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ON SPECTRA AND AFFINE STRICT POLYNOMIAL FUNCTORS

by Marcin CHAŁUPNIK (*)

ABSTRACT. — We compare derived categories of the category of strict polynomial functors over a finite field and the category of ordinary endofunctors on the category of vector spaces. We introduce two intermediate categories: the category of ∞ -affine strict polynomial functors and the category of spectra of strict polynomial functors. They provide a conceptual framework for computational theorems of Franjou–Friedlander–Scorichenko–Suslin and clarify the role of inverting the Frobenius morphism in the comparison between rational and discrete cohomology.

RÉSUMÉ. — Nous comparons des catégories dérivées de la catégorie des foncteurs strictement polynomiaux sur un domaine et la catégorie des endofoncteurs ordinaires sur la catégorie des espaces vectoriels. Nous introduisons deux catégories intermédiaires: la catégorie de foncteurs ∞ -affines strictement polynomiaux et la catégorie des spectres des foncteurs strictement polynomiaux. Ils fournissent un cadre conceptuel pour les théorèmes de calcul de Franjou–Friedlander–Scorichenko–Suslin et clarifient le rôle de l’inversion du morphisme de Frobenius dans la comparaison entre cohomologie rationnelle et discrète.

1. Introduction

The aim of the present paper is to better understand relationship between derived categories of the category \mathcal{P}_d of strict polynomial functors of degree d over a finite field \mathbf{k} and the category \mathcal{F} of usual functors on the vector spaces over \mathbf{k} . It may be thought of as an instance of a fundamental problem in algebraic geometry: comparing affine schemes with their sets of rational points over small fields. In our context, the situation is well understood at the level of abelian categories. Namely, it was shown [5, Proposition 1.4] that if $d \leq |\mathbf{k}|$, then the forgetful functor $\mathbf{f} : \mathcal{P}_d \rightarrow \mathcal{F}$ is a full embedding.

Keywords: Strict polynomial functor, Affine functor, Spectrum, Ext-group.

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Unfortunately, this is no longer true at the level of derived categories, as the forgetful functors does not induce an isomorphism on Ext-groups. However, as it was shown by Franjou–Friedlander–Scorichenko–Suslin, we still get an isomorphism on Ext-groups when we take instead of given strict polynomial functors F, G their large enough Frobenius twists $F^{(ni)}, G^{(ni)}$. To put it more precisely: since $\mathbf{f}(F) = \mathbf{f}(F^{(ni)}), \mathbf{f}(G) = \mathbf{f}(G^{(ni)})$ for $|\mathbf{k}| = p^n$, we have the induced map

$$\operatorname{colim}_i \operatorname{Ext}_{\mathcal{P}_{d p^{ni}}}^*(F^{(ni)}, G^{(ni)}) \longrightarrow \operatorname{Ext}_{\mathcal{F}}^*(F, G),$$

and [5, Theorem 3.10] says that this map is an isomorphism provided that $d \leq |\mathbf{k}|$. This result has numerous nontrivial applications, since in many cases the left hand side is much more computable. Unfortunately, neither its applications nor its proof answers a natural question: why does twisting make \mathcal{P}_d and \mathcal{F} closer to each other?

In our article we address this question by putting the above mentioned phenomena into a wider categorical context (although it should be emphasized that our work does not provide new proofs of the results of [5], since we use their computations in several places). Namely, we factorize the (derived) forgetful functor $\mathbf{f} : \mathcal{DP}_d \rightarrow \mathcal{DF}$ through certain intermediate triangulated category whose Hom-spaces are (among others) colimits of Ext-groups between twists of strict polynomial functors. In fact, we construct two apparently quite different triangulated categories which have this property.

The first construction which is described in Sections 2 and 3 uses a concept of “affine strict polynomial functor” introduced in [3]. In fact our situation resembles that considered in [3]. In both cases we would like to extend a reflective full embedding of abelian categories to their derived categories but we face an obstruction that the unit of our adjunction is not an isomorphism. Nevertheless, this unit admits an explicit description which gives us a hint how to enlarge the starting category to obtain a reflective full embedding (in fact, as shown in [13] the full embedding \mathbf{f} is a part of recollement diagram of abelian categories, however in our situation, similarly to [3], we do not get a full recollement due to appearance of categories of infinite homological dimension). By applying this procedure we get a DG functor category $\mathcal{P}_d^{af\infty}$ and then its derived category $\mathcal{DP}_d^{af\infty}$ together with a factorization of \mathbf{f} as

$$\mathcal{DP}_d \xrightarrow{z^*} \mathcal{DP}_d^{af\infty} \xrightarrow{\mathbf{f}^{af\infty}} \mathcal{DF}.$$

The important features of this factorization are that

$$\operatorname{Hom}_{\mathcal{DP}_d^{af\infty}}^*(z^*(F), z^*(G)) \simeq \operatorname{colim}_i \operatorname{Ext}_{\mathcal{P}_{d p^i}}^*(F^{(i)}, G^{(i)})$$

and that $\mathbf{f}^{af\infty}$ (when restricted to the subcategory $\mathcal{DP}_d^{faf\infty}$ of finite objects in $\mathcal{DP}_d^{af\infty}$) is a full embedding (Theorem 3.7). Thus we have achieved our goal by a rather tautological construction, since the category $\mathcal{DP}_d^{af\infty}$ is designed exactly in such a way that we obtain the desired colimits as Ext-spaces. This being said, the fact that the construction works is highly nontrivial because it utilizes in an essential way the formality phenomena observed in [2, 3].

Then in Sections 4 and 5 we take quite a different approach, which is perhaps more intuitive. It relies on an observation that an important difference between the categories \mathcal{P} and \mathcal{F} is that in the latter the Frobenius twist operation is invertible. Hence if we formally invert the Frobenius twist in \mathcal{P} we should obtain a category closer to \mathcal{F} . Moreover, when we think of classical example of applying such a construction i.e. stable homotopy category, we see how colimits enter our story: we should get them as an analog of the known description of the homotopy classes of maps between suspension spectra.

Technically, we introduce the category \mathcal{SP}_d of spectra of complexes of strict polynomial functors and, following a general approach of Hovey [8], we introduce a Quillen model structure on it. Then we put \mathcal{DSP}_d to be the homotopy category with respect to this structure and we get a factorization of \mathbf{f} as

$$\mathcal{DP}_d \xrightarrow{\mathbf{C}^\infty} \mathcal{DSP}_d \xrightarrow{\mathbf{f}^{st}} \mathcal{DF},$$

where \mathbf{C}^∞ is a functor analogous to Σ^∞ in topology. Then we have the expected description of the maps between “suspension spectra”: (Theorem 4.6):

$$\mathrm{Hom}_{\mathcal{DSP}_d}^*(\mathbf{C}^\infty(F), \mathbf{C}^\infty(G)) \simeq \mathrm{colim}_i \mathrm{Ext}_{\mathcal{P}_{d,p_i}}^*(F^{(i)}, G^{(i)})$$

and, similarly to the first approach, \mathbf{f}^{st} when restricted to the category \mathcal{DP}_d^{st} generated as triangulated category with direct summands by the image of \mathbf{C}^∞ , is a full embedding (Theorem 5.2).

Finally, in the last section we compare the two constructions. Namely, we find a full embedding $\gamma : \mathcal{DP}_d^{af\infty} \rightarrow \mathcal{DSP}_d$ which restricts to an equivalence $\mathcal{DP}_d^{faf\infty} \simeq \mathcal{DP}_d^{st}$ (Theorem 6.2). This shows that the categories $\mathcal{DP}_d^{af\infty}$ and \mathcal{DSP}_d are quite close, which is perhaps not obvious at a first sight. As a sort of heuristic explanation we can offer the following observation. Similarly to the classical context, in our category of spectra, the delooping functor Θ^∞ plays an important role. On the other hand, as it was observed in [3], the category of affine strict polynomial functors is closely related to the category of representations of the group of algebraic loops on $GL_n(\mathbf{k})$.

Thus our category $\mathcal{P}_d^{af\infty}$ should correspond to the infinite loops on $GL_n(\mathbf{k})$. Hence some sort of relation to infinite loop spaces is a feature shared by the both categories.

Now let us discuss the differences between $\mathcal{DP}_d^{af\infty}$ and \mathcal{DSP}_d . The fact that $\mathcal{DP}_d^{af\infty}$ embeds into \mathcal{DSP}_d shows that the former category is closer to \mathcal{DP}_d . This is not surprising, since we see in its very construction, that it is a possibly closest to \mathcal{DP}_d triangulated category in which the colimits of Exts of twists appear (we make no attempt to make this statement precise). In particular, we see that it is not necessary to fully invert the Frobenius twist to get these colimits. In fact, one can show that the Frobenius twist gives a full embedding $\mathcal{DP}_d^{af\infty} \subset \mathcal{DP}_{dp}^{af\infty}$ but not an equivalence.

Thus, one could think that the factorization through \mathcal{DSP}_d is something less fundamental. On the other hand however, the functor $\mathbf{f}^{st} : \mathcal{DSP}_d \rightarrow \mathcal{DF}$ has a remarkable property that it preserves (at least some) fibrant objects (Remark 5.3). This suggests that \mathcal{SP}_d (in contrast to just \mathcal{P}_d) encodes important information about injective objects in \mathcal{F} .

Let us finally point our for a recent preprint [4], which extends the Franjou–Friedlander–Scorichenko–Suslin Comparison Theorem to the case of any perfect field (I am grateful to the referee for turning my attention to that work). Assuming the results of [4], we would also have the results of our article for any infinite perfect fields.

In order to help the reader navigating in the article I provide below the list of main definitions and notations used in the paper.

| Notation | Section | Meaning |
|-----------------------------|---------|-----------------------------------------------------------------------------------------|
| \mathbf{k} | 2 | ground field, since Section 3 finite |
| A_∞ | 2 | graded algebra $\mathbf{k}[x_1, x_2, \dots]/(x_1^p, x_2^p, \dots)$ |
| \mathcal{V} | 2 | category of finite-dimensional vector spaces over \mathbf{k} |
| \mathcal{V}' | 2 | category of all vector spaces over \mathbf{k} |
| \mathcal{V}^{gr} | 2 | category of graded vector spaces over \mathbf{k} finite-dimensional in each degree |
| \mathcal{V}^{grf} | 2 | category of totally finite-dimensional graded vector spaces over \mathbf{k} |
| \mathcal{V}'^{gr} | 2 | category of all graded vector spaces over \mathbf{k} |
| \mathcal{V}_{A_∞} | 2 | subcategory of the category of free graded A_∞ -modules |
| $\Gamma^d \mathcal{V}^{gr}$ | 2 | category of divided powers over \mathcal{V}^{gr} |

| Notation | Section | Meaning |
|--------------------------------------------|---------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\Gamma^d \mathcal{V}^{grf}$ | 2 | category of divided powers over \mathcal{V}^{grf} |
| $\Gamma^d \mathcal{V}_{A_\infty}$ | 2 | category of divided powers over \mathcal{V}_{A_∞} |
| \mathcal{P}_d | 2 | category of strict polynomial functors over \mathbf{k} of degree d |
| \mathcal{P}_d^{gr} | 2 | category of graded strict polynomial functors over \mathbf{k} of degree d |
| $\mathcal{P}_d^{af\infty}$ | 2 | category of ∞ -affine strict polynomial functors of degree d |
| z^* | 2 | functor from $\mathcal{D}\mathcal{P}_d$ to $\mathcal{D}\mathcal{P}_d^{af\infty}$ induced by the forgetting |
| t^* | 2 | functor from $\mathcal{D}\mathcal{P}_d^{af\infty}$ to $\mathcal{D}\mathcal{P}_d$ induced by the scalar extension, right adjoint to z^* |
| $\Gamma^{d,U}$ | 2 | object in \mathcal{P}_d represented by $U \in \Gamma^d \mathcal{V}$ |
| $h^{U \otimes A_\infty}$ | 2 | object in $\mathcal{P}_d^{af\infty}$ represented by $U \otimes A_\infty \in \Gamma^d \mathcal{V}_{A_\infty}$ |
| \mathcal{F} | 3 | category of functors from \mathcal{V} to \mathcal{V}' |
| \mathbf{f} | 3 | forgetful functor from $\mathcal{D}\mathcal{P}_d$ to $\mathcal{D}\mathcal{F}$ |
| $\mathcal{D}\mathcal{P}_d^{af\infty}$ | 3 | smallest triangulated subcategory of $\mathcal{D}\mathcal{P}_d^{af\infty}$ containing $h^{U \otimes A_\infty}$ and closed under isomorphisms and direct summands |
| \mathbf{C} | 4 | Frobenius twist regarded as functor from $\mathcal{D}\mathcal{P}_d$ to $\mathcal{D}\mathcal{P}_{pd}$ |
| $\widehat{\mathcal{P}}$ | 4 | product category $\prod_{d>0} \mathcal{P}_d$ |
| $\mathcal{K}\mathcal{P}_d$ | 4 | category of complexes over \mathcal{P}_d |
| $\mathcal{K}\widehat{\mathcal{P}}$ | 4 | category of complexes over $\widehat{\mathcal{P}}$ |
| $\mathcal{S}\widehat{\mathcal{P}}$ | 4 | category of spectra over $\mathcal{K}\widehat{\mathcal{P}}$ |
| $\mathcal{S}\mathcal{P}_d$ | 4 | category of spectra over $\mathcal{K}\mathcal{P}_d$ |
| $\mathcal{K}\mathcal{F}$ | 5 | category of complexes over \mathcal{F} |
| $\mathcal{S}\mathcal{F}$ | 5 | category of spectra over $\mathcal{K}\mathcal{F}$ |
| $\mathcal{D}\mathcal{S}\mathcal{P}_d^{st}$ | 5 | smallest triangulated subcategory of $\mathcal{D}\mathcal{S}\mathcal{P}_d$ containing all $\mathbf{C}^\infty(F)$ and closed under isomorphisms and direct summands |
| \mathcal{P}_d^{afi} | 6 | category of i -affine strict polynomial functors of degree d |

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2. ∞ -affine functors

In this section we introduce and establish basic properties of the category $\mathcal{P}_d^{af\infty}$ of ∞ -affine strict polynomial functors. In the next section we relate $\mathcal{P}_d^{af\infty}$ to the categories \mathcal{P}_d and \mathcal{F} .

We shall frequently use graded categories and functors. There are several possible choices of the setup here, we follow the one from [10] and we briefly recall the basic definitions. We fix a ground field \mathbf{k} . By a graded (\mathbf{k} -linear) category we mean a \mathbf{k} -linear category \mathcal{C} with Hom-spaces equipped with \mathbf{Z} -grading preserved by composition. By this we mean that for $X, Y \in Ob(\mathcal{C})$, we have

$$\text{Hom}_{\mathcal{C}}(X, Y) = \bigoplus_{n \in \mathbf{Z}} \text{Hom}_{\mathcal{C}}^n(X, Y)$$

and if $f \in \text{Hom}_{\mathcal{C}}^n(X, Y)$ and $g \in \text{Hom}_{\mathcal{C}}^k(Y, Z)$ then $g \circ f \in \text{Hom}_{\mathcal{C}}^{n+k}(X, Z)$.

The basic example of graded \mathbf{k} -linear category is the category \mathcal{V}^{gr} of \mathbf{Z} -graded vector spaces over \mathbf{k} . We put decoration $(-)^{\prime}$ because in the majority of our constructions we will restrict to its subcategory \mathcal{V}^{gr} consisting of graded vector spaces finite dimensional in each degree. The Hom-spaces are given as:

$$\text{Hom}_{\mathcal{V}^{gr}}^n(V^{\bullet}, W^{\bullet}) := \prod_j \text{Hom}(V^j, W^{j+n})$$

A graded functor between graded categories is a functor which preserves the grading on Hom-spaces. Observe that for any graded \mathbf{k} -linear category \mathcal{A} the graded functors from \mathcal{A} to \mathcal{V}^{gr} form a \mathbf{k} -linear graded category. In order to define the graded Hom-spaces let us define for a graded functor F from \mathcal{A} to \mathcal{V}^{gr} its shift $F[1]$ by $F[1](A)^n := F(A)^{n+1}$. Then we define $\text{Hom}^n(F, G)$ to be $\text{Nat}(F, G[n])$ (see [10, Section 1]).

The notion of ∞ -affine strict polynomial functor is quite straightforward generalization of that of the affine strict polynomial functor introduced

in [3]. The only difference is that instead the graded algebra $A = \mathbf{k}[x]/x^p \simeq \text{Ext}_{\mathcal{P}_{dp}}^*(I^{(1)}, I^{(1)})$ we consider the graded algebra

$$A_\infty := \mathbf{k}[x_1, x_2, \dots]/(x_1^p, x_2^p, \dots)$$

with $|x_i| = 2p^i$. The appearance of the algebra A_∞ in our situation follows from the fact known from [6] that if \mathbf{k} is a finite field of characteristic p then there is an isomorphism of graded algebras $A_\infty \simeq \text{Ext}_{\mathcal{F}}^*(I, I)$.

Let \mathcal{V} (resp. \mathcal{V}') stands for the category of finite dimensional vector spaces over \mathbf{k} (resp. the category of all vector spaces over \mathbf{k}). Next, let \mathcal{V}_{A_∞} stand for the full subcategory of the category of graded free A_∞ -modules consisting of the modules $V \otimes A_\infty$ for $V \in \mathcal{V}$. Then we consider the category $\Gamma^d \mathcal{V}_{A_\infty}$ of divided powers over \mathcal{V}_{A_∞} (see e.g. [7, Section 3]). This is the graded \mathbf{k} -linear category with the objects the same as those of \mathcal{V}_{A_∞} , but the Hom-spaces are

$$\text{Hom}_{\Gamma^d \mathcal{V}_{A_\infty}}(V \otimes A_\infty, W \otimes A_\infty) := \Gamma^d(\text{Hom}_{A_\infty}(V \otimes A_\infty, W \otimes A_\infty))$$

where Γ^d stands for the space of symmetric d -tensors over \mathbf{k} . The grading comes from that on A_∞ and the standard grading on the tensor product:

$$|x_1 \otimes \dots \otimes x_d| := \sum_{s=1}^d |x_s|.$$

The composition law is given by the composition of the map:

$$\begin{aligned} &\Gamma^d(\text{Hom}_{A_\infty}(V \otimes A_\infty, W \otimes A_\infty)) \otimes \Gamma^d(\text{Hom}_{A_\infty}(W \otimes A_\infty, U \otimes A_\infty)) \\ &\longrightarrow \Gamma^d(\text{Hom}_{A_\infty}(V \otimes A_\infty, W \otimes A_\infty) \otimes \text{Hom}_{A_\infty}(W \otimes A_\infty, U \otimes A_\infty)) \end{aligned}$$

coming from the natural map $\Gamma^d(X) \otimes \Gamma^d(Y) \rightarrow \Gamma^d(X \otimes Y)$ existing for any vector spaces X, Y , with the map

$$\begin{aligned} &\Gamma^d(\text{Hom}_{A_\infty}(V \otimes A_\infty, W \otimes A_\infty) \otimes \text{Hom}_{A_\infty}(W \otimes A_\infty, U \otimes A_\infty)) \\ &\longrightarrow \Gamma^d(\text{Hom}_{A_\infty}(V \otimes A_\infty, U \otimes A_\infty)) \end{aligned}$$

coming from the composition in A_∞ -linear Homs.

DEFINITION/PROPOSITION 2.1. — *An ∞ -affine strict polynomial functor F homogeneous of degree d is a graded \mathbf{k} -linear functor*

$$F : \Gamma^d \mathcal{V}_{A_\infty} \longrightarrow \mathcal{V}'^{gr}.$$

The affine strict polynomial functors homogeneous of degree d with morphisms being natural transformations (with shifted targets) form a graded \mathbf{k} -linear category $\mathcal{P}_d^{af\infty}$ (see [3, p. 655-656]).

The reader of [3] will find there a similar construction. In fact our category $\mathcal{P}_d^{af\infty}$ may be thought of as obtained by infinitely times applying the construction producing \mathcal{P}_d^{af} from [3]. However, we alert the reader that, in contrast to [3] and also to the foundational paper on strict polynomial functors [7], we do not impose any finiteness/finite generation assumptions on values of functors.

Now we list some basic properties of the category $\mathcal{P}_d^{af\infty}$. The proofs of respective facts on affine functors from [3] in most cases carry over to the current context. Therefore we discuss more thoroughly these points only when the infinite dimension of A_∞ requires special attention. Also, the reader interested in obtaining more motivation behind the construction of affine functors is referred to [3].

Like in any functor category, for any $U \in \mathcal{V}$ we have the representable functor $h^{U \otimes A_\infty} \in \mathcal{P}_d^{af\infty}$ given by the formula

$$V \otimes A_\infty \longmapsto \text{Hom}_{\Gamma^d \mathcal{V}_{A_\infty}}(U \otimes A_\infty, V \otimes A_\infty)$$

and the co-representable functor $c_{U \otimes A_\infty}^* \in \mathcal{P}_d^{af\infty}$ given by the formula

$$V \otimes A_\infty \longmapsto \text{Hom}_{\Gamma^d \mathcal{V}_{A_\infty}}(V \otimes A_\infty, U \otimes A_\infty)^*$$

where $(-)^*$ stands for the graded \mathbf{k} -linear dual. We list the basic properties of $\mathcal{P}_d^{af\infty}$.

PROPOSITION 2.2.

- (1) *There are natural in $U \otimes A_\infty$ isomorphisms*

$$\text{Hom}_{\mathcal{P}_d^{af\infty}}(h^{U \otimes A_\infty}, F) \simeq F(U \otimes A_\infty)$$

$$\text{Hom}_{\mathcal{P}_d^{af\infty}}(F, c_{U \otimes A_\infty}^*) \simeq (F(U \otimes A_\infty))^*$$

for any $F \in \mathcal{P}_d^{af\infty}$.

- (2) *Moreover, the map $\Psi : h^{U \otimes A_\infty} \otimes F(U \otimes A_\infty) \rightarrow F$ adjoint to the map $F_{U \otimes A_\infty, V \otimes A_\infty}$ giving the action of F on morphisms is surjective, provided that $\dim(U) \geq d$.*
- (3) *If $\dim(U) \geq d$ then $h^{U \otimes A_\infty}$ is a generator of $\mathcal{P}_d^{af\infty}$, $c_{U \otimes A_\infty}^*$ is a cogenerator of $\mathcal{P}_d^{af\infty}$*

The proofs of [3, Proposition 2.5, 2.6] carry over to the current situation.

Now we turn to comparing $\mathcal{P}_d^{af\infty}$ with \mathcal{P}_d (or rather its graded variant \mathcal{P}_d^{gr} which will be defined later) which, when compared to [3], is a bit more delicate point due to infinite dimension of A_∞ . We have a pair of adjoint functors between the source categories of our functor categories. The first

is just the forgetful functor:

$$z : \Gamma^d \mathcal{V}_{A_\infty} \longrightarrow \Gamma^d \mathcal{V}^{gr}$$

(z comes from the polish word for forgetting which is *zapominanie*), where $\Gamma^d \mathcal{V}^{gr}$ stands for the graded category whose objects are graded vector spaces finite dimensional in each degree and

$$\text{Hom}_{\Gamma^d \mathcal{V}^{gr}}^\bullet(V^\bullet, W^\bullet) := \Gamma^d(\text{Hom}(V^\bullet, W^\bullet)).$$

The second is the scalar extension functor:

$$t : \Gamma^d \mathcal{V}^{gr} \longrightarrow \Gamma^d \mathcal{V}_{A_\infty}$$

explicitly given on objects as tensoring over \mathbf{k} with A_∞ : $t(V) := V \otimes A_\infty$. Then we claim that precomposing with z can be extended to a functor

$$z^* : \mathcal{P}_d^{gr} \longrightarrow \mathcal{P}_d^{af_\infty}$$

where \mathcal{P}_d^{gr} is the category of \mathbf{Z} -graded strict polynomial functors of degree d (i.e. the category of graded functors from $\Gamma^d \mathcal{V}^{gr}$ to \mathcal{V}^{gr}). This can be done in two steps. The first step was already used in [3, Section 2] (see also [15, Section 2.5]). Let $\Gamma^d \mathcal{V}^{grf}$ stands for the full subcategory of $\Gamma^d \mathcal{V}^{gr}$ consisting of totally finite dimensional vector spaces. Then any $F \in \mathcal{P}_d$ can be extended to a graded functor $F^{grf} : \Gamma^d \mathcal{V}^{grf} \rightarrow \mathcal{V}^{gr}$ (see [3, p. 657]). In the second step we shall extend F^{grf} to the graded functor F^{gr} defined on the whole graded category $\Gamma^d \mathcal{V}^{gr}$. For $V^\bullet \in \mathcal{V}^{gr}$ let $V^{\leq |j|} := \bigoplus_{|s| \leq j} V^s$. Then we define $F^{gr}(V^\bullet)$ as $\text{colim}_j F^{grf}(V^{\leq |j|})$. Now we can correctly define $z^* : \mathcal{P}_d^{gr} \rightarrow \mathcal{P}_d^{af_\infty}$ by putting

$$z^*(F)(V \otimes A_\infty) := F^{gr}(V \otimes A_\infty).$$

Analogously, precomposing with t produces the functor

$$t^* : \mathcal{P}_d^{af_\infty} \longrightarrow \mathcal{P}_d^{gr}.$$

Considering here the category \mathcal{P}_d^{gr} instead of \mathcal{P}_d will be essential in the next section where we will compare the derived categories of \mathcal{P}_d and $\mathcal{P}_d^{af_\infty}$.

We list the properties of z^* and t^* :

PROPOSITION 2.3.

- (1) z^* preserves representable objects i.e.

$$z^*(\Gamma^{d,U}) = h^{U \otimes A_\infty}.$$

where $\Gamma^{d,U} \in \mathcal{P}_d$ is defined as $V \mapsto \text{Hom}_{\Gamma^d \mathcal{V}}(U, V) = \Gamma^d(U^* \otimes V)$.

- (2) The functor t^* is right adjoint to z^* .

Again, the proof of [3, Proposition 2.4] carries over to the present situation.

Remark. — It is worth mentioning that, as pointed out by the referee, all results of the present section hold for any non-negatively graded algebra finite dimensional in each degree put instead of A_∞ . In fact we do consider in Section 6 yet another (and much simpler) variant of this construction. What is specific to A_∞ is relation of $\mathcal{P}_d^{af\infty}$ to \mathcal{P}_{pd} at the level of derived categories. We will discuss this relation in the next section.

3. Formality and ∞ -affine algebraification

In the present section we factorize the derived functor of the forgetful functor $\mathbf{f} : \mathcal{P}_d \rightarrow \mathcal{F}$ through the derived category of $\mathcal{P}_d^{af\infty}$. Starting from this section we assume that the ground field \mathbf{k} is a finite field of characteristic p . As we have mentioned in Introduction we closely follow the strategy taken in [3].

To this end we regard $\Gamma^d \mathcal{V}_{A_\infty}$ as DG category with trivial differentials. Our main reference for general facts and terminology on DG categories is still [10]. We consider the category $\text{Dif}(\Gamma^d \mathcal{V}_{A_\infty}^{op})$ consisting of DG functors from $\Gamma^d \mathcal{V}_{A_\infty}$ to the category of complexes of \mathbf{k} -modules (our strange terminology here is coherent with [10, Section 1.2], where the main focus was on contravariant functors). We are ready to introduce our main object of interest in the first part of the paper.

DEFINITION 3.1. — *Let $\mathcal{DP}_d^{af\infty}$ be the category obtained from $\text{Dif}(\Gamma^d \mathcal{V}_{A_\infty}^{op})$ by localization with respect to the class of quasi-isomorphisms (we recall again that we do not make any boundedness/finiteness assumptions).*

Let $\Gamma^d(I^* \otimes I)$ denote bifunctor given by the formula $(V, W) \mapsto \Gamma^d(V^* \otimes W)$. We regard $\Gamma^d(I^* \otimes I)$ as a contravariant strict polynomial functor of degree d in V and just a naive functor in W . We shall denote the category of such mixed bifunctors by $\mathcal{P}_{\mathcal{F}}^d$ (we put decoration d as a superscript to emphasize that our functors are contravariant with respect to the strict polynomial variable). More formally, $\mathcal{P}_{\mathcal{F}}^d$ is a full subcategory of the category of functors from $(\Gamma^d \mathcal{V})^{op} \times \mathcal{V}$ to \mathcal{V}' consisting of functors which are \mathbf{k} -linear with respect to the first variable. Thus, the assignment

$$F \longmapsto \text{Hom}_{\mathcal{F}}(\Gamma^d(I^* \otimes I), F)$$

defines the functor $\mathbf{a} : \mathcal{F} \rightarrow \mathcal{P}_d$ which we call the (right) algebraification. Then it is easy to see that \mathbf{a} is right adjoint to the forgetful functor $\mathbf{f} : \mathcal{P}_d \rightarrow \mathcal{F}$ (in fact, this adjunction is among those considered by Kuhn in [13]). Next, since \mathbf{f} is an exact functor between abelian categories, it extends degreewise to a functor between unbounded derived categories

$$\mathbf{f} : \mathcal{D}\mathcal{P}_d \longrightarrow \mathcal{D}\mathcal{F}$$

which we shall denote by the same letter, though, formally it is the derived functor of \mathbf{f} . We shall abuse notation in such a manner in several further places in the article, where functors between abelian categories thanks to exactness simply factorize to derived category. On the other hand \mathbf{a} is a left exact functor between abelian categories, hence it gives rise to a derived functor

$$\mathbf{R}\mathbf{a} : \mathcal{D}\mathcal{F} \longrightarrow \mathcal{D}\mathcal{P}_d.$$

Then it is a matter of routine verification that \mathbf{f} and $\mathbf{R}\mathbf{a}$ remain adjoint (compare [2, Theorem 2.2]). Our goal is to factorize this adjunction through $\mathcal{D}\mathcal{P}_d^{af_\infty}$.

A crucial tool for studying the formality phenomena in our context is the following DG counterpart of the graded category $\Gamma^d\mathcal{V}_{A_\infty}$.

DEFINITION 3.2. — *Let X^\bullet be a projective resolution of $\Gamma^d(I^* \otimes I)$ in the category $\mathcal{P}_\mathcal{F}^d$. We introduce the DG category $\Gamma^d\mathcal{V}_X^{op}$ whose objects are finite \mathbf{k} -vector spaces and*

$$\mathrm{Hom}_{\Gamma^d\mathcal{V}_X^{op}}(V, V') := \mathrm{Hom}_\mathcal{F}(X^\bullet(V', -), X^\bullet(V, -)[n]),$$

where $\mathrm{Hom}_\mathcal{F}$ stands for the Hom complex (ie. we do not require that the maps preserve differentials). The composition law comes from the composition in Hom complexes.

The connection between \mathcal{P}_d^{af} and $\Gamma^d\mathcal{V}_X^{op}$ over a field large enough essentially follows from the classical Ext-computations of Franjou–Friedlander–Scorichenko–Suslin:

PROPOSITION 3.3. — *Assume that $|\mathbf{k}| \geq d$. Then the assignment $V \otimes A_\infty \mapsto V$ extends to a quasi-isomorphism of graded categories $\Gamma^d\mathcal{V}_{A_\infty} \simeq H^*(\Gamma^d\mathcal{V}_X^{op})$.*

Proof. — We need to establish a natural in V, V' isomorphism of graded spaces:

$$\mathrm{Ext}_{\mathcal{F}}^*(\Gamma^{d,V'}, \Gamma^{d,V}) \simeq \Gamma^d(\mathrm{Hom}(V, V') \otimes A_\infty).$$

In fact, for $V = V' = \mathbf{k}$ it is just [5, Theorem 6.3.(3)]. The general case may be obtained by a similar reasoning. Alternatively, we can also quickly

derive Proposition 3.3 from “the derived Kan extension”. Namely, from [2, Corollary 3.7] and the Yoneda lemma we get:

$$\text{Ext}_{\mathcal{P}^{dp^i}}^*(\Gamma^{d(i),V'}, \Gamma^{d(i),V}) \simeq \Gamma^d(\text{Hom}(V, V') \otimes A_\infty).$$

Then our assertion follows from [5, Theorem 3.10]. □

However, the much stronger fact, which generalizes [3, Theorem 4.2], holds:

THEOREM 3.4. — *Assume that the ground field \mathbf{k} has $q \geq d$ elements. Then the assignment $V \otimes A_\infty \mapsto V$ extends to a quasi-isomorphism of DG categories $\phi : \Gamma^d \mathcal{V}_{A_\infty} \simeq \Gamma^d \mathcal{V}_X^{op}$.*

The proof is conceptually similar to that of [3, Theorem 4.2], but since there are some technical differences we present it in some detail. First of all we need a certain generalization of Touzé universal classes [14, 16].

LEMMA 3.5. — *There exist classes*

$$c[d]^{(i)} \in \text{Ext}_{\mathcal{P}^{dp^i}}^{2dp^{i-1}}(\Gamma^{dp^i}(I^* \otimes I), \Gamma^d(I^{*(i)} \otimes I^{(i)}))$$

such that $c[1]^{(i)} \neq 0$ for all $i \geq 1$, and are compatible with cup product i.e.

$$\Delta_*(c[d]^{(i)}) = (c[1]^{(i)})^{\cup d}$$

where $\Delta : \Gamma^d \rightarrow I^d$ is the standard embedding.

Proof of Lemma 3.5. — For $i = 1$ we have the original Touzé classes, but the proof carries over to this more general case. Indeed: it immediately follows from the degeneracy of the twisting spectral sequence [16, Proposition 17] and this degeneracy was showed also for multiple twists [16, Theorem 4]. □

Let $\Gamma^d(I^* \otimes I)_{A_\infty}$ stands for the graded bifunctor $(V, W) \mapsto \Gamma^d(V^* \otimes W \otimes A_\infty)$ with grading coming from that on A_∞ and Γ^d , regarded as an object in the derived category of $\mathcal{P}_{\mathcal{F}}^d$. Now we get an \mathcal{F} -analog of [2, Proposition 3.3].

LEMMA 3.6. — *There exist classes*

$$\tilde{e}_d \in \text{Hom}_{\mathcal{D}\mathcal{P}_{\mathcal{F}}^d}(\Gamma^d(I^* \otimes I), (\Gamma^d(I^* \otimes I))_{A_\infty^*})$$

satisfying:

- (1) $\tilde{e}_1 \in \text{Hom}_{\mathcal{D}\mathcal{P}_{\mathcal{F}}^1}(I^* \otimes I, I^* \otimes I \otimes A_\infty^*)$ is nontrivial in each degree.

(2) $\tilde{e}_1^{\otimes d} \circ \Delta = \Delta_{A_\infty^*} \circ \tilde{e}_d$ as elements of

$$\text{Hom}_{\mathcal{D}P_{\mathcal{F}}^d}(\Gamma^d(I \otimes I^*), (I^d(I \otimes I^*))_{A_\infty^*}),$$

where $\Delta : \Gamma^d(I \otimes I^*) \rightarrow I^d(I \otimes I^*)$ is the natural embedding and

$$\tilde{e}_1^{\otimes d} \in \text{Hom}_{\mathcal{D}P_{\mathcal{F}}^d}(I^d(I \otimes I^*), (I^d(I \otimes I^*))_{A_\infty^*})$$

is the d th external power of \tilde{e}_1 .

Proof of Lemma 3.6. — We obtain our classes from the classes $c[d]^{(i)}$ by applying a multitwist analog of [2, Lemma 3.4] and then pulling the obtained elements to the category \mathcal{F} . □

Proof of Theorem 3.4. — We will construct ϕ in two steps using the intermediate DG category $\Gamma^d \mathcal{V}_{X, A_\infty}^{op}$. This is yet another category with objects as in \mathcal{V}_{A_∞} and

$$\text{Hom}_{\Gamma^d \mathcal{V}_{X, A_\infty}^{op}}(V, V') := \text{Hom}_{\mathcal{F}}(X^\bullet(V' \otimes A_\infty, -), X^\bullet(V \otimes A_\infty, -)[n]),$$

where we use the fact discussed in the previous section that strict polynomial functors can be naturally extended to graded spaces of the form $V \otimes A_\infty$. The composition of morphisms is, like in $\Gamma^d \mathcal{V}_X^{op}$, given by the composition in Hom complexes.

We start with constructing the functor $\rho : \Gamma^d \mathcal{V}_{A_\infty} \rightarrow \Gamma^d \mathcal{V}_{X, A_\infty}^{op}$ being the identity on the objects. Thus we need the family of maps:

$$\begin{aligned} \rho_{V, V'} : \Gamma^d(\text{Hom}_{A_\infty}(V \otimes A_\infty, V' \otimes A_\infty)) \\ \longrightarrow \text{Hom}_{\mathcal{F}}(X^\bullet(V' \otimes A_\infty, -), X^\bullet(V \otimes A_\infty, -)) \end{aligned}$$

compatible with the compositions. For this we observe that since X^\bullet is a contravariant strict polynomial with respect to the first variable, we have a \mathbf{k} -linear map:

$$\Gamma^d(\text{Hom}(V \otimes A_\infty, V' \otimes A_\infty)) \longrightarrow \text{Hom}(X^\bullet(V' \otimes A_\infty, W), X^\bullet(V \otimes A_\infty, W))$$

for any $W \in \mathcal{V}$. By the functoriality of X^\bullet with respect to W , these maps amount to the map:

$$\Gamma^d(\text{Hom}(V \otimes A_\infty, V' \otimes A_\infty)) \longrightarrow \text{Hom}_{\mathcal{F}}(X^\bullet(V' \otimes A_\infty, -), X^\bullet(V \otimes A_\infty, -)).$$

Then our map $\rho_{V, V'}$ is the restriction of the above map to $\Gamma^d(\text{Hom}_{A_\infty}(V \otimes A_\infty, V' \otimes A_\infty))$. Now the compatibility of ρ with the compositions in the source and target categories just boils down to the fact that the action of

a functor on the morphisms commutes with their composition. Let us also observe that $H^0 \circ \rho_{V,V'}$ can be factorized as the composite:

$$\begin{aligned} & \Gamma^d(\mathrm{Hom}_{A_\infty}(V \otimes A_\infty, V' \otimes A_\infty)) \\ & \simeq \Gamma^d(\mathrm{Hom}_{A_\infty}(V'^* \otimes A_\infty^*, V^* \otimes A_\infty^*)) \subset \Gamma^d(\mathrm{Hom}(V'^* \otimes A_\infty^*, V^* \otimes A_\infty)) \\ & \simeq \mathrm{Hom}_{\mathcal{P}_d}(\Gamma^d(V'^* \otimes A_\infty^* \otimes -), \Gamma^d(V^* \otimes A_\infty^* \otimes -)) \\ & \simeq \mathrm{Hom}_{\mathcal{F}}(\Gamma^d(V'^* \otimes A_\infty^* \otimes -), \Gamma^d(V^* \otimes A_\infty^* \otimes -)) \\ & \simeq H^0(\mathrm{Hom}_{\mathcal{F}}(X^\bullet(V' \otimes A_\infty, -), X^\bullet(V \otimes A_\infty, -))), \end{aligned}$$

where the second isomorphism is the Yoneda lemma, the third one follows from the fact that $|\mathbf{k}| \geq d$. Thus our construction of ρ may be thought of as providing liftings of certain maps between objects to their projective resolutions in a canonical way (this interpretation was used in the proof of [3, Theorem 4.2] where the whole construction was presented in a less formal way).

Now we turn to constructing the functor

$$e : \Gamma^d \mathcal{V}_{X, A_\infty}^{op} \longrightarrow \Gamma^d \mathcal{V}_X^{op}$$

again being the identity on the objects. For this we choose for the element $\tilde{e}_d \in \mathrm{Hom}_{\mathcal{D}_{\mathcal{P}_d}}(\Gamma^d(I^* \otimes I), \Gamma^d(I^* \otimes A_\infty^* \otimes I))$ a representing cocycle $\tilde{e}'_d \in \mathrm{Hom}_{\mathcal{P}_d}(X^\bullet(-, -), X^\bullet(- \otimes A_\infty, -))$. Then precomposing with \tilde{e}'_d evaluated on the first variable gives for any V, V' the arrow

$$\begin{aligned} (\tilde{e}_d)^* : \mathrm{Hom}_{\mathcal{F}}(X^\bullet(V' \otimes A_\infty, -), X^\bullet(V \otimes A_\infty, -)) \\ \longrightarrow \mathrm{Hom}_{\mathcal{F}}(X^\bullet(V', -), X^\bullet(V \otimes A_\infty, -)). \end{aligned}$$

Next, let

$$X(i) : X^\bullet(V \otimes A_\infty, -) \longrightarrow X^\bullet(V, -)$$

stand for the transformation induced by the embedding $\mathbf{k} \subset A_\infty$. Then we define:

$$e_{V,V'} := X(i) \circ (\tilde{e}_d)^*.$$

Again, although the choice of representative \tilde{e}'_d is not unique, thanks to its functoriality in the first variable, the precomposition with it (followed by postcomposition with $X(i)$) is a functor from $\Gamma^d \mathcal{V}_{X, A_\infty}^{op}$ to $\Gamma^d \mathcal{V}_X^{op}$.

Finally we put ϕ to be $e \circ \rho$.

It remains to show that $\phi_{V,V'}$ is a quasi-isomorphism for any V, V' . The argument follows closely the last part of the proof of [3, Theorem 4.2] (which in turn was a reinterpretation of that of [2, Theorem 3.2]), hence we only sketch it. It uses in a crucial way the assumption $|\mathbf{k}| \geq d$, since it relies on Proposition 3.3. We start with the case $d = 1$. Here the fact that $\phi_{V,V'}$ is a

quasi-isomorphism follows from the first property of \tilde{e}_d and Proposition 3.3 (for $d = 1$). For a greater d we observe that it follows from the Kunneth formula and the second property of the classes \tilde{e}_d that $H^*(\phi_{V,V'})$ is onto. Then, since by Proposition 3.3 we know that the domain and codomain of $H^*(\phi_{V,V'})$ have equal graded dimension, the assertion follows. \square

This formality theorem, which is yet another incarnation of the phenomena observed in [2, 3] allows us to perform “an ∞ -affine extension” of the $\{\mathbf{f}, \mathbf{Ra}\}$ adjunction along the lines of [3, Section 3, 4]. We now outline this construction, which will lead us to Theorem 3.7 (referring to [3] for exhaustive details).

Let \mathcal{DP}_X be the derived category of the DG category $\text{Dif}(\Gamma^d \mathcal{V}_X^{op})$. Then by Theorem 3.4 we get an equivalence of triangulated categories

$$\mathbf{R}\phi^* : \mathcal{DP}_X \simeq \mathcal{DP}_d^{af\infty}.$$

Next, it is well known (see e.g. [13]) that \mathcal{F} may be thought of as the category of linear functors on the category $\mathbf{k}\mathcal{V}$ (we recall that $\text{Hom}_{\mathbf{k}\mathcal{V}}(V, W) = \mathbf{k}[\text{Hom}_{\mathbf{k}}(V, W)]$). Thus, in terms of formalism of [10], X is a $\mathbf{k}\mathcal{V}\text{-}\Gamma^d \mathcal{V}_X^{op}$ bi-module. Hence we can consider “the standard functors” [10, 3, Section 3]:

$$H_X : \text{Dif}(\mathbf{k}\mathcal{V}^{op}) \longrightarrow \text{Dif}(\Gamma^d \mathcal{V}_X^{op}), \quad T_X : \text{Dif}(\Gamma^d \mathcal{V}_X^{op}) \longrightarrow \text{Dif}(\mathbf{k}\mathcal{V}^{op})$$

and their derived functors

$$\mathbf{R}H_X : \mathcal{DF} \longrightarrow \mathcal{DP}_X \quad \mathbf{L}T_X : \mathcal{DP}_X \longrightarrow \mathcal{DF}.$$

We recall, that since we do not have any boundedness conditions, \mathcal{DF} stands for the unbounded derived category.

Now we define “the ∞ -affine forgetful functor”:

$$\mathbf{f}^{af\infty} : \mathcal{DP}_d^{af\infty} \longrightarrow \mathcal{DF}$$

as $\mathbf{f}^{af\infty} := \mathbf{L}T_X \circ (\mathbf{R}\phi^*)^{-1}$, and “the ∞ -affine right algebraification”:

$$\mathbf{a}^{af\infty} : \mathcal{DF} \longrightarrow \mathcal{DP}_d^{af\infty}$$

as $\mathbf{a}^{af\infty} := \mathbf{R}\phi^* \circ \mathbf{R}H_X$.

The next theorem is the main result of the first part of the paper. It is analogous to [3, Theorem 5.1], though slightly weaker. The reason is that, in contrast to [3], X^\bullet is not bounded, hence is not a finite object in \mathcal{DF} . Then we have

THEOREM 3.7. — *Functors $\mathbf{f}^{af\infty}$, $\mathbf{a}^{af\infty}$ have the following properties:*

- (1) $\mathbf{f}^{af\infty} \circ z^* \simeq \mathbf{f}$, $t^* \circ \mathbf{a}^{af\infty} \simeq \mathbf{Ra}$.
- (2) $\mathbf{a}^{af\infty}$ is right adjoint to $\mathbf{f}^{af\infty}$.

- (3) Let $\mathcal{DP}_d^{faf\infty}$ stand for the full subcategory of $\mathcal{DP}_d^{af\infty}$ consisting of finite objects. Then when we restrict $\mathbf{f}^{af\infty}$ to $\mathcal{DP}_d^{faf\infty}$, we have $\mathbf{a}^{af\infty} \circ \mathbf{f}^{af\infty} \simeq \text{Id}_{\mathcal{DP}_d^{faf\infty}}$.
- (4) $\mathbf{f}^{af\infty}$ restricted to $\mathcal{DP}_d^{faf\infty}$ is fully faithful.

Proof. — In order to get the first isomorphism in the first part we evaluate $\mathbf{f}^{af\infty} \circ z^*$ on the projective generator $\Gamma^{d,U}$ of \mathcal{P}_d . We obtain

$$\mathbf{f}^{af\infty} \circ z^*(\Gamma^{d,U}) = \mathbf{f}^{af\infty}(h^{U \otimes A_\infty}) = X(U, -) \simeq \Gamma^{d,U} = \mathbf{f}(\Gamma^{d,U}),$$

hence we get a natural in V isomorphism $\mathbf{f}^{af\infty} \circ z^*(\Gamma^{d,U}) \simeq \mathbf{f}(\Gamma^{d,U})$. This, since any object in \mathcal{DP}_d can be represented by a complex of coproducts of $\Gamma^{d,U}$ and the both functor commute with infinite coproducts, gives the first isomorphism. To get the second isomorphism we observe that

$$t^* \circ \mathbf{a}^{af\infty}(F)(V) = \text{Hom}_{\mathcal{F}}(X(V, -), F) = \mathbf{R}\mathbf{a}(F)(V)$$

for any $F \in \mathcal{DF}$.

The second part of the theorem follows from the $\{\mathbf{L}T_X, \mathbf{R}H_X\}$ adjunction and the fact that $\mathbf{R}\phi^*$ is an equivalence.

To get the third part of the theorem we first observe that by [10, Theorem 5.3], $\mathcal{DP}_d^{faf\infty}$ coincides with the smallest full triangulated subcategory of $\mathcal{DP}_d^{af\infty}$ containing representable functors and closed under isomorphisms and direct summands. Then, since

$$\mathbf{a}^{af\infty} \circ \mathbf{f}^{af\infty}(h^{U \otimes A_\infty}) \simeq h^{U \otimes A_\infty}$$

by the very definition of X , we conclude that the unit of the adjunction is an isomorphism on the whole category $\mathcal{DP}_d^{faf\infty}$.

(4) follows formally from (2), (3) (by e.g. [11, Proposition 2.3.1]). □

4. Spectra of strict polynomial functors

In this section we modify the category (of complexes over) \mathcal{P} by formally inverting the Frobenius twist operation. We achieve this goal by a general construction, known from stable homotopy theory, i.e. we consider spectra of complexes of strict polynomial functors. We follow a general approach of Hovey [8] who starts from a Quillen model category \mathcal{C} with a left Quillen endofunctor T and equips its category of spectra with an appropriate Quillen model structure. In order to conform to this context we have to slightly adjust our setup. Namely, although we are mainly interested in \mathcal{P}_d for a fixed $d > 0$, we should also allow strict polynomial functors of other degrees to make the Frobenius twisting functor \mathbf{C} into an endofunctor

(of course it would suffice to consider degrees dp^i). However, the category $\mathcal{P} := \bigoplus_{d>0} \mathcal{P}_d$ is not suitable for our purposes, since the forgetful functor $\mathbf{f} : \mathcal{P} \rightarrow \mathcal{F}$ does not possess the right adjoint. Hence we shall consider the product category

$$\widehat{\mathcal{P}} := \prod_{d>0} \mathcal{P}_d$$

and we recall again that we do not assume that the objects in \mathcal{P}_d are functors taking finite dimensional values. Then we consider the category $\mathcal{K}\widehat{\mathcal{P}}$ of unbounded complexes over $\widehat{\mathcal{P}}$. $\mathcal{K}\widehat{\mathcal{P}}$ can be equipped with the projective Quillen model structure. For the readers convenience we recall this structure. It is described in detail e.g. in [9] for the category of modules over a ring but this description readily generalizes to any AB5 category with a projective generator. In this structure all complexes are fibrant, while the cofibrant ones are those satisfying the “property P” [10]. In our situation the property P boils down to saying that a complex admits a filtration $\{M_j\}_{j \geq 0}$ such that for any $j \geq 0$:

- embedding $M_j \subset M_{j+1}$ splits in \mathcal{A} .
- M_{j+1}/M_j consists of projectives and has trivial differential.

The fibrations are the epimorphisms, the cofibrations are the monomorphisms with cofibrant cokernels which split in the underlying abelian category. The weak equivalences are the quasi-isomorphisms. Then we encounter another technical problem: since \mathbf{C} does not preserve projective objects, it is not a left Quillen functor with respect to the projective Quillen structure on $\mathcal{K}\widehat{\mathcal{P}}$. The simplest way of overcoming this obstacle is by using the following technical fact.

PROPOSITION 4.1. — *There exists a functor $\mathbf{C}' : \mathcal{K}\widehat{\mathcal{P}} \rightarrow \mathcal{K}\widehat{\mathcal{P}}$ such that*

- (1) *\mathbf{C}' is left Quillen functor with respect to the projective model structure on $\mathcal{K}\widehat{\mathcal{P}}$.*
- (2) *There is a natural isomorphism of total left derived functors $\mathbf{LC}' \simeq \mathbf{LC}$.*

Proof. — We can describe \mathbf{C}' by using an explicit construction from [3]. Namely we established in [3, Theorem 5.1(1)] an isomorphism of functors $\mathbf{C} \simeq \mathbf{C}^{af} \circ z^*$ as functors from $\mathcal{D}P_{dp}$ to $\mathcal{D}P_d$. We emphasize that z^* stands here for the functor appearing in [3] (we also remind that we stick to the convention that for exact functors we denote their derived functors by the same letter). Then we recall from [3, Sections 4, 5] that \mathbf{C}^{af} is the derived functor of the composite $T_X \circ (\phi^*)^{-1}$ of two left Quillen functors. Hence, since readily z^* is left Quillen functor, we can define \mathbf{C}' as the composite

$T_X \circ \phi^{-1} \circ z^*$. In order to obtain an equivalence $\mathbf{LC}' \simeq \mathbf{LC}$ we invoke again [3, Theorem 5.1(1)]. Namely, in the proof of [3, Theorem 5.1] we constructed a collection of quasi-isomorphisms

$$\mathbf{C}'(\Gamma^{d,U}) \longrightarrow \mathbf{C}(\Gamma^{d,U})$$

natural in U . This allows us to construct a transformation $\mathbf{C}' \rightarrow \mathbf{C}$ of functors defined on the full subcategory of $\mathcal{K}P_d$ consisting of complexes projective in each degree. Since the cofibrant objects in $\mathcal{K}P_d$ are projective in each degree, we obtain a natural isomorphism $\mathbf{LC}' \simeq \mathbf{LC}$. Finally we extend our construction degreewise to the whole product category $\mathcal{K}\widehat{\mathcal{P}}$. \square

Let $\mathbf{K}' : \mathcal{K}\widehat{\mathcal{P}} \rightarrow \mathcal{K}\widehat{\mathcal{P}}$ be the right adjoint functor to \mathbf{C}' . From now on we consider the pair of adjoint functors $\{\mathbf{C}', \mathbf{K}'\}$ instead of the original adjoint pair $\{\mathbf{C}, \mathbf{K}\}$. In fact, we will rarely refer to the specific construction of \mathbf{C}' given above. In most cases, the properties listed in Proposition 4.1 will be sufficient for our purposes.

Now we apply the machinery of [8] to the category $\mathcal{K}\widehat{\mathcal{P}}$ equipped with the projective model structure with the endofunctor \mathbf{C}' . Namely, we form the category of spectra over $\mathcal{K}\widehat{\mathcal{P}}$.

DEFINITION 4.2. — *We call a collection of complexes $F_i \in \mathcal{K}\widehat{\mathcal{P}}$ and cochain maps $\tau_i : \mathbf{C}'(F_i) \rightarrow F_{i+1}$ for all $i \geq 0$ a spectrum (of complexes of strict polynomial functors). For spectra F_\bullet, G_\bullet we call a collection of cochain maps $\phi_i : F_i \rightarrow G_i$ a map of spectra if $\tau_i \circ \mathbf{C}'(\phi_i) = \phi_{i+1} \circ \tau_i$ for all $i \geq 0$.*

Readily the spectra and maps of spectra inherit from $\mathcal{K}\widehat{\mathcal{P}}$ the structure of DG category (ie. grading comes from the shift in $\mathcal{K}\widehat{\mathcal{P}}$ and not that in spectra). We call this category the category of spectra (of complexes of strict polynomial functors) and denote it by $\mathcal{S}\widehat{\mathcal{P}}$.

For any $F \in \mathcal{K}\widehat{\mathcal{P}}$ we have a spectrum $\mathbf{C}^\infty(F)$ defined by the formula

$$\mathbf{C}^\infty(F)_i := \mathbf{C}'^i(F),$$

where \mathbf{C}'^i is the i th iteration of \mathbf{C}' . The assignment $F \mapsto \mathbf{C}^\infty(F)$ produces the functor

$$\mathbf{C}^\infty : \mathcal{K}\widehat{\mathcal{P}} \longrightarrow \mathcal{S}\widehat{\mathcal{P}}$$

which has the evaluation functor $ev(F_\bullet) := F_0$ as right adjoint.

DEFINITION 4.3. — *We call a spectrum F_\bullet a \mathbf{C} -spectrum if all the maps $\tau_i : \mathbf{C}'(F_i) \rightarrow F_{i+1}$ are quasi-isomorphisms. Similarly, we call a spectrum F_\bullet a \mathbf{K} -spectrum if all the maps $\omega_i : F_i \rightarrow \mathbf{K}'(F_{i+1})$ adjoint to τ_i are quasi-isomorphisms.*

Of course, $\mathbf{C}^\infty(F)$ is always a \mathbf{C} -spectrum. Less trivially, for a Young diagram λ , let $p\lambda$ denote the Young diagram whose rows are those of λ multiplied by p . Then by [2, Proposition 2.1], $\mathbf{K}'(S^{p\lambda}) \simeq S^\lambda$, hence the collection $\{S^{p^i\lambda}\}_{i \geq 0}$ forms a \mathbf{K} -spectrum denoted $S^{p^\bullet\lambda}$. More generally, since by [2, Proposition 4.2], $\mathbf{K}(S_{F_k(\lambda)}[h_k]) = S_\lambda$, any Schur functor S_λ gives rise to a \mathbf{K} -spectrum $\{S_{F_k^i(\lambda)}[h_k^i]\}$ (F_k^i stands for certain combinatorial operation which enlarges Young diagram, see [1]). In fact, any spectrum can be turned into a \mathbf{K} -spectrum by means of the “delooping functor”:

$$\Theta^\infty : \mathcal{S}\widehat{\mathcal{P}} \longrightarrow \mathcal{S}\widehat{\mathcal{P}}$$

given by the formula:

$$\Theta^\infty(F_\bullet)_i := \operatorname{colim}_j \mathbf{K}^{lj}(F_{i+j}).$$

Then, as it is explained in [8, Section 1], for any model category with left Quillen endofunctor one can endow its category of spectra with the obvious model structure, which is (somewhat unfortunately) also called “projective”. In order to avoid confusion with our projective model structure on $\mathcal{K}\widehat{\mathcal{P}}$ we shall call this model structure “levelwise”. Our terminology is justified by the fact that a morphism of spectra $\phi_\bullet : F_\bullet \rightarrow G_\bullet$ is a cofibration (resp. a weak equivalence) if and only if all ϕ_i are cofibrations (resp. weak equivalences). The class of fibrations is determined by the lifting property. Then the final model structure on the category of spectra, called in [8] “the stable model structure” is obtained from the levelwise model structure by the Bousfield localization process with respect to certain class of morphisms (see [8, Section 2]). We summarize below the basic properties of the stable model Quillen structure on $\mathcal{S}\widehat{\mathcal{P}}$.

PROPOSITION 4.4. — *There exists a finitely generated model structure on $\mathcal{S}\widehat{\mathcal{P}}$ with the following properties:*

- (1) *The pair of functors $\{\mathbf{C}^\infty, ev\}$ is a Quillen pair.*
- (2) *The cofibrant objects are the spectra consisting of complexes satisfying the property P with structure maps τ_i monomorphic with cokernels satisfying the property P .*
- (3) *The fibrant objects are the \mathbf{K} -spectra.*
- (4) *A map of spectra which is a levelwise quasi-isomorphism is a weak equivalence.*
- (5) *The degreewise prolongation of \mathbf{C}' on $\mathcal{S}\widehat{\mathcal{P}}$ is a Quillen equivalence with the quasi-inverse being the shift functor.*
- (6) *A natural map $X \rightarrow \Theta^\infty(X)$ is a weak equivalence for any spectrum X , hence Θ^∞ can be chosen as a fibrant replacement functor.*

- (7) The category $Ho(\mathcal{S}\widehat{\mathcal{P}})$ has a structure of triangulated category such that the total derived functor $\mathbf{LC}^\infty = \mathbf{C}^\infty : Ho(\mathcal{S}\widehat{\mathcal{P}}) \rightarrow Ho(\mathcal{S}\widehat{\mathcal{P}})$ is an exact functor.

Remark. — These properties of the model structure on $\mathcal{S}\widehat{\mathcal{P}}$ will be sufficient for our purposes. The first property allows one to compare $\mathcal{K}\widehat{\mathcal{P}}$ with $\mathcal{S}\widehat{\mathcal{P}}$. Properties (2) and (3) provide explicit descriptions of fibrant and cofibrant objects which is essential in making calculations and also shows importance of \mathbf{K} -spectra. In fact one could also extract from [8, Sections 3-4] certain descriptions of fibrations and cofibrations in $\mathcal{S}\widehat{\mathcal{P}}$ but we will not need them. Properties (4) and (5) deal with weak equivalences. Property (5) is central for the whole idea of spectra, since it shows that in $\mathcal{S}\widehat{\mathcal{P}}$ the endofunctor \mathbf{C}' becomes invertible up to homotopy. Property (6) will be crucial in the proof of Theorem 4.6.

Proof. — We apply the machinery of [8] to the projective model structure on $\mathcal{K}\widehat{\mathcal{P}}$ with the functor \mathbf{C}' as the left Quillen endofunctor. The first property is the second assertion in [8, Proposition 1.15]. The second property follows from [8, Proposition 1.14] (we recall that the Bousfield localization preserves cofibrant objects by [8, Theorem 2.2]). The third part is [8, Theorem 3.4]. The fourth property is obvious. The fifth property is [8, Theorem 3.9].

We would like to deduce the sixth property from [8, Corollary 4.11]. For this we need to show that sequential colimits in $\mathcal{K}\widehat{\mathcal{P}}$ commute with finite products and that \mathbf{K}' commutes with sequential colimits. The first fact is well known, since \mathcal{P}_d is equivalent to a module category. In order to show the second fact it suffices to show that \mathbf{K}' , when restricted to $\mathcal{K}P_{dp}$ commutes with sequential colimits. It follows from [3, Section 5] that \mathbf{K}' is explicitly given as

$$\mathbf{K}'(F)(V) = \text{Hom}_{\mathcal{P}_{dp}}(\widetilde{X}(V, -), F),$$

where \widetilde{X} is certain strict polynomial functor in two variables. Then, since \widetilde{X} is finite dimensional, \mathbf{K}' commutes with infinite sums; since $\widetilde{X}(V, -)$ consists of projectives in \mathcal{P}_{dp} , \mathbf{K}' preserves cokernels. Therefore \mathbf{K}' preserves sequential colimits.

In order to equip $Ho(\mathcal{S}\widehat{\mathcal{P}})$ with a structure of triangulated category we recall that by [9, Section 7] the homotopy category $Ho(\mathcal{C})$ of a Quillen model category \mathcal{C} is canonically a triangulated category whenever the “model theoretic suspension functor” on \mathcal{C} [9, Section 6] becomes an equivalence on

$Ho(\mathcal{C})$. The standard construction of the triangulated structure on the derived category of abelian category fits into this formalism. Hence we interpret the standard triangulated structure on $\mathcal{D}\widehat{\mathcal{P}}$ with the shift of complexes as the suspension in model theoretic terms. Then we prolong degreewise the shift functor on $\mathcal{S}\widehat{\mathcal{P}}$ and, since it remains invertible (already on $\mathcal{K}\widehat{\mathcal{P}}$), this allows one to make $Ho(\mathcal{S}\widehat{\mathcal{P}})$ with degreewise model structure into a triangulated category. At last, since the Bousfield localization preserves the cofibrations, it commutes with the suspension functor. This shows that the suspension functor remains a weak equivalence in the stable model structure on $\mathcal{S}\widehat{\mathcal{P}}$. \square

The last property justifies calling $Ho(\mathcal{S}\widehat{\mathcal{P}})$ the derived category of $\mathcal{S}\widehat{\mathcal{P}}$, hence from now on we shall use notation $\mathcal{D}\mathcal{S}\widehat{\mathcal{P}} := Ho(\mathcal{S}\widehat{\mathcal{P}})$.

Now we would like to find inside $\mathcal{D}\mathcal{S}\widehat{\mathcal{P}}$ a subcategory corresponding to $\mathcal{D}P_d$. Of course we have a decomposition of $\mathcal{K}\widehat{\mathcal{P}}$ into the product of $\mathcal{K}P_d$. However, in order to describe the situation for spectra we need more subcategories. Let $\mathbf{N}[\frac{1}{p}] := \{ep^j : e, j \in \mathbf{Z}, e > 0, (e, p) = 1\}$. Then for $ep^j \in \mathbf{N}[\frac{1}{p}]$ we define $\mathcal{S}P_{ep^j}$ to be the full subcategory of $\mathcal{S}P$ consisting of spectra X_\bullet such that $X_i = 0$ for $i < -j$ and $X_i \in \mathcal{K}P_{ep^{i+j}}$ for $i \geq -j$. Then we have a decomposition of DG categories:

$$\mathcal{S}\widehat{\mathcal{P}} = \prod_{d \in \mathbf{N}[\frac{1}{p}]} \mathcal{S}P_d.$$

Now we claim that, roughly speaking, all our constructions carry over to the subcategories $\mathcal{S}P_d$. We recall that we could not apply the ideas of [8] to $\mathcal{S}P_d$ directly, since \mathbf{C}' does not preserve these subcategories.

PROPOSITION 4.5. — *For any $f : X \rightarrow Y$ with $X, Y \in \mathcal{S}P_d$ and $d \in \mathbf{N}[\frac{1}{p}]$, its functorial factorizations into fibrations/cofibrations belong to $\mathcal{S}P_d$. Hence $\mathcal{S}P_d$ has a natural model structure inherited from $\mathcal{S}P$. The homotopy category $\mathcal{D}\mathcal{S}P_d := Ho(\mathcal{S}P_d)$ has a structure of triangulated category and there is an equivalence of triangulated categories:*

$$\mathcal{D}\mathcal{S}\widehat{\mathcal{P}} \simeq \prod_{d \in \mathbf{N}[\frac{1}{p}]} \mathcal{D}\mathcal{S}P_d.$$

Moreover, the prolongation of \mathbf{C}' restricts to a Quillen equivalence producing an equivalence of triangulated categories

$$\mathcal{D}\mathcal{S}P_d \simeq \mathcal{D}\mathcal{S}P_{pd}$$

for any $d \in \mathbf{N}[\frac{1}{p}]$.

Proof. — For $d = d'p^i \in \mathbf{N}[\frac{1}{p}]$ we consider the functor $\pi_d : \widehat{\mathcal{S}}\widehat{\mathcal{P}} \rightarrow \widehat{\mathcal{S}}\widehat{\mathcal{P}}$ given on the j th level by the projection $\mathcal{K}\widehat{\mathcal{P}} \rightarrow \mathcal{K}P_{p^j-i_d} \subset \mathcal{K}\widehat{\mathcal{P}}$ if $j \geq i$ and trivial elsewhere. Then, obviously $im(\pi_d) = \mathcal{S}P_d$ and π_d restricted to $\mathcal{S}P_d$ is the identity functor. Now, since for any $f \in \text{Mor}(\widehat{\mathcal{S}}\widehat{\mathcal{P}})$, $\pi_d(f)$ is a retract of f , π_d preserves the classes of cofibrations, fibrations and weak equivalences. This shows that $\mathcal{S}P_d$ has a natural model structure: we obtain a functorial factorization into cofibrations/fibrations in $\mathcal{S}P_d$ by applying π_d to the factorization in $\widehat{\mathcal{S}}\widehat{\mathcal{P}}$. This also shows that the decomposition $\widehat{\mathcal{S}}\widehat{\mathcal{P}} = \prod_{d \in \mathbf{N}[\frac{1}{p}]} \mathcal{S}P_d$ is an isomorphism of model categories where we consider on the right hand side the product of model structures we have just described. This gives the required decomposition of homotopy categories, which also preserves triangulated structure because π_d commutes with the cone functor.

At last, we observe that by Proposition 4.4(5) the prolongation of \mathbf{C}' is a self-equivalence of $\mathcal{D}\widehat{\mathcal{S}}\widehat{\mathcal{P}}$ which takes $\mathcal{D}\mathcal{S}P_d$ into $\mathcal{D}\mathcal{S}P_{pd}$ which proves the last part of the proposition. □

The properties of the derived category of spectra we have established so far allow us to obtain an analog of the known description of stable homotopy maps between suspension spectra. This theorem is one of the main objectives of this part of the article.

THEOREM 4.6. — *Let $F \in \mathcal{K}P_d$ be finite dimensional and $G_\bullet \in \mathcal{S}P_d$. Then there is a natural in F, G_\bullet isomorphism*

$$\text{Hom}_{\mathcal{D}\mathcal{S}P_d}(\mathbf{C}^\infty(F), G_\bullet) \simeq \text{colim}_i \text{Hom}_{\mathcal{D}P_d}(F, \mathbf{K}^{/i}(G_i)).$$

In particular, for $F, G \in \mathcal{P}_d$ we obtain:

$$\text{Hom}_{\mathcal{D}\mathcal{S}P_d}(\mathbf{C}^\infty(F), \mathbf{C}^\infty(G)[s]) \simeq \text{colim}_i \text{Ext}_{\mathcal{P}_{dp^i}}^s(F^{(i)}, G^{(i)}).$$

Proof. — We shall deduce our theorem from [8, Corollary 4.13], hence we should verify the assumptions of [8, Corollary 4.13]. Let A be a cofibrant replacement of $F \in \mathcal{K}P_d$. Then A and its cylinder are finite objects in $\mathcal{K}\widehat{\mathcal{P}}$ as being finite dimensional complexes of projectives. Additionally, we recall that any spectrum G_\bullet is fibrant in the levelwise model structure. Therefore we can apply [8, Corollary 4.13] to our A and $Y := G_\bullet$ and we obtain the first part of our theorem. The second part is just a special case which we distinguished since it is related to [5]. Indeed, we obtain:

$$\begin{aligned} \text{Hom}_{\mathcal{D}\mathcal{S}P_d}(\mathbf{C}^\infty(F), \mathbf{C}^\infty(G)[s]) &\simeq \text{colim}_i \text{Hom}_{\mathcal{D}P_d}(F, \mathbf{K}^{/i}(\mathbf{C}^{/i}(G)[s])) \\ &\simeq \text{colim}_i \text{Hom}_{\mathcal{D}P_{p^i d}}(\mathbf{C}^{/i}(F), \mathbf{C}^{/i}(G)[s]) \\ &\simeq \text{colim}_i \text{Ext}_{\mathcal{P}_{dp^i}}^s(F^{(i)}, G^{(i)}). \end{aligned} \quad \square$$

In fact, the crucial ingredient of the proof of [8, Corollary 4.13] is the fact that the delooping functor Θ^∞ is a weak equivalence. It is worth mentioning (we will use this fact in Section 6) that in our situation, thanks to the Collapsing Conjecture [2, Theorem 3.2] we have a very explicit description of the functor $\Theta^\infty \mathbf{C}^\infty$. Namely, for $F \in \mathcal{K}\widehat{\mathcal{P}}$, let $F_{A_\infty}^{(\infty)}$ denote the spectrum with

$$\left(F_{A_\infty}^{(\infty)}\right)_i := \mathbf{C}'^i(F)_{A_\infty},$$

where $\mathbf{C}'^i(F)_{A_\infty}$ is meant as precomposing of $\mathbf{C}'^i(F)$ with $- \otimes A_\infty$ which practically means that we also twist A_∞ . This is of great importance because twisting a graded space multiplies degrees of elements by p^i . When defining the map $\tau_i : \mathbf{C}'^i(F)_{A_\infty} \rightarrow \mathbf{C}'^{i+1}(F)_{A_\infty}$ we should also be careful since

$$\begin{aligned} \mathbf{C} \left(\left(F^{(i)}\right)_{A_\infty} \right) (V) &= F \left(V^{(i+1)} \otimes A_\infty^{(i)} \right) \\ &\neq F \left(V^{(i+1)} \otimes A_\infty^{(i+1)} \right) = \left(F^{(i+1)}\right)_{A_\infty} (V) \end{aligned}$$

Using these descriptions, we take τ_i as the map induced by the projection $A_\infty^{(i)} \rightarrow A_\infty^{(i+1)}$. Thus we see that τ_i is not an isomorphism. This is the reason why we do not use the notation $\mathbf{C}^\infty(F_{A_\infty})$ here. On the other hand, $F_{A_\infty}^{(\infty)}$ is a \mathbf{K} -spectrum, since $\mathbf{K}((F^{(i+1)})_{A_\infty}) = (F^{(i)})_{A_\infty}$ and, as it is easy to see, ω_i corresponds just to the identity map. Now we have

PROPOSITION 4.7. — *Let $\{F_\bullet\}$ be a \mathbf{C} -spectrum. Then there is a natural in F_\bullet weak equivalence*

$$(F_0)_{A_\infty}^{(\infty)} \simeq \Theta^\infty(F_\bullet).$$

Proof. — By the Collapsing Conjecture [2, Theorem 3.2] we have

$$\begin{aligned} \Theta^\infty(F_\bullet)_i &\simeq \operatorname{colim}_j \mathbf{K}^j \left(F_0^{(i+j)} \right) \\ &\simeq \operatorname{colim}_j \left(F_0^{(i)} \right)_{A_j} = \left(F_0^{(i)} \right)_{A_\infty} = \left((F_0)_{A_\infty}^{(\infty)} \right)_i. \quad \square \end{aligned}$$

5. Spectra of ordinary functors and factorization

The aim of the present section is to factorize the adjunction $\{\mathbf{f}, \mathbf{Ra}\}$ through $\mathcal{D}SP_d$. In order to compare the categories $\mathcal{D}SP_d$ and $\mathcal{D}F$ we take the following strategy. We introduce an intermediate category $\mathcal{S}F$ of spectra of ordinary functors. Since the Frobenius twist on \mathcal{F} is invertible,

\mathcal{DSF} and \mathcal{DF} are equivalent. Then we compare \mathcal{DSP}_d and \mathcal{DSF} by using functoriality of the construction of spectra.

Let \mathcal{KF} be the category of complexes of objects of \mathcal{F} (we admit unbounded complexes). The Frobenius twist

$$\mathbf{C} : \mathcal{KF} \longrightarrow \mathcal{KF}$$

is a self-equivalence, hence a left Quillen endofunctor for the projective model structure on \mathcal{KF} . Then we introduce the category \mathcal{SF} of spectra over \mathcal{KF} and we equip it with the stable model structure by the Hovey construction analogous to that applied in Section 4 to the category \mathcal{KP} . Now, since \mathbf{C} is a Quillen equivalence, we have:

PROPOSITION 5.1. — *The Quillen pair $\{\mathbf{C}^\infty, ev\}$ is a Quillen equivalence between \mathcal{KF} and \mathcal{SF} (i.e. their derived functors are mutually inverse equivalences between derived categories). Explicitly, we can take $\mathbf{LC}^\infty = \mathbf{C}^\infty$ and $\mathbf{Rev}(F_\bullet) = \text{colim}_j F_{nj}$ where $|\mathbf{k}| = p^n$.*

Proof. — The fact that we have a Quillen equivalence follows from [8, Theorem 5.1]. The fact that $\mathbf{LC}^\infty = \mathbf{C}^\infty$ is a consequence of the fact that \mathbf{C}^∞ preserves all weak equivalences. Then, in general, $\mathbf{Rev} = ev \circ R$ where R is a fibrant replacement functor. But we observe that in the category \mathcal{F} , the right adjoint functor to \mathbf{C} is just $\mathbf{C}^{-1} = \mathbf{C}^{n-1}$. Therefore, obviously $\Theta^\infty(F) \simeq F$ for any $F \in \mathcal{SF}$ and we can take Θ^∞ as R . Finally, we have

$$ev \circ \Theta^\infty(F_\bullet) = \text{colim}_j \mathbf{C}^{n_j-j}(F_j) \simeq \text{colim}_j F_{nj}. \quad \square$$

Now we are going to compare the categories \mathcal{DSP} and \mathcal{DF} . We introduce a temporary notation. Let us denote by $\mathbf{a}_d : \mathcal{KF} \rightarrow \mathcal{KP}_d$ the right adjoint functor to the forgetful functor $\mathbf{f} : \mathcal{KP}_d \rightarrow \mathcal{KF}$. Then it is easy to see that the functor

$$\widehat{\mathbf{a}} : \mathcal{KF} \longrightarrow \widehat{\mathcal{KP}}$$

given as the product $\widehat{\mathbf{a}}(F) := (\mathbf{a}_1, \dots, \mathbf{a}_d, \dots)$ is right adjoint to the functor

$$\widehat{\mathbf{f}} : \widehat{\mathcal{KP}} \longrightarrow \mathcal{KF}$$

sending $(F_1, \dots, F_d, \dots) \in \widehat{\mathcal{KP}}$ to the direct sum $\bigoplus_{d \geq 1} \mathbf{f}(F_d)$. Moreover, by Proposition 4.1, we have a natural transformation

$$\widehat{\mathbf{f}} \circ \mathbf{C}' \longrightarrow \mathbf{C} \circ \widehat{\mathbf{f}}$$

of functors from $\widehat{\mathcal{KP}}$ to \mathcal{KF} , which is a weak equivalence. Therefore, by [8, Proposition 5.5] we get the Quillen pair $\{\widehat{\mathbf{Sf}}, \widehat{\mathbf{S\hat{a}}}\}$ between the spectra categories \mathcal{SP} and \mathcal{SF} such that

$$\widehat{\mathbf{Sf}} \circ \mathbf{C}^\infty \simeq \mathbf{C}^\infty \circ \widehat{\mathbf{f}}.$$

Now we are ready for defining the adjunction between $\mathcal{D}\widehat{\mathcal{P}}$ and $\mathcal{D}F$. Let

$$\mathbf{f}^{st} : \mathcal{D}S\widehat{\mathcal{P}} \longrightarrow \mathcal{D}SF$$

be given as $ev \circ \Theta^\infty \circ \mathbf{L}S\widehat{\mathbf{f}}$ and

$$\mathbf{a}^{st} : \mathcal{D}SF \longrightarrow \mathcal{D}S\widehat{\mathcal{P}}$$

given as $\mathbf{R}S\widehat{\mathbf{a}} \circ \mathbf{C}^\infty$. Let $\mathcal{D}P_d^b$ be the full subcategory of $\mathcal{D}P_d$ consisting of finite dimensional complexes and let $\mathcal{D}P_d^{st}$ be the smallest full triangulated subcategory of $\mathcal{D}P_d$ containing $\mathbf{C}^\infty(\mathcal{D}P_d^b)$ and closed under isomorphisms and direct summands. Our terminology here refers to stable homotopy theory where the stable category of Spanier–Whitehead can be characterized in a similar manner as a full subcategory of the category of spectra. Then we have

THEOREM 5.2. — *The functors \mathbf{f}^{st} and \mathbf{a}^{st} satisfy the following properties:*

- (1) *The functor \mathbf{f}^{st} is left adjoint to \mathbf{a}^{st} .*
- (2) *There are isomorphisms of functors between $\mathcal{D}\widehat{\mathcal{P}}$ and $\mathcal{D}F$*

$$\begin{aligned} \mathbf{f}^{st} \circ \mathbf{C}^\infty &\simeq \widehat{\mathbf{f}}, \\ \mathbf{R}ev \circ \mathbf{a}^{st} &\simeq \mathbf{R}\widehat{\mathbf{a}}. \end{aligned}$$

- (3) *Let $d \leq |\mathbf{k}|$. Then the unit map $\text{Id} \rightarrow \mathbf{a}^{st} \circ \mathbf{f}^{st}$ is an isomorphism on the subcategory $\mathcal{D}P_d^{st}$.*
- (4) *Let $d \leq |\mathbf{k}|$. Then \mathbf{f}^{st} restricted to $\mathcal{D}P_d^{st}$ is fully faithful.*

Proof. — The first part follows from the facts that $\{S\mathbf{f}, S\mathbf{a}\}$ is a Quillen pair and Proposition 5.1. For the second part we recall that

$$S\widehat{\mathbf{f}} \circ \mathbf{C}^\infty \simeq \mathbf{C}^\infty \circ \widehat{\mathbf{f}}.$$

Therefore we obtain

$$\mathbf{f}^{st} \circ \mathbf{C}^\infty \simeq \mathbf{R}ev \circ S\widehat{\mathbf{f}} \circ \mathbf{C}^\infty \simeq \mathbf{R}ev \circ \mathbf{C}^\infty \circ \widehat{\mathbf{f}} \simeq \widehat{\mathbf{f}}.$$

In order to obtain the second isomorphism we recall, that since $S\widehat{\mathbf{a}}$ is just degreewise prolongation of \mathbf{a} by [8, Lemma 5.3], it commutes with \mathbf{C}^∞ . Hence we get

$$\mathbf{R}ev \circ \mathbf{a}^{st} \simeq \mathbf{R}ev \circ S\widehat{\mathbf{a}} \circ \mathbf{C}^\infty \simeq \mathbf{R}ev \circ \mathbf{C}^\infty \circ \widehat{\mathbf{a}} \simeq \mathbf{R}\widehat{\mathbf{a}}.$$

Parts (3) and (4) are equivalent. Since $\mathcal{D}P_d^{st}$ is generated as triangulated category with direct summands by $\mathbf{C}^\infty(\mathcal{D}P_d^b)$, it suffices to show that the

functor \mathbf{f}^{st} restricted to the objects of the form $\mathbf{C}^\infty(F)$ for $F \in \mathcal{D}P_d^{st}$ is fully faithful. To this end let us take $F, G \in \mathcal{P}_d$. Then by Theorem 4.6

$$\mathrm{Hom}_{\mathcal{D}SP_d}(\mathbf{C}^\infty(F), \mathbf{C}^\infty(G)[s]) \simeq \mathrm{colim}_i \mathrm{Ext}_{\mathcal{P}_{dp^i}}^s(F^{(i)}, G^{(i)}).$$

On the other hand

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}SF}(\mathbf{f}^{st}(\mathbf{C}^\infty(F)), \mathbf{f}^{st}(\mathbf{C}^\infty(G)[s])) &\simeq \mathrm{Hom}_{\mathcal{D}SF}(\mathbf{f}(F), \mathbf{f}(G)[s]) \\ &\simeq \mathrm{Ext}_{\mathcal{F}}^s(F, G). \end{aligned}$$

Therefore our assertion follows from [5, Theorem 3.10]. □

Remark. — The functor \mathbf{f}^{st} has an intriguing extra feature. We recall from Section 4 the fibrant spectra $S^{p^\bullet \lambda} := \{S^{p^i \lambda}\}$. Then we see that

$$\mathbf{f}^{st}(S^{p^\bullet \lambda}) = \mathrm{colim}_i S^{p^i \lambda(-i)}$$

which is nothing but the product of the Carlsson functors whose injectivity was shown by Kuhn [12]. Thus we see that \mathbf{f}^{st} preserves some important fibrant objects in contrast to the fact that the original forgetful functor $\mathbf{f} : \mathcal{D}P_d \rightarrow \mathcal{F}$ does not preserve injectives. This suggests possibility of fully reconstructing $\mathcal{D}F$ from some categories of algebrogeometric origin. We hope to extend this observation in a future work.

6. Comparison of spectra and ∞ -affine functors

In this section we construct a functor $\gamma : \mathcal{D}P_d^{af\infty} \rightarrow \mathcal{D}SP_d$ which is a full embedding and is compatible with our previous constructions.

Let $A_i := \mathbf{k}[x_1, x_2, \dots, x_i]/(x_1^p, x_2^p, \dots, x_i^p)$ for $|x_j| = 2p^j$ (we allow here also $A_0 := \mathbf{k}$). We consider the categories $\Gamma^d \mathcal{V}_{A_i}, \mathcal{P}_d^{af_i}, \mathcal{D}P_d^{af_i}$, analogous to the notions introduced in Section 2. In particular $\mathcal{P}_d^{af_0}$ means just \mathcal{P}_d^{gr} , the graded counterpart of \mathcal{P}_d (of course $\mathcal{D}P_d^{gr} \simeq \mathcal{D}P_d$). The theory of “ i -affine functors” is parallel but simpler, since A_i is finite dimensional, to that of “ ∞ -affine functors”. In particular, for $j > i$, we have the functors

$$t_{j,i}^* : \mathcal{D}P_d^{af_j} \longrightarrow \mathcal{D}P_d^{af_i}$$

induced by the embeddings $A_i \subset A_j$ and their adjoints $z_{j,i}^*$. We also consider the infinite variants:

$$t_{\infty,i}^* : \mathcal{D}P_d^{af\infty} \longrightarrow \mathcal{D}P_d^{af_i}$$

and their adjoints $z_{\infty,i}^*$. We have also the adjunction $\{\mathbf{C}^{af_i}, \mathbf{K}^{af_i}\}$ between the categories $\mathcal{D}P_d^{af_i}$ and $\mathcal{D}P_{dp^i}$.

LEMMA 6.1. — We have the following isomorphisms of functors:

- $t_{\infty,i}^* \simeq t_{i+1,i}^* \circ t_{\infty,i}^*$
- $\mathbf{C}^{af_i} \circ t_{i+1,i}^* \simeq \mathbf{K} \circ \mathbf{C}^{af_{i+1}}$

Proof. — The first isomorphism is obvious. In order to get the second one we evaluate both sides on the cofibrant generator $h^{U \otimes A_{i+1}}$ of $\mathcal{D}P_d^{af_{i+1}}$. Since $h^{U \otimes A_{i+1}} = z^*(\Gamma^{d,U})$, we have

$$\begin{aligned} \mathbf{K}(\mathbf{C}^{af_{i+1}}(h^{U \otimes A_{i+1}})) &= \mathbf{K}(\mathbf{K}^{af_{i+1}}(z^*(\Gamma^{d,U}))) \\ &= \mathbf{K}(\mathbf{C}^{i+1}(\Gamma^{d,U})) = (\Gamma^{d,U})_{A_1}^{(i+1)}, \end{aligned}$$

and we recall that

$$(\Gamma^{d,U})_{A_1}^{(i+1)}(V) := \Gamma^d(V^{(i)} \otimes A_1^{(i+1)} \otimes U^*).$$

On the other hand we have

$$\mathbf{K}^{af_i}(t_{i+1,i}^*(h^{U \otimes A_{i+1}})) = \mathbf{K}^{af_i}(h^{U \otimes A_1^{i+1} \otimes A_i}) = (\Gamma^{d,U})_{A_1}^{(i+1)}. \quad \square$$

Now for $F \in \mathcal{P}_d^{af_\infty}$ we consider the collection $\tilde{\gamma}(F)_i := \{\mathbf{C}^{af_i}(t_{\infty,i}^*(F))\}$ for $i \geq 0$. I claim that $\tilde{\gamma}(F)$ may be equipped with a structure of \mathbf{K} -spectrum. Indeed, by using the second and first part of Lemma 6.1 we obtain isomorphisms:

$$\begin{aligned} \mathbf{K}(\tilde{\gamma}(F)_{i+1}) &= \mathbf{K}(\mathbf{C}^{af_{i+1}}(t_{\infty,i+1}^*(F))) \\ &\simeq \mathbf{C}^{af_i}(t_{i+1,i}^*(t_{\infty,i+1}^*(F))) \\ &\simeq \mathbf{C}^{af_i}(t_{\infty,i}^*(F)) = \tilde{\gamma}(F)_i. \end{aligned}$$

Thus $\tilde{\gamma}$ is a functor from $\mathcal{P}_d^{af_\infty}$ to $\mathcal{S}P_d$. Then we extend $\tilde{\gamma}$ to the functor between derived categories:

$$\gamma : \mathcal{D}P_d^{af_\infty} \longrightarrow \mathcal{D}\mathcal{S}P_d.$$

by applying $\tilde{\gamma}$ to the cofibrant objects in $\mathcal{K}P_d^{af_\infty}$.

THEOREM 6.2. — The functor γ satisfies the following properties:

- (1) There are isomorphisms of functors $\gamma \circ z^* \simeq \mathbf{C}^\infty$, $\mathbf{f}^{af_\infty} \simeq \mathbf{f}^{st} \circ \gamma$.
- (2) γ is a full embedding and it restricts to an equivalence $\mathcal{D}P_d^{af_\infty} \simeq \mathcal{D}P_d^{st}$.

Remark 6.3. — Theorem 6.2 provides a comparison between the ideas of Sections 2–3 and Sections 4–5. The first part shows that γ is compatible with all our previous constructions. The crucial is the second part, which shows that, in a sense, $\mathcal{P}_d^{af_\infty}$ is a more economical construction than $\mathcal{S}P_d$, but they become equivalent when restricted to the subcategories consisting of finite objects.

Proof of Theorem 6.2. — First we observe that $t_{\infty,i}^* \circ z^* = (z_{i,0}^*)_{A_\infty^{(i)}}$. Hence for $F \in \mathcal{K}P_d$ we obtain

$$\begin{aligned} \gamma(z^*(F)) &= \{\mathbf{C}^{af_i}(t_{\infty,i}^*(z^*(F)))\} = \{\mathbf{C}^{af_i}(z_{i,0}^*(F_{A_\infty^{(i)}}))\} \\ &= \{\mathbf{C}^i(F_{A_\infty^{(i)}})\} = F_{A_\infty}^{(\infty)}. \end{aligned}$$

Now we recall that by Proposition 4.7 the spectra $F_{A_\infty}^{(\infty)}$ and $\mathbf{C}^\infty(F)$ are naturally stably quasi-isomorphic, hence equivalent in $\mathcal{D}SP_d$, which shows the first isomorphism.

In order to establish the second isomorphism we evaluate the two sides on the generator $h^{U \otimes A_\infty}$. On the one hand we have

$$\mathbf{f}^{af_\infty}(h^{U \otimes A_\infty}) = \mathbf{f}(\Gamma^{d,U}).$$

On the other hand:

$$\mathbf{f}^{st}(\gamma(h^{U \otimes A_\infty})) = \mathbf{f}^{st}((\Gamma^{d,U})_{A_\infty}^{(\infty)}) \simeq \mathbf{f}^{st}(\mathbf{C}^\infty(\Gamma^{d,U})) = \mathbf{f}(\Gamma^{d,U}).$$

In order to obtain the second part of Theorem 6.2 we recall that $s h^{U \otimes A_\infty} = z^*(\Gamma^{d,U})$, thus we have $\gamma(h^{U \otimes A_\infty}) \simeq (\Gamma^{d,U})_{A_\infty}^{(\infty)} \simeq \mathbf{C}^\infty(\Gamma^{d,U})$. Hence, since $\mathbf{C}^\infty(\Gamma^{d,U})$ form a set of finite generators of $\mathcal{D}P_d^{st}$, by [10, Lemma 4.2], it suffices to show that γ induces bijections:

$$\text{Hom}_{\mathcal{D}P_d^{af_\infty}}(h^{U \otimes A_\infty}, h^{W \otimes A_\infty}[j]) \simeq \text{Hom}_{\mathcal{D}SP_d}((\Gamma^{d,U})_{A_\infty}^{(\infty)}, (\Gamma^{d,W}[j])_{A_\infty}^{(\infty)})$$

for all spaces U, W and shifts j . It is easy to see that the source and target are abstractly isomorphic. However, since γ uses $t_{\infty,i}^*$, it may appear that it can kill morphisms. For this reason, let us look carefully how γ acts on morphisms. First of all, since $h^{U \otimes A_\infty}, h^{W \otimes A_\infty}$ are cofibrant, we have:

$$\begin{aligned} \text{Hom}_{\mathcal{D}P_d^{af_\infty}}(h^{U \otimes A_\infty}, h^{W \otimes A_\infty}[*]) &\simeq \text{Hom}_{\mathcal{P}_d^{af_\infty}}(h^{U \otimes A_\infty}, h^{W \otimes A_\infty}) \\ &\simeq \text{Hom}_{\mathcal{P}_d^{af_\infty}}(z^*(\Gamma^{d,U}), z^*(\Gamma^{d,W})). \end{aligned}$$

Now we recall that by Proposition 4.7 $(F)_{A_\infty}^\infty \simeq \Theta^\infty(\mathbf{C}^\infty(F))$. Thus by general theory of spectra (and the cofibrance of $\mathbf{C}^\infty(\Gamma^{d,U})$) we have:

$$\begin{aligned} \text{Hom}_{\mathcal{D}SP_d} \left((\Gamma^{d,U})_{A_\infty}^{(\infty)}, (\Gamma^{d,W}[*])_{A_\infty}^{(\infty)} \right) &\simeq \text{Hom}_{\mathcal{D}SP_d} \left(\mathbf{C}^\infty(\Gamma^{d,U}), (\Gamma^{d,W}[*])_{A_\infty}^{(\infty)} \right) \\ &\simeq \text{Hom}_{SP_d} \left(\mathbf{C}^\infty(\Gamma^{d,U}), (\Gamma^{d,W})_{A_\infty}^{(\infty)} \right) \\ &\simeq \text{Hom}_{\mathcal{P}_d} \left(\Gamma^{d,U}, (\Gamma^{d,W})_{A_\infty} \right) \\ &\simeq \text{Hom}_{\mathcal{P}_d} \left(\Gamma^{d,U}, t^* \circ z^*(\Gamma^{d,W}) \right). \end{aligned}$$

We point out that the first bijection is in general induced by the canonical map of spectra $\mathbf{C}^\infty(F) \rightarrow \Theta^\infty(\mathbf{C}^\infty(F))$ which on the zeroth level of spectra in our example may be identified with the unit map $\Gamma^{d,U} \rightarrow t^* \circ z^*(\Gamma^{d,U})$ by Proposition 4.7. Therefore, under these identifications, the action of γ on the morphisms can be described as the composite:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}_d^{af\infty}}(z^*(\Gamma^{d,U}), z^*(\Gamma^{d,W})) \\ \longrightarrow \mathrm{Hom}_{\mathcal{P}_d^{af\infty}}(t^* \circ z^*(\Gamma^{d,U}), t^* \circ z^*(\Gamma^{d,W})) \\ \longrightarrow \mathrm{Hom}_{\mathcal{P}_d}(\Gamma^{d,U}, t^* \circ z^*(\Gamma^{d,W})) \end{aligned}$$

where the first map is applying t^* and the second one is induced by the aforementioned unit map. This composite coincides with the adjunction isomorphism by general theory of adjunctions. \square

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