| \leqslant (1+\alpha)\|y_1 + y_2\|$$

for every $y_1 \in Y_1$, y_2 being any scalar multiple of an element of $x_2 \in N_2$, and any orthogonal operator O_1 on Y_1.

(2) For every $\alpha > 0$ and $n \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that applying Proposition 2.11 to a subspace $U \subset X_2$ of dimension $\dim U \geqslant M$ and all points x_1 in an α-net N_1 of the unit sphere of an n-dimensional subspace $Y_1 \subset X_1$, brings out a subspace $Y_2 \subset U \subset X_2$ with $\dim Y_2 \geqslant n$ such that

$$(1-\alpha)\|y_1 + y_2\| \leqslant \|y_1 + O_2 y_2\| \leqslant (1+\alpha)\|y_1 + y_2\|$$

for every y_1 being a scalar multiple of an element $x_1 \in N_1$, any $y_2 \in Y_2$, and any orthogonal operator O_2 on Y_2.

We use items (1) and (2) as follows. First we use n to find values of M and later N. Thereafter, we pick any N-dimensional subspaces X_1 and X_2 satisfying the conditions of Theorem 2.2.

After that we apply item (1) to an arbitrarily chosen M-dimensional subspace $U \subset X_2$, and get a subspace $Y_1 \subset X_1$.

Finally, with the help of item (2), for the chosen U and Y_1 constructed in the previous step we obtain Y_2.

To conclude the proof of Theorem 2.2 we need the following approximation lemma for each step of the construction. Let

$$A = \max\{\|P_1\|, \|P_2\|\},$$

where $P_1 : X_1 \oplus X_2 \to X_1$ and $P_2 : X_1 \oplus X_2 \to X_2$ are projections with kernels X_2 and X_1, respectively, and the norm is the X-norm.

LEMMA 2.12. — *The conditions of items (1) and (2) imply that, for any $y_1 \in Y_1$, $y_2 \in Y_2$, and any orthogonal operators O_1 on Y_1 and O_2 on Y_2, we have*

$$\begin{aligned}(2.10)\quad (1 - \alpha(1 + (2 - \alpha)A))^2\|y_1 + y_2\| &\leqslant \|O_1 y_1 + O_2 y_2\| \\ &\leqslant (1 + \alpha(1 + (2 + \alpha)A))^2\|y_1 + y_2\|,\end{aligned}$$

provided $(1 - \alpha(1 + (2 - \alpha)A)) > 0$.

Proof. — We may assume that $y_1 \neq 0$ and $y_2 \neq 0$. Let z_2 be a multiple of an element from N_2 such that $\|z_2 - y_2\| < \alpha\|y_2\|$. Then

$$\begin{aligned}(2.11)\qquad \|O_1 y_1 + y_2\| &\leqslant \|O_1 y_1 + z_2\| + \|z_2 - y_2\| \\ &\leqslant (1 + \alpha)\|y_1 + z_2\| + \|z_2 - y_2\| \\ &\leqslant (1 + \alpha)\|y_1 + y_2\| + (2 + \alpha)\|z_2 - y_2\| \\ &< (1 + \alpha)\|y_1 + y_2\| + (2 + \alpha)\alpha\|y_2\| \\ &\leqslant (1 + \alpha(1 + (2 + \alpha)A))\|y_1 + y_2\|.\end{aligned}$$

Similarly (assume that $\alpha < 1$),

$$\begin{aligned} \|O_1y_1 + y_2\| &\geqslant \|O_1y_1 + z_2\| - \|z_2 - y_2\| \\ &\geqslant (1-\alpha)\|y_1 + z_2\| - \|z_2 - y_2\| \\ &\geqslant (1-\alpha)\|y_1 + y_2\| - (2-\alpha)\|z_2 - y_2\| \\ &> (1-\alpha)\|y_1 + y_2\| - (2-\alpha)\alpha\|y_2\| \\ &\geqslant (1-\alpha(1+(2-\alpha)A))\|y_1 + y_2\|. \end{aligned} \tag{2.12}$$

Now we apply the same argument for any $w_1 \in Y_1$ and any z_1 in the direction of an element of N_1 satisfying $\|w_1 - z_1\| < \alpha\|w_1\|$. We get

$$\begin{aligned} (1-\alpha(1+(2-\alpha)A))\|w_1+y_2\| &\leqslant \|w_1+O_2y_2\| \\ &\leqslant (1+\alpha(1+(2+\alpha)A))\|w_1+y_2\|. \end{aligned} \tag{2.13}$$

Plugging $w_1 = O_1y_1$ and using (2.11), (2.12), and $(1-\alpha(1+(2-\alpha)A)) > 0$, we get (2.10). □

To complete the proof of Theorem 2.2 we pick $\alpha > 0$ in such a way that $(1+\alpha(1+(2+\alpha)A))^2 < 1+\varepsilon$ and $(1-\alpha(1+(2-\alpha)A))^2 > 1-\varepsilon$. □

3. Bending in unconditional sums of two spaces

Let X and Y be (possibly finite-dimensional) Banach spaces such that there exist two linear isometric embeddings $I_1 : Y \to X$ and $I_2 : Y \to X$ with distinct images $Y_1 = I_1(Y)$ and $Y_2 = I_2(Y)$.

Definition 3.1. — *Let $C \in [1,\infty)$. A mapping $T : Y \to X$ is called a* C-bending *of Y in the space X from I_1 to I_2, with parameters (r,R), $0 < r < R < \infty$, if it is a C-bilipschitz embedding such that the restriction of T to the ball of radius r coincides with I_1 and the restriction of T to the exterior of the ball of radius R in Y coincides with I_2.*

Let $Z = (\mathbb{R}^2, \|\cdot\|_Z)$ be a two-dimensional Banach space in which the unit vectors $(1,0)$ and $(0,1)$ form a normalized 1-unconditional basis. This means

$$\|(1,0)\|_Z = \|(0,1)\|_Z = 1 \text{ and } \|(a,b)\|_Z = \|(\pm a, \pm b)\|_Z. \tag{3.1}$$

Given a Banach space Y, we use $X = Y \oplus_Z Y$ to denote the Banach space consisting of pairs (u,v) with $u,v \in Y$ with the norm

$$\|(u,v)\|_X = \|(\|u\|_Y, \|v\|_Y)\|_Z.$$

When we consider a C-bending of Y in the space $X = Y \oplus_Z Y$ we restrict our attention to the case where $I_1(y) = (y,0)$ and $I_2(y) = (0,y)$ and call such bending *a C-bending of Y in the space $X = Y \oplus_Z Y$ with parameters (r, R), $0 < r < R < \infty$.*

To state the main result of this section, Theorem 3.4, we need to introduce some additional parameters. Define

$$m_Z = \min_\tau \|(\cos\tau, \sin\tau)\|_Z \text{ and } M_Z = \max_\tau \|(\cos\tau, \sin\tau)\|_Z.$$

OBSERVATION 3.2. — *It is easy to see that the unit ball of Z satisfying (3.1) contains the unit ball of ℓ_1^2 and is contained in the unit ball of ℓ_∞^2, thereby $m_Z \geqslant \frac{1}{\sqrt{2}}$ and $M_Z \leqslant \sqrt{2}$.*

Let

$$u(\tau) = \frac{(\cos\tau, \sin\tau)}{\|(\cos\tau, \sin\tau)\|_Z} \in Z.$$

Condition (3.1) implies that $u(0) = (1, 0)$ and $u(\pi/2) = (0, 1)$. We need the following

PROPOSITION 3.3. — *The set of all quotients*

$$\frac{\|u(\tau_2) - u(\tau_1)\|_Z}{\tau_2 - \tau_1}$$

for $0 \leqslant \tau_1 < \tau_2 \leqslant \frac{\pi}{2}$ is bounded. Let

$$c_Z := \sup_{0 \leqslant \tau_1 < \tau_2 \leqslant \frac{\pi}{2}} \frac{\|u(\tau_2) - u(\tau_1)\|_Z}{\tau_2 - \tau_1}. \tag{3.2}$$

Then

$$\frac{2}{\pi} \leqslant \frac{2\sqrt{2}m_Z}{\pi} \leqslant c_Z \leqslant \frac{2M_Z}{m_Z} \leqslant 4. \tag{3.3}$$

The last inequality follows from Observation 3.2. Since in this paper we do not need tight estimates for c_Z, we do not dwell on their evaluation.

Proof. — To begin with, we write:

$$\begin{aligned} u(\tau_2) - u(\tau_1) &= \frac{(\cos\tau_2 - \cos\tau_1, \sin\tau_2 - \sin\tau_1)}{\|(\cos\tau_2, \sin\tau_2)\|_Z} \\ &\quad - (\cos\tau_1, \sin\tau_1)\frac{\|(\cos\tau_2, \sin\tau_2)\|_Z - \|(\cos\tau_1, \sin\tau_1)\|_Z}{\|(\cos\tau_1, \sin\tau_1)\|_Z \, \|(\cos\tau_2, \sin\tau_2)\|_Z}. \end{aligned}$$

Applying the triangle inequality to the numerator of the norm of the second term in the right-hand side, we conclude that the norm of the second term does not exceed the norm of the first term. Therefore,

$$\|u(\tau_2) - u(\tau_1)\|_Z \leqslant \frac{2}{m_Z} \|(\cos\tau_2 - \cos\tau_1, \sin\tau_2 - \sin\tau_1)\|_Z\,.$$

Trigonometric identities imply the following vector version of a spherical Mean Value Theorem

$$(\cos\tau_2-\cos\tau_1,\sin\tau_2-\sin\tau_1)=\left(-\sin\frac{\tau_1+\tau_2}{2},\cos\frac{\tau_1+\tau_2}{2}\right)2\sin\frac{\tau_2-\tau_1}{2}.$$

Therefore

$$\|(\cos\tau_2-\cos\tau_1,\sin\tau_2-\sin\tau_1)\|_Z\leqslant M_Z(\tau_2-\tau_1),$$

and the inequality $c_Z\leqslant\frac{2M_Z}{m_Z}$ follows.

To get the bound from below on c_Z in (3.3) we substitute $\tau_1=0$ and $\tau_2=\frac{\pi}{2}$ in the quotient in (3.2). □

The main result of this section is the following theorem.

Theorem 3.4. — *Let Y be a finite-dimensional Banach space, and let Z be a 2-dimensional space satisfying* (3.1)*. Then for every $\varepsilon>0$ and every pair (r,R) of positive numbers satisfying the condition*

$$\frac{\varepsilon}{c_Z}\ln\left(\frac{R}{r}\right)=\frac{\pi}{2},\tag{3.4}$$

there is a $\left(\frac{1+\varepsilon}{1-\varepsilon}\right)$-bending T of Y into the sum $X=Y\oplus_Z Y$ with parameters (r,R). Furthermore, the bending T satisfies

$$\|Tx\|_X=\|x\|_Y\quad\textit{for all } x\in Y,$$

and

$$(1-\varepsilon)\|x-y\|_Y\leqslant\|Tx-Ty\|_X\leqslant(1+\varepsilon)\|x-y\|_Y\quad\textit{for all } x,y\in Y.\tag{3.5}$$

Remark 3.5. — Any C-bending with parameters (r,R) is also a C-bending with parameters (r_1,R_1) if $0<r_1\leqslant r<R\leqslant R_1<\infty$. For this reason, the exact value of c_Z is not important.

Proof of Theorem 3.4. — We follow the construction in [32, Section 2.2].

Let $\varepsilon\in(0,1), r>0$ be any numbers. For real numbers $t\geqslant r$, define the function

$$\tau(t)=\tau_{\varepsilon,r,Z}(t):=\frac{\varepsilon}{c_Z}\ln\left(\frac{t}{r}\right),$$

where c_Z is defined in Proposition 3.3. The function $\tau(t)$ is increasing and, by (3.4), maps the interval $[r,R]$ onto $[0,\pi/2]$. The Mean Value Theorem implies that

$$\tau(t_2)-\tau(t_1)\leqslant\frac{\varepsilon}{c_Z}\frac{t_2-t_1}{t_1}\tag{3.6}$$

for $r\leqslant t_1\leqslant t_2\leqslant R$.

We introduce the functions

$$c(x) = \begin{cases} 1 & \text{if } \|x\| \leqslant r \\ \frac{\cos\tau(\|x\|)}{\|(\cos\tau(\|x\|),\sin\tau(\|x\|))\|_Z} & \text{if } r \leqslant \|x\| \leqslant R \\ 0 & \text{if } \|x\| \geqslant R, \end{cases} \tag{3.7}$$

and

$$s(x) = \begin{cases} 0 & \text{if } \|x\| \leqslant r \\ \frac{\sin\tau(\|x\|)}{\|(\cos\tau(\|x\|),\sin\tau(\|x\|))\|_Z} & \text{if } r \leqslant \|x\| \leqslant R \\ 1 & \text{if } \|x\| \geqslant R. \end{cases} \tag{3.8}$$

It is clear that

$$\|(c(x), s(x))\|_Z = 1 \tag{3.9}$$

and

$$(c(x), s(x)) = u\,(\tau(\|x\|_Y)) \tag{3.10}$$

for every $x \in Y$ with $r \leqslant \|x\|_Y \leqslant R$.

We claim that the desired bending is the map

$$T : Y \longrightarrow X = Y \oplus_Z Y$$

given by

$$Tx = (c(x)x, s(x)x). \tag{3.11}$$

Remark 3.6. — It is worth mentioning that T is a development of the well-known in geometry logarithmic spirals in the plane, see [9, p. 4].

Equation (3.9) implies that $\|Tx\|_X = \|x\|_Y$ for every $x \in Y$. It is also clear that T satisfies the condition (3.5) whenever x, y are both in the ball of radius r or in the exterior of the ball of radius R.

When estimating $\|Tx - Ty\|_X$, from now on we assume without loss of generality that

$$\|x\|_Y \geqslant \|y\|_Y\,. \tag{3.12}$$

Next, we write

$$Tx - Ty = (c(x)x, s(x)x) - (c(y)y, s(y)y)$$

in the form

$$\begin{aligned} Tx - Ty = {} & (c(x)(x-y), s(x)(x-y)) \\ & + ((c(x)-c(y))y, (s(x)-s(y))y)\,. \end{aligned} \tag{3.13}$$

For the first summand in the right-hand side of (3.13), we have:

$$\|(c(x)(x-y), s(x)(x-y))\|_X = \|(c(x)\|x-y\|_Y, s(x)\|x-y\|_Y)\|_Z.$$

We conclude that

$$\begin{aligned}\|(c(x)(x-y), s(x)(x-y))\|_X &= \|x-y\|_Y \|(c(x), s(x))\|_Z \\ &= \|x-y\|_Y.\end{aligned} \tag{3.14}$$

For the second summand in the right-hand side of (3.13), there holds

$$\begin{aligned}&\|((c(x)-c(y))y, (s(x)-s(y))y)\|_X \\ &\qquad = \|y\|_Y \|(|c(x)-c(y)|, |s(x)-s(y)|)\|_Z.\end{aligned} \tag{3.15}$$

For $x \in Y$, set

$$U(x) \coloneqq (c(x), s(x)) \in Z.$$

According to (3.7), (3.8), and (3.10), we have

$$U(x) = \begin{cases} u(\tau(r)) & \text{if } \|x\|_Y \leqslant r, \\ u(\tau(\|x\|_Y)) & \text{if } r \leqslant \|x\|_Y \leqslant R, \\ u(\tau(R)) & \text{if } \|x\|_Y \geqslant R. \end{cases} \tag{3.16}$$

Combining the definition of U with (3.13), (3.14), and (3.15), for any $x, y \in Y$ we obtain

$$\begin{aligned}\|x-y\|_Y - \|y\|_Y \|U(x)-U(y)\|_Z& \\ &\leqslant \|Tx - Ty\|_X \\ &\leqslant \|x-y\|_Y + \|y\|_Y \|U(x)-U(y)\|_Z.\end{aligned} \tag{3.17}$$

Now, we show that for any $x, y \in Y$ satisfying (3.12) we have

$$\|y\|_Y \|U(x)-U(y)\|_Z \leqslant \varepsilon \|x-y\|_Y. \tag{3.18}$$

This inequality, together with (3.17) immediately implies (3.5), and, thus, concludes the proof of the theorem.

We prove a stronger version of (3.18), namely,

$$\|U(x)-U(y)\|_Z \leqslant \varepsilon \frac{\|x\|_Y - \|y\|_Y}{\|y\|_Y}. \tag{3.19}$$

It is clear that combining (3.19) with the triangle inequality we obtain (3.18). On the other hand,

$$\begin{aligned}(3.20)\quad &\|U(x)-U(y)\|_Z\\ &\overset{(3.16)}{=} \|u\left(\tau(\min\{R,\|x\|_Y\})\right)-u\left(\tau(\max\{r,\|y\|_Y\})\right)\|\\ &\overset{(3.2)}{\leqslant} c_Z\left(\tau(\min\{R,\|x\|_Y\})-\tau(\max\{r,\|y\|_Y\})\right)\\ &\overset{(3.6)}{\leqslant} \varepsilon\frac{\min\{R,\|x\|_Y\}-\max\{r,\|y\|_Y\}}{\max\{r,\|y\|_Y\}}\leqslant\varepsilon\frac{\|x\|_Y-\|y\|_Y}{\|y\|_Y}.\end{aligned}$$

This completes the proof of Theorem 3.4. □

4. Construction of the embedding

Proof of Theorem 1.2. — Let X be an infinite-dimensional Banach space, $\mathcal{M}$ be a locally finite subset in ℓ_2, and $\varepsilon\in(0,1)$. We assume that $0\in\mathcal{M}$. Our goal is to find an embedding of $\mathcal{M}$ into X with distortion $\leqslant 1+\varepsilon$.

To achieve the distortion $(1+\varepsilon)$ we need to introduce additional parameters $\gamma,\psi,\zeta\in(0,1)$, and $d\in\mathbb{N}$, such that the maximal quotient of the right-hand sides and respective left-hand sides in (4.1), (4.2), (4.5) does not exceed $(1+\varepsilon)$. Such values exist because the values of all coefficients go to 1 as $\gamma,\psi,\zeta\downarrow 0$ and $d\to\infty$. Also, we introduce a decreasing sequence $\{\gamma_i\}_{i=1}^\infty$, $\gamma_i>0$, such that $\prod_{i=1}^\infty(1+\gamma_i)<1+\gamma$.

Next, we define recursively an increasing sequence $\{R_i\}_{i=1}^\infty$ of positive numbers as follows:

(1) $R_1=1$.
(2) $\frac{\psi}{4}\ln\frac{R_{2i}}{R_{2i-1}}=\frac{\pi}{2}$ for all $i\in\mathbb{N}$.

Note. — We use the number 4 in this formula because it is our upper estimate for c_Z which works for every Z, see (3.3).

(3) $\frac{R_{2i+1}}{R_{2i}}=\frac{d}{\varepsilon}$ for all $i\in\mathbb{N}$.

Let $\mathcal{B}(R)\subset\mathcal{M}$ denote the ball of radius R centered at 0, while F_i denotes the subspace of ℓ_2 spanned by $\mathcal{B}(R_{4i})$ and $n_i=\dim F_i$.

To prove Theorem 1.2 we need the following lemma about FDDs (finite-dimensional Schauder decompositions) in an arbitrary infinite-dimensional Banach space X. See [17, p. 11] or [20, Section 1.g] for a basic information on FDDs.

Lemma 4.1. — *Let* $\{\gamma_i\}_{i=1}^\infty$ *and* ζ *be the numbers chosen at the beginning of Section 4, and* $\{F_i\}_{i=1}^\infty$ *be the subspaces chosen above. Introduce the sequence* $\{\mu_i\}_{i=1}^\infty$ *by* $\mu_1=\gamma_1$, $\mu_{2i}=\mu_{2i+1}=\gamma_{i+1}$.

There exists an infinite-dimensional subspace $V\subset X$ *having an FDD* $\{V_i\}_{i=1}^\infty$ *for which there exist isomorphisms* $J_i : F_{j(i)}\to V_i$, *such that*

$$j(i)=\begin{cases}(i+1)/2 & \text{if } i \text{ is odd;}\\ i/2 & \text{if } i \text{ is even,}\end{cases}$$

(that is, for each $j\in\mathbb{N}$ *there are two isomorphisms* J_i *with domain* F_j*) with the following properties*

(i) $\forall v\in F_{j(i)}\quad \|v\|_2\leqslant \|J_iv\|_X\leqslant (1+\mu_i)\|v\|_2$.

(ii) *There exist* 1*-unconditional norms* $\|\ \|_{Z_i}$ *on* $\mathbb{R}^2$ *such that the maps* $\mathcal{J}_{2i-1,2i} : F_{j(2i-1)}\oplus_{Z_i} F_{j(2i)}\to V_{2i-1}\oplus V_{2i}$ *given by* $\mathcal{J}_{2i-1,2i}(u,v)=(J_{2i-1}u, J_{2i}v)$ *satisfy*

$$\|J_{2i-1}u+J_{2i}v\|_X\leqslant \|(\|u\|_2,\|v\|_2)\|_{Z_i}\leqslant (1+\zeta)(1+\gamma_i)^2\|J_{2i-1}u+J_{2i}v\|_X.$$

(iii) *The maps* $\mathcal{J}_{2i,2i+1} : F_{j(2i)}\oplus_2 F_{j(2i+1)}\to V_{2i}\oplus V_{2i+1}$ *given by* $\mathcal{J}_{2i,2i+1}(u,v)=(J_{2i}u, J_{2i+1}v)$ *satisfy*

$$\|J_{2i}u+J_{2i+1}v\|_X\leqslant (\|u\|_2^2+\|v\|_2^2)^{1/2}\leqslant (1+\gamma_{i+1})\|J_{2i}u+J_{2i+1}v\|_X.$$

To proceed without interruption, we demonstrate how Lemma 4.1 is applied to derive Theorem 1.2, while its proof is postponed to the end of the section.

At this point, a low-distortion embedding $\Phi:\mathcal{M}\to X$ will be constructed as a piecewise defined map.

Let $R_0=0$. For any two nonnegative integers j,k $(j<k)$, consider the annulus

$$\mathcal{A}_{j,k}=\{m\in\mathcal{M}:\ R_j\leqslant d(m,0)\leqslant R_k\}.$$

Obviously when $j=0$ it is a ball. Observe that $\{F_i\}_{i=1}^\infty$ forms an increasing sequence of subspaces of ℓ_2. Consequently there exist natural isometric embeddings of F_i into F_{i+1}.

First, we define a sequence of embeddings of annuli $\mathcal{A}_{2i,2i+3}$ into sums of the form $F_i\oplus_{Z_i}F_i$ and $F_i\oplus_2 F_{i+1}$ as restrictions of bendings according to the following procedure:

- Define $\mathcal{T}_1:\mathcal{A}_{0,3}\to F_1\oplus_{Z_1}F_1$ as the restriction to $\mathcal{A}_{0,3}$ of the existing by Theorem 3.4 $\left(\frac{1+\psi}{1-\psi}\right)$-bending of F_1 into $F_1\oplus_{Z_1}F_1$ with parameters (R_1,R_2).

- Consider the restriction to $\mathcal{A}_{2,5}$ of the existing by Theorem 3.4 $\left(\frac{1+\psi}{1-\psi}\right)$-bending of F_2 into $F_2 \oplus_2 F_2$ with parameters (R_3, R_4). Observe that because $\mathcal{A}_{2,4}$ is a subset of F_1, the formula (3.11) for bending implies that the image of this map is contained in $F_1 \oplus_2 F_2$. Define $\mathcal{T}_2 : \mathcal{A}_{2,5} \to F_1 \oplus_2 F_2$ as the resulting map.
- ...
- Define $\mathcal{T}_{2i-1} : \mathcal{A}_{4i-4,4i-1} \to F_i \oplus_{Z_i} F_i$ as the restriction to $\mathcal{A}_{4i-4,4i-1}$ of the existing by Theorem 3.4 $\left(\frac{1+\psi}{1-\psi}\right)$-bending of F_i into $F_i \oplus_{Z_i} F_i$ with parameters (R_{4i-3}, R_{4i-2}).
- Consider the restriction to $\mathcal{A}_{4i-2,4i+1}$ of the existing by Theorem 3.4 $\left(\frac{1+\psi}{1-\psi}\right)$-bending of F_{i+1} into $F_{i+1} \oplus_2 F_{i+1}$ with parameters (R_{4i-1}, R_{4i}). Observe that because $\mathcal{A}_{4i-2,4i}$ is a subset of F_i, the formula (3.11) for bending implies that the image of this map is contained in $F_i \oplus_2 F_{i+1}$. Define $\mathcal{T}_{2i} : \mathcal{A}_{4i-2,4i+1} \to F_i \oplus_2 F_{i+1}$ as the resulting map.
- ...

To get embeddings into $V \subset X$, we consider compositions:

$$\Phi_{2i-1} \coloneqq \mathcal{J}_{2i-1,2i} \circ \mathcal{T}_{2i-1} : \mathcal{A}_{4i-4,4i-1} \longrightarrow V_{2i-1} \oplus V_{2i}$$

and

$$\Phi_{2i} \coloneqq \mathcal{J}_{2i,2i+1} \circ \mathcal{T}_{2i} : \mathcal{A}_{4i-2,4i+1} \longrightarrow V_{2i} \oplus V_{2i+1}.$$

Our next goal is to show that combining these maps we get a well-defined $(1+\varepsilon)$-bilipschitz map of $\mathcal{M}$ into $V \subset X$.

We start with checking that on $\mathcal{A}_{4i-2,4i-1}$, where both Φ_{2i-1} and Φ_{2i} are defined, they coincide. Similarly, we need to check that on $\mathcal{A}_{4i-4,4i-3}$ where both Φ_{2i-2} and Φ_{2i-1} are defined, they coincide. The proofs are the same. We do it only for the first case. The maps $\mathcal{T}_{2i-1}$ and $\mathcal{T}_{2i}$ map $\mathcal{A}_{4i-2,4i-1}$ isometrically into F_i. Since both $\mathcal{J}_{2i-1,2i}$ and $\mathcal{J}_{2i,2i+1}$ map F_i onto V_{2i} using J_{2i}, the maps coincide.

Therefore, the formula

$$\Phi(x) = \begin{cases} \mathcal{J}_{2i-1,2i} \circ \mathcal{T}_{2i-1}(x) \in V_{2i-1} \oplus V_{2i} & \text{if } x \in \mathcal{A}_{4i-4,4i-1}, \quad i \in \mathbb{N}, \\ \mathcal{J}_{2i,2i+1} \circ \mathcal{T}_{2i}(x) \in V_{2i} \oplus V_{2i+1} & \text{if } x \in \mathcal{A}_{4i-2,4i+1}, \quad i \in \mathbb{N}, \end{cases}$$

in which we consider $V_{2i-1} \oplus V_{2i}$ as subspaces of V, gives a well-defined map. It remains to prove that Φ is a $(1+\varepsilon)$-bilipschitz embedding. To achieve this goal it suffices to establish bilipschitz inequalities in the three cases:

Case 1. — $x, y \in \mathcal{A}_{4i-4,4i-1}$. Since $\Phi_{2i-1} = \mathcal{J}_{2i-1,2i} \circ \mathcal{T}_{2i-1}$, by Theorem 3.4,

$$(1-\psi)\|x-y\|_2 \leqslant \|\mathcal{T}_{2i-1}x - \mathcal{T}_{2i-1}y\|_{F_i \oplus_{Z_i} F_i} \leqslant (1+\psi)\|x-y\|_2,$$

and by Lemma 4.1(ii),

$$\frac{1-\psi}{(1+\zeta)(1+\gamma)^2}\|x-y\|_2 \leqslant \|\Phi_{2i-1}x - \Phi_{2i-1}y\|_X \leqslant (1+\psi)\|x-y\|_2. \tag{4.1}$$

Case 2. — $x, y \in \mathcal{A}_{4i-2,4i+1}$. Since $\Phi_{2i} = \mathcal{J}_{2i,2i+1} \circ \mathcal{T}_{2i}$, by Theorem 3.4,

$$(1-\psi)\|x-y\|_2 \leqslant \|\mathcal{T}_{2i}x - \mathcal{T}_{2i}y\|_{F_i \oplus_2 F_{i+1}} \leqslant (1+\psi)\|x-y\|_2,$$

and by Lemma 4.1(iii),

$$\frac{1-\psi}{1+\gamma}\|x-y\|_2 \leqslant \|\Phi_{2i}x - \Phi_{2i}y\|_X \leqslant (1+\psi)\|x-y\|_2. \tag{4.2}$$

Case 3. — x and y are not in the same annulus of the form $\mathcal{A}_{2i,2i+3}$. Obviously, it suffices to consider the case $\|y\| \leqslant \|x\|$. Let R_{2i} be the smallest "even" R such that $\|y\| \leqslant R_{2i}$. Then necessarily $R_{2i+1} \leqslant \|x\|$, for otherwise x and y would both be in $\mathcal{A}_{2i-2,2i+1}$. Applying condition (3) for choosing R_{2i+1}, one obtains that in this case $\|y\| \leqslant \frac{\varepsilon}{d}\|x\|$, and

$$\left(1-\frac{\varepsilon}{d}\right)\|x\| \leqslant \|x\| - \|y\| \leqslant \|x-y\| \leqslant \|x\| + \|y\| \leqslant \left(1+\frac{\varepsilon}{d}\right)\|x\|. \tag{4.3}$$

We recall the fact that $\mathcal{T}_j$ are norm-preserving. Together with inequalities for $\mathcal{J}_{j,j+1}$ in Lemma 4.1, it implies

$$\begin{aligned}\left(\frac{1}{(1+\zeta)(1+\gamma)^2} - \frac{\varepsilon}{d}\right)\|x\| \\ &\leqslant \frac{1}{(1+\zeta)(1+\gamma)^2}\|x\| - \|y\| \leqslant \|\Phi x\| - \|\Phi y\| \\ &\leqslant \|\Phi x - \Phi y\| \leqslant \|\Phi x\| + \|\Phi y\| \\ &\leqslant \|x\| + \|y\| \leqslant \left(1+\frac{\varepsilon}{d}\right)\|x\|.\end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4), we get

$$\begin{aligned}\frac{1}{1+\frac{\varepsilon}{d}}\left(\frac{1}{(1+\zeta)(1+\gamma)^2} - \frac{\varepsilon}{d}\right)\|x-y\| &\leqslant \|\Phi x - \Phi y\| \\ &\leqslant \frac{1+\frac{\varepsilon}{d}}{1-\frac{\varepsilon}{d}}\|x-y\|.\end{aligned} \tag{4.5}$$

The conclusion that Φ is a $(1+\varepsilon)$-bilipschitz embedding of $\mathcal{M}$ into X now follows from the choice of γ, ψ, ζ, and d made at the beginning of Section 4. □

To complete the picture, we need to prove Lemma 4.1. This is done in the remaining part of Section 4.

Proof of Lemma 4.1. — Let $\zeta, \gamma, \{\gamma_i\}_{i=1}^{\infty} \in (0,1)$ be the numbers, and $\{F_i\}_{i=1}^{\infty}$ be the subspaces of ℓ_2 introduced at the beginning of Section 4, and, as before, $n_i = \dim F_i$, $i \in \mathbb{N}$.

By Theorem 2.2 with $\varepsilon = \zeta$ and $A = 1$, for each n_i there exists $N_i \in \mathbb{N}$ such that any direct sum $\ell_2^{N_i} \oplus \ell_2^{N_i}$ with direct sum projections of norm 1 contains a ζ-invariant sub-sum $\ell_2^{n_i} \oplus \ell_2^{n_i}$.

In what follows, our construction of FDD uses the Mazur method for constructing basic sequences [20, p. 4]. To implement it, we need the definition below.

Definition 4.2. — *Let* $\Omega \in (0,1]$. *A subspace* $\mathcal{N} \subset X^*$ *is called* Ω-norming over a subspace $Y \subset X$ *if*

$$\forall y \in Y \ \ \sup\{f^*(y) \ : \ f^* \in \mathcal{N}, \ \|f^*\| \leqslant 1\} \geqslant \Omega\|y\|.$$

Let $N \in \mathbb{N}$ and let $\eta > 0$. Denote by $K(N,\eta) \in \mathbb{N}$ the least number for which the unit sphere of any N-dimensional normed space contains a η-net of cardinality at most $K(N,\eta)$. It is well known that such $K(N,\eta)$ exists (see, for example, [37, Lemma 9.18]).

Since X is infinite-dimensional, by the Dvoretzky Theorem, there is a subspace $U_1 \subset X$ with

$$\dim U_1 = N_1 + K(N_1, \gamma_1/(1+\gamma_1)) \ \text{ and } \ d_{\mathrm{BM}}(U_1, \ell_2^{\dim U_1}) \leqslant (1+\gamma_1).$$

We pick a finite-dimensional subspace $\mathcal{N}_1 \subset X^*$ which is $\frac{1}{1+\gamma_1}$-norming over U_1 (see, for example, [32, Lemma 4.2] for the proof of existence of such subspace).

Using the Dvoretzky Theorem again, we find a subspace $U_2 \subset (\mathcal{N}_1)_\top := \{x \in X : \ x^*(x) = 0 \ \forall x^* \in \mathcal{N}_1\}$ such that

$$\dim U_2 = N_1 + N_2 + K(N_2, \gamma_2/(1+\gamma_2))$$

and $d_{\mathrm{BM}}(U_2, \ell_2^{\dim U_2}) \leqslant (1+\gamma_2)$. Next, we pick a finite-dimensional subspace $\mathcal{N}_2 \subset X^*$ which is $\frac{1}{1+\gamma_2}$-norming over $\mathrm{lin}(U_1 \cup U_2)$. Proceeding like this, in Step k we apply the Dvoretzky Theorem to find a subspace $U_k \subset (\mathcal{N}_{k-1})_\top$ such that

$$\dim U_k = N_{k-1} + N_k + K(N_k, \gamma_k/(1+\gamma_k))$$

and $d_{\mathrm{BM}}(U_k, \ell_2^{\dim U_k}) \leqslant (1+\gamma_k)$. Next, we pick a finite-dimensional subspace $\mathcal{N}_k \subset X^*$ which is $\frac{1}{1+\gamma_k}$-norming over $\mathrm{lin}\left(\bigcup_{i=1}^k U_i\right)$, and so on.

The fact that the sequence $\{U_i\}_{i=1}^{\infty}$ forms an FDD of its closed linear span can be derived from the following lemma.

Lemma 4.3. — *Let $\mathcal{F}$ be a subspace in X. If a subspace $\mathcal{N} \subset X^*$ is Ω-norming over $\mathcal{F}$, and $\mathcal{E}$ is a subspace of $\mathcal{N}_\top$, then the projection $P : \mathcal{E} \oplus \mathcal{F} \to \mathcal{F}$ given by $P(e + f) = f$, where $e \in \mathcal{E}$, $f \in \mathcal{F}$ satisfies $\|P\| \leqslant 1/\Omega$.*

Proof. — We need to show that $\|f\| \leqslant \frac{1}{\Omega}\|e + f\|$. Let $\varepsilon > 0$ be arbitrary and $f^* \in \mathcal{N}$ be such that $\|f^*\| = 1$ and $f^*(f) \geqslant (\Omega - \varepsilon)\,\|f\|$. Since $e \in \mathcal{N}_\top$, one has

$$\|e + f\| \geqslant f^*(e + f) \geqslant (\Omega - \varepsilon)\,\|f\|.$$

Since $\varepsilon > 0$ is arbitrary, the proof is completed. □

Lemma 4.3 with $\mathcal{F} = \mathrm{lin}\{U_i\}_{i=1}^k$ and $\mathcal{E} = \mathrm{lin}\{U_i\}_{i=k+1}^\infty$ implies that for any $k \in \mathbb{N}$ the projection P_k of $\mathrm{lin}\{U_i\}_{i=1}^\infty$ onto $\mathcal{F}$ containing $\mathcal{E}$ in the kernel has norm $\leqslant (1 + \gamma_k)$. Now, the standard argument [20, p. 47] implies that $\{U_i\}_{i=1}^\infty$ forms an FDD of its linear span. It also follows that for all $i \geqslant 1$ the projections $U_i \oplus U_{i+1} \to U_i$ given by $(x_1, x_2) \to x_1$, have norm $\leqslant (1 + \gamma_i)$.

Further, we define subspaces $\{W_i\}_{i=1}^\infty$ as follows. The subspace W_{2i}, $i \in \mathbb{N}$, is picked as an arbitrary N_i-dimensional subspace of U_{i+1}.

Before defining $\{W_{2k-1}\}_{k=1}^\infty$, we endow each U_i with a Euclidean inner product and norm from a Euclidean space $\widetilde{U}_i$ on which the Banach–Mazur distance $d_{\mathrm{BM}}(U_i, \ell_2^{\dim U_i})$ is attained.

We denote this norm on U_i by $\|\cdot\|_\sim^i$ and assume that it satisfies the condition

$$\forall x \in U_i \quad \frac{1}{1 + \gamma_i}\|x\|_\sim^i \leqslant \|x\|_X \leqslant \|x\|_\sim^i. \tag{4.6}$$

Let G_{2i+1} be the orthogonal complement of W_{2i} in U_{i+1} endowed with the inner product of $\widetilde{U}_{i+1}$, and G_1 be U_1. As such, G_i is defined for odd i only.

We say that a set $\mathcal{D}$ is *η-dense* ($\eta > 0$) in a metric space $\mathcal{M}$ if, for every $m \in \mathcal{M}$, there is $x \in \mathcal{D}$ such that $\|m - x\| < \eta$.

By the definition of $K(N, \eta)$, there is a $\gamma_i/(1 + \gamma_i)$-dense set $\mathcal{D}_i$ of cardinality $K(N_i, \gamma_i/(1 + \gamma_i))$ in the unit sphere $S(W_{2i})$. For each $w \in \mathcal{D}_i$, consider a supporting functional $w_w^* \in X^*$ such that $w_w^*(w) = \|w_w^*\| = 1$. The choice of dimension of G_{2i-1} is such that the intersection

$$G_{2i-1} \bigcap \left(\bigcap_{w \in \mathcal{D}_i} \ker w_w^* \right) \tag{4.7}$$

has dimension at least N_i (it can be more because some of the supporting functionals can be linearly dependent). We pick in the intersection (4.7) a subspace of dimension N_i and denote it W_{2i-1}.

The verification that the functionals $\{w_w^*\}_{w\in\mathcal{D}_i}$ span a subspace which is $\left(1-\frac{\gamma_i}{1+\gamma_i}\right)=\frac{1}{1+\gamma_i}$-norming over W_{2i} is immediate. Applying Lemma 4.3 again, with $\mathcal{F}=W_{2i}$ and $\mathcal{E}=W_{2i-1}$, we obtain that the projections $W_{2i-1}\oplus W_{2i}\to W_{2i}$ given by $(x_1,x_2)\to x_2$, have norm $\leqslant(1+\gamma_i)$. Therefore, we conclude that the norms of both direct sum projections in the direct sum $W_{2i-1}\oplus W_{2i}$ $(i\in\mathbb{N})$ do not exceed $(1+\gamma_i)$; recall that the bound on the direct sum projection $W_{2i-1}\oplus W_{2i}\to W_{2i-1}$ follows from the bound on the projection $U_i\oplus U_{i+1}\to U_i$.

The fact that $\{W_i\}_{i=1}^\infty$ forms an FDD in its closed linear span follows from the criterion in [20, p. 47].

Finally, we prove the next auxiliary result.

Lemma 4.4. — *Let $\|\cdot\|_N^i$ be the following norm on $W_{2i-1}\oplus W_{2i}$:*

$$\|(x_1,x_2)\|_N^i=\max\{\|x_1\|_\sim^i,\|x_2\|_\sim^{i+1},\ \|(x_1,x_2)\|\},$$

where $\|(x_1,x_2)\|$ means $\|x_1+x_2\|_X$ and $\|\cdot\|_\sim^i$ is the introduced above norm on U_i. Then, the space $\left(W_{2i-1}\oplus W_{2i},\|\cdot\|_N^i\right)$ has the direct sum projections of norm 1 and

$$\|(x_1,x_2)\|\leqslant\|(x_1,x_2)\|_N^i\leqslant(1+\gamma_i)^2\|(x_1,x_2)\|.$$

Proof. — The statement about the norms of projections is immediate from the definition.

Let $(x_1,x_2)\in W_{2i-1}\oplus W_{2i}$. Recall that from the norms of the direct sum projections (as linear maps between subspaces of X with the induced norm) we have

$$\|x_1\|_X\leqslant(1+\gamma_i)\|(x_1,x_2)\|,\qquad \|x_2\|_X\leqslant(1+\gamma_i)\|(x_1,x_2)\|.$$

From the construction of $\{U_i\}_{i=1}^\infty$ we have

$$\frac{1}{1+\gamma_i}\|x_1\|_\sim^i\leqslant\|x_1\|_X\leqslant\|x_1\|_\sim^i,$$

and

$$\frac{1}{1+\gamma_i}\|x_2\|_\sim^{i+1}\leqslant\frac{1}{1+\gamma_{i+1}}\|x_2\|_\sim^{i+1}\leqslant\|x_2\|_X\leqslant\|x_2\|_\sim^{i+1}.$$

Therefore,

$$\begin{aligned}\|(x_1,x_2)\|&\leqslant\|(x_1,x_2)\|_N^i\\&=\max\{\|x_1\|_\sim^i,\|x_2\|_\sim^{i+1},\|(x_1,x_2)\|\}\\&\leqslant\max\{(1+\gamma_i)\|x_1\|_X,(1+\gamma_i)\|x_2\|_X,\|(x_1,x_2)\|\}.\end{aligned}$$

Hence,

$$\|(x_1,x_2)\|\leqslant\|(x_1,x_2)\|_N^i\leqslant(1+\gamma_i)^2\|(x_1,x_2)\|.$$ □

Next, we apply Theorem 2.2 to each of the sums $(W_{2i-1}\oplus W_{2i}, \|\cdot\|_N^i)$ for all $i\in\mathbb{N}$. We find subspaces of dimension n_i, which we denote $F_i'\subset W_{2i-1}$ and $F_i''\subset W_{2i}$ such that the norm $\|\cdot\|_N^i$ restricted to $F_i'\oplus F_i''$ is ζ-invariant.

Applying Lemma 2.3 we obtain that the norm

$$|||y_1+y_2|||^i := \sup_{O_1,O_2 \text{ orthogonal on } F_i',F_i''} \|\,(O_1y_1,O_2y_2)\,\|_N^i, \quad y_1\in F_i', y_2\in F_i''$$

on $F_i'\oplus F_i''$ satisfies

$$\|\,(y_1,y_2)\,\|_N^i \leqslant |||y_1+y_2|||^i \leqslant (1+\zeta)\|\,(y_1,y_2)\,\|_N^i \tag{4.8}$$

and

$$|||O_1y_1+O_2y_2|||^i = |||y_1+y_2|||^i$$

for every orthogonal operators O_1 on F_i' and O_2 on F_i''.

By Lemma 2.4 there exists a 1-unconditional norm $\|\cdot\|_{Z_i}$ on $\mathbb{R}^2$ such that for any $y_1\in F_i'$ and $y_2\in F_i''$

$$|||y_1+y_2|||^i = \|(\|y_1\|_\sim^i, \|y_2\|_\sim^{i+1})\|_{Z_i}.$$

For $i\in\mathbb{N}$ define V_{2i-1} as the subspace of X that coincides with F_i' and V_{2i} as the subspace of X that coincides with F_i''.

We choose isometries $I_i':F_i\to F_i'$ and $I_i'':F_i\to F_i''$ and define $J_{2i-1}: F_i\to V_{2i-1}$ and $J_{2i}:F_i\to V_{2i}$ as compositions of these isometries and the mentioned above natural maps of F_i' onto V_{2i-1} and F_i'' onto V_{2i}.

For $v\in F_i\subset \ell_2$ we have

$$\|v\|_2 = \|J_{2i-1}v\|_\sim^i \leqslant (1+\gamma_i)\,\|J_{2i-1}v\|_X \leqslant (1+\gamma_i)\,\|J_{2i-1}v\|_\sim^i = (1+\gamma_i)\,\|v\|_2\,,$$

and

$$\begin{aligned}\|v\|_2 = \|J_{2i}v\|_\sim^{i+1} &\leqslant (1+\gamma_{i+1})\,\|J_{2i}v\|_X \\ &\leqslant (1+\gamma_{i+1})\,\|J_{2i}v\|_\sim^{i+1} = (1+\gamma_{i+1})\,\|v\|_2\,.\end{aligned}$$

This is property (i) in the Lemma 4.1.

To prove property (ii), note that for $(u,v)\in F_i'\oplus F_i''$, by (4.8) we have

$$\|(J_{2i-1}u, J_{2i}v)\|_N^i \leqslant |||J_{2i-1}u+J_{2i}v|||^i \leqslant (1+\zeta)\,\|(J_{2i-1}u,J_{2i}v)\|_N^i\,.$$

Using the inequality on the left and Lemma 4.4 , we get

$$\begin{aligned}\|J_{2i-1}u+J_{2i}v\|_X &\leqslant \|(J_{2i-1}u,J_{2i}v)\|_N^i \leqslant |||J_{2i-1}u+J_{2i}v|||^i \\ &= \left\|\left(\|J_{2i-1}u\|_\sim^i\,, \|J_{2i}v\|_\sim^{i+1}\right)\right\|_{Z_i} = \|(\|u\|_2\,,\|v\|_2)\|_{Z_i}\,,\end{aligned}$$

which is the inequality on the left in (ii).

On the other hand, we have

$$\begin{aligned}\left\|(\|u\|_2\,,\|v\|_2)\right\|_{Z_i} &= \left\|\left(\|J_{2i-1}u\|_\sim^i\,,\|J_{2i}v\|_\sim^{i+1}\right)\right\|_{Z_i} = |||J_{2i-1}u+J_{2i}v|||^i\\ &\leqslant (1+\zeta)\,\|(J_{2i-1}u,J_{2i}v)\|_N^i\\ &\leqslant (1+\zeta)(1+\gamma_i)^2\,\|J_{2i-1}u+J_{2i}v\|_X\,,\end{aligned}$$

the last inequality being a consequence of Lemma 4.4.

To prove property (iii) note that because V_{2i} and V_{2i+1} are orthogonal subspaces of the Euclidean space $\widetilde{U}_{i+1}$ (i.e. the space U_{i+1} endowed with the norm $\|\cdot\|_\sim^{i+1}$), for $u\in F_i$ and $v\in F_{i+1}$ we have

$$\begin{aligned}\|J_{2i}u+J_{2i+1}v\|_X &\overset{(4.6)}{\leqslant} \|J_{2i}u+J_{2i+1}v\|_\sim^{i+1}\\ &= \left(\left(\|J_{2i}u\|_\sim^{i+1}\right)^2+\left(\|J_{2i+1}v\|_\sim^{i+1}\right)^2\right)^{\frac12}\\ &= \left(\|u\|_2^2+\|v\|_2^2\right)^{\frac12}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left(\|u\|_2^2+\|v\|_2^2\right)^{\frac12} &= \left(\left(\|J_{2i}u\|_\sim^{i+1}\right)^2+\left(\|J_{2i+1}v\|_\sim^{i+1}\right)^2\right)^{\frac12}\\ &= \|J_{2i}u+J_{2i+1}v\|_\sim^{i+1}\\ &\leqslant (1+\gamma_{i+1})\,\|J_{2i}u+J_{2i+1}v\|_X\,.\end{aligned}$$

Therefore property (iii) also holds.

In conclusion, $\{V_i\}_{i=1}^\infty$ forms an FDD of its closed linear span, satisfying all conditions of Lemma 4.1. □

5. A counterexample to the general bending problem

It would be very interesting to prove analogues of our main result, Theorem 1.2, for spaces which are different from the Hilbert space. To state some relevant problems, we recall that a Banach space W is said to be *finitely represented* in a Banach space X if, for every $\varepsilon>0$ and every finite-dimensional subspace F in W, there is a finite-dimensional subspace G in X such that $\dim G=\dim F$ and $d_{\mathrm{BM}}(F,G)\leqslant 1+\varepsilon$.

The first question of interest is the following

Problem 5.1. — *Let $\mathcal{M}$ be a locally finite subset of an infinite-dimensional Banach space W and assume that W is finitely represented in a*

Banach space X. Does it imply that, for every $\varepsilon > 0$, the space $\mathcal{M}$ admits a $(1+\varepsilon)$-bilipschitz embedding into X?

To pave a way towards solving this problem, it is desirable to obtain an affirmative answer to the problem below. Notice that its formulation uses Definition 3.1.

General Bending Problem. — *Let X and Y be finite-dimensional Banach spaces such that there exist two linear isometric embeddings $I_1 : Y \to X$ and $I_2 : Y \to X$ with distinct images, $Y_1 = I_1(Y)$ and $Y_2 = I_2(Y)$. Assume that X is the direct sum of Y_1 and Y_2 and that the direct sum projections of $X = Y_1 \oplus Y_2$ have norm 1. Does it imply that for every $\varepsilon > 0$ there exist (r, R) with $0 < r < R < \infty$ for which there exists a $(1+\varepsilon)$-bending of Y in the space X from I_1 to I_2 with parameters (r, R)?*

However, as the following theorem shows, the answer to this problem is negative even in the case where Y is a two-dimensional Euclidean space. Thence, the General Bending Problem as stated above is excessively strong, one should look for weaker statements which might be true. Also, perhaps suitable developments of Theorem 5.2 can be used to obtain the affirmative answer to the question of Problem 1.7.

Theorem 5.2. — *There exists a 4-dimensional Banach space X satisfying the conditions:*

(1) *It is a direct sum of two 2-dimensional Euclidean spaces Y_1 and Y_2 with direct sum projections having norm 1.*
(2) *There exists $\varepsilon > 0$ such that for any (r, R) satisfying $0 < r < R < \infty$ and any isometric embeddings $I_1 : \ell_2^2 \to Y_1$ and $I_2 : \ell_2^2 \to Y_2$, there is no $(1+\varepsilon)$-bending with parameters (r, R) of ℓ_2^2 in X from I_1 to I_2.*

Recall that, for a Banach space X, $S(X)$ denotes the unit sphere in X. The *spherical opening* between subspaces U and W of a Banach space X is defined as:

$$\Omega(U, W) = \max\left\{\sup_{u\in S(U)} \operatorname{dist}(u, S(W)), \sup_{w\in S(W)} \operatorname{dist}(w, S(U))\right\}.$$

It is easy to see that Ω is a metric on the set of all closed subspaces of a Banach space, and that this metric space is compact if the Banach space is finite-dimensional. We refer to [34, Section 3.12] for more properties of this metric.

Lemma 5.3. — *Let Y_1 and Y_2 be 2-dimensional Euclidean spaces and let $\delta > 0$. There exists a norm on $Y_1 \oplus Y_2$ such that the obtained normed space $(X, \|\cdot\|_X)$ satisfies the conditions:*

(1) *On each of the summands Y_1 and Y_2 the norm is isometrically equivalent to its original norm, the ℓ_2^2 norm.*
(2) *The projection onto any of the summands Y_1 or Y_2, whose kernel equals the other summand, has norm 1.*
(3) *For every sufficiently small $\gamma > 0$, there exists $\varepsilon(\gamma) > 0$ such that every two-dimensional subspace Z of X satisfying $\Omega(Z, Y_1) \geqslant \gamma$ and $\Omega(Z, Y_2) \geqslant \gamma$, satisfies $d_{\mathrm{BM}}(Z, \ell_2^2) \geqslant 1 + \varepsilon(\gamma)$, where d_{BM} is the Banach–Mazur distance.*
(4) *The norm of X is not far from the norm of $Y_1 \oplus_2 Y_2$ denoted by $\|\cdot\|_2$. Namely,*

$$\forall x \in X \quad (1 - \delta^2/2)\|x\|_X \leqslant \|x\|_2 \leqslant \|x\|_X. \tag{5.1}$$

Proof. — The main idea of our proof of Lemma 5.3 is to construct the unit ball of X as the result of cutting from the unit ball of the Euclidean space $Y_1 \oplus_2 Y_2$ some collection of symmetric pairs of caps. By *cap* centered at a unit vector w in $\mathbb{R}^4$ we mean the region of the unit ball in $\mathbb{R}^4$ separated by a hyperplane orthogonal to the line spanned by w. The *radius* of the cap is the chordal (Euclidean) distance from w to the 2-dimensional sphere that is the intersection of the hyperplane and $S(\mathbb{R}^4)$. In our construction, these radii will be small enough to satisfy inequality (5.1). In constructing the unit ball of X, sufficiently many caps will be removed so that each two-dimensional subspace G of $Y_1 \oplus_2 Y_2$, except Y_1 and Y_2, intersects the interior of at least one of the caps and, therefore, the norm of X on G will not be strictly convex; consequently G is not isometric to ℓ_2^2.

It is clear that each space X constructed as described above satisfies the conditions of items (1), (2), and (4).

Now we prove that the condition in item (3) holds. Let us assume the contrary. Then, for every $k \in \mathbb{N}$, there exists Z_k satisfying $\Omega(Z_k, Y_1) \geqslant \gamma$, $\Omega(Z_k, Y_2) \geqslant \gamma$, and $d_{\mathrm{BM}}(Z_k, \ell_2^2) < 1 + \frac{1}{k}$. Since the set of all subspaces of X is compact with respect to the metric Ω, the sequence $\{Z_k\}_{k=1}^{\infty}$ has an Ω-convergent subsequence. Let W be its limit. The fact that for finite-dimensional spaces d_{BM} is continuous with respect to Ω implies that $d_{\mathrm{BM}}(W, \ell_2^2) = 1$, and, thereupon, W is isometric to ℓ_2^2.

On the other hand, both $\Omega(W, Y_1) \geqslant \gamma$ and $\Omega(W, Y_2) \geqslant \gamma$, whence W is not the same as Y_1 or Y_2, and hence its unit sphere contains line segments. This outcome contradicts the conclusion of the previous paragraph.

We now give details on how the removed caps are to be selected. Denote by $G_2(\mathbb{R}^4)$ the set of all two-dimensional subspaces of $\mathbb{R}^4$. It is a compact space in the metric Ω. Let $\delta \in (0, \frac{1}{4})$. Using the standard approach, we find in $G_2(\mathbb{R}^4)$ a finite subset Δ such that

(1) $Y_1, Y_2 \in \Delta$,
(2) $\forall W_1, W_2 \in \Delta,\ W_1 \neq W_2,\ \Omega(W_1, W_2) \geqslant \delta$,
(3) $\forall L \in G_2(\mathbb{R}^4),\ \exists W \in \Delta,\ \Omega(W, L) < \delta$.

For each $W \in \Delta$ other than Y_1 or Y_2, we select a point $w \in S(W)$ which is at distance at least δ to both $S(Y_1)$ and $S(Y_2)$. We then cut from the unit ball of $\mathbb{R}^4$ two 0-symmetric caps of radius δ at w and $-w$. It is clear that in such a way we cut finitely many caps and that "under" any of the caps the resulting surface will be polyhedral.

Observe that the existence of w is guaranteed for every $W \in \Delta$ except Y_1 and Y_2. In fact, it is immediate that there are w_1 and w_2 in $S(W)$ with both $\operatorname{dist}(w_1, S(Y_1)) \geqslant \delta$ and $\operatorname{dist}(w_2, S(Y_2)) \geqslant \delta$. If neither w_1 nor w_2 works, meaning that both $\operatorname{dist}(w_1, S(Y_2)) < \delta$ and $\operatorname{dist}(w_2, S(Y_1)) < \delta$, then $\operatorname{dist}(w_1, S(Y_1)) > \sqrt{2} - \frac{1}{4}$ and $\operatorname{dist}(w_2, S(Y_2)) > \sqrt{2} - \frac{1}{4}$. As a consequence, moving along the sphere $S(W)$ from w_1 to w_2 we arrive at the desired point.

We are "almost" done because, for every $L \in G_2(\mathbb{R}^4)$, there is $W \in \Delta$ such that $\Omega(W, L) < \delta$. If $W \neq Y_1, Y_2$, we are done because the cap which we cut around the point $w \in S(W)$ will cut some piece under $S(L)$. The only subspaces L which are not covered by this reasoning are those that are in the set

$$\Psi := \left\{ L : \min_{W \in \Delta, W \neq Y_1, Y_2} \Omega(L, W) \geqslant \delta \right\}.$$

This is a compact set. For this reason the function

$$\omega(L) := \min\{\Omega(L, Y_1), \Omega(L, Y_2)\}$$

attains its maximum on Ψ, and this maximum μ satisfies $\mu < \delta$.

Consider an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ in $Y_1 \oplus_2 Y_2 = \mathbb{R}^4$ such that $Y_1 = \operatorname{lin}(\{e_1, e_2\})$ and $Y_2 = \operatorname{lin}(\{e_3, e_4\})$. Choose $a > 0$ in such a way that for the unit vector $f = \frac{1}{\sqrt{1+a^2}} e_1 + \frac{a}{\sqrt{1+a^2}} e_3$ we have $\|e_1 - f\| = \delta$. Specifically, this condition means that $\frac{1}{\sqrt{1+a^2}} = 1 - \frac{\delta^2}{2}$.

Let $\sigma = \frac{1}{\sqrt{1+a^2}}$ and $\tau = \frac{a}{\sqrt{1+a^2}}$. We remove 16 caps of radius δ, tangent to $S(Y_1)$, centered at the points with position vectors $(\pm\sigma e_1 \pm \tau e_3)$, $(\pm\sigma e_1 \pm \tau e_4)$, and $(\pm\sigma e_2 \pm \tau e_3)$, $(\pm\sigma e_2 \pm \tau e_4)$. Similarly, we remove the 16 caps of radius δ, tangent to $S(Y_2)$, centered at the points with position vectors $(\pm\sigma e_3 \pm \tau e_1)$, $(\pm\sigma e_3 \pm \tau e_2)$, and $(\pm\sigma e_4 \pm \tau e_1)$, $(\pm\sigma e_4 \pm \tau e_2)$.

We now prove that for each $L \in \Psi$ there will be some part cut out of $S(L)$ by some of the caps described above.

Let us choose $L \in \Psi$ and since one (and only one) of the conditions $\Omega(L, Y_1) < \delta$ or $\Omega(L, Y_2) < \delta$ holds we can assume that $\Omega(L, Y_1) < \delta$. First, we argue that $S(L)$ intersects the hyperplane $\operatorname{lin}(\{e_1, e_3, e_4\})$ at unique point of position vector l so that $\|e_1 - l\| < \delta$. Note that L cannot be a subspace of $\operatorname{lin}(\{e_1, e_3, e_4\})$ since that would imply $\Omega(L, Y_1) = \sqrt{2}$.

Since $S(L)$ is symmetric about the origin, if $(x_1, x_2, x_3, x_4) \in S(L)$, then so is its opposite, and because the coordinate functions are continuous we necessarily have two diametrically opposite points with the coordinate $x_2 = 0$ (there are only two such points, for otherwise $L \subset \operatorname{lin}(\{e_1, e_3, e_4\})$). Let $\pm l$ be the position vectors of the two points $\pm(x_1, 0, x_3, x_4) \in S(L)$. Since $\Omega(L, Y_1) < \delta$ we have that $\operatorname{dist}(l, S(Y_1)) < \delta$ and, therefore,

$$\min_t \left\{(x_1 - \cos t)^2 + (0 - \sin t)^2 + x_3^2 + x_4^2\right\} < \delta^2,$$

i.e. $\min_t \{2 - 2x_1 \cos t\} < \delta^2$. Note that $\operatorname{dist}(l, S(Y_1))$ is achieved when $x_1 \cos t = |x_1|$ and without loss of generality we will assume that $x_1 > 0$ and therefore $t = 0$, i.e. the vector on $S(Y_1)$ closest to l is e_1. Moreover, we may assume without loss of generality that $l = \frac{1}{\sqrt{1+b^2+c^2}}(e_1 + be_3 + ce_4)$ for coefficients $b \geqslant c \geqslant 0$ where at least b is positive. Indeed, if $b = c = 0$, then $l = e_1$ and in this case we repeat the argument near the vector e_2 where we search for points in $S(L) \cap \operatorname{lin}(\{e_2, e_3, e_4\})$. Again, this intersection consists of a vector and its opposite. This time the vector near e_2 cannot coincide with e_2 for this would imply $L = Y_1$. If this happens, then we swap the labels of e_1 and e_2 and we are in the situation claimed above, with $l \neq e_1$ and $b > 0$.

To show that a nonempty part will be cut out of $S(L)$, we show that l is in the open cap of radius δ centered at $f = \sigma e_1 + \tau e_3$. For this it suffices to show the inequality $\langle f, l\rangle > \langle f, e_1\rangle$ between inner products of unit vectors. It is equivalent to

$$\frac{1 + ab}{\sqrt{1 + b^2 + c^2}} > 1. \tag{5.2}$$

We remark that $\|e_1 - f\| = \delta > \|e_1 - l\|$ is equivalent to $\langle e_1, l\rangle > \langle e_1, f\rangle$ which means

$$\frac{1}{\sqrt{1 + b^2 + c^2}} > \frac{1}{\sqrt{1 + a^2}},$$

and therefore $a > b$.

We thus have

$$(1+ab)^2 > 1+2ab > 1+2b^2 \geqslant 1+b^2+c^2,$$

which implies (5.2).

Deleting these 32 caps together with caps centered at $w \in S(W)$ chosen above from the unit ball of $\mathbb{R}^4$, we get the unit ball of X satisfying all of the conditions of Lemma 5.3. □

Proof of Theorem 5.2. — We are going to prove that there exists $\varepsilon > 0$ such that the space X constructed in Lemma 5.3 does not admit a $(1+\varepsilon)$-bending of $Y = \ell_2^2$ with parameters (r, R) for any $0 < r < R < \infty$.

To prove the statement by contradiction, select

$$\sqrt[6]{2} - 1 > \gamma > 0,$$

so that

$$1 > \frac{(1+\gamma)^3}{\sqrt{2}}.$$

Let $\varepsilon(\gamma)$ be the value given by item (3) in Lemma 5.3. We pick $\varepsilon > 0$ so that

$$\varepsilon < \min\{\gamma, \varepsilon(\gamma)\}.$$

Finally, we choose $\delta > 0$ such that

$$1 - \frac{\delta^2}{2} > \frac{(1+\gamma)^3}{\sqrt{2}}. \tag{5.3}$$

Next, assume that there exists a $(1+\varepsilon)$-bending $T : Y \to X$ with parameters (r, R), $0 < r < R < \infty$. Conforming to the notation above, we write $T = (T_1, T_2)$ meaning

$$T_1 : Y \longrightarrow Y_1 \quad \text{and} \quad T_2 : Y \longrightarrow Y_2.$$

In view of the Rademacher theorem, this map is differentiable almost everywhere. By a standard argument, the derivative $DT(y)$, whenever it exists, is a $(1+\varepsilon)$-bilipschitz linear embedding of Y into X (see [7, Chapter 7, Section 1]).

Remark 5.4. — Our construction of X yields that, for $\varepsilon < \varepsilon(\gamma)$, item (3) in Lemma 5.3 implies that at every point of differentiability $y \in Y$, either

$$\Omega(DT(y)Y, Y_1) < \gamma, \quad \text{or} \quad \Omega(DT(y)Y, Y_2) < \gamma.$$

Indeed, if both $\Omega(DT(y)Y, Y_1)$ and $\Omega(DT(y)Y, Y_2)$ are $\geqslant \gamma$, then Lemma 5.3 item (3) implies that $d_{BM}(DT(y)Y, \ell_2^2) \geqslant 1 + \varepsilon(\gamma)$, which contradicts the fact that T is a $(1+\varepsilon)$-bending of $Y = \ell_2^2$ with $\varepsilon < \varepsilon(\gamma)$.

Let us paint Y in three colors:

- blue for the points where $DT(y)Y$ is close to Y_1,
- yellow for the points where $DT(y)Y$ is close to Y_2,
- red for the points where $DT(y)$ does not exist.

Note that since γ is such that a two-dimensional subspace Z of X cannot have simultaneously $\Omega(Z, Y_1) < \gamma$ and $\Omega(Z, Y_2) < \gamma$, it follows that points of differentiability of T cannot be simultaneously blue and yellow.

We continue by proving the following statement. There exists a line segment in Y such that:

(1) Almost all of its points are either blue or yellow.
(2) The set of points which are blue takes half of its measure.

To prove this statement consider the 0-centered disc of radius r in Y. We fix Cartesian coordinates (x, y) in Y and denote by u the unit vector in the positive y-direction. Consider the set of all vertical (parallel to u) x-axis-symmetric line segments I_x of length $2R+r$, whose intersection with the disc are of length at least r (see Figure 5.1).

The Fubini theorem (e.g. Theorem 14.1 page 147 in [11]) applied to the characteristic function of the set of differentiability points of T in the rectangle $\bigcup_{x\in[-\frac{\sqrt{3}}{2}r, \frac{\sqrt{3}}{2}r]} I_x$ implies that the set of red points on I_x has measure 0 for almost all x.

Also, the intersections of I_x with the blue and yellow sets are measurable for almost all x. Hence, we can pick x for which the "vertical" line segment is blue or yellow almost everywhere and blue-yellow pieces are measurable. Consider a moving subsegment of length $r/2$ along this I_x line segment. We claim that there is a position at which the measure of yellow points on this segment is exactly $r/4$. This can be done as follows. For $0 \leqslant t \leqslant R$, consider a line segment $[t, t + \frac{r}{2}]$ and the integral $F(t) := \int_t^{t+\frac{r}{2}} c(s)\, \mathrm{d}s$, where $c(s) = -1$ if (x, s) is blue and $c(s) = 1$ if (x, s) is yellow. Then $F(t)$ is a continuous function which varies from $-r/2$ to $r/2$ as t ranges from 0 to R. This is because for $s \in [0, r/2]$ we have $\|(x, s)\| \leqslant r$ and therefore $DT(x, s)Y = Y_1$ and $c(s) = -1$, while for $s \in [R, R + r/2]$ we have $\|(x, s)\| \geqslant R$ and $DT(x, s)Y = Y_2$ and $c(s) = 1$. Therefore F attains value 0 for some $0 \leqslant t_0 \leqslant R$.

The argument will be completed in the following way. Since T is a Lipschitz function, the norm equivalence (5.1) implies that each one of its four components is also Lipschitz. Since the Fundamental Theorem of Calculus holds for absolutely continuous functions (e.g. Proposition 7.2 in [11]), it holds for Lipschitz functions. We use $[t_0, t_0 + r/2]$ to parameterize the interval above (with the measure of blue set equal to the measure of the

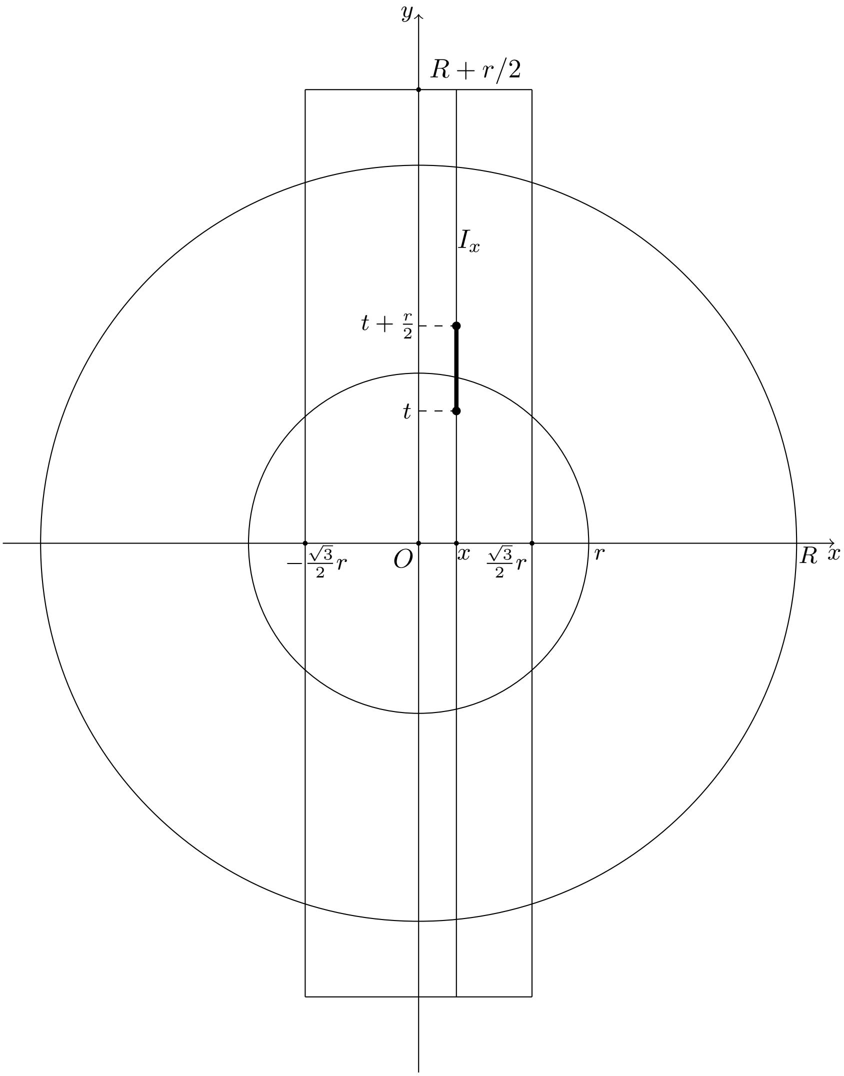

Figure 5.1. Looking for a suitable interval

yellow set equal to $r/4$) as

$$t_0 \leqslant t \leqslant t_0 + r/2 \longrightarrow p(t) = (x, t).$$

Let $a = p(t_0)$ be the bottom endpoint and $b = p(t_0 + r/2)$ be the top endpoint of the interval. Denote by I the set of those $t \in [t_0, t_0 + r/2]$ for which T is differentiable at $p(t)$. I is not necessarily an interval but it has 1-dimensional Lebesgue measure $|I| = r/2$. Applying the Fundamental Theorem of Calculus to T, one obtains:

$$T(b) - T(a) = \int_I DT(p(t))u \, \mathrm{d}t. \tag{5.4}$$

We claim that the X-norm of this integral cannot be $(1+\varepsilon)$-equivalent to $\|b - a\| = r/2$. Splitting the integral as

$$\int_I DT(p(t))u \, \mathrm{d}t = \int_{I_1} DT(p(t))u \, \mathrm{d}t + \int_{I_2} DT(p(t))u \, \mathrm{d}t, \tag{5.5}$$

where on the right-hand side we consider integrals over values $t \in I_1$ for which $p(t)$ is in the blue set and values $t \in I_2$ for which $p(t)$ is in the yellow set. Note that I_1 and I_2 are measurable subsets of I and that $|I_1| = |I_2| = r/4$ by the previous step. Now, we estimate the norm of the integral in (5.5) from above.

With the notation $T = (T_1, T_2)$, one has:

$$DT(p(t))u = DT_1(p(t))u + DT_2(p(t))u \in Y_1 \oplus Y_2.$$

For $t \in I_1$, the definition of I_1 implies that

$$DT(p(t))u \in DT(p(t))Y \quad \text{with} \quad \Omega\left(DT(p(t))Y, Y_1\right) < \gamma.$$

Further, we need the following

Observation 5.5. — *For any vector $y = (y_1, y_2) \in Z$ in some 2-dimensional subspace Z of X for which $\Omega(Z, Y_1) \leqslant \gamma$, it holds $\|y_2\| \leqslant \gamma \|y\|_X$. Similarly if $\Omega(Z, Y_2) \leqslant \gamma$ then $\|y_1\| \leqslant \gamma \|y\|_X$.*

Proof. — Assume that $y = (y_1, y_2) \in Z$, where Z is a 2-dimensional subspace of X such that $\Omega(Z, Y_1) \leqslant \gamma$. This implies that $d_X(y, Y_1) \leqslant \gamma \|y\|_X$. Let w be a vector in Y_1 such that

$$\|y - w\|_X = d_X(y, Y_1).$$

Then,

$$\|y_2\| = \|y - y_1\| \leqslant \|y - w\|_2 \overset{(5.1)}{\leqslant} \|y - w\|_X \leqslant \gamma \|y\|_X . \qquad \square$$

Using this observation, we obtain that for every $t \in I_1$,

$$\|DT_2(p(t))u\| \leqslant \gamma \|DT(p(t))u\|_X \leqslant \gamma(1+\varepsilon).$$

Similarly, for every $t \in I_2$, we have:

$$DT(p(t))u \in DT(p(t))Y \quad \text{with} \quad \Omega\left(DT(p(t))Y, Y_2\right) < \gamma,$$

and hence

$$\|DT_1(p(t))u\| \leqslant \gamma \|D_T(p(t))u\|_X \leqslant \gamma(1+\varepsilon).$$

Re-write (5.4) and (5.5) as

$$\begin{aligned} T(b)-T(a) = &\left(\int_{I_1} DT_1(p(t))u\,\mathrm{d}t + \int_{I_2} DT_1(p(t))u\,\mathrm{d}t\right) \\ &+ \left(\int_{I_1} DT_2(p(t))u\,\mathrm{d}t + \int_{I_2} DT_2(p(t))u\,\mathrm{d}t\right). \end{aligned}$$

The first parentheses contain a vector v_1 in Y_1 with norm bounded by

$$\begin{aligned} \|v_1\| &= \left\|\int_{I_1} DT_1(p(t))u\,\mathrm{d}t + \int_{I_2} DT_1(p(t))u\,\mathrm{d}t\right\| \\ &\leqslant \int_{I_1} \|DT_1(p(t))u\|\ \mathrm{d}t + \int_{I_2} \|DT_1(p(t))u\|\ \mathrm{d}t \\ &\leqslant \int_{I_1} (1+\varepsilon)\,\mathrm{d}t + \int_{I_2} \gamma(1+\varepsilon)\,\mathrm{d}t = (1+\gamma)(1+\varepsilon)\frac{r}{4}. \end{aligned}$$

Similarly, the second parentheses contain a vector v_2 in Y_2 with the same upper bound for the norm.

Therefore,

$$\begin{aligned} \|T(b)-T(a)\|_X = \|v_1+v_2\|_X &\overset{(5.1)}{\leqslant} \frac{1}{1-\delta^2/2}\,\|v_1+v_2\|_2 \\ &\leqslant \frac{1}{1-\delta^2/2}\sqrt{2}(1+\gamma)(1+\varepsilon)\frac{r}{4}, \end{aligned}$$

where the last inequality follows from the Pythagorean Theorem and the estimates on the norms of v_1 and v_2.

Since

$$\frac{1}{1+\varepsilon}\frac{r}{2} = \frac{1}{1+\varepsilon}\,\|b-a\| \leqslant \|T(b)-T(a)\|_X\,,$$

we obtain

$$\frac{1}{1+\varepsilon}\frac{r}{2} \leqslant \frac{1}{1-\delta^2/2}\sqrt{2}(1+\gamma)(1+\varepsilon)\frac{r}{4}.$$

Thus,

$$1-\frac{\delta^2}{2} \leqslant \frac{(1+\gamma)(1+\varepsilon)^2}{\sqrt{2}}.$$

As ε was chosen strictly less than γ, we derive:

$$1-\frac{\delta^2}{2} < \frac{(1+\gamma)^3}{\sqrt{2}}.$$

However, this contradicts (5.3) and, thus, it contradicts the existence of the function T with the required properties. □

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Florin CATRINA
Department of Mathematics
and Computer Science,
St. John's University,
8000 Utopia Parkway,
Queens, NY 11439 (USA)
catrinaf@stjohns.edu

Sofiya OSTROVSKA
Department of Mathematics,
Atilim University,
06830 Incek, Ankara (Turkey)
sofia.ostrovska@atilim.edu.tr

Mikhail I. OSTROVSKII
Department of Mathematics
and Computer Science,
St. John's University,
8000 Utopia Parkway,
Queens, NY 11439 (USA)
ostrovsm@stjohns.edu