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# CRYSTALLINE REPRESENTATIONS AND WACH MODULES IN THE RELATIVE CASE

**by ABHINANDAN**

Abstract. — We study the notion of Wach modules in the relative setting, generalizing the arithmetic case. Over an unramified base, for a $p$-adic representation admitting such structure, we examine the relationship between its relative Wach module and filtered $(\varphi, \partial)$-module. Moreover, we show that such a representation is crystalline (in the sense of Faltings-Brinon), and one can recover its filtered $(\varphi, \partial)$-module from the relative Wach module. Conversely, for low Hodge-Tate weights $[0, p-2]$, we construct relative Wach modules from free relative Fontaine-Laffaille modules (in the sense of Faltings).

Résumé. — Nous étudions la notion de module de Wach dans le cas relatif en généralisant le cas arithmétique. Sur une base non-ramifiée, pour une représentation $p$-adique admettant une telle structure, nous examinons la relation entre son module de Wach relatif et son $(\varphi, \partial)$-module filtré. De plus, nous montrons qu'une telle représentation est cristalline (au sens de Faltings–Brinon) et que l'on récupère son $(\varphi, \partial)$-module filtré à partir du module de Wach relatif. Réciproquement, pour les poids faibles de Hodge–Tate $[0, p-2]$, nous construisons des modules de Wach relatifs à partir de modules libres de Fontaine–Laffaille relatifs (au sens de Faltings).

## 1. Introduction

The theory of Wach modules for $p$-adic crystalline representations of the absolute Galois group of a finite unramified extension of $\mathbb{Q}_p$ was introduced in the paper of Fontaine [21]. This notion was further developed by Wach [37, 38] and Berger [7]. Over the years, this theory has found many applications, for example, to the Iwasawa theory of crystalline representations in [5, 6], and in the study of the $p$-adic local Langlands program [8]. Wach modules were also among one of the motivations for Scholze's idea of

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$q$-deformations [34], which in turn paved the way for the theory of prisms and prismatic cohomology of Bhatt and Scholze developed in [11].

Our goal in this article is to upgrade the notion of Wach modules to the relative case by which we mean certain étale algebras over a formal torus (see Section 1.4 for precise setup). But before examining the relative case, let us recall the relation between Wach modules and crystalline representations in the arithmetic case.

## 1.1. The arithmetic case

Let $p$ be a fixed prime number and let $\kappa$ denote a finite field of characteristic $p$; set $O_F = W(\kappa)$ to be the ring of $p$-typical Witt vectors with coefficients in $\kappa$ and $F = \mathrm{Fr}\,(O_F)$. Let $\overline{F}$ denote a fixed algebraic closure of $F$, $\mathbb{C}_p := \widehat{\overline{F}}$ the $p$-adic completion, and $G_F = \mathrm{Gal}(\overline{F}/F)$ the absolute Galois group of $F$. Further, let $F_\infty = \bigcup_n F(\mu_{p^n})$ with $\Gamma_F := \mathrm{Gal}(F_\infty/F)$ and $H_F := \mathrm{Gal}(\overline{F}/F_\infty)$. Finally, let $\mathbb{C}_p^\flat$ denote the tilt of $\mathbb{C}_p$.

### 1.1.1. $(\varphi, \Gamma_F)$-modules

Using a certain period ring $\mathbf{A} \subset W(\mathbb{C}_p^\flat)$ stable under the Frobenius on Witt vectors and the $G_F$-action (see Section 3.1 for precise definition), Fontaine functorially attached to any $\mathbb{Z}_p$-representation $T$ of $G_F$ (i.e. finitely generated $\mathbb{Z}_p$-modules equipped with a linear and continuous $G_F$-action), the module $\mathbf{D}(T) = (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_F}$ over the two dimensional local ring $\mathbf{A}_F = \mathbf{A}^{H_F}$. The module $\mathbf{D}(T)$ is equipped with a (induced from $\mathbf{A}$) Frobenius-semilinear operator $\varphi$ such that the image of $\varphi$ generates $\mathbf{D}(T)$, i.e. $\mathbf{D}(T)$ is étale. Moreover, $\mathbf{D}(T)$ is equipped with a continuous and semilinear action of $\Gamma_F$ and if $T$ is free the $\mathbf{A}_F$-rank of $\mathbf{D}(T)$ equals the $\mathbb{Z}_p$-rank of $T$. In [21] Fontaine estalished an equivalence of categories between $\mathbb{Z}_p$-representations of $G_F$ and étale $(\varphi, \Gamma_F)$-modules over $\mathbf{A}_F$. Furthermore, this construction naturally extends to $p$-adic representations of $G_F$. Namely, using the period ring $\mathbf{B} = \mathbf{A}\left[\frac{1}{p}\right]$, Fontaine functorially attached to any $p$-adic representation $V$ of $G_F$ an étale $(\varphi, \Gamma_F)$-module $\mathbf{D}(V) = (\mathbf{B} \otimes_{\mathbb{Q}_p} V)^{H_F}$ over $\mathbf{B}_F = \mathbf{B}^{H_F}$ (i.e. there exists a $\mathbb{Z}_p$-lattice $T \subset V$ such that $\mathbf{D}(T)$ is an étale $(\varphi, \Gamma_F)$-module over $\mathbf{A}_F$). Moreover, he showed that this induces an equivalence between $p$-adic representations of $G_F$ and étale $(\varphi, \Gamma_F)$-modules over $\mathbf{B}_F$.

#### 1.1.2. Crystalline representations of $G_F$

Using another period ring $\mathbf{B}_{\mathrm{cris}}$ also equipped with a Frobenius and continuous $G_F$-action (see Section 2.2 for precise definition), Fontaine functorially attached to any $p$-adic representation $V$ of $G_F$ an $F$-vector space $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_F}$. The $F$-vector space $\mathbf{D}_{\mathrm{cris}}(V)$ is a filtered $\varphi$-module, i.e. it is equipped with a (induced from $\mathbf{B}_{\mathrm{cris}}$) Frobenius-semilinear operator $\varphi$ and a filtration. In case $\dim_F \mathbf{D}_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p} V$, such a representation is said to be crystalline (the terminology *crystalline* comes from the fact that for a smooth proper scheme $X/O_F$ and $i \in \mathbb{N}$ the $p$-adic étale cohomology group of the generic fiber $V_i = H^i_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)$ is crystalline as a $G_F$-representation and the crystalline cohomology group of the special fiber $H^i_{\mathrm{cris}}(X_\kappa/F)$ is naturally isomorphic to $\mathbf{D}_{\mathrm{cris}}(V_i)$). Restricting the functor $\mathbf{D}_{\mathrm{cris}}$ to the subcategory of crystalline representations, in [20] Fontaine observed that the associated filtered $\varphi$-modules are weakly admissible (a property relating the endomorphism $\varphi$ and filtration on $\mathbf{D}_{\mathrm{cris}}(V)$ in a non-trivial manner). In fact, in [16] Colmez and Fontaine showed that crystalline representations of $G_F$ are equivalent to weakly admissible filtered $\varphi$-modules.

#### 1.1.3. Arithmetic Wach modules

From the discussion above, it is a natural question to ask: Does there exist some direct relation between the étale $(\varphi, \Gamma)$-module of a crystalline representation and its associated weakly admissible filtered $\varphi$-module? For a fixed representation, this question could be rephrased in terms of comparing certain elements of the period rings $\mathbf{B}$ and $\mathbf{B}_{\mathrm{cris}}$. However, the rings $\mathbf{B}$ and $\mathbf{B}_{\mathrm{cris}}$ are not comparable. So to answer this question, Fontaine considered a smaller period ring $\mathbf{B}^+ \subset \mathbf{B}$ stable under Frobenius and $G_F$-action and such that $\mathbf{B}^+ \rightarrowtail \mathbf{B}_{\mathrm{cris}}$ stable under Frobenius and $G_F$-action. Using $\mathbf{B}^+$ he defined: a $p$-adic representation $V$ of $G_F$ is said to be of finite height if the associated $(\varphi, \Gamma_F)$-module $\mathbf{D}(V)$ admits a $(\varphi, \Gamma_F)$-stable lattice over the subring $\mathbf{B}_F^+ = (\mathbf{B}^+)^{H_F} \subset \mathbf{B}_F$ (see Section 4.1 for precise definitions).

In [21] Fontaine conjectured that for a crystalline representation $V$ of $G_F$ there exist lattices inside $\mathbf{D}(V)$ over which the action of $\Gamma_F$ admits a simpler form. More precisely, finite height and crystalline representations of $G_F$ are related as follows:

Theorem 1.1 (Wach [37], Colmez [15], Berger [7]). — *Let $V$ be a $p$-adic representation of $G_F$. Then $V$ is crystalline if and only if it is of finite height and there exists $r \in \mathbb{Z}$ and a free $\mathbf{B}_F^+$-submodule $N \subset \mathbf{D}(V)$ of rank*

$= \dim_{\mathbb{Q}_p} V$, *stable under the action of* $\Gamma_F$ *and such that* $\Gamma_F$ *acts trivially over* $(N/\pi N)(-r)$.

Here $(-r)$ denotes the Tate twist. Note that in the situation of Theorem 1.1, the module $N$ is not unique. A functorial construction was given by Berger in [7], i.e. to any $p$-adic crystalline representation $V$ of $G_F$ he attached a canonical $\mathbf{B}_F^+$-submodule $\mathbf{N}(V) \subset \mathbf{D}(V)$ which he called the Wach module of $V$. Moreover, Berger established an equivalence of categories between crystalline representations of $G_F$ and Wach modules over $\mathbf{B}_F^+$. Furthermore, Berger obtained an integral version of his result by considering the period ring $\mathbf{A}^+ = \mathbf{A} \cap \mathbf{B}^+ \subset \mathbf{B}$ stable under Frobenius and $G_F$-action. He showed that for a crystalline representation $V$ of $G_F$, there exists a bijection between $G_F$-stable $\mathbb{Z}_p$-lattices $T \subset V$ and integral Wach modules $\mathbf{N}(T) \subset \mathbf{N}(V)$ where $\mathbf{N}(T)$ is defined over the integral subring $\mathbf{A}_F^+ = (\mathbf{A}^+)^{H_F}$. Finally, given $\mathbf{N}(V)$ one can canonically recover the other linear algebraic object attached to $V$, i.e. $\mathbf{D}_{\mathrm{cris}}(V)$ (see [7, Propositions II.2.1 & III.4.4]).

## 1.2. The relative case

The motivation for defining Wach modules in the relative case and exploring its relation with $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ (see Section 2 for notations) comes from the hope of computing Galois cohomology of $p$-adic representations using syntomic complexes with coefficients in $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$. Using syntomic complexes and techniques from the theory of $(\varphi, \Gamma)$-modules, this was done for the trivial representation by Colmez and Nizioł [17]. A generalization of these complexes to non-trivial coefficients can be found in [2] and [1, Chapter 5].

In this article, we are interested in the $p$-adic Hodge theory of an étale algebra over a formal torus defined over $O_F$. More precisely, let $d \in \mathbb{N}$ and $X = (X_1, X_2, \dots, X_d)$ be some indeterminates, $O_F\{X, X^{-1}\}$ the $p$-adic completion of a $d$-dimensional torus over $O_F$ and let $R$ denote the $p$-adic completion of an étale algebra over $O_F\{X, X^{-1}\}$ with non-empty and geometrically integral special fiber. Next, let $G_R$ denote the étale fundamental group of $R\left[\frac{1}{p}\right]$ and $\Gamma_R$ the Galois group of the cyclotomic tower over $R$ and $H_R = \mathrm{Ker}\ (G_R \to \Gamma_R)$ (see Section 3.1 for precise definitions). In the relative setting, on one hand Brinon has developed the theory of crystalline representations of $G_R$ [14], while on the other hand Andreatta, Brinon and Iovita have developed the theory of $(\varphi, \Gamma_R)$-modules in [3, 4].

*Remark 1.2.* — Note that in Theorem 1.1 it is important to restrict to an unramified extension $F/\mathbb{Q}_p$. For ramified extensions, such a statement does not hold in general. Therefore, in the relative setting we consider an analogue of “unramified extension of $\mathbb{Q}_p$” (indeed, by removing the geometric coordinates one obtains $R = O_F$).

### 1.2.1. $(\varphi, \Gamma_R)$-modules

Analogous to the arithmetic case, we have relative period rings $\mathbf{A} \subset \mathbf{B} \supset \mathbf{B}^+$ and $\mathbf{A}^+ = \mathbf{A} \cap \mathbf{B}^+ \subset \mathbf{B}$ (see Section 3.1 for precise definition) equipped with Frobenius and a continuous action of $G_R$. Let $V$ be a $p$-adic representation of $G_R$, then one can functorially attach to $V$ a projective and étale $(\varphi, \Gamma_R)$-module $\mathbf{D}(V) = (\mathbf{B} \otimes_{\mathbb{Q}_p} V)^{H_R}$ over $\mathbf{B}_R = \mathbf{B}^{H_R}$ of rank $= \dim_{\mathbb{Q}_p} V$ equipped with a Frobenius-semilinear operator $\varphi$ and a semilinear and continuous action of $\Gamma_R$. This induces an equivalence of categories between $p$-adic representations of $G_R$ and étale $(\varphi, \Gamma_R)$-modules over $\mathbf{B}_R$. Similarly, using the period ring $\mathbf{A}$ one can functorially attach to any $\mathbb{Z}_p$-representation $T$ of $G_R$ an étale $(\varphi, \Gamma_R)$-module $\mathbf{D}(T) = (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_R}$ over the period ring $\mathbf{A}_R = \mathbf{A}^{H_R}$. Again, this induces an equivalence between $\mathbb{Z}_p$-representations of $G_R$ and étale $(\varphi, \Gamma_R)$-modules over $\mathbf{A}_R$.

### 1.2.2. Relative Wach modules

Using the period ring $\mathbf{A}^+$ we set $\mathbf{D}^+(T) = (\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T)^{H_R}$, which is a $(\varphi, \Gamma_R)$-module over $\mathbf{A}_R^+ = (\mathbf{A}^+)^{H_R}$ and let $q = \frac{\varphi(\pi)}{\pi}$, where $\pi$ is the usual element in Fontaine's constructions (see Section 2.1 for notations). Note that for a finite free $\mathbb{Z}_p$-representation $T$ of $G_R$ the $\mathbf{A}_R$-module $\mathbf{D}(T)$ is finite projective, however it is not known whether $\mathbf{D}^+(T)$ is projective. So, we introduce the following definition:

Definition 1.3. — *A* positive finite $q$-height *representation is a $p$-adic representation $V$ of $G_R$ admitting a $\mathbb{Z}_p$-lattice $T \subset V$ such that there exists a finite projective $\mathbf{A}_R^+$-submodule $\mathbf{N}(T) \subset \mathbf{D}^+(T)$ of rank* $= \dim_{\mathbb{Q}_p} V$ *satisfying the following conditions:*

(i) *$\mathbf{N}(T)$ is stable under the action of $\varphi$ and $\Gamma_R$ and $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} \mathbf{N}(T) \xrightarrow{\sim} \mathbf{D}(T)$;*
(ii) *The $\mathbf{A}_R^+$-module $\mathbf{N}(T)/\varphi^*(\mathbf{N}(T))$ is killed by $q^s$ for some $s \in \mathbb{N}$;*
(iii) *The action of $\Gamma_R$ is trivial on $\mathbf{N}(T)/\pi\mathbf{N}(T)$;*
(iv) *There exists $R' \subset \overline{R}$ finite étale over $R$ such that the $\mathbf{A}_{R'}^+$-module $\mathbf{A}_{R'}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ is free.*

*The module* $\mathbf{N}(T)$ *is a* Wach module *associated to* $T$ *and we set* $\mathbf{N}(V) := \mathbf{N}(T)\left[\frac{1}{p}\right]$ *which satisfies analogous properties. The* height *of* $V$ *is the smallest* $s \in \mathbb{N}$ *satisfying* (ii) *above.*

*Remarks 1.4.*

(i) A finite $q$-height representation is twist of a positive one by some power of the $p$-adic cyclotomic character (see Definition 4.9 for details). The terminology "positive" refers to the fact that the Wach module $\mathbf{N}(T)$ is stable under the Frobenius-semilinear operator $\varphi$. It is motivated by the fact (and as we will show) that $V$ is positive crystalline (see Theorem 1.6).

(ii) In the arithmetic case, i.e. $R = O_F$, the notion of finite height representations in Theorem 1.1 and finite $q$-height representations in Definition 1.3 are related. In fact, in the arithmetic case using Definition 1.3 one obtains the functorial Wach module of Berger mentioned above (see [7, Proposition II.1.1]).

### 1.2.3. Crystalline representations of $G_R$

Using the period ring $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ Brinon functorially attaches to any $p$-adic representation $V$ of $G_R$ an $R\left[\frac{1}{p}\right]$-module

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) := \left(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V\right)^{G_R}.$$

The module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is called a filtered $(\varphi, \partial)$-module, i.e. it is equipped with a filtration, a Frobenius-semilinear endomorphism $\varphi$ and a quasi-nilpotent integrable connection $\partial$ satisfying Griffiths transversality with respect to the filtration (see Section 2.3 for precise definitons). The representation $V$ is said to be crystalline if the natural map is an isomorphism

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V,$$

compatible with Frobenius, filtration, connection and the action of $G_R$ on each side. Moreover, Brinon also defined the notion of weak admissibility in the relative case and showed that $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is weakly admissible for crystalline representations (see [14, Chapitre 8] for more details).

*Notation 1.5.* — We use period rings such as $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ which is a modified version of Fontaine's relative period ring $\mathbf{B}_{\mathrm{cris}}(\overline{R})$ (see Section 2.2 for details). The notation $\mathcal{O}$ here indicates that apart from Frobenius, filtration and $G_R$-action, we have a connection over $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ and we will call such rings fat relative period rings. However, note that in [14] Brinon denotes these rings as $B_{\mathrm{cris}}(R)$ and $B_{\mathrm{cris}}^{\nabla}(R)$, respectively. Similarly, we will use the notation $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ and $\mathbf{D}_{\mathrm{cris}}(V)$ for modules instead of Brinon's $D_{\mathrm{cris}}(V)$ and $D_{\mathrm{cris}}^{\nabla}(V)$, respectively. We hope it is not confusing for the reader.

### 1.2.4. Main result

Our aim is to show that for positive finite $q$-height representations, the $\mathbf{B}_R^+$-module $\mathbf{N}(V)$ and the $R\big[\frac{1}{p}\big]$-module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ are related in a precise manner and the latter can be recovered from the former. To relate these objects we consider the ring $R[\varpi]$ where $\varpi = \zeta_p - 1$ for a primitive $p$-th root of unity $\zeta_p$ (take $\varpi = \zeta_{p^2} - 1$ if $p = 2$ for a primitve $p^2$-th root of unity $\zeta_{p^2}$), and using this ring we construct a fat relative period ring $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ equipped with compatible Frobenius, filtration, connection and the action of $\Gamma_R$ (see Section 4.3 for precise definitions). The main result of this article is as follows:

Theorem 1.6 (see Theorem 4.25). — *Let $V$ be a positive finite $q$-height representation of $G_R$, then*

(i) *$V$ is a positive crystalline representation.*

(ii) *Let $M := \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\big)^{\Gamma_R}$, then after extending scalars to $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and inverting $p$, we obtain a natural isomorphism*

$$\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M\left[\frac{1}{p}\right] \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V),$$

*compatible with Frobenius, filtration, connection and the action of $\Gamma_R$ on each side.*

(iii) *We have an isomorphism of $R\big[\frac{1}{p}\big]$-modules*

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xleftarrow{\sim} \Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\Big)^{\Gamma_R}\left[\frac{1}{p}\right],$$

*compatible with Frobenius, filtration, and connection on each side. Therefore, we obtain a comparison isomorphism*

$$\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V),$$

*compatible with Frobenius, filtration, connection and the action of $\Gamma_R$ on each side.*

Let us mention the idea of the proof. In case $\mathbf{N}(T)$ is free, we proceed in two steps: First, we describe a process (see Proposition 4.28 for details) by which we can recover a submodule of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ starting with the Wach module $\mathbf{N}(T)$, establishing a comparison over $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ between the submodule obtained and the Wach module. Next, the claims made in the theorem are shown by exploiting properties of Wach modules and the comparison obtained in the first step. In the first step, one can take two approaches to obtain generators of the promised submodule of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$: either by taking $\Gamma_R$-fixed points of $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ (by successively approximating for $\Gamma_R$-action on a basis of $\mathbf{N}(T)$); or by taking elements killed by differential operators defined using topological generators of $\Gamma_R$ (see Lemma 4.41 for details). In this paper, we take the latter approach whereas the former approach is detailed in [1, Chapter 3]. In the general case when $\mathbf{N}(T)$ is projective, using property (iv) in Definition 1.3 one can pass to an extension $\mathbf{A}_R^+ \subset \mathbf{A}_{R'}^+$ to obtain a free Wach module, then use the preceding argument and finally apply Galois descent to obtain the theorem (see Proposition 4.28 for details). Finally, we also show that all one-dimensional crystalline representations are of finite $q$-height and for such representations one can directly compare $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ and the Wach module $\mathbf{N}(V)$.

### 1.3. Relative Fontaine–Laffaille modules

After obtaining Theorem 1.6 above, it is natural to wonder if a converse statement could be true, i.e. starting with a lattice $T \subset V$ of a crystalline representation $G_R$, is it possible to construct the Wach module $\mathbf{N}(T)$? In the arithmetic setting, for $p$-adic crystalline representations of $G_F$, this was shown to be true by Wach [37], and the statement was refined by Berger [7]. In the relative case, the picture is quite encouraging when we restrict to Hodge–Tate length $\leqslant p-2$ (also see Remark 1.9).

For a $p$-adic crystalline representation of $G_F$ with Hodge–Tate length $\leqslant p-1$, there exists a canonical $O_F$-lattice inside $\mathbf{D}_{\mathrm{cris}}(V)$ called the Fontaine–Laffaille module defined in [23]. In this case, Wach constructed Wach modules out of Fontaine–Laffaille data in [38]. In the relative setting, Faltings studied relative Fontaine–Laffaille modules in [19] and used them to functorially recover $\mathbb{Z}_p$-lattices inside crystalline representations of $G_R$. Recently, for free relative Fontaine–Laffaille modules of filtration length $\leqslant p-2$, adapting techniques from Wach's computations, Tsuji has constructed generalized representations of $G_R$ over $\mathbf{A}_{\mathrm{inf}}(\overline{R})$ (see [36]). In fact, it is possible to show that starting with a free relative Fontaine–Laffaille module, one can obtain a free relative Wach module over $\mathbf{A}_R^+$.

THEOREM 1.7 (see Theorem 5.5). — *Let $M$ be a free relative Fontaine–Laffaille module over $R$ of level $[0, p-2]$, and let $T_{\mathrm{cris}}(M)$ denote the associated $\mathbb{Z}_p$-representation of $G_R$. Then, the $p$-adic representation $V_{\mathrm{cris}}(M) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_{\mathrm{cris}}(M)$ is a positive finite $q$-height representation.*

Twisting the representation thus obtained by powers of the cyclotomic character, generalizes the statement to all free Fontaine–Laffaille modules with filtration length $\leqslant p-2$.

The proof of the theorem crucially exploits the computation of Fontaine [22], Wach [38] and Tsuji [36]. It follows in three steps: First, starting with a Fontaine–Laffaille module, we obtain an $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$-module using formal properties of crystalline site for maps $\theta : \mathbf{A}_{R,\varpi}^{\mathrm{PD}} \twoheadrightarrow R$ and $\theta_R : \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \twoheadrightarrow R$ (see Section 5.3.1 for details). Next, we exploit equivalence of categories obtained in Theorem 5.21 by extending scalars along $\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \twoheadrightarrow \mathbf{A}_{R,\varpi}^{\mathrm{PD}}/I^{(p-1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \xleftarrow{\sim} \mathbf{A}_{R,\varpi}^{+}/I^{(p-1)}\mathbf{A}_{R,\varpi}^{+} \twoheadleftarrow \mathbf{A}_{R,\varpi}^{+}$ (see Proposition 5.12 for explanations). This gives us an $\mathbf{A}_{R,\varpi}^{+}$-module with precise description of the Frobenius and the action of $\Gamma_R$. Finally, we descend over to the ring $\mathbf{A}_R^{+}$ by exploiting the Frobenius and $\Gamma_R$-action, thus obtaining a Wach module over $\mathbf{A}_R^{+}$ and proving the theorem (see Section 5.3.2).

*Remark 1.8.* — In a recent work, Morrow and Tsuji have developed a theory of coefficients for integral $p$-adic Hodge theory in [32]. Extending scalars of relative Wach modules along $O_F[\![\pi]\!] \to \mathbf{A}_{\mathrm{inf}}(O_{\overline{F}})$ would yield generalized represention over $\mathbf{A}_{\mathrm{inf}}^{\square}(R)$ in the sense of Morrow–Tsuji.

*Remark 1.9.* — Recent developments in the theory of prismatic crystals [12, 18, 28], indicate that to obtain a full converse statement, i.e. to construct Wach modules from lattices inside crystalline representations, one needs to generalize Definition 1.3 slightly. This is a work in progress and we will report further on this line of investigation in future.

## 1.4. Setup and notations

In this section we will describe the setup for the rest of the text and fix some notations.

CONVENTION. — *We will work under the convention that $0 \in \mathbb{N}$, the set of natural numbers.*

Let $p$ be a fixed prime number, $\kappa$ a finite field of characteristic $p$, $W := W(\kappa)$ the ring of $p$-typical Witt vectors with coefficients in $\kappa$ and $F :=$

$W\left[\frac{1}{p}\right]$, the fraction field of $W$. In particular, $F$ is an unramified extension of $\mathbb{Q}_p$ with ring of integers $O_F = W$. Let $\overline{F}$ be a fixed algebraic closure of $F$ so that its residue field, denoted as $\overline{\kappa}$, is an algebraic closure of $\kappa$. Further, we denote by $G_F = \mathrm{Gal}(\overline{F}/F)$, the absolute Galois group of $F$.

Let $Z = (Z_1, \dots, Z_s)$ denote a set of indeterminates and $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}^s$ be a multi-index, then we write $Z^{\mathbf{k}} := Z_1^{k_1} \cdots Z_s^{k_s}$. For $\mathbf{k} \to +\infty$ we will mean that $\sum k_i \to +\infty$. Now for a topological algebra $\Lambda$ we define

$$\Lambda\{Z\} := \left\{ \sum_{\mathbf{k}\in\mathbb{N}^s} a_{\mathbf{k}} Z^{\mathbf{k}}, \text{ where } a_{\mathbf{k}} \in \Lambda \text{ and } a_{\mathbf{k}} \longrightarrow 0 \text{ as } \mathbf{k} \longrightarrow +\infty \right\}.$$

We fix $d \in \mathbb{N}$ and let $X = (X_1, X_2, \dots, X_d)$ be some indeterminates. Let $R$ be the $p$-adic completion of an étale algebra over $O_F\{X, X^{-1}\}$ with non-empty geometrically integral special fiber. In particular, we have a presentation

$$R = O_F\{X, X^{-1}\}\{Z_1, \dots, Z_s\}/(Q_1, \dots, Q_s),$$

where $Q_i(Z_1, \dots, Z_s) \in O_F\{X, X^{-1}\}[Z_1, \dots, Z_s]$ for $1 \leqslant i \leqslant s$ are multivariate polynomials such that $\det\left(\frac{\partial Q_i}{\partial Z_j}\right)_{1\leqslant i,j\leqslant s}$ is invertible in $R$. The algebra $R\left[\frac{1}{p}\right]$ is the relative analogue of "finite unramified extension of $\mathbb{Q}_p$" (indeed, by removing the geometric coordinates we will obtain $R\left[\frac{1}{p}\right] = F$).

*Remark 1.10.* — Note that Theorem 1.1 serves as our main motivation for the theory developed in this article. The assumptions we put on $R$ generalizes the fact that "$F$ is unramified over $\mathbb{Q}_p$".

The $p$-adic Hodge theory over $R$ is the study of $p$-adic representations of the étale fundamental group of $R\left[\frac{1}{p}\right]$, which we introduce next. We fix an algebraic closure $\overline{\mathrm{Fr}\,(R)}$ of $\mathrm{Fr}\,(R)$ containing $\overline{F}$. Let $\overline{R}$ denote the union of finite $R$-subalgebras $S \subset \overline{\mathrm{Fr}\,(R)}$, such that $S\left[\frac{1}{p}\right]$ is étale over $R\left[\frac{1}{p}\right]$. Let $\overline{\eta}$ denote a geometric point of the generic fiber $\mathrm{Spec}\,R\left[\frac{1}{p}\right]$ and let $G_R := \pi_1^{\text{ét}}\big(\mathrm{Spec}\,R\left[\frac{1}{p}\right], \overline{\eta}\big)$ denote the étale fundamental group. By [27, Exposé V, Section 8], we can write this étale fundamental group as the Galois group (of the fraction field of $\overline{R}\left[\frac{1}{p}\right]$ over the fraction field of $R\left[\frac{1}{p}\right]$)

$$G_R = \pi_1^{\text{ét}}\left(\mathrm{Spec}\,R\left[\frac{1}{p}\right], \overline{\eta}\right) = \mathrm{Gal}\left(\overline{R}\left[\frac{1}{p}\right]/R\left[\frac{1}{p}\right]\right).$$

For $n \in \mathbb{N}$, let $F_n := F(\mu_{p^n})$. From now onwards, we will fix some $m \in \mathbb{N}_{\geqslant 1}$ (take $m \in \mathbb{N}_{\geqslant 2}$ if $p = 2$) and set $K := F_m$, with its ring of integers $O_K$. The element $\varpi = \zeta_{p^m} - 1 \in O_K$ is a uniformizer of $K$, and its minimal polynomial $P_\varpi(X) = \frac{(1+X)^{p^m}-1}{(1+X)^{p^{m-1}}-1}$ is an Eisenstein polynomial in $W[X]$ of degree $e := [K : F] = p^{m-1}(p-1)$. Finally, for $S = R[\varpi] = O_K \otimes_{O_F} R$ we have that $R[\varpi]$ is totally ramified at the prime ideal $(p) \subset R[\varpi]$. And similar to above, we obtain Galois groups $G_K \lhd G_F$ and $G_S \lhd G_R$ respectively, such that $G_R/G_S = G_F/G_K = \mathrm{Gal}(K/F)$. Finally, we have that $R$ and $R[\varpi]$ are *small* algebras in the sense of Faltings (see [19, Section II(a)]).

For $k \in \mathbb{N}$, let $\Omega^k_R$ denote the $p$-adic completion of module of $k$-differentials of $R$ relative to $\mathbb{Z}$. Then, we have

$$\Omega^1_R = \bigoplus_{i=1}^{d} R\, d\log X_i, \quad \text{and} \quad \Omega^k_R = \bigwedge_R^k \Omega^1_R.$$

We also have that $R/pR \xrightarrow{\sim} S/\varpi S$ and for all $n \in \mathbb{N}$, $R/p^nR$ is a smooth $\mathbb{Z}/p^n\mathbb{Z}$-algebra. Finally, we have a unique lift $\varphi : R \to R$ of the absolute Frobenius $x \mapsto x^p$ over $R/pR$ such that $\varphi(X_i) = X_i^p$, for all $1 \leqslant i \leqslant d$ (in general, a lift of Frobenius modulo $p$ need not be unique, see [14, p. 9]).

Convention. — *Let $A$ be a ring and $I \subsetneq A$ an ideal. We say that an $A$-module $M$ is $I$-adically complete if and only if $M \xrightarrow{\sim} \lim_n M/I^nM$.*

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## 2. $p$-adic Hodge theory

In this section we will recall some constructions and results in relative $p$-adic Hodge theory developed in [14], albeit in a simpler setting compared to Brinon's book. As we will be using different notations compared to Brinon, we will make most of the definitions explicit.

We are interested in exploring the relationship between $p$-adic crystalline representations and finite height representations of $G_R$. This will be detailed in Section 4 and Section 5. To carry out some computations in the aforementioned sections, we will need to extend our base field (hence the base ring) by adjoining some $p$-power roots of unity (see the field $K$ and the ring $S = R[\varpi]$ in Section 1.4). As a consequence, we will also require the corresponding period rings defined for such rings. However, in Section 2.1, Section 2.2 & Section 2.3 we will only recall results from [14] by fixing our base as $R$. As we shall see the period rings will only depend on $\overline{R}$ and we have $\overline{S} = \overline{R} \subset \overline{\mathrm{Fr}\,(R)} = \overline{\mathrm{Fr}\,(S)}$, therefore fixing our base as $R$ is sufficient (see [14] for general constructions).

## 2.1. The de Rham period ring

We will recall definitions and properties of the relative version of Fontaine's period ring $\mathbf{B}_{\mathrm{dR}}$ (see [22] for classical case).

### 2.1.1. The ring $\mathbb{C}^+(\overline{R})$ and its tilt

Let $\mathbb{C}_p$ denote the $p$-adic completion of $\overline{F}$. Recall that $\overline{R}$ is the union of finite $R$-subalgebras $S \subset \overline{\mathrm{Fr}\,(R)} = \overline{\mathrm{Fr}\,(R[\varpi])}$, such that $S\left[\frac{1}{p}\right]$ is étale over $R\left[\frac{1}{p}\right]$. Let $\mathbb{C}^+(\overline{R})$ denote the $p$-adic completion of $\overline{R}$ and $\mathbb{C}(\overline{R}) = \mathbb{C}^+(\overline{R})\left[\frac{1}{p}\right]$. We define the tilt $\mathbb{C}^+(\overline{R})^\flat := \lim_{x\mapsto x^p} \mathbb{C}^+(\overline{R})/p = \lim_{x\mapsto x^p} \overline{R}/p$ and equip it with the inverse limit topology (where we equip $\overline{R}/p$ with the discrete topology) and let $\mathbb{C}(\overline{R})^\flat = \mathbb{C}^+(\overline{R})^\flat\left[\frac{1}{p^\flat}\right]$ for $p^\flat := (p, p^{1/p}, p^{1/p^2}, \ldots) \in \mathbb{C}^+(\overline{R})^\flat$ and equipped with the coarsest ring topology such that $\mathbb{C}^+(\overline{R})$ is an open subring. Note that an element $x \in \mathbb{C}(\overline{R})^\flat$ can be described as a sequence $(x_n)_{n\in\mathbb{N}}$, with $x_n \in \mathbb{C}(\overline{R})$ and $x_{n+1}^p = x_n$ for all $n \in \mathbb{N}$. These rings admit a continuous $G_R$-action for the topology described.

We fix some choices of compatible $p$-power roots which will appear in the sequel. Let $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots) \in \mathbb{C}_p^\flat$, $X_i^\flat := \left(X_i, X_i^{1/p}, X_i^{1/p^2}, \ldots\right) \in \mathbb{C}(\overline{R})^\flat$ for $1 \leqslant i \leqslant d$. We set $\mathbf{A}_{\mathrm{inf}}(\overline{R}) := W(\mathbb{C}^+(\overline{R})^\flat)$, the ring of $p$-typical Witt vectors with coefficients in $\mathbb{C}^+(\overline{R})^\flat$ equipped with weak topology (see [4, Section 2.10]). The absolute Frobenius on $\mathbb{C}^+(\overline{R})^\flat$ lifts to an endomorphism $\varphi : \mathbf{A}_{\mathrm{inf}}(\overline{R}) \to \mathbf{A}_{\mathrm{inf}}(\overline{R})$ and the $G_R$-action extends to $\mathbf{A}_{\mathrm{inf}}(\overline{R})$ such that the action is continuous for the weak topology. For $x \in \mathbb{C}^+(\overline{R})^\flat$, let $[x] =$

$(x, 0, 0, \ldots) \in \mathbf{A}_{\mathrm{inf}}(\overline{R})$ denote its Teichmüller representative. So we have $[\varepsilon] \in \mathbf{A}_{\mathrm{inf}}(\overline{R})$ with $g[\varepsilon] = [\varepsilon]^{\chi(g)}$ for $g \in G_R$ and $\chi : G_R \to \mathbb{Z}_p^\times$ the $p$-adic cyclotomic character and $\varphi([\varepsilon]) = [\varepsilon]^p$. Now any element $x \in \mathbf{A}_{\mathrm{inf}}(\overline{R})$ can be uniquely written as $x = \sum_{k\in\mathbb{N}} p^k[x_k]$ for $x_k \in \mathbb{C}^+(\overline{R})^\flat$. So we set $\pi := [\varepsilon] - 1$, $\pi_1 := \varphi^{-1}(\pi) = [\varepsilon^{1/p}] - 1$, and $\xi := \frac{\pi}{\pi_1}$. Clearly we have $g(\pi) = (1+\pi)^{\chi(g)} - 1$ for $g \in G_R$ and $\varphi(\pi) = (1+\pi)^p - 1$.

### 2.1.2. Definition of $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$

We have Fontaine's $\theta$-map defined as $\theta : \mathbf{A}_{\mathrm{inf}}(\overline{R}) \to \mathbb{C}^+(\overline{R})$ sending $\sum_{k\in\mathbb{N}} p^k[x_k] \mapsto \sum_{k\in\mathbb{N}} p^k x_k^\sharp$, it is a $G_R$-equivariant surjective ring homomorphism whose kernel is principal and generated by, for example, $p - [p^\flat]$ or $\xi$ (see [20, Proposition 2.4(ii)]). By $\mathbb{Q}_p$-linearity, the map $\theta$ can be extended to $\theta : \mathbf{A}_{\mathrm{inf}}(\overline{R})\left[\frac{1}{p}\right] \to \mathbb{C}(\overline{R})$ and we define

$$\mathbf{B}_{\mathrm{dR}}^+(\overline{R}) := \lim_n \mathbf{A}_{\mathrm{inf}}(\overline{R}) \left[\frac{1}{p}\right] / (\mathrm{Ker}\ \theta)^n,$$

as the $(\mathrm{Ker}\ \theta)$-adic completion of $\mathbf{A}_{\mathrm{inf}}(\overline{R})\left[\frac{1}{p}\right]$. The ring $\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$ is an $F$-algebra equippedw with an action of $G_R$. The map $\theta$ further extends to a $G_R$-equivariant surjective ring homomorphism $\theta : \mathbf{B}_{\mathrm{dR}}^+(\overline{R}) \to \mathbb{C}(\overline{R})$ with $\mathrm{Ker}\ \theta = t\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$, where $t := \log[\varepsilon] = \log(1+\pi) = \sum_{k\in\mathbb{N}} (-1)^k \frac{\pi^{k+1}}{k+1} \in \mathbf{B}_{\mathrm{dR}}^+(\overline{R})$ such that $g \in G_R$ acts by $g(t) = \chi(g)t$. By functoriality of the construction of $\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$, the homomorphism $O_{\overline{F}} \to \overline{R}$ induces an injection $\mathbf{B}_{\mathrm{dR}}^+(O_{\overline{F}}) \to \mathbf{B}_{\mathrm{dR}}^+(\overline{R})$. The ring $\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$ is $t$-torsion free, so we set $\mathbf{B}_{\mathrm{dR}}(\overline{R}) := \mathbf{B}_{\mathrm{dR}}^+(\overline{R})\left[\frac{1}{t}\right]$. The $G_R$-action extends to $\mathbf{B}_{\mathrm{dR}}(\overline{R})$ and the ring $\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$ admits a natural $G_R$-stable filtration given as $\mathrm{Fil}^r\mathbf{B}_{\mathrm{dR}}(\overline{R}) := t^r\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$ for $r \in \mathbb{Z}$ and we equip $\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$ with the induced filtration (see [14, Section 5.1] for details).

We can extend the map $\theta : \mathbf{A}_{\mathrm{inf}}(\overline{R}) \to \mathbb{C}^+(\overline{R})$ by $R$-linearity to obtain a ring homomorphism $\theta_R : R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(\overline{R}) \to \mathbb{C}^+(\overline{R})$. Let $\mathcal{O}\mathbf{A}_{\mathrm{inf}}(\overline{R})$ denote the $\theta_R^{-1}(p\mathbb{C}^+(\overline{R}))$-adic completion of $R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(\overline{R})$ (the ideal $\theta_R^{-1}(p\mathbb{C}^+(\overline{R}))$ is generated by $p$ and $\mathrm{Ker}\ \theta_R$), then $\theta_R$ extends to a surjective homomorphism $\theta_R : \mathcal{O}\mathbf{A}_{\mathrm{inf}}(\overline{R})\left[\frac{1}{p}\right] \to \mathbb{C}(\overline{R})$. Define

$$\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R}) := \lim_n \mathcal{O}\mathbf{A}_{\mathrm{inf}}(\overline{R}) \left[\frac{1}{p}\right] / (\mathrm{Ker}\ \theta_R)^n,$$

as the $(\mathrm{Ker}\ \theta_R)$-adic completion of $\mathcal{O}\mathbf{A}_{\mathrm{inf}}(\overline{R})\left[\frac{1}{p}\right]$. The ring $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$ is an $R\left[\frac{1}{p}\right]$-algebra and admits a $G_R$-action. The homomorphism $\theta_R$ extends to a $G_R$-equivariant surjective homomorphism $\theta_R : \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R}) \to \mathbb{C}(\overline{R})$.

The ring $\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$ is $t$-torsion free and we set $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \coloneqq \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R})\big[\frac{1}{t}\big]$. Moreover, the $G_R$-action extends to $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$.

### 2.1.3. Structure and properties of $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$

A more explicit description of the ring $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$ can be given. Note that $X_i \otimes 1 - 1 \otimes [X_i^\flat] \in \mathrm{Ker}\ \theta_R \subset R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(\overline{R})$ for $1 \leqslant i \leqslant d$ and let $z_i$ denote its image in $\mathcal{O}\mathbf{A}_{\mathrm{inf}}(\overline{R}) \subset \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$. Since $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$ is complete for the $(\mathrm{Ker}\ \theta_R)$-adic topology, the homomorphism $\mathbf{B}_{\mathrm{dR}}^+(\overline{R}) \to \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$ extends to a homomorphism

$$\begin{aligned} f : \mathbf{B}_{\mathrm{dR}}^+(\overline{R})[\![T_1, \ldots, T_d]\!] &\longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R}) \\ T_i &\longmapsto z_i, \qquad \text{for } 1 \leqslant i \leqslant d. \end{aligned}$$

In fact, $f$ is an isomorphism and $\mathrm{Ker}\ \theta_R = (t, z_1, \ldots, z_d) \subset \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$. Therefore, one can identify $\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$ with a subring of $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$. There is a natural $G_R$-stable filtration on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$ given by $\mathrm{Fil}^r \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R}) = (\mathrm{Ker}\ \theta_R)^r$ for $r \in \mathbb{N}$. We set $\mathrm{Fil}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \coloneqq \sum_{n=0}^{+\infty} t^{-n} \mathrm{Fil}^n \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R}) = \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R})\big[\frac{z_1}{t}, \ldots, \frac{z_d}{t}\big]$ and $\mathrm{Fil}^r \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \coloneqq t^r \mathrm{Fil}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$ for $r \in \mathbb{Z}$, satisfying the same conditions. Moreover, the induced filtrations on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$, $\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$ and $\mathbf{B}_{\mathrm{dR}}(\overline{R})$ match with the ones defined before. Finally, we have $\big(\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})\big)^{G_R} = R\big[\frac{1}{p}\big]$ (see [14, Section 5.2] for details).

We can equip the rings $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$ and $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$ with a connection. Let $N_i$ denote the unique $(\mathrm{Ker}\ \theta_R)$-adically continuous, $\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$-linear derivation on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R})$ given as $N_i(z_j) = \delta_{ij} X_j$ for $1 \leqslant i, j \leqslant d$, where $\delta_{ij}$ denotes the Kronecker delta symbol. Furthremore, the derivation $N_i$ extends to a $\mathbf{B}_{\mathrm{dR}}(\overline{R})$-linear derivation on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$, since $N_i(t) = 0$. Define a connection

$$\begin{aligned} \partial : \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) &\longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \otimes_{R[\frac{1}{p}]} \Omega^1_R \left[\frac{1}{p}\right] \\ x &\longmapsto \sum_{i=1}^{d} N_i(x) \otimes\ d\log X_i. \end{aligned}$$

The connection $\partial$ is $G_R$-equivariant and satisfies Griffiths transversality with respect to the filtration, i.e.

$$\partial\big(\mathrm{Fil}^r \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})\big) \subset \mathrm{Fil}^{r-1} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \otimes_{R[\frac{1}{p}]} \Omega^1_R \left[\frac{1}{p}\right].$$

Its restriction to $R\big[\frac{1}{p}\big]$ is the canonical differential operator. Moreover, we have $\big(\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\overline{R})\big)^{\partial=0} = \mathbf{B}_{\mathrm{dR}}^+(\overline{R})$ and $\big(\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})\big)^{\partial=0} = \mathbf{B}_{\mathrm{dR}}(\overline{R})$ (see [14, Section 5.3] for details).

## 2.2. The crystalline period ring

In this section, we will recall the definition and properties of crystalline period rings following [14]. Note that Brinon defines these rings under a certain assumption on his base ring (see condition (BR) on [14, p. 9]) which is always true in our setting.

### 2.2.1. Definition of $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$

Let us consider the map $\theta : \mathbf{A}_{\mathrm{inf}}(\overline{R}) \rightarrow \mathbb{C}^+(\overline{R})$ whose kernel is a principal ideal generated by $\xi$ or $p - [p^\flat]$. Let us denote $x^{[k]} := \frac{x^k}{k!}$ for $x \in \mathrm{Ker}\ \theta \subset \mathbf{A}_{\mathrm{inf}}(\overline{R})$ and $k \in \mathbb{N}$. The divided power envelope of $\mathbf{A}_{\mathrm{inf}}(\overline{R})$ with respect to Ker $\theta$ is given as $\mathbf{A}_{\mathrm{inf}}(\overline{R})\big[x^{[k]},\ x \in \mathrm{Ker}\ \theta\big]_{k\in\mathbb{N}} = \mathbf{A}_{\mathrm{inf}}(\overline{R})\big[\xi^{[k]}\big]_{k\in\mathbb{N}}$. We define

$$\mathbf{A}_{\mathrm{cris}}(\overline{R}) := p\text{-adic completion of } \mathbf{A}_{\mathrm{inf}}(\overline{R})\big[\xi^{[k]}\big]_{k\in\mathbb{N}}.$$

This is a $W(\kappa)$-algebra equipped with a continuous action of $G_R$. The ring $\mathbf{A}_{\mathrm{cris}}(\overline{R})$ is $p$-torsion free and the Frobenius on $\mathbf{A}_{\mathrm{inf}}(\overline{R})$ extends to $\mathbf{A}_{\mathrm{cris}}(\overline{R})$. The homomorphism $\theta$ in Section 2.1.2 extends to a surjective homomorphism $\theta : \mathbf{A}_{\mathrm{cris}}(\overline{R}) \rightarrow \mathbb{C}^+(\overline{R})$. Also, we have $t = \log(1+\pi) \in \mathrm{Ker}\ \theta \subset \mathbf{A}_{\mathrm{cris}}(\overline{R})$ and the Frobenius $\varphi$ on this element is given as $\varphi(t) = pt$. Moreover, Ker $\theta \subset \mathbf{A}_{\mathrm{cris}}(\overline{R})$ is a divided power ideal. Further, the ring $\mathbf{A}_{\mathrm{cris}}(\overline{R})$ is $t$-torsion free, so we set $\varphi\big(\frac{1}{t}\big) = \frac{1}{pt}$ and define $\mathbf{B}^+_{\mathrm{cris}}(\overline{R}) := \mathbf{A}_{\mathrm{cris}}(\overline{R})\big[\frac{1}{p}\big]$ and $\mathbf{B}_{\mathrm{cris}}(\overline{R}) := \mathbf{B}^+_{\mathrm{cris}}(\overline{R})\big[\frac{1}{t}\big]$. These are $F$-algebras, equipped with a continuous action of $G_R$ and the Frobenius $\varphi$ (see [14, Section 6.1 and Section 6.2] for details).

Next, let us consider the map $\theta_R : R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(\overline{R}) \rightarrow \mathbb{C}^+(\overline{R})$. The kernel of this map is an ideal generated by $\{1 \otimes \xi, z_1, \ldots, z_d\}$, where $z_i = X_i \otimes 1 - 1 \otimes [X_i^\flat]$ for $1 \leqslant i \leqslant d$. The divided power envelope of $R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(\overline{R})$ with respect to Ker $\theta_R$ is given as $R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(\overline{R})\big[x^{[k]},\ x \in \mathrm{Ker}\ \theta_R\big]_{k\in\mathbb{N}}$. We define

$$\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) := p\text{-adic completion of } R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(\overline{R})\big[x^{[k]},\ x \in \mathrm{Ker}\ \theta_R\big]_{k\in\mathbb{N}}.$$

This is an $R$-algebra equipped with a continuous action of $G_R$. Taking the diagonal action of the Frobenius on $R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(\overline{R})$ it can be shown that the Frobenius extends to $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ and we denote this extension again by $\varphi$. The homomorphism $\theta_R$ from Section 2.1 extends to surjective homomorphism $\theta_R : \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \rightarrow \mathbb{C}^+(\overline{R})$ (see [14, p. 64] for details).

### 2.2.2. Structure and properties of $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$

Let $T = (T_1, \ldots, T_d)$ be some indeterminates as in Section 2.1.3 and let us denote by $\mathbf{A}_{\mathrm{cris}}(\overline{R})\langle T\rangle^\wedge$ the $p$-adic completion of the divided power polynomial algebra in indeterminates $T$ and coefficients in $\mathbf{A}_{\mathrm{cris}}(\overline{R})$. Then we obtain an isomorphism of $\mathbf{A}_{\mathrm{cris}}(\overline{R})$-algebras (see [14, Proposition 6.1.5])

$$\begin{aligned} f_{\mathrm{cris}} : \mathbf{A}_{\mathrm{cris}}(\overline{R})\langle T\rangle^\wedge &\longrightarrow \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \\ T_i &\longmapsto z_i \quad \text{for } 1 \leqslant i \leqslant d. \end{aligned}$$

The ring $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ is $p$-torsion free as well as $t$-torsion free, so we set $\mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(\overline{R}) := \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})\left[\frac{1}{p}\right]$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) := \mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(\overline{R})\left[\frac{1}{t}\right]$. These $R\left[\frac{1}{p}\right]$-algebras are equipped with a continuous action of $G_R$ and the action of Frobenius extends to these rings and we denote this extension again by $\varphi$ (see [14, Section 6.1 and Section 6.2] for details).

Note that there exist natural morphisms of rings $\mathbf{A}_{\mathrm{cris}}(\overline{R}) \to \mathbf{B}^+_{\mathrm{dR}}(\overline{R})$ and $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \to \mathcal{O}\mathbf{B}^+_{\mathrm{dR}}(\overline{R})$. So we obtain induced homomorphisms $\mathbf{B}^+_{\mathrm{cris}}(\overline{R}) \to \mathbf{B}^+_{\mathrm{dR}}(\overline{R})$, $\mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(\overline{R}) \to \mathcal{O}\mathbf{B}^+_{\mathrm{dR}}(\overline{R})$, $\mathbf{B}_{\mathrm{cris}}(\overline{R}) \to \mathbf{B}_{\mathrm{dR}}(\overline{R})$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \to \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$, which are injective and $G_R$-equivariant. Using this, we get induced filtrations on crystalline period rings as $\mathrm{Fil}^r\mathbf{B}_{\mathrm{cris}}(\overline{R}) := \mathbf{B}_{\mathrm{cris}}(\overline{R}) \cap \mathrm{Fil}^r\mathbf{B}_{\mathrm{dR}}(\overline{R})$ and $\mathrm{Fil}^r\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) := \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \cap \mathrm{Fil}^r\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$ for $r \in \mathbb{Z}$ (see [14, Section 6.2] for details).

Next, we will consider a connection on $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ induced from the connection on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$. For $n \in \mathbb{N}$, we have $\partial(z_i^{[n]}) = z_i^{[n-1]} \otimes \mathrm{d}X_i$ for $1 \leqslant i \leqslant d$, so we get that for any $x \in \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) = \mathbf{A}_{\mathrm{cris}}(\overline{R})\langle T\rangle^\wedge$, we have $\partial(x) \in \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \otimes_R \Omega^1_R$. This gives us an induced connection

$$\partial : \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[\frac{1}{p}]} \Omega^1_R\left[\frac{1}{p}\right].$$

The connection $\partial$ is $G_R$-equivariant and satisfies Griffiths transversality with respect to the filtration, since the same is true over $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$. Its restriction to $R\left[\frac{1}{p}\right]$ is the canonical differential operator. Moreover, taking horizontal sections we get $\left(\mathcal{O}\mathbf{A}^+_{\mathrm{cris}}(\overline{R})\right)^{\partial=0} = \mathbf{A}_{\mathrm{cris}}(\overline{R})$, $\left(\mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(\overline{R})\right)^{\partial=0} = \mathbf{B}^+_{\mathrm{cris}}(\overline{R})$ and $\left(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})\right)^{\partial=0} = \mathbf{B}_{\mathrm{cris}}(\overline{R})$. We equip $\Omega^1_R\left[\frac{1}{p}\right]$ with the unique Frobenius-linear map $\varphi$ satisfying $\varphi(\mathrm{d}x) = \mathrm{d}\varphi(x)$ for $x \in R$. Then, over $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ the Frobenius operator commutes with the connection, i.e. $\varphi\partial = \partial\varphi$ (see [14, Proposition 6.2.5]). Furthermore, we have $\left(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})\right)^{G_R} = R\left[\frac{1}{p}\right]$. Finally, the natural map $R\left[\frac{1}{p}\right] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ is faithfully flat (see [14, Section 6.2 and Section 6.3] for details).

## 2.3. $p$-adic representations

In this section we will recall results on linear algebra data associated to $p$-adic de Rham and crystalline representations of the Galois group $G_R$. We will use the $G_R$-regularity of a topological $\mathbb{Q}_p$-algebra $B$ in the sense of [14, p. 106]. If $V$ is a $p$-adic representation of $G_R$, we set

$$\mathbf{D}_B(V) := \left(B \otimes_{\mathbb{Q}_p} V\right)^{G_R}.$$

This is a $B^{G_R}$-module and we have a natural morphism of $B$-modules, functorial in $V$

$$\begin{aligned} \alpha_B(V) : B \otimes_{B^{G_R}} \mathbf{D}_B(V) &\longrightarrow B \otimes_{\mathbb{Q}_p} V \\ b \otimes d &\longmapsto bd. \end{aligned}$$

The representation $V$ is said to be *$B$-admissible* if $\alpha_B$ is an isomorphism.

### 2.3.1. Unramified representations

Let $R^{\mathrm{ur}}$ denote the union of finite étale $R$-subalgebras $S \subset \overline{R}$, and let $\widehat{R^{\mathrm{ur}}}$ denote its $p$-adic completion. It is an $R$-subalgebra of $\mathbb{C}(\overline{R})$ equipped with a continuous action of $G_R$. Further, we have $\left(\widehat{R^{\mathrm{ur}}}\left[\frac{1}{p}\right]\right)^{G_R} = R\left[\frac{1}{p}\right]$ and $\widehat{R^{\mathrm{ur}}}\left[\frac{1}{p}\right]$ is $G_R$-regular. Let us set $G_R^{\mathrm{ur}} := \mathrm{Gal}(R^{\mathrm{ur}}/R)$ which is a quotient of $G_R$. A $p$-adic representation $\rho : G_R \to \mathrm{GL}(V)$ is said to be *unramified*, if $\rho$ factorizes through $G_R \to G_R^{\mathrm{ur}}$.

Let $V$ be a $p$-adic representation of $G_R$ and we set

$$\mathbf{D}_{\mathrm{ur}}(V) := \left(\widehat{R^{\mathrm{ur}}}\left[\frac{1}{p}\right] \otimes_{\mathbb{Q}_p} V\right)^{G_R},$$

which is an $R\left[\frac{1}{p}\right]$-module and we say that $V$ is unramified if and only if $V$ is $\widehat{R^{\mathrm{ur}}}\left[\frac{1}{p}\right]$-admissible (see [14, Section 8.1]).

*Remark 2.1.* — Let $V$ be an $h$-dimensional $p$-adic representation of $G_R$ and $T \subset V$ a $\mathbb{Z}_p$-lattice stable under the action of $G_R$ such that the action is trivial modulo $p$. Consider the associated continuous cocycle $f : G_R^{\mathrm{ur}} \to \mathrm{GL}_h(\widehat{R^{\mathrm{ur}}})$ describing the action of $G_R^{\mathrm{ur}}$ over $\widehat{R^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} T$. Since $V$ is unramified, $f$ is cohomologous to the trivial cocycle and from [14, proof of Proposition 8.1.2], there exists $b \in 1 + p \cdot \mathrm{Mat}(h, \widehat{R^{\mathrm{ur}}})$ such that $f$ is given as $g \mapsto f(g) = g(b)b^{-1}$ for $g \in G_R$. In this case, we say that $f$ is *trivialised* by $b \in 1 + p \cdot \mathrm{Mat}(h, \widehat{R^{\mathrm{ur}}})$.

2.3.2. de Rham representations

Note that $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$ is a $G_R$-regular $R\left[\frac{1}{p}\right]$-algebra. We set

$$\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V) \coloneqq \left(\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \otimes_{\mathbb{Q}_p} V\right)^{G_R}.$$

The representation $V$ is said to be de Rham if it is $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$-admissible. The $R\left[\frac{1}{p}\right]$-module $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ is equipped with a decreasing, separated and exhaustive filtration induced from the filtration on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \otimes_{\mathbb{Q}_p} V$ where we consider the $G_R$-stable filtration on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$ from Section 2.1.3. Moreover, the module $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ is equipped with an integrable connection, induced from the $G_R$-equivariant integrable connection

$$\begin{aligned} \partial : V \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) &\longrightarrow V \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \otimes_{R[\frac{1}{p}]} \Omega^1_R \left[\frac{1}{p}\right] \\ v \otimes b &\longmapsto v \otimes \partial(b). \end{aligned}$$

We denote the induced connection on $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ again by $\partial$. Since the connection $\partial$ on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$ satisfies Griffiths transversality, the same is true for $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$, i.e. $\partial(\mathrm{Fil}^r \mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)) \subset \mathrm{Fil}^{r-1}\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V) \otimes_{R[\frac{1}{p}]} \Omega^1_R\left[\frac{1}{p}\right]$. Further, $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ is projective of rank $\leqslant \dim(V)$ over $(\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}))^{G_R} = R\left[\frac{1}{p}\right]$. If $V$ is de Rham then for all $r \in \mathbb{Z}$, the $R\left[\frac{1}{p}\right]$-modules $\mathrm{Fil}^r\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ and $\mathrm{gr}^r\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ are projective of finite type and for such a representation the collection of integers $r_i$ for $1 \leqslant i \leqslant \dim_{\mathbb{Q}_p}(V)$ such that $\mathrm{gr}^{-r_i}\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V) \neq 0$ are called *Hodge–Tate weights* of $V$. Moreover, we say that $V$ is positive if and only if $r_i \leqslant 0$ for all $1 \leqslant i \leqslant \dim_{\mathbb{Q}_p}(V)$ (see [14, Section 8.3] for details).

2.3.3. Crystalline representations

Note that $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ is a $G_R$-regular $R\left[\frac{1}{p}\right]$-algebra. We set

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \coloneqq \left(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V\right)^{G_R}.$$

The representation $V$ is said to be crystalline if it is $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$-admissible and we denote the category of all crystalline representations of $G_R$ by $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_R)$. The $R\left[\frac{1}{p}\right]$-module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is equipped with a Frobenius-semilinear operator $\varphi$ induced from the Frobenius on $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V$, where we consider the $G_R$-equivariant Frobenius on $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$. Further, $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is an $R\left[\frac{1}{p}\right]$-submodule of $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$, and we equip the former with induced filtration and connection which satisfies Griffiths transversality with respect to the filtration. Additionally, we have $\partial\varphi = \varphi\partial$ over $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ (see [14, Section 8.3] for details).

The $R\left[\frac{1}{p}\right]$-module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is projective of rank $\leqslant \dim(V)$. If $V$ is crystalline, then the $R\left[\frac{1}{p}\right]$-linear homomorphism

$$1 \otimes \varphi : R\left[\frac{1}{p}\right] \otimes_{R[\frac{1}{p}],\varphi} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \longrightarrow \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$$

is an isomorphism and $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is called a filtered $(\varphi, \partial)$-module. The inclusion $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \rightarrowtail \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$ induces the inclusion $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \rightarrowtail \mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$. Let $V$ be a non-trivial de Rham representation of $G_R$, then the inclusion $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \rightarrowtail \mathcal{O}\mathbf{D}_{\mathrm{dR}}(V) \neq 0$ is surjective if and only if $V$ is crystalline (see [14, Section 8.2 and Section 8.3] for details).

In conclusion, we have a functor

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathcal{O}\mathrm{cris}}(G_R) \longrightarrow \text{filtered } (\varphi, \partial)\text{-modules over } R\left[\frac{1}{p}\right].$$

Objects in the essential image are called *admissible* filtered $(\varphi, \partial)$-modules and the functor induces an equivalence of categories with the essential image (see [14, Théorèmes 8.4.2, 8.5.1]).

*Remark 2.2.* — In the arithmetic case, the essential image of $\mathbf{D}_{\mathrm{cris}}$, i.e. admissible filtered $\varphi$-modules can be described more explicitly. In particular, using certain invariants attached to filtered $\varphi$-modules one considers the full subcategory of *weakly admissible* filtered $\varphi$-modules and it is a result of Colmez and Fontaine that weakly admissible filtered $\varphi$-modules are admissible (in the sense above, see [16, Théorème A]). In the relative case, Brinon gave a definition of weakly admissible filtered $(\varphi, \partial)$-modules (see [14, p. 136]). However, the notion is not completely satisfactory as one does not obtain an equivalence between admissible and weakly admissible filtered $(\varphi, \partial)$-modules (see [31, Theorem 1.3]).

### 2.3.4. One dimensional de Rham and crystalline representations

In the 1-dimensional case, it is possible to classify all crystalline representations:

Proposition 2.3 ([14, Propositions 8.4.1, 8.6.1]). — *Let $\eta : G_R \to \mathbb{Z}_p^\times$ be a continuous character.*

(i) *$\eta$ is de Rham if and only if we can write $\eta = \eta_{\mathrm{f}}\eta_{\mathrm{ur}}\chi^n$ where $\eta_{\mathrm{f}}$ is a finite character, $\eta_{\mathrm{ur}}$ is an unramified character taking values in $1 + p\mathbb{Z}_p$ (therefore trivialized $\alpha \in 1 + p\widehat{R^{\mathrm{ur}}}$, see Remark 2.1) and $\chi$ is the $p$-adic cyclotomic character and $n \in \mathbb{Z}$.*

(ii) *$\eta$ is crystalline if and only if we can write $\eta = \eta_{\mathrm{f}}\eta_{\mathrm{ur}}\chi^n$ where $\eta_{\mathrm{f}}$ is a finite unramified character, $\eta_{\mathrm{ur}}$ is an unramified character taking values in $1 + p\mathbb{Z}_p$ (therefore trivialized by some $\alpha \in 1 + p\widehat{R^{\mathrm{ur}}}$, see Remark 2.1) and $\chi$ is the $p$-adic cyclotomic character and $n \in \mathbb{Z}$.*

*In particular, a 1-dimensional de Rham representation is potentially crystalline.*

(iii) *Let $V = \mathbb{Q}_p(\eta)$ be a one-dimensional crystalline representation. Then there exists a finite étale extension $R \to R'$ such that the $R'\left[\frac{1}{p}\right]$-module $R'\left[\frac{1}{p}\right] \otimes_{R[\frac{1}{p}]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is free. In particular, if $\eta_{\mathrm{f}}$ is trivial then $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is a free $R\left[\frac{1}{p}\right]$-module of rank 1.*

## 3. $(\varphi, \Gamma)$-modules and crystalline coordinates

We will keep the setting and notations of Section 2. In particular, we have that $F$ is a finite unramified extension of $\mathbb{Q}_p$ and $K = F(\mu_{p^m})$ for a fixed $m \in \mathbb{N}_{\geqslant 1}$ (fix $m \in \mathbb{N}_{\geqslant 2}$ if $p = 2$). Recall that $R$ is étale over $O_F\{X, X^{-1}\}$ and we have multivariate polynomials $Q_i(Z_1, \dots, Z_s) \in O_F\{X, X^{-1}\}[Z_1, \dots, Z_s]$ for $1 \leqslant i \leqslant s$ such that $\det\left(\frac{\partial Q_i}{\partial Z_j}\right)$ is invertible in $R$. In particular, the ring $O_F\{X, X^{-1}\}$ provides a system of coordinates for $R$.

### 3.1. $(\varphi, \Gamma)$-modules

In this section, we briefly recall the theory of relative $(\varphi, \Gamma)$-modules from [3, 4].

Let $F_n = F(\mu_{p^n})$ for $n \in \mathbb{N}$ and $F_\infty = \bigcup_n F_n$. We take $R_n$ to be the integral closure of $R \otimes_{O_F[X^{\pm 1}]} O_{F_n}\left[X_1^{p^{-n}}, \dots, X_d^{p^{-n}}\right]$ inside $\overline{R}\left[\frac{1}{p}\right]$ and set $R_\infty \coloneqq \bigcup_{n \geqslant m} R_n$ noting that $F_\infty \subset R_\infty\left[\frac{1}{p}\right]$. From Section 2.1.2 recall that $\mathbb{C}(\overline{R}) = \mathbb{C}^+(\overline{R})\left[\frac{1}{p}\right]$ and $\mathbb{C}(\overline{R})^\flat$ denotes its tilt. The ring $\mathbb{C}(\overline{R})^\flat$ is perfect of characteristic $p$ and we set $\mathbf{A}_{\overline{R}} \coloneqq W(\mathbb{C}(\overline{R})^\flat)$, the ring of $p$-typical Witt vectors with coefficients in $\mathbb{C}(\overline{R})^\flat$ and endowed with the weak topology (see [4, Section 2.10]). The absolute Frobenius over $\mathbb{C}(\overline{R})^\flat$ lifts to an endomorphism $\varphi : \mathbf{A}_{\overline{R}} \to \mathbf{A}_{\overline{R}}$, which we again call the Frobenius. The action of $G_R$ on $\mathbb{C}(\overline{R})^\flat$ extends to a continuous action on $\mathbf{A}_{\overline{R}}$ commuting with the Frobenius. The inclusion $\overline{F} \subset \overline{R}\left[\frac{1}{p}\right]$ induces inclusions $\mathbb{C}_p^\flat \subset \mathbb{C}(\overline{R})^\flat$ and $\mathbf{A}_{\overline{F}} \subset \mathbf{A}_{\overline{R}}$. Recall that we set $\mathbf{A}_{\mathrm{inf}}(\overline{R}) \coloneqq W(\mathbb{C}^+(\overline{R})^\flat)$. The inclusion $O_{\overline{F}} \subset \overline{R}$ induces inclusions $O_{\mathbb{C}_p}^\flat \subset \mathbb{C}^+(\overline{R})^\flat$ and $\mathbf{A}_{\mathrm{inf}}(O_{\overline{F}}) \subset \mathbf{A}_{\mathrm{inf}}(\overline{R})$.

### 3.1.1. The group $\Gamma_R$

The ring $R_\infty\big[\frac{1}{p}\big]$ is a Galois extension of $R\big[\frac{1}{p}\big]$ with Galois group $\Gamma_R \coloneqq \mathrm{Gal}\big(R_\infty\big[\frac{1}{p}\big]/R\big[\frac{1}{p}\big]\big)$ isomorphic to the semidirect product of $\Gamma_F$ and $\Gamma'_R$, where $\Gamma_F = \mathrm{Gal}(F_\infty/F)$ and $\Gamma'_R = \mathrm{Gal}\big(R_\infty\big[\frac{1}{p}\big]/F_\infty R\big[\frac{1}{p}\big]\big)$. In particular, we have an exact sequence

$$1 \longrightarrow \Gamma'_R \longrightarrow \Gamma_R \longrightarrow \Gamma_F \longrightarrow 1, \tag{3.1}$$

where (see [14, p. 9] and [3, Section 2.4])

$$\begin{aligned} \Gamma'_R = \mathrm{Gal}\left(R_\infty\left[\frac{1}{p}\right]/F_\infty R\left[\frac{1}{p}\right]\right) &\xrightarrow{\sim} \mathbb{Z}_p^d, \\ \chi : \Gamma_F = \mathrm{Gal}(F_\infty/F) &\xrightarrow{\sim} \mathbb{Z}_p^\times. \end{aligned}$$

The group $\Gamma_F$ can be viewed as a subgroup of $\Gamma_R$, i.e. we can take a section of the projection map in (3.1) such that for $\gamma \in \Gamma_F$ and $g \in \Gamma'_R$, we have $\gamma g \gamma^{-1} = g^{\chi(\gamma)}$. So we can choose topological generators $\{\gamma, \gamma_1, \ldots, \gamma_d\}$ of $\Gamma_R$ such that

$$\begin{aligned} &\gamma(\varepsilon) = \varepsilon^{\chi(\gamma)}, && \gamma_i(\varepsilon) = \varepsilon && \text{for } 1 \leqslant i \leqslant d, \\ &\gamma_i(X_i^\flat) = \varepsilon X_i^\flat, && \gamma_i(X_j^\flat) = X_j^\flat && \text{for } i \neq j \text{ and } 1 \leqslant j \leqslant d, \end{aligned}$$

and that $\gamma_0 = \gamma^e$ with $\chi(\gamma_0) = \exp(p^m)$, is a topological generator of $\Gamma_K = \mathrm{Gal}(K_\infty/K)$, where $K_\infty = F_\infty$ and $e = [K : F]$. It follows that $\{\gamma_1, \ldots, \gamma_d\}$ are topological generators of $\Gamma'_R$, $\gamma$ is a lift of a topological generator of $\Gamma_F$, and $\gamma_0$ is a topological generator of $\Gamma_K$. In particular,

$$\chi : \Gamma_K = \mathrm{Gal}(F_\infty/K) \xrightarrow{\sim} 1 + p^m\mathbb{Z}_p.$$

### 3.1.2. Setup

In [24, 25, 39], using the field-of-norms functor, Fontaine and Wintenberger constructed a non-archimedean complete discrete valuation field $\mathbf{E}_K \subset \widehat{K}_\infty^\flat$ of characteristic $p$ with residue field $\kappa$ and admitting a continuous action of $\Gamma_K$ (notation is a bit unfortunate as $\mathbf{E}_K$ depends only on $K_\infty$). Utilizing the isomorphism of Galois groups $\mathrm{Gal}(\overline{F}/K_\infty) \xrightarrow{\sim} \mathrm{Gal}(\mathbf{E}_K^{\mathrm{sep}}/\mathbf{E}_K)$ (also see tilting correspondence in [33] for a modern treatment), Fontaine classified mod-$p$ representations of $G_K$ in terms of étale $(\varphi, \Gamma_K)$-modules over $\mathbf{E}_K$. By some technical considerations one can then lift this to the classification of $\mathbb{Z}_p$-representations of $G_F$ in terms of étale $(\varphi, \Gamma_K)$-modules over a certain two dimensional regular local ring $\mathbf{A}_K \subset W(\widehat{K}_\infty^\flat)$ (see [21] for details).

We have an analogous theory in the relative setting, to describe which we need to consider generically étale algebras over finite extensions of $R$ in the cyclotomic tower $R_\infty/R$. More precisely, let $S \subset \overline{R}$ be a finite $R_n$-algebra with $S\left[\frac{1}{p}\right]$ étale over $R_n\left[\frac{1}{p}\right]$. For $k \geqslant n$ denote by $S_k$ the integral closure of $S \otimes_{R_n} R_k$ in $\overline{R}\left[\frac{1}{p}\right]$ and set $S_\infty := \bigcup_{k \geqslant n} S_k$. We have that $S_\infty$ is a normal $R_\infty$-algebra and an integral domain as a subring of $\overline{R}$. As in the case of $R$, for $S$ we define $G_S := \mathrm{Gal}\big(\overline{R}\left[\frac{1}{p}\right]/S\left[\frac{1}{p}\right]\big)$, $\Gamma_S := \mathrm{Gal}\big(S_\infty\left[\frac{1}{p}\right]/S\left[\frac{1}{p}\right]\big)$ and $H_S := \mathrm{Ker}\ (G_S \to \Gamma_S)$. Again, $\Gamma_S$ is isomorphic to the semidirect product of $\Gamma_{F_n}$ and $\Gamma'_S$, where $\Gamma'_S = \mathrm{Gal}\big(S_\infty\left[\frac{1}{p}\right]/F_\infty S\left[\frac{1}{p}\right]\big)$ is a finite index subgroup of $\Gamma'_R \xrightarrow{\sim} \mathbb{Z}_p^d$.

### 3.1.3. Rings in characteristic $p$

In the relative setting, Andreatta in [3] constructed an analogue of the subfield $\mathbf{E}_K \subset \widehat{K}_\infty^\flat$, i.e. to any $S$ as above, he associated a ring $\mathbf{E}_S \subset \mathrm{Fr}\, \widehat{S}_\infty^\flat$ functorial in $S_\infty$. Let us recall his definition: Let $\mathbf{E}_F^+$ denote the valuation ring of $\mathbf{E}_F$ and we have $\pi \in W\big(\widehat{F}_\infty^\flat\big)$ such that its reduction modulo $p$, denoted as $\overline{\pi} = \varepsilon - 1$, is a uniformizer of $\mathbf{E}_F^+$. Depending on $S$, let $\delta \in \mathbb{Q} \cap [0,1]$ small enough and $N \in \mathbb{N}$ large enough (see [3, Definition 4.2] for precise formulations of $\delta$ and $N$), and define the ring

$$\mathbf{E}_S^+ := \Big\{(a_0, \ldots, a_k, \ldots) \in \widehat{S}_\infty^\flat, \text{ such that } a_k \in S_k/p^\delta S_k \text{ for all } k \geqslant N\Big\}.$$

The ring $\mathbf{E}_S^+$ is finite and torsion free as an $\mathbf{E}_R^+$-module. It is a reduced Noetherian ring which is $\overline{\pi}$-adically complete. By construction, it is endowed with a $\overline{\pi}$-adically continuous action of $\Gamma_S$ and a Frobenius endomorphism $\varphi$, commuting with each other and compatible with respective structures on $\widehat{S}_\infty^\flat$. Moreover, $\mathbf{E}_S^+$ is a normal extension of $\mathbf{E}_R^+$, étale after inverting $\overline{\pi}$ and of degree equal to the generic degree of $R_m \subset S$. Further, the set of elements $\{\overline{\pi}, X_1^\flat, \ldots, X_d^\flat\}$ form an absolute $p$-basis of $\mathbf{E}_R^+$ (see [3, Proposition 4.5, Corollaries 5.3 & 5.4]). The ring $\widehat{S}_\infty^\flat$ coincides with the $\overline{\pi}$-adic completion of the perfect closure of $\mathbf{E}_S^+$ and the extension $\mathbf{E}_S^+ \to \widehat{S}_\infty^\flat$ is faithfully flat. Finally, set $\mathbf{E}_S := \mathbf{E}_S^+\left[\frac{1}{\overline{\pi}}\right]$.

Definition 3.1. — *Define $\mathbf{E}^+ := \bigcup_S \mathbf{E}_S^+$, where the union runs over $R_n$-subalgebras $S \subset \overline{R}$ for some $n \in \mathbb{N}$ such that $S$ is normal and finite as an $R_n$-module and $S\left[\frac{1}{p}\right]$ is étale over $R_n\left[\frac{1}{p}\right]$. Also, we set $\mathbf{E} := \mathbf{E}^+\left[\frac{1}{\overline{\pi}}\right]$. These rings are $\overline{\pi}$-adically complete and equipped with a Frobenius and a continuous $G_R$-action.*

*Remark 3.2.* — From [4, Proposition 2.9], we have $\big(\mathbb{C}^+(\overline{R})\big)^{H_R} = \widehat{R}_\infty$, $\big(\mathbb{C}^+(\overline{R})^\flat\big)^{H_R} = \widehat{R}_\infty^\flat$, $\big(\mathbb{C}(\overline{R})^\flat\big)^{H_R} = \widehat{R}_\infty^\flat\big[\frac{1}{\pi}\big]$, $(\mathbf{E}^+)^{H_R} = \mathbf{E}_R^+$ and $\mathbf{E}^{H_R} = \mathbf{E}_R$.

*Remark 3.3.* — We will describe $\mathbb{C}^+(\overline{R})^\flat$ as the ring of power-bounded elements inside $\mathbb{C}(\overline{R})^\flat$ (for the spectral norm). Recall that $\overline{R}$ is the union of finite $R$-subalgebras $S \subset \overline{\mathrm{Fr}\,(R)}$ such that $S\big[\frac{1}{p}\big]$ is étale over $R\big[\frac{1}{p}\big]$. Since $\overline{R}$ is an integral domain and $p$-adically separated, i.e. $\cap_{k\in\mathbb{N}} p^k\overline{R} = 0$, we obtain that the filtration by powers of the ideal $p\overline{R} \subset \overline{R}$ induces a sub-multiplicative norm (see [13, Section 1.3.3, Proposition 1]) which extends to $\overline{R}\big[\frac{1}{p}\big]$. A further "smoothening" of the aforementioned norm yields a power-multiplicative norm on $\overline{R}\big[\frac{1}{p}\big]$ (see [13, Section 1.3.2]) which we call the *spectral norm* on $\overline{R}\big[\frac{1}{p}\big]$. Let $C$ denote the completion of $\overline{R}\big[\frac{1}{p}\big]$ for the spectral norm and $C^\circ$ its power-bounded elements.

Next, one can show that under the spectral norm the power-bounded elements (or equivalently, the closed unit ball) of $\overline{R}\big[\frac{1}{p}\big]$ written as $\big(\overline{R}\big[\frac{1}{p}\big]\big)^\circ$ is exactly $\overline{R}$. Indeed, we have the obvious inclusion $\overline{R} \subset \big(\overline{R}\big[\frac{1}{p}\big]\big)^\circ$ and for the converse taking $x \in \big(\overline{R}\big[\frac{1}{p}\big]\big)^\circ$, one can reduce the claim to a finite $R$-subalgebra $S \subset \overline{R}$ integrally closed in $\overline{R}\big[\frac{1}{p}\big]$ and such that $x \in S\big[\frac{1}{p}\big]$. Then it easily follows that $S = \big(S\big[\frac{1}{p}\big]\big)^\circ = S\big[\frac{1}{p}\big] \cap \big(\overline{R}\big[\frac{1}{p}\big]\big)^\circ \subset \overline{R}\big[\frac{1}{p}\big]$. So we obtain that the topology induced by the spectral norm is equivalent to the $p$-adic topology on $\overline{R}\big[\frac{1}{p}\big]$, therefore $C = \mathbb{C}(\overline{R})$ and $C^\circ = \mathbb{C}^+(\overline{R})$ and $(\mathbb{C}(\overline{R}), \mathbb{C}^+(\overline{R}))$ is a uniform adic Banach $\mathbb{Q}_p$-algebra (see [29, Definitions 2.4.1 and 2.8.1]).

Finally, by the perfectoid correspondence of uniform adic Banach algebras in [29, Theorem 3.6.5], we obtain that $(\mathbb{C}(\overline{R})^\flat, \mathbb{C}^+(\overline{R})^\flat)$ is a uniform adic Banach $\mathbb{F}_p$-algebra such that the topology induced by the spectral norm (arising from the sub-multiplicative norm induced by the ideal $p^\flat\mathbb{C}^+(\overline{R})^\flat \subset \mathbb{C}^+(\overline{R})^\flat$) is equivalent to the topology on $(\mathbb{C}(\overline{R})^\flat, \mathbb{C}^+(\overline{R})^\flat)$ described in Section 2.1.2. Finally, since $\mathbb{C}^+(\overline{R})$ is the ring of power-bounded elements in $\mathbb{C}(\overline{R})$ we obtain that the its tilt $\mathbb{C}^+(\overline{R})^\flat$ is the ring of power-bounded elements in $\mathbb{C}(\overline{R})^\flat$.

*Remark 3.4.* — Let us denote the natural valuation on $\mathbb{C}_p^\flat$ by $\upsilon^\flat$. Then one can show that $\upsilon^\flat(\overline{\pi}) = \frac{p}{p-1} > 0$, i.e. $\overline{\pi}$ is not invertible in $O_{\mathbb{C}_p}^\flat$. Since $O_{\mathbb{C}_p}^\flat = \mathbb{C}_p^\flat \cap \mathbb{C}^+(\overline{R})^\flat \subset \mathbb{C}(\overline{R})^\flat$, we obtain that $\overline{\pi}$ is not invertible in $\mathbb{C}^+(\overline{R})^\flat$. Moreover, as $\mathbb{C}^+(\overline{R})^\flat$ is the ring of power-bounded elements in $\mathbb{C}^+(\overline{R})^\flat$ (see Remark 3.3) we conclude that $\mathbf{E}^+ = \mathbf{E} \cap \mathbb{C}^+(\overline{R})^\flat \subset \mathbb{C}(\overline{R})^\flat$.

### 3.1.4. Rings in characteristic 0

We have liftings of the rings discussed above to characteristic 0. In other words, there exists a Noetherian regular domain $\mathbf{A}_R \subset W\big(\widehat{R}^\flat_\infty\big[\frac{1}{\pi}\big]\big)$, complete for the weak topology and endowed with a continuous action of $\Gamma_R$ and a Frobenius such that $\mathbf{A}_R/p\mathbf{A}_R = \mathbf{E}_R$. Moreover, $\mathbf{A}_R$ contains a subring $\mathbf{A}_R^+$ lifting $\mathbf{E}_R^+$ complete for the weak topology with $\pi, [X_1^\flat], \ldots, [X_d^\flat] \in \mathbf{A}_R^+$ (see [3, Appendix C]). Furthermore, for $S$ as in Definition 3.1 let $\mathbf{A}_S$ denote the unique finite étale $\mathbf{A}_R$-algebra lifting the finite étale extension $\mathbf{E}_R \subset \mathbf{E}_S$. It is a Noetherian regular domain, complete for the weak topology and endowed with a continuous action of $\Gamma_S$ and a Frobenius, lifting the ones defined on $\mathbf{E}_S$. Moreover, it contains a subring $\mathbf{A}_S^+$ lifting $\mathbf{E}_S^+$ so that the former is complete for the weak topology. In characteristic 0, we set $\mathbf{B}_{\overline{R}} := \mathbf{A}_{\overline{R}}\big[\frac{1}{p}\big] = \bigcup_{j\in\mathbb{N}} p^{-j}\mathbf{A}_{\overline{R}}$ equipped with the direct limit topology (see [3, Section 7] for details).

Definition 3.5. — *Define $\mathbf{A}$ := completion of $\bigcup_S \mathbf{A}_S \subset \mathbf{A}_{\overline{R}}$ for the $p$-adic topology, where the union runs over all $R_n$-subalgebras $S \subset \overline{R}$ as in Definition 3.1. Equip $\mathbf{A}$ with the weak topology induced by the inclusion $\mathbf{A} \subset \mathbf{A}_{\overline{R}}$. Moreover, we set $\mathbf{A}^+ := \mathbf{A} \cap \mathbf{A}_{\inf}(\overline{R})$, $\mathbf{B}^+ := \mathbf{A}^+\big[\frac{1}{p}\big]$ and $\mathbf{B} := \mathbf{A}\big[\frac{1}{p}\big]$ equipped with induced weak topology. These rings are stable under $\varphi$ and admit a continuous $G_R$-action.*

*Remark 3.6.* — In Definition 3.5 one can take the base ring as $R[\varpi]$ instead of $R$ to obtain period rings $\mathbf{A}_\varpi^+ \subset \mathbf{A}_\varpi$ (instead of $\mathbf{A}^+ \subset \mathbf{A}$). In particular, one has that $\pi_m = \varphi^{-m}(\pi) \in \mathbf{A}_\varpi^+$ and it easily follows that $\mathbf{A}^+ \subset \mathbf{A}_\varpi^+ \subset \mathbf{A}_{\inf}(\overline{R})$ compatible with Frobenius and $G_R$-action.

*Remarks 3.7.*

(i) It follows from definitions that

$$p\mathbf{A}^+ = p\mathbf{A} \cap \mathbf{A}_{\inf}(\overline{R}) = \mathbf{A} \cap p\mathbf{A}_{\inf}(\overline{R}) = p(\mathbf{A} \cap \mathbf{A}_{\inf}(\overline{R})).$$

Therefore, from Remark 3.4 it easily follows that $\mathbf{A}^+/p\mathbf{A}^+ = \mathbf{E}^+$.

(ii) From [4, Lemma 2.11] we have $\mathbf{A}^{H_R} = \mathbf{A}_R$ and $(\mathbf{A}^+)^{H_R} = \mathbf{A}_R^+$.

### 3.1.5. Some lemmas on matrices

Let us note some results which will be useful in the proof of Proposition 4.11.

Lemma 3.8. — *Let $h \in \mathbb{N}$ and matrices $Y \in \mathrm{Mat}(h, \mathbf{E})$ and $X, Z, W \in \mathrm{Mat}(h, \mathbf{E}^+)$ such that $\varphi(Y) = XYZ + W$, then $Y \in \mathrm{Mat}(h, \mathbf{E}^+)$.*

*Proof.* — From Remark 3.4 we have $\mathbf{E}^+ = \mathbf{E} \cap \mathbb{C}^+(\overline{R})^\flat$. So it is enough to show that $Y \in \mathrm{Mat}(h, \mathbb{C}^+(\overline{R})^\flat)$. Recall that we have $\mathbb{C}(\overline{R})^\flat = \mathbb{C}^+(\overline{R})^\flat\big[\frac{1}{p^\flat}\big]$. Therefore, for some smallest $k \in \mathbb{N}$, we can write $Y = \frac{1}{(p^\flat)^k} Y_1$ with $Y_1 \in \mathrm{Mat}(h, \mathbb{C}^+(\overline{R})^\flat)$. Now, applying $\varphi$ we get that $\varphi\big(\frac{1}{(p^\flat)^k} Y_1\big) = \frac{1}{(p^\flat)^k} XY_1Z + W$, which can be rewritten as $\frac{(p_1^\flat)^k}{(p^\flat)^k} Y_1 = \varphi^{-1}(XY_1Z + (p^\flat)^k W)$, where $p_1^\flat = \varphi^{-1}(p^\flat)$. In the last equality, note that the expression on the left $\frac{(p_1^\flat)^k}{(p^\flat)^k} Y_1 \in \mathrm{Mat}(h, \mathbb{C}(\overline{R})^\flat)$, whereas the expression on the right $\varphi^{-1}(XY_1Z + (p^\flat)^k W) \in \mathrm{Mat}(h, \varphi^{-1}(\mathbb{C}^+(\overline{R})^\flat)) = \mathrm{Mat}(h, \mathbb{C}^+(\overline{R})^\flat)$ since $\mathbb{C}^+(\overline{R})^\flat$ is perfect. So we obtain that $\frac{(p_1^\flat)^k}{(p^\flat)^k} Y_1 \in \mathrm{Mat}(h, \mathbb{C}^+(\overline{R})^\flat)$, i.e. $Y = \frac{1}{(p^\flat)^k} Y_1 \in \mathrm{Mat}\big(h, \frac{1}{(p_1^\flat)^k} \mathbb{C}^+(\overline{R})^\flat\big)$. Next, write $Y = \frac{1}{(p_1^\flat)^k} Y_2$ with $Y_2 \in \mathrm{Mat}(h, \mathbb{C}^+(\overline{R})^\flat)$. Again, applying $\varphi$ and arguing as above, one gets $Y \in \mathrm{Mat}\big(h, \frac{1}{(p_2^\flat)^k} \mathbb{C}^+(\overline{R})^\flat\big)$, where $p_2^\flat = \varphi^{-2}(p^\flat)$. Now, it easily follows by induction on $n \in \mathbb{N}$ that $Y \in \mathrm{Mat}\big(h, \frac{1}{(p_n^\flat)^k} \mathbb{C}^+(\overline{R})^\flat\big)$, where $p_n^\flat = \varphi^{-n}(p^\flat)$. Therefore, we get that

$$Y \in \mathrm{Mat}\left(h, \cap_{n \in \mathbb{N}} \frac{1}{(p_n^\flat)^k} \mathbb{C}^+(\overline{R})^\flat\right) \subset \mathrm{Mat}(h, \mathbb{C}(\overline{R})^\flat).$$

But since $\mathbb{C}^+(\overline{R})^\flat$ is the ring of power-bounded elements in $\mathbb{C}(\overline{R})^\flat$, we obtain that

$$\cap_{n \in \mathbb{N}} \frac{1}{(p_n^\flat)^k} \mathbb{C}^+(\overline{R})^\flat = \mathbb{C}^+(\overline{R})^\flat.$$

Hence, we get $Y \in \mathrm{Mat}(h, \mathbb{C}^+(\overline{R})^\flat)$ as desired. □

Lemma 3.9. — *Let $h \in \mathbb{N}$ and matrices $Y \in \mathrm{Mat}(h, \mathbf{A})$ and $X, Z, W \in \mathrm{Mat}(h, \mathbf{A}^+)$ such that $\varphi(Y) = XYZ + W$, then $Y \in \mathrm{Mat}(h, \mathbf{A}^+)$.*

*Proof.* — Reducing the equation modulo $p$ we have $\varphi(\overline{Y}) = \overline{X}\,\overline{Y}\,\overline{Z} + \overline{W}$, with $\overline{Y} \in \mathrm{Mat}(h, \mathbf{E})$ and $\overline{X}, \overline{Z}, \overline{W} \in \mathrm{Mat}(h, \mathbf{E}^+)$. Therefore, from Lemma 3.8 we obtain that $\overline{Y} \in \mathrm{Mat}(h, \mathbf{E}^+)$. As we have $\mathbf{A}^+/p\mathbf{A}^+ = \mathbf{E}^+$ (see Remark 3.7(ii)), let $V_0 \in \mathrm{Mat}(h, \mathbf{A}^+)$ such that $\overline{Y} = \overline{V}_0$ and $\varphi(\overline{V}_0) = \overline{X}\,\overline{V}_0\,\overline{Z} + \overline{W}$. So we can write $Y = V_0 + pY_1$ with $Y_1 \in \mathrm{Mat}(h, \mathbf{A})$, and obtain that $\varphi(V_0 + pY_1) = X(V_0 + pY_1)Z + W$. Simplifying the latter expression, we have $\varphi(V_0) - (XV_0Z + W) = p(XY_1Z - \varphi(Y_1))$. Since $\varphi(V_0) - (XV_0Z + W) \in \mathrm{Mat}(h, p\mathbf{A}^+)$, we conclude that

$$\varphi(Y_1) - XY_1Z \in \mathrm{Mat}(h, \mathbf{A}^+).$$

In other words, we have an equality $\varphi(Y_1) = XY_1Z + W_1$ with $Y_1 \in \mathrm{Mat}(h, \mathbf{A})$ and $X, Z, W_1 \in \mathrm{Mat}(h, \mathbf{A}^+)$. Repeating the argument as above, we get that $\overline{Y}_1 \in \mathrm{Mat}(h, \mathbf{E}^+)$ and we can take a lift to write $Y_1 = V_1 + pY_2$ with $V_1 \in \mathrm{Mat}(h, \mathbf{A}^+)$ and $Y_2 \in \mathrm{Mat}(h, \mathbf{A})$. This gives us that $Y = V_0 + pV_1 + p^2Y_2$. Now, it easily follows by induction on $n \in \mathbb{N}$ that $Y = V_0 + pV_1 + \cdots + p^{n-1}V_{n-1} + p^nY_n$ with $V_i \in \mathrm{Mat}(h, \mathbf{A}^+)$ for $0 \leqslant i \leqslant n-1$ and $Y_n \in \mathrm{Mat}(h, \mathbf{A})$. Letting $n \to +\infty$ and noting that $\mathbf{A}^+$ is $p$-adically complete, we obtain that $Y \in \mathrm{Mat}(h, \mathbf{A}^+)$ as desired. □

### 3.1.6. Étale $(\varphi, \Gamma_R)$-modules

Definition 3.10. — *A $(\varphi, \Gamma_R)$-module $D$ over $\mathbf{A}_R$ is a finitely generated module equipped with*

(i) *A semilinear action of $\Gamma_R$, continuous for the weak topology,*
(ii) *A $\Gamma_R$-equivariant Frobenius-semilinear endomorphism $\varphi$.*

*We say that $D$ is* étale *if the natural map $1 \otimes \varphi : \mathbf{A}_R \otimes_{\mathbf{A}_R, \varphi} D \to D$ is an isomorphism of $\mathbf{A}_R$-modules.*

Denote by $(\varphi, \Gamma_R)\text{-}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_R}$ the category of étale $(\varphi, \Gamma_R)$-modules over $\mathbf{A}_R$ with morphisms between objects being continuous, $(\varphi, \Gamma_R)$-equivariant morphisms of $\mathbf{A}_R$-modules. Next, denote by $\mathrm{Rep}_{\mathbb{Z}_p}(G_R)$ the category of finitely generated $\mathbb{Z}_p$-modules equipped with a linear and continuous action of $G_R$, with morphisms between objects being continuous and $G_R$-equivariant morphisms of $\mathbb{Z}_p$-modules.

Let $T$ be a $\mathbb{Z}_p$-representation of $G_R$. The $\mathbf{A}_R$-module $\mathbf{D}(T) := (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_R}$ is equipped with a semilinear operator $\varphi$ and a continuous (for the weak topology) and semilinear action of $\Gamma_R$, commuting with each other. Moreover, $\mathbf{D}(T)$ is an étale $(\varphi, \Gamma_R)$-module. Furthermore, if $T$ is free of finite rank, then $\mathbf{D}(T)$ is a projective module of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$ (see [3, Theorem 7.11]). The functor

$$\mathbf{D} : \mathrm{Rep}_{\mathbb{Z}_p}(G_R) \longrightarrow (\varphi, \Gamma_R)\text{-}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_R}, \tag{3.2}$$

induces an equivalence of categories (see [3, Theorem 7.11]), and the natural map $\mathbf{A} \otimes_{\mathbf{A}_R} \mathbf{D}(T) \xrightarrow{\sim} \mathbf{A} \otimes_{\mathbb{Z}_p} T$ is an isomorphism of $\mathbf{A}$-modules compatible with Frobenius and the action of $G_R$ on each side.

## 3.2. Crystalline coordinates

In this section we will introduce certain "coordinate" rings. As we shall see in the next section, these rings are related to period rings appearing in Section 2 and Section 3.1.

Let $r_\varpi^+$ and $r_\varpi$ denote the algebras $O_F[\![X_0]\!]$ and $O_F[\![X_0]\!]\{X_0^{-1}\}$. Sending $X_0$ to $\varpi$ induces a surjective homomorphism $r_\varpi^+ \twoheadrightarrow O_K$. Let $R_{\varpi,\square}^+$ denote the completion of $O_F[X_0, X, X^{-1}]$ for the $(p, X_0)$-adic topology. Sending $X_0$ to $\varpi$ induces a surjective homomorphism $R_{\varpi,\square}^+ \twoheadrightarrow O_K\{X, X^{-1}\}$, whose kernel is generated by $P = P_\varpi(X_0)$. This provides a closed embedding of $\operatorname{Spf} O_K\{X, X^{-1}\}$ into a formal scheme $\operatorname{Spf} R_{\varpi,\square}^+$, which is smooth over $O_F$. Recall that $R$ is étale over $O_F\{X, X^{-1}\}$ and we have multivariate polynomials $Q_i(Z_1, \ldots, Z_s) \in O_F\{X, X^{-1}\}[Z_1, \ldots, Z_s]$ for $1 \leqslant i \leqslant s$ such that $\det\left(\frac{\partial Q_i}{\partial Z_j}\right)$ is invertible in $R$. So we can set $R_\varpi^+$ to be the quotient by $(Q_1, \ldots, Q_s)$ of the completion of $R_{\varpi,\square}^+[Z_1, \ldots, Z_s]$ for $(p, X_0)$-adic topology. Again, we have that $\det\left(\frac{\partial Q_i}{\partial Z_j}\right)$ is invertible in $R_\varpi^+$ (since $R \rightarrowtail R_\varpi^+$). Hence, $R_\varpi^+$ is étale over $R_{\varpi,\square}^+$ and smooth over $O_F$. Sending $X_0$ to $\varpi$ induces a surjective homomorphism $R_\varpi^+ \twoheadrightarrow R[\varpi]$ whose kernel is generated by $P = P_\varpi(X_0)$. This can be summarized by the commutative diagram

$$\begin{tikzcd}
\operatorname{Spf} R[\varpi] \arrow[rr, tail] \arrow[dr] \arrow[ddd] & & \operatorname{Spf} R_\varpi^+ \arrow[dl] \arrow[ddd] \\
& \operatorname{Spf} R \arrow[d] & \\
& \operatorname{Spf} O_F\{X, X^{-1}\} & \\
\operatorname{Spf} O_K\{X, X^{-1}\} \arrow[rr, tail] \arrow[ur] & & \operatorname{Spf} R_{\varpi,\square}^+, \arrow[ul]
\end{tikzcd}$$

where the vertical arrows are étale extensions and the horizontal maps are obtained by sending $X_0 \mapsto \varpi$, and the rest are natural maps. Finally, we set $R_\varpi = p$-adic completion of $R_\varpi^+\left[\frac{1}{X_0}\right]$.

Next, since $P \equiv X_0^e \mod p$, we have

$$R_\varpi^+\left[\frac{P^k}{k!}\right]_{k\in\mathbb{N}} = R_\varpi^+\left[\frac{X_0^k}{[k/e]!}\right]_{k\in\mathbb{N}}.$$

So, we set $R_\varpi^{\mathrm{PD}} := p$-adic completion of $R_\varpi^+\left[\frac{P^k}{k!}\right]_{k\in\mathbb{N}}$. In summary, we have a diagram of formal schemes where the horizontal arrows are closed embeddings into formal schemes smooth over $O_F$, obtained by sending $X_0 \mapsto \varpi$

on the level of algebras,

$$
\begin{tikzcd}
& \mathrm{Spf}\ R_{\varpi}^{\mathrm{PD}} \arrow[dr] & \\
\mathrm{Spf}\ R[\varpi] \arrow[ur, tail] \arrow[rr, tail] \arrow[d] & & \mathrm{Spf}\ R_{\varpi}^{+} \arrow[d] \\
\mathrm{Spf}\ O_K\{X, X^{-1}\} \arrow[rr, tail] \arrow[d] & & \mathrm{Spf}\ R_{\varpi,\square}^{+} \arrow[d] \\
\mathrm{Spf}\ O_K \arrow[rr, tail] \arrow[d] & & \mathrm{Spf}\ r_{\varpi}^{+} \arrow[dll] \\
\mathrm{Spf}\ O_F. & &
\end{tikzcd}
$$

Recall that $P$ generates the kernel of the surjective map $R_{\varpi}^{+} \twoheadrightarrow R[\varpi]$ and divided powers of $P$ generate the kernel of the surjective map $R_{\varpi}^{\mathrm{PD}} \twoheadrightarrow R[\varpi]$.

Definition 3.11. — *Endow the ring $R_{\varpi}^{\mathrm{PD}}$ with a filtration by divided power ideals as*

$$\mathrm{Fil}^k R_{\varpi}^{\mathrm{PD}} = (P^{[n]},\ n \geqslant k) \subset R_{\varpi}^{\mathrm{PD}} \ \ \textit{for}\ \ k \in \mathbb{N}.$$

*In other words, the filtration on $R_{\varpi}^{\mathrm{PD}}$ is given by divided powers of the kernel of $R_{\varpi}^{\mathrm{PD}} \twoheadrightarrow R[\varpi]$. Furthermore, the ring $R_{\varpi}^{+}$ is endowed with the induced filtration*

$$\mathrm{Fil}^k R_{\varpi}^{+} := R_{\varpi}^{+} \cap \mathrm{Fil}^k R_{\varpi}^{\mathrm{PD}} = P^k R_{\varpi}^{+} \ \ \textit{for}\ \ k \in \mathbb{N},$$

*where the last equality follows since $P$ generates the kernel of $R_{\varpi}^{+} \twoheadrightarrow R[\varpi]$.*

### 3.3. Cyclotomic embedding

In this section, we will describe the relationship between $R_{\varpi}^{\star}$ for $\star \in \{\ , +, \mathrm{PD}\}$ and the period rings discussed in Section 2 and Section 3.1. We start by defining the (cyclotomic) Frobenius endomorphism on the former rings. Over $R_{\varpi,\square}^{+}$ define a lift of the absolute Frobenius on $R_{\varpi,\square}^{+}/p$ by

$$
\begin{aligned}
\varphi : R_{\varpi,\square}^{+} &\longrightarrow R_{\varpi,\square}^{+} \\
X_0 &\longmapsto (1+X_0)^p - 1 \\
X_i &\longmapsto X_i^p, \ \ \text{for}\ \ 1 \leqslant i \leqslant d,
\end{aligned}
$$

which we will call the (cyclotomic) Frobenius. Clearly, $\varphi(x) - x^p \in pR^+_{\varpi,\square}$ for $x \in R^+_{\varpi,\square}$. Using the implicit function theorem for topological rings [17, Proposition 2.1], we can extend the Frobenius homomorphism to $\varphi : R^+_{\varpi} \to R^+_{\varpi}$. By continuity, the Frobenius endomorphism $\varphi$ admits unique extensions $\varphi : R^{\mathrm{PD}}_{\varpi} \to R^{\mathrm{PD}}_{\varpi}$ and $\varphi : R_{\varpi} \to R_{\varpi}$.

### 3.3.1. The rings $\mathbf{A}^{\star}_{R,\varpi}$

We will describe the (cyclotomic) embeddings of $R^{\star}_{\varpi}$ into various period rings discussed in Section 2 and Section 3.1. Define an embedding

$$\begin{aligned}\iota_{\mathrm{cycl}} : R^+_{\varpi,\square} &\longrightarrow \mathbf{A}_{\mathrm{inf}}(\overline{R})\\ X_0 &\longmapsto \pi_m = \varphi^{-m}(\pi),\\ X_i &\longmapsto [X_i^{\flat}], \quad \text{for } 1 \leqslant i \leqslant d.\end{aligned}$$

Lemma 3.12. — *The map $\iota_{\mathrm{cycl}}$ has a unique extension to an embedding $R^+_{\varpi} \to \mathbf{A}_{\mathrm{inf}}(\overline{R})$ such that $\theta \circ \iota_{\mathrm{cycl}}$ is the projection $R^+_{\varpi} \to R[\varpi]$.*

*Proof.* — We can use the implicit function theorem [17, Proposition 2.1] to extend the embedding to $\iota_{\mathrm{cycl}} : R^+_{\varpi} \to \mathbf{A}_{\mathrm{inf}}(\overline{R})$. Next, from defintions we already have that $\theta \circ \iota_{\mathrm{cycl}} : R^+_{\varpi,\square} \twoheadrightarrow O_K\{X, X^{-1}\}$ coincides with the canonical projection and $R^+_{\varpi}$ is étale over $R^+_{\varpi,\square}$, hence the second claim follows. $\square$

This embedding commutes with Frobenius on either side, i.e. $\iota_{\mathrm{cycl}} \circ \varphi = \varphi \circ \iota_{\mathrm{cycl}}$. By continuity, the morphism $\iota_{\mathrm{cycl}}$ extends to embeddings $\iota_{\mathrm{cycl}} : R^{\mathrm{PD}}_{\varpi} \rightarrowtail \mathbf{A}_{\mathrm{cris}}(\overline{R})$ and $\iota_{\mathrm{cycl}} : R_{\varpi} \rightarrowtail \mathbf{A}_{\overline{R}}$. Denote by $\mathbf{A}^+_{R,\varpi}$ and $\mathbf{A}_{R,\varpi}$ the image in $\mathbf{A}_{\overline{R}}$ of $R^+_{\varpi}$ and $R_{\varpi}$ respectively, under the map $\iota_{\mathrm{cycl}}$. Similarly, let $\mathbf{A}^{\mathrm{PD}}_{R,\varpi} := \iota_{\mathrm{cycl}}\big(R^{\mathrm{PD}}_{\varpi}\big) \subset \mathbf{A}_{\mathrm{cris}}(\overline{R})$. These rings are stable under the action of $\Gamma_R$ (see [17, Section 2.5.3]). Moreover, these embeddings induce a filtration on $\mathbf{A}^{\star}_{R,\varpi}$ for $\star \in \{+, \mathrm{PD}\}$ and $r \in \mathbb{Z}$ (use Definition 3.11).

*Remark 3.13.* — Note that we write $\mathbf{A}^+_{R,\varpi}$ and so on instead of slightly cumbersome notation $\mathbf{A}^+_{R[\varpi]}$ or simpler notation $\mathbf{A}^+_S$ for $S = R[\varpi]$, in order to emphasize the choice of root of unity in the definition.

We note a simple lemma that will be useful later.

Lemma 3.14. — *$\frac{t}{\pi}$ is a unit in $\mathbf{A}^{\mathrm{PD}}_{F,\varpi} \subset \mathbf{A}^{\mathrm{PD}}_{R,\varpi}$.*

*Proof.* — We can write the fraction

$$\frac{t}{\pi} = \frac{\log(1+\pi)}{\pi} = \sum_{k \geqslant 0} (-1)^k \frac{\pi^k}{k+1}.$$

Formally, we can write

$$\frac{\pi}{t} = \frac{\pi}{\log(1+\pi)} = 1 + b_1\pi + b_2\pi^2 + b_3\pi^3 + \cdots,$$

where $\upsilon_p(b_k) \geqslant -\frac{k}{p-1}$ for all $k \geqslant 1$. Since $\pi = (1+\pi_m)^{p^m} - 1$, we get that $\pi \in (p, \pi_m^{p^m})\mathbf{A}_{F,\varpi}^+$ (as $m \geqslant 1$). By induction over $k$, we can easily conclude that $\pi^k \in \big(p, \pi_m^{p^m}\big)^k \mathbf{A}_{F,\varpi}^{\mathrm{PD}}$. Using this, we can re-express the series $\sum_k b_k\pi^k$ as a power series in $\pi_m$, written as $\sum_i c_i\pi_m^i$. We need to check that this re-expressed series converges in $\mathbf{A}_{F,\varpi}^{\mathrm{PD}}$. To do this, we collect the terms with coefficients having the smallest $p$-adic valuation for each power of $\pi_m^{p^m}$ in the re-expressed series. For $k \geqslant 1$, $b_k$ has the smallest $p$-adic valuation among the coefficients of $\pi_m^{p^m k}$, and therefore it has the least $p$-adic valuation among coefficients of $\pi_m^i$ for $p^m k \leqslant i < p^m(k+1)$. We write the collection of these terms as

$$\sum_{k\geqslant 1}(-1)^{k+1}b_k\pi_m^{p^m k} = \sum_{k\geqslant 1}(-1)^{k+1}b_k\left\lfloor\frac{p^m k}{e}\right\rfloor!\frac{\pi_m^{p^m k}}{\lfloor p^m k/e\rfloor!},\tag{3.3}$$

and by the preceding discussion it is sufficient to show that these coefficients go to 0 as $k \to +\infty$. Moreover, for (3.3) it would suffice to check the estimate for $k = (p-1)j$ as $j \to +\infty$ (this gets rid of the floor function above). With the observation in Remark 3.15, we have

$$\begin{aligned}\upsilon_p\left(b_k\left\lfloor\frac{p^m k}{e}\right\rfloor!\right) &= \upsilon_p(b_k) + \upsilon_p((pj)!)\\ &\geqslant -\frac{(p-1)j}{p-1} + \frac{pj - s_p(pj)}{p-1} = \frac{j - s_p(j)}{p-1} = \upsilon_p(j!),\end{aligned}$$

which goes to $+\infty$ as $j \to +\infty$. Hence, $\frac{\pi}{t}$ converges in $\mathbf{A}_{F,\varpi}^{\mathrm{PD}}$ and is an inverse to $\frac{t}{\pi}$. □

The following elementary observation was used above,

*Remark 3.15.* — Let $n \in \mathbb{N}$, so we can write $n = \sum_{i=0}^k n_i p^i$ for some $k \in \mathbb{N}$, where $0 \leqslant n_i \leqslant p-1$ for $0 \leqslant i \leqslant k$. Let us set $s_p(n) = \sum_{i=0}^k n_i$. Then we have

$$\begin{aligned}\upsilon_p(n!) = \sum_{j\geqslant 1}\left\lfloor\frac{n}{p^j}\right\rfloor &= \sum_{j\geqslant 0}\left\lfloor\frac{\sum_{i=0}^k n_i p^i}{p^j}\right\rfloor = \sum_{j=1}^k\sum_{i=j}^k n_i p^{i-j}\\ &= \sum_{i=1}^k n_i\sum_{j=1}^i p^j = \sum_{i=1}^k n_i\frac{p^i-1}{p-1} = \frac{n - s_p(n)}{p-1}.\end{aligned}$$

Also, note that we have $s_p(pn) = s_p(n)$ for any $n \in \mathbb{N}$.

Lemma 3.16. — *Let $i \in \{0, 1, \ldots, d\}$. Then $(\gamma_i - 1)\mathbf{A}^{\bigstar}_{R,\varpi} \subset \pi\mathbf{A}^{\bigstar}_{R,\varpi}$ for $\bigstar \in \{+, \mathrm{PD}\}$;*

*Proof.* — First, let $i = 0$. Then we have

$$\begin{aligned}(\gamma_0 - 1)\pi_m &= (1 + \pi_m)\big((1 + \pi_m)^{\chi(\gamma_0)-1} - 1\big)\\ &= (1 + \pi_m)\big((1 + \pi_m)^{p^m a} - 1\big) = (1 + \pi_m)((1 + \pi)^a - 1)\\ &= (1 + \pi_m)\left(a\pi + \frac{a(a-1)}{2!}\pi^2 + \frac{a(a-1)(a-2)}{3!}\pi^3 + \cdots\right) = \pi x,\end{aligned}$$

for some $x \in \mathbf{A}^+_{F,\varpi}$, i.e. $(\gamma_0 - 1)\pi_m \in \pi\mathbf{A}^+_{F,\varpi}$. Then it follows that

$$(\gamma_0 - 1)\mathbf{A}^{\bigstar}_{F,\varpi} \subset \pi\mathbf{A}^{\bigstar}_{F,\varpi} \text{ for } \bigstar \in \{+, \mathrm{PD}\}.$$

Next, for $i \in \{1, \ldots, d\}$ we have $(\gamma_i - 1)[X_i^\flat] = \pi[X_i^\flat] \in \pi\mathbf{A}^+_{R,\varpi}$ and $(\gamma_i - 1)\big([X_i^\flat]^{-1}\big) = -\pi(1 + \pi)^{-1}[X_i^\flat]^{-1} \in \pi\mathbf{A}^+_{R,\varpi}$. Therefore, we get the claim. $\square$

### 3.3.2. The ring $\mathbf{A}^+_R$

The preceding discussion works well for $R[\varpi]$ where $\varpi = \zeta_{p^m} - 1$ for $m \in \mathbb{N}_{\geqslant 1}$ ($m \in \mathbb{N}_{\geqslant 2}$ if $p = 2$). For $R$ one can repeat the construction above to obtain the period ring $\mathbf{A}^+_R \subset \mathbf{A}^+_{R,\varpi}$ (the embedding $R^+_\varpi \rightarrowtail \mathbf{A}_{\mathrm{inf}}(\overline{R})$ for $R$ sends $X_0 \mapsto \pi$). Moreover, restriction of the map $\theta$ gives us a surjective map $\theta : \mathbf{A}^+_R \twoheadrightarrow R$ whose kernel is principal and generated by $\pi$ (since $\theta \circ \iota_{\mathrm{cycl}} = id$ on $R$). Next, over $\mathbf{A}^+_{R,\varpi}$ the filtration is given as $\mathrm{Fil}^k\mathbf{A}^+_{R,\varpi} = \xi^k\mathbf{A}^+_{R,\varpi}$, where $\xi = \frac{\pi}{\pi_1}$. However, $\xi \notin \mathbf{A}^+_R$. Therefore, we equip $\mathbf{A}^+_R$ with the induced filtration $\mathrm{Fil}^k\mathbf{A}^+_R = \mathbf{A}^+_R \cap \mathrm{Fil}^k\mathbf{A}^+_{R,\varpi}$. Then describing the filtration as kernel of the $\theta$ map, we obtain

Lemma 3.17. — $\mathrm{Fil}^k\mathbf{A}^+_R = \pi^k\mathbf{A}^+_R$.

*Remark 3.18.* — Let $\mathbf{A}^+$ be the ring from Definition 3.5 and $\mathbf{A}^+_\varpi$ be the ring defined in Remark 3.6. From the definitions it follows that $\mathbf{A}^+_{R,\varpi} \otimes_{\mathbf{A}^+_R} \mathbf{A}^+ \xrightarrow{\sim} \mathbf{A}^+_\varpi$ compatible with Frobenius and $G_R$-action. Moreover, we have $\mathbf{A}^+_R = (\mathbf{A}^+)^{H_R}$ and $\mathbf{A}^+_{R,\varpi} = (\mathbf{A}^+_\varpi)^{H_{R,\varpi}}$ where $H_{R,\varpi} = H_R$. Now, if we equip $\mathbf{A}^+ \subset \mathbf{A}^+_\varpi \subset \mathbf{A}_{\mathrm{inf}}(\overline{R})$ with the induced filtration, then we see that the isomorphism $\mathbf{A}^+_{R,\varpi} \otimes_{\mathbf{A}^+_R} \mathbf{A}^+ \xrightarrow{\sim} \mathbf{A}^+_\varpi$ is compatible with filtrations as well (where on the left we consider the tensor product filtration).

### 3.4. Fat period rings

In this section we will introduce an alternative construction of fat period rings. This will be helpful in constructing some auxiliary rings in the proof of Proposition 4.28. Let $S$ and $\Lambda$ be $p$-adically complete filtered $O_F$-algebras. Let $\iota : S \to \Lambda$ be a continuous injective morphism of filtered $O_F$-algebras and let $f : S \otimes \Lambda \to \Lambda$ be the morphism sending $x \otimes y \mapsto \iota(x)y$.

Definition 3.19. — *Define $S\Lambda$ to be the $p$-adic completion of the divided power envelope of $S \otimes \Lambda$ with respect to* $\operatorname{Ker} f$.

Now, let $S = R, R_{\varpi}^{\mathrm{PD}}$, where over $R$ we consider the trivial filtration, whereas over $R_{\varpi}^{\mathrm{PD}}$ we consider the filtration described in Definition 3.11. Then we have,

*Remarks 3.20.*

(i) The ring $S\Lambda$ is the $p$-adic completion of $S \otimes \Lambda$ adjoined $(x \otimes 1 - 1 \otimes \iota(x))^{[k]}$, for $x \in S$ and $n \in \mathbb{N}$ and $(V_i - 1)^{[k]}$ for $1 \leqslant i \leqslant d$ and $k \in \mathbb{N}$, where $V_i = \frac{X_i \otimes 1}{1 \otimes \iota(X_i)}$ for $1 \leqslant i \leqslant d$.

(ii) The morphism $f : S \otimes \Lambda \to \Lambda$ extends uniquely to a continuous morphism $f : S\Lambda \to \Lambda$.

(iii) There is a natural filtration over $S\Lambda$ where we define $\operatorname{Fil}^r S\Lambda$ to be the topological closure of the ideal generated by the products of the form $x_1 x_2 \prod (V_i - 1)^{[k_i]}$, with $x_1 \in \operatorname{Fil}^{r_1} S$, $x_2 \in \operatorname{Fil}^{r_2} \Lambda$ and $r_1 + r_2 + \sum k_i \geqslant r$.

(iv) From [17, Lemma 2.36], we have that any element $x \in S\Lambda$ can be uniquely written as $x = \sum_{\mathbf{k} \in \mathbb{N}^d} x_{\mathbf{k}} (1 - V_1)^{[k_1]} \cdots (1 - V_d)^{[k_d]}$ with $x_{\mathbf{k}} \in \Lambda$ for all $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ and $x_{\mathbf{k}} \to 0$ as $|\mathbf{k}| = \sum_{i=1}^{d} k_i \to +\infty$. Moreover, an element $x \in \operatorname{Fil}^r S\Lambda$ if and only if $x_{\mathbf{k}} \in \operatorname{Fil}^{r-|\mathbf{k}|}\Lambda$ for all $\mathbf{k} \in \mathbb{N}^d$.

## 4. Finite height representations

In this section we will study Wach modules and their relationship with crystalline modules for crystalline representations.

### 4.1. The arithmetic case

Recall that we have $G_F = \operatorname{Gal}(\overline{F}/F)$ as the absolute Galois group of $F$, $\Gamma_F \coloneqq \operatorname{Gal}(F_\infty/F)$ and $H_F \coloneqq \operatorname{Gal}(\overline{F}/F_\infty)$, where $F_\infty = \bigcup_n F(\mu_{p^n})$. From the theory of $(\varphi, \Gamma_F)$-modules, we have a two dimensional local ring $\mathbf{A}_F$ given as the $p$-adic completion of $O_F[\![\pi]\!]\left[\frac{1}{\pi}\right]$ and $\mathbf{B}_F := \mathbf{A}_F\left[\frac{1}{p}\right]$ is a complete discrete valuation field with uniformizer $p$ and residue field $\kappa((\overline{\pi}))$, the field of Laurent series with uniformizer $\overline{\pi}$ (the reduction of $\pi$ modulo $p$).

Next, we have certain subrings $\mathbf{A}_F^+ := O_F[\![\pi]\!] \subset \mathbf{A}_F$ and $\mathbf{B}_F^+ = \mathbf{A}_F^+\left[\frac{1}{p}\right] \subset \mathbf{B}_F$, stable under the action of $\varphi$ and $\Gamma_F$. Let $V$ be a $p$-adic representation of $G_F$, then $\mathbf{D}^+(V) = (\mathbf{B}^+ \otimes_{\mathbb{Q}_p} V)^{H_F}$ is a free module over the principal domain $\mathbf{B}_F^+$ of rank $\leqslant \dim_{\mathbb{Q}_p} V$, equipped with a Frobenius-semilinear endomorphism $\varphi$ and a continuous and semilinear action of $\Gamma_F$. Further, let $\mathbf{D}(V) = (\mathbf{B} \otimes_{\mathbb{Q}_p} V)^{H_F}$ be the associated $(\varphi, \Gamma_F)$-module which is a $\mathbf{B}_F$-vector space of dimension $= \dim_{\mathbb{Q}_p} V$, equipped with a Frobenius-semilinear endomorphism $\varphi$ and a continuous and semilinear action of $\Gamma_F$. We have a $\mathbf{B}_F^+$-linear inclusion $\mathbf{D}^+(V) \subset \mathbf{D}(V)$ compatible with the action of $\varphi$ and $\Gamma_F$. We say that $V$ is of *finite height* if $\mathbf{D}^+(V)$ is a $\mathbf{B}_F^+$-lattice inside $\mathbf{D}(V)$.

Similarly, if $T \subset V$ is a free $\mathbb{Z}_p$-lattice, stable under the action of $G_F$, then $\mathbf{D}^+(T) = (\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T)^{H_F}$ is a free $\mathbf{A}_F^+$-module of rank $\leqslant \dim_{\mathbb{Q}_p} V$, stable under the action of $\varphi$ and $\Gamma_F$ (see [21, Section B.1.2]). Moreover, $\mathbf{D}(T) = (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_F}$ is a free $\mathbf{A}_F$-module of rank $= \dim_{\mathbb{Q}_p} V$ equipped with a Frobenius-semilinear operator $\varphi$ and a continuous and semilinear action of $\Gamma_F$, and we have $\mathbf{D}^+(T) \subset \mathbf{D}(T)$.

Fontaine showed that $V$ is of finite height if and only if there exists a finite free $\mathbf{B}_F^+$-submodule of $\mathbf{D}(V)$ of rank $= \dim_{\mathbb{Q}_p} V$, stable under the operator $\varphi$ (see [21, Section B.2.1] and [15, Section III.2]). Moreover, if $T \subset V$ is a free $\mathbb{Z}_p$-lattice as above and $V$ of finite height, then $\mathbf{D}^+(T)$ is a free $\mathbf{A}_F^+$-module of rank $= \dim_{\mathbb{Q}_p} V$ such that $\mathbf{A}_F \otimes_{\mathbf{A}_F^+} \mathbf{D}^+(T) \xrightarrow{\sim} \mathbf{D}(T)$ (see [21, Théorème B.1.4.2]).

For crystalline representations there exist submodules of $\mathbf{D}^+(V)$ admitting a simpler action of $\Gamma_F$. Finite height and crystalline representations of $G_F$ are related by the following result:

Theorem 4.1 ([7, 15, 37]). — *Let $V$ be a $p$-adic representation of $G_F$. Then $V$ is crystalline if and only if it is of finite height and there exists $r \in \mathbb{Z}$ and a $\mathbf{B}_F^+$-submodule $N \subset \mathbf{D}^+(V)$ of rank $= \dim_{\mathbb{Q}_p} V$, stable under the action of $\Gamma_F$, such that $\Gamma_F$ acts trivially over $(N/\pi N)(-r)$.*

In the situation of Theorem 4.1, the module $N$ is not unique. A functorial construction was given by Berger:

Proposition 4.2 ([7, Proposition II.1.1]). — *Let $V$ be a positive crystalline representation of $G_F$, i.e. all Hodge–Tate weights of $V$ are $\leqslant 0$. Let*

*$T \subset V$ be a free $\mathbb{Z}_p$-lattice, stable under the action of $G_F$. Then there exists a unique $\mathbf{A}_F^+$-module $\mathbf{N}(T) \subset \mathbf{D}(T)$, which is free of rank $= \dim_{\mathbb{Q}_p} V$, stable under the action of $\varphi$ and $\Gamma_F$, and the action of $\Gamma_F$ is trivial over $\mathbf{N}(T)/\pi\mathbf{N}(T)$. Moreover, there exists $s \in \mathbb{N}$ such that $\pi^s\mathbf{D}^+(T) \subset \mathbf{N}(T)$. Finally, set $\mathbf{N}(V) := \mathbf{B}_F^+ \otimes_{\mathbf{A}_F^+} \mathbf{N}(T)$, then $\mathbf{N}(V)$ is a unique $\mathbf{B}_F^+$-submodule of $\mathbf{D}^+(V)$ satisfying analogous properties.*

*Notation 4.3.* — For an algebra $S$ admitting an action of the Frobenius and an $S$-module $M$ admitting a Frobenius-semilinear endomorphism $\varphi : M \to M$, we denote by $\varphi^*(M) \subset M$ the $S$-submodule generated by the image of $\varphi$.

*Remarks 4.4.*

(i) In Proposition 4.2 for positive crystalline representations, Berger applies Theorem 4.1 with $r = 0$ to define $\mathbf{N}(V) := \mathbf{D}^+(V) \cap N\big[\frac{1}{\varphi^{n-1}(q)}\big]_{n \geqslant 1}$, where $q = \frac{\varphi(\pi)}{\pi}$. Using this one can take $\mathbf{N}(T) := \mathbf{N}(V) \cap \mathbf{D}(T)$ and it can be shown to satisfy the desired properties.

(ii) Berger further showed that in the setup of Proposition 4.2, if we take $s$ to be the maximum among the absolute values of Hodge–Tate weights of $V$, then $\mathbf{N}(T)/\varphi^*(\mathbf{N}(T))$ is killed by $q^s$ and we have that $\pi^s\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T \subset \mathbf{A}^+ \otimes_{\mathbf{A}_F^+} \mathbf{N}(T)$ (see [7, Théorème III.3.1]). The former observation can be thought of as a finite $q$-height property of Wach modules. We will impose it as one of the main conditions for defining finite $q$-height representations in the relative case (see 4.9).

Definition 4.5. — *Let $a, b \in \mathbb{Z}$ with $b \geqslant a$. A* Wach module *with weights in the interval $[a, b]$ is a finite free $\mathbf{A}_F^+$-module or a $\mathbf{B}_F^+$-module $N$, equipped with a continuous and semilinear action of $\Gamma_F$ such that the action of $\Gamma_F$ is trivial on $N/\pi N$ and a Frobenius-semilinear operator $\varphi : N\big[\frac{1}{\pi}\big] \to N\big[\frac{1}{\varphi(\pi)}\big]$ commuting with the action of $\Gamma_F$, $\varphi(\pi^b N) \subset \pi^b N$ and $\pi^b N/\varphi^*(\pi^b N)$ is killed by $q^{b-a}$.*

*Remark 4.6.* — The definition of the functor $\mathbf{N}$ can be extended to crystalline representations of arbitrary Hodge–Tate weights quite easily. Indeed, let $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_F)$ with Hodge–Tate weights in the interval $[a, b]$ and let $T \subset V$ a free $\mathbb{Z}_p$-lattice, stable under the action of $G_F$. Then $\mathbf{N}(T) = \pi^{-b}\mathbf{N}(T(-b)) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(b)$ is a Wach module over $\mathbf{A}_F^+$ with weights in the interval $[a, b]$.

As it turns out, one can recover the crystalline representation from a given Wach module:

Proposition 4.7 ([7, Proposition III.4.2]). — *The functor*

$$\begin{aligned}\mathbf{N} : \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_F) &\longrightarrow \text{ Wach modules over } \mathbf{B}^+_F\\ V &\longmapsto \mathbf{N}(V),\end{aligned}$$

*establishes an equivalence of categories with a quasi-inverse given by* $N \mapsto (\mathbf{B}\otimes_{\mathbf{B}^+_F} N)^{\varphi=1}$*. These functors are compatible with tensor products, duality and preserve exact sequences. Moreover, for a crystalline representation* $V$*, the map* $T \mapsto \mathbf{N}(T)$ *induces a bijection between* $\mathbb{Z}_p$*-lattices inside* $V$ *and Wach modules over* $\mathbf{A}^+_F$ *contained in* $\mathbf{N}(V)$*.*

We have a natural filtration on Wach modules given as

$$\mathrm{Fil}^k\mathbf{N}(V) = \{x \in \mathbf{N}(V) \text{ such that } \varphi(x) \in q^k\mathbf{N}(V)\} \text{ for } k \in \mathbb{Z}.$$

If $V$ is positive crystalline, i.e. all its Hodge–Tate weights are $\leqslant 0$, then for $r \in \mathbb{N}$ we have

$$\mathrm{Fil}^k\mathbf{N}(V(r)) = \mathrm{Fil}^k\pi^{-r}\mathbf{N}(V)(r) = \pi^{-r}\mathrm{Fil}^{k+r}\mathbf{N}(V)(r).$$

Using this filtration on $\mathbf{N}(V)$, one can also recover the other linear algebraic object associated to $V$, i.e. the filtered $\varphi$-module $\mathbf{D}_{\mathrm{cris}}(V)$: Let $\mathbf{B}^+_{\mathrm{rig},F} \subset F[\![\pi]\!]$ denote the subring of convergent power series over the open unit disc. Then we have $\mathbf{D}_{\mathrm{cris}}(V) \subset \mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V)$ and this gives $\mathbf{D}_{\mathrm{cris}}(V) = \big(\mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V)\big)^{\Gamma_F}$ (see [7, Proposition II.2.1]). Moreover, the induced map

$$\mathbf{D}_{\mathrm{cris}}(V) \longrightarrow \Big(\mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V)\Big)/\pi\Big(\mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V)\Big) = \mathbf{N}(V)/\pi\mathbf{N}(V),$$

is an isomorphism of filtered $\varphi$-modules (see [7, Proposition III.4.4]).

## 4.2. The relative case

In this section, we will introduce the notion of relative Wach modules and study representations of finite height. Recall that we fixed $m \in \mathbb{N}_{\geqslant 1}$ (fix $m \in \mathbb{N}_{\geqslant 2}$ if $p = 2$) and we have $K = F_m = F(\zeta_{p^m})$. The element $\varpi = \zeta_{p^m} - 1$ is a uniformizer of $K$. We have $X = (X_1, \ldots, X_d)$ a set of indeterminates and we defined $R$ to be the $p$-adic completion of an étale algebra over $O_F[X, X^{-1}]$ having non-empty and geometrically integral special fiber and $R[\varpi] = O_K \otimes_{O_F} R$. For $R$ and $R[\varpi]$, we can use the $(\varphi, \Gamma)$-module theory discussed in Section 3.1, as well as the constructions in Section 3.2 and Section 3.3.

Setting $q = \frac{\varphi(\pi)}{\pi}$ and using the formulation in Definition 4.5, we define relative Wach modules:

Definition 4.8. — *Let $a$, $b \in \mathbb{Z}$ with $b \geqslant a$. A* Wach module *over $\mathbf{A}_R^+$ (resp. $\mathbf{B}_R^+$) with weights in the interval $[a, b]$ is a finite projective $\mathbf{A}_R^+$-module (resp. $\mathbf{B}_R^+$-module) $N$, equipped with a continuous and semilinear action of $\Gamma_R$ such that the action of $\Gamma_R$ is trivial on $N/\pi N$. Further, there is a Frobenius-semilinear operator $\varphi : N\big[\frac{1}{\pi}\big] \to N\big[\frac{1}{\varphi(\pi)}\big]$ commuting with the action of $\Gamma_R$ such that $\varphi(\pi^b N) \subset \pi^b N$ and $\pi^b N/\varphi^*(\pi^b N)$ is killed by $q^{b-a}$.*

Let $V$ be a $p$-adic representation of the Galois group $G_R$ admitting a $\mathbb{Z}_p$-lattice $T \subset V$ stable under the action of $G_R$. Then we have the finitely generated $\mathbf{A}_R^+$-module $\mathbf{D}^+(T) := (\mathbf{A}^+ \otimes_{\mathbb{Q}_p} T)^{H_R}$. We introduce the following definition:

Definition 4.9. — *A* positive finite $q$-height *representation is a $p$-adic representation $V$ of $G_R$ admitting a $\mathbb{Z}_p$-lattice $T \subset V$ such that there exists a finite projective $\mathbf{A}_R^+$-submodule $\mathbf{N}(T) \subset \mathbf{D}^+(T)$ of rank $= \dim_{\mathbb{Q}_p} V$ satisfying the following conditions:*

(i) *$\mathbf{N}(T)$ is stable under the action of $\varphi$ and $\Gamma_R$, and $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} \mathbf{N}(T) \xrightarrow{\sim} \mathbf{D}(T)$;*

(ii) *The $\mathbf{A}_R^+$-module $\mathbf{N}(T)/\varphi^*(\mathbf{N}(T))$ is killed by $q^s$ for some $s \in \mathbb{N}$;*

(iii) *The action of $\Gamma_R$ is trivial on $\mathbf{N}(T)/\pi\mathbf{N}(T)$;*

(iv) *There exists a $R' \subset \overline{R}$ finite étale over $R$ such that the $\mathbf{A}_{R'}^+$-module $\mathbf{A}_{R'}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ is free.*

*The module $\mathbf{N}(T)$ is a* Wach module *associated to $T$ with weights in the interval $[-s, 0]$ and we set $\mathbf{N}(V) := \mathbf{N}(T)\big[\frac{1}{p}\big]$ satisfying properties analogous to* (i)-(iv) *above. The* height *of $V$ is defined to be the smallest $s \in \mathbb{N}$ satisfying* (ii) *above.*

For $r \in \mathbb{Z}$, we set $V(r) := V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r)$ and $T(r) := T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r)$. We will call these twists as representations of *finite $q$-height* and define

$$\mathbf{N}(T(r)) := \frac{1}{\pi^r}\mathbf{N}(T)(r) \ \text{ and } \ \mathbf{N}(V(r)) := \frac{1}{\pi^r}\mathbf{N}(V)(r).$$

Since $\mathbf{N}(V)$ and $\mathbf{N}(T)$ are Wach modules with weights in the interval $[-s, 0]$, twisting by $r$ gives us Wach modules in the sense of Definition 4.8 with weights in the interval $[r - s, r]$. We will say that height of $V(r) =$ (height of $V$) $- r$.

*Remarks 4.10.*

(i) In the arithmetic case, i.e. $R = O_F$, the notion of finite height representations in Theorem 4.1 and finite $q$-height representations in Definition 4.9 are related. In fact, in the arithmetic case using

Definition 4.9 one obtains the functorial object of Berger mentioned above (see [7, Proposition II.1.1]).

(ii) In Definition 4.9 conditions (i), (ii) and (iii) are motivated from the definition of finite height representations of $G_F$ admitting a Wach module structure. The last condition, i.e. (iv) is inspired by Brinon's definition of weak admissibility in the relative case (see [14, p. 136]).

(iii) In Definition 4.9 following Remark 4.4(i), one can first define Wach module for the representation $V$ and then consider the module $\mathbf{N}(T) = \mathbf{N}(V) \cap \mathbf{D}(T)$ associated to $T$. However, it is not clear whether the latter module, defined in this fashion, is a projective $\mathbf{A}_R^+$-module. Therefore, we impose the condition on $\mathbf{N}(T)$ to be projective, which is required in establishing several results in this section.

### 4.2.1. Some properties of Wach modules

Let us note some important properties of Wach modules associated to finite $q$-height representations

Proposition 4.11. — *Let $V$ be a positive finite $q$-height representation and $T \subset V$ a $G_R$-stable $\mathbb{Z}_p$-lattice. Then we have $\pi^s \mathbf{A}^+ \otimes_{\mathbb{Z}_p} T \subset \mathbf{A}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$, where $s \in \mathbb{N}$ is the height of the representation $V$.*

*Proof.* — To show the claim, we can assume that $\mathbf{N}(T)$ is free by base changing to the period ring corresponding to the finite étale extension $R'$ of $R$. Then $\mathbf{A}^+ \otimes_{\mathbf{A}_{R'}^+} \big(\mathbf{A}_{R'}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\big) = \mathbf{A}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ is free. Since the discussion of previous chapters hold for the $p$-adic completion of a finite étale extension of $R$ (see [14, Chapitre 2] and [4, Section 2] for more on this), base changing to $R'$ is harmless. So with a slight abuse of notation, below we will replace $R'$ obtained in this manner by $R$ and assume $\mathbf{N}(T)$ to be free of rank $h = \dim_{\mathbb{Q}_p} V$ over $\mathbf{A}_R^+$.

Note that by definition we have $\mathbf{N}(T) \subset \mathbf{D}^+(T) = (\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T)^{H_R} \subset \mathbf{A}^+ \otimes_{\mathbb{Z}_p} T$. So let $A \in \mathrm{Mat}(h, \mathbf{A}^+)$ be the matrix obtained by expressing a basis of $\mathbf{N}(T)$ in a chosen basis of $T$. Also, let $P \in \mathrm{Mat}(h, \mathbf{A}_R^+)$ be the matrix of $\varphi$ in the basis of $\mathbf{N}(T)$. Then we have $\varphi(A) = AP$ and therefore $\varphi(\pi^s A^{-1}) = (q^s P^{-1})(\pi^s A^{-1})$. The fact that $\mathbf{N}(T)/\varphi^*(\mathbf{N}(T))$ is killed by $q^s$ implies that $q^s P^{-1} \in \mathrm{Mat}(h, \mathbf{A}_R^+)$, therefore from Lemma 3.9 we obtain that $\pi^s A^{-1} \in \mathrm{Mat}(h, \mathbf{A}^+)$. Hence, we conclude that $\pi^s \mathbf{A}^+ \otimes_{\mathbb{Z}_p} T \subset \mathbf{A}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$. □

Corollary 4.12. — *By taking $H_R$-invariants in Proposition 4.11 it follows that $\pi^s \mathbf{D}^+(T) \subset \mathbf{N}(T)$.*

Proposition 4.13. — *Let $V$ be a finite $q$-height representation $G_R$. The Wach module $\mathbf{N}(V)$ over $\mathbf{B}_R^+$ is unique. Same holds true for the $\mathbf{A}_R^+$-module $\mathbf{N}(T)$.*

*Proof.* — The argument carries over from the classical case [7, p. 13]. First note that we can assume that $V$ is positive, since by definition the uniquess of Wach module for such a representation is equivalent to uniqueness for all its Tate twists. In this case, let $N_1$ and $N_2$ be two $\mathbf{A}_R^+$-modules satisfying the conditions of Definition 4.9 (the proof stays the same for $\mathbf{N}(V)$). By symmetry, it is enough to show that $N_1 \subset N_2$. Since we have $\pi^s N_1 \subset \pi^s \mathbf{D}^+(T) \subset N_2$ (see Corollary 4.12) and $N_2$ is $\pi$-torsion free, therefore for any $x \in N_1$ there exists $k \leqslant s$ such that $\pi^k x \in N_2$ but $\pi^k x \notin \pi N_2$. Varying over all $x \in N_1 \setminus \pi N_1$, we can take $k \leqslant s$ to be the minimal integer such that $\pi^k N_1 \subset N_2$. Since $\pi^k x \in N_2$ and $\Gamma_R$ acts trivially on $N_2/\pi N_2$, we have that $(\gamma_0 - 1)(\pi^k x) \in \pi N_2$. So we can write

$$(\gamma_0 - 1)(\pi^k x) = \gamma_0(\pi^k)(\gamma_0(x) - x) + (\gamma_0(\pi^k) - \pi^k)x.$$

Since $\Gamma_R$ also acts trivially on $N_1/\pi N_1$ and $\pi^k N_1 \subset N_2$, we see that $\gamma_0(\pi^k)(\gamma_0(x) - x) \in \pi N_2$, therefore $(\gamma_0(\pi^k) - \pi^k)x \in \pi N_2$, which means that $(\chi(\gamma_0)^k - 1)\pi^k x \in \pi N_2$. But $\pi \nmid (\chi(\gamma_0)^k - 1)$ if $k \geqslant 1$, and $\pi^k x \notin \pi N_2$. Hence, we must have $k = 0$, i.e. $N_1 \subset N_2$. $\square$

The uniqueness of Wach modules helps us in establishing compatibility with usual operations:

Proposition 4.14. — *Let $V$ and $V'$ be two finite $q$-height representations of $G_R$. Then we have that $\mathbf{N}(V \oplus V') = \mathbf{N}(V) \oplus \mathbf{N}(V')$ and $\mathbf{N}(V \otimes V') = \mathbf{N}(V) \otimes \mathbf{N}(V')$. Similar statements hold for $\mathbf{N}(T)$ and $\mathbf{N}(T')$.*

*Proof.* — We note similar to previous lemma that it is enough to show the statement for $V$ and $V'$ such that both representations are positive. By uniqueness of Wach modules proved in Proposition 4.13, it is enough to show that direct sum and tensor product of finite $q$-height representations are again of finite $q$-height.

First, it is straightforward to see that $\mathbf{N}(T) \oplus \mathbf{N}(T') \subset \mathbf{D}^+(T \oplus T')$ is a projective $\mathbf{A}_R^+$-module of rank $\mathrm{rk}_{\mathbb{Z}_p}(T \oplus T')$ such that $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} (\mathbf{N}(T) \oplus \mathbf{N}(T')) \xrightarrow{\sim} \mathbf{D}(T) \oplus \mathbf{D}(T')$. Similarly, we have that $\mathbf{N}(T) \otimes \mathbf{N}(T') \subset \mathbf{D}^+(T \otimes T')$ is a projective $\mathbf{A}_R^+$-module of rank $\mathrm{rk}_{\mathbb{Z}_p}(T \otimes T')$ such that

$$\mathbf{A}_R \otimes_{\mathbf{A}_R^+} (\mathbf{N}(T) \otimes \mathbf{N}(T')) \xrightarrow{\sim} \mathbf{D}(T) \otimes \mathbf{D}(T').$$

Next, let $s$ and $s'$ denote the height of representations $V$ and $V'$ respectively and let $i := \max(s, s')$. Then we see that $(\mathbf{N}(T) \oplus \mathbf{N}(T'))/\varphi^*(\mathbf{N}(T) \oplus$

$\mathbf{N}(T'))$ is killed by $q^i$ and $(\mathbf{N}(T)\otimes\mathbf{N}(T'))/\varphi^*(\mathbf{N}(T)\otimes\mathbf{N}(T'))$ is killed by $q^{s+s'}$. Further, $\Gamma_R$ acts trivially modulo $\pi$ on $\mathbf{N}(T)\oplus\mathbf{N}(T')$ and $\mathbf{N}(T)\otimes\mathbf{N}(T')$. This verifies conditions (i), (ii) and (iii) for these modules. For condition (iv), note that given any two finite étale extensions $R'$ and $R''$ of $R$, there exists a finite étale extension $S$ over $R$ such that $S$ is finite étale over $R'$ as well as $R''$. Hence, we get the claim. $\square$

Corollary 4.15. — *Let $V$ be a finite $q$-height representation of $G_R$. Then, for $k \in \mathbb{N}$ the representations $\mathrm{Sym}^k(V)$ and $\wedge^k V$ are of finite $q$-height.*

*Proof.* — Note that the compatibility with tensor products in Proposition 4.14 is enough to establish the compatibility with symmetric powers and exterior powers because then we can set

$$\mathbf{N}\big(\mathrm{Sym}^k(T)\big) := \mathrm{Sym}^k(\mathbf{N}(T)), \ \text{ and } \ \mathbf{N}\left(\wedge^k T\right) := \wedge^k\mathbf{N}(T).$$

We have $\mathbf{N}\big(\mathrm{Sym}^k(T)\big) \subset \mathrm{Sym}^k(\mathbf{D}^+(T)) \subset \mathbf{D}^+\big(\mathrm{Sym}^k(T)\big)$, since

$$\mathbf{A}^+\otimes_{\mathbf{A}_R^+}\mathrm{Sym}^k(\mathbf{D}^+(T)) \subset \mathbf{A}^+\otimes_{\mathbf{A}_R^+}\mathbf{D}^+\big(\mathrm{Sym}^k(T)\big).$$

Similarly, $\mathbf{N}\big(\wedge^k T\big) \subset \mathbf{D}^+\big(\wedge^k T\big)$. Rest of the assumptions of Definition 4.9 follows in a same manner as in the proof of Proposition 4.14. This establshes that $\mathrm{Sym}^k(V)$ and $\wedge^k V$ are finite $q$-height representations and gives us the corresponding Wach modules. $\square$

### 4.2.2. Filtration on Wach modules

There is a natural filtration on Wach modules associated to finite $q$-height representations. We will introduce this filtration next and prove a lemma concerning this filtration.

Definition 4.16. — *Let $V$ be a positive finite $q$-height represenation of $G_R$ and $r \in \mathbb{N}$. Then there is a natural filtration on the associated Wach modules given as*

$$\mathrm{Fil}^k\mathbf{N}(V(r)) := \{x \in \mathbf{N}(V(r)),\ \textit{such that}\ \varphi(x) \in q^k\mathbf{N}(V(r))\}\ \ \textit{for}\ k \in \mathbb{Z},$$

*and we set $\mathrm{Fil}^k\mathbf{N}(T(r)) := \mathrm{Fil}^k\mathbf{N}(V(r)) \cap \mathbf{N}(T(r))$, where the intersection is taken inside $\mathbf{N}(V(r))$.*

Lemma 4.17. — *With notations as above, we have*

(i) $\mathrm{Fil}^k\mathbf{N}(T(r)) = \{x \in \mathbf{N}(T(r)),\ \textit{such that}\ \varphi(x) \in q^k\mathbf{N}(T(r))\}$.

(ii) $\mathrm{Fil}^k\mathbf{N}(V(r)) = \mathrm{Fil}^k\pi^{-r}\mathbf{N}(V)(r) = \pi^{-r}\mathrm{Fil}^{k+r}\mathbf{N}(V)(r)$ *and similarly for* $\mathrm{Fil}^k\mathbf{N}(T(r))$.

*Proof.* — To show (i), note that for $k \leqslant 0$, the claim is obvious, so we assume that $k > 0$. Then we are reduced to showing that $q^k \mathbf{N}(V(r)) \cap \mathbf{N}(T(r)) = q^k \mathbf{N}(T(r))$. To prove this claim, note that it is enough to work under the assumption that $\mathbf{N}(T(r))$ is free. Indeed, for any finite $q$-height representation $V(r)$, there exists a finite étale $R$-algebra $R'$ such that $\mathbf{A}_{R'}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T(r))$ is free. Since $\mathbf{A}_{R'}^+$ is faithfully flat over $\mathbf{A}_R^+$, the claim is equivalent to showing that $\mathbf{A}_{R'}^+ \otimes_{\mathbf{A}_R^+} (q^k \mathbf{N}(V) \cap \mathbf{N}(T)) = q^k \mathbf{A}_{R'}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$. But one can easily obtain that

$$\mathbf{A}_{R'}^+ \otimes_{\mathbf{A}_R^+} (q^k \mathbf{N}(V) \cap \mathbf{N}(T)) = \Big(q^k \mathbf{A}_{R'}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)\Big) \cap \Big(\mathbf{A}_{R'}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\Big)$$

(or see [30, Theorem 7.4(i)]) as submodules of $\mathbf{A}_{R'}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)$. So below we will assume that $\mathbf{N}(T(r))$ is free over $\mathbf{A}_R^+$ with a basis $\{f_1, \ldots, f_h\}$, where $h = \dim_{\mathbb{Q}_p} V(r)$. Let $x = \sum_{i=1}^h x_i f_i \in q^k \mathbf{N}(V(r)) \cap \mathbf{N}(T(r))$ with $x_i \in \mathbf{A}_R^+$. Since $\{f_1, \ldots, f_h\}$ is also a $\mathbf{B}_R^+$-basis of $\mathbf{N}(V(r))$, we can write $x = q^k \sum_{i=1}^h y_i f_i$ with $y_i \in \mathbf{B}_R^+$. Comparing the two expressions for $x$ we obtain that $q^k y_i = x_i \in \mathbf{A}_R^+$, i.e. $y_i \in \mathbf{A}_R$ for $1 \leqslant i \leqslant h$. But this just means that $y_i \in \mathbf{B}_R^+ \cap \mathbf{A}_R = \mathbf{A}_R^+$, therefore $x_i = q^k y_i \in q^k \mathbf{A}_R^+$ for $1 \leqslant i \leqslant h$. Hence, $x \in q^k \mathbf{N}(T(r))$ as desired. The other inclusion is obvious.

To show (ii), note that the inclusion $\pi^{-r} \mathrm{Fil}^{k+r} \mathbf{N}(V)(r) \subset \mathrm{Fil}^k \pi^{-r} \mathbf{N}(V)(r)$ is obvious. To show the converse let $\pi^{-r} x \otimes \epsilon^{\otimes r} \in \mathrm{Fil}^k \pi^{-r} \mathbf{N}(V)(r)$, with $x \in \mathbf{N}(V)$ and $\epsilon^{\otimes r}$ being a basis of $\mathbb{Q}_p(r)$. Then we have that $\varphi(\pi^{-r} x \otimes \epsilon^{\otimes r}) = q^{-r} \pi^{-r} \varphi(x) \otimes \epsilon^{\otimes r} \in q^k \pi^{-r} \mathbf{N}(V)(r)$. Therefore, we obtain that $\varphi(x) \in q^{k+r} \mathbf{N}(V)$, i.e. $x \in \mathrm{Fil}^{k+r} \mathbf{N}(V)$. $\square$

*Remark 4.18.* — For $V = \mathbb{Q}_p$ the filtration in Definition 4.16 coincides with the filtration in Lemma 3.17

*Proof.* — We have $T = \mathbb{Z}_p$ and $\mathbf{N}(T) = \mathbf{A}_R^+$ and let $\varpi = \zeta_p - 1$ (let $\varpi = \zeta_{p^2} - 1$ if $p = 2$) in this proof. Since $\pi^k \mathbf{A}_R^+ \subset \mathrm{Fil}^k \mathbf{N}(T)$ (where the term on right is the filtration in Definiton 4.16), we only need to show that $\mathrm{Fil}^k \mathbf{N}(T) \subset \pi^k \mathbf{A}_R^+ = \mathbf{A}_R^+ \cap \xi^k \mathbf{A}_{R,\varpi}^+$. Let $x \in \mathbf{A}_R^+$ such that $\varphi(x) = q^k y$ for some $y \in \mathbf{A}_R^+$. As we have $\mathbf{A}_R^+ \subset \mathbf{A}_{R,\varpi}$, we can also write $\varphi(x) = \varphi(\xi^k) y \in \varphi(\mathbf{A}_{R,\varpi}) \subset \mathbf{A}_R$, i.e. $y \in \varphi(\mathbf{A}_{R,\varpi}) \cap \mathbf{A}_R^+ = \varphi(\mathbf{A}_{R,\varpi}^+)$ (where the intersection is taken inside $\mathbf{A}_R$). Therefore, we obtain that $y = \varphi(z)$ for some $z \in \mathbf{A}_{R,\varpi}^+$. Since $\varphi : \mathbf{A}_{R,\varpi}^+ \to \mathbf{A}_{R,\varpi}^+$ is injective, we must have $x = \xi^k z \in \mathbf{A}_R^+ \cap \xi^k \mathbf{A}_{R,\varpi}^+$, as desired. $\square$

## 4.3. Statement of the main result

In this section, we will relate the notion of crystalline and finite $q$-height representations. As we will see, we can recover the $R\big[\frac{1}{p}\big]$-module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ from the $\mathbf{A}_R^+$-module $\mathbf{N}(T)$ after passing to a larger period ring and inverting $p$. We begin by introducing this ring below.

Recall from Section 1.4 that we have $F$ as a finite unramified extenion of $\mathbb{Q}_p$ with ring of integers $O_F$ and we take $K = F(\zeta_{p^m})$ for a fixed $m \in \mathbb{N}_{\geqslant 1}$ (fix $m \in \mathbb{N}_{\geqslant 2}$ if $p = 2$). Note that the formulation of the results and proofs depend on $m$ and it is necessary to have $m \geqslant 1$ ($m \geqslant 2$ if $p = 2$) for the discussion below to make sense.

### 4.3.1. The ring $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$

In this section, we will work with the ring $\mathbf{A}_{R,\varpi}^+$ defined in Section 3.3, equipped with an action of the Frobenius $\varphi$ and a continuous action of $\Gamma_R$. Since we have a natural injection $\mathbf{A}_{R,\varpi}^+ \rightarrowtail \mathbf{A}_{\mathrm{inf}}(\overline{R})$, we obtain a $G_R$-equivariant commutative diagram

$$\begin{tikzcd}
\mathbf{A}_{R,\varpi}^+ \arrow[r, two heads, "\theta"] \arrow[d, tail] & R[\varpi] \arrow[d, tail] \\
\mathbf{A}_{\mathrm{inf}}(\overline{R}) \arrow[r, two heads, "\theta"] & \mathbb{C}^+(\overline{R}).
\end{tikzcd}$$

By $R$-linearity, extending scalars for the map $\theta$ above, we obtain a ring homomorphism

$$\theta_R : R \otimes_{\mathbb{Z}} \mathbf{A}_{R,\varpi}^+ \longrightarrow R[\varpi],$$

sending $X_i \otimes 1 \mapsto X_i$, $1 \otimes [X_i^\flat] \mapsto X_i$ for $1 \leqslant i \leqslant d$ and $1 \otimes \pi_m \mapsto \zeta_{p^m} - 1$. Note that we have inclusion of ideals $\big(\xi, X_i \otimes 1 - 1 \otimes [X_i^\flat],\ \text{for } 1 \leqslant i \leqslant d\big) \subset \operatorname{Ker} \theta_R \subset R \otimes_{\mathbb{Z}} \mathbf{A}_{R,\varpi}^+$, where $\xi = \frac{\pi}{\pi_1}$. We have $\mathbf{A}_{R,\varpi}^+ \subset \mathbf{A}_{\mathrm{inf}}(\overline{R})$ and $\theta_R$ above is the restriction of $\theta_R : R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(\overline{R}) \twoheadrightarrow \mathbb{C}^+(\overline{R})$ (see Section 2.2.1). So similar to $\mathcal{O}\mathbf{A}_{\mathrm{inf}}(\overline{R})$ in Section 2.1.3 and $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ in Section 2.2.2 we define the following rings:

Definition 4.19.

(i) *Define $\mathcal{O}\mathbf{A}_{R,\varpi}^+$ to be $\theta_R^{-1}(pR[\varpi])$-adic completion of $R \otimes_{\mathbb{Z}} \mathbf{A}_{R,\varpi}^+$.*

(ii) *Let $x^{[n]} := x^n/n!$ for $x \in \operatorname{Ker} \theta_R$. Define $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ to be the $p$-adic completion of the divided power envelope of $R \otimes_{\mathbb{Z}} \mathbf{A}_{R,\varpi}^+$ with respect to* $\operatorname{Ker} \theta_R$.

*Note that we have $\mathcal{O}\mathbf{A}_{R,\varpi}^+ = \mathcal{O}\mathbf{A}_{\mathrm{inf}}(\overline{R}) \cap \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$.*

Taking the divided power envelope of $\theta_R/p^n$, note that $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/p^n \rightarrowtail \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})/p^n$. Since we have $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} = \lim_n \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/p^n$ and $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) = \lim_n \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})/p^n$, and (projective) limit is left exact, it follows that for the $p$-adic completion of divided power envelope of $\theta_R$, we have $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$. Now, over the ring $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ we can consider the induced action of $\Gamma_R$ under which it is stable, and it admits a Frobenius endomorphism arising from the Frobenius on each component of the tensor product. In particular, from the diagram above we obtain a $G_R$-equivariant commutative diagram

$$\begin{tikzcd}
\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \arrow[r, two heads, "\theta_R"] \arrow[d, tail] & R[\varpi] \arrow[d, tail] \\
\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \arrow[r, two heads, "\theta_R"] & \mathbb{C}^+(\overline{R}).
\end{tikzcd}$$

Note that the left vertical arrow is Frobenius-equivariant.

Next, we will give an alternative description of the ring $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. Let $T = (T_1, \ldots, T_d)$ denote a set of indeterminates and let $\mathbf{A}_{\mathrm{cris}}(\overline{R})\langle T\rangle^\wedge$ denote the $p$-adic completion of the divided power polynomial algebra $\mathbf{A}_{\mathrm{cris}}(\overline{R})\langle T\rangle = \mathbf{A}_{\mathrm{cris}}(\overline{R})[T_i^{[n]},\ n \in \mathbb{N},\ 1 \leqslant i \leqslant d]$. Recall from Section 2.2.2 that we have an isomorphism of rings

$$\begin{aligned}
f_{\mathrm{cris}} : \mathbf{A}_{\mathrm{cris}}(\overline{R})\langle T\rangle^\wedge &\xrightarrow{\sim} \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \\
T_i &\longmapsto X_i \otimes 1 - 1 \otimes [X_i^\flat], \quad \text{for } 1 \leqslant i \leqslant d.
\end{aligned}$$

Now recall that $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is the $p$-adic completion of the divided power envelope of the surjective map $\theta : \mathbf{A}_{R,\varpi}^{+} \twoheadrightarrow R[\varpi]$ with respect to its kernel (see Section 3.2). Next, let $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\langle T\rangle^\wedge$ denote the $p$-adic completion of the divided power polynomial algebra $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\langle T\rangle = \mathbf{A}_{R,\varpi}^{\mathrm{PD}}[T_i^{[n]},\ n \in \mathbb{N},\ 1 \leqslant i \leqslant d]$. Then via the isomorphism $f^{\mathrm{PD}}$ (see Lemma 4.20 below), we will show that the preimage of $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, under $f_{\mathrm{cris}}$ is exactly $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\langle T\rangle^\wedge$. In other words,

Lemma 4.20. — *The morphism of rings*

$$\begin{aligned}
f^{\mathrm{PD}} : \mathbf{A}_{R,\varpi}^{\mathrm{PD}}\langle T\rangle^\wedge &\longrightarrow \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \\
T_i &\longmapsto X_i \otimes 1 - 1 \otimes [X_i^\flat], \quad \textit{for } 1 \leqslant i \leqslant d,
\end{aligned}$$

*is an isomorphism.*

*Proof.* — The proof follows [14, Proposition 6.1.5] closely.

Recall that we have a surjective ring homomorphism $\theta : \mathbf{A}^{\mathrm{PD}}_{R,\varpi} \twoheadrightarrow R[\varpi]$, which is the restriction of the map $\theta : \mathbf{A}_{\mathrm{cris}}(\overline{R}) \twoheadrightarrow \mathbb{C}^+(\overline{R})$ defined in Section 2.2. This can be extended in a unique manner into the homomorphism $\theta : \mathbf{A}_{\mathrm{cris}}(\overline{R})\langle T\rangle^\wedge \twoheadrightarrow \mathbb{C}^+(\overline{R})$. Restriction of the latter map gives us $\theta : \mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle^\wedge \twoheadrightarrow R[\varpi]$ such that $\theta(T_i^{[n]}) = 0$ for $1 \leqslant i \leqslant d$ and $n \geqslant 1$.

First, we will show that the $O_F\{X^{\pm 1}\}$-algebra structure on $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle^\wedge$ given by $X_i \mapsto [X_i^\flat] + T_i$, extends uniquely to an $R$-algebra structure. Let $\mathcal{A} \coloneqq (\mathbf{E}^+_{R,\varpi}/\overline{\pi}^{p-1}\mathbf{E}^+_{R,\varpi})[T_1, \ldots, T_d]/(T_1^p, \ldots, T_d^p)$. We have a surjective map $\theta : \mathbf{A}^+_{R,\varpi} \twoheadrightarrow R[\varpi]$ and its reduction modulo $p$ is given as $\bar{\theta} : \mathbf{E}^+_{R,\varpi} \twoheadrightarrow R[\varpi]/pR[\varpi]$. Since $\xi^p \equiv \overline{\pi}^{p-1} \mod p$, where $\xi = \frac{\pi}{\pi_1}$ is a generator of $\operatorname{Ker}\theta \subset \mathbf{A}^+_{R,\varpi}$, we obtain that $\bar{\theta}$ factors as $\bar{\theta} : \mathbf{E}^+_{R,\varpi}/\overline{\pi}^{p-1}\mathbf{E}^+_{R,\varpi} \twoheadrightarrow R[\varpi]/pR[\varpi]$. This can be extended to a map $\bar{\theta} : \mathcal{A} \twoheadrightarrow R[\varpi]/pR[\varpi]$ by setting $\bar{\theta}(T_i) = 0$ for $1 \leqslant i \leqslant d$. The kernel $\mathcal{I} = \operatorname{Ker}\bar{\theta} \subset \mathcal{A}$ is generated by $\xi \equiv \overline{\pi}_1^{p-1} \mod p$ and $\{T_i\}_{1\leqslant i\leqslant d}$. Now from the natural inclusion $R/pR \rightarrowtail R[\varpi]/pR[\varpi]$ and the isomorphism $\mathcal{A}/\mathcal{I} \xrightarrow{\sim} R[\varpi]/pR[\varpi]$ via $\bar{\theta}$, we obtain a map $\bar{g} : R/pR \to \mathcal{A}/\mathcal{I}$ such that $\bar{g}(X_i) = X_i$, which is the image of $X_i^\flat \in \mathcal{A}$ under the map $\bar{\theta}$. So we obtain a commutative diagram

$$\begin{tikzcd}
\kappa[X^{\pm 1}] \arrow[r] \arrow[d] & \mathcal{A} \arrow[d, two heads] \\
R/pR \arrow[r, tail] \arrow[ur, dashed, "\bar{g}"] & \mathcal{A}/\mathcal{I}
\end{tikzcd}$$

where the top horizontal arrow is the map $X_i \mapsto X_i^\flat + T_i$. Note that $\mathcal{I}^{(d+1)p} = 0$. Since $R/pR$ is étale over $\kappa[X^{\pm 1}]$, there exists a unique lift of $\bar{g} : R/pR \to \mathcal{A}/\mathcal{I}$ to a homomorphism $\bar{g} : R/pR \to \mathcal{A}$ (which we again denote by $\bar{g}$ by slight abuse of notations).

Further, by the description of divided power envelope in [14, Proposition 6.1.1] we have that

$$\begin{aligned}
\mathbf{A}^+_{R,\varpi}[Y_0, Y_1, \ldots]/(pY_0 - \xi^p, pY_{n+1} - Y_n^p)_{n\geqslant 1} &\xrightarrow{\sim} \mathbf{A}^{\mathrm{PD}}_{R,\varpi} \\
Y_n &\longmapsto \frac{\xi^{p^{n+1}}}{p^{n+1}}.
\end{aligned}$$

Therefore,

$$(\mathbf{E}^+_{R,\varpi}/\overline{\pi}^{p-1}\mathbf{E}^+_{R,\varpi})[Y_0, Y_1, \ldots]/(Y_n^p)_{n\geqslant 1} \xrightarrow{\sim} \mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p\mathbf{A}^{\mathrm{PD}}_{R,\varpi}.$$

Similarly, we have that $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle$ is isomorphic to

$$(\mathbf{A}^{\mathrm{PD}}_{R,\varpi}[T_1, \ldots, T_d])[T_{i,0}, T_{i,1}, \ldots]/(pT_{i,0} - T_i^p, pT_{i,n+1} - T_{i,n}^p)_{1\leqslant i\leqslant d,\, n\in\mathbb{N}}$$

Therefore, $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle/p\mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle$ is isomorphic to

$$(\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p\mathbf{A}^{\mathrm{PD}}_{R,\varpi})[T_1,\ldots,T_d][T_{i,0},T_{i,1},\ldots]/(T_i^p,T_{i,n}^p)_{1\leqslant i\leqslant d,\, n\in\mathbb{N}}.$$

In conclusion, we have

$$\mathcal{A}[Y_0,Y_1,\ldots,T_{i,0},T_{i,1},\ldots]/(Y_n^p,T_{i,n}^p)_{1\leqslant i\leqslant d,\, n\in\mathbb{N}} \xrightarrow{\sim} \mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle/p\mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle.$$

From the discussion above we obtain a natural map of $\kappa[X^{\pm 1}]$-algebras by composition $\bar{g}_1 : R/pR \to \mathcal{A} \to \mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle/p\mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle$.

Now let $n\in\mathbb{N}$, then modulo $p^n$ we have a natural map

$$O_F\{X^{\pm 1}\}/p^nO_F\{X^{\pm 1}\} \longrightarrow \mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle/p^n\mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle.$$

Again, since $R/p^nR$ is étale over $O_F\{X^{\pm 1}\}/p^nO_F\{X^{\pm 1}\}$, we have a unique lift of $\bar{g}_n : R/p^nR \to \mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle/p^n\mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle$ in the commutative diagram

$$\begin{tikzcd}
O_F\{X^{\pm 1}\}/p^nO_F\{X^{\pm 1}\} \arrow[r] \arrow[d] & \mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle/p^n\mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle \arrow[d, two heads] \\
R/p^nR \arrow[r] \arrow[ur, dashed, "\bar{g}_n"] & \mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle/p\mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle.
\end{tikzcd}$$

Via this lifting, the following diagram commutes

$$\begin{tikzcd}
R/p^{n+1}R \arrow[r] \arrow[d] & \mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle/p^{n+1}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle \arrow[d] \\
R/p^nR \arrow[r] & \mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle/p^n\mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle,
\end{tikzcd}$$

where the vertical arrows are natural projection maps. From the universal property of inverse limit of the right side of the diagram, we obtain a natural map of $O_F\{X^{\pm 1}\}$-algebras

$$g : R \longrightarrow \lim_n \mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle/p^n\mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle = \mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle^\wedge.$$

Now, let $\bar{\theta} : \mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle/p\mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle \to R[\varpi]/pR[\varpi]$ denote the reduction of $\theta$ modulo $p$. Recall that by construction, $\bar{\theta}\circ\bar{g}$ is the inclusion of $R/pR$ in $R[\varpi]/pR[\varpi]$. Therefore, the reduction modulo $p$ of $\theta\circ g$ and the natural inclusion $R \rightarrowtail R[\varpi]$ coincide. As $R$ is $p$-torsion free, arguing as above we get that for each $n\in\mathbb{N}$, the natural inclusion and $\theta\circ g$ coincide modulo $p^n$.

Next, by $\mathbf{A}^+_{R,\varpi}$-linearity, $g$ can be extended to a map $g : R\otimes_{O_F}\mathbf{A}^+_{R,\varpi} \to \mathbf{A}^{\mathrm{PD}}_{R,\varpi}\langle T\rangle^\wedge$. From the discussion above and the definition of $\theta_R$, we have that $\theta_R$ coincides with the homomorphism $\theta\circ g : R\otimes_{O_F}\mathbf{A}^+_{R,\varpi} \to R[\varpi]$. In particular, $g(\text{Ker } \theta_R) \subset \text{Ker } \theta \subset \mathbf{A}_{R,\varpi}^{\mathrm{PD}}\langle T\rangle^\wedge$. Since Ker $\theta$ contains divided powers, the map $g$ extends to a map

$$g : (R \otimes_{O_F} \mathbf{A}_{R,\varpi})\, [x^{[n]}, x \in \text{Ker } \theta_R, n \in \mathbb{N}] \longrightarrow \mathbf{A}_{R,\varpi}^{\mathrm{PD}}\langle T\rangle^\wedge.$$

Finally, since $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\langle T\rangle^\wedge$ is $p$-adically complete, $g$ extends to a map $g : \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \to \mathbf{A}_{R,\varpi}^{\mathrm{PD}}\langle T\rangle^\wedge$.

Now by uniqueness of $g : R \to \mathbf{A}_{R,\varpi}^{\mathrm{PD}}\langle T\rangle^\wedge$, the composition

$$\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \xrightarrow{\ g\ } \mathbf{A}_{R,\varpi}^{\mathrm{PD}}\langle T\rangle^\wedge \xrightarrow{\ f^{\mathrm{PD}}\ } \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}},$$

coincides with the identity over $R \subset \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. Since it also coincides with identity on the image of $\mathbf{A}_{R,\varpi}^+$ (by $\mathbf{A}_{R,\varpi}^+$-linearity), we obtain that $f^{\mathrm{PD}} \circ g = \text{id}$ over $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. Similarly, the homomorphism $g \circ f^{\mathrm{PD}}$ coincides with identity over $\mathbf{A}_{R,\varpi}^+$ as well as over $O_F\{X^{\pm 1}\}$ (since $g$ lifts the map $O_F\{X^{\pm 1}\} \to \mathbf{A}_{R,\varpi}^{\mathrm{PD}}\langle T\rangle^\wedge$), therefore it is identity over $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\langle T\rangle^\wedge$. This establishes that $f^{\mathrm{PD}}$ is an isomorphism of rings. □

*Remark 4.21.* — We can give an alternative construction of the ring $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. Note that we have a ring homomorphism $\iota : R \to \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, where $X_i \mapsto [X_i^\flat]$ for $1 \leqslant i \leqslant d$. As in Definition 3.19, we define a map $g : R \otimes_{\mathbb{Z}} \mathbf{A}_{R,\varpi}^{\mathrm{PD}} \to \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, where $x \otimes y \mapsto \iota(x)y$. We obtain that $\text{Ker } g = \big(X_i \otimes 1 - 1 \otimes [X_i^\flat],\ \text{for } 1 \leqslant i \leqslant d\big) \subset \text{Ker } \theta_R \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$. Since $R \otimes_{\mathbb{Z}} \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ already contains divided powers of $\xi$, from Definition 4.19 we obtain that the $p$-adic completion of the divided power envelope of $R \otimes_{\mathbb{Z}} \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ with respect to Ker $g$ is the same as $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$.

There is a natural filtration over $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ by $\Gamma_R$-stable submodules:

Definition 4.22. — *Let $U_i := \frac{1 \otimes [X_i^\flat]}{X_i \otimes 1}$ for $1 \leqslant i \leqslant d$ and $r \in \mathbb{Z}$, define the filtration over $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ as*

$$\mathrm{Fil}^r \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} := \Big\langle (a \otimes b) \prod_{i=1}^{d} (U_i - 1)^{[k_i]} \in \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}},\ \textit{such that}$$
$$a \in R, b \in \mathrm{Fil}^j \mathbf{A}_{R,\varpi}^{\mathrm{PD}},\ \textit{and } j + \sum_i k_i \geqslant r \Big\rangle.$$

*Remark 4.23.* — The filtration over $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ (via its identification with $R_\varpi^{\mathrm{PD}}$, see Section 3.3 and Definition 3.11) coincides with the filtration induced from its embedding in $\mathbf{A}_{\mathrm{cris}}(\overline{R})$. Indeed, in both cases we have $\mathrm{Fil}^r \mathbf{A}_{R,\varpi}^{\mathrm{PD}} = \big(\xi^{[k]},\ k \leqslant r\big) \subset \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ for $r \geqslant 0$, whereas $\mathrm{Fil}^r \mathbf{A}_{R,\varpi}^{\mathrm{PD}} = \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$

for $r < 0$. Next, the filtration on $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ is defined as the induced filtration from its embedding inside $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(\overline{R})$ and the filtration on the latter ring is given by powers of Ker $\theta_R$ (see Section 2.1 & 2.2 for definition and notation). The induced filtration over $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ is therefore given by divided powers of the ideal Ker $\theta_R \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$. Since the filtration over $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ in Definition 4.22 is again given by divided powers of the ideal Ker $\theta_R \subset \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, we infer that this filtration coincides with the one induced by its embedding into $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$.

Lemma 4.24.

(i) *The action of* $\Gamma_{R,\varpi}$ *is trivial on* $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/\pi$, *whereas* $\Gamma_R/\Gamma_{R,\varpi}$ *acts trivially over* $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/\pi_m$.

(ii) *We have* $(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}})^{\Gamma_R} = R$ *and* $(\mathrm{Fil}^1\mathcal{O}\mathbf{A}_{R}^{\mathrm{PD}})^{\Gamma_R} = 0$.

*Proof.*

(i). — The first part follows from the definition of $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and the action of $\Gamma_{R,\varpi}$ on $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ (see Lemma 3.16). The second part follows from observing that $\Gamma_R/\Gamma_{R,\varpi} = \Gamma_F/\Gamma_K$ is a finite cyclic group of order $[K : F] = p^{m-1}(p-1)$, and a lift $g \in \Gamma_R$ of a generator of $\Gamma_R/\Gamma_{R,\varpi}$ acts as $g(\pi_m) = (1+\pi_m)^{\chi(g)} - 1$.

(ii). — This is straightforward, since

$$R \subset \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\big)^{\Gamma_R} \subset \big(\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})\big)^{G_R} = R$$

and

$$(\mathrm{Fil}^1\mathcal{O}\mathbf{A}_{R}^{\mathrm{PD}})^{\Gamma_R} \subset (\mathrm{Fil}^1\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}))^{G_R} \subset (\mathrm{Fil}^1\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}))^{G_R} = 0$$

(for last equality see the proof of [14, Proposition 5.2.12]). □

Next we consider a connection over $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ induced by the connection on $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$,

$$\partial : \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \longrightarrow \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes \Omega_R^1,$$

where we have $\partial\big(X_i \otimes 1 - 1 \otimes [X_i^\flat]\big)^{[n]} = \big(X_i \otimes 1 - 1 \otimes [X_i^\flat]\big)^{[n-1]}\, dX_i$. This connection over $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ satisfies Griffiths transversality with respect to the filtration since it does so over $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$.

### 4.3.2. Main result

Theorem 4.25. — *Let* $V$ *be a positive finite* $q$*-height representation of* $G_R$, *then*

(i) $V$ *is a positive crystalline representation.*

(ii) *Let* $M := \big(\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\big)^{\Gamma_R}$, *then after extending scalars to* $\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$ *and inverting* $p$, *we obtain a natural isomorphism*

$$\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi} \otimes_R M\left[\frac{1}{p}\right] \xrightarrow{\sim} \mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V),$$

*compatible with Frobenius, filtration, connection and the action of* $\Gamma_R$ *on each side.*

(iii) *We have an isomorphism of* $R\big[\frac{1}{p}\big]$*-modules*

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xleftarrow{\sim} \Big(\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\Big)^{\Gamma_R}\left[\frac{1}{p}\right],$$

*compatible with Frobenius, filtration, and connection on each side. Therefore, we obtain a comparison isomorphism*

$$\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V),$$

*compatible with Frobenius, filtration, connection and the action of* $\Gamma_R$ *on each side.*

*Remark 4.26.* — The statement of Theorem 4.25 can be seen an analogue of the result of Berger [7, Proposition II.2.1] (see the discussion after Proposition 4.7).

Recall that from Definition 4.9 any finite $q$-height representation is a twist of a positive finite $q$-height representation by $\mathbb{Q}_p(r)$, for $r \in \mathbb{N}$. Since twist by $\mathbb{Q}_p(r)$ of crystalline representations are again crystalline, we obtain:

Corollary 4.27. — *All finite* $q$*-height representations of* $G_R$ *are crystalline.*

The proof of Theorem 4.25 will proceed in two steps: First, we will describe a process by which we can recover a submodule of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ starting from the Wach module (see Proposition 4.28), here we establish the comparison displayed in (ii). Next, the remaining claims made in the theorem are shown by exploiting some properties of Wach modules and the comparison obtained in the first step.

In Section 4.6, we will explicitly state the structure of Wach module attached to a one-dimensional finite $q$-height representation and we will also show that all one-dimensional crystalline representations are of finite $q$-height and one can recover $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ starting with the Wach module $\mathbf{N}(V)$. Combining this with the theorem above, we will obtain that the notion of crystalline representations and finite $q$-height representations coincide in dimension 1.

### 4.4. From $(\varphi, \Gamma)$-modules to $(\varphi, \partial)$-modules

The objective of this section is to prove the following:

Proposition 4.28. — *Let $V$ be an $h$-dimensional positive finite $q$-height representation of $G_R$, $T \subset V$ a $\mathbb{Z}_p$-lattice of rank $h$ stable under the action of $G_R$ and $\mathbf{N}(T)$ the associated Wach module. Then*

(i) $M := \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\big)^{\Gamma_R}$ *is a finitely generated $R$-module contained in $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$.*

(ii) $M\big[\frac{1}{p}\big]$ *is a finitely generated projective $R\big[\frac{1}{p}\big]$-module of rank $h$ and the natural inclusion*
$$\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M\left[\frac{1}{p}\right] \longrightarrow \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V),$$
*is an isomorphism compatible with Frobenius, filtration, connection and the action of $\Gamma_R$.*

(iii) *If $\mathbf{N}(T)$ is free over $\mathbf{A}_R^+$ then there exists a free $R$-module $M_0 \subset M$ such that $M_0\big[\frac{1}{p}\big] = M\big[\frac{1}{p}\big]$ are free modules of rank $h$ over $R\big[\frac{1}{p}\big]$.*

*Proof.* — We will use the notation of Definition 4.9 without repeating them. The first claim is easy to establish. Since $H_R = \mathrm{Gal}\big(\overline{R}\big[\frac{1}{p}\big]/R_\infty\big[\frac{1}{p}\big]\big)$, therefore $M = \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\big)^{\Gamma_R}$ is contained in

$$\begin{aligned}\Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{D}^+(T)\Big)^{\Gamma_R} &\subset \Big(\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})^{H_R} \otimes_{\mathbf{A}_R^+} \mathbf{D}^+(T)\Big)^{\Gamma_R} \\ &\subset \Big(\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})^{H_R} \otimes_{\mathbf{A}_R^+} \big(\mathbf{A}^+ \otimes_{\mathbf{Z}_p} T\big)^{H_R}\Big)^{\Gamma_R} \\ &\subset \big(\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Z}_p} T\big)^{G_R} \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V).\end{aligned} \tag{4.1}$$

The module $\big(\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Z}_p} T\big)^{G_R}$ is finitely generated over $R$. Since $R$ is Noetherian, $M$ is finitely generated.

Independently, we have that $R\big[\frac{1}{p}\big]$ is Noetherian and $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is a finitely generated $R\big[\frac{1}{p}\big]$-module, therefore $M\big[\frac{1}{p}\big] \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is finitely generated over $R\big[\frac{1}{p}\big]$. Moreover, the module $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ is equipped with an $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$-linear and integrable connection $\partial_N = \partial \otimes 1$, where $\partial$ is the connection on $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ described after Lemma 4.24. Therefore, we can consider the induced connection on $M\big[\frac{1}{p}\big]$, which is integrable since it is integrable over $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$. This connection is compatible with the one on $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ since the connection over $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is induced from the connection over $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$. So by [14, Proposition 7.1.2] we obtain that $M\big[\frac{1}{p}\big]$ must be projective of rank $\leqslant h$. Furthermore, the inclusion $M\big[\frac{1}{p}\big] \subset$

$\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is compatible with natural Frobenius on each module since all the inclusions in (4.1) are compatible with Frobenius.

Next, we will show that the rank of $M\left[\frac{1}{p}\right]$ as a projective $R\left[\frac{1}{p}\right]$-module is exactly $h$. But first let us prove that it is enough to show that the rank is $h$ after a finite étale extension of $R$. Let us consider $R'$ to be a finite étale extension of $R$ such that the corresponding scalar extension $\mathbf{A}_{R'}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ is a free module of rank $h$ (see Definition 4.9) and $R'\left[\frac{1}{p}\right]/R\left[\frac{1}{p}\right]$ is Galois. The discussion of previous chapters hold for $R'$ (see [14, Chapitre 2] and [4, Section 2] for more on this). In particular, for $R'[\varpi]$ we have rings $\mathbf{A}_{R'}^+$, $\mathbf{A}_{R',\varpi}^+$, $\mathbf{A}_{R',\varpi}^{\mathrm{PD}}$ and $\mathcal{O}\mathbf{A}_{R',\varpi}^{\mathrm{PD}}$. Let $R'_\infty\left[\frac{1}{p}\right]$ denote the cyclotomic tower over $R'\left[\frac{1}{p}\right]$

$$\Gamma_{R'} = \mathrm{Gal}\left(R'_\infty\left[\frac{1}{p}\right]/R'\left[\frac{1}{p}\right]\right) \text{ and } H_{R'} = \mathrm{Ker}\ (G_{R'} \longrightarrow \Gamma_{R'}).$$

Similarly, we have Galois groups $\Gamma_{R'}$ and $H_{R'}$. Let us define

$$\begin{aligned} G' = \mathrm{Gal}\left(R'_\infty\left[\frac{1}{p}\right]/R_\infty\left[\frac{1}{p}\right]\right) &= \mathrm{Gal}\left(R'[\varpi]\left[\frac{1}{p}\right]/R[\varpi]\left[\frac{1}{p}\right]\right) \\ &= \mathrm{Gal}\left(R'\left[\frac{1}{p}\right]/R\left[\frac{1}{p}\right]\right), \end{aligned}$$

then we have that $H_{R,\varpi}/H_{R',\varpi} = H_R/H_{R'} = G'$. So we obtain that

$$\mathbf{A}_R^+ = (\mathbf{A}^+)^{H_R} = \left((\mathbf{A}^+)^{H_{R'}}\right)^{H_R/H_{R'}} = \left(\mathbf{A}_{R'}^+\right)^{G'}.$$

Moreover, for the base ring $R[\varpi]$ (instead of $R$) one can consider the ring $\mathbf{A}_\varpi^+$ as in Remark 3.6. Then we have

$$\mathbf{A}_{R,\varpi}^+ = (\mathbf{A}_\varpi^+)^{H_{R,\varpi}} = \left((\mathbf{A}_\varpi^+)^{H_{R',\varpi}}\right)^{H_{R,\varpi}/H_{R',\varpi}} = \left(\mathbf{A}_{R',\varpi}^+\right)^{G'}.$$

From these equalities and the description of the action of $\Gamma_R$ on $\xi = \frac{\pi}{\pi_1}$, it is clear that

$$\mathbf{A}_{R,\varpi}^{\mathrm{PD}} = \left(\mathbf{A}_{R',\varpi}^{\mathrm{PD}}\right)^{G'}, \text{ and therefore } \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} = \left(\mathcal{O}\mathbf{A}_{R',\varpi}^{\mathrm{PD}}\right)^{G'}.$$

Now, since $\mathbf{N}(T)$ is projective and $G'$ acts trivially on it, we obtain that

$$\begin{aligned} \left(\mathcal{O}\mathbf{A}_{R',\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_{R'}^+} \left(\mathbf{A}_{R'}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\right)\right)^{G'} &= \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T) \\ \left(\mathcal{O}\mathbf{A}_{R',\varpi}^{\mathrm{PD}} \otimes_{R'} \left(R' \otimes_R M\left[\frac{1}{p}\right]\right)\right)^{G'} &= \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M\left[\frac{1}{p}\right]. \end{aligned}$$

In particular, base changing to $\mathbf{A}_{R'}^+$ to obtain $\mathbf{N}(T)$ as a free module is harmless. For the convenience in notation, below we will replace $R'$ obtained in this manner by $R$ and assume $\mathbf{N}(T)$ to be free over $\mathbf{A}_R^+$.

In order to show that the rank of $M\left[\frac{1}{p}\right]$ is at least $h$, we will find $\Gamma_R$-fixed elements of $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ corresponding to a basis of $\mathbf{N}(T)$, which are linearly independent elements of $M\left[\frac{1}{p}\right]$. To carry this out, first we will define several new rings following [37, Section B.1] and examine their relation with $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. After extending scalars of $\mathbf{N}(T)$, we will define differential operators on the obtained module, corresponding to the topological generators of $\Gamma_R$. Next, for any element of $\mathbf{N}(T)$, we will write down a corresponding element killed by the differential operators, i.e. an element fixed by $\Gamma_R$.

*Remark 4.29.* — Note that the $\Gamma_R$-fixed elements of $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ can be obtained by successive approximation as well. This computation was carried out in [1, Section 3.2.3].

### 4.4.1. Auxiliary rings and modules

For $n \in \mathbb{N}$, let us define a $p$-adically complete ring

$$S_n^{\mathrm{PD}} := \mathbf{A}_R^+\left\{\frac{\pi}{p^n}, \frac{\pi^2}{2!p^{2n}}, \ldots, \frac{\pi^k}{k!p^{kn}}, \ldots\right\}.$$

Let $I_n^{[i]}$ denote the ideal of $S_n^{\mathrm{PD}}$ generated by $\frac{\pi^k}{k!p^{kn}}$ for $k \geqslant i$ and we set

$$\widehat{S}_n^{\mathrm{PD}} := \lim_i S_n^{\mathrm{PD}}/I_n^{[i]}. \tag{4.2}$$

Note that $\widehat{S}_n^{\mathrm{PD}}$ is $p$-adically complete as well. Further, note that we can write $\varphi(\pi) = (1+\pi)^p - 1 = \pi^p + p\pi x$ for some $x \in \mathbf{A}_F^+$, therefore

$$\begin{aligned}\frac{\varphi(\pi^k)}{k!p^{kn}} &= \frac{(\pi^p + p\pi x)^k}{k!p^{kn}} = \frac{\sum_{i=0}^k \binom{k}{i}\pi^{pi}(p\pi x)^{k-i}}{k!p^{kn}}\\ &= \sum_{i=0}^k \frac{(k+(p-1)i)!p^{i(n(p-1)-p)}}{i!(k-i)!} \frac{\pi^{k+(p-1)i}x^{k-i}}{(k+(p-1)i)!p^{(k+(p-1)i)(n-1)}} \in \widehat{S}_{n-1}^{\mathrm{PD}}\end{aligned}$$

Using this, the Frobenius operator on $S$ can be extended to a map $\varphi : \widehat{S}_n^{\mathrm{PD}} \to \widehat{S}_{n-1}^{\mathrm{PD}}$, which we will again call Frobenius. The ring $\widehat{S}_n^{\mathrm{PD}}$ readily admits a continuous action of $\Gamma_R$ which commutes with the Frobenius.

Lemma 4.30. — *The ring $\widehat{S}_0^{\mathrm{PD}}$ is a subring of $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, and therefore $\varphi^n\big(\widehat{S}_n^{\mathrm{PD}}\big) \subset \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$.*

*Proof.* — The first claim is true because we have

$$\pi_1^p \equiv \pi \bmod p\mathbf{A}_{F,\varpi}^+, \quad \text{which gives} \quad \pi_1^{p^i} \equiv \pi^{p^{i-1}} \bmod p^i\mathbf{A}_{F,\varpi}^+.$$

So for $k \geqslant p^i$ we can write

$$\begin{aligned}\frac{\pi^k}{k!} &= \frac{\xi^k \pi_1^k}{k!} = \frac{\xi^k}{k!}\pi_1^{k-p^i}\left(\pi^{p^{i-1}} + p^i a\right)\\ &= p^i a \pi_1^{k-p^i}\frac{\xi^k}{k!} + p^{i-1}\pi_1^{p^{i-1}}\frac{(k+p^{i-1})!}{k!p^{i-1}}\frac{\xi^{k+p^{i-1}}}{(k+p^{i-1})!} \in p^{i-1}\mathbf{A}_{F,\varpi}^{\mathrm{PD}},\end{aligned}$$

for some $a \in \mathbf{A}_{F,\varpi}^{+}$. Therefore, we get that $I_0^{[p^i]} \subset p^{i-1}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and hence $\widehat{S}_0^{\mathrm{PD}} \subset \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. The second claim is obvious. ☐

In the relative setting, we need slightly larger rings. Let us consider the $O_F$-linear homomorphism of rings

$$\begin{aligned}\iota : R &\longrightarrow \widehat{S}_n^{\mathrm{PD}}\\ X_j &\longmapsto [X_j^\flat] \text{ for } 1 \leqslant j \leqslant d.\end{aligned}$$

Using $\iota$ we can define an $O_F$-linear morphism of rings

$$\begin{aligned}f : R \otimes_{O_F} \widehat{S}_n^{\mathrm{PD}} &\longrightarrow \widehat{S}_n^{\mathrm{PD}}\\ a \otimes b &\longmapsto \iota(a)b.\end{aligned}$$

Let $\mathcal{O}\widehat{S}_n^{\mathrm{PD}}$ denote the $p$-adic completion of the divided power envelope of $R \otimes_{O_F} \widehat{S}_n^{\mathrm{PD}}$ with respect to Ker $f$. Further, the morphism $f$ extends uniquely to a continuous morphism $f : \mathcal{O}\widehat{S}_n^{\mathrm{PD}} \to \widehat{S}_n^{\mathrm{PD}}$. Now, it easily follows from the discussion in Section 3.4 that the kernel of the morphism $f$ is generated by divided powers of the ideal generated by $(1 - V_1, \ldots, 1 - V_d)$, where $V_j = \frac{X_j \otimes 1}{1 \otimes [X_j^\flat]}$ for $1 \leqslant j \leqslant d$. The Frobenius operator extends to $\mathcal{O}\widehat{S}_n^{\mathrm{PD}}$ as well as the continuous action of $\Gamma_R$. From the discussion above we have $\varphi^n(\widehat{S}_n^{\mathrm{PD}}) \subset \widehat{S}_0^{\mathrm{PD}} \subset \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, and following the description of $\mathcal{O}\widehat{S}_0^{\mathrm{PD}}$ in Section 3.4 and of $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ from Remark 4.21, we obtain that

$$\mathcal{O}\widehat{S}_0^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \text{ and } \varphi^n\big(\mathcal{O}\widehat{S}_n^{\mathrm{PD}}\big) \subset \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}.$$

Moreover, we have a canonical inclusion of $\widehat{S}_n^{\mathrm{PD}} \subset \mathcal{O}\widehat{S}_n^{\mathrm{PD}}$ compatible with all the structures.

Now let us take $n \in \mathbb{N}_{\geqslant 1}$ and consider the ring $\mathcal{O}\widehat{S}_n^{\mathrm{PD}}$ below. Set

$$J := \left(\frac{\pi}{p^n}, 1 - V_1, \ldots, 1 - V_d\right) \subset \mathcal{O}\widehat{S}_n^{\mathrm{PD}},$$

and its divided power inside $\mathcal{O}\widehat{S}_n^{\mathrm{PD}}$ as

$$J^{[i]} := \left\langle \frac{\pi^{[k_0]}}{p^{nk_0}}\prod_{j=1}^{d}(1-V_j)^{[k_j]}, \mathbf{k} = (k_0, k_1, \ldots, k_d) \in \mathbb{N}^{d+1} \text{ with } \sum_{j=0}^{d} k_j \geqslant i \right\rangle.$$

By construction of $\mathcal{O}\widehat{S}_n^{\mathrm{PD}}$, it is clear that a summation $\sum_{i\in\mathbb{N}} x_i a_i$ with $a_i \in J^{[i]}$ and $x_i \in \widehat{S}_n^{\mathrm{PD}}$ goes to 0 as $i \to +\infty$, converges in $\mathcal{O}\widehat{S}_n^{\mathrm{PD}}$. Moreover, every $x \in \mathcal{O}\widehat{S}_n^{\mathrm{PD}}$ has a presentation as $x = \sum_{\mathbf{k}\in\mathbb{N}^{d+1}} x_{\mathbf{k}} \frac{\pi^{[k_0]}}{p^{nk_0}} \prod_{j=1}^{d}(1-V_j)^{[k_j]}$, where $x_{\mathbf{k}} \in \mathbf{A}_R^+$ goes to 0 as $|\mathbf{k}| = \sum_j k_j \to +\infty$.

Next, we set

$$\mathcal{O}N_n^{\mathrm{PD}} := \mathcal{O}\widehat{S}_n^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T).$$

Again, $\mathcal{O}N_n^{\mathrm{PD}}$ is $p$-adically complete and it is equipped with a Frobenius-semilinear operator $\varphi : \mathcal{O}\widehat{S}_n^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T) \to \mathcal{O}\widehat{S}_{n-1}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ and a continuous and semilinear action of $\Gamma_R$. Now recall that we fixed $m \in \mathbb{N}_{\geqslant 1}$ (fix $m \in \mathbb{N}_{\geqslant 2}$ if $p = 2$) such that $K = F(\zeta_{p^m})$. So we take

$$M' := \big(\mathcal{O}N_m^{\mathrm{PD}}\big)^{\Gamma_R'} \text{ and } M'' := (M')^{\Gamma_F} = \big(\mathcal{O}N_m^{\mathrm{PD}}\big)^{\Gamma_R}.$$

Since we assumed $\mathbf{N}(T)$ to be free, therefore $\mathcal{O}N_m^{\mathrm{PD}}$ is a free $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$-module of rank $h$. As we have $\varphi^m\big(\mathcal{O}\widehat{S}_m^{\mathrm{PD}}\big) \subset \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ so we get that $\varphi^m(M'') \subset \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\big)^{\Gamma_R} = M$. Therefore, to show that the $R\big[\frac{1}{p}\big]$-rank of $M\big[\frac{1}{p}\big]$ is at least $h$, it is enough to show that for each $x \in \mathbf{N}(T)$ there exists unique $x'' \in M'' \subset \mathcal{O}N_m^{\mathrm{PD}}$ fixed by $\Gamma_R$ and $x \equiv x'' \mod J^{[1]}\mathcal{O}N_m^{\mathrm{PD}}$ (see Lemma 4.43).

### 4.4.2. Infinitesimal action of $\Gamma_R$

From Section 3.1 recall that we have $\{\gamma, \gamma_1, \ldots, \gamma_d\}$ as a set of topological generators of $\Gamma_R$ such that $\{\gamma_1, \ldots, \gamma_d\}$ generate $\Gamma_R'$ topologically, and $\gamma$ is a lift of a topological generator of $\Gamma_F$ where $\gamma^e = \gamma_0$ is a lift of a topological generator of $\Gamma_K$, $e = [K : F]$ and $\chi(\gamma_0) = \exp(p^m)$ where we fixed $m \in \mathbb{N}_{\geqslant 1}$ (fix $m \in \mathbb{N}_{\geqslant 2}$ if $p = 2$). Further, we have the identity $\gamma_0\gamma_i = \gamma_i^{\chi(\gamma_0)}\gamma_0$ for $1 \leqslant i \leqslant d$. In this section we will study the infinitesimal action of $\Gamma_R$ on the rings and modules constructed in previous section.

Lemma 4.31. — *Let $k \in \mathbb{N}$, $n \geqslant m$ and $i \in \{0, 1, \ldots, d\}$. Then $(\gamma_i - 1)(p^m, \pi)^k \widehat{S}_n^{\mathrm{PD}} \subset (p^m, \pi)^{k+1} \widehat{S}_n^{\mathrm{PD}}$.*

*Proof.* — First, let $i = 0$. Recall that we have $\chi(\gamma_0) = \exp(p^m) = 1 + p^m a \in 1 + p^m\mathbb{Z}_p$. So we can write

$$\begin{aligned}(\gamma_0 - 1)\pi &= (1+\pi)^{\chi(\gamma_0)} - (1+\pi) \\ &= \left(\chi(\gamma_0)\pi + \frac{\chi(\gamma_0)(\chi(\gamma_0)-1)}{2!}\pi^2 + \cdots\right) - \pi = (\chi(\gamma_0)u - 1)\pi,\end{aligned}$$

for some $u = 1 + \pi x \in 1 + \pi\mathbf{A}_R^+$. Therefore, $\chi(\gamma_0)u - 1 = p^m a + \pi x + p^m a\pi x \in (p^m, \pi)\mathbf{A}_R^+$ which gives us that $(\gamma_0 - 1)\pi \in (p^m, \pi)\pi\mathbf{A}_R^+$. Now we have $(\gamma_0 - 1)\mathbf{A}_R^+ \subset \pi\mathbf{A}_R^+ \subset (p^m, \pi)\mathbf{A}_R^+$, so proceeding by induction on $k \geqslant 1$ and using the fact that $\gamma_0 - 1$ acts as a twisted derivation (i.e. $(\gamma_0 - 1)xy = (\gamma_0 - 1)x \cdot y + \gamma_0(x)(\gamma_0 - 1)y$ for $x, y \in \mathbf{A}_R^+$), we conclude that

$$(\gamma_0 - 1)(p^m, \pi)^k\mathbf{A}_R^+ \subset (p^m, \pi)^{k+1}\mathbf{A}_R^+.$$

Next, any $f \in \widehat{S}_n^{\mathrm{PD}}$ can be written as $f = \sum_{s\in\mathbb{N}} f_s \frac{\pi^s}{s!p^{ns}}$ such that $f_s \in \mathbf{A}_R^+$ goes to 0 as $s \to +\infty$. Clearly we have

$$(\gamma_0 - 1)\frac{\pi^s}{s!p^{ns}} = \frac{(\chi(\gamma_0)^s u^s - 1)\pi^s}{s!p^{ns}} \in (p^m, \pi)\frac{\pi^s}{s!p^{ns}}\widehat{S}_n^{\mathrm{PD}}.$$

Combining the discussion for $\mathbf{A}_R^+$ and $\frac{\pi^s}{s!p^{ns}}$, using induction on $k \geqslant 1$ and using the fact that $\gamma_0 - 1$ acts as a twisted derivation, we conclude that

$$(\gamma_0 - 1)(p^m, \pi)^k\widehat{S}_n^{\mathrm{PD}} \subset (p^m, \pi)^{k+1}\widehat{S}_n^{\mathrm{PD}}.$$

Finally, for $i \in \{1, \ldots, d\}$ we have $(\gamma_i - 1)[X_i^\flat] = \pi[X_i^\flat] \in (p^m, \pi)\mathbf{A}_R^+$ and $(\gamma_i - 1)\big([X_i^\flat]^{-1}\big) = -\pi(1+\pi)^{-1}[X_i^\flat]^{-1} \in (p^m, \pi)\mathbf{A}_R^+$. Again by induction on $k \geqslant 1$ and using the fact that $\gamma_i - 1$ acts as a twisted derivation, we get that

$$(\gamma_i - 1)(p^m, \pi)^k\mathbf{A}_R^+ \subset (p^m, \pi)^{k+1}\mathbf{A}_R^+.$$

Now any $f \in \widehat{S}_n^{\mathrm{PD}}$ can be written as $f = \sum_{s\in\mathbb{N}} f_s \frac{\pi^s}{s!p^{ns}}$ such that $f_s \in \mathbf{A}_R^+$ goes to 0 as $s \to +\infty$, and $\gamma_i$ acts trivially on $\pi$ for $1 \leqslant i \leqslant d$, so we conclude that

$$(\gamma_i - 1)(p^m, \pi)^k\widehat{S}_n^{\mathrm{PD}} \subset (p^m, \pi)^{k+1}\widehat{S}_n^{\mathrm{PD}}. \qquad \square$$

Lemma 4.32. — *For $n \geqslant m$ and $i \in \{0, 1, \ldots, d\}$ the operators*

$$\nabla_i := \log\gamma_i = \sum_{k\in\mathbb{N}}(-1)^k\frac{(\gamma_i - 1)^{k+1}}{k+1},$$

*converge as series of operators on $\widehat{S}_n^{\mathrm{PD}}$.*

*Proof.* — From Lemma 4.31, we have that for $k \in \mathbb{N}$

$$(\gamma_i - 1)(p^m, \pi)^k\widehat{S}_n^{\mathrm{PD}} \subset (p^m, \pi)^{k+1}\widehat{S}_n^{\mathrm{PD}}.$$

Therefore, using the fact that $\gamma_i - 1$ acts as a twisted derivation (i.e. $(\gamma_i - 1)xy = (\gamma_i - 1)x \cdot y + \gamma_i(x)(\gamma_i - 1)y$ for $x, y \in \widehat{S}_n^{\mathrm{PD}}$), we obtain that for $x \in \widehat{S}_n^{\mathrm{PD}}$

$$(\gamma_i - 1)^{k+1}(x) \subset (p^m, \pi)^{k+1}\widehat{S}_n^{\mathrm{PD}}. \tag{4.3}$$

Therefore, the following series converges in $\widehat{S}_n^{\mathrm{PD}}$

$$\nabla_i(x) = \sum_{k\in\mathbb{N}}(-1)^k \frac{(\gamma_i - 1)^{k+1}(x)}{k+1}.$$

This allows us to conlcude. $\square$

*Remark 4.33.* — Note that $\Gamma_R$ acts trivially modulo $\pi$ on $\mathbf{A}_R^+$. Therefore, we also get that it acts trivially modulo $\pi$ over $\widehat{S}_n^{\mathrm{PD}}$. Hence, for $0 \leqslant i \leqslant d$ we have $\nabla_i(\widehat{S}_n^{\mathrm{PD}}) \subset \pi\widehat{S}_n^{\mathrm{PD}} = t\widehat{S}_n^{\mathrm{PD}}$, where the last equality follow from the fact that $\frac{t}{\pi}$ is a unit in $\widehat{S}_n^{\mathrm{PD}}$ (see Lemma 4.35 below).

*Remark 4.34.* — The operators $\nabla_i$ for $0 \leqslant i \leqslant d$, defined in Lemma 4.32, describe the action of the Lie algebra Lie $\Gamma_R$ on $\widehat{S}_n^{\mathrm{PD}}$, i.e. $\nabla_i$ acts as a differential operator on $\widehat{S}_n^{\mathrm{PD}}$.

Lemma 4.35. — *$\frac{t}{\pi}$ is a unit in $\widehat{S}_n^{\mathrm{PD}}$ for $n \geqslant m$.*

*Proof.* — We can write the fraction

$$\frac{t}{\pi} = \frac{\log(1+\pi)}{\pi} = \sum_{k\geqslant 0}(-1)^k \frac{\pi^k}{k+1}.$$

Formally, we can write

$$\frac{\pi}{t} = \frac{\pi}{\log(1+\pi)} = b_0 + b_1\pi + b_2\pi^2 + b_3\pi^3 + \cdots,$$

where $b_0 = 1$ and $\upsilon_p(b_k) \geqslant -\frac{k}{p-1}$ for all $k \geqslant 1$. But rewriting the series as a power series in $\frac{\pi^k}{k!p^{nk}}$, we get that

$$\frac{\pi}{t} = \sum_{k\in\mathbb{N}} b_k k! p^{nk} \frac{\pi^k}{k!p^{nk}}.$$

The $p$-adic valuation of coefficients in the series above is given as

$$\upsilon_p(b_k k! p^{nk}) \geqslant \frac{-k}{p-1} + nk + \upsilon_p(k!) = \frac{p-2}{p-1}nk + \upsilon_p(k!),$$

which clearly goes to $+\infty$ as $k \to +\infty$. Hence, $\frac{\pi}{t}$ converges in $\widehat{S}_n^{\mathrm{PD}}$ and is an inverse to $\frac{t}{\pi}$. $\square$

Now let us consider the ring $\mathcal{O}\widehat{S}_n^{\mathrm{PD}}$ and divided power ideals

$$J^{[i]} := \left\langle \frac{\pi^{[k_0]}}{p^{nk_0}} \prod_{j=1}^{d}(1-V_j)^{[k_j]}, \mathbf{k} = (k_0, k_1, \ldots, k_d) \in \mathbb{N}^{d+1} \text{ with } \sum_{j=0}^{d} k_j \geqslant i \right\rangle.$$

Arguments similar to Lemmas 4.31 and 4.32 show that for $0 \leqslant i \leqslant d$ the series of operators $\nabla_i = \log\gamma_i = \sum_{k\in\mathbb{N}}(-1)^k \frac{\pi^{k+1}}{k+1}$ converge over $\mathcal{O}\widehat{S}_n^{\mathrm{PD}}$. Moreover, from Remark 4.33 we obtain that for $0 \leqslant i \leqslant d$, we have

$\nabla_i(\mathcal{O}\widehat{S}_n^{\mathrm{PD}}) \subset t\mathcal{O}\widehat{S}_n^{\mathrm{PD}}$. Also, it is easy to observe that we have $\nabla_0(t) = \log(\chi(\gamma_0))t = p^m t$ and $\nabla_i(V_i) = tV_i$ for $1 \leqslant i \leqslant d$. Finally, recall that $\gamma_i\gamma_j = \gamma_j\gamma_i$ for $1 \leqslant i, j \leqslant d$ and $\gamma_0\gamma_i = \gamma_i^{\chi(\gamma_0)}\gamma_0$, therefore we conclude that

$$\begin{aligned}[]
[\nabla_i, \nabla_j] &= 0,\\
[\nabla_i, \nabla_0] &= \log(\chi(\gamma_0))\nabla_i = p^m\nabla_i.
\end{aligned}$$

Now we will adapt the discussion above to scalar extension of Wach module $\mathbf{N}(T)$ to $\mathcal{O}\widehat{S}_n^{\mathrm{PD}}$, i.e. for $\mathcal{O}N_n^{\mathrm{PD}} := \mathcal{O}\widehat{S}_n^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$.

Lemma 4.36. — *For $n \geqslant m$ and $i \in \{0, 1, \ldots, d\}$ the operators*

$$\nabla_i = \log \gamma_i = \sum_{k\in\mathbb{N}} (-1)^{k+1} \frac{(\gamma_i - 1)^{k+1}}{k+1}$$

*converge as series of operators on $\mathcal{O}N_n^{\mathrm{PD}}$.*

*Proof.* — For $0 \leqslant i \leqslant d$, observe that $\gamma_i - 1$ acts as a twisted derivation, i.e. for $a \in \mathcal{O}\widehat{S}_n^{\mathrm{PD}}$ and $x \in \mathbf{N}(T)$, we have

$$(\gamma_i - 1)(ax) = (\gamma_i - 1)a \cdot x + \gamma_i(a)(\gamma_i - 1)x.$$

The action of $\Gamma_R$ is trivial on $\mathbf{N}(T)/\pi\mathbf{N}(T)$, so we can write $(\gamma_i - 1)x = \pi y$, for some $y \in \mathbf{N}(T)$, i.e. $(\gamma_i - 1)\mathcal{O}N_n^{\mathrm{PD}} \subset (p^m, \pi)\mathcal{O}N_n^{\mathrm{PD}}$. From the proof of Lemma 4.32 and (4.3) and induction over $k \geqslant 1$, it follows that

$$(\gamma_i - 1)(p^m, \pi)^k\mathcal{O}N_n^{\mathrm{PD}} \subset (p^m, \pi)^{k+1}\mathcal{O}N_n^{\mathrm{PD}}.$$

Next, using the fact that $\gamma_i - 1$ acts as a twisted derivation, we obtain that

$$(\gamma_i - 1)^{k+1}(ax) \subset (p^m, \pi)^{k+1}\mathcal{O}N_n^{\mathrm{PD}}.$$

Therefore, the following series converges in $\mathcal{O}N_n^{\mathrm{PD}}$

$$\nabla_i(ax) = \sum_{k\in\mathbb{N}} (-1)^k \frac{(\gamma_i - 1)^{k+1}(ax)}{k+1}.$$

This allows us to conlcude. ☐

*Remark 4.37.* — Note that $\Gamma_R$ acts trivially modulo $\pi$ on $\mathcal{O}\widehat{S}_n^{\mathrm{PD}}$ and $\mathbf{N}(T)$. Therefore, we also get that it acts trivially modulo $\pi$ over $\mathcal{O}N_n^{\mathrm{PD}}$. Hence, for $0 \leqslant i \leqslant d$ we have $\nabla_i(\mathcal{O}N_n^{\mathrm{PD}}) \subset \pi\mathcal{O}N_n^{\mathrm{PD}} = t\mathcal{O}N_n^{\mathrm{PD}}$, where the last equality follows from the fact that $\frac{t}{\pi}$ is a unit in $\mathcal{O}\widehat{S}_n^{\mathrm{PD}}$ (see Lemma 4.35).

Again, over $\mathcal{O}N_n^{\mathrm{PD}}$ we have

$$\begin{aligned}[]
[\nabla_i, \nabla_j] &= 0,\\
[\nabla_i, \nabla_0] &= \log(\chi(\gamma_0))\nabla_i = p^m\nabla_i,
\end{aligned}$$

which enables us to define differential operators $\partial_i$ over $\mathcal{O}N_n^{\mathrm{PD}}$ using the formula

$$\partial_i = \begin{cases} -t^{-1}\nabla_0 & \text{for } i = 0, \\ t^{-1}V_i^{-1}\nabla_i & \text{for } 1 \leqslant i \leqslant d, \end{cases}$$

where $V_i = \frac{X_i \otimes 1}{1 \otimes [X_i^\flat]}$ for $1 \leqslant i \leqslant d$. Note that $\partial_i$ is well defined since $\nabla_i(\mathcal{O}N_n^{\mathrm{PD}}) \subset t\mathcal{O}N_n^{\mathrm{PD}}$ (see Remark 4.37).

Lemma 4.38. — *For $n \geqslant m$, the differential operators defined on $\mathcal{O}N_n^{\mathrm{PD}}$ commute, i.e. $\partial_i \circ \partial_j = \partial_j \circ \partial_i$ for $0 \leqslant i, j \leqslant d$.*

*Proof.* — From above we have $[\nabla_i, \nabla_j] = 0$ for $1 \leqslant i, j \leqslant d$, whereas $[\nabla_0, \nabla_i] = p^m \nabla_i$, for $1 \leqslant i \leqslant d$. So it follows that over $\mathcal{O}N_n^{\mathrm{PD}}$ we have the composition of operators

$$\begin{aligned} t^2 V_i V_j (\partial_i \circ \partial_j - \partial_j \circ \partial_i) &= tV_i\partial_i \circ tV_j\partial_j - tV_j\partial_j \circ tV_i\partial_i \\ &= \nabla_i \circ \nabla_j - \nabla_j \circ \nabla_i = 0, \quad \text{for } 1 \leqslant i, j \leqslant d. \end{aligned}$$

Next, for $1 \leqslant i \leqslant d$, we have

$$\begin{aligned} \nabla_0 \circ \nabla_i - \nabla_i \circ \nabla_0 &= -t\partial_0 \circ (tV_i\partial_i) + tV_i\partial_i \circ (t\partial_0) \\ &= -p^m tV_i\partial_i - t^2 V_i \partial_0 \circ \partial_i + t^2 V_i \partial_i \circ \partial_0 \\ &= p^m \nabla_i - t^2 V_i(\partial_0 \circ \partial_i - \partial_i \circ \partial_0). \end{aligned}$$

In particular, $\partial_i \circ \partial_j - \partial_j \circ \partial_i = 0$ for $0 \leqslant i, j \leqslant d$ since $\mathcal{O}N_n^{\mathrm{PD}}$ is $t$-torsion free. ☐

For the rest of the section, let us now assume $n = m$.

Lemma 4.39. — *Let $1 \leqslant i \leqslant d$ and $x \in \mathbf{N}(T)$, then we have that $\partial_i^k(x) \to 0$ in $\mathcal{O}N_m^{\mathrm{PD}}$ as $k \to +\infty$.*

*Proof.* — First, let us note that since $\partial_i(V_i) = 1$, $\partial_i(V_j) = 0$ for $j \neq i$ and $\partial_i(\pi) = 0$, so we have that $\partial_i^p(\mathcal{O}\widehat{S}_m^{\mathrm{PD}}) \subset p\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$. Moreover, an easy computation shows that for $x \in \mathbf{N}(T)$ we have

$$\begin{aligned} \partial_i(\varphi(x)) &= \frac{\nabla_i(\varphi(x))}{tV_i} = \frac{\varphi(\nabla_i(x))}{tV_i} \\ &= pV_i^{p-1}\varphi(\partial_i(x)) \in \mathcal{O}\widehat{S}_m^{\mathrm{PD}} \otimes_{\varphi(\mathbf{A}_R^+)} \varphi(\mathbf{N}(T)), \end{aligned}$$

where note that we have

$$\varphi(\partial_i(x)) \in \varphi\left(\mathcal{O}\widehat{S}_{m+1}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\right) \subset \mathcal{O}\widehat{S}_m^{\mathrm{PD}} \otimes_{\varphi(\mathbf{A}_R^+)} \varphi(\mathbf{N}(T))$$

since $\partial_i(x)$ converges over $\mathcal{O}\widehat{S}_{m+1}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ by Lemma 4.36.

Next, from Definiton 4.9 recall that we have $q^s\mathbf{N}(T) \subset \varphi^*(\mathbf{N}(T))$. Let us write $q^s x = \sum_{j=1}^h a_j\varphi(e_j)$ for $a_j \in \mathbf{A}_R^+$ and $\{e_1, \ldots, e_h\}$ an $\mathbf{A}_R^+$-basis of

$\mathbf{N}(T)$. Then it follows that $\partial_i^p(q^s x) \in pq^s\big(\mathcal{O}\widehat{S}_m^{\mathrm{PD}} \otimes_{\varphi(\mathbf{A}_R^+)} \varphi(\mathbf{N}(T))\big)$, therefore $\partial_i^p(x) \in p\big(\mathcal{O}\widehat{S}_m^{\mathrm{PD}} \otimes_{\varphi(\mathbf{A}_R^+)} \varphi(\mathbf{N}(T))\big)$. By induction on $k$ we see that $\partial_i^{pk}(x) \in p^k\big(\mathcal{O}\widehat{S}_m^{\mathrm{PD}} \otimes_{\varphi(\mathbf{A}_R^+)} \varphi(\mathbf{N}(T))\big) \subset p^k\mathcal{O}N_m^{\mathrm{PD}}$ and the claim follows. □

*Remark 4.40.* — Note that one can recover the action of $\gamma_i$ using the differential operator $\partial_i$. For $i \in \{1, \ldots, d\}$ we have $\gamma_i = \exp(tV_i\partial_i)$, whereas for $i = 0$ we have $\gamma_0 = \exp(-t\partial_0)$.

From the remark above it is clear that for $0 \leqslant i \leqslant d$ and $x \in \mathcal{O}N_m^{\mathrm{PD}}$ we have $\gamma_i(x) = x$ if and only if $\partial_i(x) = 0$.

Lemma 4.41. — *For any $x \in \mathbf{N}(T)$ there exists a unique $x'' \in \mathcal{O}N_m^{\mathrm{PD}}$ such that*

$$\begin{aligned} x'' &\equiv x \mod J^{[1]}\mathcal{O}N_m^{\mathrm{PD}}, \\ \gamma_i(x'') &= x'' \quad \text{for } 0 \leqslant i \leqslant d. \end{aligned}$$

*In particular, $x'' \in M'' = \big(\mathcal{O}N_m^{\mathrm{PD}}\big)^{\Gamma_R}$.*

*Proof.* — For $x \in \mathbf{N}(T)$, we set

$$x' = \sum_{\mathbf{k} \in \mathbb{N}^d} \partial_1^{k_1} \circ \cdots \circ \partial_d^{k_d}(x)(1-V_1)^{[k_1]} \cdots (1-V_d)^{[k_d]} \in \mathcal{O}N_m^{\mathrm{PD}}$$

The summation converges since for $1 \leqslant i \leqslant d$ we have that $\partial_0^{k_0} \circ \partial_1^{k_1} \circ \cdots \circ \partial_d^{k_d}(x) \to 0$ as $|\mathbf{k}| = \sum_{i=1}^d k_i \to +\infty$ from Lemma 4.39. Note that we have an isomorphism of rings $\widehat{S}_m^{\mathrm{PD}} \xrightarrow{\sim} (\mathcal{O}\widehat{S}_m^{\mathrm{PD}})^{\Gamma_{R'}}$ compatible with $\Gamma_R/\Gamma_{R'} = \Gamma_F$-action. Therefore, by the description of $\widehat{S}_m^{\mathrm{PD}}$ in (4.2) and since $x' \in (\mathcal{O}N_m^{\mathrm{PD}})^{\Gamma_R'}$ we see that the following sum converges

$$x'' = \sum_{k_0 \in \mathbb{N}} \partial_0^{k_0}(x')\frac{t^{[k_0]}}{p^{mk_0}} \in \mathcal{O}N_m^{\mathrm{PD}}.$$

Since the differential operators on $\mathcal{O}N_m^{\mathrm{PD}}$ commute by Lemma 4.38, we get that

$$x'' = \sum_{\mathbf{k} \in \mathbb{N}^{d+1}} \partial_0^{k_0} \circ \partial_1^{k_1} \circ \cdots \circ \partial_d^{k_d}(x) \frac{t^{[k_0]}}{p^{mk_0}}(1-V_1)^{[k_1]} \cdots (1-V_d)^{[k_d]} \in \mathcal{O}N_m^{\mathrm{PD}} \tag{4.4}$$

By the definition of $x''$ it is clear that $x'' \equiv x \mod J^{[1]}\mathcal{O}N_m^{\mathrm{PD}}$. Next, using the fact that $\partial_i \circ \partial_j = \partial_j \circ \partial_i$ for $0 \leqslant i, j \leqslant d$ (see Lemma 4.38) as well as $\partial_0(t) = -p^m$ and $\partial_i(V_i) = 1$ for $1 \leqslant i \leqslant d$, it is easy to deduce that $\partial_i(x'') = 0$ for $0 \leqslant i \leqslant d$. So by Remark 4.40, we get that $\gamma_i(x'') = x''$ for $0 \leqslant i \leqslant d$.

Uniqueness of $x''$ follows from Lemma 4.43. Finally, let $g \in \Gamma_F$ be a lift of a generator of the cyclic group $\Gamma_F/\Gamma_K$. Then we have that $g(x'') \in \mathcal{O}N_m^{\mathrm{PD}}$ satisfies the conditions of the claim (since $(g-1)x \in \pi\mathbf{N}(T) \subset J^{[1]}\mathcal{O}N_m^{\mathrm{PD}}$). But by uniqueness, we obtain that $g(x'') = x''$, i.e. $x'' \in \big(\mathcal{O}N_m^{\mathrm{PD}}\big)^{\Gamma_R} = M''$. $\square$

*Remark 4.42.* — Note that the lemma above can also be obtained by a "successive approximation" argument (see [1, Lemmas 3.33 & 3.37]).

Following claim was used above:

Lemma 4.43. — *For any $x \in \mathbf{N}(T)$ suppose there exists $x'' \in \mathcal{O}N_m^{\mathrm{PD}}$ such that*

$$\begin{aligned} x'' &\equiv x \quad \mod J^{[1]}\mathcal{O}N_m^{\mathrm{PD}},\\ \gamma_i(x'') &= x'' \quad \text{for } 0 \leqslant i \leqslant d. \end{aligned}$$

*Then $x''$ is unique.*

*Proof.* — Let $\{f_1, \dots, f_h\}$ denote an $\mathbf{A}_R^+$-basis of $\mathbf{N}(T)$. Then $\{f_1, \dots, f_h\}$ is also an $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$-basis of $\mathcal{O}N_m^{\mathrm{PD}}$. Now using the formula in (4.4), for all $1 \leqslant i \leqslant h$ let

$$f_i'' = \sum_{\mathbf{k} \in \mathbb{N}^{d+1}} \partial_0^{k_0} \circ \partial_1^{k_1} \circ \cdots \circ \partial_d^{k_d}(f_i) \frac{t^{[k_0]}}{p^{mk_0}} (1-V_1)^{[k_1]} \cdots (1-V_d)^{[k_d]} \in \mathcal{O}N_m^{\mathrm{PD}}.$$

We want to show that $\{f_1'', \dots, f_h''\}$ also form an $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$-basis of $\mathcal{O}N_m^{\mathrm{PD}}$ Let us write $f_i'' = f_i + \sum_{j=1}^h a_{ij} f_j$ with $a_{ij} \in J^{[1]}\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$ and let $A = id_h + (a_{ij}) \in \mathrm{Mat}(h, \mathcal{O}\widehat{S}_m^{\mathrm{PD}})$ denote the $h \times h$ matrix thus obtained. We have that $\det A = 1 + x$ with $x \in J^{[1]}\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$ and $1 - x + x^2 - x^3 + \cdots = \sum_{n \in \mathbb{N}} (-1)^n n! x^{[n]}$ converges in $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$ as an inverse of $1 + x$, i.e. $\det A$ is invertible in $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$. Therefore, $\{f_1'', \dots, f_h''\}$ form a basis of $\mathcal{O}N_m^{\mathrm{PD}}$.

Now for any $x \in \mathbf{N}(T)$, writing $x = \sum_{i=1}^h x_i f_i''$ and plugging into the formula (4.4) we obtain $x'' \in \mathcal{O}N_m^{\mathrm{PD}}$ such that $x'' \equiv x \mod J^{[1]}\mathcal{O}N_m^{\mathrm{PD}}$ and $\gamma_j(x'') = x''$ for all $0 \leqslant j \leqslant d$. By linear independence of $\{f_1'', \dots, f_h''\}$ over $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$ we obtain that $x''$ is unique. $\square$

*Remark 4.44.* — The uniquess claim can also be established by a "successive approximation" argument (see [1, p. 63- 65]).

Lemma 4.45. — *We have $\mathcal{O}\widehat{S}_m^{\mathrm{PD}} \otimes_R M'' \xrightarrow{\sim} \mathcal{O}\widehat{S}_m^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$.*

*Proof.* — Let $\{f_1, \dots, f_h\}$ denote an $\mathbf{A}_R^+$-basis of $\mathbf{N}(T)$. Then $\{f_1, \dots, f_h\}$ is also an $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$-basis of $\mathcal{O}N_m^{\mathrm{PD}}$. From the proof of Lemmas 4.41 & 4.43 we have $f_i'' \in M''$ for all $1 \leqslant i \leqslant h$, such that $\{f_1'', \dots, f_d''\}$ also form an $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$-basis of $\mathcal{O}N_m^{\mathrm{PD}}$. Therefore, $\mathcal{O}\widehat{S}_m^{\mathrm{PD}} \otimes_R M'' \xrightarrow{\sim} \mathcal{O}N_m^{\mathrm{PD}}$. $\square$

### 4.4.3. Finishing the proof of Proposition 4.28

Recall that at the beginning of the proof we assumed $\mathbf{N}(T)$ to be free of rank $h$ (after extension of scalars to $\mathbf{A}_{R'}^{+}$, which we again wrote as $\mathbf{A}_R^{+}$ by abusing notations), therefore $\mathcal{O}N_m^{\mathrm{PD}}$ is free of rank $h$. Further, we have $M = \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^{+}} \mathbf{N}(T)\big)^{\Gamma_R}$ and since $M\big[\frac{1}{p}\big]$ is equipped with an integrable connection, it is projective of rank $\leqslant h$ (see the beginning of the proof). So applying Lemma 4.41 to a basis of $\mathbf{N}(T)$, we obtain that the rank of $M\big[\frac{1}{p}\big]$ as an $R\big[\frac{1}{p}\big]$-module is exactly $h$.

Next, we want to show that the natural inclusion $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M\big[\frac{1}{p}\big] \rightarrowtail \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^{+}} \mathbf{N}(V)$ is bijective. To show this claim, we require the following lemma:

Lemma 4.46. — *We have a natural isomorphism*

$$\varphi^{*}\Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^{+}} \mathbf{N}(V)\Big) \xrightarrow{\sim} \Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^{+}} \mathbf{N}(V)\Big).$$

*Proof.* — Recall that we are working under the assumption that $\mathbf{N}(V)$ is free and by definition of a positive finite $q$-height representation we have that the cokernel of the inclusion $\varphi^{*}(\mathbf{N}(V)) \to \mathbf{N}(V)$ is killed by $q^s$ where $s \in \mathbb{N}$ is the height of the representation $V$. Extending scalars to $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, we obtain that the cokernel of the inclusion $\varphi^{*}\big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^{+}} \mathbf{N}(V)\big) \to \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^{+}} \mathbf{N}(V)\big)$ is killed by $q^s$. Now note that we have $q = \frac{\varphi(\pi)}{\pi} = p\varphi\big(\frac{\pi}{t}\big)\frac{t}{\pi}$ where $\frac{t}{\pi}$ is a unit in $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ (see Lemma 3.14), i.e. $p$ and $q$ are associates in $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. Therefore, the cokernel of the inclusion in the claim is killed by $p^s$. But, $p$ is invertible in $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\big[\frac{1}{p}\big]$. Hence, we obtain that $\varphi^{*}\big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^{+}} \mathbf{N}(V)\big) \xrightarrow{\sim} \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^{+}} \mathbf{N}(V)\big)$. $\square$

Since we assumed $\mathbf{N}(T)$ to be a free module, let $\{f_1, \ldots, f_h\}$ be its $\mathbf{A}_R^{+}$-basis. Let $P \in \mathrm{Mat}(h, \mathbf{A}_R^{+})$ denote the matrix for the action of Frobenius on $\mathbf{N}(T)$ in the chosen basis. In Lemma 4.46 we obtained that $\det P$ is invertible in $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\big[\frac{1}{p}\big]$.

Now, recall that $\mathcal{O}N_m^{\mathrm{PD}} = \mathcal{O}\widehat{S}_m^{\mathrm{PD}} \otimes_{\mathbf{A}_R^{+}} \mathbf{N}(T)$ and $M'' = \big(\mathcal{O}N_m^{\mathrm{PD}}\big)^{\Gamma_R}$. So we consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}\widehat{S}_m^{\mathrm{PD}} \otimes_R M'' \arrow[r, "\sim"] \arrow[d, "\varphi^m \otimes \varphi^m"'] & \mathcal{O}N_m^{\mathrm{PD}} \arrow[d, "\varphi^m"] \\
\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \arrow[r] & \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^{+}} \mathbf{N}(T),
\end{tikzcd}$$

where the top horizontal arrow is bijective (see Lemma 4.45) and all other arrows are injective. We also have that $\{f_1, \ldots, f_h\}$ is an $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$-basis of $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ as well as an $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$-basis of $\mathcal{O}N_m^{\mathrm{PD}}$. From Lemmas 4.41 & 4.45 and the discussion above, for $1 \leqslant i \leqslant h$ we have $f_i'' \in M''$ such that $f_i'' = f_i + \sum_{i=1}^h a_{ij} f_j$ for $a_{ij} \in J^{[1]}\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$ and let $A \coloneqq id_h + (a_{ij}) \in \mathrm{Mat}(h, \mathcal{O}\widehat{S}_m^{\mathrm{PD}})$ denote the $h \times h$ matrix obtained in this manner. From the proof of Lemma 4.43 we have that $\det A$ is invertible in $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$.

Now let $v_i = (\varphi^m \otimes \varphi^m) f_i'' = \varphi^m(f_i) + \sum_{j=1}^h \varphi^m(a_{ij})\varphi^m(f_j) \in M$ and let $M_0$ be the free $R$-submodule of $M$ generated by $\{v_1, \ldots, v_h\}$. From the expression of $\{v_1, \ldots, v_h\}$ in the basis of $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$, we get that the determinant of the inclusion $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M_0 \rightarrowtail \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ is given by $\varphi^m(\det A)\varphi^{m-1}(\det P)\varphi^{m-2}(\det P) \cdots \varphi(\det P)(\det P)$. Since $\det A$ is invertible in $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$, we have that $\varphi^m(\det A)$ is invertible in $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and from above we already have that $\det P$ is invertible in $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\left[\frac{1}{p}\right]$. Therefore, the natural inclusions

$$\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M_0 \left[\frac{1}{p}\right] \longrightarrow \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \left[\frac{1}{p}\right] \longrightarrow \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V), \tag{4.5}$$

are bijective. The maps above are compatible with Frobenius, connection and the action of $\Gamma_R$ on each side and compability of the second map with filtrations follows from Corollary 4.54. This shows the second claim of Proposition 4.28.

Finally, note that above we assumed $\mathbf{N}(T)$ to be free of rank $h$, therefore we obtain a free $R$-submodule $M_0 \subset M$ such that

$$\begin{aligned} M_0\left[\frac{1}{p}\right] = \left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M_0 \left[\frac{1}{p}\right]\right)^{\Gamma_R} \\ \xrightarrow{\sim} \left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \left[\frac{1}{p}\right]\right)^{\Gamma_R} = M\left[\frac{1}{p}\right], \end{aligned}$$

which are free of rank $h$ over $R\left[\frac{1}{p}\right]$. This shows the last claim of Proposition 4.28. In general, when $\mathbf{N}(T)$ is projective of rank $h$, we obtain that $M\left[\frac{1}{p}\right]$ is projective of rank $h$. This sums up our proof. $\square$

## 4.5. Proof of Theorem 4.25

Let $M = \left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\right)^{\Gamma_R}$. From Proposition 4.28 we already have the isomorphism of $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\left[\frac{1}{p}\right]$-modules

$$\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \left[\frac{1}{p}\right] \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V),$$

compatible with Frobenius, filtration (see Corollary 4.54), connection and the action of $\Gamma_R$ on each side. This proves the second claim and we are left to show that $V$ is crystalline and $M\big[\frac{1}{p}\big] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ compatible with supplementary structures. Also note from Proposition 4.28 that we already have the inclusion of projective $R\big[\frac{1}{p}\big]$-modules of rank $h = \dim_{\mathbb{Q}_p} V$, $M\big[\frac{1}{p}\big] \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$. So we are left to show that this inclusion is bijective and compatible with supplementary structures.

First, we will show that $V$ is crystalline and the inclusion described above is in fact bijective. Extending scalars along $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\big[\frac{1}{p}\big] \rightarrowtail \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ for the isomorphism $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M\big[\frac{1}{p}\big] \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)$, we obtain an isomorphism of $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$-modules

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[\frac{1}{p}]} M\left[\frac{1}{p}\right] \xrightarrow{\sim} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbf{B}_R^+} \mathbf{N}(V),$$

compatible with Frobenius, connection and $G_R$-action. Now, recall that from the definitions we have a natural inclusion of free $\mathbf{A}^+$-modules $\mathbf{A}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(V) \rightarrowtail \mathbf{A}^+ \otimes_{\mathbf{A}_R^+} V$ compatible with supplementary structures and the cokernel of this inclusion is killed by $\pi^s$ (see Proposition 4.11). Since $\pi$ is invertible in $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$, extending scalars along $\mathbf{A}^+ \rightarrowtail \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$, we obtain an isomorphism of $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$-modules

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbf{B}_R^+} \mathbf{N}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V,$$

compatible with Frobenius, connection and $G_R$-action. Finally, since $R\big[\frac{1}{p}\big] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ is faithfully flat (see [14, Théorème 6.3.8]), we obtain an inclusion of $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$-modules

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[\frac{1}{p}]} M\left[\frac{1}{p}\right] \subset \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[\frac{1}{p}]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V),$$

compatible with Frobenius, connection and the action of $G_R$. In particular, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[\frac{1}{p}]} M\big[\frac{1}{p}\big] \arrow[r, "\simeq"] \arrow[d, tail] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbf{B}_R^+} \mathbf{N}(V) \arrow[d, "\simeq"] \\
\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[\frac{1}{p}]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \arrow[r, tail] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V,
\end{tikzcd}$$

compatible with Frobenius, connection and $G_R$-action. As the top horizontal arrow and right vertical arrow are bijections, it is immediately clear from the diagram that the left vertical arrow and bottom horizontal arrow must be bijective as well. The bijection of bottom horizontal arrow implies that $V$ is a crystalline representation of $G_R$. Moreover, since $R\left[\frac{1}{p}\right] \rightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ is faithfully flat (see [14, Théorème 6.3.8]), we obtain an isomorphism of $R\left[\frac{1}{p}\right]$-modules $M\left[\frac{1}{p}\right] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$.

Finally, we note that the isomorphism $M\left[\frac{1}{p}\right] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is compatible with supplementary structures. From Proposition 4.28 it is clear that this isomorphism is compatible with Frobenius and connection. Combining Proposition 4.49 with observations made before, we obtain that the isomorphism of $R\left[\frac{1}{p}\right]$-modules $M\left[\frac{1}{p}\right] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is compatible with Frobenius, filtration and connection on each side.

Finally, we can compose these natural maps as

$$\begin{aligned}\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xleftarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)\Big)^{\Gamma_R} \\ \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V),\end{aligned}$$

where the second map is compatible with the Frobenius, filtration (see Corollary 4.54), connection and the action of $\Gamma_R$ on each side (see Proposition 4.28). This proves the theorem.

*Remark 4.47.* — In the case when $\mathbf{N}(T)$ is a free $\mathbf{A}_R^+$- module of rank $h$, from Proposition 4.28 we obtain that $M\left[\frac{1}{p}\right] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is a free $R\left[\frac{1}{p}\right]$-module of rank $h$. In particular, for finite $q$-height representations there exists a finite étale extension $R'$ over $R$ such that

$$R'\left[\frac{1}{p}\right] \otimes_{R[\frac{1}{p}]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$$

is free of rank $h$.

*Remark 4.48.* — For $0 \leqslant i \leqslant d$, one can define $[\varepsilon]$-derivatives by the formula $\frac{\gamma_i - 1}{\pi} : \mathbf{N}(T) \rightarrow \mathbf{N}(T)$. Considering the reduction modulo $\pi$ of Frobenius, filtration and $[\varepsilon]$-connection on $\mathbf{N}(T)$ defined above, we conjecture that we have $(\mathbf{N}(T)/\pi\mathbf{N}(T))\left[\frac{1}{p}\right] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ as filtered $(\varphi, \partial)$-modules over $R\left[\frac{1}{p}\right]$. Details on this line of thought and its connection with [11] and [26] will appear elsewhere.

### 4.5.1. Compatibility between filtrations

Note that using Definition 4.16 and Remark 4.21, the filtration on $M\left[\frac{1}{p}\right]$ is given as

$$\mathrm{Fil}^k M\left[\frac{1}{p}\right] = \Big(\sum_{i\in\mathbb{N}} \mathrm{Fil}^i \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathrm{Fil}^{k-i}\mathbf{N}(V)\Big)^{\Gamma_R}.$$

Proposition 4.49. — *We have* $\mathrm{Fil}^k M\left[\frac{1}{p}\right] = \mathrm{Fil}^k \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ *for* $k \in \mathbb{Z}$.

*Proof.* — We only need to show the claim for $k \geqslant 1$. Note that from (4.1), Remark 4.23 and Lemma 4.53 we have

$$\begin{aligned}\mathrm{Fil}^k M\left[\frac{1}{p}\right] &= \left(\mathrm{Fil}^k\left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)\right)\right)^{\Gamma_R} \\ &\subset \left(\mathrm{Fil}^k\left(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V\right)\right)^{G_R} \\ &= \mathrm{Fil}^k \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V).\end{aligned}$$

Conversely, let $\{e_1, \ldots, e_h\}$ denote a $\mathbb{Q}_p$-basis of $V$ and we take $x \in \mathrm{Fil}^k \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \setminus \mathrm{Fil}^{k+1} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$. Since $x \neq 0$, we can write

$$x = \sum_{i=1}^{h} b_i e_i$$

where either $b_i = 0$ or $b_i \in \mathrm{Fil}^k \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \setminus \mathrm{Fil}^{k+1} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ for each $1 \leqslant i \leqslant h$ and at least one $b_i \neq 0$. Moreover, we have $M\left[\frac{1}{p}\right] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ as $R\left[\frac{1}{p}\right]$-modules, so we take $r \leqslant k$ to be the largest integer such that $x \in \mathrm{Fil}^r M\left[\frac{1}{p}\right]$, in particular $x \notin \mathrm{Fil}^{r+1} M\left[\frac{1}{p}\right]$. Let us write $x = \sum_{j\in\mathbb{N}} c_j \otimes f_{r-j}$ with $c_j \in \mathrm{Fil}^j \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and $f_{r-j} \in \mathrm{Fil}^{r-j}\mathbf{N}(V)$ for all $j \in \mathbb{N}$. By assumption on $x$ there exists $\emptyset \neq I \subset \mathbb{N}$ such that for each $j \in I$ we have $c_j \in \mathrm{Fil}^j \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \setminus \mathrm{Fil}^{j+1} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, $f_{r-j} \in \mathrm{Fil}^{r-j}\mathbf{N}(V) \setminus \mathrm{Fil}^{r-j+1}\mathbf{N}(V)$ with

$$\sum_{j\in I} c_j \otimes f_{r-j} \in \mathrm{Fil}^r\left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)\right) \setminus \mathrm{Fil}^{r+1}\left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)\right),$$

$$\sum_{j\in\mathbb{N}\setminus I} c_j \otimes f_{r-j} \in \mathrm{Fil}^{r+1}\left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)\right).$$

Equip $\mathbf{B}^+$ with the induced filtration

$$\mathrm{Fil}^n \mathbf{B}^+ := \mathbf{B}^+ \cap \mathrm{Fil}^n \mathbf{B}_{\mathrm{cris}}(\overline{R}) = \mathbf{B}^+ \cap \mathrm{Fil}^n\left(\mathbf{A}_{\mathrm{inf}}(\overline{R})\left[\frac{1}{p}\right]\right) \quad \text{for } n \in \mathbb{N}.$$

Using the definition of filtration on $\mathbf{N}(V)$ (see Definition 4.16) and Lemma 4.53, we have that $\mathrm{Fil}^{r-j}\mathbf{N}(V) = (\mathrm{Fil}^{r-j}\mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap \mathbf{N}(V)$ for all $j \in \mathbb{N}$. Therefore, in the expression $\sum_{j\in I} c_j \otimes f_{r-j}$ we must have

$$f_{r-j} \in \left(\mathrm{Fil}^{r-j}\mathbf{B}^+ \otimes_{\mathbb{Q}_p} V\right) \setminus \left(\mathrm{Fil}^{r-j+1}\mathbf{B}^+ \otimes_{\mathbb{Q}_p} V\right) \text{ for all } j \in I.$$

This implies that in the basis of $V$ we can write

$$f_{r-j} = \sum_{i=1}^{h} f_{r-j}^{(i)} e_i$$

with $f_{r-j}^{(i)} \in \mathrm{Fil}^{r-j}\mathbf{B}^+ \setminus \mathrm{Fil}^{r-j+1}\mathbf{B}^+$ for all $j \in I$ and all $1 \leqslant i \leqslant h$. In conclusion, we obtain

$$\begin{aligned} x - \sum_{j\in\mathbb{N}\setminus I} c_j \otimes f_{r-j} &= \sum_{j\in I} c_j \otimes \left(\sum_{i=1}^{h} f_{r-j}^{(i)} e_i\right) \\ &= \sum_{i=1}^{h} \left(\sum_{j\in I} c_j \otimes f_{r-j}^{(i)}\right) e_i, \end{aligned} \tag{4.6}$$

with $c_j \in \mathrm{Fil}^j \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \setminus \mathrm{Fil}^{j+1}\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and $f_{r-j}^{(i)} \in \mathrm{Fil}^{r-j}\mathbf{B}^+ \setminus \mathrm{Fil}^{r-j+1}\mathbf{B}^+$ for all $1 \leqslant i \leqslant h$ and $j \in I$.

Let us set $g_i = \sum_{j\in I} c_j \otimes f_{r-j}^{(i)}$ for $1 \leqslant i \leqslant h$. Then by the discussion above we have that $g_i \in \mathrm{Fil}^r(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{B}^+)$ for $1 \leqslant i \leqslant h$, where $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{B}^+$ is equipped with the tensor product filtration. Note that

$$x \in \mathrm{Fil}^r M\left[\frac{1}{p}\right] \setminus \mathrm{Fil}^{r+1} M\left[\frac{1}{p}\right]$$

and

$$\sum_{j\in\mathbb{N}\setminus I} c_j \otimes f_{r-j} \in \mathrm{Fil}^{r+1}\left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)\right).$$

Moreover, from Lemma 4.50 and Remark 4.51 we deduce that for $n \in \mathbb{N}$ and inside $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V$ we have

$$\begin{aligned} &\mathrm{Fil}^n\left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)\right) \\ &\qquad = \left(\mathrm{Fil}^n\left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{B}^+\right) \otimes_{\mathbb{Q}_p} V\right) \cap \left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)\right). \end{aligned}$$

Therefore, we conclude that we must have at least one $i = i_0$ such that

$$g_{i_0} \in \mathrm{Fil}^r(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{B}^+) \setminus \mathrm{Fil}^{r+1}(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{B}^+).$$

Now using Lemma 4.52 and Remark 4.51 we further note that for $n \in \mathbb{N}$ and inside $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ we have

$$\mathrm{Fil}^n \left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{B}^+\right) = \left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{B}^+\right) \cap \mathrm{Fil}^n \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}).$$

Therefore, we get that $g_i \in \mathrm{Fil}^r \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ for all $1 \leqslant i \leqslant h$ and $g_{i_0} \in \mathrm{Fil}^r \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \setminus \mathrm{Fil}^{r+1} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$. For convenience, let us write

$$\sum_{j \in \mathbb{N} \setminus I} c_j \otimes f_{r-j} = \sum_{i=1}^{h} d_i e_i$$

with $d_i \in \mathrm{Fil}^{r+1} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ for all $1 \leqslant i \leqslant h$. In particular, comparing (4.6) with the expression $x = \sum_{i=1}^{h} b_i e_i$ at the start of the proof, we get $b_{i_0} = g_{i_0} + d_{i_0}$.

Finally, since $r \leqslant k$, consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Fil}^{k+1} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) & \longrightarrow & \mathrm{Fil}^{k} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) & \longrightarrow & \mathrm{gr}^{k} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Fil}^{r+1} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) & \longrightarrow & \mathrm{Fil}^{r} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) & \longrightarrow & \mathrm{gr}^{r} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) & \longrightarrow & 0,
\end{array}$$

where the left and middle vertical arrows are injective and the right vertical arrow is non-trivial if and only if $r = k$. From the fact that $g_{i_0} \in \mathrm{Fil}^r \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \setminus \mathrm{Fil}^{r+1} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$, we see that the image of $b_{i_0}$ is non-zero in $\mathrm{gr}^r \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$. But we already have that image of $b_{i_0}$ is non-zero in $\mathrm{gr}^k \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$. Therefore, the right vertical arrow must be non trivial, i.e. $r = k$. Hence $x \in \mathrm{Fil}^k M\left[\frac{1}{p}\right]$. This proves the claim. ☐

Lemma 4.50. — *For $k \in \mathbb{N}$, inside $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Z}_p} T$ we have*

$$\mathrm{Fil}^k \left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\right) = \left(\mathrm{Fil}^k \left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{A}^+\right) \otimes_{\mathbb{Z}_p} T\right) \cap \left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\right).$$

*Proof.* — From Section 3.1 we have rings $\mathbf{A}^+ \subset \mathbf{A}_{\varpi}^+ \subset \mathbf{A}_{\mathrm{inf}}(\overline{R})$ equipped with an induced filtration from $\mathbf{A}_{\mathrm{cris}}(\overline{R})$ and from Remark 3.18 we have an isomorphism $\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} \mathbf{A}^+ \xrightarrow{\sim} \mathbf{A}_{\varpi}^+$ compatible with Frobenius, filtration and $G_R$-action. Since $\mathbf{A}_R^+ \to \mathbf{A}_{R,\varpi}^+$ is flat and $\mathrm{Fil}^i \mathbf{A}_{R,\varpi}^+ = \xi^i \mathbf{A}_{R,\varpi}^+$, using Lemma 4.53 we note that

$$
\begin{aligned}
(4.7)\quad & \mathrm{Fil}^k \left(\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\right) \\
&= \sum_{i+j=k} \mathrm{Fil}^i \mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} \left(\left(\mathrm{Fil}^j \mathbf{A}^+ \otimes_{\mathbb{Z}_p} T\right) \cap \mathbf{N}(T)\right) \\
&= \left(\sum_{i+j=k} \mathrm{Fil}^i \mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} \mathrm{Fil}^j \mathbf{A}^+ \otimes_{\mathbb{Z}_p} T\right) \cap \left(\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\right) \\
&= \left(\mathrm{Fil}^k \mathbf{A}_\varpi^+ \otimes_{\mathbb{Z}_p} T\right) \cap \left(\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\right).
\end{aligned}
$$

From Definition 4.19 recall that we have the ring $\mathcal{O}\mathbf{A}_{R,\varpi}^+$ and we claim that it is flat over $\mathbf{A}_{R,\varpi}^+$. Indeed, let $T_1, \ldots, T_d$ denote a set of indeterminates and define a map $\mathbf{A}_{R,\varpi}^+[T_1, \ldots, T_d] \to \mathcal{O}\mathbf{A}_{R,\varpi}^+$ via $T_i \mapsto X_i - [X_i^\flat]$, then the target may be identified with $(p, \xi, T_1, \ldots, T_d)$-adic completion of the source which is noetherian, in particular, $\mathcal{O}\mathbf{A}_{R,\varpi}^+$ is flat over $\mathbf{A}_{R,\varpi}^+$. Let us set $\mathcal{O}\mathbf{A}_\varpi^+ := \mathcal{O}\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} \mathbf{A}^+ \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_{R,\varpi}^+} \mathbf{A}_\varpi^+$ equipped with tensor product filtration, Frobenius and $G_R$-action. Let $J = (X_1 - [X_1^\flat], \ldots, X_d - [X_d^\flat])\mathcal{O}\mathbf{A}_{R,\varpi}^+$ then the filtration on $\mathcal{O}\mathbf{A}_\varpi^+$ can also be given as $\mathrm{Fil}^k \mathcal{O}\mathbf{A}_\varpi^+ = \sum_{i+j=k} J^i \mathcal{O}\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_{R,\varpi}^+} \xi^j \mathbf{A}_\varpi^+$. Let us set $N_{R,\varpi}^+ = \mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ equipped with tensor product filtration. Then since $J$ is flat as an $\mathbf{A}_{R,\varpi}^+$-module an argument similar to (4.7) gives us that

$$
\begin{aligned}
(4.8)\quad & \mathrm{Fil}^k \left(\mathcal{O}\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\right) \\
&= \mathrm{Fil}^k \left(\mathcal{O}\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_{R,\varpi}^+} N_{R,\varpi}^+\right) \\
&= \sum_{i+j=k} J^i \mathcal{O}\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_{R,\varpi}^+} \left(\left(\mathrm{Fil}^j \mathbf{A}_\varpi^+ \otimes_{\mathbb{Z}_p} T\right) \cap N_{R,\varpi}^+\right) \\
&= \left(\sum_{i+j=k} J^i \mathcal{O}\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_{R,\varpi}^+} \xi^j \mathbf{A}_\varpi^+ \otimes_{\mathbb{Z}_p} T\right) \cap \left(\mathcal{O}\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_{R,\varpi}^+} N_{R,\varpi}^+\right) \\
&= \left(\mathrm{Fil}^k \mathcal{O}\mathbf{A}_\varpi^+ \otimes_{\mathbb{Z}_p} T\right) \cap \left(\mathcal{O}\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\right).
\end{aligned}
$$

Let us set $\mathcal{O}\mathbf{A}_\varpi^{\mathrm{PD}} := \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_{R,\varpi}^+} \mathbf{A}_\varpi^+ \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{A}^+$ where the isomorphism is compatible with Frobenius, filtration, connection and $G_R$-action. We will show our claim that

$$
\mathrm{Fil}^k \left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\right) = \left(\mathrm{Fil}^k \mathcal{O}\mathbf{A}_\varpi^{\mathrm{PD}} \otimes_{\mathbb{Z}_p} T\right) \cap \left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\right).
$$

Let $f \in \{\xi, X_1 - [X_1^\flat], \ldots, X_d - [X_d^\flat]\}$ be one of the generators of the ideal $(\xi, X_1 - [X_1^\flat], \ldots, X_d - [X_d^\flat])\mathcal{O}\mathbf{A}_\varpi^{\mathrm{PD}}$ and $x \in \mathcal{O}\mathbf{A}_\varpi^{\mathrm{PD}} \otimes_{\mathbb{Z}_p} T$. Then to obtain our claim, it is enough to show that if $f^{[k]}x \in \big(f^{[k]}\mathcal{O}\mathbf{A}_\varpi^{\mathrm{PD}} \otimes_{\mathbb{Z}_p} T\big) \setminus \big(f^{[k+1]}\mathcal{O}\mathbf{A}_\varpi^{\mathrm{PD}} \otimes_{\mathbb{Z}_p} T\big)$ such that

$$f^{[k]}x \in \Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\Big)$$

then $f^{[k]}x \in \mathrm{Fil}^k\big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\big)$. Note that the claim is true for $k = 0$. So let $k \geqslant 1$ and $f$ as above. Let $f^{[k]}x \in \big(f^{[k]}\mathcal{O}\mathbf{A}_\varpi^{\mathrm{PD}} \otimes_{\mathbb{Z}_p} T\big) \setminus \big(f^{[k+1]}\mathcal{O}\mathbf{A}_\varpi^{\mathrm{PD}} \otimes_{\mathbb{Z}_p} T\big)$ such that $f^{[k]}x \in \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\big)$. Since $x \neq 0$, by induction on $k$ we may assume that $x = \sum_{i=1}^h x_i e_i \in \mathcal{O}\mathbf{A}_\varpi^+ \otimes_{\mathbb{Z}_p} T$ with either $x_i = 0$ or $x_i \in f^k\mathcal{O}\mathbf{A}_\varpi^+ \setminus f^{k+1}\mathcal{O}\mathbf{A}_\varpi^+$ for each $1 \leqslant i \leqslant h$ and at least one $x_i \neq 0$. Recall that we have $\pi^s\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T \subset \mathbf{A}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$, therefore $\pi^s x \in \mathcal{O}\mathbf{A}_\varpi^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$. But then inside $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ we must have

$$\begin{aligned} f^k x = k! f^{[k]}x &\in \Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\Big) \cap \frac{1}{\pi^s}\Big(\mathcal{O}\mathbf{A}_\varpi^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\Big) \\ &= \mathcal{O}\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T). \end{aligned}$$

Therefore,

$$f^k x \in \Big(\mathrm{Fil}^k\mathcal{O}\mathbf{A}_\varpi^+ \otimes_{\mathbb{Z}_p} T\Big) \cap \Big(\mathcal{O}\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\Big) = \mathrm{Fil}^k\Big(\mathcal{O}\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\Big)$$

where the last equality follows from (4.8). Hence, inside $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ we have

$$\begin{aligned} f^{[k]}x \in \frac{1}{k!}\mathrm{Fil}^k\Big(\mathcal{O}\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\Big) &\cap \Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\Big) \\ &\subset \mathrm{Fil}^k\Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\Big) \end{aligned}$$

as desired. □

*Remark 4.51.* — From Lemma 4.52 below, it easily follows that inside $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ we have

$$\mathrm{Fil}^k\Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{B}^+\Big) = \Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{B}^+\Big) \cap \mathrm{Fil}^k\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}).$$

So from Lemma 4.50 we get that the corresponding rational version of the statement is also true, i.e. inside $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V$ we have

$$\begin{aligned} &\mathrm{Fil}^k\Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)\Big) \\ &\quad = \Big(\mathrm{Fil}^k\Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{B}^+\Big) \otimes_{\mathbb{Q}_p} V\Big) \cap \Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)\Big). \end{aligned}$$

Lemma 4.52. — *For $k \in \mathbb{N}$, inside $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ we have*

$$\mathrm{Fil}^k \left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{A}^+\right) = \left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{A}^+\right) \cap \mathrm{Fil}^k \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}).$$

*Proof.* — Recall that filtrations on $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ are compatible (see Remark 4.23). Moreover, from Section 3.1 the inclusion of rings $\mathbf{A}^+ \subset \mathbf{A}_\varpi^+ \subset \mathbf{A}_{\mathrm{inf}}(\overline{R})$ is compatible with induced filtration from $\mathbf{A}_{\mathrm{cris}}(\overline{R})$. From the proof of Lemma 4.50 we have an isomorphism of rings $\mathcal{O}\mathbf{A}_\varpi^{\mathrm{PD}} = \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{A}^+ \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_{R,\varpi}^+} \mathbf{A}_\varpi^+$ compatible with tensor product filtrations. Now by the description of filtration on the rightmost term we get that $\mathcal{O}\mathbf{A}_\varpi^{\mathrm{PD}}$ is equipped with filtration by divided powers of the ideal $(\xi, X_1 - [X_1^\flat], \ldots, X_d - [X_d^\flat])\mathcal{O}\mathbf{A}_\varpi^{\mathrm{PD}}$. Finally, the natural multiplication map $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_{R,\varpi}^+} \mathbf{A}_\varpi^+ \to \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ is injective. Hence, it follows that for $k \in \mathbb{N}$, inside $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ we have

$$\mathrm{Fil}^k \left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{A}^+\right) = \left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{A}^+\right) \cap \mathrm{Fil}^k \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}). \qquad \square$$

Lemma 4.53. — *For $k \in \mathbb{N}$ we have* $(\mathrm{Fil}^k \mathbf{A}^+ \otimes_{\mathbb{Z}_p} T) \cap \mathbf{N}(T) = \mathrm{Fil}^k \mathbf{N}(T)$ *and* $(\mathrm{Fil}^k \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap \mathbf{N}(V) = \mathrm{Fil}^k \mathbf{N}(V)$.

*Proof.* — It is enough to show that $(\mathrm{Fil}^k \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap \mathbf{N}(V) = \mathrm{Fil}^k \mathbf{N}(V)$. Indeed, from Definition 4.16 we have $\mathrm{Fil}^k \mathbf{N}(T) = \mathrm{Fil}^k \mathbf{N}(V) \cap \mathbf{N}(T) = (\mathrm{Fil}^k \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap \mathbf{N}(V) \cap \mathbf{N}(T) = (\mathrm{Fil}^k \mathbf{A}^+ \otimes_{\mathbb{Q}_p} T) \cap \mathbf{N}(T)$ since $\mathrm{Fil}^k \mathbf{B}^+ \cap \mathbf{A}^+ = \mathrm{Fil}^k \mathbf{A}^+$. Next, the inclusion $\mathrm{Fil}^k \mathbf{N}(V) \subset (\mathrm{Fil}^k \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V)$ is obvious. For the converse, we claim that it is enough to show that $(q^k \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap \mathbf{N}(V) = q^k \mathbf{N}(V)$. Indeed, if we have $x \in (\mathrm{Fil}^k \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap \mathbf{N}(V)$ then $\varphi(x) \in (q^k \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap \mathbf{N}(V) = q^k \mathbf{N}(V)$, i.e. $x \in \mathrm{Fil}^k \mathbf{N}(V)$.

The inclusion $q^k \mathbf{N}(V) \subset (q^k \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap \mathbf{N}(V)$ is obvious. To show the converse, first let us assume that $\mathbf{N}(V)$ is free with $\{f_1, f_2, \ldots, f_h\}$ as a $\mathbf{B}_R^+$-basis, and let $\{e_1, \ldots, e_h\}$ be a $\mathbb{Q}_p$-basis of $V$. Now let $q^k x \in (q^k \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap \mathbf{N}(V)$ for $x = \sum_{i=1}^h x_i e_i \in \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V$. We can also write $q^k x = \sum_{i=1}^h y_i f_i \in \mathbf{N}(V)$ with $y_i \in \mathbf{B}_R^+$. Next, from Proposition 4.11 we have $\pi^s \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V \subset \mathbf{B}^+ \otimes_{\mathbf{B}_R^+} \mathbf{N}(V)$, so we can write

$$q^k x = \pi^{-s} q^k \sum_{i=1}^h x_i \pi^s e_i = \pi^{-s} q^k \sum_{i=1}^h x_i \sum_{j=1}^h z_{ij} f_j = \pi^{-s} q^k \sum_{i=1}^h \left(\sum_{j=1}^h x_j z_{ji}\right) f_i,$$

with $z_{ij} \in \mathbf{B}^+$. But then we must have $\pi^{-s} q^k \sum_{j=1}^h x_j z_{ji} = y_i$ for all $1 \leqslant i \leqslant h$. Since $H_R$ acts trivially on $\pi$, $q$ and $y_i$, we get that $w_i := \sum_{j=1}^h x_j z_{ji} \in \mathbf{B}_R^+$. But $y_i \in \mathbf{B}_R^+$ and $\pi$ and $q$ are coprime in $\mathbf{B}_R^+$ (since $q \equiv p \mod \pi \mathbf{B}_R^+$), therefore we obtain that $w_i \in \pi^s \mathbf{B}_R^+$. In particular, $y_i \in q^k \mathbf{B}_R^+$, therefore $q^k x = \sum_{i=1}^h y_i f_i \in q^k \mathbf{N}(V)$. Hence, $(q^k \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap \mathbf{N}(V) = q^k \mathbf{N}(V)$.

Next, if $\mathbf{N}(V)$ is projective (and not free) over $\mathbf{B}_R^+$, let $R'$ be the $p$-adic completion of a finite étale algebra over $R$ such that the scalar extension $\mathbf{B}_{R'}^+ \otimes_{\mathbf{B}_R^+} \mathbf{N}(V)$ is a free module over $\mathbf{B}_{R'}^+$ and $R'\big[\frac{1}{p}\big]/R\big[\frac{1}{p}\big]$ is Galois (see Definition 4.9). Then we can argue as above and conclude by taking $\mathrm{Gal}\big(R'\big[\frac{1}{p}\big]/R\big[\frac{1}{p}\big]\big)$-invariants of $q^k \mathbf{B}_{R'}^+ \otimes_{\mathbf{B}_R^+} \mathbf{N}(V)$. ☐

Corollary 4.54. — *For $k \in \mathbb{N}$ we have* $\mathrm{Fil}^k\big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M\big[\frac{1}{p}\big]\big) \xrightarrow{\sim} \mathrm{Fil}^k(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V))$ *under the isomorphism* (4.5).

*Proof.* — From the definition of filtration on the left term we know that the map in claim is injective. To check the surjectivity, using Proposition 4.49, it is enough to show that under the isomorphisms

$$\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xleftarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M\left[\frac{1}{p}\right] \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)$$

we have

$$\mathrm{Fil}^k\mathbf{N}(V) \subset \mathrm{Fil}^k\left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)\right) \text{ for all } k \in \mathbb{N}.$$

Using Lemma 4.53, note that inside $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V$, we have

$$\mathrm{Fil}^k\mathbf{N}(V) \subset \mathrm{Fil}^k\left(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V\right) \cap \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V).$$

We claim that the last term equals $\mathrm{Fil}^k(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V))$, i.e. the induced filtration on $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is the tensor product filtration (or equivalently, on $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\big[\frac{1}{p}\big] \otimes_{R[\frac{1}{p}]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ since $\mathrm{Fil}^k(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}})\big[\frac{1}{p}\big] = \mathrm{Fil}^k\big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\big[\frac{1}{p}\big]\big)$). Indeed, from Section 2.3 recall that $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$, $\mathrm{Fil}^k\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ and $\mathrm{gr}^k\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ are projective $R\big[\frac{1}{p}\big]$-modules for all $k \in \mathbb{N}$. Then it easily follows that for $i, j \in \mathbb{N}$ such that $i + j = k$, inside $\mathrm{Fil}^k\big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\big[\frac{1}{p}\big] \otimes_{R[\frac{1}{p}]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)\big)$ we have

$$\begin{aligned}\left(\mathrm{Fil}^i\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\left[\frac{1}{p}\right] \otimes_{R[\frac{1}{p}]} \mathrm{Fil}^j\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)\right)&\\ \cap \mathrm{Fil}^{k+1}\left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\left[\frac{1}{p}\right] \otimes_{R[\frac{1}{p}]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)\right)&\\ = \mathrm{Fil}^i\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\left[\frac{1}{p}\right] \otimes_{R[\frac{1}{p}]} \mathrm{Fil}^{j+1}\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)&\\ + \mathrm{Fil}^{i+1}\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\left[\frac{1}{p}\right] \otimes_{R[\frac{1}{p}]} \mathrm{Fil}^j\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V).&\end{aligned}$$

Therefore, we get that

$$\begin{aligned}\mathrm{gr}^k\left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\left[\frac{1}{p}\right]\bigotimes_{R[\frac{1}{p}]}\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)\right)\\ =\bigoplus_{i+j=k}\mathrm{gr}^i\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\left[\frac{1}{p}\right]\bigotimes_{R[\frac{1}{p}]}\mathrm{gr}^j\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V).\end{aligned}$$

Similarly, one can also show that

$$\begin{aligned}\mathrm{gr}^k\left(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})\bigotimes_{R[\frac{1}{p}]}\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)\right)\\ =\bigoplus_{i+j=k}\mathrm{gr}^i\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})\bigotimes_{R[\frac{1}{p}]}\mathrm{gr}^j\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V).\end{aligned}$$

Since the filtration on $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\left[\frac{1}{p}\right]$ is induced from the filtration on $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ (see Remark 4.23), the natural map $\mathrm{gr}^k\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\left[\frac{1}{p}\right]\to\mathrm{gr}^k\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ is injective. Therefore, the natural map $\mathrm{gr}^k\big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\left[\frac{1}{p}\right]\otimes_R\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)\big)\to$ $\mathrm{gr}^k(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})\otimes_{R[\frac{1}{p}]}\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V))=\mathrm{gr}^k(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})\otimes_{QQ_p}V)$ is injective as well. Hence, we have $\mathrm{Fil}^k(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})\otimes_{\mathbb{Q}_p}V)\cap\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\otimes_R\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)=$ $\mathrm{Fil}^k(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\otimes_R\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V))$ for all $k\in\mathbb{N}$. □

### 4.6. One-dimensional representations

In this section we will show that all one-dimensional crystalline representations are of finite $q$-height by writing down the corresponding Wach modules precisely.

Proposition 4.55. — *All one-dimensional crystalline representations of $G_R$ are of finite $q$-height. Furthermore, for a one-dimensional crystalline representation $V$ we have an isomorphism of $R\left[\frac{1}{p}\right]$-modules*

$$\left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\otimes_{\mathbf{A}_R^+}\mathbf{N}(V)\right)^{\Gamma_R}\xrightarrow{\sim}\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V).$$

*Therefore, there exists natural isomorphisms*

$$\begin{aligned}\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\otimes_R\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)\xleftarrow{\sim}\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\otimes_R\left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\otimes_{\mathbf{A}_R^+}\mathbf{N}(V)\right)^{\Gamma_R}\\ \xrightarrow{\sim}\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\otimes_{\mathbf{A}_R^+}\mathbf{N}(V),\end{aligned}$$

*compatible with Frobenius, filtration and the action of $\Gamma_R$.*

*Proof.* — The structure of one-dimensional crystalline representations of $G_R$ is well-known (see [14, Section 8.6]). From Proposition 2.3 we have that for $\eta : G_R \to \mathbb{Z}_p^\times$, a continuous character, $V = \mathbb{Q}_p(\eta)$ is crystalline if and only if we can write $\eta = \eta_{\mathrm{f}}\eta_{\mathrm{ur}}\chi^n$ with $n \in \mathbb{Z}$, and where $\eta_{\mathrm{f}}$ is a finite unramified character, $\eta_{\mathrm{ur}}$ is an unramified character taking values in $1 + p\mathbb{Z}_p$ and trivialized by an element $\alpha \in 1 + p\widehat{R^{\mathrm{ur}}}$, and $\chi$ is the $p$-adic cyclotomic character. Recall that a $p$-adic representation of $G_R$ is unramified if the action of $G_R$ factorizes through the quotient $G_R^{\mathrm{ur}}$ (see Section 2.3). Moreover, if $\eta_{\mathrm{f}}$ is trivial then $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is a free $R\big[\frac{1}{p}\big]$-module of rank 1.

In Lemma 4.56 below, we show that crystalline representations $V_1 := \mathbb{Q}_p(\eta_{\mathrm{f}}\eta_{\mathrm{ur}})$ and $V_2 := \mathbb{Q}_p(\chi^n)$ are of finite $q$-height. For a one-dimensional crystalline representation $V := \mathbb{Q}_p(\eta) = \mathbb{Q}_p(\eta_{\mathrm{f}}\eta_{\mathrm{ur}}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\chi^n) = V_1 \otimes_{\mathbb{Q}_p} V_2$ as above, by compatibility of tensor products in Propositions 4.14 we get that $V$ is a finite $q$-height representation as well with

$$\mathbf{N}(V) = \mathbf{N}(V_1) \otimes_{\mathbf{B}_R^+} \mathbf{N}(V_2).$$

From the isomorphisms of $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$-modules in Lemma 4.56, compatibility of Wach modules with tensor product in Proposition 4.14 and compatibility of the functor $\mathcal{O}\mathbf{D}_{\mathrm{cris}}$ with tensor products in Section 2.3 (see also [14, Théorème 8.4.2]), we get a string of isomorphisms of $\mathcal{O}\mathbf{B}_{R,\varpi}^{\mathrm{PD}} := \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\big[\frac{1}{p}\big]$-modules compatible with Frobenius, filtration and the action of $\Gamma_R$,

$$\begin{aligned}
&\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)\\
&\qquad \xrightarrow{\sim} \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_1)\big) \otimes_{\mathcal{O}\mathbf{B}_{R,\varpi}^{\mathrm{PD}}} \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_2)\big)\\
&\qquad \xleftarrow{\sim} \Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V_1)\Big) \otimes_{\mathcal{O}\mathbf{B}_{R,\varpi}^{\mathrm{PD}}} \Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V_2)\Big)\\
&\qquad \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V_1) \otimes_{\mathbf{B}_R^+} \mathbf{N}(V_2)\\
&\qquad \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}\left(V_1 \otimes_{\mathbb{Q}_p} V_2\right) \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V).
\end{aligned}$$

Taking $\Gamma_R$-invariants of the first and the last term gives us that $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{\sim} \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)\big)^{\Gamma_R}$, compatible with Frobenius and filtration. ☐

Following claim was used above:

Lemma 4.56.

(i) *Let $\eta : G_R \to \mathbb{Z}_p^\times$ be a continuous unramified character. Then the $p$-adic representation $\mathbb{Q}_p(\eta)$ is a finite $q$-height representation.*

(ii) *Let $\chi$ be the $p$-adic cyclotomic character then for $n \in \mathbb{Z}$, the $p$-adic representation $\mathbb{Q}_p(n)$ is a finite $q$-height representation.*

*Further, for* $V = \mathbb{Q}_p(\eta), \mathbb{Q}_p(n)$ *we have an isomorphism of* $R\big[\frac{1}{p}\big]$*-modules*

$$\Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)\Big)^{\Gamma_R} \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V).$$

*Therefore, there exists natural isomorphisms*

$$\begin{aligned}\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xleftarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)\Big)^{\Gamma_R} \\ \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V),\end{aligned}$$

*compatible with Frobenius, filtration and the action of* $\Gamma_R$*.*

*Proof. —* Let $\eta = \eta_{\mathrm{f}}\eta_{\mathrm{ur}}$, where $\eta_{\mathrm{f}}$ is an unramified character of finite order and $\eta_{\mathrm{ur}}$ is an unramified character taking values in $1 + p\mathbb{Z}_p$ and trivialised by an element $\alpha \in 1 + p\widehat{R^{\mathrm{ur}}}$ (see Proposition 2.3).

First, let us consider the finite unramified character $\eta_{\mathrm{f}}$. Set $T = \mathbb{Z}_p(\eta_{\mathrm{f}}) = \mathbb{Z}_p e$, such that $g(e) = \eta_{\mathrm{f}}(g)e$. We have

$$\begin{aligned}\mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big) &= \big(\mathbf{A}^+ \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\eta_{\mathrm{f}})\big)^{H_R} \\ &\xrightarrow{\sim} \big\{a \otimes e, \text{ with } a \in \mathbf{A}^+ \text{ such that } g(a) = \eta_{\mathrm{f}}^{-1}(g)a, \text{ for } g \in H_R\big\}.\end{aligned}$$

Since $\eta_{\mathrm{f}}$ is a finite unramified character, it trivializes over a finite Galois extension $S$ over $R$ (see [14, Proposition 8.6.1]), and we have that $\mathrm{Gal}\big(S\big[\frac{1}{p}\big]/R\big[\frac{1}{p}\big]\big) = G_R/G_S = H_R/H_S = \Gamma_R/\Gamma_S$. As $S$ is finite étale over $R$ the construction of previous chapters apply and we obtain that the $\mathbf{A}_S^+$-module $\mathbf{D}_S^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big) = \big(\mathbf{A}^+ \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\eta_{\mathrm{f}})\big)^{H_S} = \mathbf{A}_S^+(\eta_{\mathrm{f}}) = \mathbf{A}_S^+ e$ is free of rank 1. Further, we know that $\mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big) = \mathbf{D}_S^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big)^{H_R/H_S}$, which implies that the natural inclusion

$$\mathbf{A}_S^+ \otimes_{\mathbf{A}_R^+} \mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big) \longrightarrow \mathbf{D}_S^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big),$$

is bijective. Since $\mathbf{A}_R^+ \to \mathbf{A}_S^+$ is faithfully flat, we obtain that $\mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big)$ is projective of rank 1. Moreover, $\mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big)$ admits a Frobenius-semilinear endomorphism $\varphi$ such that $\mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big) \xrightarrow{\sim} \varphi^*\big(\mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big)\big)$ (one can obtain this after faithfully flat scalar extension $\mathbf{A}_R^+ \to \mathbf{A}_S^+$ and applying descent as above, since $\varphi$ commutes with $G_R$-action). The action of $\Gamma_R$ is trivial on $\mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big)$. Now, we can take $\mathbf{N}\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big) = \mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big)$. From the discussion above, $\mathbf{N}\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big)$ clearly satisfies the conditions of Definition 4.9. Also, we have that $\mathbf{N}(\mathbb{Q}_p(\eta_{\mathrm{f}})) = \mathbf{D}^+(\mathbb{Q}_p(\eta_{\mathrm{f}}))$. On the other hand, we have

$$\begin{aligned}\mathcal{O}\mathbf{D}_{\mathrm{cris}}\big(\mathbb{Q}_p(\eta_{\mathrm{f}})\big) &= \big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta_{\mathrm{f}})\big)^{G_R} \\ &= \big\{b \otimes e, \text{ with } b \in \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \text{ such that } g(b) = \eta_{\mathrm{f}}(g)b\big\}.\end{aligned}$$

Since $\eta_{\mathrm{f}}$ trivializes over the finite Galois extension $S$ over $R$, we have

$$\Big(\mathcal{O}\mathbf{A}_{S,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}\big(\mathbb{Q}_p(\eta_{\mathrm{f}})\big)\Big)^{\Gamma_S} = S_0\left[\frac{1}{p}\right] e = \big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{S}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta_{\mathrm{f}})\big)^{G_S},$$

where the rings $\mathcal{O}\mathbf{A}_{S,\varpi}^{\mathrm{PD}}$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{S})$ are defined for $S$ over which all the construction of previous sections apply (since $S$ is finite étale over $R$). Now taking invariants under the finite Galois group $\mathrm{Gal}\big(S\big[\frac{1}{p}\big]/R\big[\frac{1}{p}\big]\big) = G_R/G_S$, gives us

$$\Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}\big(\mathbb{Q}_p(\eta_{\mathrm{f}})\big)\Big)^{\Gamma_R} = \mathcal{O}\mathbf{D}_{\mathrm{cris}}\big(\mathbb{Q}_p(\eta_{\mathrm{f}})\big).$$

Clearly, the natural maps

$$\begin{aligned}\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}\big(\mathbb{Q}_p(\eta_{\mathrm{f}})\big) &\xleftarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}\big(\mathbb{Q}_p(\eta_{\mathrm{f}})\big)\Big)^{\Gamma_R}\\ &\xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}\big(\mathbb{Q}_p(\eta_{\mathrm{f}})\big),\end{aligned}$$

are isomorphisms compatible with Frobenius, filtration and the action of $\Gamma_R$.

Next, let us consider the unramified character $\eta_{\mathrm{ur}}$ which takes values in $1 + p\mathbb{Z}_p$ and trivialised by an element $\alpha \in 1 + p\widehat{R^{\mathrm{ur}}}$ (see Proposition 2.3). Set $T = \mathbb{Z}_p(\eta_{\mathrm{ur}}) = \mathbb{Z}_p e$, such that $g(e) = \eta_{\mathrm{ur}}(g)e$. We have

$$\mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{ur}})\big) = \big(\mathbf{A}^+ \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\eta_{\mathrm{ur}})\big)^{H_R} = \mathbf{A}_R^+\alpha e.$$

So we take $\mathbf{N}\big(\mathbb{Z}_p(\eta_{\mathrm{ur}})\big) = \mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{ur}})\big) = \mathbf{A}_R^+\alpha e$. This clearly satisfies the conditions of Definition 4.9. Also, we have that $\mathbf{N}(\mathbb{Q}_p(\eta_{\mathrm{ur}})) = \mathbf{D}^+(\mathbb{Q}_p(\eta_{\mathrm{ur}}))$. On the other hand, we have

$$\begin{aligned}\mathcal{O}\mathbf{D}_{\mathrm{cris}}\big(\mathbb{Q}_p(\eta_{\mathrm{ur}})\big) &= \big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta_{\mathrm{ur}})\big)^{G_R}\\ &= \big\{b \otimes e, \text{ with } b \in \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \text{ such that } g(b) = \eta_{\mathrm{ur}}(g)b\big\}\\ &= R\left[\frac{1}{p}\right]\alpha e.\end{aligned}$$

Therefore, we obtain

$$\Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}\big(\mathbb{Q}_p(\eta_{\mathrm{ur}})\big)\Big)^{\Gamma_R} = R\left[\frac{1}{p}\right]\alpha e = \big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta_{\mathrm{ur}})\big)^{G_R}.$$

Clearly, the natural maps

$$\begin{aligned}\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}\big(\mathbb{Q}_p(\eta_{\mathrm{ur}})\big) &\xleftarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}\big(\mathbb{Q}_p(\eta_{\mathrm{ur}})\big)\big)^{\Gamma_R}\\ &\xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}\big(\mathbb{Q}_p(\eta_{\mathrm{ur}})\big),\end{aligned}$$

are isomorphisms compatible with Frobenius, filtration and the action of $\Gamma_R$.

Finally, let $T = \mathbb{Z}_p(n) = \mathbb{Z}_p e_n$ such that $g(e_n) = \chi(g)^n e_n$, then $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$ is a crystalline representation and we can take $\mathbf{N}\big(\mathbb{Z}_p(n)\big) = \mathbf{A}_R^+\pi^{-n}e_n$. Note that for $n \leqslant 0$, we have that $\mathbf{N}\big(\mathbb{Z}_p(n)\big)/\varphi^*\big(\mathbf{N}\big(\mathbb{Z}_p(n)\big)\big)$ is killed by $q^{-n}$, where $q = \frac{\varphi(\pi)}{\pi}$. It can easily be verified that $\Gamma_R$ acts trivially modulo $\pi$ on $\mathbf{N}(T)$. So, we set $\mathbf{N}\big(\mathbb{Q}_p(n)\big) = \mathbf{B}_R^+\pi^{-n}e_n$. Similarly,

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}}\big(\mathbb{Q}_p(n)\big) = \big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)\big)^{G_R} = R\left[\frac{1}{p}\right] t^{-n}e_n,$$

and $\big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}\big(\mathbb{Q}_p(n)\big)\big)^{\Gamma_R} = R\big[\frac{1}{p}\big]t^{-n}e_n = \mathcal{O}\mathbf{D}_{\mathrm{cris}}\big(\mathbb{Q}_p(n)\big)$ compatible with Frobenius, filtration and connection on each side. Finally, the map

$$\begin{aligned}\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}\big(\mathbb{Q}_p(n)\big) &\longrightarrow \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}\big(\mathbb{Q}_p(n)\big)\\ t^{-n}e_n &\longmapsto \frac{\pi^n}{t^n}\pi^{-n}e_n.\end{aligned}$$

is trivially an isomorphism compatible with Frobenius, filtration and the action of $\Gamma_R$, since $\frac{\pi^n}{t^n} \in \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ are units for $n \in \mathbb{Z}$ (see Lemma 3.14). This proves the lemma. □

*Remark 4.57.* — Note that for $T = \mathbb{Z}_p\big(\eta_{\mathrm{f}}\eta_{\mathrm{ur}}\big)$ or $\mathbb{Z}_p(n)$, we even have an isomorphism on the integral level

$$\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \Big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\big)\Big)^{\Gamma_R} \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T).$$

## 5. Relative Fontaine–Laffaille modules

In this section we will consider relative Fontaine–Laffaille data and construct Wach modules given such data. Carrying out such a process would involve starting with a module over $R$ and constructing modules over the ring $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and $\mathbf{A}_{R,\varpi}^+$, and finally descending over to the ring $\mathbf{A}_R^+$.

Explicitly, we will work with objects of the category $\mathrm{MF}_{[0,p-2],\,\mathrm{free}}(R,\Phi,\partial)$, defined by [36, Section 4] as a full subcategory of the abelian category $\mathfrak{MF}_{[0,p-2]}^{\nabla}(R)$ introduced by Faltings in [19, Section II]. In particular,

Definition 5.1. — *Define the category of* free relative Fontaine–Laffaille *modules of level* $[0, p-2]$, *denoted by* $\mathrm{MF}_{[0,p-2],\,\mathrm{free}}(R,\Phi,\partial)$, *as follows: An object with weights in the interval* $[0,p–2]$ *is a quadruple* $(M, \mathrm{Fil}^{\bullet}M, \partial, \Phi)$ *such that,*

(i) *$M$ is a free $R$-module of finite rank.*
(ii) *$M$ is equipped with a decreasing filtration $\{\mathrm{Fil}^k M\}_{k\in\mathbb{Z}}$ by finite $R$-submodules with $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{s+1} M = 0$ such that $\mathrm{gr}_{\mathrm{Fil}}^k M$ is a finite free $R$-module for every $k \in \mathbb{Z}$.*

(iii) *The connection* $\partial : M \to M \otimes_R \Omega^1_R$ *is* $p$*-adically quasi-nilpotent and integrable, and satisfies Griffiths transversality with respect to the filtration, i.e.* $\partial(\mathrm{Fil}^k M) \subset \mathrm{Fil}^{k-1} M \otimes_R \Omega^1_R$ *for* $k \in \mathbb{Z}$*.*

(iv) *Let* $(\varphi^*(M), \varphi^*(\partial))$ *denote the pullback of* $(M, \partial)$ *by* $\varphi : R \to R$*, and equip it with a decreasing filtration defined as* $\mathrm{Fil}^k_p(\varphi^*(M)) = \sum_{i \in \mathbb{N}} p^{[i]} \varphi^*(\mathrm{Fil}^{k-i} M)$ *for* $k \in \mathbb{Z}$*. We suppose that there is an* $R$*-linear morphism* $\Phi : \varphi^*(M) \to M$ *such that* $\Phi$ *is compatible with connections,* $\Phi\big(\mathrm{Fil}^k_p(\varphi^*(M))\big) \subset p^k M$ *for* $0 \leqslant k \leqslant s$*, and we have* $\sum_{k=0}^{s} p^{-k} \Phi\big(\mathrm{Fil}^k_p(\varphi^*(M))\big) = M$*. We denote the composition* $M \to \varphi^*(M) \xrightarrow{\Phi} M$ *by* $\varphi$*.*

*A morphism between two objects of the category* $\mathrm{MF}_{[0,p-2],\,\mathrm{free}}(R, \Phi, \partial)$ *is a continuous* $R$*-linear map compatible with the homomorphism* $\Phi$*, the connection* $\partial$ *and filtration on each side.*

*Notation 5.2.* — Abusing notations, we will denote $(M, \mathrm{Fil}^k M, \partial, \Phi) \in \mathrm{MF}_{[0,p-2],\,\mathrm{free}}(R, \Phi, \partial)$ by $M$ and say that it is of level $[0, p-2]$.

To an object $M \in \mathrm{MF}_{[0,p-2],\,\mathrm{free}}(R, \Phi, \partial)$, we associate a $\mathbb{Z}_p$-module as

$$T^*_{\mathrm{cris}}(M) := \mathrm{Hom}_{R,\,\mathrm{Fil},\,\varphi,\,\partial}(M, \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})), \tag{5.1}$$

i.e. $R$-linear maps from $M$ to $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ compatible with Frobenius, filtration and connection, where we have $\varphi : M \to \varphi^*(M) \xrightarrow{\Phi} M$.

Proposition 5.3.

(i) *For a free Fontaine–Laffaille module* $M$ *of level* $[0, p-2]$*, the* $\mathbb{Z}_p$*-module* $T^*_{\mathrm{cris}}(M)$ *is a free module of rank* $= \mathrm{rk}_R M$ *equipped with a continuous action of* $G_R$*. Further, the* $p$*-adic representation* $V^*_{\mathrm{cris}}(M) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T^*_{\mathrm{cris}}(M)$ *is a crystalline representation of* $G_R$ *with Hodge–Tate weights in the interval* $[0, p-2]$*.*

(ii) *The contravariant* $\mathbb{Z}_p$*-linear functor*

$$T^*_{\mathrm{cris}} : \mathrm{MF}_{[0,p-2],\,\mathrm{free}}(R, \Phi, \partial) \longrightarrow \mathrm{Rep}_{\mathbb{Z}_p,\,\mathrm{free}}(G_R),$$

*is fully faithful. Here* $\mathrm{Rep}_{\mathbb{Z}_p,\,\mathrm{free}}(G_R)$ *denotes the category of finite free* $\mathbb{Z}_p$*-modules equipped with a continuous action of* $G_R$*.*

*Proof.* — The claim in (i) follows from [19, Theorem 2.4] and [36, Proposition 66]. Further, the claim in (ii) follows from [19, Theorem 2.4] and [36, Theorem 77]. □

Definition 5.4. — *Let* $M$ *be a free relative Fontaine–Laffaille module of level* $[0, p-2]$*, and set*

$$T_{\mathrm{cris}}(M) := \mathrm{Hom}_{\mathbb{Z}_p}(T^*_{\mathrm{cris}}(M), \mathbb{Z}_p),$$

*which is a free $\mathbb{Z}_p$-module of rank* $= \mathrm{rk}_R M$*, admitting a continuous action of* $G_R$*.*

The main result of this section is as follows:

Theorem 5.5. — *For a free relative Fontaine–Laffaille module $M$ over $R$ of level* $[0, p-2]$*, the associated representation* $V_{\mathrm{cris}}(M) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_{\mathrm{cris}}(M)$ *is a positive finite $q$-height representation (in the sense of Definition 4.9).*

The proof crucially exploits the computation of Fontaine [22], Wach [38] and Tsuji [36]. It follows in three steps: First, starting with a Fontaine–Laffaille module, we obtain an $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$-module using formal consequences of crystalline site for maps $\theta : \mathbf{A}_{R,\varpi}^{\mathrm{PD}} \twoheadrightarrow R[\varpi]$, and $\theta_R : \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \twoheadrightarrow R[\varpi]$ (see Proposition 5.25, we also give an alternate proof of the proposition). Next, we exploit equivalence of categories obtained in Theorem 5.21 by extending scalars along $\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \twoheadrightarrow \mathbf{A}_{R,\varpi}^{\mathrm{PD}}/I^{(p-1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \xleftarrow{\sim} \mathbf{A}_{R,\varpi}^{+}/I^{(p-1)}\mathbf{A}_{R,\varpi}^{+} \twoheadleftarrow \mathbf{A}_{R,\varpi}^{+}$. This gives us an $\mathbf{A}_{R,\varpi}^{+}$-module with precise description of the Frobenius and the action of $\Gamma_R$ (see Proposition 5.30). Finally, we descend over to the ring $\mathbf{A}_R^{+}$ by exploiting the Frobenius and $\Gamma_R$-action, thus obtaining a Wach module over $\mathbf{A}_R^{+}$ and proving the theorem (see Section 5.3.2).

For clarity of exposition and notational convenience in explaining the result of the first step, we start with preliminaries on some ideals of $\mathbf{A}_{R,\varpi}^{+}$ and $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ (appearing in the second step in the paragraph above) which will help us in proving categorical equivalence between certain modules over the concerned rings.

## 5.1. Some ideals of $\mathbf{A}_{R,\varpi}^{+}$ and $\mathbf{A}_{R,\varpi}^{\mathbf{PD}}$

In this section, we will collect some technical results about the rings $\mathbf{A}_{R,\varpi}^{+}$ and $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and some of their ideals. The results are motivated by the corresponding results over $\mathbf{A}_{\mathrm{inf}}(\overline{R})$ and $\mathbf{A}_{\mathrm{cris}}(\overline{R})$ and their respective ideals, studied in [22, Section 5].

Lemma 5.6. — *Let $a \in \mathbf{A}_{R,\varpi}^{+}$ such that $\mathbf{A}_{R,\varpi}^{+}/p\mathbf{A}_{R,\varpi}^{+}$ is $a$-torsion free and $a$-adically complete. Then,*

(i) *$\mathbf{A}_{R,\varpi}^{+}$ is $(p, a)$-adically complete.*

(ii) *For $n \in \mathbb{N}$, the rings $\mathbf{A}_{R,\varpi}^{+}/a^n\mathbf{A}_{R,\varpi}^{+}$ are $p$-torsion free and $p$-adically complete.*

(iii) *For $n \in \mathbb{N}$, $\mathbf{A}_{R,\varpi}^{+}$ and $\mathbf{A}_{R,\varpi}^{+}/p^n\mathbf{A}_{R,\varpi}^{+}$ are $a$-torsion free and $a$-adically complete.*

(iv) *The $(p, a)$-adic topology coincides with $(p, \pi_m)$-adic topology.*

*Proof.* — As $\mathbf{A}_{R,\varpi}^+$ is a flat $\mathbb{Z}_p$-algebra, claims (i), (ii) and (iii) follow from [36, Lemma 2]. The last claim follows from [36, Lemma 1] and the fact that $\mathbf{A}_{R,\varpi}^+ \subset \mathbf{A}_{\mathrm{inf}}(\overline{R})$, where the former ring is equipped with the induced topology. □

For $n \in \mathbb{N}$, let us write $n = (p-1)f(n) + r(n)$, with $r(n), f(n) \in \mathbb{N}$ and $0 \leqslant r(n) < p-1$. Let $t^{\{n\}} := \frac{t^n}{p^{f(n)} f(n)!}$.

Lemma 5.7. — *We have $t^{p-1} \in p\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, therefore $t^{\{n\}} \in \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$.*

*Proof.* — Note that we have $q = \frac{\varphi(\pi)}{\pi} = p\varphi\big(\frac{\pi}{t}\big)\frac{t}{\pi}$. Since $\frac{t}{\pi}$ is a unit in $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ (see Lemma 3.14), we get that $q$ and $p$ are associates in $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. But also, $q = \frac{\varphi(\pi)}{\pi} = \pi^{p-1} + p(\pi^{p-2} + \cdots + 1)$, i.e. $\pi^{p-1} \in p\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. Again, using Lemma 3.14, we get that $t^{p-1} \in p\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. □

Note that we also have $\pi = \exp(t) - 1 = \sum_{n \geqslant 1} \frac{t^n}{n!} = \sum_{n \geqslant 1} c_n t^{\{n\}}$, where $c_n = \frac{p^{f(n)} f(n)!}{n!}$ such that $c_n \to 0$ as $n \to +\infty$ (see [22, Section 5.2.4]). Let

$$\Lambda := \left\{ \sum_{n \in \mathbb{N}} a_n t^{\{n\}} \text{ with } a_n \in O_F \text{ such that } a_n \longrightarrow 0 \text{ as } n \longrightarrow +\infty \right\}$$

be a ring and let $z = \sum_{a \in \mathbb{F}_p} [\varepsilon]^{[a]}$ and $\pi_0 = z - p$. then we have $\pi_0 = (p-1)\sum_{n \geqslant 1,\, (p-1)|n} \frac{t^n}{n!} \in \Lambda$. Further, we have that $\pi_0 \in p\Lambda$ and there exists $v \in \Lambda^\times$ such that $\frac{\pi_0}{p} = v\frac{t^{p-1}}{p}$, see [22, Section 5.2.5].

Next, recall that the filtration on $\mathbf{A}_{\mathrm{cris}}(\overline{R})$ is given as $\mathrm{Fil}^k \mathbf{A}_{\mathrm{cris}}(\overline{R}) = \langle \xi^{[n]},\, n \geqslant k \rangle \subset \mathbf{A}_{\mathrm{cris}}(\overline{R})$, for $k \in \mathbb{N}$ (see Section 2.2). The filtration on $\mathbf{A}_{\mathrm{inf}}(\overline{R})$ is defined as the induced filtration, i.e. $\mathrm{Fil}^k \mathbf{A}_{\mathrm{inf}}(\overline{R}) = \mathrm{Fil}^k \mathbf{A}_{\mathrm{cris}}(\overline{R}) \cap \mathbf{A}_{\mathrm{inf}}(\overline{R}) = \xi^k \mathbf{A}_{\mathrm{inf}}(\overline{R})$. Similarly, the filtration on $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is again given by divided powers of $\xi$, i.e. $\mathrm{Fil}^k \mathbf{A}_{R,\varpi}^{\mathrm{PD}} = \langle \xi^{[n]},\, n \geqslant k \rangle \subset \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, for $k \in \mathbb{N}$ (see Definition 3.11). The filtration on $\mathbf{A}_{R,\varpi}^+$ is defined as the induced filtration, i.e. $\mathrm{Fil}^k \mathbf{A}_{R,\varpi}^+ = \mathrm{Fil}^k \mathbf{A}_{R,\varpi}^{\mathrm{PD}} \cap \mathbf{A}_{R,\varpi}^+ = \xi^k \mathbf{A}_{\mathrm{inf}}(\overline{R})$.

Now, for $k \in \mathbb{N}$ let us define an ideal of $\mathbf{A}_{\mathrm{inf}}(\overline{R})$ as

$$I^{(k)}\mathbf{A}_{\mathrm{inf}}(\overline{R}) = \{x \in \mathbf{A}_{\mathrm{inf}}(\overline{R}) \text{ such that } \varphi^n(x) \in \mathrm{Fil}^k \mathbf{A}_{\mathrm{inf}}(\overline{R}) \text{ for } n \in \mathbb{N}\}.$$

Similarly, we define respective ideals $I^{(k)}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \subset \mathbf{A}_{\mathrm{cris}}(\overline{R})$, $I^{(k)}\mathbf{A}_{R,\varpi}^+ \subset \mathbf{A}_{R,\varpi}^+$ and $I^{(k)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. We have

$$\mathbf{A}_{R,\varpi}^+ = \mathbf{A}_{\mathrm{inf}}(\overline{R}) \cap \mathbf{A}_{R,\varpi} \subset W(\mathbb{C}(\overline{R})^\flat),$$

so we obtain that

Lemma 5.8.

(i) *The ideal* $I^{(k)}\mathbf{A}_{R,\varpi}^{+}$ *is generated by* $\pi^k$.

(ii) *The element* $\pi_0$ *is a generator of* $I^{(p-1)}\mathbf{A}_{R,\varpi}^{+}$.

(iii) *Let* $S_0 = W[\![\pi_0]\!]$ *then there exists a unit* $u \in S_0$ *such that* $\varphi(\pi_0) = u\pi_0 z^{p-1}$.

*Proof.* — From the definitions it is clear that $I^{(k)}\mathbf{A}_{\mathrm{inf}}(\overline{R}) \cap \mathbf{A}_{R,\varpi}^{+} = I^{(k)}\mathbf{A}_{R,\varpi}^{+}$, where we take the intersection inside $\mathbf{A}_{\mathrm{inf}}(\overline{R})$. Now, from [22, Section 5.1.3, Proposition] we have that $I^{(k)}\mathbf{A}_{\mathrm{inf}}(\overline{R}) = \pi^k\mathbf{A}_{\mathrm{inf}}(\overline{R})$. Take $x \in I^{(k)}\mathbf{A}_{R,\varpi}^{+} \subset I^{(k)}\mathbf{A}_{\mathrm{inf}}(\overline{R})$ and write $x = \pi^k y$ for some $y \in \mathbf{A}_{\mathrm{inf}}(\overline{R})$. But then we have $x = \pi^k y \in \mathbf{A}_{R,\varpi}$, i.e. $y \in \mathbf{A}_{\mathrm{inf}}(\overline{R}) \cap \mathbf{A}_{R,\varpi} = \mathbf{A}_{R,\varpi}^{+}$. So we obtain that $I^{(k)}\mathbf{A}_{R,\varpi}^{+} = \pi^k\mathbf{A}_{R,\varpi}^{+}$. This shows (i). For (ii), note that $\pi_0 \in \mathbf{A}_{R,\varpi}^{+}$, and $\mathbf{A}_{R,\varpi}^{+} = \mathbf{A}_{\mathrm{inf}}(\overline{R}) \cap \mathbf{A}_{R,\varpi}$. So arguing as above we get $\pi_0\mathbf{A}_{\mathrm{inf}}(\overline{R}) \cap \mathbf{A}_{R,\varpi}^{+} = \pi_0\mathbf{A}_{R,\varpi}^{+}$. Now, from [22, Section 5.2.6, Proposition (i)] we have that $I^{(p-1)}\mathbf{A}_{\mathrm{inf}}(\overline{R}) = \pi_0\mathbf{A}_{\mathrm{inf}}(\overline{R})$. So we obtain that $I^{(p-1)}\mathbf{A}_{R,\varpi}^{+} = I^{(p-1)}\mathbf{A}_{\mathrm{inf}}(\overline{R}) \cap \mathbf{A}_{R,\varpi}^{+} = \pi_0\mathbf{A}_{\mathrm{inf}}(\overline{R}) \cap \mathbf{A}_{R,\varpi}^{+} = \pi_0\mathbf{A}_{R,\varpi}^{+}$. Claim in (iii) follows from [22, Section 5.2.6, Proposition (ii)]. $\square$

Proposition 5.9. — *The continuous morphism of* $\mathbf{A}_{R,\varpi}^{+}$*-algebras*

$$\alpha : \mathbf{A}_{R,\varpi}^{+} \widehat{\otimes}_{S_0} \Lambda \longrightarrow \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$$
$$\sum_{n\in\mathbb{N}} a_n \otimes \left(\frac{\pi_0}{p}\right)^{[n]} \longmapsto \sum_{n\in\mathbb{N}} a_n \left(\frac{\pi_0}{p}\right)^{[n]},$$

*is an isomorphism.*

*Proof.* — The proof follows in a manner similar to the proof of [22, Section 5.2.7, Théorème]. The homomorphism $\alpha$ in the claim is well defined and continuous since $\frac{\pi_0}{p} \in \mathrm{Fil}^1\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. So we are left to show that $\alpha$ is an isomorphism. Since the source and targets are $p$-adically complete $p$-torsion-free rings, it is enough to show that $\alpha$ is an isomorphism modulo $p$.

Let $z_1 = \varphi^{-1}(z) \in \mathbf{A}_{R,\varpi}^{+}$. Note that $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ modulo $p$ is the divided power envelope of $\mathbf{E}_{R,\varpi}^{+}$ with respect to the ideal generated by $\overline{z_1} \equiv \bar{\xi}$ mod $p$. Therefore, it is a free module over $\mathbf{E}_{R,\varpi}^{+}/\overline{z_1}^p$ with basis the images of $z_1^{[pn]}$, or equivalently $\left(\frac{z_1^p}{p}\right)^{[n]}$. From Lemma 5.8(iii), we have that $\varphi(\pi_0) = u\pi_0 z^{p-1}$, with $u \in S_0^{\times}$. Therefore, $\pi_0 = \varphi^{-1}(u)\varphi^{-1}(\pi_0)z_1^{p-1} = \varphi^{-1}(u)(z_1 - p)z_1^{p-1}$, which implies that $\mathbf{E}_{R,\varpi}^{+}/\overline{z_1}^p = \mathbf{E}_{R,\varpi}^{+}/\overline{\pi}_0$ and $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ modulo $p$ is a free module over $\mathbf{E}_{R,\varpi}^{+}/\overline{\pi}_0$ with basis the images of $\left(\frac{\pi_0}{p}\right)^{[n]}$. Since it is immediate that the same is true for $\mathbf{A}_{R,\varpi}^{+} \otimes_{S_0} \Lambda$ modulo $p$, we get the claim. $\square$

Lemma 5.10. — *For $k \in \mathbb{N}$ the ideal $I^{(k)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is a divided power ideal which is the associated $\mathbf{A}_{R,\varpi}^{+}$-submodule of $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ generated by $t^{\{n\}}$ for $n \geqslant k$.*

*Proof.* — The proof follows in a manner similar to the proof of [22, Section 5.3.5, Proposition]. Let $J^{(k)}$ be the $\mathbf{A}_{R,\varpi}^{+}$-submodule of $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ generated by $t^{\{n\}}$ for $n \geqslant k$. It is straightforward to check that $J^{(k)} \subset I^{(k)}$, and $J^{(k)}$ is a divided power ideal. Thus it remains to show that $I^{(k)} \subset J^{(k)}$. We will show this by induction on $k$. The case $k = 0$ is trivial.

Now suppose $k \geqslant 1$ and $x \in I^{(k)}$. The induction hypothesis allows us to write $x = \sum_{n \geqslant k-1} a_n t^{\{n\}}$ where $a_n \in \mathbf{A}_{R,\varpi}^{+}$ goes to 0 as $n \to +\infty$. If $b = a_{n-1}$, we have $a = bt^{\{n-1\}} + a'$ where $a' \in J^{(k)} \subset I^{(k)}$, thus $bt^{\{k-1\}} \in I^{(k)}$. But $\varphi^s(bt^{\{k-1\}}) = p^{(k-1)s}\varphi^s(b)t^{\{k-1\}} = c_{k,s}\varphi^s(b)t^{\{k-1\}}$, where $c_{k,s}$ is a nonzero rational number. Since $t^{k-1} \in \mathrm{Fil}^{k-1}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \setminus \mathrm{Fil}^{k}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, one has $b \in I^{(1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \cap \mathbf{A}_{R,\varpi}^{+}$, which is the principal ideal generated by $\pi$. Thus $bt^{\{k-1\}}$ belongs to an ideal of $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ generated by $\pi t^{\{k-1\}}$. But $\frac{t}{\pi} \in \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is a unit (see Lemma 3.14). Hence, $bt^{\{k-1\}}$ belongs to an ideal generated by $t \cdot t^{\{k-1\}}$, which is contained in $J^{(k)}$. □

Following is an immediate consequence of Lemma 5.10:

Corollary 5.11. — *For $k \in \mathbb{N}$, consider the homomorphism $\mathbf{A}_{R,\varpi}^{+} \to I^{(k)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ sending $x \mapsto x \cdot t^{\{k\}}$. Then, the induced map $\mathbf{A}_{R,\varpi}^{+}/I^{(1)}\mathbf{A}_{R,\varpi}^{+} \to I^{(k)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/I^{(k+1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is bijective.*

Now, from [22, Section 5.3.5, Proposition], we have a natural isomorphism $\mathbf{A}_{\mathrm{inf}}(\overline{R})/I^{(k)}\mathbf{A}_{\mathrm{inf}}(\overline{R}) \xrightarrow{\sim} \mathbf{A}_{\mathrm{cris}}(\overline{R})/I^{(k)}\mathbf{A}_{\mathrm{cris}}(\overline{R})$, for $0 \leqslant k \leqslant p-1$. A similar statement is true in our setting:

Proposition 5.12. — *For $k \in \mathbb{N}$, $\mathbf{A}_{R,\varpi}^{+}/I^{(k)}\mathbf{A}_{R,\varpi}^{+}$ and $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/I^{(k)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ are $p$-torsion free. Moreover, if $0 \leqslant k \leqslant p-1$, then the natural map $\mathbf{A}_{R,\varpi}^{+}/I^{(k)}\mathbf{A}_{R,\varpi}^{+} \to \mathbf{A}_{R,\varpi}^{\mathrm{PD}}/I^{(k)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is an isomorphism.*

*Proof.* — The proof follows from arguments similar to the proof of [22, Section 5.3.5, Proposition]. First, note that for every $k \in \mathbb{N}$, $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/\mathrm{Fil}^{k}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is torsion free. Further, the kernel of the map

$$\begin{aligned} \mathbf{A}_{R,\varpi}^{\mathrm{PD}} &\longrightarrow (\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/\mathrm{Fil}^{k}\mathbf{A}_{R,\varpi}^{\mathrm{PD}})^{\mathbb{N}} \\ x &\longmapsto (\varphi^n(x) \mod \mathrm{Fil}^{k}\mathbf{A}_{R,\varpi}^{\mathrm{PD}})_{k\in\mathbb{N}}, \end{aligned}$$

is $I^{(k)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. Therefore, we get $\mathbf{A}_{R,\varpi}^{+}/I^{(k)}\mathbf{A}_{R,\varpi}^{+} \rightarrowtail \mathbf{A}_{R,\varpi}^{\mathrm{PD}}/I^{(k)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \rightarrowtail (\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/\mathrm{Fil}^{k}\mathbf{A}_{R,\varpi}^{\mathrm{PD}})^{\mathbb{N}}$, which implies that the former two rings are torsion free.

From Proposition 5.9 and Lemma 5.10, it follows that as $\mathbf{A}_{R,\varpi}^{+}$-module, $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/I^{(k)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is generated by the images of $\left(\frac{\pi_0}{p}\right)^{[n]}$ for $0 \leqslant (p-1)n < k$. For $0 \leqslant k \leqslant p-1$, we have that $\left(\frac{\pi_0}{p}\right)^{[n]} \in \mathbf{A}_{R,\varpi}^{+}$, hence we get the claim. □

Next, we mention a lemma useful for the proof of Proposition 5.20.

Lemma 5.13.

(i) *For $0 \leqslant k < j$, we have that $I^{(k)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/I^{(j)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is $p$-torsion free.*
(ii) *For $k \in \mathbb{N}$, we have that $I^{(k)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is $p$-adically complete.*

*Proof.* — The proof (i) is similar to the proof of [35, Lemma A3.19(1)]. Let $x \in I^{(k)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and assume that $px \in I^{(j)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. Then $p\varphi^i(x) \in \mathrm{Fil}^j\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ for all $i \in \mathbb{N}$. Since $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/\mathrm{Fil}^j\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathbf{A}_{\mathrm{cris}}(\overline{R})/\mathrm{Fil}^j\mathbf{A}_{\mathrm{cris}}(\overline{R})$ is $p$-torsion free (see [35, Lemma A2.11(2)]), we get that $\varphi^i(x) \in \mathrm{Fil}^j\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ for all $i \in \mathbb{N}$, i.e. $x \in I^{(j)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$.

The proof of (ii) is similar to the proof of [35, Lemma A3.27]. We will prove the statement by induction on $k$. For $k = 0$, the statement is trivial by the definition of $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. Next, from part (i) and Corollary 5.11, we have that $I^{(k)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/I^{(k+1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is $p$-torsion free and $p$-adically complete. Therefore, we obtain exact sequences

$$0 \longrightarrow \lim_n \left(I^{(k+1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes \mathbb{Z}/p^n\mathbb{Z}\right) \longrightarrow \lim_n \left(I^{(k)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes \mathbb{Z}/p^n\mathbb{Z}\right)$$
$$\longrightarrow I^{(k)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/I^{(k+1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \longrightarrow 0.$$

The statement now follows by induction on $k$. □

## 5.2. Equivalence of categories

In [36], Tsuji has established a relationship between free relative Fontaine–Laffaille modules (see Definition 5.1) and $\mathbf{A}_{\mathrm{inf}}(\overline{R})$-representations as well as $\mathbf{A}_{\mathrm{cris}}(\overline{R})$-representations of $G_R$ (in a precise functorial manner). Tsuji's computations are motivated by computations of Wach in [38] for the arithmetic case.

Recall from Section 5.1 that for $k \in \mathbb{N}$ we have the ideal

$$I^{(k)}\mathbf{A}_{\mathrm{inf}}(\overline{R}) = \{x \in \mathbf{A}_{\mathrm{inf}}(\overline{R}) \text{ such that } \varphi^n(x) \in \mathrm{Fil}^k\mathbf{A}_{\mathrm{inf}}(\overline{R}) \text{ for } n \in \mathbb{N}\}.$$

Similarly, we define respective ideals $I^{(k)}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \subset \mathbf{A}_{\mathrm{cris}}(\overline{R})$, $I^{(k)}\mathbf{A}_{R,\varpi}^{+} \subset \mathbf{A}_{R,\varpi}^{+}$ and $I^{(k)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. Given a free Fontaine–Laffaille module, in [36, Section 5] Tsuji functorially obtains an $\mathbf{A}_{\mathrm{cris}}(\overline{R})$-module (in a manner similar to Proposition 5.25). Furthermore, he exploits the natural isomorphism $\mathbf{A}_{\mathrm{inf}}(\overline{R})/I^{(p-1)}\mathbf{A}_{\mathrm{inf}}(\overline{R}) \xrightarrow{\sim} \mathbf{A}_{\mathrm{cris}}(\overline{R})/I^{(p-1)}\mathbf{A}_{\mathrm{cris}}(\overline{R})$, to construct an $\mathbf{A}_{\mathrm{inf}}(\overline{R})$-representation of $G_R$. The last step is carried out by establishing certain equivalence of categories. Tsuji's computations are general and follows from certain assumptions on the structure of the rings and modules, one is studying. In this section, we will recall and verify those assumptions in our case, which would help us in establishing equivalence between several categories (see Theorem 5.21).

Let $A = \mathbf{A}^{+}_{R,\varpi}$, $\mathbf{A}^{+}_{R,\varpi}/I^{(p-1)}\mathbf{A}^{+}_{R,\varpi}$, $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$, or $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/I^{(p-1)}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$.

Lemma 5.14. — *Let $q = \frac{\varphi(\pi)}{\pi} \in A$, then $q$ is a non-zero-divisor in $A$.*

*Proof.* — For $A = \mathbf{A}^{+}_{R,\varpi}$ and $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$, the claim follows from the definitions. Next, note that we have $q = \frac{\varphi(\pi)}{\pi} = \pi^{p-1} + pu \in \mathbf{A}^{+}_{R,\varpi}$ for some unit $u \in \mathbf{A}^{+}_{R,\varpi}$, in particular, $q \equiv pu \mod \pi^{p-1}$. As $I^{(p-1)}\mathbf{A}^{+}_{R,\varpi} = \pi^{p-1}\mathbf{A}^{+}_{R,\varpi}$ by Lemma 5.8(ii), we obtain that $q$ and $p$ are associates in $\mathbf{A}^{+}_{R,\varpi}/I^{(p-1)}\mathbf{A}^{+}_{R,\varpi} \xrightarrow{\sim} \mathbf{A}^{\mathrm{PD}}_{R,\varpi}/I^{(p-1)}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$. Since $\mathbf{A}^{+}_{R,\varpi}/I^{(p-1)}\mathbf{A}^{+}_{R,\varpi}$ and $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/I^{(p-1)}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$ are $p$-torsion free, we conclude by Proposition 5.12. ☐

Next, note that we have $\mathrm{Fil}^0 A = A$ and $\mathrm{Fil}^i A \cdot \mathrm{Fil}^j A \subset \mathrm{Fil}^{i+j} A$ for $i, j \in \mathbb{Z}$, and $\varphi(\mathrm{Fil}^k A) \subset q^k A$ for $k \in \mathbb{N}$. In particular, we see that our choice of $A$ and $q$ satisfies [36, Condition 39].

Definition 5.15. — *Define the category $\mathrm{MF}^{q}_{[0,p-2],\,\mathrm{free}}(A, \varphi, \Gamma_R)$ as follows: An object is a triplet $(N, \mathrm{Fil}^k N, \varphi)$ such that,*

(i) *$N$ is a free $A$-module of rank $h$.*
(ii) *The filtration $\mathrm{Fil}^k N$ is decreasing and there exists an $A$-basis of $N$ as $\{e_1, \ldots, e_h\}$ and integers $k_1, \ldots, k_h \in \mathbb{N}_{\leqslant p-2}$ such that we have $\mathrm{Fil}^k N = \sum_{i=1}^{h} \mathrm{Fil}^{k-k_i} A e_i$ for $0 \leqslant k \leqslant p-2$.*
(iii) *A Frobenius-semilinear endomorphism $\varphi : N \to N$ such that we have $\varphi(\mathrm{Fil}^k N) \subset q^k N$ for $0 \leqslant k \leqslant p-2$ and*

$$\sum_{k=0}^{p-2} A \cdot q^{-k}\varphi(\mathrm{Fil}^k N) = N.$$

(iv) *$N$ is equipped with a continuous action of $\Gamma_R$ such that $\mathrm{Fil}^k N$ is stable under this action, and the endomorphism $\varphi$ commutes with the action of $\Gamma_R$.*

*A morphism between two objects of the category $\mathrm{MF}^{q}_{[0,p-2],\,\mathrm{free}}(A, \varphi, \Gamma_R)$ is a continuous $A$-linear morphism commuting with the endomorphism $\varphi$ and the action of $\Gamma_R$ on each side.*

*Notation 5.16.* — Abusing notations, we will denote $(N, \mathrm{Fil}^k N, \partial, \Phi) \in \mathrm{MF}^q_{[0,p-2],\,\mathrm{free}}(A, \varphi, \Gamma_R)$ by $N$ and say that it has filtration of level $[0, p-2]$.

*Remark 5.17.* — In $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$, note that we can write $q = p\frac{t}{\pi}\varphi\big(\frac{\pi}{t}\big)$, and since $\frac{t}{\pi}$ is a unit in $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$ (see Lemma 3.14), we obtain that $q$ and $p$ are associates in $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$. Therefore, for $A = \mathbf{A}^{\mathrm{PD}}_{R,\varpi}$ in Definition 5.15, we can replace $q$ by $p$. Further, since $q = \pi^{p-1} + pu$ for $u \in (\mathbf{A}^+_{R,\varpi})^\times$ and $\pi^{p-1}$ generates $I^{(p-1)}\mathbf{A}^+_{R,\varpi}$ (see Lemma 5.8(ii)), we obtain that $q \equiv pu \mod I^{(p-1)}\mathbf{A}^+_{R,\varpi}$, i.e. $q$ and $p$ are associates in $\mathbf{A}^+_{R,\varpi}/I^{(p-1)}\mathbf{A}^+_{R,\varpi} \xrightarrow{\sim} \mathbf{A}^{\mathrm{PD}}_{R,\varpi}/I^{(p-1)}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$. Therefore, for $A = \mathbf{A}^+_{R,\varpi}/I^{(p-1)}\mathbf{A}^+_{R,\varpi}$ in Definition 5.15, we can replace $q$ by $p$, and similarly for $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/I^{(p-1)}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$.

Lemma 5.18 ([36, Lemma 41]). — *K Let $(N, \mathrm{Fil}^k N)$ be as in Definition 5.15(i), (ii). Then a Frobenius-semilinear endomorphism $\varphi : N \to N$ satisfies the conditions in Definition 5.15(iii) if and only if $\varphi(e_i) \in q^{k_i}N$ for $1 \leqslant i \leqslant h$ and $\{q^{-k_1}\varphi(e_1), \ldots, q^{-k_h}\varphi(e_h)\}$ is an $A$-basis of $N$.*

*Proof.* — Let us assume that $(N, \mathrm{Fil}^k N)$ satisfies the condition in Definition 5.15(iii). Then, since $e_i \in \mathrm{Fil}^{k_i} N$, we have $\varphi(e_i) \in q^{k_i}N$ for $1 \leqslant i \leqslant h$. Now for $0 \leqslant k \leqslant p-2$, we have

$$\varphi(\mathrm{Fil}^{k-k_i} Ae_i) = \varphi(\mathrm{Fil}^{k-k_i} A)\varphi(e_i) \subset q^k A \cdot q^{-k_i}\varphi(e_i) \subset q^k N.$$

Therefore, from the identity $\sum_{k=0}^{p-2} A \cdot q^{-k}\varphi(\mathrm{Fil}^k N) = N$, we obtain that $\{q^{-k_1}\varphi(e_1), \ldots, q^{-k_h}\varphi(e_h)\}$ generate $N$ as an $A$-module. Since $N$ is free of rank $h$ over $A$, we get that $\{q^{-k_1}\varphi(e_1), \ldots, q^{-k_h}\varphi(e_h)\}$ is indeed a basis.

Conversely, assume $\varphi(e_i) \in q^{k_i}N$ for $1 \leqslant i \leqslant h$ such that the elements $\{q^{-k_1}\varphi(e_1), \ldots, q^{-k_h}\varphi(e_h)\}$ form an $A$-basis of $N$. Then, from Definition 5.15(ii), we have

$$\begin{aligned}\varphi(\mathrm{Fil}^k N) = \varphi\left(\sum_{i=1}^h \mathrm{Fil}^{k-k_i} Ae_i\right) &\subset \sum_{i=1}^h q^{k-k_i} A\varphi(e_i)\\ &= q^k \sum_{i=1}^h A \cdot q^{-k_i}\varphi(e_i) = q^k N.\end{aligned}$$

Further, since $\{q^{-k_1}\varphi(e_1), \ldots, q^{-k_h}\varphi(e_h)\} \in \sum_{k=0}^{p-2} A \cdot q^{-k}\varphi(\mathrm{Fil}^k N)$, we obtain the last equality in Definition 5.15(iii). $\square$

*Remark 5.19.* — We introduce some necessary conditions in order to adapt Tsuji's results from [36, Section 4-Section 8]. Let $A = \mathbf{A}^+_{R,\varpi}, \mathbf{A}^{\mathrm{PD}}_{R,\varpi}$ and $q = \frac{\varphi(\pi)}{\pi} \in A$. Consider the projection map $A \twoheadrightarrow A/J$ for some ideal $J \subset A$ and assume that

(i) The ideal $J$ is contained in the Jacobson radical of $A$, and $J \subset \mathrm{Fil}^{p-2}A$. Moreover, $\varphi(J) \subset J$ and $\varphi(J) \subset q^{p-1}A$. Further, the ideal $J$ is preserved under the action of $\Gamma_R$.
(ii) The ideal $J$ is closed as a submodule of $A$.
(iii) There exists a decreasing sequence of ideals $\cdots \subset H_{n+1} \subset H_n \subset \cdots \subset H_0 \subset A$ for $n \in \mathbb{N}$, such that $H_n$ form a fundamental system of neighborhoods of 0 in $A$, the homomorphism $A \to \lim_n A/H_n$ is an isomorphism, and $q^{-(p-1)}\varphi(H_n \cap J) \subset H_n \cap J$ for every $n \in \mathbb{N}$.
(iv) The image of $q$ in $A/J$ is a non-zero-divisor. Moreover, the sequence $\prod_{k=0}^{n} \varphi^k(q) \in A$ converges to 0 as $n \to +\infty$.
(v) The homomorphism $\varphi : A \to A$ is continuous and multiplication by $q$ induces a homeomorphism $A \to qA$, where the latter is equipped with the induced topology.

Proposition 5.20.

(i) *Let*
$$A = \mathbf{A}^+_{R,\varpi}$$
*with* $J = I^{(p-1)}\mathbf{A}^+_{R,\varpi}$, *and* $H_n = p^n\mathbf{A}^+_{R,\varpi} + \pi^{n+p-1}\mathbf{A}^+_{R,\varpi}$. *Then* $\mathbf{A}^+_{R,\varpi}$ *satisfies conditions in Remark 5.19.*
(ii) *Let* $A = \mathbf{A}^{\mathrm{PD}}_{R,\varpi}$ *with* $J = I^{(p-1)}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$, *and* $H_n = p^n\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$. *Then* $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$ *satisfies conditions in Remark 5.19.*

*Proof.* — The proof follows in a manner similar to the proof of [36, Proposition 59]. For (i), note that the ring $\mathbf{A}^+_{R,\varpi}$ is $\pi$-adically complete, and since $I^{(p-1)}\mathbf{A}^+_{R,\varpi} = \pi^{p-1}\mathbf{A}^+_{R,\varpi} \subset \pi\mathbf{A}^+_{R,\varpi}$, we see that $I^{(p-1)}\mathbf{A}^+_{R,\varpi}$ is contained in the Jacobson radical of $\mathbf{A}^+_{R,\varpi}$. Moreover, note that we have inclusions $\varphi(\pi^{p-1}\mathbf{A}^+_{R,\varpi}) \subset q^{p-1}\pi^{p-1}\mathbf{A}^+_{R,\varpi} \subset \pi^{p-1}\mathbf{A}^+_{R,\varpi}$, therefore we get $I^{(p-1)}\mathbf{A}^+_{R,\varpi} \subset \mathrm{Fil}^{p-2}\mathbf{A}^+_{R,\varpi}$. It is clear that $I^{(p-1)}\mathbf{A}^+_{R,\varpi}$ is stable under the action of $\Gamma_R$. Therefore, the condition in Remark 5.19(i) is satisfied.

Now we have $H_n = p^n\mathbf{A}^+_{R,\varpi} + \pi^{n+p-1}\mathbf{A}^+_{R,\varpi}$ for $n \in \mathbb{N}$, which is a fundamental system of neighborhoods of $0 \in \mathbf{A}^+_{R,\varpi}$ and $\mathbf{A}^+_{R,\varpi} = \lim_n \mathbf{A}^+_{R,\varpi}/H_n$ (see Lemma 5.6). Further, since $\mathbf{A}^+_{R,\varpi}/I^{(p-1)}\mathbf{A}^+_{R,\varpi}$ is $p$-torsion free, we obtain that $H_n \cap I^{(p-1)}\mathbf{A}^+_{R,\varpi} = (p^n\mathbf{A}^+_{R,\varpi} + \pi^{n+p-1}\mathbf{A}^+_{R,\varpi}) \cap I^{(p-1)}\mathbf{A}^+_{R,\varpi} = p^nI^{(p-1)}\mathbf{A}^+_{R,\varpi} + \pi^nI^{(p-1)}\mathbf{A}^+_{R,\varpi}$. The condition in Remark 5.19(iii) now follows from this. Moreover, $I^{(p-1)}\mathbf{A}^+_{R,\varpi}$ is a free $\mathbf{A}^+_{R,\varpi}$-module of rank 1, so it follows that $J$ is a closed submodule of $\mathbf{A}^+_{R,\varpi}$ by Lemma 5.6(i) & (iv), verifying the condition in Remark 5.19(ii). Next, from Lemma 5.14 we have that $q \in \mathbf{A}^+_{R,\varpi}/I^{(p-1)}\mathbf{A}^+_{R,\varpi}$ is a non-zero-divisor. Moreover, we have $\varphi^k(q) = \varphi^{k+1}(\xi) \in \varphi^{k+1}(\mathrm{Fil}^1\mathbf{A}^+_{R,\varpi}) \subset \varphi^{k+1}(p\mathbf{A}^+_{R,\varpi} + \pi_1\mathbf{A}^+_{R,\varpi}) \subset$

$p\mathbf{A}_{R,\varpi}^{+} + \pi\mathbf{A}_{R,\varpi}^{+}$, for $k \in \mathbb{N}$. Therefore, $\prod_{k=0}^{n} \varphi^k(q)$ converges to 0 as $n \to +\infty$, and the condition in Remark 5.19(iv) has been verified. By the definition of $\varphi$ in Section 3.3, we see that it is continuous. Further, from Lemma 5.6(iii), it follows that $\mathbf{A}_{R,\varpi}^{+}/q\mathbf{A}_{R,\varpi}^{+}$ is $p$-torsion free. Therefore, we have $(p^n\mathbf{A}_{R,\varpi}^{+} + q^{n+1}\mathbf{A}_{R,\varpi}^{+}) \cap q\mathbf{A}_{R,\varpi}^{+} = p^n(q\mathbf{A}_{R,\varpi}^{+}) + q^n(q\mathbf{A}_{R,\varpi}^{+})$. By Lemma 5.6(i), it follows that $\mathbf{A}_{R,\varpi}^{+} \xrightarrow{\times q} q\mathbf{A}_{R,\varpi}^{+}$ is a homeomorphism, verifying the condition in Remark 5.19(v). This shows the claim in (i).

For the claim in (ii), note that we have $q = p\varphi\big(\frac{\pi}{t}\big)\frac{t}{\pi}$, which implies that $q$ and $p$ are associates in $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ (see Lemma 3.14). Therefore, it is enough to verfiy the conditions in Remark 5.19, with $q$ replaced by $p$ everywhere.

We have $I^{(p-1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathrm{Fil}^1\mathbf{A}_{R,\varpi}^{\mathrm{PD}} + p\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is $p$-adically complete and $\mathrm{Fil}^1\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/p\mathrm{Fil}^1\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is a nil ideal of $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/p\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. Therefore, $I^{(p-1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is contained in the Jacobson radical of $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. Moreover, we have $I^{(p-1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathrm{Fil}^{p-2}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and $\varphi\big(I^{(p-1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\big) \subset I^{(p-1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. Also, $\varphi\big(I^{(p-1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\big) \subset q^{p-1}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} = p^{p-1}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. It is clear that $I^{(p-1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is stable under the action of $\Gamma_R$. Therefore, the condition in Remark 5.19(i) is satisfied. Next, we know that $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is $p$-adically complete and the ring $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/I^{(p-1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is $p$-torsion free by Proposition 5.12, therefore $p^n\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \cap I^{(p-1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} = p^n I^{(p-1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. This gives us the condition in Remark 5.19(iii). Further, $I^{(p-1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is $p$-adically complete by Lemma 5.13(ii), so we get the condition in Remark 5.19(ii). Conditions in Remark 5.19(iv) & (v) follow trivially from the fact that $\mathbf{A}_{R,\varpi}^{\mathrm{PD}} = \lim_n \mathbf{A}_{R,\varpi}^{\mathrm{PD}}/p^n\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. □

Finally, we come to the main result of this section. Note that categories $\mathrm{MF}^p$ below are defined by combining Definition 5.15 and Remark 5.17.

Theorem 5.21. — *The natural maps $\mathbf{A}_{R,\varpi}^{+} \twoheadrightarrow \mathbf{A}_{R,\varpi}^{+}/I^{(p-1)}\mathbf{A}_{R,\varpi}^{+} \xrightarrow{\sim} \mathbf{A}_{R,\varpi}^{\mathrm{PD}}/I^{(p-1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \twoheadleftarrow \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, induce equivalence of categories:*

$$\begin{aligned}
\mathrm{MF}^p_{[0,p-2],\,\mathrm{free}}(\mathbf{A}_{R,\varpi}^{\mathrm{PD}}, \varphi, \Gamma_R) \xrightarrow[\simeq]{(1)} & \ \mathrm{MF}^p_{[0,p-2],\,\mathrm{free}}(\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/I^{(p-1)}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}, \varphi, \Gamma_R)\\
\xleftarrow[\simeq]{(2)} & \ \mathrm{MF}^p_{[0,p-2],\,\mathrm{free}}(\mathbf{A}_{R,\varpi}^{+}/I^{(p-1)}\mathbf{A}_{R,\varpi}^{+}, \varphi, \Gamma_R)\\
\xleftarrow[\simeq]{(3)} & \ \mathrm{MF}^q_{[0,p-2],\,\mathrm{free}}(\mathbf{A}_{R,\varpi}^{+}, \varphi, \Gamma_R).
\end{aligned}$$

*Proof.* — The natural projection map $\mathbf{A}_{R,\varpi}^{+} \twoheadrightarrow \mathbf{A}_{R,\varpi}^{+}/I^{(p-1)}\mathbf{A}_{R,\varpi}^{+}$ is compatible with Frobenius and the action of $\Gamma_R$ and we have $q \equiv pu \mod I^{(p-1)}\mathbf{A}_{R,\varpi}^{+}$ for $u \in (\mathbf{A}_{R,\varpi}^{+})^{\times}$ (see also Remark 5.17), i.e. $q$ and $p$ are associates in $\mathbf{A}_{R,\varpi}^{+}/I^{(p-1)}\mathbf{A}_{R,\varpi}^{+}$. Further, $\mathbf{A}_{R,\varpi}^{+}$ satisfies the conditions in Remark 5.19. Therefore, from [36, Proposition 56], we obtain that the functor in (3) is an equivalence of categories.

From Proposition 5.12, we have an isomorphism $\mathbf{A}^+_{R,\varpi}/I^{(p-1)}\mathbf{A}^+_{R,\varpi} \xrightarrow{\sim} \mathbf{A}^{\mathrm{PD}}_{R,\varpi}/I^{(p-1)}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$, compatible with Frobenius and the action of $\Gamma_R$. Therefore, we obtain that the functor in (2) is an equivalence of catgeories.

Finally, the natural projection map $\mathbf{A}^{\mathrm{PD}}_{R,\varpi} \twoheadrightarrow \mathbf{A}^{\mathrm{PD}}_{R,\varpi}/I^{(p-1)}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$ is compatible with Frobenius and the action of $\Gamma_R$ and $q \equiv pu \mod I^{(p-1)}\mathbf{A}^+_{R,\varpi}$ for some $u \in (\mathbf{A}^+_{R,\varpi})^\times$ (see also Remark 5.17), i.e. $q$ and $p$ are associates in $\mathbf{A}^+_{R,\varpi}/I^{(p-1)}\mathbf{A}^+_{R,\varpi} \xrightarrow{\sim} \mathbf{A}^{\mathrm{PD}}_{R,\varpi}/I^{(p-1)}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$. Further, $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$ satisfies the conditions in Remark 5.19. Therefore, from [36, Proposition 56], we obtain that the functor in (1) is an equivalence of categories. □

## 5.3. Wach modules from Fontaine–Laffaille data

In this section, we will work with objects of $\mathrm{MF}_{[0,p-2],\,\mathrm{free}}(R,\Phi,\partial)$ (see Definition 5.1) and using these objects we will construct Wach modules over $\mathbf{A}^+_R$ (see Definition 4.8). In Section 5.3.1, starting with a Fontaine–Laffaille module, we will first obtain a free module over $\mathbf{A}^+_{R,\varpi}$ with desired properties and in Section 5.3.2 we will descend over to $\mathbf{A}^+_R$. Note that in Section 5.3.1, we will first establish a mod $p^n$ statement (see (5.2)) and as a consequence deduce a $p$-adic statement (see Proposition 5.25). However, it is possible to prove the $p$-adic statement directly (see proof of Proposition 5.25). Readers interested only in the $p$-adic statement may directly skip to Proposition 5.25.

### 5.3.1. From Fontaine–Laffaille modules to $\mathbf{A}^+_{R,\varpi}$-modules

Following [36, Section 4], set $X_n = \mathrm{Spec}\,(R/p^n)$ and $\Sigma_n = \mathrm{Spec}\,(O_F/p^n)$ for $n \in \mathbb{N}_{>0}$ and consider big crystalline sites $\mathrm{CRIS}(X_n,\Sigma_n)$ and $\mathrm{CRIS}(X_1,\Sigma_n)$ and respective toposes $(X_n/\Sigma_n)_{\mathrm{CRIS}}$ and $(X_1/\Sigma_n)_{\mathrm{CRIS}}$, with the PD-ideal $(p(O_F/p^n),[\ ])$. Let $F_{\Sigma_n} : \Sigma_n \to \Sigma_n$ denote a lifting of the absolute Frobenius of $\Sigma_1$, such that it is a PD-morphism with repsect to the PD-structure. The absolute Frobenius $F_{X_1}$ of $X_1$ and $F_{\Sigma_n}$ define a morphism of PD-ringed topos $F_{X_1/\Sigma_n,\mathrm{CRIS}} : (X_1/\Sigma_n)_{\mathrm{CRIS}} \to (X_1/\Sigma_n)_{\mathrm{CRIS}}$.

Let $(M, \mathrm{Fil}^\bullet M, \partial, \Phi) \in \mathrm{MF}_{[0,p-2],\,\mathrm{free}}(R,\Phi,\partial)$ be a free relative Fontaine–Laffaille module (see Definition 5.1), and let $(M_n, \mathrm{Fil}^\bullet M_n, \partial, \Phi)$ denote its modulo $p^n$ reduction. Then, by [36, Definition 26, Theorems 17 & 29] this data corresponds to a quasi-coherent filtered crystal $(\mathcal{F}_n, \mathrm{Fil}^\bullet\mathcal{F}_n)$ on $\mathrm{CRIS}(X_n/\Sigma_n)$. Similarly, by [36, Definition 26, Theorems 22 & 29] this data also corresponds to a quasi-coherent crystal $\mathcal{G}_n$ on $\mathrm{CRIS}(X_1/\Sigma_n)$.

The reduction modulo $p^n$ of $\Phi : \varphi^*M \to M$ equip $\mathcal{G}_n$ with a morphism $\Phi_{\mathcal{G}_n} : F^*_{X_1/\Sigma_n,\mathrm{CRIS}}(\mathcal{G}_n) \to \mathcal{G}_n$. Further, for the morphism of ringed topos $i_{n,\mathrm{CRIS}} : (X_1/\Sigma_n)_{\mathrm{CRIS}} \to (X_n/\Sigma_n)_{\mathrm{CRIS}}$ induced by the closed immersion $i_n : X_1 \to X_n$ over $\mathrm{id}_{\Sigma_n}$, we have $i^*_{n,\mathrm{CRIS}}(\mathcal{F}_n) = \mathcal{G}_n$ (see [36, Propositions 25 & 32]). Moreover, we have similar statements for the morphism of ringed topos induced by $X_n \to X_{n+1}$ and $\Sigma_n \to \Sigma_{n+1}$.

Now, for $n \in \mathbb{N}_{>0}$ let $X'_n \coloneqq \mathrm{Spec}\,(R[\varpi]/p^n)$, $D_n \coloneqq \mathrm{Spec}\,(\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n)$ and $F_{D_n} : D_n \to D_n$ be the lifting of the absolute Frobenius on $D_1$ defined by $\varphi$ of $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$. We have the surjective map $\theta : \mathbf{A}^+_{R,\varpi} \twoheadrightarrow R[\varpi]$. So taking mod $p^n$ reduction, we obtain an embedding $X'_n \rightarrowtail \mathrm{Spec}\,(\mathbf{A}^+_{R,\varpi}/p^n)$ and taking divided power envelope, we obtain a closed immersion $X'_n \rightarrowtail D_n$ (resp. $X'_1 \rightarrowtail D_n$) which can naturally be regarded as an object of the site $\mathrm{CRIS}(X_n/\Sigma_n)$ (resp. $\mathrm{CRIS}(X_1/\Sigma_n)$), endowed with a right action of $\Gamma_R$.

Definition 5.22. — *Define an* $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$*-module by setting* $N^{\mathrm{PD}}_n \coloneqq \Gamma(X'_n \rightarrowtail D_n, \mathcal{F}_n) \xrightarrow{\sim} \Gamma(X'_1 \rightarrowtail D_n, \mathcal{G}_n)$.

The right action of $\Gamma_R$ on $D_n$ induces a left action on $N^{\mathrm{PD}}_n$. The filtration on $\mathcal{F}_n$ induces a filtration by $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$-submodules on $N^{\mathrm{PD}}_n$, which is stable under the $\Gamma_R$-action. Then $N^{\mathrm{PD}}_n$ is a finite free filtered $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$-module of level $[0, p-2]$ (see [36, Lemma 20]). The Frobenius $\Phi_{\mathcal{G}_n}$ of $\mathcal{G}_n$ and the lifting of Frobenius $F_{D_n}$ on $D_n$ define a semilinear $\Gamma_R$-equivariant endomorphism of $\Gamma(X'_1 \rightarrowtail D_n, \mathcal{G}_n)$ and hence that of $N^{\mathrm{PD}}_n$ as $\Gamma(X'_1 \rightarrowtail D_n, \mathcal{G}_n) \to \Gamma(X'_1 \rightarrowtail D_n, F^*_{X_1,\mathrm{CRIS}}\mathcal{G}_n) \xrightarrow{\Phi_{\mathcal{G}_n}} \Gamma(X'_1 \rightarrowtail D_n, \mathcal{G}_n)$, where the first homomorphism is induced by $F_{X'_1}$ and $F_{D_n}$.

Let $[\,]$ denote the PD-structure on the ideal $p(\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n) + \mathrm{Fil}^1\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$ of $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$. Then we have the big crystalline sites $\mathrm{CRIS}(X'_n/D_n)$ and $\mathrm{CRIS}(X'_1/D_n)$, and the respective topos $(X'_n/D_n)_{\mathrm{CRIS}}$ and $(X'_1/D_n)_{\mathrm{CRIS}}$ with the PD-ideal $(p(\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n) + \mathrm{Fil}^1\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n, [\,])$ of $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$. By taking the pullback of $(\mathcal{F}_n, \mathrm{Fil}^\bullet\mathcal{F}_n)$ (resp. $\mathcal{G}_n$) under the morphism of ringed topos $(X'_n/D_n)_{\mathrm{CRIS}} \to (X_n/\Sigma_n)_{\mathrm{CRIS}}$ (resp. $(X'_1/D_n)_{\mathrm{CRIS}} \to (X_1/\Sigma_n)_{\mathrm{CRIS}}$), we obtain a quasi-coherent filtered crystal $(\mathcal{F}'_n, \mathrm{Fil}^\bullet\mathcal{F}'_n)$ (resp. a quasi-coherent crystal $\mathcal{G}'_n$ with a morphism $\Phi_{\mathcal{G}'_n} : F^*_{X'_1/D_n,\mathrm{CRIS}}(\mathcal{G}'_n) \to \mathcal{G}'_n$), endowed with compatible $\Gamma_R$-action. Since $X'_n \rightarrowtail D_n$ (resp. $X'_1 \rightarrowtail D_n$) is a final object of $\mathrm{CRIS}(X'_n/D_n)$ (resp. $\mathrm{CRIS}(X'_1/D_n)$), we have canonical $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$-linear isomorphisms $N^{\mathrm{PD}}_n \xrightarrow{\sim} \Gamma((X'_n/D_n)_{\mathrm{CRIS}}, \mathcal{F}'_n) \xrightarrow{\sim} \Gamma((X'_1/D_n)_{\mathrm{CRIS}}, \mathcal{G}'_n)$ compatible with supplementary structures (see [36, p. 188-189]).

Next, for $n \in \mathbb{N}_{>0}$, similar to above let $E_n \coloneqq \mathrm{Spec}\,(\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n)$ and $F_{E_n} : E_n \to E_n$ be the lifting of the absolute Frobenius on $E_1$ defined by $\varphi$ of $\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$. We have the surjective map $\theta_R : R \otimes_{\mathbb{Z}} \mathbf{A}^+_{R,\varpi} \twoheadrightarrow R[\varpi]$. So taking mod $p^n$ reduction, we have an embedding

$$X'_n \rightarrowtail \mathrm{Spec}\left(R \otimes_{\mathbb{Z}} \mathbf{A}^+_{R,\varpi}/p^n\right)$$

and taking divided power envelope, we obtain a closed immersion $X'_n \rightarrowtail E_n$ (resp. $X'_1 \rightarrowtail E_n$) which can naturally be regarded as an object of the site $\mathrm{CRIS}(X'_n/D_n)$ (resp. $\mathrm{CRIS}(X'_1/D_n)$), endowed with a right action of $\Gamma_R$.

Definition 5.23. — *Define an $\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$-module by setting $\mathcal{O}N^{\mathrm{PD}}_n := \Gamma(X'_n \rightarrowtail E_n, \mathcal{F}'_n) \xrightarrow{\sim} \Gamma(X'_1 \rightarrowtail E_n, \mathcal{G}'_n)$.*

The right action of $\Gamma_R$ on $E_n$ induces a left action on $\mathcal{O}N^{\mathrm{PD}}_n$. The filtration on $\mathcal{F}'_n$ induces a filtration by $\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$-submodules on $\mathcal{O}N^{\mathrm{PD}}_n$, which is stable under the $\Gamma_R$-action. Then $\mathcal{O}N^{\mathrm{PD}}_n$ is a finite free filtered $\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$-module of level $[0, p-2]$ (see [36, Lemma 20]). Further, by [36, Theorem 29, Proposition 32] $\mathcal{O}N^{\mathrm{PD}}_n$ is equipped with a $\Gamma_R$-equivariant integrable connection compatible with the connection on $\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$ and satisfying Griffiths transversality with the respect to the filtration. Moreover, this the $\Gamma_R$-action and connection are compatible with the respective structures on $\Gamma(X'_1 \rightarrowtail E_n, \mathcal{G}'_n)$ (see [36, Propositions 25 & 32]). The Frobenius $\Phi_{\mathcal{G}'_n}$ of $\mathcal{G}'_n$ and the lifting of Frobenius $F_{E_n}$ on $E_n$ define a semilinear $\Gamma_R$-equivariant endomorphism $\varphi$ of $\Gamma(X'_1 \rightarrowtail E_n, \mathcal{G}'_n)$ and hence that of $\mathcal{O}N^{\mathrm{PD}}_n$. Further, the Frobenius-semilinear endomorphism $\varphi$ commutes with the connection on $\mathcal{O}N^{\mathrm{PD}}_n$.

From [9, Proposition 4.1.4] and [10, Theorem 7.1], we have a description of the global sections of a crystal in terms of horizontal sections of the corresponding module with an integrable connection on the PD-envelope of an embedding into a smooth scheme. In other words, we have an $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$-linear isomorphism $N^{\mathrm{PD}}_n \xrightarrow{\sim} \left(\mathcal{O}N^{\mathrm{PD}}_n\right)^{\partial=0}$, compatible with filtration, Frobenius and the action of $\Gamma_R$ on each side (see [36, p. 190]). Since $X'_n \rightarrowtail D_n$ (resp. $X'_1 \rightarrowtail D_n$) is a final object of $\mathrm{CRIS}(X'_n/D_n)$ (resp. $\mathrm{CRIS}(X'_1/D_n)$), we obtain a canonical $\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$-linear ismorphism

$$\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n \otimes_{\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n} N^{\mathrm{PD}}_n \xrightarrow{\sim} \mathcal{O}N^{\mathrm{PD}}_n,$$

compatible with Frobenius, filtration, connection and the action of $\Gamma_R$ on each side. Here the connection on the tensor product on the left is given as $\partial_{\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}} \otimes 1$. Moreover, from [36, Propositions 24, 25 & 32], we obtain an $\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$-linear ismorphism

$$\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n \otimes_{R/p^n} M/p^n \xrightarrow{\sim} \mathcal{O}N^{\mathrm{PD}}_n,$$

compatible with Frobenius, filtration, connection and the action of $\Gamma_R$ on each side (see [36, p. 191]). Here the connection on the tensor product on the left is given as $\partial_{\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}} \otimes 1 + 1 \otimes \partial_M$. Combining the two isomorphisms above, we obtain an $\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$-linear isomorphism

$$\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n \otimes_{\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n} N^{\mathrm{PD}}_n \xrightarrow{\sim} \mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n \otimes_{R/p^n} M/p^n, \tag{5.2}$$

compatible with Frobenius, filtration, connection and the action of $\Gamma_R$ on each side. Therefore, we also have an $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n$-linear ismorphism

$$N^{\mathrm{PD}}_n \xrightarrow{\sim} \left(\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}/p^n \otimes_{R/p^n} M/p^n\right)^{\partial=0}$$

compatible with Frobenius, filtration and the action of $\Gamma_R$ on each side.

Definition 5.24. — *Define an $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$-module as $N^{\mathrm{PD}}(M) := \lim_n N^{\mathrm{PD}}_n$, equipped with a semilinear and continuous action of $\Gamma_R$, a filtration given as $\mathrm{Fil}^k N^{\mathrm{PD}}(M) := \lim_n \mathrm{Fil}^k N^{\mathrm{PD}}_n$, which is stable under the action of $\Gamma_R$, and a Frobenius-semilinear $\Gamma_R$-equivariant endomorphism $\varphi$.*

Passing to the limit in (5.2) we obtain an $\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$-linear isomorphism

$$\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi} \otimes_{\mathbf{A}^{\mathrm{PD}}_{R,\varpi}} N^{\mathrm{PD}}(M) \xrightarrow{\sim} \mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi} \otimes_R M,$$

compatible with Frobenius, filtration, connection and the action of $\Gamma_R$ on each side. Therefore, we have the following conclusion:

Proposition 5.25. — *Let $M$ be a free relative Fontaine–Laffaille module. Then*

$$N^{\mathrm{PD}}(M) := \left(\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi} \otimes_R M\right)^{\partial=0},$$

*is a finite free $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$-module equipped with a decreasing filtration of level $[0, p-2]$, a Frobenius-semilinear endomorphism $\varphi : N^{\mathrm{PD}}(M) \to N^{\mathrm{PD}}(M)$ and a continuous action of $\Gamma_R$ on each side. In particular, $N^{\mathrm{PD}}(M) \in \mathrm{MF}^p_{[0,p-2],\,\mathrm{free}}(\mathbf{A}^{\mathrm{PD}}_{R,\varpi}, \varphi, \Gamma_R)$. Further, we have a natural isomorphism*

$$\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi} \otimes_{\mathbf{A}^{\mathrm{PD}}_{R,\varpi}} N^{\mathrm{PD}}(M) \xrightarrow{\sim} \mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi} \otimes_R M, \tag{5.3}$$

*compatible with the Frobenius, filtration, connection and the action of $\Gamma_R$ on each side.*

*Another proof of Proposition 5.25.* — Let us consider a Frobenius-equivaraint injective map $O_F\{X_1^{\pm 1}, \ldots, X_d^{\pm 1}\} \to \mathbf{A}^{\mathrm{PD}}_{R,\varpi}$ by sending $X_i \to [X_i^\flat]$ and we extend it uniquely, by étaleness of $O_F\{X_1^{\pm 1}, \ldots, X_d^{\pm 1}\} \to R$ to obtain a Frobenius-equivariant injective map $R \to \mathbf{A}^{\mathrm{PD}}_{R,\varpi}$.

Lemma 5.26. — *We have an $\mathbf{A}^{\mathrm{PD}}_{R,\varpi}$-linear isomorphism $\mathbf{A}^{\mathrm{PD}}_{R,\varpi} \otimes_R M \xrightarrow{\sim} \left(\mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi} \otimes_R M\right)^{\partial=0}$.*

*Proof.* — Let $J = ([X_1^\flat] - X_1, \ldots, [X_d^\flat] - X_d)\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and let $J^{[n]}$ denote its $n$-th divided power for $n \geqslant 1$. We have the projection map,

$$\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \longrightarrow \mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M,$$

via the map $X_i \mapsto [X_i^\flat]$ and the kernel is given as $J^{[1]} \otimes_R M$. Moreover, we have an $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$-linear section of the projection above given as

$$\begin{aligned}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M &\longrightarrow \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M\\ 1 \otimes d &\longmapsto \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^d \partial_i^{k_i}(d) \prod_{i=1}^d ([X_i^\flat] - X_i)^{[k_i]}.\end{aligned} \tag{5.4}$$

Note that the image of the section lies in $(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M)^{\partial=0}$. Now let $Q = J^{[1]} \otimes_R M$ and $Q' = (\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M)/(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M)^{\partial=0}$ and we consider the following diagram with exact rows (the top row is split exact)

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \arrow[r] \arrow[d] & \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \arrow[r] \arrow[d, equal] & Q \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M)^{\partial=0} \arrow[r] & \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \arrow[r] & Q' \arrow[r] & 0. \end{tikzcd}$$

Note that the left vertical arrow is an injection and the right vertical arrow is a surjection. To get that the left vertical arrow is a bijection we need to show that the right vertical arrow is an injection. We have

$$\begin{aligned}(J^{[1]} \otimes_R M)^{\partial=0} &\subset \Big(J\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[\frac{1}{p}]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)\Big)^{\partial=0}\\ &= \big(J\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V\big)^{\partial=0},\end{aligned}$$

where $V = V_{\mathrm{cris}}(M)$ is crystalline (see Proposition 5.3(i)) and the ideal $J\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \subset \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ is generated by $([X_1^\flat] - X_1, \ldots, [X_d^\flat] - X_d)$. Then it easily follows that $\big(J\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V\big)^{\partial=0} = 0$ and we conclude that $\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \xrightarrow{\sim} \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M\big)^{\partial=0}$. □

From the identification $N^{\mathrm{PD}}(M) = \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M\big)^{\partial=0} \xrightarrow{\sim} \mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M$ (where the rightmost term is equipped with a $\Gamma_R$-action as in Remark 5.27), it easily follows that $N^{\mathrm{PD}}(M) \in \mathrm{MF}^p_{[0,p-2],\,\mathrm{free}}(\mathbf{A}_{R,\varpi}^{\mathrm{PD}}, \varphi, \Gamma_R)$. Next, we can $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$-linearly extend the map in (5.4) to obtain

$$\begin{aligned}\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_{R,\varpi}^{\mathrm{PD}}} \big(\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M\big) &\longrightarrow \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M\\ 1 \otimes d &\longmapsto \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^d \partial_i^{k_i}(d) \prod_{i=1}^d ([X_i^\flat] - X_i)^{[k_i]}.\end{aligned} \tag{5.5}$$

We equip the left term with a $\Gamma_R$-action as in Remark 5.27. Choosing a basis of $M$ it is easy to see that the determinant of the map in (5.5) is invertible in $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, i.e. the map (5.5) is bijective. Moreover, it is compatible with Frobenius, filtration, connection and the action of $\Gamma_R$. Now we have a natural injective map

$$\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_{R,\varpi}^{\mathrm{PD}}} N^{\mathrm{PD}}(M) \longrightarrow \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M,$$

compatible with the Frobenius, filtration, connection and the action of $\Gamma_R$ on each side. The map above is bijective because of the following commutative diagram

$$\begin{tikzcd} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_{R,\varpi}^{\mathrm{PD}}} N^{\mathrm{PD}}(M) \arrow[r, tail] \arrow[d, "\wr"] & \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \arrow[d, equal] \\ \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_{R,\varpi}^{\mathrm{PD}}} (\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M) \arrow[r, "\sim"] & \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M, \end{tikzcd}$$

where the bottom horizontal arrow is the isomorphism in (5.5). This concludes the proof. □

*Remark 5.27.* — Using (5.5) and the $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$-linear isomorphism in Lemma 5.26, $\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \xrightarrow{\sim} \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M\big)^{\partial=0}$, we can describe the action of $\Gamma_R$ on the left term explitcitly. The action is given by the formula $g(a \otimes d) = g(a) \otimes \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^d \partial_i^{k_i}(d) \prod_{i=1}^d (g([X_i^\flat]) - [X_i^\flat])^{[k_i]}$, for $g \in \Gamma_R$.

Lemma 5.28. — *Let $N^{\mathrm{PD}}(M)$ as in Proposition 5.25. Then, the action of $\Gamma_{R,\varpi}$ is trivial on $N^{\mathrm{PD}}(M)/\pi N^{\mathrm{PD}}(M)$, whereas $\Gamma_R/\Gamma_{R,\varpi}$ acts trivially over $N^{\mathrm{PD}}(M)/\pi_m N^{\mathrm{PD}}(M)$.*

*Proof.* — This follows from the $\Gamma_R$-equivariant isomorphism in (5.3) (or from Lemma 5.26 and Remark 5.27) and the action of $\Gamma_R$ on $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ (see Lemma 4.24(i)). □

Proposition 5.29. — *The following functor is fully faithful*

$$\begin{aligned} N^{\mathrm{PD}} : \mathrm{MF}_{[0,p-2],\,\mathrm{free}}(R, \Phi, \partial) &\longrightarrow \mathrm{MF}^p_{[0,p-2],\,\mathrm{free}}(\mathbf{A}_{R,\varpi}^{\mathrm{PD}}, \varphi, \Gamma_R) \\ M &\longmapsto N^{\mathrm{PD}}(M) = \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M\big)^{\partial=0}. \end{aligned}$$

*Proof.* — By taking $\Gamma_R$-invariants in (5.3), we obtain an $R$-linear isomorphism $\big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_{R,\varpi}^{\mathrm{PD}}} N^{\mathrm{PD}}(M)\big)^{\Gamma_R} \xrightarrow{\sim} M$ compatible with Frobenius, filtration, connection on each side, and functorial in $M$. □

Having obtained a finite free module with desired structures over the ring $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, we will now pass to the ring $\mathbf{A}_{R,\varpi}^{+}$. Let $M \in \mathrm{MF}_{[0,p-2],\,\mathrm{free}}(R,\Phi,\partial)$ and $N^{\mathrm{PD}}(M) \in \mathrm{MF}_{[0,p-2],\,\mathrm{free}}^{p}(\mathbf{A}_{R,\varpi}^{\mathrm{PD}},\varphi,\Gamma_R)$ the $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$-module obtained under the functor of Proposition 5.29.

Next, from Theorem 5.21 we have that $\mathrm{MF}_{[0,p-2],\,\mathrm{free}}^{p}(\mathbf{A}_{R,\varpi}^{\mathrm{PD}},\varphi,\Gamma_R) \xrightarrow{\sim} \mathrm{MF}_{[0,p-2],\,\mathrm{free}}^{q}(\mathbf{A}_{R,\varpi}^{+},\varphi,\Gamma_R)$ sending $N^{\mathrm{PD}}(M) \mapsto N^{+}(M)$ for $M$ as above. Combining this with Propositions 5.25 & 5.29, we obtain:

Proposition 5.30. — *The following functor is fully faithful*

$$\begin{aligned} N^{+} : \mathrm{MF}_{[0,p-2],\,\mathrm{free}}(R,\Phi,\partial) &\longrightarrow \mathrm{MF}_{[0,p-2],\,\mathrm{free}}^{q}(\mathbf{A}_{R,\varpi}^{+},\varphi,\Gamma_R) \\ M &\longmapsto N^{+}(M). \end{aligned}$$

*For $M$ and $N^{+}(M)$ as above, we have a natural isomorphism*

$$\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_{R,\varpi}^{+}} N^{+}(M) \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M, \tag{5.6}$$

*compatible with the Frobenius, filtration, connection and the action of $\Gamma_R$ on each side.*

Lemma 5.31. — *Let $N^{+}(M)$ as in Proposition 5.30. Then, the action of $\Gamma_{R,\varpi}$ is trivial on $N^{+}(M)/\pi N^{+}(M)$, whereas $\Gamma_R/\Gamma_{R,\varpi}$ acts trivially over $N^{+}(M)/\pi_m N^{+}(M)$.*

*Proof.* — This follows from the $\Gamma_R$-equivariant isomorphism in (5.6) and the action of $\Gamma_R$ on $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ (see Lemma 4.24(i)). ☐

### 5.3.2. Obtaining Wach modules

For the rest of this section we will fix $m = 1$ (fix $m = 2$ if $p = 2$), i.e. we take $K = F(\zeta_p)$ (take $K = F(\zeta_{p^2})$ if $p = 2$). Consider the localization $S = \mathbf{A}_{R,\varpi}^{+}\big[\frac{1}{\pi_1}\big]$. Let $M$ and $M'$ be free relative Fontaine–Laffaille modules and $N^{+}(M)$ and $N^{+}(M')$ the respective $\mathbf{A}_{R,\varpi}^{+}$-modules obtained by the functor in Proposition 5.30.

Lemma 5.32. — *We have a natural bijection*

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{A}_{R,\varpi}^{+},\Gamma_R}(N^{+}(M),\, N^{+}(M')) \\ &\qquad \xrightarrow{\sim} \mathrm{Hom}_{S,\Gamma_R}\left(N^{+}(M)\left[\frac{1}{\pi_1}\right], N^{+}(M')\left[\frac{1}{\pi_1}\right]\right). \end{aligned} \tag{5.7}$$

*Proof.* — As we are working with free modules and the morphism of rings $\mathbf{A}_{R,\varpi}^{+} \to \mathbf{A}_{R,\varpi}^{+}\big[\frac{1}{\pi_1}\big] = S$ is flat, we obtain that (5.7) is injective. To check surjectivity, let $f : N^{+}(M)\big[\frac{1}{\pi_1}\big] \to N^{+}(M')\big[\frac{1}{\pi_1}\big]$ be an $S$-linear and

$\Gamma_R$-equivariant morphism. We need to show that $f(N^+(M)) \subset N^+(M')$. Assume $f(N^+(M)) \subset \pi_1^{-k} N^+(M')$ for $k \in \mathbb{N}$, and consider the reduction of $f$ modulo $\pi$, which is again $\Gamma_R$-equivariant. Now from Lemma 5.31, we have that $\Gamma_R$ acts trivially over $N^+(M)/\pi_1 N^+(M)$, whereas the action of $\Gamma_R$ is non-trivial over $\pi_1^{-k} N^+(M')/\pi_1^{-k+1} N^+(M')$ for $k \neq 0$ (the action of $\gamma_0 \in \Gamma_K$ is non-trivial for $k \neq 0$). Hence, we must have $k = 0$, i.e. $f(N^+(M)) \subset N^+(M')$, which shows the claim. □

Note that we have a morphism $\varphi : S = \mathbf{A}^+_{R,\varpi}\big[\frac{1}{\pi_1}\big] \to \mathbf{A}^+_{R,\varpi}\big[\frac{1}{\pi}\big]$. The respective Frobenius-semilinear endomorphisms $\varphi$ on $N^+(M)$ and $N^+(M')$ induce semilinear morphisms $\varphi : N^+(M)\big[\frac{1}{\pi_1}\big] \to N^+(M)\big[\frac{1}{\pi}\big]$ and $\varphi : N^+(M')\big[\frac{1}{\pi_1}\big] \to N^+(M')\big[\frac{1}{\pi}\big]$. Let $f \in \mathrm{Hom}_{S,\Gamma_R}\big(N^+(M)\big[\frac{1}{\pi_1}\big], N^+(M')\big[\frac{1}{\pi_1}\big]\big)$ be a morphism, such that the following diagram commutes

$$\begin{tikzcd} N^+(M)\big[\frac{1}{\pi_1}\big] \arrow[r, "f"] \arrow[d, "\varphi"] & N^+(M')\big[\frac{1}{\pi_1}\big] \arrow[d, "\varphi"] \\ N^+(M)\big[\frac{1}{\pi}\big] \arrow[r, "f"] & N^+(M')\big[\frac{1}{\pi}\big], \end{tikzcd}$$

where the bottom horizontal arrow is well-defined due to Lemma 5.32. We will call such a morphism $f$ to be $(\varphi, \Gamma_R)$-equivariant.

Lemma 5.33. — *We have a natural bijection*

$$\mathrm{Hom}_{\mathbf{A}^+_{R,\varpi},\varphi,\Gamma_R}(N^+(M), N^+(M')) \xrightarrow{\sim} \mathrm{Hom}_{S,\varphi,\Gamma_R}\left(N^+(M)\left[\frac{1}{\pi_1}\right], N^+(M')\left[\frac{1}{\pi_1}\right]\right).$$

*Proof of Theorem 5.5.* — Let $M \in \mathrm{MF}_{[0,p-2],\,\mathrm{free}}(R, \Phi, \partial)$ and let $N^+(M)$ denote the $\mathbf{A}^+_{R,\varpi}$-module obtained from $M$ from the functor of Proposition 5.30. We will show that a basis of $N^+(M)$ descends over to $\mathbf{A}^+_R$.

In the notation of Definition 5.15, let $\{e_1, \ldots, e_h\}$ denote an $\mathbf{A}^+_{R,\varpi}$-basis of $N^+(M)$. Then from Lemma 5.18, we get that $\{q^{-k_1}\varphi(e_1), \ldots, q^{-k_h}\varphi(e_h)\}$ is also an $\mathbf{A}^+_{R,\varpi}$-basis of $N^+(M)$. Without loss of generality, we may further assume that $k_h \leqslant k_{h-1} \leqslant \cdots \leqslant k_1$. Let us set $s := k_1$, so we get that $N^+(M)/\varphi^*(N^+(M))$ is killed by $q^s$ and $s \in \mathbb{N}$ is the smallest such number.

Let $D(M) := N^+(M)\big[\frac{1}{\pi_1}\big]^\wedge$, where $^\wedge$ denotes the $p$-adic completion. Then $D(M)$ is an étale $(\varphi, \Gamma_{R,\varpi})$-module over $\mathbf{A}_{R,\varpi} = \mathbf{A}^+_{R,\varpi}\big[\frac{1}{\pi_1}\big]^\wedge$, free of rank $h$. Since $N^+(M)$ is free, it follows that $N^+(M)\big[\frac{1}{\pi_1}\big]/p^n \xrightarrow{\sim} D(M)/p^n$. Similar to the proof of Lemma 5.32 and using dévissage we obtain that the functor $N^+(M) \mapsto D(M)$ is fully faithful. Therefore, using Proposition 5.30

we conclude that the functor

$$\begin{aligned}\mathrm{MF}_{[0,p-2],\,\mathrm{free}}(R,\Phi,\partial) &\longrightarrow (\varphi,\Gamma_R)\text{-}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_{R,\varpi}}\\ M &\longmapsto N^+(M)\left[\frac{1}{\pi_1}\right]^{\wedge},\end{aligned}$$

is fully faithful.

Now, from Proposition 5.3 and Definition 5.4 we have that $T := T_{\mathrm{cris}}(M)$ is a free $\mathbb{Z}_p$-representation of $G_R$. Considering $T$ as a representation of $G_{R,\varpi}$, we have the associated $(\varphi,\Gamma_{R,\varpi})$-module $\mathbf{D}_{R,\varpi}(T)$ over $\mathbf{A}_{R,\varpi}$. By the full faithfullness of the functor above and equivalence of categories in (3.2), we conclude that $D(M) \xrightarrow{\sim} \mathbf{D}_{R,\varpi}(T)$ as étale $(\varphi,\Gamma_{R,\varpi})$-module over $\mathbf{A}_{R,\varpi}$. Also, we have $\varphi(\mathbf{D}_{R,\varpi}(T)) \subset \mathbf{D}(T)$, where the latter module is an étale $(\varphi,\Gamma_R)$-module over $\mathbf{A}_R$, free of rank $h$.

Next, let $N := N^+(M) \cap \mathbf{D}(T)$ where we take the intersection inside $\mathbf{D}_{R,\varpi}(T)$. Note that $N$ is equipped with a Frobenius-semilinear endomorphism $\varphi$ and it is stable under the action of $\Gamma_R$. We claim that

Lemma 5.34. — *The elements $\{q^{-k_1}\varphi(e_1),\dots,q^{-k_h}\varphi(e_h)\}$ form a basis of $N$.*

*Proof.* — Let us set $N' := \sum_{i=1}^h \mathbf{A}_R^+ q^{-k_i}\varphi(e_i)$. Since we have $q^{-k_i}\varphi(e_i) \in N^+(M)\cap \mathbf{D}(T) = N$, therefore $N' \subset N$. This also implies that $\varphi(e_i) \in q^{k_i}N$. Extending scalars along the faithfully flat morphism of rings $\mathbf{A}_R^+ \to \mathbf{A}_{R,\varpi}^+$, we get that $N^+(M) = \mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} N' \subset \mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} N \subset N^+(M)$. Therefore, $\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} N' \xrightarrow{\sim} \mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} N$. But since the map $\mathbf{A}_R^+ \to \mathbf{A}_{R,\varpi}^+$ is faithfully flat, we obtain that $N' \xrightarrow{\sim} N$. $\square$

We will now verify the conditions of Definition 4.9 for $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$. Since $V$ arises from a Fontaine–Laffaille module of level $[0,p-2]$, we have that $V$ is crystalline with non-positive Hodge–Tate weights. We have that $N$ is a free $\mathbf{A}_R^+$-module of rank $h$ stable under $\varphi$ and $\Gamma_R$, and such that $N \subset \mathbf{D}^+(T)$ as well as $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} N \xrightarrow{\sim} \mathbf{D}(T)$. Next, we want to show that $q^sN \subset \varphi^*(N)$ as $\mathbf{A}_R^+$-modules, where $s = k_1$. Since $\mathbf{A}_R^+ \to \mathbf{A}_{R,\varpi}^+$ is faithfully flat, it is equivalent to showing that $q^s\mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} N \subset \mathbf{A}_{R,\varpi}^+ \otimes_{\mathbf{A}_R^+} \varphi^*(N)$. But the latter inclusion can be re-expressed as $q^sN^+(M) \subset \varphi^*(N^+(M))$ as $\mathbf{A}_{R,\varpi}^+$-modules, which was established above by showing that $N^+(M)/\varphi^*(N^+(M))$ is killed by $q^s$. Therefore, we conclude that $N/\varphi^*(N)$ is killed by $q^s$ and $s \in \mathbb{N}$ is the smallest such number.

Next, we look at the action of $\Gamma_R$ over $N$. Recall from Section 3.1 that we have $\{\gamma_0,\gamma_1,\dots,\gamma_d\}$ as topological generators of $\Gamma_{R,\varpi}$, where $\gamma_0$ is a lift of a topological generator of $\Gamma_K$. The action of $\gamma_j$ on the basis elements of $N^+(M)$ can be given as

$$\gamma_j(e_i) = e_i + \pi x_{i,j} \ \text{ for } \ 1 \leqslant i \leqslant h,\ 0 \leqslant j \leqslant d \ \text{ and } \ x_{i,j} \in \mathbf{A}^+_{R,\varpi}.$$

Since $\varphi$ is $\Gamma_R$-equivariant, we get that $\gamma_j(\varphi(e_i)) = \varphi(e_i) + q\pi\varphi(x_{i,j})$, where $\varphi(x_{i,j}) \in \varphi(N^+(M)) \subset N^+(M) \cap \mathbf{D}(T) = N$. Now $\varphi(e_i) \in q^{k_i}N$, so we must have that $q\pi\varphi(x_{i,j}) \in q^{k_i}N \cap q\pi N = q^{k_i}\pi N \subset N$, for $1 \leqslant i \leqslant h$ and $0 \leqslant j \leqslant d$. Therefore, we get that

$$\gamma_j(q^{-k_i}\varphi(e_i)) \equiv q^{-k_i}\varphi(e_i) \mod \pi N \ \text{ for } \ 1 \leqslant j \leqslant d.$$

For $j = 0$, recall that $\gamma_0(\pi) = \chi(\gamma_0)\pi u$ for some unit $u \in 1 + \pi\mathbf{A}^+_R$. Therefore, we have $\gamma_0(q) = q\varphi(u)u^{-1}$ and $\gamma_0(q^{-1}) = q^{-1}\varphi(u^{-1})u$. So we obtain

$$\begin{aligned}\gamma_0(q^{-k_i}\varphi(e_i)) = \gamma_0(q^{-k_i})\gamma_0(\varphi(e_i)) &= q^{-k_i}\varphi(u^{-k_i})u^{k_i}(\varphi(e_i) + q\pi\varphi(x_{i,j})) \\ &\equiv q^{-k_i}\varphi(e_i) \mod \pi N.\end{aligned}$$

Finally, let $g \in \Gamma_R$ be a lift of a generator $\bar{g} \in \Gamma_R/\Gamma_{R,\varpi}$, a finite group of order $p-1$. Then we have $g(e_i) = e_i + \pi_1 y_i$ for $1 \leqslant i \leqslant h$ and $y_i \in N^+(M)$. Since $\varphi$ is $\Gamma_R$-equivariant, we get that $g(\varphi(e_i)) = \varphi(e_i) + \pi\varphi(y_i)$, where $\varphi(y_i) \in \varphi(N^+(M)) \subset N^+(M) \cap \mathbf{D}(T) = N$. Now $\varphi(e_i) \in q^{k_i}N$, so we must have that $\pi\varphi(y_i) \in q^{k_i}N \cap \pi N = q^{k_i}\pi N \subset N$, for $1 \leqslant i \leqslant h$. Further, we know that $g(\pi) = \chi(g)\pi v$ for some unit $v \in 1 + \pi\mathbf{A}^+_R$, which gives us that $g(q) = q\varphi(v)v^{-1}$. Therefore, $g(q^{-k_i}\varphi(e_i)) = q^{-k_i}\varphi(u^{-k_i})u^{k_i}(\varphi(e_i) + \pi\varphi(y_i)) \equiv q^{-k_i}\varphi(e_i) \mod \pi N$, for $1 \leqslant i \leqslant h$. Hence, $\Gamma_R$ acts trivially over $N/\pi N$.

Setting $\mathbf{N}(T) \coloneqq N$, we see that conditions of Definition 4.9 have been satisfied. In particular, $V$ is a positive finite $q$-height representation. □

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