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GOOD SEQUENCES WITH UNCOUNTABLE SPECTRUM AND SINGULAR ASYMPTOTIC DISTRIBUTION

by Christophe CUNY & François PARREAU

Abstract. — We construct a good sequence with uncountable spectrum. The construction also allows us to exhibit a continuous and singular probability measure representable by a good sequence in the sense of the recent work of Lesigne, Quas, Rosenblatt and Wierdl.

Résumé. — Nous construisons une bonne suite à spectre non dénombrable. La construction nous permet également d'exhiber une probabilité continue singulière représentable par une bonne suite au sens du travail récent de Lesigne, Quas, Rosenblatt et Wierdl.

1. Good sequences with uncountable spectrum

Let $S = (s_n)_{n\geqslant 1}$ be an increasing sequence of positive integers. We say that S is a *good sequence* if the following limit exists for every $\lambda \in \mathbb{S}^1$ ($\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$)

$$c(\lambda) = c_S(\lambda) \coloneqq \lim_{N\to+\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda^{s_n} . \tag{1.1}$$

Equivalently, S is good if, for every $\lambda \in \mathbb{S}^1$, the following limit exists

$$\lim_{N\to+\infty} \frac{1}{\pi_S(N)} \sum_{1\leqslant k\leqslant N,\, k\in S} \lambda^k, \tag{1.2}$$

where $\pi_S(N) = \#\,(S \cap [1, N])$.

Good sequences have been studied by many authors. See for instance Rosenblatt and Wierdl [13] who introduced that notion, Rosenblatt [12],

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Boshernitzan, Kolesnik, Quas and Wierdl [3], Lemańczyk, Lesigne, Parreau, Volný and Wierdl [9] or Cuny, Eisner and Farkas [4].

Given a good sequence S, we define its spectrum as the set

$$\Lambda_S := \{\lambda \in \mathbb{S}^1 \,:\, c(\lambda) \neq 0\}\,. \tag{1.3}$$

By [13, Theorem 2.22] (due to Weyl), for any good sequence S, Λ_S has Lebesgue measure 0. If moreover S has positive upper density, i.e. satisfies $\limsup_{N\to+\infty}(\pi_S(N)/N) > 0$, then Λ_S is countable. See [4, Proposition 2.12 and Corollary 2.13] for a proof based on a result of Boshernitzan published in [12]. See also [8] for more general results of that type.

On another hand, up to our knowledge, no good sequence with uncountable spectrum is known.

In [4], good sequences have been studied in connection with Wiener's lemma. In particular, the authors of [4] obtained the following results for good sequences, see their Proposition 2.6 and Theorem 2.10. Recall that if τ is a finite measure on $\mathbb{S}^1$, then $\widehat{\tau}(n) = \int_{\mathbb{S}^1} \lambda^n \,\mathrm{d}\tau(\lambda)$, for every $n \in \mathbb{Z}$.

Proposition 1.1. — *Let $S = (s_n)_{n\geqslant 1}$ be a good sequence. Then, for every probability measure μ on $\mathbb{S}^1$, we have*

$$\frac{1}{N}\sum_{n=1}^{N} |\widehat{\mu}(s_n)|^2 \underset{N\to+\infty}{\longrightarrow} \int_{(\mathbb{S}^1)^2} c(\lambda_1\overline{\lambda}_2)\,\mathrm{d}\mu(\lambda_1)\,\mathrm{d}\mu(\lambda_2)\,.$$

In particular, if S has countable spectrum and μ is continuous

$$\frac{1}{N}\sum_{n=1}^{N} |\widehat{\mu}(s_n)|^2 \underset{N\to+\infty}{\longrightarrow} 0\,. \tag{1.4}$$

Remark. — (1.4) implies that $\widehat{\mu}(s_n)$ converges in density to 0, by the Koopman-von Neumann Lemma (see e.g. [4, Lemma 2.1]).

The above considerations yield and put into perspective the following question: does there exist a good sequence with uncountable spectrum?

We answer positively to that question below. To state the result, we need some more notation.

Let $(m_j)_{j\geqslant 1}$ be an increasing sequence of positive integers such that $m_{j+1}/m_j \geqslant 3$ for every $j \geqslant 1$.

We associate with $(m_j)_{j\geqslant 1}$ the sequence $S = (s_n)_{n\geqslant 1}$ made out of the integers (an empty sum is assumed to be 0)

$$\left\{ m_k + \sum_{1\leqslant j\leqslant k-1} \omega_j m_j \,:\, k \geqslant 1,\, (\omega_1,\ldots,\omega_{k-1}) \in \{-1,0,1\}^{k-1} \right\} \tag{1.5}$$

in increasing order. Notice that our assumption on $(m_j)_{j\geqslant 1}$ implies that all the integers in (1.5) are positive and distinct.

Denote by $\|\cdot\|$ the distance to the nearest integer: $\|t\| := \min\{|m-t| : m\in\mathbb{Z}\}$ for every $t\in\mathbb{R}$.

THEOREM 1.2. — *Let $(m_j)_{j\geqslant 1}$ be an increasing sequence of positive integers such that $m_{j+1}/m_j \geqslant 3$ for every $j\geqslant 1$, and define S as above. Then S is a good sequence and*

$$\Lambda := \left\{ \mathrm{e}^{2\mathrm{i}\pi\theta} \,:\, \theta\in[0,1)\backslash\mathbb{Q},\ \sum_{j\geqslant 1}\|m_j\theta\|^2 < \infty \right\} \subset \Lambda_S\,. \tag{1.6}$$

Proof. — For every $k\geqslant 1$, consider the following set of integers

$$M_k := \left\{ \sum_{1\leqslant j\leqslant k-1} \omega_j m_j \,:\, (\omega_\ell)_{1\leqslant \ell<k} \in \{-1,0,1\}^{k-1} \right\}.$$

For every $k\geqslant 1$ and every $\theta\in[0,1)$, set

$$L_k(\theta) := \prod_{1\leqslant j\leqslant k-1} \frac{1}{3}(1+2\cos(2\pi m_j\theta)) \tag{1.7}$$

$$= \frac{1}{3^{k-1}} \prod_{1\leqslant j\leqslant k-1} (1+\mathrm{e}^{-2\mathrm{i}\pi m_j\theta}+\mathrm{e}^{2\mathrm{i}\pi m_j\theta})$$

$$= \frac{1}{3^{k-1}} \sum_{x\in M_k} \mathrm{e}^{2\mathrm{i}\pi x\theta}\,. \tag{1.8}$$

Let $\theta\in[0,1)$. As $-1/3\leqslant(1+2\cos(2\pi\theta m_j))/3\leqslant 1$ for all j, if $1+2\cos(2\pi\theta m_j)$ is infinitely often non positive, then $(L_k(\theta))_{k\geqslant 1}$ converges to 0.

Assume now that $1+2\cos(2\pi\theta m_j)>0$ for $j\geqslant J$, for some integer J. Then, the convergence of $(L_k(\theta))_{k\geqslant 1}$ follows from the convergence of $(\prod_{j=J}^{k}(1+2\cos(2\pi\theta m_j))/3)_{k\geqslant J}$ which is clear since we have an infinite product of positive terms less than or equal to 1. Moreover this infinite product converges, i.e. the limit is non-zero, if and only if

$$\sum_{k=J}^{\infty}\left[1-\frac{1}{3}(1+2\cos(2\pi m_k\theta))\right] = \sum_{k=J}^{\infty}\frac{2}{3}\big(1-\cos(2\pi m_k\theta)\big) < +\infty,$$

which is equivalent to $\sum_{k=J}^{\infty}\|m_k\theta\|^2<+\infty$.

If $\mathrm{e}^{2\mathrm{i}\pi\theta}$ is in the set Λ defined by (1.6) the above condition is satisfied and moreover, as θ is then irrational, the product $\prod_{j=1}^{J-1}(1+2\cos(2\pi\theta m_j))/3$ does not vanish.

Hence in any case $(L_k(\theta))_{k\geqslant 1}$ converges, say to $L(\theta)$, and L does not vanish on Λ.

We wish to prove that $(\frac{1}{N}\sum_{n=1}^N \mathrm{e}^{2\mathrm{i}\pi s_n\theta})_{N\geqslant 1}$ converges to $L(\theta)$ for every $\theta\in[0,1)$.

Let $N\geqslant 1$. Since $(s_n)_{n\geqslant 1}$ is the increasing sequence made out of the numbers given by (1.5), we can write $s_{N+1}=m_{k_N}+\sum_{1\leqslant j\leqslant k_N-1}\omega_j(N)m_j$.

The integers $s_1,\dots,s_N$ may be split into consecutive blocks

$$m_1+M_1,\dots,m_{k_N-1}+M_{k_N-1},\, W_N,$$

where $W_N=\{\ell\in m_{k_N}+M_{k_N} \,:\, \ell\leqslant s_N\}$.

As each block M_k consists in 3^{k-1} integers, we have

$$\frac{3^{k_N-1}-1}{2}\leqslant N<\frac{3^{k_N}-1}{2}\,. \tag{1.9}$$

We may furthermore split W_N into translates of blocks M_k. Namely, if $\omega_{k_N-1}(N)\neq -1$, then W_N begins with $m_{k_N}-m_{k_N-1}+M_{k_N-1}$, if $\omega_{k_N-1}(N)=1$ another block $m_{k_N}+0\times m_{k_N-1}+M_{k_N-1}$ follows, and so on. More precisely, W_N is the disjoint union

$$W_N=\bigcup_{1\leqslant j\leqslant k_N-1}\ \bigcup_{\omega<\omega_j(N)}\left(m_{k_N}+\sum_{\ell=j+1}^{k_N-1}\omega_\ell(N)m_\ell+\omega m_j+M_j\right).$$

Hence, by (1.8),

$$\begin{aligned}\sum_{n=1}^N \mathrm{e}^{2\mathrm{i}\pi s_n\theta}\\ &=\sum_{j=1}^{k_N-1}3^{j-1}\,\mathrm{e}^{2\mathrm{i}\pi m_j\theta}\,L_j(\theta)+\sum_{j=1}^{k_N-1}\sum_{\omega<\omega_j(N)}3^{j-1}\,\mathrm{e}^{2\mathrm{i}\pi u_j(\omega)\theta}\,L_j(\theta),\end{aligned} \tag{1.10}$$

where $u_j(\omega)=m_{k_N}+\sum_{\ell=j+1}^{k_N-1}\omega_\ell(N)m_\ell+\omega m_j$.

Let us first assume that $L(\theta)=0$, that is $L_j(\theta)\to 0$ as $j\to+\infty$. Then we have

$$\frac{1}{N}\left|\sum_{n=1}^N \mathrm{e}^{2\mathrm{i}\pi s_n\theta}\right|\leqslant\frac{1}{N}\sum_{j=1}^{k_N-1}3^{j-1}|L_j(\theta)|+\frac{1}{N}\sum_{j=1}^{k_N-1}\sum_{\omega<\omega_j(N)}3^{j-1}|L_j(\theta)|\underset{N\to+\infty}{\longrightarrow}0,$$

where the convergence follows from (1.9).

Assume now that $L(\theta)\neq 0$. Then $\mathrm{e}^{2\mathrm{i}\pi m_n\theta}\underset{n\to+\infty}{\longrightarrow}1$.

Fix $\varepsilon>0$. Let $r\geqslant 1$ be such that $\mathrm{e}^{-r}<\varepsilon$, and let $d\geqslant 1$ be such that $|1-\mathrm{e}^{2\mathrm{i}\pi m_j\theta}|<\varepsilon/(r+1)$ and $|L(\theta)-L_j(\theta)|<\varepsilon$ for every $j\geqslant d$.

For every N such that $k_N \geqslant d+r$, we have on one hand, since $(L_n(\theta))_{n\geqslant 1}$ is bounded by 1,

$$\sum_{j=1}^{k_N-r-1} 3^{j-1}|\,\mathrm{e}^{2\mathrm{i}\pi m_j\theta}\,L_j(\theta)-L(\theta)| + \sum_{j=1}^{k_N-r-1}\sum_{\omega<\omega_j(N)} 3^{j-1}|\,\mathrm{e}^{2\mathrm{i}\pi u_j(\omega)\theta}\,L_j(\theta)-L(\theta)| \leqslant \sum_{j=1}^{k_N-r-1} 3^{j-1}[2+2\times 2] \leqslant 3^{k_N-r} < 3^{k_N}\varepsilon. \tag{1.11}$$

And on the other hand, as $k_N-r\geqslant d$, when $k_N-r\leqslant j\leqslant k_N$ we have $|1-\mathrm{e}^{2\mathrm{i}\pi m_j\theta}\,| < \varepsilon/(r+1)$ and

$$|1-\mathrm{e}^{2\mathrm{i}\pi u_j(\omega)\theta}\,| \leqslant \sum_{\ell=k_N-r}^{k_N} |1-\mathrm{e}^{2\mathrm{i}\pi m_\ell\theta}\,| < \varepsilon$$

for every choice of ω. So,

$$\sum_{j=k_N-r}^{k_N-1} 3^{j-1}|\,\mathrm{e}^{2\mathrm{i}\pi m_j\theta}\,L_j(\theta)-L(\theta)| + \sum_{j=k_N-r}^{k_N-1}\sum_{\omega<\omega_j(N)} 3^{j-1}|\,\mathrm{e}^{2\mathrm{i}\pi u_j(\omega)\theta}\,L_j(\theta)-L(\theta)| < \sum_{j=k_N-r}^{k_N-1} 3^{j-1}[2\varepsilon+2\times 2\varepsilon] < 3^{k_N}\varepsilon. \tag{1.12}$$

Gathering (1.11) and (1.12), it follows from (1.10) that

$$\left|\sum_{n=1}^{N}\mathrm{e}^{2\mathrm{i}\pi s_n\theta} - NL(\theta)\right| < 2\cdot 3^{k_N}\varepsilon.$$

Finally, in view of (1.9),

$$\limsup_{N\to+\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\mathrm{e}^{2\mathrm{i}\pi s_n\theta} - L(\theta)\right| \leqslant 12\varepsilon\,,$$

and the announced result follows since ε may be chosen arbitrarily small. $\square$

It follows from Theorem 1.2 that, in order to produce a good sequence with uncountable spectrum, it is sufficient to exhibit an increasing sequence of integers $(m_j)_{j\geqslant 1}$ with $m_{j+1}/m_j \geqslant 3$ for every $j \geqslant 1$ and such that the subgroup of $\mathbb{S}^1$

$$H_2 = H_2((m_j)_{j\geqslant 1}) := \left\{ \mathrm{e}^{2\mathrm{i}\pi\theta} \, : \, \theta \in [0,1), \sum_{j\geqslant 1} \|m_j\theta\|^2 < \infty \right\} \tag{1.13}$$

be uncountable.

It turns out that those type of subgroups have been studied in [7] (see also [11] and [1]).

A similar subgroup, defined by $H_1 := \{\mathrm{e}^{2\mathrm{i}\pi\theta} \, : \, \theta \in [0,1), \sum_{j\geqslant 1} \|m_j\theta\| < \infty\}$, studied in [7] in connection with H_2, has also been considered by Erdős and Taylor [5] and, in connection with IP-rigidity, by Bergelson & al. [2] and Aaronson & al. [1].

In the above papers, sufficient conditions have been obtained for H_2 or H_1 to be uncountable.

To state the results concerning H_2 subgroups, we shall need a strengthening of the lacunarity condition. We say that $(m_j)_{j\geqslant 1}$ satisfies assumption (A) if one of the conditions (A_1) or (A_2) below is satisfied:

$$\sum_{j\geqslant 1} \left(\frac{m_j}{m_{j+1}} \right)^2 < \infty \tag{A_1}$$

$$\forall j \geqslant 1 \quad m_j | m_{j+1} \quad \text{and} \quad m_{j+1}/m_j \underset{j\to+\infty}{\longrightarrow} \infty\,. \tag{A_2}$$

Proposition 1.3. — *Let $(m_j)_{j\geqslant 1}$ be a sequence of integers satisfying assumption (A). Then, $H_2((m_j)_{j\geqslant 1})$ is uncountable.*

The proposition was proved by the second author [11] (see also [7, Section 4.2]) under (A_1) (notice that the condition $\inf_{j\geqslant 1} m_{j+1}/m_j \geqslant 3$ used in [11] and [7] is not restrictive for the uncountability of H_2). Actually, it is proved in [11] and [7] that H_2 supports a continuous (singular) probability measure given by a symmetric Riesz product. A proof of the uncountability of H_2 under (A_1) can also be derived from the proof of [5, Theorem 5], which states that H_1 is uncountable when $\sum_{j\geqslant 1} m_j/m_{j+1} < \infty$.

Under condition (A_2), the proposition follows from [5, Theorem 3] which states that $H_1 \subset H_2$ is uncountable. We use their argument below in the proofs of Proposition 1.5 and Theorem 2.1.

See also [1, Propositions 3 and 4] for more precise versions of Proposition 1.3.

We are now able to state our main result, which follows in a straightforward way from Proposition 1.3 and Theorem 1.2.

THEOREM 1.4. — *Let $(m_j)_{j\geqslant 1}$ be an increasing sequence of positive integers such that $m_{j+1}/m_j \geqslant 3$ for every $j \geqslant 1$, and define S as above. If assumption (A) is satisfied then S is a good sequence and it has uncountable spectrum.*

We also derive the following proposition which complements Proposition 1.1. It can be shown as an abstract consequence of the existence of a good sequence with uncountable spectrum, but we shall give explicit examples.

PROPOSITION 1.5. — *There exist a good sequence $(s_n)_{n\geqslant 1}$ and a continuous measure μ on $\mathbb{S}^1$ such that $(\frac{1}{N}\sum_{n=1}^{N}|\widehat{\mu}(s_n)|^2)_{N\geqslant 1}$ converges to some positive number.*

Proof. — We construct such a measure for each sequence S associated with a sequence $(m_j)_{j\geqslant 1}$ satisfying (A_2) and $\inf_{j\geqslant 1} m_{j+1}/m_j \geqslant 3$.

Under this assumption, choose a subsequence $(m_{j_k})_{k\geqslant 1}$ such that $j_1 > 1$ and $m_j/m_{j-1} > 2^{k+2}$ for all $j \geqslant j_k$. For every sequence $\eta = (\eta_k)_{k\geqslant 1} \in \{0,1\}^{\mathbb{N}^*}$, let

$$\theta(\eta) := \sum_{k=1}^{\infty} \frac{\eta_k}{m_{j_k}} \ .$$

Given $j \geqslant 1$, let k be the smallest integer such that $j_k > j$. Since m_j/m_{j_ℓ} is an integer when $\ell < k$, we have

$$\|m_j\theta(\eta)\| \leqslant m_j \sum_{\ell\geqslant k} \frac{1}{m_{j_\ell}} \leqslant 2\,\frac{m_j}{m_{j_k}}, \tag{1.14}$$

and in particular $\|m_j\theta(\eta)\| \leqslant 1/4$, which yields that all the terms in the products (1.7) are positive.

We also have $\sum_{j<j_k} m_j^2 < 2m_{j_k-1}^2$, so if we sum up the $\|m_j\theta(\eta)\|^2$ by blocks from j_{k-1} to $j_k - 1$ (or from 1 to $j_1 - 1$ for the first one), we get that each partial sum is less than $8(m_{j_k-1}/m_{j_k})^2$,

$$\sum_{j=1}^{\infty} \|m_j\theta(\eta)\|^2 < 8\sum_{k=1}^{\infty} \frac{m_{j_k-1}^2}{m_{j_k}^2} < \sum_{k=1}^{\infty} \frac{1}{4^k} < +\infty.$$

and $L(\theta(\eta)) > 0$ follows.

Now, let $\xi = (\xi_j)_{j\geqslant 1}$ be a sequence of i.i.d. random variables with $\mathbb{P}(\xi_1 = 0) = \mathbb{P}(\xi_1 = 1) = \frac{1}{2}$ and let μ be the probability distribution of $\mathrm{e}^{2\mathrm{i}\pi\theta(\xi)}$. Then, as the mapping $\eta \mapsto \mathrm{e}^{2\mathrm{i}\pi\theta(\eta)}$ is one-to-one, μ is a continuous probability measure concentrated on Λ_S. Moreover $\widehat{\mu}(s) = \int_{\mathbb{S}^1} \lambda^s \,\mathrm{d}\mu(\lambda) = \mathbb{E}(\mathrm{e}^{2\mathrm{i}\pi s\theta(\xi)})$ for every integer s, and thus

$$\frac{1}{N}\sum_{n=1}^{N} \widehat{\mu}(s_n) = \mathbb{E}\left(\frac{1}{N}\sum_{n=1}^{N} \mathrm{e}^{2\mathrm{i}\pi s_n\theta(\xi)}\right) \longrightarrow \mathbb{E}\big(L(\theta(\xi))\big) > 0 \quad \text{as } N \longrightarrow +\infty.$$

Finally, Proposition 1.1 ensures the convergence of $\frac{1}{N}\sum_{n=1}^{N} |\widehat{\mu}(s_n)|^2$ and the positivity of the limit follows the inequality

$$\frac{1}{N}\sum_{n=1}^{N} |\widehat{\mu}(s_n)|^2 \geqslant \left|\frac{1}{N}\sum_{n=1}^{N} \widehat{\mu}(s_n)\right|^2 . \qquad \square$$

Remark. — Under assumption (A_1) and $\inf_{j\geqslant 1} m_{j+1}/m_j \geqslant 3$, the result holds for the measure μ constructed in [11] or [7]. Indeed then μ is a generalized Riesz product, weak*-limit of products of trigonometric polynomials P_j with coefficients in blocks $\{km_j\,;\, -k_j \leqslant k \leqslant k_j\}$ and $\widehat{P}_j(m_j) = \widehat{P}_j(-m_j) = \cos(\pi/(m_j+2))$. Then for every $s = \sum_{1\leqslant j\leqslant n} \omega_j m_j$ where $|\omega_j| \leqslant k_j$ for all j, we have $\widehat{\mu}(s) = \Pi_{1\leqslant j\leqslant n} \widehat{P}_j(\omega_j m_j)$ (see [7]). From there, the convergence of $\frac{1}{N}\sum_{n=1}^{N} |\widehat{\mu}(s_n)|^2$ and the positivity of the limit can be proven as in Theorem 1.2 (we skip the details).

2. Singular asymptotic distribution

We now turn to a matter adressed by Lesigne, Quas, Rosenblatt and Wierdl in the preprint [10].

Let $S = (s_n)_{n\geqslant 1}$ be a good sequence. Let $\lambda \in \mathbb{S}^1$. Since S is good, the sequence $\big(\frac{1}{N}\sum_{n=1}^{N} \widehat{\delta}_{\lambda^{s_n}}(m)\big)_{N\in\mathbb{N}} = \big(\frac{1}{N}\sum_{n=1}^{N} \lambda^{ms_n}\big)_{N\in\mathbb{N}}$ converges towards $c(\lambda^m)$ for any integer m, that is for any character on $\mathbb{S}^1$, so that $(\frac{1}{N}\sum_{n=1}^{N} \delta_{\lambda^{s_n}})_{N\in\mathbb{N}}$ converges weakly to some probability measure $\nu_{S,\lambda}$.

Given a probability measure ν on $\mathbb{S}^1$, if there exist a good sequence S and $\lambda \in \mathbb{S}^1$ such that $\nu_{S,\lambda} = \nu$, we say according to [10] that S represents the measure ν at the point λ.

Lesigne & al. proved several interesting results concerning the measures that can be represented by a good sequence at some point $\lambda \in \mathbb{S}^1$. For instance, they proved that if λ is not a root of unity then $\nu_{S,\lambda}$ is continuous (see their Theorem 1.5). They also proved that if a given probability measure ν on $\mathbb{S}^1$ is not Rajchman (i.e. its Fourier coefficients do not vanish at infinity) then, for almost every λ with respect to the Haar measure, there does not exist any good sequence representing ν at λ (see their Theorem 1.6). On the opposite, if ν is absolutely continuous with respect to the Haar measure, then for every $\lambda \in \mathbb{S}^1$ which is not a root of unity there exists a good sequence S representing ν at λ (see their Theorem 1.8).

The above results raise the following questions. Does there exist a continuous but singular probability measure ν on $\mathbb{S}^1$ that can be represented by a good sequence? If so, can one take ν to be non Rajchman?

It turns out that Theorem 1.2 allows us to exhibit a good sequence S and a point λ such that $\nu_{S,\lambda}$ is a non Rajchman probability measure.

THEOREM 2.1. — *Let $(m_j)_{j\geqslant 1}$ be an increasing sequence of integers satisfying (A_2) and $\inf_{j\geqslant 1} m_{j+1}/m_j \geqslant 3$, and let S be the sequence associated with it. There are uncountably many $\lambda \in \Lambda_S$ such that the weak*-limit $\nu_{S,\lambda}$ of $(\frac{1}{N}\sum_{n=1}^{N}\delta_{\lambda^{s_n}})_{N\geqslant 1}$ satisfies $\limsup_{j\to+\infty}|\widehat{\nu}_{S,\lambda}(m_j)| = 1$.*

Proof. — We proceed as in the proof of Proposition 1.5, except that we require a stronger condition on the subsequence $(m_{j_k})_{k\geqslant 1}$, namely $m_j/m_{j-1} > 2^{k+2}m_{j_{k-1}}$ for all $j \geqslant j_k$ if $k > 1$.

For $\eta \in \{0,1\}^{\mathbb{N}^*}$, we still define $\theta(\eta) = \sum_{k\geqslant 1}\eta_k/m_{j_k}$. By the proof of Proposition 1.5, this yields an uncountable family of $\lambda = \mathrm{e}^{2\mathrm{i}\pi\theta(\eta)}$ in Λ_S.

For each such $\theta = \theta(\eta)$ we have $\widehat{\nu}_{S,\lambda}(m) = c(\mathrm{e}^{2\mathrm{i}\pi m\theta}) = L(m\theta)$ for all $m \in \mathbb{Z}$. So, it will be sufficient to show that $L(m_{j_n}\theta) \to 1$ as $n \to +\infty$. Clearly, from the expression of $L(\theta)$ as an infinite product, it is equivalent to prove that $\sum_{j\geqslant 1}^{\infty}\|m_{j_n}m_j\theta\|^2$ converges to 0 as $n \to +\infty$.

Fix $n > 1$. We may apply the inequality (1.14) either to $\|m_j\theta\|$ or to $\|m_{j_n}\theta\|$. For $j < j_n$ we get $\|m_{j_n}m_j\theta\| \leqslant m_j\|m_{j_n}\theta\| \leqslant 2\,m_{j_n}m_j/m_{j_{n+1}}$, and in the opposite case $\|m_{j_n}m_j\theta\| \leqslant m_{j_n}\|m_j\theta\| \leqslant 2\,m_{j_n}m_j/m_{j_k}$ where k is the smallest integer such that $j_k > j$. So,

$$\sum_{j=1}^{j_n-1}\|m_{j_n}m_j\theta\|^2 \leqslant 4\,\frac{m_{j_n}^2}{m_{j_{n+1}}^2}\sum_{j=1}^{j_n-1}m_j^2 \leqslant 8\,\frac{m_{j_n}^2}{m_{j_{n+1}}^2}\,m_{j_n-1}^2 < \frac{1}{4^n}\,.$$

For $j \geqslant j_n$, summing again by blocks from j_{k-1} to $j_k - 1$ for $k > n$, we get

$$\sum_{j_{k-1}}^{j_k-1}\|m_{j_n}m_j\theta\|^2 \leqslant 4\,\frac{m_{j_n}^2}{m_{j_k}^2}\sum_{j_{k-1}}^{j_k-1}m_j^2 \leqslant 8\,\frac{m_{j_n}^2}{m_{j_k}^2}\,m_{j_k-1}^2 < \frac{1}{4^k}\,\frac{m_{j_n}^2}{m_{j_{k-1}}^2} \leqslant \frac{1}{4^k}$$

and finally

$$\sum_{j=1}^{\infty}\|m_{j_n}m_j\theta\|^2 < \sum_{k=n}^{\infty}\frac{1}{4^k} \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \qquad \square$$

Let S and $\lambda \in \mathbb{S}^1$ be as in Theorem 2.1 and write $\nu = \nu_{S,\lambda}$.

The property $\limsup_{j\to+\infty}|\widehat{\nu}(m_j)| = 1$ means precisely that ν is a *Dirichlet measure*, see [6] and [7] for properties of Dirichlet measures.

In particular there is then a subsequence $(n_j)_{j\geqslant 1}$ such that λ^{n_j} converges towards a constant of modulus 1 in the $L^1(\nu)$ topology, and it follows that any measure absolutely continuous with respect to ν is itself a Dirichlet measure.

On the other hand, any probability measure absolutely continuous with respect to some Rajchman measure is itself a Rajchman measure.

Hence, we infer that ν is singular with respect to any Rajchman probability measure on $\mathbb{S}^1$.

This result sheds light on the problem posed by Lesigne, Lesigne, Quas, Rosenblatt and Wierdl in [10], Question 1.7: can all singular continuous Borel probability measures on $\mathbb{S}^1$ be represented by a good sequence at some $\lambda \in \mathbb{S}^1$? Failing to solve it in all generality, the following questions arise now.

Questions. — *In view of Theorem 2.1, one may wonder if it is possible to find a good sequence S and $\lambda \in \mathbb{S}^1$ such that*

$$0 < \limsup_{n\to+\infty} |\widehat{\nu}_{S,\lambda}(n)| < 1\,.$$

Another question is whether one can have $\nu_{S,\lambda}$ Rajchman and singular with respect to the Lebesgue measure.

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